Continuity 2

Proposition: A function $f : X \to Y$ is continuous if and only if for every open set $O \subseteq Y$, the preimage $f^{-1}(O)$ is open in X.

Proof:

- $\Rightarrow$ Let us assume $f : X \to Y$ is continuous and $O \subseteq Y$ is open in Y. To show that $f^{-1}(O)$ is open, we have to show: $\forall x \in f^{-1}(O) \exists U_x$ open in X such that $U_x \subseteq f^{-1}(O)$. First, if $f^{-1}(O) = \emptyset$, nothing is left to show because $\emptyset$ is open in X. Let's take $x \in f^{-1}(O)$. Then there exists at least one $y \in O$ with $f(x) = y$. Because O is open, O is a neighborhood of y. Because f is continuous, there exists a $\tilde{U}_x$ neighborhood with $f(\tilde{U}_x) \subseteq O$, because of the definition of neighborhood $\tilde{U}_x$ contains an open U_x with $U_x \subseteq \tilde{U}_x$, and so $f(U_x) \subseteq O$, but because of this every $\hat{x} \in U_x$ is also an element of $f^{-1}(O)$, and therefore $U_x \subseteq f^{-1}(O)$.

- $\Leftarrow$ Now suppose $\tilde{O}$ is a neighborhood of an element y in Y, then there exists an open set O in Y with $O \subseteq \tilde{O}$ and $y \in O$. We want to show the preimage of this open set $f^{-1}(O)$ is open and it contains x with $f(x) = y$ if $y \in f(X)$, because $f^{-1}(O)$ is open it is a neighborhood of x and $f(f^{-1}(O)) \subseteq O \subseteq \tilde{O}$. Therefore f is continuous. (The second part is somehow quirky, it should start with $x \in X$ and then take $\tilde{O}$ from Y with $f(x) \in \tilde{O}$, then all works the same way, and we show that f is continuous in X for all $x \in X$).

Recommended Reading

- **Writing Proofs in Analysis** by Jonathan M. Kane (Match: 0.67)
- **Introduction to Mathematical Analysis** by Igor Kriz, Aleš Pultr (Match: 0.66)
- **Basic Analysis I** by James K. Peterson (Match: 0.65)