Accumulation Points of a Set in (M, d)

Proof

Definition (Accumulation Points of a Set in (M, d))**.** Let $A \subset M$ and $x \in M$. A point x is an accumulation point of A if every open set U containing x contains some point of A which is not x. Alternatively: $\forall \varepsilon > 0, (B_\varepsilon(x) \setminus \{x\}) \cap A \neq \emptyset$.

Examples: All accumulation points of $\{\left(x, \sin\left(\frac{1}{x}\right)\right) : x > 0\} \subset \mathbb{R}^2$ are $A \cup \{(0, y) : -1 \leq y \leq 1\}$.

Lemma. A set is closed if and only if the accumulation points of A are in A. Or, there are no accumulation points of A in A^c.

Proof. A closed $\iff$ A^c open $\iff \forall x \in A^c, \exists \varepsilon > 0 : B_\varepsilon(x) \subset A^c$

$$\iff \forall x \in A^c, \exists \varepsilon > 0 : B_\varepsilon(x) \cap A = \emptyset$$

$$\iff x$$

is no accumulation point. □ □

The Closure of a Set in (M, d)

Definition (The Closure of a Set in (M, d))**.** The closure of A (denoted $\mathrm{cl}(A)$ or $\overline{A}$) is the intersection of all closed sets containing A.

Lemma. $\mathrm{cl}(A)$ is closed • $A \subseteq \mathrm{cl}(A)$ • $A = \mathrm{cl}(A) \iff A$ is closed.

Proof. A intersection of closed sets is closed. Every set in the intersection contains A, therefore A is in the intersection. If A is closed, A is in the intersection, so A is the intersection. □

Example:

$$\mathrm{cl}\left(\{\frac{1}{n} : n \in \mathbb{N}\} \subset \mathbb{R}\right) = \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup \{0\}$$

Lemma. Given $A \subset M$, let L be the set of all accumulation points of A. The closure of A is:

$$\mathrm{cl}(A) = A \cup L. \quad (\iff \mathrm{cl}(A)^c = A^c \cap L^c)$$

Proof. □

Recommended Reading

- **Writing Proofs in Analysis** by Jonathan M. Kane (Match: 0.69)
- **Lecture Notes** by Topologie (2018) - Andreas Cap (Match: 0.68)
- **Basic Analysis I** by James K. Peterson (Match: 0.67)