Series 4

Theorem (Ratio Test): Let $\sum x_k$ be a series in E and K_0 be such that $x_k \neq 0$ for all $k \geq K_0$. The the following hold:

1. If there are $q \in (0,1)$ and $K \geq K_0$ such that $\left|\frac{x_{k+1}}{x_k}\right| \leq q$ for all $k \geq K$, then the series $\sum x_k$ converges absolutely.
2. If there is some $K \geq K_0$ such that $\left|\frac{x_{k+1}}{x_k}\right| \geq 1$ for all $k \geq K$, then the series $\sum x_k$ diverges.

Proof:

1. $|x_{k+1}| \leq q|x_k|$ for all $k \geq K$. Therefore: $|x_k| \leq q^{k-K}|x_K| = \frac{|x_K|}{q^K} q^k$, for $k \geq K$. Set $c := |x_K|/q^K$. Then $c\sum q^k$ is a convergent majorant for the series $\sum |x_k|$, and $\sum x_k$ converges absolutely absolutely.
2. $|x_{k+1}| \geq |x_k|$ implies that x_k is no null sequence, and so $\sum x_k$ diverges. $\square$

Rearrangement of Series:
Def.: Let $\sigma : \mathbb{N} \to \mathbb{N}$ be a permutation. Then the series $\sum_k x_{\sigma(k)}$ is called a rearrangement of $\sum_k x_k$.
Theorem (Rearrangement Theorem): Every rearrangement of an absolutely absolutely series $\sum x_k$ is absolutely convergent and has the same value as $\sum x_k$.
Proof: For each $\varepsilon > 0$, there is by Cauchy criterion a $N \in \mathbb{N}$ such that $\sum_{k=N+1}^{m} |x_k| < \varepsilon$ for all $m > N$. Taking the limit $m \to \infty$: $\sum_{k=N+1}^{\infty} |x_k| \leq \varepsilon$.

Let σ be a permutation of $\mathbb{N}$. For $M := \max\{\sigma^{-1}(0), \ldots, \sigma^{-1}(N)\}$, we have

$$\left|\sum_{k=0}^{m} x_{\sigma(k)} - \sum_{k=0}^{N} x_k\right| \leq \sum_{k=N+1}^{m} |x_k| \leq \varepsilon$$

$\{\sigma(0), \ldots, \sigma(M)\} \supseteq \{0, \ldots, N\}$, this means we choose M so large that all indices up to N are in the images up to M. Thus, for each $m \geq N$,

$$\left|\sum_{k=0}^{m} x_{\sigma(k)} - \sum_{k=0}^{N} x_k\right| \leq \sum_{k=N+1}^{m} |x_k| \leq \varepsilon \quad \text{and also} \quad \left|\sum_{k=0}^{m} x_{\sigma(k)} - \sum_{k=0}^{N} x_k\right| \leq \varepsilon$$

Recommended Reading

- **Writing Proofs in Analysis** by Jonathan M. Kane (Match: 0.70)
- **Analysis I** by H. Amann (Match: 0.70)
- **Introduction to Real Analysis** by Christopher Heil (Match: 0.69)