

So the metric induced by the norm $\|\cdot\|_\infty$ is defined by $d : E \times E \rightarrow R^+$:

$(x, y) \mapsto \|x - y\|_\infty$; $(x, y) \mapsto \max_{1 \leq j \leq m} \|x_j - y_j\|_j$, and with the induced metric

for all j , $d_j(x, y) := \|x_j - y_j\|_j$, we get the product metric.

So for normed vector spaces with norm $\|\cdot\|_\infty$ and the induced metric d ,

$x_n \rightarrow x$ in $E \Leftrightarrow \forall \epsilon > 0, \exists N \in \mathbb{N} : \|x_n - x\|_\infty \leq \epsilon \forall n \geq N$ is equivalent to $x_n \in B(a, \epsilon) \forall n \geq N$.

For each $\epsilon > 0$, there is some N such that we now show:

Let X be the product of the metric spaces $(X_j, d_j), 1 \leq j \leq m$. Then the sequence $(x_n) = ((x_n^1, \dots, x_n^m))_{n \in \mathbb{N}}$ converges in X to the point (a^1, \dots, a^m) if and only if, for each $j \in \{1, \dots, m\}$, the sequence $(x_n^j)_{n \in \mathbb{N}}$ converges in X_j to $a^j \in X_j$.

Proof: Let $\epsilon > 0$. If almost all $x_n \in B_X(a, \epsilon) = \prod_{j=1}^m B_{X_j}(a^j, \epsilon) \Leftrightarrow$ almost all $x_n^j \in B_{X_j}(a^j, \epsilon)$ for all $j \in \{1, \dots, m\}$.

The key was: "The open balls of the product is the cartesian product of the balls of the components".

This is not true for all norms on K^n , first let us state that the proof states the equivalence of 3.14 for the norm $\|\cdot\|$.

We now show that the euclidean norm and the 1-norm are equivalent to the $\|\cdot\|_\infty$ norm, and that the convergence in equivalent norms is similar. This proves that 3.14 is true for all these norms.

Recommended Reading

- **Calculus On Normed Vector Spaces** by Rodney Coleman (Match: 0.70)
- **Real Analysis** by Patrick Fitzpatrick (Match: 0.69)
- **Analysis** by Jean-Paul Penot (Match: 0.69)