Eigenvalues & Eigenvectors 1

Def.: A linear map from a vector space to itself is called an operator.
Def.: Suppose $T \in \mathcal{L}(V)$. A subspace U of V is called invariant under T if $T(U) \subseteq U$ for all $U \in U$.
Def.: Suppose $T \in \mathcal{L}(V)$. A number $\lambda \in \mathbb{F}$ is called an eigenvalue of T if there exists $v \in V$ such that $v \neq 0$ and $Tv = \lambda v$. The vector v is called the corresponding eigenvector.

Prop.: Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$, and $\lambda \in \mathbb{F}$. Then the following are equivalent:

1. λ is an eigenvalue of T.
2. $T - \lambda I$ is not injective.
3. $T - \lambda I$ is not surjective.
4. $T - \lambda I$ is not invertible.

Proof: (a) and (b) are equivalent because $Tv = \lambda v$ is equivalent to the equation $(T - \lambda I)v = 0$. (b), (c), (d) are equivalent because V is finite-dimensional.

Def.: Suppose $T \in \mathcal{L}(V)$ and $\lambda \in \mathbb{F}$ is an eigenvalue of T. A vector $v \in V$ is called an eigenvector of T corresponding to λ if $v \neq 0$ and $Tv = \lambda v$.

Prop.: Suppose $T \in \mathcal{L}(V)$. Then every list of eigenvectors of T corresponding to distinct eigenvalues of T is linearly independent.
Proof: Assume there exists a linearly dependent list of the given kind, and denote m as the minimal number of such vectors. Thus there exist $a_1, \dots, a_m$ such that $\sum_{i=1}^m a_i v_i = 0$, but none of them 0 (because of the minimality of m). Apply $T - \lambda_m I$ to both sides of the equation $\sum_{i=1}^m a_i(Tv_i - I\lambda_m v_i) = \sum_{i=1}^m a_i(\lambda_i v_i - \lambda_m v_i) = \sum_{i=1}^m a_i v_i(\lambda_i - \lambda_m)$. Because the eigenvalues are distinct but $\lambda_k - \lambda_m = 0$ we get $\sum_{i=1}^{m-1} a_i v_i(\lambda_i - \lambda_m)$, hence then $v_1, \dots, v_{m-1}$ is linearly dependent, in contradiction to the minimality of m.

Prop.: Operators can not have more eigenvalues than the dimension of the vector space: Suppose V is finite dimensional. Then each operator on V has at most $\dim V$ distinct eigenvalues.

Proof: Let $T \in \mathcal{L}(V)$. Suppose $\lambda_1, \ldots, \lambda_m$ are distinct eigenvalues of T. Let $v_1, \ldots, v_m$ be corresponding eigenvectors. Then the proposition above implies $v_1, \ldots, v_m$ are linearly independent and this implies $m \leq \dim V$.

Prop.: Suppose $T \in \mathcal{L}(V)$ and $p \in \mathbb{P}(\mathbb{F})$. Then $\mathrm{null}(p(T))$ and $\mathrm{range}(p(T))$ are invariant under T.
Proof: Suppose $v \in \mathrm{null}(p(T))$. Then $p(T)v = 0$. Thus $(p(T))(Tv) = T(p(T)v) = T(0) = 0$. Suppose $v \in \mathrm{range}(p(T))$. Then there exists $w \in V$ such that $v = p(T)w$. Thus $Tv = T(p(T)w) = p(T)(Tw)$.

Recommended Reading

- **Lineare Algebra 1** by Menny-Akka (Match: 0.74)
- **Linear algebra problem book** by Paul R. Halmos (Match: 0.73)
- **Linear Algebra** by Meckes Meckes (Match: 0.73)