Inner Product Spaces

Let E be a vector space over $\mathbb{R}$ or $\mathbb{C}$. A function $(\cdot|\cdot) : E \times E \to \mathbb{R} \setminus \mathbb{C}$ is called a scalar product or inner product on E if:

SP1 $(x|y) = \overline{(y|x)}$

SP2 $(\lambda x + \mu y|z) = \lambda(x|z) + \mu(y|z)$

SP3 $(x|x) \geq 0$ and $(x|x) = 0 \Leftrightarrow x = 0$

Notes:

1. Cauchy-Schwarz Inequality: $|(x|y)|^2 \leq (x|x)(y|y)$
2. $||x|| := \sqrt{(x|x)}$ is a norm on E, the norm induced from the scalar product $(\cdot|\cdot)$.
3. $|(x,y)| \leq ||x||||y||$
4. $2(||x||^2 + ||y||^2) = ||x+y||^2 + ||x-y||^2$, $x, y \in E$
5. $(x \pm y|x \pm y) = (x|x) \pm \mathrm{Re}(x|y) + (y,y)$

Two elements in an inner product space are called orthogonal if $(x|y) = 0$. A subset $M \subseteq E$ is called an orthogonal system if $(x|y) = 0$ for all $x, y \in M$ with $x \neq y$. An orthogonal system is called orthonormal if $||x|| = 1$ for all $x \in M$.

Recommended Reading

- **Linear Algebra** by Seymour Lipschutz (Match: 0.69)
- **Introduction to Real Analysis** by Christopher Heil (Match: 0.69)
- **Real Analysis** by Patrick Fitzpatrick (Match: 0.69)