Definition: For $T \in \mathcal{L}(V, W)$, the null space of T is the subset of V with $T(v) = 0$.
null $T = \{v \in V \mid T(v) = 0\}$.

Definition: A function $T : V \to W$ is called injective if $T(v) = T(w)$ implies $v = w$, or equivalently $v \neq w \implies T(v) \neq T(w)$.

Definition: For $T \in \mathcal{L}(V, W)$, the range of T is the subset of W consisting of vectors that are equal to $T(v)$ for a $v \in V$.
range $T = \{w \in W \mid Tv = w\}$

Definition: A function $T : V \to W$ is called surjective if $W =$ range T.

Theorem: Suppose $T \in \mathcal{L}(V, W)$, then null T is a subspace of V.
Proof: Because of $T(0) = 0$, 0 is an element of null T. For $v, w \in$ null T we have $T(v + w) = T(v) + T(w) = 0 + 0 = 0$, so $v + w \in$ null T. And with $\lambda \in K$ we get $T(\lambda w) = \lambda T(w) = \lambda 0 = 0$, so λv is also an element of null T, and null T therefore is a subspace of V.

Theorem: For $T \in \mathcal{L}(V, W)$, null $T = \{0\}$ if and only if T is injective.
Proof: Suppose T is injective and $v \in V$ is an element of null T, then $T(v) = 0 = T(0)$ and because of injectivity $v = 0$. Therefore null $T = \{0\}$.

Suppose now null $T = \{0\}$ and $T(v) = T(w)$ for two elements of V, then $T(v) - T(w) = T(v - w) = 0$. This means $v - w \in$ null T, but so $v - w$ must be 0, and this yields $v = w$, therefore T is injective.

Lemma (because we used it, twice already): $0 \in$ null T.
Proof: $T(0) = T(v - v) = T(v) - T(v) = 0$.

Theorem: Suppose $T \in \mathcal{L}(V, W)$, then range T is a subspace of W.
Proof: 1) $0 \in$ range W because $T(0) = 0$, if $w_1, w_2 \in$ range T, then there exist v_1, v_2 with $T(v_1) = w_1$ and $T(v_2) = w_2$ and so $T(v_1 + v_2) = T(v_1) + T(v_2) = w_1 + w_2$, therefore $w_1 + w_2 \in$ range W, if $w \in$ range T and $\lambda \in K$, then there exist a $v \in V$ with $T(v) = w$, and so $T(\lambda v) = \lambda T(v) = \lambda w$ and $\lambda w \in$ range T.

Recommended Reading

- **Prüfungstraining Lineare Algebra : Band I** by Thomas C. T. Michaels (Match: 0.73)
- **Lineare Algebra 1** by Menny-Akka (Match: 0.73)
- **Prüfungstraining Lineare Algebra : Band II** by Thomas C. T. Michaels, Marcel Liechti (Match: 0.71)