So the metric induced by the norm $||\cdot||_\infty$ is defined by: $d : E \times E \to R^+$, $(x, y) \mapsto ||x-y||_\infty$; $(x, y) \mapsto \max_{1\leq j\leq m} ||x_j - y_j||_j$ and with the induced metrics for all j: $d_j(x, y) := ||x_j - y_j||_j$ we get the product metric.

So for normed vector spaces with norm $||\cdot||_\infty$ and the induced metric d: $x_n \to x$ in $E \Leftrightarrow \forall\epsilon > 0 \exists N \in N : ||x_n - x||_\infty \leq \epsilon \forall n \geq N$ is equivalent.

Now: For each $\epsilon > 0$, there is some N such that $x_n \in B(a, \epsilon)\forall n \geq N$.

We now show: Let X be the product of the metric spaces (X_j, d_j), $1 \leq j \leq m$. Then the sequence $(x_n) = ((x_n^1, ..., x_n^m))_{n\in N}$ converges in X to the point $(a^1, ..., a^m)$ if and only if, for each $j \in \{1, ..., m\}$, the sequence $(x_n^j)_{n\in N}$ converges in X_j to $a^j \in X_j$.

Proof: Let $\epsilon > 0$. If almost all $x_n \in B_X(a, \epsilon) = \prod_{j=1}^m B_{X_j}(a^j, \epsilon) \Leftrightarrow$ almost all $x_n^j \in B_{X_j}(a^j, \epsilon)$ for all $j \in \{1, ..., m\}$.

The key was: "The open balls of the product is the cartesian product of the balls of the components".

This is not true for all norms on K^m, first let us state that the proof states the equivalence of 3.14 for the norm $||\cdot||_1$.

We now show that the Euclidean norm and the 1-norm are equivalent to the $||\cdot||_\infty$ norm and that the convergence in equivalent norms is similar. This proof that 3.14 is true for all the norms.

Recommended Reading

- **Calculus On Normed Vector Spaces** by Rodney Coleman (Match: 0.70)
- **Real Analysis** by Patrick Fitzpatrick (Match: 0.70)
- **Introduction to Real Analysis** by Christopher Heil (Match: 0.69)