Linear Dependence of Vectors Shifted by w

February 11, 2026

Problem 12

Suppose $\{V_1, \ldots, V_m\}$ is linearly independent in V and $w \in V$.

Prove that if $\{V_1+w, \ldots, V_m+w\}$ is linearly dependent, then $w \in \text{span}(\{V_1, \ldots, V_m\})$.

Solution:

Assume $\{V_1 + w, \ldots, V_m + w\}$ is linearly dependent.

There exist $a_1, \ldots, a_m \in \mathbb{F}$, not all zero, such that:

$$a_1(V_1 + w) + \ldots + a_m(V_m + w) = 0$$

Rearranging the terms:

$$(a_1 + \ldots + a_m)w = -a_1V_1 - \ldots - a_mV_m$$

If $(a_1 + \ldots + a_m) = 0$, then $0 = -a_1V_1 - \ldots - a_mV_m$. This is impossible because $\{V_1, \ldots, V_m\}$ is linearly independent (since this implies $a_1 = a_2 = \ldots = a_m = 0$, which contradicts the assumption that not all a_i are zero).

Therefore, $a_1 + \ldots + a_m \neq 0$. Let $b = a_1 + \ldots + a_m$. Since $b \neq 0$, we can divide by b:

$$w = -\frac{a_1}{b}V_1 - \ldots - \frac{a_m}{b}V_m$$

And thus $w \in \text{span}(\{V_1, \ldots, V_m\})$.

Recommended Reading

- **Linear algebra problem book** by Paul R. Halmos (Match: 0.69)
- **Prüfungstraining Lineare Algebra : Band I** by Thomas C. T. Michaels (Match: 0.68)
- **Linear Algebra** by Seymour Lipschutz (Match: 0.68)