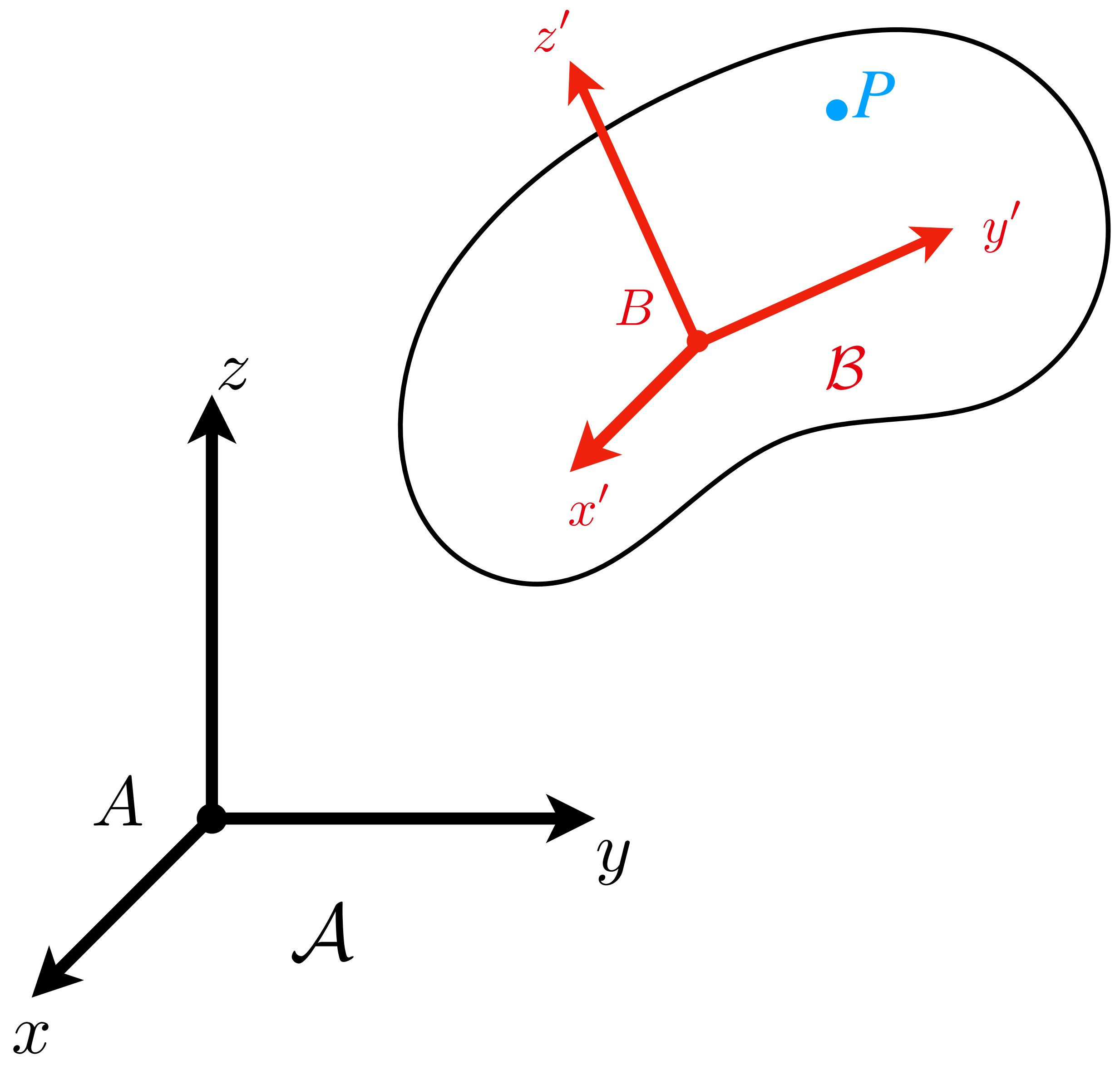
Robot Dynamics Exercise Session 02

linear and angular velocity

$_{\mathcal{A}}\vec{v}_{AP} \in \mathbb{R}^3$: relative linear velocity of Point P w.r.t. A expressed in $\mathcal{A}$

$_{\mathcal{A}}\vec{\omega}_{\mathcal{A}\mathcal{B}} \in \mathbb{R}^3$: relative angular velocity of frame $\mathcal{B}$ w.r.t. frame $\mathcal{A}$ expressed in $\mathcal{A}$



$$[_{\mathcal{A}}\vec{\omega}_{\mathcal{A}\mathcal{B}}]_\times = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} = \dot{C}_{\mathcal{A}\mathcal{B}}\, C_{\mathcal{A}\mathcal{B}}^\top \qquad \left(\text{why?} \quad \begin{bmatrix} \dot{e}_1^\top \\ \dot{e}_2^\top \\ \dot{e}_3^\top \end{bmatrix} \begin{bmatrix} e_1 & e_2 & e_3 \end{bmatrix} \right)$$

identities: $[_{\mathcal{A}}\vec{\omega}_{\mathcal{A}\mathcal{B}}]_\times^\top = -[_{\mathcal{A}}\vec{\omega}_{\mathcal{A}\mathcal{B}}]_\times$ $\qquad [_{\mathcal{A}}\vec{\omega}_{\mathcal{A}\mathcal{B}}]_\times = [C_{\mathcal{A}\mathcal{B}}\, {_{\mathcal{B}}}\vec{\omega}_{\mathcal{A}\mathcal{B}}]_\times = C_{\mathcal{A}\mathcal{B}}\, [_{\mathcal{B}}\vec{\omega}_{\mathcal{A}\mathcal{B}}]_\times\, C_{\mathcal{A}\mathcal{B}}^\top$

if $\mathcal{A}$ is an inertial frame (e.g. $\mathcal{A} \equiv \mathcal{I}$):

$_{\mathcal{A}}\vec{v}_{AP} \equiv {_{\mathcal{A}}}\vec{v}_A$: absolute linear velocity

$_{\mathcal{A}}\vec{\omega}_{\mathcal{A}\mathcal{B}} \equiv {_{\mathcal{A}}}\vec{\omega}_{\mathcal{B}}$: absolute angular velocity

behave as "expected":

change of representation	addition	negation
$_{\mathcal{A}}\vec{v}_{BP} = C_{\mathcal{A}\mathcal{B}}\, {_{\mathcal{B}}}\vec{v}_{BP}$	$\vec{v}_{AC} = \vec{v}_{AB} + \vec{v}_{BC}$	$\vec{v}_{AB} = -\vec{v}_{BA}$
$_{\mathcal{A}}\vec{\omega}_{\mathcal{A}\mathcal{B}} = C_{\mathcal{A}\mathcal{B}}\, {_{\mathcal{B}}}\vec{\omega}_{\mathcal{A}\mathcal{B}}$	$\vec{\omega}_{\mathcal{A}\mathcal{C}} = \vec{\omega}_{\mathcal{A}\mathcal{B}} + \vec{\omega}_{\mathcal{B}\mathcal{C}}$	$\vec{\omega}_{\mathcal{A}\mathcal{B}} = -\vec{\omega}_{\mathcal{B}\mathcal{A}}$

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relating velocities to time derivatives

$$\vec{v}_{BP} = \frac{d}{dt}\vec{r}_{BP}$$

$${}_{\mathcal{A}}\vec{v}_{BP} = {}_{\mathcal{A}}\left(\frac{d}{dt}\vec{r}_{BP}\right) \overset{\text{unless } \mathcal{A} \text{ is a non-rotating (or inertial frame)}}{\neq} \frac{d}{dt}\left({}_{\mathcal{A}}\vec{r}_{BP}\right) = {}_{\mathcal{A}}\dot{\vec{r}}_{BP} = \begin{bmatrix} {}_{\mathcal{A}}\dot{r}_{BP,x} \\ {}_{\mathcal{A}}\dot{r}_{BP,y} \\ {}_{\mathcal{A}}\dot{r}_{BP,z} \end{bmatrix}$$

But what is $\frac{d}{dt}\left({}_{\mathcal{A}}\vec{r}_{BP}\right)$?

=> change of $\vec{r}_{BP}$ <u>including</u> the change of $\mathcal{A}$ due to the representation in $\mathcal{A}$

Usually, ${}_{\mathcal{B}}\dot{\vec{r}}_{BP} = \frac{d}{dt}\left({}_{\mathcal{B}}\vec{r}_{BP}\right)$ is available and ${}_{\mathcal{B}}\vec{v}_{BP}$ is unknown. In the following, assume $\mathcal{A}$ is an inertial frame.

$${}_{\mathcal{B}}\vec{v}_{BP} = C_{\mathcal{BA}}\ {}_{\mathcal{A}}\vec{v}_{BP} \overset{\mathcal{A} \text{ is inertial}}{=} C_{\mathcal{BA}}\ \frac{d}{dt}\left({}_{\mathcal{A}}\vec{r}_{BP}\right) = C_{\mathcal{BA}}\ \frac{d}{dt}\left(C_{\mathcal{AB}}\ {}_{\mathcal{B}}\vec{r}_{BP}\right)$$

$$\overset{\text{product rule}}{=} C_{\mathcal{BA}}\left(\frac{d}{dt}\left(C_{\mathcal{AB}}\right){}_{\mathcal{B}}\vec{r}_{BP} + C_{\mathcal{AB}}\ \frac{d}{dt}\left({}_{\mathcal{B}}\vec{r}_{BP}\right)\right) = \underbrace{C_{\mathcal{BA}}\ \dot{C}_{\mathcal{AB}}}_{*}\ {}_{\mathcal{B}}\vec{r}_{BP} + \underbrace{C_{\mathcal{BA}}\ C_{\mathcal{AB}}}_{\mathbb{I}}\ {}_{\mathcal{B}}\dot{\vec{r}}_{BP}$$

$$= \left[{}_{\mathcal{B}}\vec{\omega}_{\mathcal{AB}}\right]_\times\ {}_{\mathcal{B}}\vec{r}_{BP} + {}_{\mathcal{B}}\dot{\vec{r}}_{BP}$$

$$* \quad C_{\mathcal{BA}}\ \dot{C}_{\mathcal{AB}} = C_{\mathcal{BA}}\underbrace{\left[{}_{\mathcal{A}}\vec{\omega}_{\mathcal{AB}}\right]_\times}_{\dot{C}_{\mathcal{AB}}\ C_{\mathcal{AB}}^\top} C_{\mathcal{AB}} = \left[C_{\mathcal{BA}}\ {}_{\mathcal{A}}\vec{\omega}_{\mathcal{AB}}\right]_\times = \left[{}_{\mathcal{B}}\vec{\omega}_{\mathcal{AB}}\right]_\times$$

$${}_{\mathcal{B}}\vec{v}_{BP} = \underbrace{{}_{\mathcal{B}}\dot{\vec{r}}_{BP}}_{\substack{\text{easy to express in } \mathcal{B} \\ \text{e.g. prismatic joint}}} + \underbrace{{}_{\mathcal{B}}\vec{\omega}_{\mathcal{AB}} \times {}_{\mathcal{B}}\vec{r}_{BP}}_{\substack{\text{contribution of} \\ \text{rotating frame } \mathcal{B}}}$$

⚠ If $\mathcal{B}$ rotates w.r.t. $\mathcal{A}$, ${}_{\mathcal{B}}\vec{v}_{BP}$ is not zero, even if frame $\mathcal{B}$ and point P have a fixed relationship

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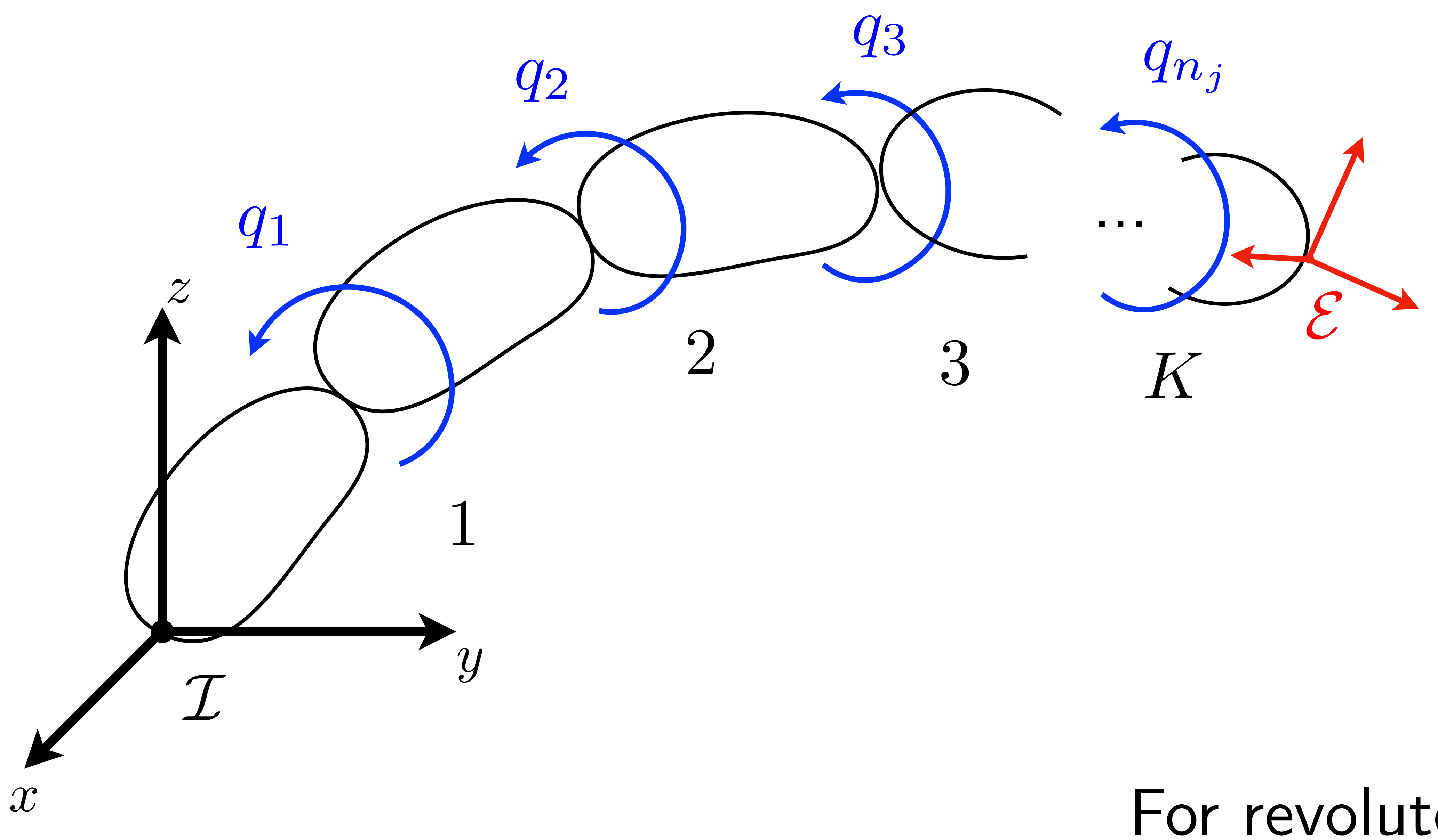
Robot Dynamics Exercise Session 02

Jacobians



$$\begin{bmatrix} {}_{\mathcal{I}}\vec{v}_{IE} \\ {}_{\mathcal{I}}\vec{\omega}_{\mathcal{I}\mathcal{E}} \end{bmatrix} = {}_{\mathcal{I}}J_{IE}\left(\vec{q}\right) \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \vdots \\ \dot{q}_{n_j} \end{bmatrix}$$

$J_{IE}\left(\vec{q}\right) \in \mathbb{R}^{6\times n_j}$ geometric Jacobian maps joint space into Euclidean velocity space

$J_{IE} = \begin{bmatrix} J_{IE,\mathcal{P}} \\ J_{IE,\mathcal{R}} \end{bmatrix}$ positional Jacobian / rotational Jacobian

For revolute joints: easy structure

rotation axis: ${}_{\mathcal{I}}\vec{n}_1 = C_{\mathcal{I}1}\ {}_1\vec{n}_1$

$J_{IE,\mathcal{R}} = \begin{bmatrix} {}_{\mathcal{I}}\vec{n}_1 & {}_{\mathcal{I}}\vec{n}_2 & {}_{\mathcal{I}}\vec{n}_3 & \dots & {}_{\mathcal{I}}\vec{n}_K \end{bmatrix}$

Why? ${}_{\mathcal{I}}\vec{\omega}_{\mathcal{I}\mathcal{E}} = {}_{\mathcal{I}}\vec{\omega}_{\mathcal{I}1} + {}_{\mathcal{I}}\vec{\omega}_{12} + \dots + {}_{\mathcal{I}}\vec{\omega}_{\mathcal{K}\mathcal{E}}$ and ${}_{\mathcal{I}}\vec{\omega}_{\mathcal{I}1} = {}_{\mathcal{I}}\vec{n}_1\ \dot{q}_1$ etc.

$J_{IE,\mathcal{P}} = \begin{bmatrix} {}_{\mathcal{I}}\vec{n}_1 \times {}_{\mathcal{I}}\vec{r}_{1E} & {}_{\mathcal{I}}\vec{n}_2 \times {}_{\mathcal{I}}\vec{r}_{2E} & {}_{\mathcal{I}}\vec{n}_3 \times {}_{\mathcal{I}}\vec{r}_{3E} & \dots & {}_{\mathcal{I}}\vec{n}_K \times {}_{\mathcal{I}}\vec{r}_{KE} \end{bmatrix}$

Why? ${}_{\mathcal{I}}\vec{v}_{IE} = {}_{\mathcal{I}}\vec{v}_{IK} + {}_{\mathcal{I}}\vec{\omega}_{\mathcal{I}\mathcal{K}} \times {}_{\mathcal{I}}\vec{r}_{KE}$, recursive.

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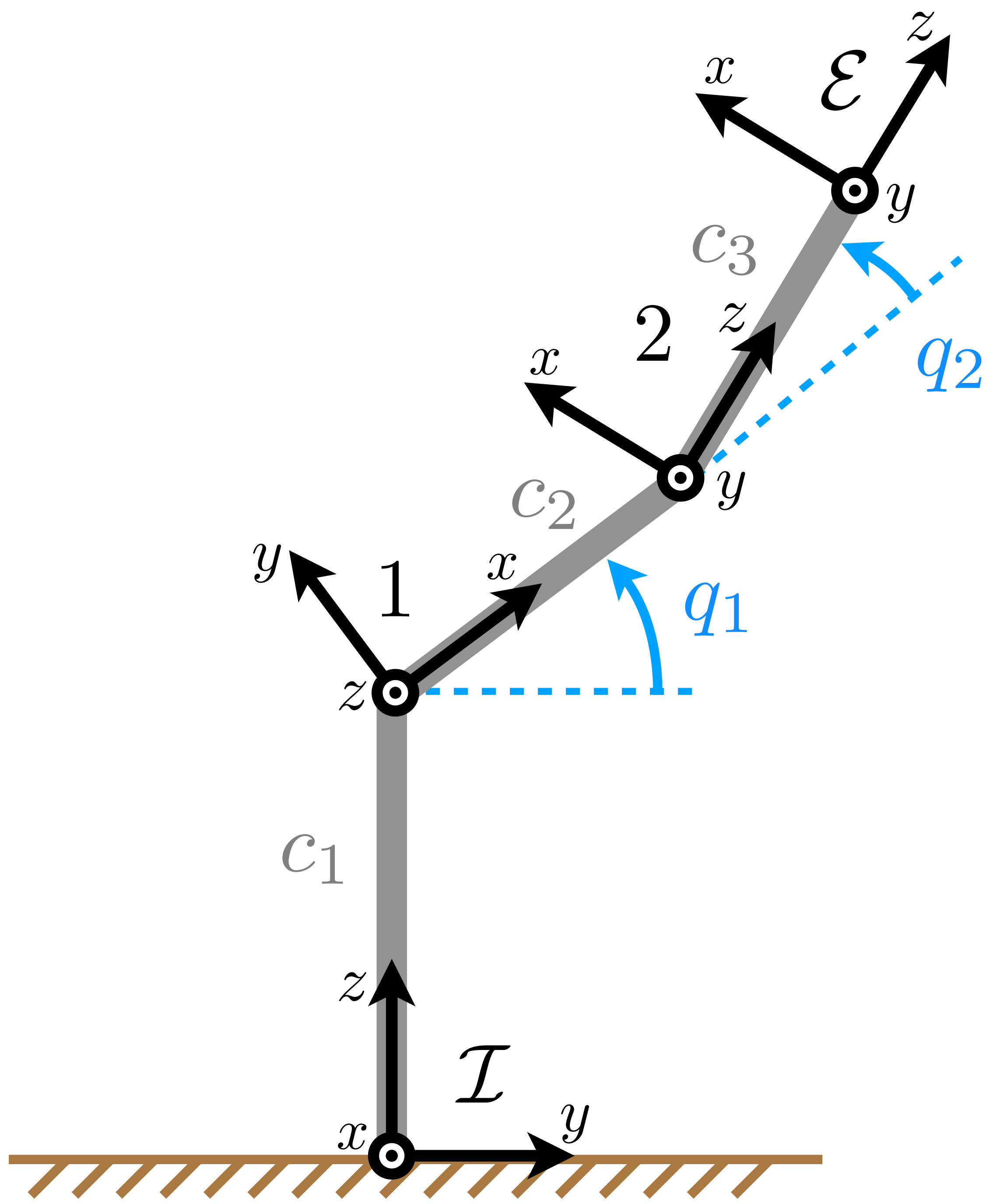
example: **differential** kinematics of a planar manipulator

given:	$T_{\mathcal{I}1},\ T_{12},\ T_{2\mathcal{E}},\ \dot{\vec{q}}$
find:	${}_\mathcal{I}\vec{v}_{IE},\ {}_\mathcal{I}\vec{\omega}_{\mathcal{I}\mathcal{E}}$
	$J_{IE,\mathcal{P}},\ J_{IE,\mathcal{R}}$



1)
$$
\begin{aligned}
{}_\mathcal{I}\vec{\omega}_{\mathcal{I}\mathcal{E}} &= {}_\mathcal{I}\vec{\omega}_{\mathcal{I}1} + {}_\mathcal{I}\vec{\omega}_{12} + {}_\mathcal{I}\vec{\omega}_{2\mathcal{E}} \\
&= C_{\mathcal{I}1}\ {}_1\vec{\omega}_{\mathcal{I}1} + C_{\mathcal{I}2}\ {}_2\vec{\omega}_{12} + \vec{0} \\
&= C_{\mathcal{I}1}\begin{bmatrix}0\\0\\\dot{q}_1\end{bmatrix} + C_{\mathcal{I}2}\begin{bmatrix}0\\\dot{q}_2\\0\end{bmatrix}
\end{aligned}
$$

2)
$$
\begin{aligned}
{}_\mathcal{I}\vec{v}_{IE} &= {}_\mathcal{I}\vec{v}_{I1} + {}_\mathcal{I}\vec{v}_{12} + {}_\mathcal{I}\vec{v}_{2E} \\
&= \vec{0} + \frac{d}{dt}\left(C_{\mathcal{I}1}\ {}_1\vec{r}_{12}\right) + \frac{d}{dt}\left(C_{\mathcal{I}2}\ {}_2\vec{r}_{2E}\right) \\
&= {}_\mathcal{I}\vec{\omega}_{\mathcal{I}1} \times {}_\mathcal{I}\vec{r}_{12} + C_{\mathcal{I}1}\ \underbrace{{}_1\dot{\vec{r}}_{12}}_{\vec{0}} + {}_\mathcal{I}\vec{\omega}_{\mathcal{I}2} \times {}_\mathcal{I}\vec{r}_{2E} + C_{\mathcal{I}2}\ \underbrace{{}_2\dot{\vec{r}}_{2E}}_{\vec{0}} \\
&= {}_\mathcal{I}\vec{\omega}_{\mathcal{I}1} \times {}_\mathcal{I}\vec{r}_{12} + {}_\mathcal{I}\vec{\omega}_{\mathcal{I}2} \times {}_\mathcal{I}\vec{r}_{2E}
\end{aligned}
$$

3)
$$
\begin{aligned}
{}_\mathcal{I}J_{IE,\mathcal{R}} &= \begin{bmatrix}{}_\mathcal{I}\vec{e}_1^{\,z} & {}_\mathcal{I}\vec{e}_2^{\,y}\end{bmatrix} \\
{}_\mathcal{I}J_{IE,\mathcal{P}} &= \begin{bmatrix}{}_\mathcal{I}\vec{e}_1^{\,z} \times {}_\mathcal{I}\vec{r}_{1E} & {}_\mathcal{I}\vec{e}_2^{\,y} \times {}_\mathcal{I}\vec{r}_{2E}\end{bmatrix}
\end{aligned}
$$

where $\quad {}_\mathcal{I}\vec{e}_1^{\,z} = {}_\mathcal{I}\vec{e}_2^{\,y} = \begin{bmatrix}1\\0\\0\end{bmatrix}$