Continuity 1

Definitions and Propositions

Definition: Let $f\colon X \to Y$ be a function between metric spaces (X, d_X) and (Y, d_Y). Then f is continuous at $x_0 \in X$ if for each neighborhood V of $f(x_0)$ in Y, there is a neighborhood U of x_0 in X such that $f(U) \subseteq V$.

The function $f\colon X \to Y$ is continuous if it is continuous at each point of X.

Proposition: (ε, δ) A function $f\colon X \to Y$ is continuous at $x_0 \in X$ if and only if for each $\varepsilon > 0$, there is some $\delta := \delta(x_0, \varepsilon) > 0$ with the property that

$$d(f(x), f(x_0)) < \varepsilon \quad \forall x \in X\colon d(x_0, x) < \delta \quad (*)$$

Proof

Proof:

1. Let f be (ε, δ)-continuous and $\varepsilon > 0$. By the definition of continuity, $B_Y(f(x_0), \varepsilon)$ is a neighborhood of $f(x_0)$. Then there exists a U of x_0 such that $f(U) \subseteq B_Y(f(x_0), \varepsilon)$, but for this U exists a $\delta > 0$ such that $B_X(x_0, \delta) \subset U$ so we have

$$f(B_X(x_0, \delta)) \subseteq f(U) \subseteq B_Y(f(x_0), \varepsilon)$$

which translates to the statement $(*)$.

2. Now suppose $(*)$ is true, and $V \in \mathcal{N}(f(x_0))$. Choose a $\varepsilon > 0$ such that $B_Y(f(x_0), \varepsilon) \subset V$, then with $(*)$ we have $U := B_X(x_0, \delta)$ with $f(U) \subseteq B_Y(f(x_0), \varepsilon) \subset V$.

[width=0.8]diagram.png

Recommended Reading

- **Writing Proofs in Analysis** by Jonathan M. Kane (Match: 0.70)
- **Basic Analysis I** by James K. Peterson (Match: 0.68)
- **Real Analysis** by Patrick Fitzpatrick (Match: 0.67)