Series 1

E stands for a Banach space $(E, \|\cdot\|)$. $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$.
Definition: Let (x_k) be a sequence in E. Then $S_n := \sum_{k=0}^{n} x_k$ for $n \in \mathbb{N}$ defines a new sequence (S_n) in E, called the series in E. The element S_n is called the n-th partial sum, and x_n is called the n-th summand of the series $\sum x_k$.

The series $\sum x_k$ converges if (S_n) converges. The limit of (S_n) is called the value of the series $\sum x_k$ and is written $\sum_{k=0}^{\infty} x_k$. The series diverges if the sequence (S_n) diverges in E.
Proposition: If the series $\sum x_k$ converges, then (x_k) is a null sequence. **Proof:** Because (S_n) converges, (S_n) is a Cauchy sequence: $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ such that $|S_n - S_m| < \varepsilon$ for all $m, n \geq N$. In particular, $|S_{n+1} - S_n| = |\sum_{k=0}^{n+1} x_k - \sum_{k=0}^{n} x_k| = |x_{n+1}| < \varepsilon$ for large enough n.

Convergence Tests: Theorem (Cauchy criterion): For a series $\sum x_k$ in E the following are equivalent:

1. $\sum x_k$ converges.
2. For each $\varepsilon > 0$, there is some $N \in \mathbb{N}$ such that $|\sum_{k=n+1}^{m} x_k| < \varepsilon$ for $m > n \geq N$.

Proof: $S_m - S_n = \sum_{k=n+1}^{m} x_k$. Thus (S_n) is a Cauchy sequence in E if and only if (ii) is true, and in a Banach space a Cauchy sequence converges.
Theorem: Let $\sum x_k$ be a series in $\mathbb{R}$ such that $x_k > 0$ for all $k \in \mathbb{N}$. Then $\sum x_k$ converges if and only if (S_n) is bounded. In this case, the series has the value $\sup_{n \in \mathbb{N}} S_n$. **Proof:** Since the summands are nonnegative, the sequence (S_n) is increasing, by the Bolzano-Weierstrass theorem (S_n) only converges if (S_n) is bounded.
Definition: The series $\sum x_k$ converges absolutely or is absolutely convergent if $\sum |x_k|$ converges in $\mathbb{R}$ (that is $\sum_{k=0}^{\infty} |x_k| < \infty$).

Recommended Reading

- **Analysis I** by H. Amann (Match: 0.71)
- **Writing Proofs in Analysis** by Jonathan M. Kane (Match: 0.70)
- **Analysis in Banach Spaces : Volume I** by Tuomas Hytönen, Jan van Neerven, Mark Veraar, Lutz Weis (Match: 0.70)