Open Sets in a Metric Space (M, d)

1) Definition: Given $\varepsilon > 0$, the ε-neighborhood (nbhd) around $x \in M$ is the set

$$B_\varepsilon(x) = \{y \in M : d(x, y) < \varepsilon\}$$

2) Definition: For any $A \subseteq M$, we say A is open if for any $x \in A$ there exists an $\varepsilon > 0$ so that the ε-neighborhood of x is a subset of A.

$$\forall x \in A : \exists \varepsilon > 0 : B_\varepsilon(x) \subseteq A$$

3) Definition: A neighborhood of $A \subseteq M$, $x \in M$, is any open set containing x.

Lemma: For any $x \in M$ and $\varepsilon > 0$, $B_\varepsilon(x)$ is open.

Proof: Given any $x \in M$ and $\varepsilon > 0$, choose $z \in B_\varepsilon(x)$. Define $\delta_z = \varepsilon - d(x, z)$. Consider any $y \in B_{\delta_z}(z)$, notice

$$d(x, y) \leq d(x, z) + d(z, y) < d(x, z) + \delta_z = d(x, z) + (\varepsilon - d(x, z)) = \varepsilon$$

Thus $y \in B_\varepsilon(x)$ and therefore $B_{\delta_z}(z) \subseteq B_\varepsilon(x)$. $\square$

Lemma: In a metric space (M, d):

1. M and $\emptyset$ are open.
2. If A_λ is open for each index $\lambda \in \Lambda$, then $\bigcup_{\lambda \in \Lambda} A_\lambda$ is open.
3. If $A_1, \ldots, A_N$ are open, then $\bigcap_{k=1}^{N} A_k$ is open.

Proof:

2. Suppose $x \in \bigcup_{\lambda \in \Lambda} A_\lambda$. Then $\exists \lambda_0 \in \Lambda$ with $x \in A_{\lambda_0}$. Since A_{λ_0} is open, $\exists \varepsilon > 0$ such that $B_\varepsilon(x) \subseteq A_{\lambda_0} \subseteq \bigcup_{\lambda \in \Lambda} A_\lambda$.
3. Suppose $x \in \bigcap_{k=1}^{N} A_k$. For each $k \in \{1, \ldots, N\}$, $x \in A_k$, and A_k is open: $\exists \varepsilon_k > 0 : B_{\varepsilon_k}(x) \subseteq A_k$. Let $\varepsilon = \min\{\varepsilon_1, \ldots, \varepsilon_N\}$. Then $B_\varepsilon(x) \subseteq A_k$ for all $k \in \{1, \ldots, N\}$. Thus $B_\varepsilon(x) \subseteq \bigcap_{k=1}^{N} A_k$. $\square$

Recommended Reading

- **Writing Proofs in Analysis** by Jonathan M. Kane (Match: 0.70)
- **Basic Analysis I** by James K. Peterson (Match: 0.68)
- **Real Analysis** by Patrick Fitzpatrick (Match: 0.68)