

Theorem: A continuous function f defined on an interval $[a, b]$ takes on absolute minimum and absolute maximum values.

For some $x_0, x_1 \in [a, b]$ and for all $x \in [a, b]$ we have

$$f(x_0) \leq f(x) \leq f(x_1)$$

Proof:

- $M = \text{least upper bound from } f(t) \text{ as } t \text{ varies in } [a, b]$

$$M = \sup\{f(x) : x \in [a, b]\}$$

This M exists, because M is bounded and not empty.

- Consider the set $X = \{x \in [a, b] : \sup V_x < M\}$, with

$$V_x = \{y \in \mathbb{R} : \text{for some } t \in [a, x] \text{ we have } y = f(t)\}$$

- $-V_x$ is the set of all images of f from a until x .
- $-X$ is the set of all points in $[a, b]$, such that all elements from a to x have images under f that are smaller than M , more so even $\sup V_x < M$.

- Case 1: $f(a) = M$. Then f takes on a maximum at a and the theorem is proved.
- Case 2: $f(a) < M$. Then $a \in X$ and therefore $X \neq \emptyset$ and because X is bounded by b , X has a least upper bound. Define $c = \sup X$.

If $f(c) < M$, we choose $\varepsilon > 0$ with $\varepsilon < M - f(c)$. By continuity at c , there exists a $\delta > 0$ such that

$$|t - c| < \delta \implies |f(t) - f(c)| < \varepsilon$$

Thus $\sup V_c < M$. If $c < b$ this implies there exist points t to the right of c at which $\sup V_t < M$, contrary to the fact that c is the sup of such points. Therefore $c = b$, which implies that $M < M$, a contradiction. So $f(c)$ is not less than M , and so $f(c) = M$, and f assumes its maximum at c .

Recommended Reading

- **Writing Proofs in Analysis** by Jonathan M. Kane (Match: 0.68)
- **Problems in real analysis** by Teodora-Liliana Rădulescu (Match: 0.67)
- **Mathematical Analysis** by Tom M. Apostol (Match: 0.67)