13.) Suppose $V_1, \ldots, V_m$ is linearly independent in V and $W \in V$. Show that

$$V_1, \ldots, V_m, W \text{ linearly independent} \iff W \notin \operatorname{span}(V_1, \ldots, V_m)$$

Solution

Forward direction ($\Rightarrow$)

This follows straight from the definitions of linearly independent.

Backward direction ($\Leftarrow$)

Suppose $V_1, \ldots, V_m$ linearly independent and $W \notin \operatorname{span}(V_1, \ldots, V_m)$. We assume there exist scalars $a_1, \ldots, a_m, b$ such that

$$a_1 V_1 + \ldots + a_m V_m + bW = 0$$

We show $a_1 = \ldots = a_m = b = 0$.

1. If $b \neq 0$:
$$W = -\frac{a_1}{b} V_1 - \ldots - \frac{a_m}{b} V_m$$
This implies $W \in \operatorname{span}(V_1, \ldots, V_m)$ in contradiction to our premise.

2. So, $b = 0$ and
$$a_1 V_1 + \ldots + a_m V_m + 0W = 0$$
$$a_1 V_1 + \ldots + a_m V_m = 0$$
But now the linear independence of $V_1, \ldots, V_m$ implies $a_1 = \ldots = a_m = 0$.

Thus, $a_1 = \ldots = a_m = b = 0$.

Recommended Reading

- **Prüfungstraining Lineare Algebra : Band I** by Thomas C. T. Michaels (Match: 0.67)
- **Lineare Algebra 1** by Menny-Akka (Match: 0.66)
- **Groups, Matrices, and Vector Spaces** by James B. Carrell (Match: 0.66)