Limit Superior and Limit Inferior

Let (x_n) be a sequence in $\mathbb{R}$.

$$Y_n := \sup_{k \geq n} x_k := \sup\{x_k : k \geq n\}$$
$$Z_n := \inf_{k \geq n} x_k := \inf\{x_k : k \geq n\}$$

Y_n and Z_n are monotone sequences, so they converge in $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$.

$$\limsup_{n\to\infty} x_n := \lim_{n\to\infty} (\sup_{k\geq n} x_k) \quad \text{and}$$
$$\liminf_{n\to\infty} x_n := \lim_{n\to\infty} (\inf_{k\geq n} x_k)$$

Or,

$$\limsup_{n\to\infty} x_n = \inf_{n\in\mathbb{N}} (\sup_{k\geq n} x_k) \quad \text{and}$$
$$\liminf_{n\to\infty} x_n = \sup_{n\in\mathbb{N}} (\inf_{k\geq n} x_k)$$

Notes:

- Any sequence (x_n) in $\mathbb{R}$ has a smallest cluster point x_* and a greatest cluster point x^* in $\overline{\mathbb{R}}$ and these satisfy:

$$\liminf_{n\to\infty} x_n = x_*$$
$$\limsup_{n\to\infty} x_n = x^*$$

- (x_n) sequence in $\mathbb{R}$. Then

  (x_n) converges in $\overline{\mathbb{R}} \Leftrightarrow \limsup_{n\to\infty} x_n \leq \liminf_{n\to\infty} x_n$

  (x_n) converges in $\mathbb{R} \Leftrightarrow x = \lim_{n\to\infty} x_n = \limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n$

- (Bolzano-Weierstrass):

  Every bounded sequence in $\mathbb{R}^m$ and $\mathbb{C}^m$ has a convergent subsequence, that is, a cluster point.

Recommended Reading

- **Writing Proofs in Analysis** by Jonathan M. Kane (Match: 0.68)
- **Problems in real analysis** by Teodora-Liliana Rădulescu (Match: 0.68)
- **Introduction to real analysis** by Robert Gardner Bartle (Match: 0.67)