Metric Spaces

Let X be a set. A function $d : X \times X \to \mathbb{R}_0^+$ is called a metric on X, if:

M1 $d(x, y) = 0 \iff x = y$

M2 $d(x, y) = d(y, x), \quad$ for all $x, y \in X$

M3 $d(x, y) \leq d(x, z) + d(z, y), \quad$ for all $x, y, z \in X$

Notes:

1. The open ball is $B(a, r) := \{x \in X : d(a, x) < r\}$.
2. The closed ball is $\overline{B}(a, r) := \{x \in X : d(a, x) \leq r\}$.
3. Let (X_j, d_j), $1 \leq j \leq m$, be metric spaces, and $X := X_1 \times \cdots \times X_m$. Then

$$d(x, y) := \max_{1 \leq j \leq m} d_j(x_j, y_j)$$

   is a metric on X, and

$$B_X(a, r) = \prod_{j=1}^{m} B_{X_j}(a_j, r), \quad \overline{B}_X(a, r) = \prod_{j=1}^{m} \overline{B}_{X_j}(a_j, r)$$

4. For all $x, y, z \in X$ of a metric space:

$$d(x, y) \geq |d(x, z) - d(z, y)|$$