Boundary of a Set in (M, d)

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Definition: For a given set $A \subset M$, the boundary of A is

$$\partial A = \overline{A} \cap \overline{(A^c)}$$

(Interpreting cl(A) as $\overline{A}$)

Examples:

$$\partial M = \emptyset, \quad \partial \mathbb{Z} = \emptyset, \quad \partial \mathbb{Q} = \mathbb{R}$$

Lemma 1: $x \in \partial A \iff \forall \varepsilon > 0, B_\varepsilon(x) \cap A \neq \emptyset$ and $B_\varepsilon(x) \cap A^c \neq \emptyset$.
Proof (for $\Rightarrow$):

$$x \in \partial A \iff x \in \overline{A} \text{ and } x \in \overline{(A^c)}$$

Since $x \in \overline{A}$ means that for every $\varepsilon > 0$, $B_\varepsilon(x) \cap A \neq \emptyset$, and $x \in \overline{(A^c)}$ means that for every $\varepsilon > 0$, $B_\varepsilon(x) \cap A^c \neq \emptyset$.

Lemma 2: $\partial A = \overline{A} \setminus \text{int}(A) = \overline{A} \cap (\text{int}(A))^c$.
Proof (for $\Leftarrow$ direction of showing $\overline{A} \cap (\textbf{int}(A))^c$): We use the fact that $(\text{int}(A))^c = \overline{(A^c)}$ because

$$x \in (\text{int}(A))^c \iff x \notin \text{int}(A)$$

$$\iff \forall \varepsilon > 0 : B_\varepsilon(x) \cap A^c \neq \emptyset$$

$$\iff x \in \overline{A^c}$$

Recommended Reading

- **Writing Proofs in Analysis** by Jonathan M. Kane (Match: 0.67)
- **Measure Theory and Fine Properties of Functions** by Gariepy Evans (Match: 0.66)
- **Real Analysis** by Patrick Fitzpatrick (Match: 0.66)