Column-Row Factorization

Suppose A is an $m \times n$ matrix with entries in a field F and column rank $c \geq 1$. Then there exist an $m \times c$ matrix C and a $c \times n$ matrix R, both with entries in F, such that $A = CR$.

Proof:

The list $A_{:,1}, A_{:,2}, \ldots, A_{:,n}$ of columns of A can be reduced thus to a basis of the span of the columns of A. This basis has the length c. The c columns in this basis can be put together to form an $m \times c$ matrix C. If $k \in \{1, \ldots, n\}$, then column k of A is a linear combination of the columns of C. Make the coefficients of this linear combination the k-th column of an $c \times n$ matrix that we call R. Then $A = CR$.

Suppose $A \in F^{m \times n}$. Then the column rank of A equals the row rank of A.

Proof: Let c denote the column rank of A. Let $A = CR$ be the column-row factorization of A. Then every row of A is then a linear combination of the rows of R. Because R has c rows, this implies that the row rank of A is less or equal to the column rank c of A. To prove the inequality in the other direction, apply the result in the previous paragraph to A^T, getting:

$$\text{column rank of } A = \text{row rank of } A^T$$

$$\leq \text{column rank of } A^T$$

$$\leq \text{row rank of } A.$$

Thus the column rank of A equals the row rank of A.

Recommended Reading

- **Second Course in Linear Algebra** by Stephan Ramon Garcia, Roger A. Horn (Match: 0.69)
- **Groups, Matrices, and Vector Spaces** by James B. Carrell (Match: 0.68)
- **Prüfungstraining Lineare Algebra : Band I** by Thomas C. T. Michaels (Match: 0.68)