CONNECTED SUBSETS OF $\mathbb{R}$ ARE INTERVALS

Theorem: A subset of $\mathbb{R}$ is connected if and only if it is an interval.
Proof:
$\Rightarrow:$ Let $X \subseteq \mathbb{R}$ be connected. Since X has at least two elements, set $a := \inf(X) \in \overline{\mathbb{R}}$ and $b := \sup(X) \in \overline{\mathbb{R}}\{$, and (a, b) is not empty and $X \subseteq [a, b] \cup \{a, b\}$.

We show now that $(a, b) \subseteq X$. Suppose to the contrary that $(a, b) \nsubseteq X$. Then there is some $c \in (a, b)$ with $c \notin X$.

Let $O_1 := X \cap (-\infty, c)$ and $O_2 := X \cap (c, \infty)$. Then O_1 and O_2 are open in X. Off course $O_1 \cup O_2 = X$ and $O_1 \cap O_2 = \emptyset$.

By our choice of a, b and c, there are elements of $X : x, y$ with $x < c$ and $y > c$. This means that $x \in O_1$ and $y \in O_2$, and so O_1 and O_2 are nonempty. Hence X is not connected, contradicting the hypothesis.

Since we have shown $(a, b) \subseteq X$ and we know $X \subseteq [a, b] \cup \{a, b\}$, we have shown X is an interval.

$\Leftarrow:$ Suppose to the contrary that X is an interval and there are open nonempty subsets O_1 and O_2 of X with $O_1 \cap O_2 = \emptyset$ and $O_1 \cup O_2 = X$.

Choose $x \in O_1$ and $y \in O_2$ and suppose $x < y$. Define $z := \sup(O_1 \cap [x, y])$. The element z cannot be in O_1 because O_1 is open in X and X is an interval, and so there is some $\varepsilon > 0$ such that $[z, z+\varepsilon) \subseteq O_1 \cap [x, y]$. This contradicts the supremum property of z. Similarly, z cannot be in O_2, since otherwise there is some $\varepsilon > 0$ such that $(z - \varepsilon, z] \subseteq O_2 \cap [x, y]$, which contradicts $O_1 \cap O_2 = \emptyset$ and the definition of z. Thus $z \notin O_1 \cup O_2 = X$.

On the other hand, $[x, y] \subseteq X$ is contained in X because X is an interval. Since $x \in O_1$ and $y \in O_2$, we have $z = \sup(O_1 \cap [x, y]) \in [x, y] \subseteq X$.

This leads to the contradiction $z \in [x, y] \subseteq X$ and $z \notin X$.

Recommended Reading

- **Writing Proofs in Analysis** by Jonathan M. Kane (Match: 0.69)
- **Introduction to real analysis** by Robert Gardner Bartle (Match: 0.66)
- **Real Analysis** by Patrick Fitzpatrick (Match: 0.66)