Finding a Subspace Complement for U in F^4

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20) Suppose U $= \{$(x, x, y, y) $\in \mathbb{F}^4 : x, y \in \mathbb{F}\}$.
Find a subspace $\mathcal{W}$ of $\mathbb{F}^4$ such that $\mathbb{F}^4 = \mathcal{U} \oplus \mathcal{W}$.
Solution: There are many possible solutions. This is one of them:$\mathcal{W} = \{(x, 0, y, 0) \in \mathbb{F}^4 : x, y \in \mathbb{F}\}$

$\mathcal{U} + \mathcal{W} = \mathbb{F}^4$ Let b $\in \mathbb{F}^4$, so $b = (b_1, b_2, b_3, b_4)$.
Choose $a \in \mathcal{U}$ with $a = (b_2, b_2, b_4, b_4)$ and
$\overline{a} \in \mathcal{W}$ with $\overline{a} = (b_1 - b_2, 0, b_3 - b_4, 0)$, so
$b = a + \overline{a}$.
We show that the only possible way to write 0 as a sum of u $\in \mathcal{U}$ and $w \in \mathcal{W}$ is to write $0 = 0 + 0$:
Let $u = (x_u, x_u, y_u, y_u) \in \mathcal{U}$ and $w = (x_w, 0, y_w, 0) \in \mathcal{W}$.
$0 = u + w = (x_u + x_w, x_u, y_u + y_w, y_u)$.This implies $x_u = 0$ and $y_u = 0$.
Let $x_u + x_w = 0 \implies 0 + x_w = 0 \implies x_w = 0$
and $y_u + y_w = 0 \implies 0 + y_w = 0 \implies y_w = 0$.
Therefore $u = 0$ and $w = 0$.
So U $\oplus \mathcal{W} = \mathbb{F}^4$.

Recommended Reading

- **Lineare Algebra 1** by Menny-Akka (Match: 0.64)
- **Linear Algebra Done Right** by Sheldon Axler (Match: 0.64)
- **Groups, Matrices, and Vector Spaces** by James B. Carrell (Match: 0.63)