Isomorphic Vector Spaces

Definition: An isomorphism is an invertible linear map.

- Two vector spaces are called isomorphic if there is an isomorphism from one vector space onto the other one.

Theorem: Two finite-dimensional vector spaces over a field $\mathbb{F}$ are isomorphic if and only if they have the same dimension.**Proof:**

1. First suppose V and W are isomorphic finite-dimensional vector spaces, and T the isomorphism.
   - Because T is invertible, we have $\text{null}(T) = \{0\}$ and $\text{range}(T) = W$. Thus $\dim(\text{null } T) = 0$ and $\dim(\text{range } T) = \dim W$.
   - Thus $\dim V = \dim(\text{null } T) + \dim(\text{range } T) = 0 + \dim W = \dim W$.
2. Suppose V and W are vector spaces with $\dim(V) = \dim(W) = n$. Let $\{v_1, \ldots, v_n\}$ be a basis of V and $\{w_1, \ldots, w_n\}$ be a basis of W.
3. Define $T \in \mathcal{L}(V, W)$ by:

$$T(c_1 v_1 + \cdots + c_n v_n) = c_1 w_1 + \cdots + c_n w_n$$

4. Thus, T is a well-defined isomorphism between V and W.

Recommended Reading

- **Groups, Matrices, and Vector Spaces** by James B. Carrell (Match: 0.68)
- **Linear Algebra** by Meckes Meckes (Match: 0.67)
- **Lineare Algebra 1** by Menny-Akka (Match: 0.66)