Proof of the Lemma:

Let $U \subset X$, and suppose first that U satisfies the basis criterion. Let $V = \bigcup\{B \in \mathcal{B} : B \subset U\}$.

Then $V \in \mathcal{T}$, because V is a union of sets of $\mathcal{B}$. Thus if $U = V$ it will follow that $U \in \mathcal{T}$.

$V \subset U$ because V is a union of subsets of U.

To show that $U \subset V$, let $x \in U$ be an arbitrary element. Since U satisfies the basis criterion, there must exist a $B \in \mathcal{B}$ such that $x \in B \subset U$, and therefore $x \in V$.

Conversely, suppose that $U \in \mathcal{T}$. This means

$$U = \bigcup_{\alpha \in A} B_\alpha \quad \text{with } B_\alpha \in \mathcal{B}.$$

In other words, $x \in U$ if and only if $x \in B$ for some $B \in \mathcal{B}$. In particular, $x \in B \subset U$, so U satisfies the basis criterion. $\square$

Proof of the Proposition:

1) Suppose $U_1, U_2 \in \mathcal{T}$. Then, for any $x \in U_1 \cap U_2$, there exists (because of the basis criterion) $B_1, B_2 \in \mathcal{B}$ such that $x \in B_1 \subset U_1$ and $x \in B_2 \subset U_2$. For these two elements of $\mathcal{B}$ there exists a $B_3 \in \mathcal{B}$ such that $x \in B_3 \subset B_1 \cap B_2 \subset U_1 \cap U_2$. Thus $U_1 \cap U_2$ satisfies the basis criterion, so it is also in $\mathcal{T}$.

Induction shows this is also true for $\bigcap_{i=1}^{n} U_i$ with $U_i \in \mathcal{T}$.

2) X is per definition $X = \bigcup_{B \in \mathcal{B}} B$, and therefore a union of elements of $\mathcal{B}$, and so $X \in \mathcal{T}$.

3) $U_\alpha \in \mathcal{T}$ for all $\alpha \in A$. $U = \bigcup_{\alpha \in A} U_\alpha$ is a union of unions of elements of $\mathcal{B}$, and therefore also in $\mathcal{T}$.

Recommended Reading

- **Writing Proofs in Analysis** by Jonathan M. Kane (Match: 0.66)
- **Basic Analysis I** by James K. Peterson (Match: 0.66)
- **Lecture Notes** by Topologie - Andreas Kriegl (Match: 0.66)