Properties of the Dual Basis

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2.M Axler

If $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is a basis of V, then the dual basis of $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is the list $\varphi_1, \ldots, \varphi_n$ of elements of V', where each φ_j is the linear functional on V such that

$$\varphi_j(\mathbf{v}_k) = \begin{cases} 1 & \text{if } k = j \\ 0 & \text{if } k \neq j \end{cases}$$

Suppose $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is a basis of V and $\varphi_1, \ldots, \varphi_n$ is the dual basis. Then, for each $\mathbf{v} \in V$, we have

$$\mathbf{v} = \varphi_1(\mathbf{v})\mathbf{v}_1 + \cdots + \varphi_n(\mathbf{v})\mathbf{v}_n$$

Suppose $\mathbf{v} \in V$. We write $\mathbf{v}$ in terms of the basis $\{\mathbf{v}_k\}$:

$$\mathbf{v} = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n$$

Applying φ_j to both sides yields

$$\varphi_j(\mathbf{v}) = \varphi_j(c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n) = c_1\varphi_j(\mathbf{v}_1) + \cdots + c_n\varphi_j(\mathbf{v}_n) = c_j$$

Thus, $\mathbf{v} = \varphi_1(\mathbf{v})\mathbf{v}_1 + \cdots + \varphi_n(\mathbf{v})\mathbf{v}_n$.

Suppose V is finite-dimensional. Then the dual of a basis of V is a basis of V'. Let $\varphi_1, \ldots, \varphi_n$ denote the dual basis.

Suppose $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is a basis of V. Let $\varphi_1, \ldots, \varphi_n$ denote the dual basis.

First, we show that $\varphi_1, \ldots, \varphi_n$ are linearly independent. Suppose $a_1\varphi_1 + \cdots + a_n\varphi_n = 0$ (the zero linear map).

Then, for each $\mathbf{v}_k$:

$$(a_1\varphi_1 + \cdots + a_n\varphi_n)(\mathbf{v}_k) = a_1\varphi_1(\mathbf{v}_k) + \cdots + a_n\varphi_n(\mathbf{v}_k) = a_k$$

Since the linear map is zero, we must have $a_k = 0$ for each k.

Because the length of the dual basis is n, this linearly independent list must be a basis.

Recommended Reading

- **Linear Algebra Done Right** by Sheldon Axler (Match: 0.71)
- **Linear algebra problem book** by Paul R. Halmos (Match: 0.70)
- **Linear Algebra** by Seymour Lipschutz (Match: 0.70)