3)

Suppose that $T \in \mathcal{L}(F^n, F^m)$.
Show that there exists scalars $A_{j,k} \in \mathbb{F}$ for $j = 1, \ldots, m$ and $k = 1, \ldots, n$ such that:

$$T(x_1, \ldots, x_n) = (A_{1,1}x_1 + \cdots + A_{1,n}x_n, \; \ldots, \; A_{m,1}x_1 + \cdots + A_{m,n}x_n)$$

for every $(x_1, \ldots, x_n) \in F^n$.
Solution
Let $B_n = \{e_1, \ldots, e_n\}$ be a basis for F^n, and $B_m = \{e'_1, \ldots, e'_m\}$ be a basis for F^m.
Since T is linear, we have:

$$T(x_1, \ldots, x_n) = T\left(e_1x_1 + \cdots + e_nx_n\right) = T(e_1)x_1 + \cdots + T(e_n)x_n$$

Since $T(e_k)$ are elements of F^m, therefore $A_{j,k}$ exist such that:

$$\begin{aligned} T(e_1) &= A_{1,1}e'_1 + \cdots + A_{m,1}e'_m \\ &\vdots \\ T(e_n) &= A_{1,n}e'_1 + \cdots + A_{m,n}e'_m \end{aligned}$$

and so:

$$T(x_1, \ldots, x_n) = \left(A_{1,1}e'_1 + \cdots + A_{m,1}e'_m\right)x_1 + \cdots + \left(A_{1,n}e'_1 + \cdots + A_{m,n}e'_m\right)x_n$$

Rearranging by the basis vectors e'_j:

$$T(x_1, \ldots, x_n) = \left(A_{1,1}x_1 + A_{1,2}x_2 + \cdots + A_{1,n}x_n\right)e'_1 + \cdots + \left(A_{m,1}x_1 + A_{m,2}x_2 + \cdots + A_{m,n}x_n\right)e'_m$$

If we let e'_j be the standard basis vectors in F^m, i.e., $e'_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, and so on, then the components of the resulting vector are exactly the terms in the parentheses:

$$T(x_1, \ldots, x_n) = A_{1,1}x_1 + A_{1,2}x_2 + \cdots + A_{1,n}x_n, \; \ldots, \; A_{m,1}x_1 + A_{m,2}x_2 + \cdots + A_{m,n}x_n$$

(The original note seems to use a slightly different notation for the resulting vector components in the first display, implying e'_j corresponds to the j-th component, which is standard when e'_j is the standard basis. We interpret

$T(x_1, \ldots, x_n)$ as a vector in F^m whose j-th component is the expression shown in the j-th position of the first displayed equation.)

Recommended Reading

- **Linear algebra problem book** by Paul R. Halmos (Match: 0.68)
- **Lineare Algebra 1** by Menny-Akka (Match: 0.68)
- **Prüfungstraining Lineare Algebra : Band I** by Thomas C. T. Michaels (Match: 0.68)