Theorem: Let V be finite dimensional and $T \in \mathcal{L}(V)$. The zeros of the minimal polynomial of T are the eigenvalues of T.

Proof: Assume $p(x)$ is the minimal polynomial of T and λ is a zero of this polynomial. Then we find $p(x) = (x - \lambda) \cdot q(x)$ and $p(T) = (T - \lambda I) \cdot q(T) = 0$. Because $q(T)$ is of degree smaller than $p(T)$, there must exist a $v \in V$ with $q(T)v \neq 0$. This means $(T - \lambda I)(q(T)v) = 0$ and this shows λ is an eigenvalue of T. For the other direction assume that λ is an eigenvalue of T, therefore $Tv = \lambda v$ for $\alpha v \neq 0$. By successive guesses we get $T^k v = \lambda^k v$. Thus $p(T)v = p(\lambda)v$. Because p is the minimal polynomial of T we get $p(\lambda)v = 0$ but this means $p(\lambda) = 0$, which identifies λ as a zero of p.

Proposition: Suppose V is a finite dimensional vector space and $T \in \mathcal{L}(V)$ and $q \in \mathcal{P}(\mathbb{F})$. Then $q(T) = 0$ if and only if q is a polynomial multiple of the minimal polynomial p of T.

Proof: First suppose $q(T) = 0$, then by polynomial division algorithm there are polynomials s and r such that $q = p \cdot s + r$, with degree$(r) <$ degree(p). We have $q(T) = p(T)s(T) + r(T) = r(T)$, but this implies $r(T) = 0$ and because $p(T)$ is the minimal polynomial of T, r must be of degree 0 or $r = 0$. This leads to the equation $q(T) = p(T) \cdot s(T)$ which is what we want to show. If $q(T)$ is a polynomial multiple of $p(T)$ then we have $q(T) = s(T) \cdot p(T) = s(T) \cdot 0 = 0$ as desired. $\square$

Corollary: Suppose V is finite dimensional, $T \in \mathcal{L}(V)$, and U is a subspace of V that is invariant under T. Then the minimal polynomial of $T|_U$ is a polynomial multiple of the minimal polynomial of T.

Proof: Let ρ the minimal polynomial of $T|_U$ on U. Then $\rho(T|_U) = 0$ and therefore $\rho = s \cdot p$ with $s \in \mathcal{P}(\mathbb{F})$.

Theorem: Suppose V is finite dimensional and $T \in \mathcal{L}(V)$. Then T is not invertible if and only if the constant term of the polynomial of T is 0.

Proof: Suppose $T \in \mathcal{L}(V)$ and ρ is the minimal polynomial of T. Then T is not invertible $\iff$ 0 is no eigenvalue of T $\iff$ 0 is no zero of ρ $\iff$ the constant term of ρ is 0.

Recommended Reading

- **Linear algebra problem book** by Paul R. Halmos (Match: 0.72)
- **Lineare Algebra 1** by Menny-Akka (Match: 0.71)
- **Linear Algebra Done Right** by Sheldon Axler (Match: 0.71)