Theorem: Let f be a C^n function on $(-a, a)$ for some $\alpha > 0$. Then for all $x \in (-a, a)$ we have that

$$f(x) = \sum_{j=0}^{n-1} \frac{x^j}{j!} f^{(j)}(x) + \int_0^x \frac{(x-y)^{n-1}}{(n-1)!} f^{(n)}(y) dy \tag{1}$$

Proof: Let $g(x)$ be the integral in (1). So long as $n - 1 > 0$, the integrand vanishes at $y = x$, so there is no contribution from the change of upper limit, that is,

$$g'(x) = \int_0^x \frac{(x-y)^{n-2}}{(n-2)!} f^{(n)}(y) dy \tag{2}$$

Iterating, we get for $j = 1, 2, \ldots, n-1$,

$$\varphi^{(j)}(x) = \int_0^x \frac{(x-y)^{n-1-j}}{(n-1-j)!} f^{(n)}(y) dy \tag{3}$$

Finally, at $j = n - 1$, the integrand is x-independent, so

$$\varphi^{(n)}(x) = f^{(n)}(x) \tag{4}$$

Iterating the fact that $f^{(j)} = 0 \Rightarrow f = $ constant, we see that $(f - g)(x)$ is a polynomial of degree at most $n - 1$. Since (3) implies $\varphi(0) = \varphi'(0) = \cdots = \varphi^{(n-1)}(0) = 0$, we see the polynomial is the sum.

Recommended Reading

- **Basic Analysis I** by James K. Peterson (Match: 0.66)

- **Analysis I** by H. Amann (Match: 0.65)

- **Introduction to Mathematical Analysis** by Igor Kriz, Aleš Pultr (Match: 0.65)