Least Upper Bounds and Greatest Lower Bounds

A set $A \subseteq R$ is bounded above if there exists a number b such that $a \leq b$ for all $a \in A$. b is called an upper bound for A.

A set $A \subseteq R$ is bounded above if there exists a number $l \in R$ such that $l \leq a$ for all $a \in A$. l is called a lower bound.

A real number s is the least upper bound for a set $A \subseteq R$, denoted $\sup(A)$.

1. s is an upper bound for A.
2. If b is any upper bound of A, then $s \leq b$.

Similarly, the greatest lower bound for A is defined as ...

Example: $A = \{\frac{1}{n} : n \in N\}$. Claim: $\sup(A) = 1$.

1. For all $n \in N$, $n \geq 1$, so $\frac{1}{n} \leq 1$.
2. If b is an upper bound, then for all $n \in N$, $\frac{1}{n} \leq b$ must be true. $1 \in A$, therefore $1 \leq b$.

A real number a_0 is a maximum of the set A if a_0 is an element of A and $a_0 \geq a$ for all $a \in A$.

Similarly, a number a_1 is a minimum of A if $a_1 \leq a$ for all $a \in A$ and $a_1 \in A$.

Lemma: Assume $s \in R$ is an upper bound for a set $A \subseteq R$. Then $s = \sup A$ if and only if, for every choice of $\epsilon > 0$, there exists an element $a \in A$ satisfying $s - \epsilon < a$.

Proof: $\implies A$

Recommended Reading

- **Writing Proofs in Analysis** by Jonathan M. Kane (Match: 0.70)
- **Basic Analysis I** by James K. Peterson (Match: 0.70)
- **Undergraduate Analysis** by McCluskey McMaster (Match: 0.69)