Continuity of Power Series and Composition

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1.7. Proposition:

Let $\alpha = \sum_{k=0}^{\infty} a_k x^k$ be a power series with positive radius of convergence ρa. Then the function α represented by α is continuous on $\mathbb{R}/B_\rho$.

Proof: Let $x_0 \in \mathbb{R}B_\rho$, $\varepsilon > 0$, and assume $|x_0| < \rho$. The series $\sum_{k=0}^{\infty} a_k x^k$ converges.
Therefore exists a $K \in \mathbb{N}$, such that $\sum_{k=K+1}^{\infty} |a_k|\rho^k < \frac{\varepsilon}{4}$, thus for $|x| \leq r$, we have

$$|\alpha(x) - \alpha(x_0)| \leq \left|\sum_{k=0}^{K} a_k x^k - \sum_{k=0}^{K} a_k x_0^k\right| + \sum_{k=K+1}^{\infty} |a_k||x|^k + \sum_{k=K+1}^{\infty} |a_k||x_0|^k \leq \dots$$

$$|\rho(x) - \rho(x_0)| + \frac{\varepsilon}{2} \quad \text{with } \rho(x) := \sum_{k=0}^{K} a_k x^k \text{ and}$$

because $\rho(x)$ is a polynomial it is continuous and there is some $\delta \in (0, \rho - |x_0|)$ such that $|\rho(x) - \rho(x_0)| < \frac{\varepsilon}{2}$, $|x - x_0| < \delta$. Therefore we have:

$$|\alpha(x) - \alpha(x_0)| < \varepsilon \quad \text{for all } |x - x_0| < \delta.$$

Since $B(x_0, \delta) \subseteq \mathbb{R}B_\rho$ we have proved the claim. ■

1.8. Theorem:

Let X, Y and Z be metric spaces. Suppose that $f : X \to Y$ is continuous at $x \in X$ and $g : Y \to Z$ is continuous at $f(x) \in Y$. Then the composition $g \circ f : X \to Z$ is continuous at x.

Proof: Let U_z be a neighborhood of an element of Z with $g(y) = z$, then there exists a neighborhood in Y, U_y around y with $g(U_y) \subset U_z$. For this U_y with $y = f(x)$ exists a neighborhood of x, U_x with $f(U_x) \subset U_y$. Therefore: $g \circ f(U_x) \subset U_z$.

Recommended Reading

- **Analysis I** by H. Amann (Match: 0.71)
- **Writing Proofs in Analysis** by Jonathan M. Kane (Match: 0.70)
- **Analysis 1** by Alessio Figalli (Match: 0.70)