24.10. Axler

Lemma: Suppose that $P, s \in \mathcal{P}(\mathbb{F})$, with $s \neq 0$. Then there exist unique polynomials $q, r \in \mathcal{P}(\mathbb{F})$ such that:

$$P = sq + r$$

and

$\deg r < \deg s$.

Proof: Let $n = \deg p$ and $m = \deg s$. If $m > n$, take $q = 0$ and $r = p$. Thus assume $n \geq m$. The list $\{1, z, \ldots, z^{m-1}\}$ is linearly independent in $\mathcal{P}_n(\mathbb{F})$, also the list $\{1, z, \ldots, z^n\}$ is a basis of $\mathcal{P}_n(\mathbb{F})$. Hence, it is a basis of $\mathcal{P}_n(\mathbb{F})$. Because $p \in \mathcal{P}_n(\mathbb{F})$ there exist unique constants $a_0, \ldots, a_n \in \mathbb{F}$ such that

$$p = a_0 + a_1 z + \cdots + a_{m-1} z^{m-1} + b_0 s + b_1 zs + \cdots + b_{n-m} z^{n-m} s$$

where the first part is r and the second part is qs. (Note: the handwriting suggests a slight reinterpretation of the coefficients a_i and b_i grouping, leading to the structure below based on the standard Division Algorithm proof structure where one constructs r and qs. Let's follow the visible structure which seems to imply p is written as a sum of two parts.)

Let's re-examine the structure implied by the underlining/grouping in the image:

$$p = \underbrace{a_0 + a_1 z + \cdots + a_{m-1} z^{m-1}}_{\text{This part seems to be part of } rextorrelated} + \underbrace{b_0 s + b_1 zs + \cdots + b_{n-m} z^{n-m} s}_{qs} \quad \text{where } b_i \in \mathbb{F}$$

Let $q = b_0 + b_1 z + \cdots + b_{n-m} z^{n-m}$. Since $p \in \mathcal{P}_n(\mathbb{F})$ and $s \in \mathcal{P}_m(\mathbb{F})$, then $\deg(qs) \leq (n - m) + m = n$. If we define $r = p - qs$, then $r \in \mathcal{P}_n(\mathbb{F})$.

We need $\deg r < \deg s = m$. If $r \neq 0$, let $k = \deg r$. If $k \geq m$, we could perform another step of division, contradicting minimality/uniqueness.

We are led to find q, r such that $p = sq + r$ and $\deg r < \deg s$. The coefficients $b_0, b_1, \ldots, b_{n-m} \in \mathbb{F}$ are determined by comparing leading terms, leading to the uniqueness of q and r.

We assert that $p = sq + r$ with $\deg r < \deg s$.

The uniqueness of $q, r \in \mathcal{P}(\mathbb{F})$ follows from the uniqueness of the constants $a_0, a_1, \ldots, a_{m-1} \in \mathbb{F}$ and $b_0, b_1, \ldots, b_{n-m} \in \mathbb{F}$.

Theorem?: Every nonconstant polynomial with complex coefficients has a zero in $\mathbb{C}$.

Recommended Reading

- **Abstract Algebra** by Paul B. Garrett (Match: 0.70)
- **Algebra Notes from the Underground** by Paolo Aluffi (Match: 0.68)
- **Abstract Algebra (3rd Ed)** by Dummit Foote (Match: 0.67)