Problems on Linear Operators and Eigenvalues

February 11, 2026

1. Suppose $P \in \mathcal{L}(V)$ is such that $P^2 = P$. Prove that if λ is an eigenvalue of P, then $\lambda = 0$ or $\lambda = 1$. Let $v \in V$ be an eigenvector corresponding to the eigenvalue λ, so $Pv = \lambda v$ and $v \neq 0$. Applying P again gives $P(Pv) = P(\lambda v) = \lambda(Pv) = \lambda(\lambda v) = \lambda^2 v$. Since $P^2 = P$, we have $P^2 v = Pv = \lambda v$. Thus, $\lambda^2 v = \lambda v$, which implies $(\lambda^2 - \lambda)v = 0$. Since $v \neq 0$, we must have $\lambda^2 - \lambda = 0$, so $\lambda(\lambda - 1) = 0$. Therefore, $\lambda = 0$ or $\lambda = 1$.

2. Define $T : \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$ by $T_P = P'$. Find all eigenvalues and eigenvectors of T. We look for an eigenvalue λ and a non-zero eigenvector P such that $TP = \lambda P$, which means $P' = \lambda P$. If P is a non-zero polynomial, let $n = \deg(P)$. Then $\deg(P') = n - 1$ (if $n \geq 1$) or $P' = 0$ (if $n = 0$).

   If $\lambda \neq 0$, then $\deg(\lambda P) = \deg(P) = n$. However, $\deg(P') < n$ (or $P' = 0$ if $n = 0$). Thus, $P' = \lambda P$ can only hold if $P' = 0$, which implies P is a constant polynomial, $P(x) = c \neq 0$. If $P(x) = c$, then $P'(x) = 0$. The equation becomes $0 = \lambda c$. Since $c \neq 0$, we must have $\lambda = 0$. This contradicts the assumption $\lambda \neq 0$. Therefore, the only possible eigenvalue is $\lambda = 0$.

   If $\lambda = 0$, the equation is $P' = 0$, which means P must be a constant polynomial, $P(x) = c$, where $c \neq 0$ because P is an eigenvector. The corresponding eigenspace is $E_0 = \operatorname{span}\{1\}$.

   (Note: The handwritten solution mentions P^0 is an eigenvector with $\lambda = 0$ and discusses degree, aligning with this conclusion that $\lambda = 0$ is the only eigenvalue, with eigenvectors being constant polynomials.)

3. Define $T \in \mathcal{L}(P_4(\mathbb{R}))$ by $(TP)(x) = xP'(x)$ for all $x \in \mathbb{R}$. Find all eigenvalues and eigenvectors. We test basis elements $P(x) = x^k$ for $k \in \{0, 1, 2, 3, 4\}$. Note that $P_4(\mathbb{R})$ has dimension 5. We define $P^0 := 1$. For $P(x) = x^k$:

   $$(TP)(x) = xP'(x) = x(kx^{k-1}) = kx^k = kP(x)$$

   Thus, $P(x) = x^k$ is an eigenvector with eigenvalue $\lambda = k$. The distinct eigenvalues are $\lambda \in \{0, 1, 2, 3, 4\}$. The corresponding eigenvectors are $P_k(x) = x^k$, for $k = 0, 1, 2, 3, 4$. Since we have found 5 linearly independent eigenvectors in a 5-dimensional space, these are all the eigenvalues and eigenvectors. The set of eigenvectors $\{x^0, x^1, x^2, x^3, x^4\}$ forms a basis of $P_4(\mathbb{R})$.

4. Suppose V is finite dimensional, $T \in \mathcal{L}(V)$, and $\alpha \in \mathbb{F}$. Prove that there exists $\delta > 0$ such that $T - \lambda I$ is invertible for all $\lambda \in \mathbb{F}$ such that $0 < |\alpha - \lambda| < \delta$. Because V is finite dimensional, $T - \lambda I$ is invertible if and only if λ is not an eigenvalue of T. Let $\sigma(T)$ be the set of eigenvalues of T. Since V is finite dimensional, $\sigma(T)$ is a finite set.

   We consider two cases for α:

   (a) Case 1: α is not an eigenvalue of T, i.e., $\alpha \notin \sigma(T)$. Since $\sigma(T)$ is finite, the set $\{|\alpha - \lambda| : \lambda \in \sigma(T)\}$ is a finite set of non-negative real numbers, and since $\alpha \notin \sigma(T)$, $\alpha - \lambda \neq 0$ for all $\lambda \in \sigma(T)$. Define $\delta_1 := \min\{|\alpha - \lambda| : \lambda \in \sigma(T)\}$. Since the minimum is taken over a finite set of positive numbers, $\delta_1 > 0$. If we choose $\delta = \delta_1$, then for any λ such that $0 < |\alpha - \lambda| < \delta$, we have $|\alpha - \lambda| < \delta_1$, which implies λ cannot be an eigenvalue of T. Thus, $T - \lambda I$ is invertible.

   (b) Case 2: α is an eigenvalue of T, i.e., $\alpha \in \sigma(T)$. Let $\sigma'(T) = \sigma(T) \setminus \{\alpha\}$ be the set of eigenvalues of T other than α. If $\sigma'(T)$ is empty, then $\sigma(T) = \{\alpha\}$. Define $\delta = 1$. Then for any λ such that $0 < |\alpha - \lambda| < \delta$, $\lambda \neq \alpha$, so $\lambda \notin \sigma(T)$, and $T - \lambda I$ is invertible. If $\sigma'(T)$ is non-empty, define $\delta_2 := \min\{|\alpha - \lambda| : \lambda \in \sigma'(T)\}$. Since $\sigma'(T)$ is a finite set of eigenvalues different from α, $\delta_2 > 0$. If we choose $\delta = \delta_2$, then for any λ such that $0 < |\alpha - \lambda| < \delta$, we have $|\alpha - \lambda| < \delta_2$, which implies $\lambda \neq \alpha$ and $\lambda \notin \sigma'(T)$. Thus $\lambda \notin \sigma(T)$, and $T - \lambda I$ is invertible.

   In both cases, we can find such a $\delta > 0$. (The handwritten text seems to conflate α being an eigenvalue or not, and uses Δ_α which corresponds to δ_2 or δ_1 depending on context, finally settling on $\delta = \Delta_\alpha/2$ in the second case, which is a standard way to ensure the open ball around α contains no other eigenvalues.)

Recommended Reading

- **Linear algebra problem book** by Paul R. Halmos (Match: 0.72)
- **Prüfungstraining Lineare Algebra : Band II** by Thomas C. T. Michaels, Marcel Liechti (Match: 0.71)
- **Linear Algebra Done Right** by Sheldon Axler (Match: 0.71)