Problem 1

Find a list of four distinct vectors in $\mathbb{F}^3$ whose span equals $V = \{(x, y, z) \in \mathbb{F}^3 : x + y + z = 0\}$.

Solution: For every $\mathbf{v} \in V$ holds: $x+y = -z$, and so every vector of V must be of the form $\mathbf{v} = (x, y, -x - y)$. Such vectors can be constructed by a linear combination of the two vectors $\mathbf{v}_1 = (1, 0, -1)$ and $\mathbf{v}_2 = (0, 1, -1)$ because of

$$\mathbf{v} = x \cdot \mathbf{v}_1 + y \cdot \mathbf{v}_2.$$

So $V \subseteq \operatorname{span}(\mathbf{v}_1, \mathbf{v}_2)$. On the other side, it holds that every linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$ fulfills $x + y + z = 0$, and so $\operatorname{span}(\mathbf{v}_1, \mathbf{v}_2) \subseteq V$. Therefore $V = \operatorname{span}(\mathbf{v}_1, \mathbf{v}_2)$. To get the solution just add two linear combinations of $\mathbf{v}_1$ and $\mathbf{v}_2$, say $\mathbf{v}_3 = \mathbf{v}_1 + \mathbf{v}_2$ and $\mathbf{v}_4 = \mathbf{v}_1 - \mathbf{v}_2$.

Problem 2

Prove or give a counterexample: If $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ spans V, then the list $\mathbf{v}_1 - \mathbf{v}_2, \mathbf{v}_2 - \mathbf{v}_3, \mathbf{v}_3 - \mathbf{v}_4, \mathbf{v}_4$ also spans V.

Solution: The list also spans V because:

$$\mathbf{v}_4 = \mathbf{v}_4$$

$$\mathbf{v}_3 = (\mathbf{v}_3 - \mathbf{v}_4) + \mathbf{v}_4$$
$$\mathbf{v}_2 = (\mathbf{v}_2 - \mathbf{v}_3) + (\mathbf{v}_3 - \mathbf{v}_4) + \mathbf{v}_4$$
$$\mathbf{v}_1 = (\mathbf{v}_1 - \mathbf{v}_2) + (\mathbf{v}_2 - \mathbf{v}_3) + (\mathbf{v}_3 - \mathbf{v}_4) + \mathbf{v}_4$$

Therefore every linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ can also be written in a linear combination of the new list.

Recommended Reading

- **Prüfungstraining Lineare Algebra : Band I** by Thomas C. T. Michaels (Match: 0.71)
- **Linear algebra problem book** by Paul R. Halmos (Match: 0.70)
- **Lineare Algebra 1** by Menny-Akka (Match: 0.70)