22.10. Algebra.
Definition: A number $\lambda \in \mathbb{F}$ is called a zero (or root) of a polynomial $p \in \mathbb{P}(\mathbb{F})$ if

$$p(\lambda) = 0.$$

Lemma: Suppose m is a positive integer and $p \in \mathbb{P}(\mathbb{F})$ is a polynomial of degree m. Suppose $\lambda \in \mathbb{F}$. Then $p(\lambda) = 0$ if and only if there exists a polynomial $q \in \mathbb{P}(\mathbb{F})$ of degree $m-1$ such that

$$p(z) = (z - \lambda) \cdot q(z).$$

Proof: $\Rightarrow$) Suppose $p(\lambda) = 0$. And $p(z) = a_0 + a_1 z + \cdots + a_n z^n$ for all $z \in \mathbb{F}$. Then $p(z) = p(z) - p(\lambda) = a_0 + a_1 z + \cdots + a_n z^n - (a_0 + a_1 \lambda + \cdots + a_n \lambda^n)$. For each $k \in \{1, \ldots, m\}$, the equation $z^k - \lambda^k = (z - \lambda) \cdot \sum_{j=0}^{k-1} z^j \lambda^{k-1-j}$ holds. Thus $p(z) = (z - \lambda) \cdot q(z)$ for some polynomial $q(z)$ of degree $m - 1$.

$\Leftarrow Suppose such a $q(z)$ exists. Then $p(\lambda) = (\lambda - \lambda) \cdot q(\lambda) = 0 \cdot q(\lambda) = 0$. □
Lemma: Suppose m is a positive integer and $p \in \mathbb{P}(\mathbb{F})$ is a polynomial of degree m. Then p has at most m zeros in $\mathbb{F}$.
Proof: By induction: $m = 1$: because $a_1 \neq 0$, $z = -a_0/a_1$ is the only zero.

$m - 1 \Rightarrow m$: If p has no zeros in $\mathbb{F}$, nothing is to show. If λ is a zero of p, then $q \in \mathbb{P}(\mathbb{F})$ exists with degree $m-1$ such that $p(x) = (x - \lambda) \cdot q(x)$. $q(x)$ has at most $m - 1$ zeros, and therefore $p(x)$ at most m zeros. □
Remark: This implies that the coefficients of a polynomial are uniquely determined, because if a polynomial had two different sets of coefficients, then subtracting the two representations would give a polynomial with non-zero degree but infinitely many zeros.

("Herausheben von NS" & max. m zeros)

$$\to \text{Grad} - 1$$

Recommended Reading

- **Abstract Algebra** by Paul B. Garrett (Match: 0.69)
- **Abstract Algebra** by Marco Hien (Match: 0.69)
- **Lineare Algebra 1** by Menny-Akka (Match: 0.69)