Every $v \in K^n$ defines a $v'(w) := v_1w_1 + \cdots + v_nw_n$, with $w_1, \ldots, w_n$ basis of V.
Example: $V = P_{n-1}(R)$ and $v = (v_1, \ldots, v_n) \in K$.
If V is a finite dimensional vector space and $B = v_1, \ldots, v_n$ is a basis of V, then the linear map $\varphi_B(v) = (x_1, \ldots, x_n)$ with $v = x_1v_1 + \cdots + x_nv_n$ is an isomorphism from V to K^n. This isomorphism of course depends on B. If $B' = v'_1, \ldots, v'_n$ is another basis of V with $\varphi_{B'}$ as its coordinate isomorphism, we get

$$\psi_{B'B} = \varphi_{B'}^{-1}(\varphi_B(v)) \text{ is a function from } V \to V.$$

1. Is this function linear?

- $\psi_{B'B}(v_1+v_2) = \varphi_{B'}^{-1}(\varphi_B(v_1+v_2)) = \varphi_{B'}^{-1}(\varphi_B(v_1)+\varphi_B(v_2)) = \psi_{B'B}(v_1) + \psi_{B'B}(v_2)$
- $\alpha \cdot \psi_{B'B}(v_1) = \cdots = \psi_{B'B}(v_1\alpha)$.

$\Longrightarrow$

2. What is the null space of the operator $\psi_{B'B}$?

$$\psi_{B'B}(v) = 0 \quad \varphi_{B'}^{-1}(v') = 0 \iff v' = 0 \text{ because } \varphi_{B'}^{-1} \text{ is an isomorphism.}$$

$$\varphi_B(v) = 0 \iff v = 0.$$

Therefore, null $\psi = \{0\}$.

Therefore, $\psi_{B'B}$ is an isomorphism from V to V. $\quad (n = 0{+}\text{range} \implies \text{range} = n)$

3. What is the "meaning" of this isomorphism? The isomorphism from $K^n \to K^n$ with $\varphi_{B'}(\varphi_B^{-1}(x))$ is the change of basis operator. The inverse $\psi_{B'B}^{-1}$ is $\psi_{B'B}^{-1} = \varphi_B^{-1}(\varphi_B(v))$.

Is there any use of this?

Recommended Reading

- **Lineare Algebra 1** by Menny-Akka (Match: 0.68)
- **Prüfungstraining Lineare Algebra : Band I** by Thomas C. T. Michaels (Match: 0.68)
- **Linear algebra problem book** by Paul R. Halmos (Match: 0.68)