The Limit Superior and Limit Inferior

[Definition] Let (x_n) be a sequence in $\mathbb{R}$. We can define two new sequences (y_n) and (z_n) by

$$y_n := \sup_{k \geq n} x_k := \sup\{x_k; k \geq n\}$$

$$z_n := \inf_{k \geq n} x_k := \inf\{x_k; k \geq n\}$$

Since (x_n) is decreasing and (z_n) is increasing in $\mathbb{R}$, therefore these sequences converge in $\mathbb{R}$.

The limit superior is defined as:

$$\limsup_{n\to\infty} x_n := \lim_{n\to\infty} y_n := \lim_{n\to\infty} \left(\sup_{k \geq n} x_k \right)$$

The limit inferior is defined as:

$$\liminf_{n\to\infty} x_n := \lim_{n\to\infty} z_n := \lim_{n\to\infty} \left(\inf_{k \geq n} x_k \right)$$

We also have:

$$\limsup_{n\to\infty} x_n = \inf_{n\in\mathbb{N}} y_n = \inf_{n\in\mathbb{N}} \left(\sup_{k \geq n} x_k \right) \quad \text{and} \quad \liminf_{n\to\infty} x_n = \sup_{n\in\mathbb{N}} z_n = \sup_{n\in\mathbb{N}} \left(\inf_{k \geq n} x_k \right)$$

Any sequence (x_n) in $\mathbb{R}$ has a smallest cluster point x_* and a greatest cluster point x^* in $\overline{\mathbb{R}}$ and these satisfy:

$$\liminf_{n\to\infty} x_n = x_* \quad \text{and} \quad \limsup_{n\to\infty} x_n = x^*$$

Proof. Let $x^* = \limsup_{n\to\infty} x_n$ and $y_n := \sup_{k\geq n} x_k$ for $n \in \mathbb{N}$. Then (y_n) is a decreasing sequence such that

$$x^* = \inf_{n\in\mathbb{N}} y_n$$

We consider 3 cases:

1. $x^* = -\infty$. Then for each $K > 0$, there is some n such that $-K > y_n = \sup_{k\geq n} x_k$. Otherwise we would have $x_k \geq -K_0$ for some $K_0 \geq 0$. Hence $x_k \in (-\infty, K)$ for all $k \geq n$, that is $x^* = -\infty$ is the only cluster point of (x_n).

2. Suppose that $x^* > -\infty$. For each $\varepsilon > x^*$, we have some n such that $y_n = \sup_{k\geq n} x_k \geq x^*$. The terms x_k for $k \geq n$ are all smaller than $x^* + \varepsilon$. So no cluster point of (x_n) is larger than x^*. Since $\sup_{k\geq n} x_k = y_n \geq x^*$ for all $n \in \mathbb{N}$, we have for each n, some $k \geq n$ such that $x_k > x^* - \varepsilon$. Since we already know that no cluster point of (x_n) is larger than x^*, the interval $(x^* - \varepsilon, x^* + \varepsilon)$ must contain infinitely many terms of (x_n). That is x^* is a cluster point of (x_n).

3. $x^* = \infty$. Because of $x^* = \inf_{n\in\mathbb{N}} y_n$, we have $y_n = \infty$ for all $n \in \mathbb{N}$. Since for each $K > 0$ and any n, there is some $k \geq n$ such that $x_k > K$. This means that $x^* = \infty$ is a cluster point of (x_n), and clearly the largest such point.

The proof that $x_* = \liminf_{n\to\infty} x_n$ is similar. $\square$

Recommended Reading

- **Writing Proofs in Analysis** by Jonathan M. Kane (Match: 0.68)
- **Problems in real analysis** by Teodora-Liliana Rădulescu (Match: 0.67)
- **Undergraduate Analysis** by McCluskey McMaster (Match: 0.65)