Differentiability 1

Conventions

$X \subseteq \mathbb{K}$ is a set, $a \in X$ is a limit point of X and $E = (E, \|\cdot\|_E)$ is a normed vector space over $\mathbb{K}$.

The Derivative

Def: A function $f : X \to E$ is called differentiable at a if the limit

$$f'(a) := \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

exists exists in E. When this occurs, $f'(a) \in E$ is called the derivative of f at a. Besides the symbols $f'(a)$ many other notations for the derivative are used:

$$\dot{f}(a), \quad \partial f(a), \quad Df(a), \quad \frac{df}{dx}(a)$$

Theorem: For $f : X \to E$ the following are equivalent:

1. f is differentiable at a
2. There is some $m_a \in E$ such that
$$\lim_{x \to a} \frac{f(x) - f(a) - m_a(x - a)}{x - a} = 0$$
3. There are $m_a \in E$ and a function $r : X \to E$ which is continuous at a such that $r(a) = 0$ and
$$f(x) = f(a) + m_a(x - a) + r(x)(x - a), \quad x \in X$$

in cases (ii) and (iii) $\implies$ $m_a = f'(a)$.
Proof:

i) $\Rightarrow$ ii) $m_a := f'(a)$.

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} - \frac{m_a(x - a)}{x - a} = m_a - m_a = 0$$

ii) $\Rightarrow$ iii) Define $r(x) :=$

$$\begin{cases} 0 & , x = a \\ \frac{f(x)-f(a)-m_a(x-a)}{x-a} & , x \neq a \end{cases}$$

because of (ii) $\lim_{x\to a} r(x) = 0 = r(a)$ and so r is continuous at a.

We need to show $f(x) = f(a) + m_a(x-a) + r(x)(x-a)$.

For $x = a$:

$$f(a) + m_a(a-a) + r(a)(a-a) = f(a) + 0 + 0 = f(a)$$

For $x \neq a$:

$$\begin{aligned} &f(a) + m_a(x-a) + r(x)(x-a) \\ &= f(a) + m_a(x-a) + \left(\frac{f(x)-f(a)-m_a(x-a)}{x-a}\right)(x-a) \\ &= f(a) + m_a(x-a) + f(x) - f(a) - m_a(x-a) \\ &= f(x) \quad \text{for } x \neq a. \end{aligned}$$

iii) $\Rightarrow$ i) From (iii):

$$\frac{f(x)-f(a)}{x-a} = m_a + r(x)$$

Taking the limit as $x \to a$:

$$\lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{x\to a} m_a + \lim_{x\to a} r(x) = m_a$$

because $r(x)$ is continuous and $r(a) = 0$.

Recommended Reading

- **Analysis I** by H. Amann (Match: 0.71)
- **Fréchet differentiability of Lipschitz functions and porous sets in Banach spaces** by Joram Lindenstrauss (Match: 0.71)
- **Calculus On Normed Vector Spaces** by Rodney Coleman (Match: 0.70)