Theorem: Let X be a topological space and Y a metric space. Let f_n be a sequence of bounded functions of X to Y, each of which is continuous. Let f be a function from X to Y so that $\overline{\rho}(f_n, f) \to 0$ then f is bounded and continuous.
Proof: Because $\overline{\rho}(f_n, f) \to 0$ there exists a $n \in \mathbb{N}$ such that $\overline{\rho}(f_n, f) < \frac{\varepsilon}{3}$. Pick an open set A containing x_0, so $x \in A \implies \rho(f_n(x), f_n(x_0)) < \frac{\varepsilon}{3}$ which is possible because f_n is continuous. For such x we have

$$\begin{aligned}\rho(f(x), f(x_0)) &\leq \rho(f(x), f_n(x)) + \rho(f_n(x), f_n(x_0)) + \rho(f_n(x_0), f(x_0)) \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon\end{aligned}$$

Thus f is continuous at x_0.

For bounded observe $\exists n \in \mathbb{N} : \overline{\rho}(f_n, f) < 1$ and because f_n is bounded: $\sup_{x \in X} \rho(f_n(x), y) < \infty$ and therefore:

$$\sup_{x \in X} \rho(f(x), y) \leq \sup_{x \in X} \rho(f_n(x), f(x)) + \sup_{x \in X} \rho(f_n(x), y) \leq 1 + \infty$$

Def: $\rho(x, y)$ is a metric on Y, $f : X \to Y$ is bounded if and only if for one y in Y: $\sup_{x \in X}(\rho(f(x), y)) < \infty$. Given two bounded functions from X to Y : $\overline{\rho}(f, g) := \sup_{x \in X}(\rho(f(x), g(x)))$.

Recommended Reading

- **Writing Proofs in Analysis** by Jonathan M. Kane (Match: 0.67)
- **Lecture Notes** by Topologie - Andreas Kriegl (Match: 0.66)
- **Lecture Notes** by Topologie (2018) - Andreas Cap (Match: 0.66)