Linear Algebra Problem Set Snippet

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Problem 2

Suppose $S, T \in \mathcal{L}(V)$ are such that range $S \subseteq$ null T. Prove that $(ST)^2 = 0$. To prove that $(ST)^2 = 0$, we have to show $(ST)^2 v = 0$ for all $v \in V$. Let $v \in V$ then $(ST)^2 v = S(T(S(T(v))))$. Let $\bar{v} = T(v)$, then $S(\bar{v}) \in$ range S and also an element of null T, therefore $T(S(T(v))) = 0$ and because S is linear $S(T(S(T(v)))) = 0$.

Problem 3

Suppose $v_1, \ldots, v_m$ is a list of vectors in V. Define $T \in \mathcal{L}(\mathbb{F}^m, V)$ by

$$T(z_1, \ldots, z_m) = z_1 v_1 + \cdots + z_m v_m$$

1. What property of T corresponds to $v_1, \ldots, v_m$ spanning V? If $\operatorname{span}(v_1, \ldots, v_m) = V$, every $v \in V$ can be written as $v = z_1 v_1 + \cdots + z_m v_m$ and therefore $\operatorname{range}(T) = V$ and T is surjective.
2. What property of T corresponds to the list $v_1, \ldots, v_m$ being linearly independent? If $v_1, \ldots, v_m$ is linearly independent then $z_1 v_1 + \cdots + z_m v_m = 0$ follows $z_1 = \cdots = z_m = 0$, therefore only $0_{\mathbb{F}^m}$ is an element of null T, and T is injective.

Problem 4

Show that $\{T \in \mathcal{L}(\mathbb{R}^5, \mathbb{R}^4) : \dim \operatorname{null} T > 2\}$ is not a subspace of $\mathcal{L}(\mathbb{R}^5, \mathbb{R}^4)$. Let

$$T_1 : (x_1, x_2, x_3, x_4, x_5) \mapsto (0, 0, 0, x_4, x_5)$$
$$T_2 : (x_1, x_2, x_3, x_4, x_5) \mapsto (x_1, x_2, 0, 0, 0)$$

Then

$$(T_1 + T_2) : (x_1, x_2, x_3, x_4, x_5) \mapsto (x_1, x_2, 0, x_4, x_5)$$
$$\dim \operatorname{null}\ (T_1 + T_2) = 1$$

Since $1 \not> 2$, the set is not closed under addition.

Recommended Reading

- **Linear algebra problem book** by Paul R. Halmos (Match: 0.72)
- **Lineare Algebra 1** by Menny-Akka (Match: 0.68)
- **Prüfungstraining Lineare Algebra : Band I** by Thomas C. T. Michaels (Match: 0.68)