Proposition

Let $\mathcal{B}$ be a basis in a set X, and let $\mathcal{T}$ be the collection of all unions of $\mathcal{B}$. Then $\mathcal{T}$ is a topology on X.

For the proof we first observe

Remark: Suppose $\mathcal{B}$ is a basis of X. Then the collection $\mathcal{T}$ defined in the proposition is precisely the set of all subsets of X that satisfy:

- For every $x \in \mathcal{T}$, there exists $B \in \mathcal{B}$ such that $x \in B \subset \mathcal{T}$. (Note: The inclusion $B \subset \mathcal{T}$ seems to be a transcription error in the original note, likely meaning $x \in B \subset U$ for some $U \in \mathcal{T}$, or perhaps $x \in B \subset U$ where U is the set being tested for membership in $\mathcal{T}$. Given the context of the basis criterion below, the intended structure is usually that the set U containing x is covered by a basis element B. I will transcribe the visible text exactly but use the standard form for the basis criterion definition in the Lemma).

Basis criterion with respect to $\mathcal{B}$

Given X and a collection $\mathcal{B}$ of subsets of X, we say that a subset $U \subset X$ satisfies the basis criterion with respect to $\mathcal{B}$ if, for every $x \in U$, there exists $B \in \mathcal{B}$ such that $x \in B \subset U$.

Lemma:

Suppose $\mathcal{B}$ is a basis of X. Then the collection $\mathcal{T}$ defined in the proposition is precisely the set of all subsets of X that satisfy the basis criterion with respect to $\mathcal{B}$.

Recommended Reading

- **Basic Analysis I** by James K. Peterson (Match: 0.68)
- **Writing Proofs in Analysis** by Jonathan M. Kane (Match: 0.68)
- **Lecture Notes** by Topologie (2018) - Andreas Cap (Match: 0.68)