Double Series

Consider a function $X : \mathbb{N} \times \mathbb{N} \to E$, we abbreviate $X(i,k)$ by X_{iK}. The set $\mathbb{N} \times \mathbb{N}$ is countable, that is, there is a bijection $\alpha : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$. If α is such a bijection, $\sum_n X_{\alpha(n)}$ is an ordering of the double series $\sum_{i,k} X_{ik}$.

If we fix $j \in \mathbb{N}$ (or $k \in \mathbb{N}$), then $\sum_k X_{jk}$ (or $\sum_j X_{jk}$) is called the j^{th} row series (or j^{th} column series) of $\sum_{j,k} X_{jk}$.

If every row series (or column series) converges, then we consider the series of row sums $\sum_j \left(\sum_{k=0}^{\infty} X_{jk}\right)$ (or the series of column sums $\sum_k \left(\sum_{j=0}^{\infty} X_{jk}\right)$).

The double series $\sum_{j,k} X_{jk}$ is summable if $\sup_{n\in\mathbb{N}} \sum_{j,k=0}^{n} |X_{jk}| < \infty$.

Theorem: Let $\sum_{j,k} X_{jk}$ be a summable double series.

i) Every ordering $\sum_n X_{\alpha(n)}$ of $\sum_{j,k} X_{jk}$ converges absolutely to a value $s \in E$ which is independent of α.

ii) The series of row sums $\sum_j \left(\sum_{k=0}^{\infty} X_{jk}\right)$ and column sums $\sum_k \left(\sum_{j=0}^{\infty} X_{jk}\right)$ converges absolutely, and

$$\sum_{j=0}^{\infty} \left(\sum_{k=0}^{\infty} X_{jk} \right) = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{\infty} X_{jk} \right) = S$$

Proof: Set $M = \sup_{n\in\mathbb{N}} \sum_{j,k=0}^{n} |X_{jk}| < \infty$. Let $\alpha : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ be a bijection and $N \in \mathbb{N}$.

Then there is some $K \in \mathbb{N}$ such that

$$\{\alpha(0), \ldots, \alpha(N)\} \subseteq \{(0,0), (1,0), \ldots, (K,0), \ldots, (0,K), \ldots, (K,K)\}$$

This implies

$$\sum_{n=0}^{N} |X_{\alpha(n)}| \leq \sum_{j,k=0}^{K} |X_{jk}| \leq M$$

and therefore $\sum_n X_{\alpha(n)}$ is absolutely convergent.

Now let $\beta : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ be an other bijection. Then $\sigma := \alpha^{-1} \circ \beta$ is a permutation of $\mathbb{N}$. Set $Y_m := X_{\beta(m)}$ for all $m \in \mathbb{H}$. Then

$$Y_{\sigma(n)} = X_{\beta(\sigma(n))} = X_{\alpha(n)} \quad n \in \mathbb{N}$$

Thus $\sum_n X_{\beta(n)}$ is a rearrangement of $\sum_n X_{\alpha(n)}$. Since we already know that $\sum_n X_{\alpha(n)}$ converges absolutely, the rest of the statement follows from the rearrangement theorem.

Recommended Reading

- **Writing Proofs in Analysis** by Jonathan M. Kane (Match: 0.68)
- **Real Analysis** by Patrick Fitzpatrick (Match: 0.68)
- **Analysis I** by H. Amann (Match: 0.68)