Subspace Union Properties

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1. Prove that the union of three subspaces A, B, C of V is a subspace of V if and only if one of the subspaces contains the other two.
   - $\Longleftarrow$ Easy: $A \cup B \cup C = A$ if $B \subseteq A$ and $C \subseteq A$, and A is a subspace.
   - $\Longrightarrow$ If $A \cup B \cup C$ is a subspace and there exists $x \in A$ but $x \notin B$ or $x \in A$ but $x \notin C$:
2. Prove that the union of two subspaces of V is a subspace of V if and only if one of the subspaces is contained in the other.
   (a) Proof: 1) 0 has to be in A and B, because both are subsets.
   (b) 2) $a \in A \cup B$ implies $a \in A$ or $a \in B$.
   (c) $\Longrightarrow$ 2) If none of the both is contained in the other then there exist two elements with the following conditions:

   $$a \in A \text{ and } a \notin B, \quad b \in B \text{ and } b \notin A.$$

   Because $A \cup B$ is a subspace: $a + b$ must be $\in A \cup B$

   $$\text{so } a + b \in A \text{ or } a + b \in B;$$

   but if $a + b \in A$ and $a \in A$ and so $-a \in A$:

   $$b = (a + b) + (-a) \text{ also must be element of } A.$$

   The same argument holds for $a + b \in B$, therefore. A must be contained in B, or B must be contained in A.

   $$\Longleftarrow \text{ If } A \subseteq B \text{ or } B \subseteq A \text{ then } A \cup B = A \text{ or } A \cup B = B.$$

Recommended Reading

- **Lineare Algebra 1** by Menny-Akka (Match: 0.63)
- **Prüfungstraining Lineare Algebra : Band I** by Thomas C. T. Michaels (Match: 0.63)
- **Groups, Matrices, and Vector Spaces** by James B. Carrell (Match: 0.62)