Interior and Closed Sets in Metric Spaces

February 11, 2026

The Interior of a Set in a Metric Space (M, d)

Let $A \subseteq M$. A point $x \in A$ is an interior point of A if there exists an open set U such that $x \in U \subseteq A$.

The interior of A, $\text{int}(A)$, is the set of all interior points of A.

Example:

$$\text{int}(\mathbb{R}^n) = \mathbb{R}^n, \quad \text{int}((0,1)) = (0,1), \quad \text{int}(\Gamma \cup \{1\}) = (0,1),$$

$\text{int}(\mathbb{Z}) = \emptyset$. If $A \subseteq \mathbb{R}$ is any set such that d is the discrete metric, $\text{int}(A) = A$.

The interior of $A \subseteq M$ is the union of all open subsets of A.

Proof: Let $\mathcal{B}$ call the union of all open subsets of A. If $x \in \text{int}(A) \implies \exists$ open U such that $x \in U \subseteq A$. If $x \in \mathcal{B} \implies x$ lies in one element of $\mathcal{B} \implies x \in \text{int}(A)$.

$\text{int}(A)$ is open.

A is open $\iff \text{int}(A) = A$.

Proof: $(\Rightarrow)$: Suppose A is open. Then for all $x \in A$, $x \in A \subseteq A$. So $x \in \text{int}(A)$. Thus $A \subseteq \text{int}(A)$. Since $\text{int}(A) \subseteq A$, we have $\text{int}(A) \subseteq A$. $(\Leftarrow)$: If $A = \text{int}(A)$ and $\text{int}(A)$ is open $\implies A$ is open.

Closed Sets in a Metric Space (M, d)

Let (M, d) be a metric space. A set $B \subseteq M$ is closed if its complement $M \setminus B$ is open.

In a metric space (M, d):

1. M and $\emptyset$ are closed.
2. If A_λ is closed for each index $\lambda \in \Lambda$, then $\bigcap_{\lambda \in \Lambda} A_\lambda$ is closed.
3. If $A_1, \ldots, A_N$ are closed, then $\bigcup_{k=1}^{N} A_k$ is closed.

Proof:

1. $M^c = \emptyset$; $\emptyset^c = M$.

2. A_λ closed $\implies$ A_λ^c open. $\bigcap_{\lambda\in\Lambda} A_\lambda = \left(\bigcup_{\lambda\in\Lambda} A_\lambda^c\right)^c$ closed;

3. $A_1, \ldots, A_N$ closed $\implies$ $\bigcap_{i=1}^N A_i^c$ open; $\bigcup_{i=1}^N A_i = \left(\bigcap_{i=1}^N A_i^c\right)^c$ closed.

Recommended Reading

- **Writing Proofs in Analysis** by Jonathan M. Kane (Match: 0.69)
- **Introduction to Real Analysis** by Christopher Heil (Match: 0.68)
- **Real Analysis** by Patrick Fitzpatrick (Match: 0.67)