

**Definition:** A linear map is a function  $T : V \rightarrow W$ , with the following properties:

1. For all  $u, v \in V$ :  $T(u + v) = T(u) + T(v)$
2. For all  $v \in V$  and  $\lambda \in K$ :  $T(\lambda v) = \lambda T(v)$

**Linear Map Lemma:** If  $v_1, \dots, v_n$  is a basis of  $V$ , and  $w_1, \dots, w_n \in W$ , then there exists a unique linear map  $f$  from  $V$  to  $W$  with  $f(v_i) = w_i$ , for all  $i \in \{1, \dots, n\}$ .

**Proof:** Every  $v \in V$  can be written as  $v = v_1 c_1 + \dots + v_n c_n$ . We define  $f : V \rightarrow W$  such that  $f(v) = c_1 w_1 + \dots + c_n w_n$ . If we take  $c_i = 1$  and  $c_j = 0$  for  $j \neq i$ , we have  $f(v_i) = w_i$ , so  $f$  fulfills the hypothesis. We now show that  $f$  is a linear function: With  $v = v_1 c_1 + \dots + v_n c_n$  and  $u = v_1 \bar{c}_1 + \dots + v_n \bar{c}_n$ , we have

$$\begin{aligned} f(v + u) &= w_1(c_1 + \bar{c}_1) + \dots + w_n(c_n + \bar{c}_n) = w_1 c_1 + \dots + w_n c_n + w_1 \bar{c}_1 + \dots + w_n \bar{c}_n \\ &= f(v) + f(u) \end{aligned}$$

And with  $\lambda \in K$ ,

$$f(\lambda v) = w_1 \lambda c_1 + \dots + w_n \lambda c_n = \lambda(w_1 c_1 + \dots + w_n c_n) = \lambda f(v).$$

The uniqueness of  $f$ : Let  $\tilde{f}$  be a linear function with the  $f(v_i) = w_i$  property. Then because  $v_1, \dots, v_n$  is a basis, we have

$$f(v) = f(c_1 v_1 + \dots + c_n v_n) = f(v_1) c_1 + \dots + f(v_n) c_n = w_1 c_1 + \dots + w_n c_n.$$

This means  $f = \tilde{f}$  for all  $v \in V$ .

**Definition:** If  $V$  and  $W$  are vector spaces, we define  $\mathcal{L}(V, W)$  as the set of all linear functions from  $V$  to  $W$ . Suppose  $S, T \in \mathcal{L}(V, W)$  and  $\lambda \in F$ . We define the sum  $S + T$  and the product  $\lambda T$  by:

$$\begin{aligned} (S + T)(v) &= S(v) + T(v) \text{ and} \\ (\lambda T)(v) &= \lambda T(v). \end{aligned}$$

Then  $\mathcal{L}(V, W)$  is a vector space.

## Recommended Reading

- **Lineare Algebra 1** by Menny-Akka (Match: 0.71)
- **Prüfungstraining Lineare Algebra : Band I** by Thomas C. T. Michaels (Match: 0.70)
- **Lineare Algebra** by Dirk Werner (Match: 0.70)