Series 3

The geometric series $\sum_{k=0}^{\infty} a^k$ where $a \in \mathbb{K}$ converges for $|a| < 1$ to $\frac{1}{1-a}$.

Proof:

1. **Lemma:** The sequence a^n converges for $|a| < 1$ to 0. If a^n converges, it converges to 0 or $a = 1$, because $\lim_{n\to\infty} a^n = \lim_{n\to\infty} a^{n+1} = a \cdot \lim_{n\to\infty} a^n$. Because $|a| < 1$: $|a^{n+1}| < |a^n|$ and so $|a^n|$ is monotonically decreasing. Because $|a^n|$ is also bounded, it converges to its infimum: $\lim_{n\to\infty} |a^n| \to 0$. But this yields $a^n \to 0$.

2. **Lemma:** $\sum_{k=0}^{n} a^k = \frac{1-a^{n+1}}{1-a}$. $(1-a)\sum_{k=0}^{n} a^k = \sum_{k=0}^{n} a^k - \sum_{k=0}^{n} a^{k+1} = 1 - a^{n+1}$.

   So $s_n = \sum_{k=0}^{n} a^k = \frac{1-a^{n+1}}{1-a}$ if $|a| < 1$: $\lim_{n\to\infty} \frac{1-a^{n+1}}{1-a} = \frac{1}{1-a}$.

Theorem (Root Test): Let $\sum x_k$ be a series in $\mathbb{E}$ and $\alpha := \lim_{k\to\infty} \sqrt[k]{|x_k|}$. Then following holds:

3. $\sum x_k$ converges absolutely if $\alpha < 1$.
4. $\sum x_k$ diverges if $\alpha > 1$.
5. $\sum x_k$ converges or diverges if $\alpha = 1$ (both cases exist).

Proof:

1. If $\alpha < 1$, then the intervall $(\alpha, 1)$ is not empty and we can choose some $q \in (\alpha, 1)$. α is the greatest cluster point of the sequence $(\sqrt[k]{|x_k|})$. Hence there is some K such that $\sqrt[k]{|x_k|} < q$ for all $k \geq K$, that is, for all $k \geq K$ we have $|x_k| < q^k$. Therefore the geometric series $\sum q^k$ is a convergent majorant for $\sum x_k$, and $\sum x_k$ converges absolutely.

2. If $\alpha > 1$, then there are infinitely many $k \in \mathbb{N}$ such that $\sqrt[k]{|x_k|} \geq 1$. Thus $|x_k| \geq 1$ for infinitely many $k \in \mathbb{N}$. In particular, (x_k) is not a null sequence and the series $\sum x_k$ diverges.

3. As example: $\sum x_k$ with $x_k := (-\frac{1}{k})^{k+1}/k$, for this series we have:

   $$\sqrt[k]{|x_k|} = \sqrt[k]{\frac{1}{k}} = \frac{1}{\sqrt[k]{k}} \to 1 \quad (k \to \infty).$$

   Thus $\alpha = \limsup_{k\to\infty} \sqrt[k]{|x_k|} = 1$. This series converges by the Leibniz criterion but $\sum |x_k|$ diverges and $\alpha = 1$ still.

Recommended Reading

- **Analysis I** by H. Amann (Match: 0.69)
- **Writing Proofs in Analysis** by Jonathan M. Kane (Match: 0.68)
- **Analysis I Script** by ETH (Match: 0.67)