Robot Dynamics Exercise Session 03

previous weeks: Ex. 1 a – forward kinematics $\{\vec{r}_{IE},\, C_{\mathcal{IE}}\} = \mathrm{FK}(\vec{q})$ unique mapping

Ex. 1 b – differential forward kinematics $\{\vec{v}_{IE},\, \vec{\omega}_{\mathcal{IE}}\} = \mathrm{DFK}(\vec{q},\, \dot{\vec{q}}) = J(\vec{q})\,\dot{\vec{q}}$ unique mapping

today: Ex. 1 c – inverse kinematics $\vec{q} = \mathrm{IK}\left(\vec{r}_{IE},\, C_{\mathcal{IE}}\right)$ non-unique mapping

issue: Difficult to solve analytically

solution: Resort to numerical solutions based on solving the differential inverse kinematics

1) IK & DIK

Let $\vec{q} \in \mathbb{R}^n$ be the joint variables, e.g., $q_1,\, q_2,\, \dots$

Let $\vec{\chi} \in \mathbb{R}^m$ be the task variables, e.g., $\vec{r}_{IE},\, C_{\mathcal{IE}},\, \dots$

IK problem: find $\vec{q}$ s.t. $\vec{\chi}_d = \vec{f}(\vec{q})$

DIK problem: find $\dot{\vec{q}}$ s.t. $\dot{\vec{\chi}}_d = J(\vec{q})\,\dot{\vec{q}} \longrightarrow \dot{\vec{q}} = J^{+}(\vec{q})\,\dot{\vec{\chi}}_d$

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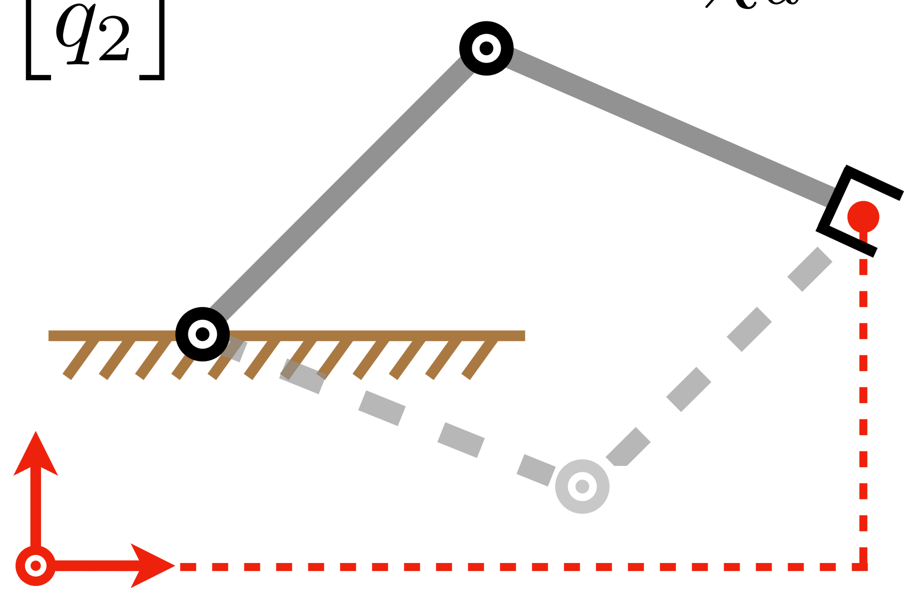
Assuming no kinematic singularities, we define the following three cases:

<u>Case 1:</u> $m = n$

$$\vec{q} = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \qquad \vec{\chi}_d = \begin{bmatrix} x_d \\ y_d \end{bmatrix}$$



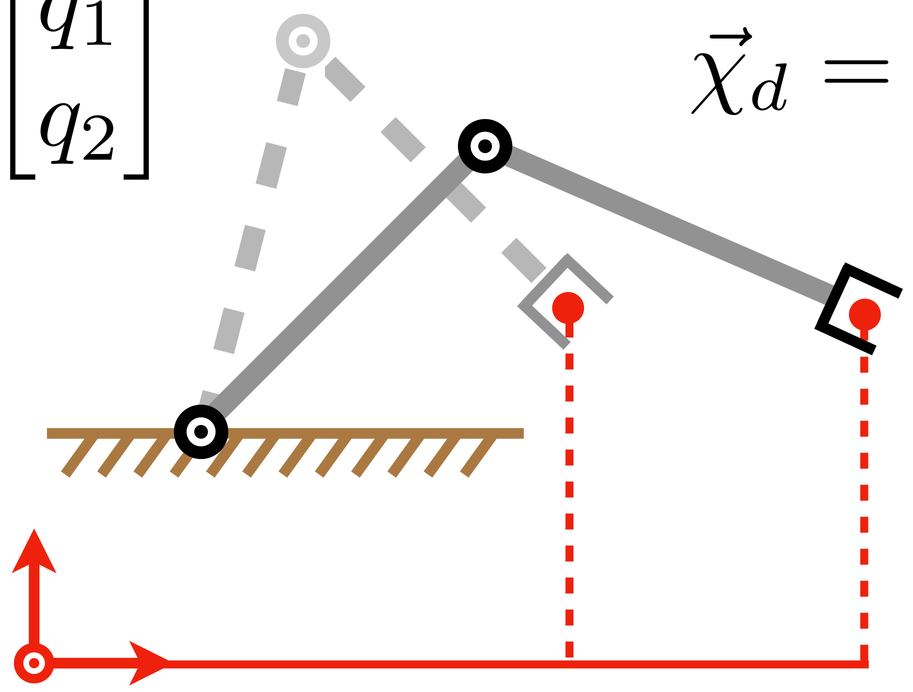
non unique but finitely many solutions for IK

$$\boxed{J^+(\vec{q}) = J^{-1}(\vec{q})}$$

J^+ is just J^{-1} as it is square

<u>Case 2:</u> $m < n$ (redundant)

$$\vec{q} = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \qquad \vec{\chi}_d = \begin{bmatrix} y_d \end{bmatrix}$$



infinitely many solutions for IK

$$\boxed{J^+ = J^\top \left(J J^\top\right)^{-1}}$$

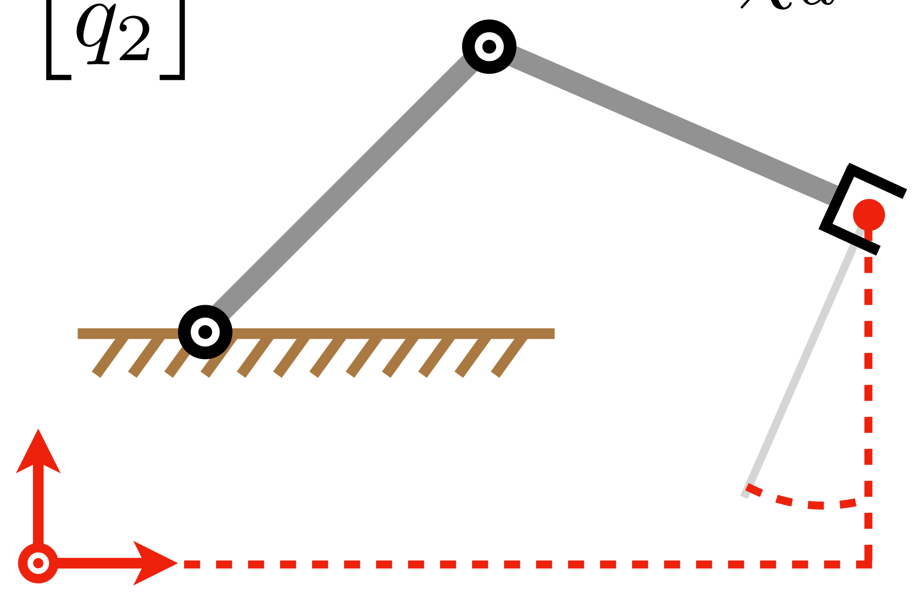
right pseudo-inverse

is constructed s.t.

$$\dot{\vec{q}}^* = \begin{cases} \underset{\dot{\vec{q}}}{\operatorname{argmin}} \ \frac{1}{2} \left\|\dot{\vec{q}}\right\|_2^2 \\ \text{s.t.} \quad \dot{\vec{\chi}}_d = J(\vec{q})\,\dot{\vec{q}} \end{cases}$$

<u>Case 3:</u> $m > n$

$$\vec{q} = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \qquad \vec{\chi}_d = \begin{bmatrix} x_d \\ y_d \\ \alpha_d \end{bmatrix}$$



no exact solution for IK

$$\boxed{J^+ = \left(J^\top J\right)^{-1} J^\top}$$

left pseudo-inverse

is constructed s.t.

$$\dot{\vec{q}}^* = \underset{\dot{\vec{q}}}{\operatorname{argmin}} \ \frac{1}{2} \left\|\dot{\vec{\chi}}_d - J\,\dot{\vec{q}}\right\|_2^2$$

For kinematics singularities: Add $\lambda^2\,\mathbb{I}$ before taking the inverse (i.e. damping)

(Jacobian will have linearly dependent columns in case of singularity)

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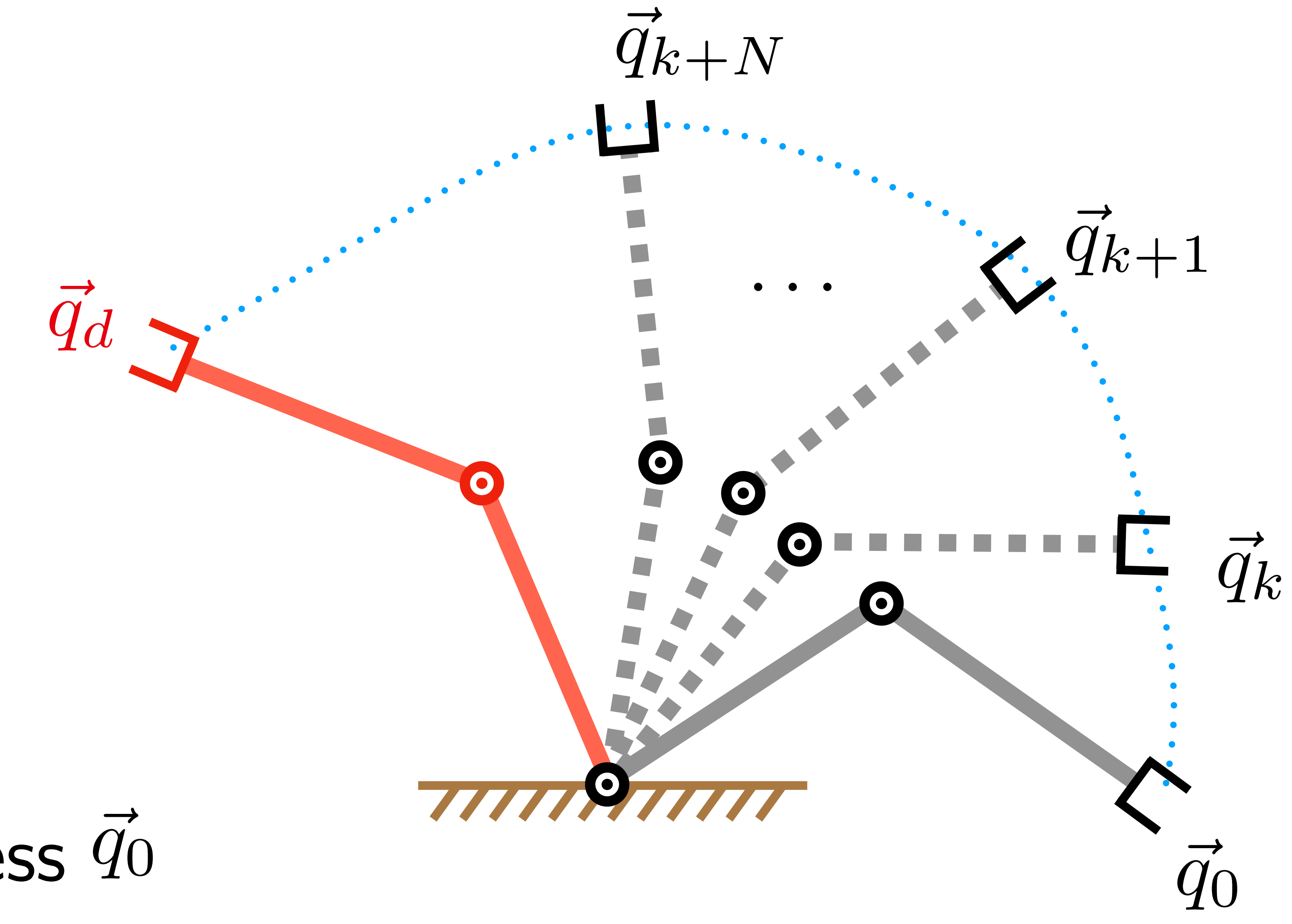
2) Iterative Inverse Kinematics

Solving IK is equivalent to finding the roots of $\vec{g}(\vec{q}) = \vec{\chi}_d - \vec{f}(\vec{q}) = \vec{0}$

Use of numerical technique analogous to Newton's method:



$$\vec{q}_{k+1} = \vec{q}_k - \left(\frac{\partial \vec{g}}{\partial \vec{q}}\Big|_{\vec{q}_k}\right)^+ \vec{g}(\vec{q}_k)$$

$$\vec{q}_{k+1} = \vec{q}_k + \left(\frac{\partial \vec{f}}{\partial \vec{q}}\Big|_{\vec{q}_k}\right)^+ \left(\vec{\chi}_d - \vec{f}(\vec{q}_k)\right)$$

for which we need an initial guess $\vec{q}_0$
and a stopping criterion $\|\vec{g}(\vec{q}_d)\|_2 \leq \varepsilon$

$$\vec{q}_{k+1} = \vec{q}_k + J_A^+(\vec{q}_k) \underbrace{\left(\vec{\chi}_d - \vec{f}(\vec{q}_k)\right)}_{\Delta\vec{\chi}_{\mathcal{I}\mathcal{E}}}$$

pose error ⚠

⚠ How to properly compute the pose error: $\Delta\vec{\chi}_{\mathcal{I}\mathcal{E}} = \begin{bmatrix} {}_I\vec{r}_{IE^*} - {}_I\vec{r}_{IE}(\vec{q}) \\ \underbrace{\Phi_{\mathcal{I}\mathcal{E}^*} \boxminus \Phi_{\mathcal{I}\mathcal{E}}(\vec{q})}_{{}_\mathcal{I}\varphi_{\mathcal{E}\mathcal{E}^*}} \end{bmatrix} \propto \begin{bmatrix} {}_\mathcal{I}\vec{v}_{IE} \\ {}_\mathcal{I}\vec{\omega}_{\mathcal{I}\mathcal{E}} \end{bmatrix}$ => use J_0 instead of J_A

$\boxminus$: Boxminus: shortest "path" between orientations (minus does not work in orientation space)

Given $C_{\mathcal{I}\mathcal{E}}, C_{\mathcal{I}\mathcal{E}^*}$:

$$\begin{aligned} {}_\mathcal{E}\vec{\varphi}_{\mathcal{E}\mathcal{E}^*} &= \mathrm{RotMat2RotVec}(C_{\mathcal{E}\mathcal{E}^*}) \\ {}_\mathcal{I}\vec{\varphi}_{\mathcal{E}\mathcal{E}^*} &= C_{\mathcal{I}\mathcal{E}}\, {}_\mathcal{E}\vec{\varphi}_{\mathcal{E}\mathcal{E}^*} = \mathrm{RotMat2RotVec}(C_{\mathcal{I}\mathcal{E}^*}\, C_{\mathcal{I}\mathcal{E}}^\top(\vec{q})) \end{aligned}$$

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3) Kinematics Motion Control

Given a reference trajectory defined by $\vec{\chi}_d(t) = \begin{bmatrix} {}_{\mathcal{I}}\vec{r}_{IE^*} \\ \Phi_{\mathcal{I}\mathcal{E}^*} \end{bmatrix}$, $\vec{w}_d(t) = \begin{bmatrix} {}_{\mathcal{I}}\vec{v}_{IE^*} \\ {}_{\mathcal{I}}\vec{\omega}_{\mathcal{I}\mathcal{E}^*} \end{bmatrix}$

Control law $\dot{\vec{q}} = J^+ \underbrace{\begin{bmatrix} {}_{\mathcal{I}}\vec{v}_{IE} \\ {}_{\mathcal{I}}\vec{\omega}_{\mathcal{I}\mathcal{E}} \end{bmatrix}}_{\vec{w}(t)}$, where $\vec{w}(t) = \vec{w}_d(t) + \underbrace{K_p}_{>0} \Delta\vec{\chi}_{\mathcal{I}\mathcal{E}}$

we require feed forward reference if the desired pose is not constant

relates "changes": $\dot{(\,)} \sim \Delta$

Reason why this works: error dynamics $\Delta\dot{\vec{\chi}}_{\mathcal{I}\mathcal{E}} + K_p \Delta\vec{\chi}_{\mathcal{I}\mathcal{E}} = \vec{0}$

$\Delta\vec{\chi}_{\mathcal{I}\mathcal{E}} \rightarrow \vec{0}$

$\vec{\chi}_{\mathcal{I}\mathcal{E}}(t) \rightarrow \vec{\chi}_d(t)$

as $t \rightarrow \infty$