

Continuity 3

Lemma 1. Suppose that $X \rightarrow \mathbb{R}$ is continuous at $x_0 \in X$ and $f(x_0) > 0$. Then there is a neighborhood U of x_0 such that $f(x) > 0$ for all $x \in U$.

Proof. Set $\varepsilon := f(x_0)/2 > 0 \Rightarrow \exists \delta > 0 : |f(x) - f(x_0)| \leq f(x) - f(x_0) < \varepsilon = \frac{f(x_0)}{2} \quad \forall x \in B_X(x_0, \delta) := U \Rightarrow f(x) > f(x_0) - \frac{f(x_0)}{2} = \frac{f(x_0)}{2} > 0$. \square

Definition 1. A function $f : X \rightarrow Y$ is Lipschitz continuous with Lipschitz constant $\alpha > 0$ if

$$d(f(x), f(y)) \leq \alpha d(x, y) \quad \forall x, y \in X.$$

Lemma 2. Every Lipschitz continuous function is continuous.

Proof. Let $\varepsilon > 0$. $x_0 \in X$: with $\delta := \frac{\varepsilon}{\alpha} > 0$ we have $d(f(x), f(y)) \leq \alpha d(x, y) < \alpha \cdot \frac{\varepsilon}{\alpha} = \varepsilon$ for $y \in B_X(x_0, \delta)$.

Note that, in this case δ is independent of $x_0 \in X$. \square

Examples of Lipschitz continuous functions:

1. If E_1, \dots, E_m are normed vector spaces, then $E := E_1 \times \dots \times E_m$ is a normed vector space with respect to the product norm $\|\cdot\|_\infty$. The canonical projections $\text{pr}_k : E \rightarrow E_k$, $x = (x_1, \dots, x_m) \mapsto x_k$ are Lipschitz continuous. In particular, the projections $\text{pr}_k : \mathbb{K}^m \rightarrow \mathbb{K}$ are Lipschitz continuous.

Proof. For $x = (x_1, \dots, x_m)$ and $y = (y_1, \dots, y_m)$ we have

$$\|\text{pr}_k(x) - \text{pr}_k(y)\|_{E_k} = \|x_k - y_k\|_{E_k} \leq \|x - y\|_\infty$$

\square

2. $z \mapsto \text{Re}(z)$, $z \mapsto \text{Im}(z)$ and $z \mapsto \bar{z}$ is Lipschitz continuous on \mathbb{C} .

Proof. $\max\{|\text{Re}(z_1) - \text{Re}(z_2)|, |\text{Im}(z_1) - \text{Im}(z_2)|\} \leq |z_1 - z_2| = |\bar{z}_1 - \bar{z}_2|$. \square

3. Let E be a normed vector space. Then the norm function $\|\cdot\| : E \rightarrow \mathbb{R}$, $x \mapsto \|x\|$ is Lipschitz continuous.

Proof. $|||x| - |y||| \leq \|x - y\| \quad \forall x, y \in E$. \square

4. For any inner product space $(E, \langle \cdot, \cdot \rangle)$, the scalar product $\langle \cdot, \cdot \rangle : E \times E \rightarrow \mathbb{K}$ is continuous.

Proof. Let $(x, y), (x_0, y_0) \in E \times E$ and $\varepsilon \in (0, 1)$: $|\langle x, y \rangle - \langle x_0, y_0 \rangle| = |\langle x - x_0, y \rangle + \langle x_0, y - y_0 \rangle| \leq |\langle x - x_0, y \rangle| + |\langle x_0, y - y_0 \rangle| \leq$

$$\|x - x_0\| \|y\| + \|x_0\| \|y - y_0\| \leq d((x, y), (x_0, y_0))(\|x\| + \|x_0\|) \|y - y_0\|$$

(this line seems incomplete/miswritten in original, continuing with next line structure)

- Let $d := \max d_E(x_1, y_1), (x_2, y_2) = \max\{\|x_1\|, \|y_1\|, \|x_2\|, \|y_2\|\}$ (The product norm on $E \times E$).
- Set $M := \max\{1, \|x_0\|, \|y_0\|\}$ and $\delta := \frac{\varepsilon}{(1+2M)}$. Then, for all $(x, y) \in B_{((x_0, y_0), \delta)}$:

$$|\langle x, y \rangle - \langle x_0, y_0 \rangle| \leq \delta(2M + \delta) < \varepsilon$$

which proves the continuity of the scalar product $\langle \cdot, \cdot \rangle$ at the point (x_0, y_0) . \square

(Note: $\|f(y) - f(x)\|$ is roughly $\leq C\|x - y\|$ through some bounding with $\|x - y\|$ and α if it is Lipschitz continuous). \square

Recommended Reading

- **Fréchet differentiability of Lipschitz functions and porous sets in Banach spaces** by Joram Lindenstrauss (Match: 0.71)
- **Real Analysis** by Patrick Fitzpatrick (Match: 0.70)
- **Writing Proofs in Analysis** by Jonathan M. Kane (Match: 0.70)