Series 2

Proposition: Every absolutely convergent series converges.

Proof: Let $\sum x_k$ be an absolutely convergent series in E. Then $\sum |x_k|$ converges in $\mathbb{R}$. Therefore $\sum |x_k|$ satisfies the Cauchy criterion, that is, for all $\varepsilon > 0$ there is some N such that

$$\sum_{k=n+1}^{m} |x_k| < \varepsilon \quad \forall m > n \geq N$$

Since

$$\left| \sum_{k=n+1}^{m} x_k \right| \leq \sum_{k=n+1}^{m} |x_k| < \varepsilon \quad \forall m > n \geq N$$

The series $\sum x_k$ also satisfies the Cauchy criterion.

Def: Let $\sum x_k$ be a series in E and $\sum a_k$ a series in $\mathbb{R}^+$. Then the series $\sum a_k$ is called a majorant (or numerant) for $\sum x_k$ if there is some $K \in \mathbb{N}$ such that $|x_k| \leq a_k$ (or $a_k \leq |x_k|$) for all $k \geq K$.

Theorem: (Majorant criterion) If a series in a Banach space has a convergent majorant, then it converges absolutely.

Proof: Let $\sum x_k$ be a series in E and $\sum a_k$ a convergent majorant. Then there is some K such that $|x_k| \leq a_k$ for all $k \geq K$. By Cauchy criterion for $\varepsilon > 0$ there is some $N \geq K$ such that

$$\sum_{k=n+1}^{m} a_k < \varepsilon \quad \text{for all } m > n \geq N$$

So we have:

$$\left| \sum_{k=n+1}^{m} x_k \right| \leq \sum_{k=n+1}^{m} |x_k| \leq \sum_{k=n+1}^{m} a_k < \varepsilon \quad m > n \geq N$$

Therefore $\sum |x_k|$ converges and $\sum x_k$ converges absolutely.

Recommended Reading

- **Analysis in Banach Spaces (Vol III)** by Hytoenen et al (Match: 0.69)
- **Analysis in Banach Spaces : Volume II** by Tuomas Hytönen, Jan van Neerven, Mark Veraar, Lutz Weis (Match: 0.69)
- **Analysis I** by H. Amann (Match: 0.68)