The Generalized Intermediate Value Theorem

Connected sets have the property that their images under continuous functions are also connected. This important fact can be proved easily using the results of Section 2.

4.5 Theorem *Let X and Y be metric spaces and $f : X \to Y$ continuous. If X is connected, then so is $f(X)$. That is,* continuous images of connected sets are connected.

Proof Suppose, to the contrary, that $f(X)$ is not connected. Then there are nonempty subsets V_1 and V_2 of $f(X)$ such that V_1 and V_2 are open in $f(X)$, $V_1 \cap V_2 = \emptyset$ and $V_1 \cup V_2 = f(X)$. By Proposition 2.26, there are open sets O_j in Y such that $V_j = O_j \cap f(X)$ for $j = 1, 2$. Set $U_j := f^{-1}(O_j)$. Then, by Theorem 2.20, U_j is open in X for $j = 1, 2$. Moreover

$$U_1 \cup U_2 = X \ , \quad U_1 \cap U_2 = \emptyset \qquad \text{and} \qquad U_j \neq \emptyset \ , \quad j = 1, 2 \ ,$$

which is not possible for the connected set X. ■

4.6 Corollary *Continuous images of intervals are connected.*

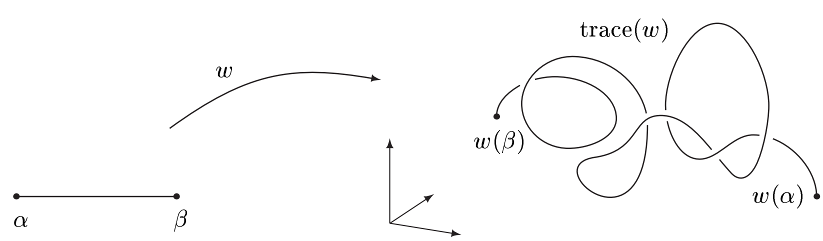
We will demonstrate in the next two sections that Theorems 4.4 and 4.5 are extremely useful tools for the investigation of real functions. Already we note the following easy consequence of these theorems.

4.7 Theorem (generalized intermediate value theorem) *Let X be a connected metric space and $f : X \to \mathbb{R}$ continuous. Then $f(X)$ is an interval. In particular, f takes on every value between any two given function values.*

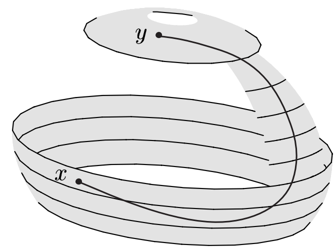
Proof This follows directly from Theorems 4.4 and 4.5. ■

Path Connectivity

Let $\alpha, \beta \in \mathbb{R}$ with $\alpha < \beta$. A continuous function $w : [\alpha, \beta] \to X$ is called a **continuous path** connecting the **end points** $w(\alpha)$ and $w(\beta)$.



A metric space X is called **path connected** if, for each pair $(x, y) \in X \times X$, there is a continuous path in X connecting x and y. A subset of a metric space is called **path connected** if it is a path connected metric space with respect to the induced metric.



4.8 Proposition *Any path connected space is connected.*

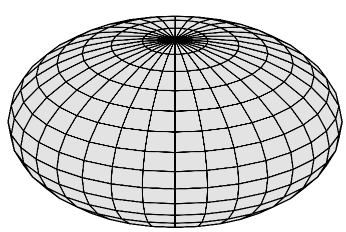
Proof Suppose, to the contrary, that there is a metric space X which is path connected, but not connected. Then there are nonempty open sets O_1, O_2 in X such that $O_1 \cap O_2 = \emptyset$ and $O_1 \cup O_2 = X$. Choose $x \in O_1$ and $y \in O_2$. By hypothesis, there is a path $w\colon [\alpha, \beta] \to X$ such that $w(\alpha) = x$ and $w(\beta) = y$. Set $U_j := w^{-1}(O_j)$. Then, by Theorem 2.20, U_j is open in $[\alpha, \beta]$. We now have α in U_1 and β in U_2, as well as $U_1 \cap U_2 = \emptyset$ and $U_1 \cup U_2 = [\alpha, \beta]$, and so the interval $[\alpha, \beta]$ is not connected. This contradicts Theorem 4.4. ■

Let E be a normed vector space and $a, b \in E$. The linear structure of E allows us to consider 'straight' paths in E:

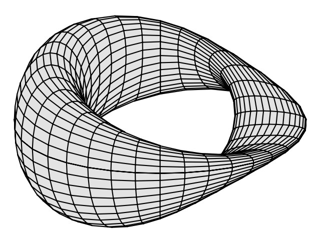
$$v\colon [0,1] \to E\ , \quad t \mapsto (1-t)a + tb\ . \tag{4.1}$$

We denote the image of the path v by $[\![a, b]\!]$.

A subset X of E is called **convex** if, for each pair $(a, b) \in X \times X$, $[\![a, b]\!]$ is contained in X.



Convex



Not convex

4.9 Remarks Let E be a normed vector space.

(a) Every convex subset of E is path connected and connected.

Proof Let X be convex and $a, b \in X$. Then (4.1) defines a path in X connecting a and b. Thus X is path connected. Proposition 4.7 then implies that X is connected. ■

(b) For all $a \in E$ and $r > 0$, the balls $\mathbb{B}_E(a, r)$ and $\bar{\mathbb{B}}_E(a, r)$ are convex.

Proof For $x, y \in \mathbb{B}_E(a,r)$ and $t \in [0,1]$ we have

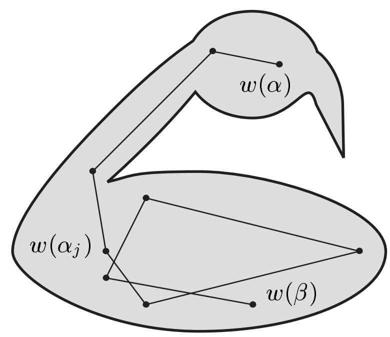
$$\begin{aligned}\|(1-t)x + ty - a\| &= \|(1-t)(x-a) + t(y-a)\| \\ &\le (1-t)\,\|x-a\| + t\,\|y-a\| < (1-t)r + tr = r\ .\end{aligned}$$

This inequality implies that $[\![x,y]\!]$ is in $\mathbb{B}_E(a,r)$. The second claim can be proved similarly. ■

(c) A subset of $\mathbb{R}$ is convex if and only if it is an interval.

Proof Let $X \subseteq \mathbb{R}$ be convex. Then, by (a), X is connected and so, by Theorem 4.4, X is an interval. The claim that intervals are convex is clear. ■

In $\mathbb{R}^2$ there are simple examples of connected sets which are not convex. Even so, in such cases, it seems plausible that any pair of points in the set can be joined with a path which consists of finitely many straight line segments. The following theorem shows that this holds, not just in $\mathbb{R}^2$, but in any normed vector space, so long as the set is open.



Let X be a subset of a normed vector space. A function $w\colon [\alpha,\beta] \to X$ is called a **polygonal path**[2] in X if there are $n \in \mathbb{N}$ and real numbers $\alpha_0, \ldots, \alpha_{n+1}$ such that $\alpha = \alpha_0 < \alpha_1 < \cdots < \alpha_{n+1} = \beta$ and

$$w\big((1-t)\alpha_j + t\alpha_{j+1}\big) = (1-t)w(\alpha_j) + tw(\alpha_{j+1})$$

for all $t \in [0,1]$ and $j = 0, \ldots, n$.

4.10 Theorem *Let X be a nonempty, open and connected subset of a normed vector space. Then any pair of points of X can be connected by a polygonal path in X.*

Proof Let $a \in X$ and

$$M := \big\{\, x \in X\ ;\ \text{there is a polygonal path in } X \text{ connecting } x \text{ and } a \big\}.$$

We now apply the proof technique described in Remark 4.3.

(i) Because $a \in M$, the set M is not empty.

(ii) We next prove that M is open in X. Let $x \in M$. Since X is open, there is some $r > 0$ such that $\mathbb{B}(x,r) \subseteq X$. By Remark 4.9(b), for each $y \in \mathbb{B}(x,r)$, the set $[\![x,y]\!]$ is contained in $\mathbb{B}(x,r)$ and so also in X. Since $x \in M$, there is a polygonal path $w\colon [\alpha,\beta] \to X$ such that $w(\alpha) = a$ and $w(\beta) = x$.

[2]The function $w : [\alpha,\beta] \to X$ is clearly left and right continuous at each point, and so, by Proposition 1.12, is continuous. Thus a polygonal path is, in particular, a path.