(b) Now let $\{\, O_\alpha \;;\; \alpha \in \mathsf{A} \,\}$ be an open cover of K. Suppose, contrary to our claim, that there is no finite subcover of $\{\, O_\alpha \;;\; \alpha \in \mathsf{A} \,\}$. Since K is totally bounded, for each $k \in \mathbb{N}^\times$, there is a finite set of open balls of radius $1/k$ and center in K which forms a cover of K. Then one of these open balls, B_k say, has the property that no finite subset of $\{\, O_\alpha \;;\; \alpha \in \mathsf{A} \,\}$ is a cover of $K \cap B_k$. Let x_k be the center of B_k for $k \in \mathbb{N}^\times$. By hypothesis, the sequence (x_k) has a cluster point $\overline{x}$ in K.

Now let $\overline{\alpha} \in \mathsf{A}$ be such that $\overline{x} \in O_{\overline{\alpha}}$. Since $O_{\overline{\alpha}}$ is open, there is some $\varepsilon > 0$ such that $\mathbb{B}(\overline{x}, \varepsilon) \subseteq O_{\overline{\alpha}}$. Since $\overline{x}$ is a cluster point of the sequence (x_k), there is some $M > 2/\varepsilon$ such that $d(x_M, \overline{x}) < \varepsilon/2$. Thus, for each $x \in B_M$, we have

$$ d(x, \overline{x}) \leq d(x, x_M) + d(x_M, \overline{x}) < \frac{1}{M} + \frac{\varepsilon}{2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \;, $$

that is, $B_M \subseteq \mathbb{B}(\overline{x}, \varepsilon) \subseteq O_{\overline{\alpha}}$. This contradicts our choice of B_M and so the cover $\{\, O_\alpha \;;\; \alpha \in \mathsf{A} \,\}$ must have a finite subcover. ∎

Sequential Compactness

We say that a subset $K \subseteq X$ is **sequentially compact** if every sequence in K has a subsequence which converges to an element of K.

The relationship between the cluster points of a sequence and convergent subsequences (see Proposition II.1.17) makes possible a reformulation of Theorem 3.3 in terms of sequential compactness.

3.4 Theorem *A subset of a metric space is compact if and only if it is sequentially compact.*

As an important application of Theorem 3.3 we describe the compact subsets of $\mathbb{K}^n$.

3.5 Theorem (Heine-Borel) *A subset of $\mathbb{K}^n$ is compact if and only if it is closed and bounded. In particular, an interval is compact if and only if it is closed and bounded.*

Proof   By Proposition 3.2, any compact set is closed and bounded. The converse follows from the Bolzano-Weierstrass theorem (see Theorem II.5.8), Proposition 2.11 and Theorem 3.4. ∎

Continuous Functions on Compact Spaces

The following theorem shows that compactness is preserved under continuous functions.

3.6 Theorem *Let X and Y be metric spaces and $f : X \to Y$ continuous. If X is compact, then $f(X)$ is compact. That is,* continuous images of compact sets are compact.

Proof Let $\{ O_\alpha \; ; \; \alpha \in \mathsf{A} \}$ be an open cover of $f(X)$ in Y. By Theorem 2.20, for each $\alpha \in \mathsf{A}$, $f^{-1}(O_\alpha)$ is an open subset of X. Hence $\{ f^{-1}(O_\alpha) \; ; \; \alpha \in \mathsf{A} \}$ is an open cover of the compact space X and there are $\alpha_0, \ldots, \alpha_m \in \mathsf{A}$ such that $X = \bigcup_{k=0}^{m} f^{-1}(O_{\alpha_k})$. It follows that $f(X) \subseteq \bigcup_{k=0}^{m} O_{\alpha_k}$, that is, $\{O_{\alpha_0}, \ldots, O_{\alpha_m}\}$ is a finite subcover of $\{ O_\alpha \; ; \; \alpha \in \mathsf{A} \}$. Hence $f(X)$ is compact. ∎

3.7 Corollary *Let X and Y be metric spaces and $f : X \to Y$ continuous. If X is compact, then $f(X)$ is bounded.*

Proof This follows directly from Theorem 3.6 and Proposition 3.2. ∎

The Extreme Value Theorem

For real valued functions, Theorem 3.6 has the important consequence that a real valued continuous function on a compact set attains its minimum and maximum values.

3.8 Corollary (extreme value theorem) *Let X be a compact metric space and $f : X \to \mathbb{R}$ continuous. Then there are $x_0, x_1 \in X$ such that*

$$f(x_0) = \min_{x \in X} f(x) \quad and \quad f(x_1) = \max_{x \in X} f(x) \; .$$

Proof From Theorem 3.6 and Proposition 3.2 we know that $f(X)$ is closed and bounded in $\mathbb{R}$. Thus $m := \inf\big(f(X)\big)$ and $M := \sup\big(f(X)\big)$ exist in $\mathbb{R}$. By Proposition I.10.5, there are sequences (y_n) and (z_n) in $f(X)$ which converge to m and M in $\mathbb{R}$. Since $f(X)$ is closed, Proposition 2.11 implies that m and M are in $f(X)$, that is, there are $x_0, x_1 \in X$ such that $f(x_0) = m$ and $f(x_1) = M$. ∎

The importance of this result can be seen in the following examples.

3.9 Examples **(a)** *All norms on $\mathbb{K}^n$ are equivalent.*

Proof (i) Let $|\cdot|$ be the Euclidean norm and $\|\cdot\|$ an arbitrary norm on $\mathbb{K}^n$. Then it suffices to show the equivalence of these two norms, that is, the existence of a positive constant C such that

$$C^{-1} |x| \leq \|x\| \leq C |x| \; , \qquad x \in \mathbb{K}^n \; . \tag{3.1}$$

(ii) Set $S := \{ x \in \mathbb{K}^n \; ; \; |x| = 1 \}$. From Example 1.3(j) we know that the function $|\cdot| : \mathbb{K}^n \to \mathbb{R}$ is continuous, and so, by Example 2.22(a), S is closed in $\mathbb{K}^n$. Of course, S is also bounded in $\mathbb{K}^n$. By the Heine-Borel theorem, S is a compact subset of $\mathbb{K}^n$.