19) Let V be a finite dimensional vector space. $T \in \mathcal{L}(V, W)$

$$T \text{ injective} \iff \exists S \in \mathcal{L}(W, V) : ST = \mathrm{id}_V$$

Proof:

$\Longrightarrow$ because T is injective: $\dim V = \dim \operatorname{range}(T) \leq \dim W$, so V is finite dimensional too.

Choose a basis of V: $B_V = \{v_1, \ldots, v_n\}$, because T is injective $T(v_1), \ldots, T(v_n)$ are linearly independent, and we can extend that list to a basis of W: $\{T(v_1), \ldots, T(v_n), w_1, \ldots, w_m\}$. Define $S \in \mathcal{L}(W, V)$:

$$S(T(v_i)) = v_i, \quad S(T(v_n)) = v_n, \quad S(w_i) = 0$$

$$\ldots S(w_m) = 0$$

S is linear and unique.

For $v \in V$:

$$ST(v) = ST\left(a_1 v_1 + \cdots + a_n v_n\right) =$$

$$S\left(a_1 T(v_1) + \cdots + a_n T(v_n)\right) =$$

$$a_1 ST(v_1) + \cdots + a_n ST(v_n) = a_1 v_1 + \cdots + a_n v_n = v$$

$\Longleftarrow$ If $T(v_1) = T(v_2)$ then $ST(v_1) = ST(v_2)$ and with the assumption given $v_1 = v_2$.

Recommended Reading

- **Prüfungstraining Lineare Algebra : Band I** by Thomas C. T. Michaels (Match: 0.66)
- **Linear algebra problem book** by Paul R. Halmos (Match: 0.66)
- **Groups, Matrices, and Vector Spaces** by James B. Carrell (Match: 0.66)