Eigenvalues and Eigenvectors

Theorem: Every operator on a finite-dimensional nonzero complex vector space has an eigenvalue.
Proof: Suppose V is a finite-dimensional complex vector space of dimension $n > 0$ and $T \in \mathcal{L}(V)$. Choose $v + 0 \in V$, then the list $\{v, Tv, \ldots, T^n v\}$ because the dimension of V is n is linearly dependent: $v, Tv, \ldots, T^n v$ There exists $k \in \{1, \ldots, n\}$ such that $T^k v$ is a linear combination of $\{v, Tv, \ldots, T^{k-1}v\}$. Since $\dim(V) = n$, the list $\{v, Tv, \ldots, T^n v\}$ is linearly dependent, and the length of this list is $n + 1$, and the dimension of V is n. Therefore,

$$\sum_{k=0}^{n} a_k T^k v = 0 \quad \text{with some } a_k \neq 0.$$

This defines a polynomial $p(T)$ with $p(T)v = 0$, if there are more possible $p(t)$ we choose the one with the smallest degree. Because of the fundamental theorem of algebra there exists $\lambda \in \mathbb{C}$ with $p(\lambda) = 0$, and so we can write $p(z) = (z-\lambda)q(z)$, and $p(T)v = (T - \lambda I) \cdot q(T)v = 0$, but because $q(T)$ is of smaller degree as $p(T)$, $q(T)v \neq 0$ and so $(T - \lambda I)q(T)v = 0$. This implies x is an eigenvalue and $q(T)v$ is the associated eigenvector. $q(T)v$ is an eigenvector.
Theorem: For every $T \in \mathcal{L}(V)$ of a finite dimensional vector space yields a unique monic polynomial p with smallest degree and $p(T) = 0$ and $\dim V \geq$ degree p.
Proof: Suppose $\dim(V) = n$. We prove the theorem through induction over the dimension of V. 1) $n = 0$: because for $\dim V = 0$, I is the null operator, we have $p = I$. 2) We now take on the case of $\dim V > 0$ and know therefore that for every vector space not with dimension smaller than V and for every operator defined on them the theorem holds. Let's start with a vector $v \in V$, $v \neq 0$ and define the list $v, Tv, \ldots, T^n v$. This list is linearly dependent and with the linear dependence lemma we find a $m \leq n$ such that $\alpha_0 v + \alpha_1 Tv + \cdots + \alpha_{m-1}T^{m-1}v + \alpha_m T^m v = T^m v$ and so with $p(z) = z^m + \alpha_{m-1}z^{m-1} + \cdots + \alpha_0$ we have a polynomial of degree $m \leq \dim V$ with $p(T)(v) = 0$. Observe that for all $k \in \mathbb{N}$ we have

$$p(T^k(v)) = T^k(p(v)) = T^k(0) = 0$$

and that the list $\{v, \ldots, T^{m-1}v\}$ is linearly independent, this means $\dim \operatorname{null}(p(T)) \geq m$ and therefore $\operatorname{range}(p(T)) \leq n - m$. Because $\operatorname{range}(p(T))$ is invariant under T, we can define an operator $T|_{\operatorname{range}(p(T))}$ on the subspace $\operatorname{range}(p(T))$. Because of $\operatorname{range}(p(T)) = \dim V - \dim(\operatorname{null}(p(T)) \leq n - m$ we can use the induction hypothesis and have a polynomial $s(t)$ with degree $s \leq n - m$ and $s(T) = 0$ for all $v \in \operatorname{range}(p(T))$. When we now define $q(z) = s(p(z))$ we have $s \cdot p(T)(v) = s(T) \cdot p(T)(v) = 0$ Thus sp is a monic polynomial with degree smaller than or equal then n and $sp(T) = 0$, this proves the existence of the polynomial.

Recommended Reading

- **Linear Algebra Done Right** by Sheldon Axler (Match: 0.73)
- **Linear Algebra** by Meckes Meckes (Match: 0.72)
- **Lineare Algebra 1** by Menny-Akka (Match: 0.72)