T is surjective iff there exists S such that TS = id_W

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20. Suppose W is finite dimensional and T $\in \mathcal{L}(V, W)$. Prove that T is surjective if and only if there exis
$\mathcal{L}(W, V)$ such that TS is the identity operation on W.
Proof:
$\Rightarrow)(T$ is surjective$):$ range$(T) = W$
Let $B_W = \{w_1, \ldots, w_n\}$ be a basis of W. Then because range$(T) = W,$ for every $w_i \in B_W$ there exists a $v_i \in V$ with $T(v_i) = w_i$.
Define $S:$
$Sw_i = v_i,$
S is in $\mathcal{L}(W, V)$ and with this definition
$TS(w) = T(a_1 S(w_1) + \cdots + a_n S(w_n))$
$= a_1 TS(w_1) + \cdots + a_n TS(w_n)$
$= a_1 w_1 + \cdots + a_n w_n =$
therefore $TS(w) = \mathrm{id}_W$
$\Leftarrow (S$ exists$\ldots):$
$w = T(Sw),$ but this implies $v = Sw : T(v) = w$ and
therefore range$(T) = W$

Recommended Reading

- **Linear algebra problem book** by Paul R. Halmos (Match: 0.63)
- **Linear Algebra Done Right** by Sheldon Axler (Match: 0.62)
- **Linear Algebra** by Meckes Meckes (Match: 0.62)