Product Metric

D: Product Metric

Definition: Let (X_j, d_j), $1 \leq j \leq m$ be metric spaces and $X := X_1 \times \cdots \times X_m$. Then the function

$$d(x, y) := \max_{1 \leq j \leq m} d_j(x_j, y_j)$$

where $x = (x_1, \ldots, x_m) \in X$ and $y = (y_1, \ldots, y_m) \in X$, is called the product metric. The metric space $X = (X, d)$ is called the product of the metric spaces $(X_1, d_1), \ldots, (X_m, d_m)$.

Proposition: For $a = (a_1, \ldots, a_m) \in X$ and $r > 0$, we have

$$\overline{B}_X(a, r) = \prod_{j=1}^{m} \overline{B}_{X_j}(a_j, r)$$

Proof: $x \in \overline{B}_X(a, r) \iff d(a, x) < r$

$$\iff x \in \{x \in X \mid d(a, x) < r\}$$

$$\iff \max_{1 \leq j \leq m} d_j(a_j, x_j) < r$$

$$\iff d_j(a_j, x_j) < r \quad \forall j \in \{1, \ldots, m\}$$

$$\iff x_j \in \overline{B}_{X_j}(a_j, r) \quad \forall j \in \{1, \ldots, m\}$$

$$\iff x \in \prod_{j=1}^{m} \overline{B}_{X_j}(a_j, r)$$

Proposition: Let X be the product of the metric spaces (X_j, d_j), $1 \leq j \leq m$. Then the sequence $(x_n) = ((x_n^1, \ldots, x_n^m))_{n \in \mathbb{N}}$ converges in X to the point $a = (a^1, \ldots, a^m)$ if and only if, for each $j \in \{1, \ldots, m\}$, the sequence $(x_n^j)_{n \in \mathbb{N}}$ converges in X_j to $a^j \in X_j$.

Proof: $\boxed{\forall \epsilon > 0 : \exists N_j \in \mathbb{N} : d_j(x_n^j, a^j) < \epsilon \quad \forall j \in \{1, \ldots, m\}, \text{ for } N := \max N_j \text{ we have } d(x_n, a) < \epsilon \text{ for}}$

On the other hand, if then for all $\epsilon > 0$ there exists a $N \in \mathbb{N}$ such that $d(x_n, a) = \max_{1 \leq j \leq m} d_j(x_n^j, a^j) < \epsilon$ for all $n > N$. Then of course $d_j(x_n^j, a^j) < \epsilon$ for all $n > N$.

Recommended Reading

- **Real Analysis** by Patrick Fitzpatrick (Match: 0.69)

- **Writing Proofs in Analysis** by Jonathan M. Kane (Match: 0.68)

- **Introduction to Mathematical Analysis** by Igor Kriz, Aleš Pultr (Match: 0.68)