Linear Independence Exercises

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Problem 8

Suppose $\{v_1, v_2, v_3, v_4\}$ is linearly independent in V. Prove that the list $\{v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4\}$ is also linearly independent.

Solution:

We set the linear combination to zero:

$$0 = \alpha_1(v_1 - v_2) + \alpha_2(v_2 - v_3) + \alpha_3(v_3 - v_4) + \alpha_4 v_4$$

Distributing and collecting terms for each v_i:

$$0 = \alpha_1 v_1 + (\alpha_2 - \alpha_1)v_2 + (\alpha_3 - \alpha_2)v_3 + (\alpha_4 - \alpha_3)v_4$$

Because $\{v_1, v_2, v_3, v_4\}$ is linearly independent, we must have all coefficients equal to zero:

$$\alpha_1 = 0\alpha_2 - \alpha_1 = 0\alpha_3 - \alpha_2 = 0\alpha_4 - \alpha_3 = 0$$

From the first equation, $\alpha_1 = 0$. Substituting this into the second gives $\alpha_2 = 0$. Continuing this process shows that $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$.

Thus, the list $\{v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4\}$ is linearly independent.

Problem 9

Prove or give a counterexample: If $\{v_1, v_2, \ldots, v_n\}$ is a linearly independent list of vectors in V, then the list $\{5v_1 - 4v_2, v_2, v_3, \ldots, v_n\}$ is linearly independent.

Solution:

We set the linear combination to zero:

$$\alpha_1(5v_1 - 4v_2) + \alpha_2 v_2 + \alpha_3 v_3 + \cdots + \alpha_n v_n = 0$$

Rearranging terms:

$$5\alpha_1 v_1 + (\alpha_2 - 4\alpha_1)v_2 + \alpha_3 v_3 + \cdots + \alpha_n v_n = 0$$

With the linear independence of $\{v_1, v_2, \ldots, v_n\}$, it follows that all coefficients must be zero:

$$5\alpha_1 = 0\alpha_2 - 4\alpha_1 = 0\alpha_3 = 0\vdots\alpha_n = 0$$

From $5\alpha_1 = 0$, we get $\alpha_1 = 0$. Substituting into the second equation gives $\alpha_2 - 4(0) = 0$, so $\alpha_2 = 0$. The rest follow immediately ($\alpha_3 = \cdots = \alpha_n = 0$).

So $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$, and the list is therefore linearly independent.

Problem 11

Prove or give a counterexample: If $\{v_1, \ldots, v_m\}$ and $\{w_1, \ldots, w_m\}$ are linearly independent lists of vectors in V, then the list $\{v_1 + w_1, v_2 + w_2, \ldots, v_m + w_m\}$ is linearly independent.

Solution: Counterexample

Let $m = 1$. Let $V = \mathbb{R}^2$. Let $\{v_1\} = \{(1, 0)\}$ and $\{w_1\} = \{(-1, 0)\}$. Both lists $\{v_1\}$ and $\{w_1\}$ are linearly independent (since they are single non-zero vectors).

Consider the new list $\{v_1 + w_1\}$.

$$v_1 + w_1 = (1, 0) + (-1, 0) = (0, 0)$$

Since $v_1 + w_1 = 0$, the list $\{v_1 + w_1\}$ is linearly dependent (the scalar 1 times the vector is zero).

Alternatively, using the notation from the handwritten image for $m = 1$: If we take $v_1 = -w_1$, then $v_1 + w_1 = 0$, which is linearly dependent. The assertion is false.

Recommended Reading

- **Linear algebra problem book** by Paul R. Halmos (Match: 0.71)
- **Linear Algebra** by Seymour Lipschutz (Match: 0.69)
- **Groups, Matrices, and Vector Spaces** by James B. Carrell (Match: 0.69)