Let X and Y be normed spaces. A mapping $f : E \to Y$ of a set $E \subset X$ into Y is differentiable at an interior point $x \in E$ if there exists a continuous linear transformation $\mathcal{L}(x) : X \to Y$ such that

$$f(x+h) - f(x) = \mathcal{L}(x) \cdot h + \alpha(x;h) \quad (*)$$

where $\alpha(x,h) = o(\|h\|)$ as $h \to 0$, $x + h \in E$.

Proposition: If a mapping $f : E \to Y$ is differentiable at an interior point x of a set $E \subset X$, its differential $\mathcal{L}(x)$ at that point is uniquely determined.

Proof: Let $\mathcal{L}_1(x)$ and $\mathcal{L}_2(x)$ be linear mappings satisfying $(*)$, that is

$$f(x+h) - f(x) = \mathcal{L}_1(x) \cdot h + \alpha_1(x,h) f(x+h) - f(x) = \mathcal{L}_2(x) \cdot h + \alpha_2(x,h)$$

where $\alpha_i(x;h) = o(\|h\|)$ as $h \to 0$, $x + h \in E$, $i = 1, 2$.

Then $\mathcal{L}(x) = \mathcal{L}_2(x) - \mathcal{L}_1(x)$ and $\alpha(x;h) = \alpha_2(x;h) - \alpha_1(x;h)$ we obtain

$$\mathcal{L}(x)h = \alpha(x;h)$$

Here $\mathcal{L}(x)$ is a mapping that is linear with respect to h, and $\alpha(x;h) = o(\|h\|)$ as $h \to 0$, $x + h \in E$.

Taking an auxiliary numerical parameter λ, we can now write

$$\|\mathcal{L}(x)h\| = \frac{\|\mathcal{L}(x)(\lambda h)\|}{\|\lambda\|} = \frac{\|\alpha(x;\lambda h)\|}{\|\lambda h\|} \frac{\|h\|}{\|\lambda\|} = \frac{\|\alpha(x;\lambda h)\|}{\|\lambda h\|} \|h\|$$

Thus $\mathcal{L}(x)h = 0$ for any $h \neq 0$.

Recommended Reading

- **Linear Functional Analysis** by Alt Nuernberg (Match: 0.71)
- **Fréchet differentiability of Lipschitz functions and porous sets in Banach spaces** by Joram Lindenstrauss (Match: 0.70)
- **Linear Functional Analysis** by Hans Wilhelm Alt (Match: 0.70)