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Stone Duality for First Order Logic

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INTRODUCTION

The most interesting phenomena in model theory are conclusions concerning the syntactical structure of a first order theory drawn from the examination of the models of the theory. With these phenomena in mind, it is natural to ask if it is possible to endow the collection of models of the theory with a natural abstract structure so that from the resulting entity one can fully recover the theory as a syntactical structure. We report here on results intended to constitute a positive answer to this question.

Consider a first order theory T. The models of T form a category Mod T with morphisms the elementary embeddings. One oberves that every formula ϕ of the theory, say with one free variable, gives rise to a functor $[\phi]$ from Mod T to SET, the category of sets. $[\phi]$ associates with any model M the extension $\phi(M)$ of ϕ in M, and which every morphism h: $M \to N$ the map $a \mapsto h(a)$; since h is elementary, the latter is a function $\phi(M) \to \phi(N)$. Let us call functors from Mod T to SET of the form $[\phi]$ *standard.* We would like to find properties of functors from Mod T to SET that are characteristic of standard functors.

By Los' theorem, ultraproducts of models of T are again models of T. Moreover, the operation of taking ultraproducts, $\prod_{i\in I}(\ )_i/U$, with a given ultrafilter U on a set I, can be construed as a functor $(\text{Mod } T)^I \to \text{Mod } T$, by defining the ultraproduct of elementary maps in a natural way. Also, one has an ultraproduct functor $\prod_{i\in I}(\ )_i/U$: $(\text{SET})^I \to \text{SET}$ defined similarly. By Los' theorem again, the standard functors are seen to preserve ultraproducts, at least up to isomorphism; the precise statement of this asserts the existence of a natural isomorphism of two composite functors.

There are canonically defined maps between various ultraproducts of sets; the simplest one is the diagonal map of a set into an ultrapower of it.

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A general notion of such canonical maps, called ultramorphisms, is the main new concept of the present work. We can lift ultramorphisms from SET to Mod T and talk about functors from Mod T to SET preserving them. The content of our main result (Theorem 4.1) is that the structure preserving functors from Mod T to SET, with respect to all the aforementioned structure put on Mod T and SET, will be essentially only the standard ones. A more precise statement says that the category of structure preserving functors from Mod T to SET with morphisms the structure preserving natural transformations is equivalent to the pretopos completion of the theory; cf. [MR].

The results of this paper should be compared to the Stone duality theory for Boolean algebras and Stone spaces [H]. If T is a theory in (finitary) propositional logic (given by a "language" which is a set of of propositional atoms, and a set of axioms in the given language), one has the associated Lindenbaum–Tarski (L-T) algebra $B = B(T)$ of T; the elements of B are equivalence classes $\phi/\sim$ of propositional formulas under the equivalence relation

$$\phi \sim \psi \Leftrightarrow T \vdash \phi \leftrightarrow \psi.$$

A model is an assignment of truthvalues ("true" or "false") to the propositional atoms that makes all axioms true; the models of T are in one-to-one correspondence with the ultrafilters on B. The well-known Stone space $S(B)$ of ultrafilters on B can therefore be construed as "the space of models of T." Upon identifying propositional theories with their L-T algebras (which means an abstraction from the "presentation" given by the language and the axioms) and identifying interpretations of a propositional theory in another (in the obvious sense) by Boolean homomorphisms between the corresponding L-T algebras, $\mathscr{B}\mathit{ool}\mathit{e}$, the category of Boolean algebras is construed as the category of propositional theories. On the other hand, for any $B \in |\mathscr{B}\mathit{ool}\mathit{e}|$, $S(B)$ is an object of **Stone**, the category of Stone (compact totally disconnected) spaces. $B \mapsto S(B)$ is the object function of a functor

$$S\colon \quad \mathscr{B}\mathit{ool}\mathit{e}^{op} \to \mathbf{Stone}$$

and the duality theorem asserts that S is an equivalence of categories.

In our theory, propositional theories are replaced by theories in first order predicate logic (with equality); the L-T algebra of a propositional theory is replaced by the pretopos completion $P(T)$ of a first order theory T. (The notion of pretopos was introduced in [SGA4]. Part of the subject matter of [MR] is the justification of the "identification" of coherent theories with pretoposes; the pretopos completion of a coherent theory is the pretopos of coherent objects of the classifying topos of the theory. In the last section of [M1] it is explained in detail how to construe any first order theory as a coherent one by changing the language and what the resulting pretopos completion will be.) Just as propositional models of T (a propositional theory) became 2-valued homomorphisms $B(T) \to 2$, models of T (a first order theory) become identified with certain functors $P(T) \to$ SET where SET is the category of all sets and functions (these functors are called "elementary" in this paper, "logical" in [MR]; for definitions, see also Sect. 1 below). The category $\mathcal{B}ool\!e$ is thus replaced by $\mathcal{PT}$, the category of pretoposes and elementary functors.

The counterpart of **Stone** is of course the new contribution of this paper; it is **UC**. the category of *ultracategories*. A *pre-ultracategory* **K** is a category K together with (arbitrary) functors ("ultraproducts")

$$[U]:\quad K^I \to K$$

associated with all ultrafilters (I, U). An *ultracategory* is a pre-ultracategory together with a certain additional structure consisting of *ultramorphisms* (the definition of ultramorphisms is given in Section 3; later in this Introduction some remarks concerning them will be given). A *pre-ultrafunctor* is a functor between pre-ultracategories preserving the pre-ultrastructure ("ultraproducts") up to *specified* natural isomorphisms; these natural isomorphisms form a part of the data defining a pre-ultrafunctor (see Sect. 2). An *ultrafunctor* is a pre-ultrafunctor also "preserving ultramorphisms" (see Sect. 3). The objects of **UC** are the ultracategories, the morphisms the ultrafunctors.

The Stone functor S: $\mathcal{B}ool\!e^{\mathrm{op}} \to \mathbf{Stone}$ is naturally replaced by the functor

$$\mathbf{Mod}:\quad \mathcal{PT}^{\mathrm{op}} \to \mathbf{UC};$$

for a pretopos T, **Mod** T is the *ultracategory of models* of T; this is the category Mod T of all elementary functors $T \to$ SET endowed with the usual ultraproducts and with the ultramorphisms as already mentioned above. **Mod** is not an equivalence of categories however, its properties are explained as follows.

The first point is that both $\mathcal{PT}$ and **UC** are 2-categories (cf., e.g., [CWM]): the hom-"sets" $\mathcal{PT}(\mathcal{T}, \mathcal{T}')$, $\mathbf{UC}(\mathbf{K}, \mathbf{K}')$ are both categories. (In Sect. 7, there is a self-contained introduction to 2-categories.) The morphisms of the latter categories (2-cells of the 2-categories) are all the natural transformations in the first case, and the *ultratransformations* (ultraproduct preserving natural transformations) in the second case (see Sect. 2).

The functor $G = \mathbf{Mod}$ is in fact a 2-functor. Moreover, it has a left adjoint F: $\mathbf{UC} \to \mathcal{PT}$; F is a 2-functor, and the adjunction is meant in the straightforward "strict" sense for 2-functors (see Sect. 7). For $\mathbf{K}$ an ultracategory, $F(\mathbf{K})$ is the category (a pretopos) of all ultrafunctors from $\mathbf{K}$ to SET, and ultratransformations between them.

The counit of this adjunction, a 2-transformation

$$\varepsilon\colon\ \mathrm{Id}_{\mathscr{P}\mathscr{T}} \to FG,$$

has the property that for any small pretopos T,

$$\varepsilon_T\colon\ T \to FGT$$

is equivalence of categories; this is the main result of the paper, Theorem 4.1 (and in a sharpened form, Theorem 5.2). In other words, the "theory" (pretopos) T is recovered, up to equivalence, as the value of the functor F at $\mathbf{Mod}\,T$, the ultracategory of models. The situation is analogous to an ordinary (1-) adjunction $F \dashv G$ of functors $G\colon \mathscr{C} \to \mathscr{D}$, $F\colon \mathscr{D} \to \mathscr{C}$ in which the right adjoint G is a full embedding (see Theorem 1, p. 88 in [CWM]) and thus C becomes a full reflective subcategory of $\mathscr{D}$ in the terminology of [CWM]. Our result can also be paraphrased somewhat imprecisely that $\mathscr{P}\mathscr{T}^{\mathrm{op}}$ is a full reflective sub-2-category of $\mathbf{UC}$ at least as far as small objects of $\mathscr{P}\mathscr{T}$ are concerned. More precisely, we say that the adjunction $F \dashv G$ is a "reflection in the small"; see Section 8.

The author has made the observation that the construction of the adjunction $F \dashv G$ is based solely on a simple basic property of SET, the category of sets. This property is that SET is a pretopos and at the same time an ultracategory in a standard way; moreover the two structures on SET "commute" in a sense which is expressed by what we call Los' theorem: any ultraproduct functor $[U]\colon \mathrm{SET}^I \to \mathrm{SET}$ is an elementary functor between pretoposes. In Section 8 we explain in somewhat more detail the general idea of what we call the Stone adjunction based on a pair of "structures" $(\mathscr{A}, \mathbf{A})$ with identical "underlying objects" A such that $\mathscr{A}$ "commutes" with $\mathbf{A}$. The Stone duality turns out to come from the Stone adjunction based on the two-element set endowed with the standard Boolean algebra structure on the one hand, and the discrete topological space structure on the other.

Professor Michael Barr has informed me that the general idea of what I called Stone adjunction was found by William Lawvere many years ago; Lawvere also pointed out that important instances of duality in mathematics (such as the Gelfand duality between Banach algebras and compact T_2-spaces) fall under the general scheme of Lawvere (Stone) adjunction. (I emphasize that it is only the adjunction that follows from general principles; the additional properties of being a reflection, or even an equivalence, are proved in each case by detailed examination of the concrete situation.) A general theory of duality, including a precise general formulation of Stone adjunctions for the case the "left-hand-side" category is operational (essentially: a category of algebras defined by operations on underlying sets), is given in [11].

We are not aware of instances in the literature of the Stone adjunction for structures with underlying *categories* (such as our case) instead of structures with underlying *sets*. However, we can point out a simple instance as follows.

Let Lex denote the 2-category of left exact categories: its objects (0-cells) are the categories having all finite (left) limits: its 1-cells are the finite-limit-preserving functors, and its 2-cells are all natural transformations between them. On the other hand, **UD** is the 2-category of categories with arbitrary small limits, and directed colimits; again, the 1-cells are the structure-preserving functors, and the 2-cells are all the natural transformations. Given $R \in |\text{Lex}|$, $G(R)$ is the category Lex(R, SET) of all left exact functors $R \to \text{SET}$; this category has all small limits and directed colimits, i.e., $G(R) \in |\mathbf{LD}|$. G is a part of a Stone adjunction $F \dashv G$ (based again on SET); moreover, it turns out that (similarly to our main result) this adjunction is a reflection in the small. This is equivalent to saying that the canonical "evaluation" functor

$$R \to \mathbf{LD}(\text{Lex}(R, \text{SET}), \text{SET})$$

is an equivalence of categories, for any small left exact category R. (Originally, the author found this fact as a consequence of the present work; Michael Barr has pointed out a simple direct proof to the author.)

Next, we point out a third "2-categorical" Stone adjunction, this time involving exact categories (cf. [EC]). We will then make a comparison between the three, to emphasize novel aspects of the one for pretoposes.

Let $\mathscr{E}x$ be the 2-category of (finitely complete) exact categories (cf. [EC]) with exact functors as 1-cells and all natural transformations as 2-cells. Let **ΠD** denote the 2-category of all categories having all small products, and directed colimits. Since ultraproducts can be defined in terms of products and directed colimits, any object of **ΠD** can be considered in a natural way to be a pre-ultracategory; a **ΠD**-u-*category* is by definition an ultracategory whose pre-ultracategory part is an object of **ΠD**. For the same reason, a 1-cell X: $\mathbf{K} \to \mathbf{K}'$ in **ΠD** can be regarded in a natural way to be a pre-ultrafunctor; then, if $\mathbf{K}, \mathbf{K}'$ are in addition **ΠD**-u-categories, it makes sense to say of X that it is an ultra-functor (preserves ultramorphisms). Let $\mathbf{\Pi D}_u$ be the 2-category whose 0-cells are the **ΠD**-u-categories, and such that $\mathbf{\Pi D}_u(\mathbf{K}, \mathbf{K}')$ is the full subcategory $\mathbf{\Pi D}(\mathbf{K}, \mathbf{K}')$ with objects the ultrafunctors $\mathbf{K} \to \mathbf{K}'$. For an exact category S, $G(S) = \mathscr{E}x(S, \text{SET})$ turns out to be an object of $\mathbf{\Pi D}_u$ in a natural way (in particular, $G(S)$ has products and directed colimits). Corollary 6.1 in Section 6 below expresses the fact that G is a part of a Stone adjunction $F \dashv G$ which is a reflection in the small.

Comparison of the three "2-categorical" Stone adjunctions so far mentioned shows, first, that the one for pretoposes and ultracategories concerns, on the "space" side, categories with *genuine* additional structure (ultracategories) in the sense that the ultraproduct functors are not defined (by something like a universal property) on the basis of the category structure: the category of models of a first order theory does not have products in general (although it does have directed colimits). On the other hand, part of the additional structure on a **ΠD**-u-category (products and directed colimits) are defined on the basis of the category structure alone, though another part (ultramorphisms) is not. (We note in passing that ultramorphisms are not as extraneous in the context of exact categories as they seem at first; the general idea of an ultramorphism can be applied to the product and directed colimit functors directly instead of to ultraproducts, resulting in a natural notion "**ΠD**-morphism;" moreover, it turns out that the resulting **ΠD**-morphisms are more general than ultramorphisms, and therefore the "natural" version of the reflection result for exact categories is a consequence (weakening) of the version stated above). Finally, in the simplest case of left exact categories, all structure on the "space" side is definable by the category structure alone.

The difference between the pretopos case and the exact category case accounts for the fact that, in the exact case, there is no need for a notion corresponding to *ultratransformations*. Because of this, $\mathbf{\Pi D}_u(\mathbf{K}, \mathbf{K}')$ is a full subcategory of $\mathbf{\Pi D}(\mathbf{K}, \mathbf{K}')$; therefore Corollary 6.1 contains Michael Barr's full exact embedding theorem (see [EC]) in the sharpened version given in [M2]. In the context of general first order theories (pretoposes), it is natural that on the category of models one needs, besides the ultraproduct functors, some additional structure expressing connections between the various ultraproduct functors; it is somewhat surprising, however, that such additional structure (ultramorphisms) becomes relevant even in the case of exact categories when the functor-operations on the "category of models" are categorically definable.

There are at least two topics in this paper that ideally should be discussed in much greater generality. One is the setting-up of the Stone adjunction for pretoposes and ultracategories, begun in Section 2, continued in Section 3, and completed in Section 8. It would have been nicer to subsume those straightforward but tedious computations under a general theory of Stone adjunctions for structures with underlying objects in an arbitrary Cartesian closed category (SET and Cat being the chief examples for the latter). Although the organization of the material reflects an attempt at such a general theory, the author could not find a satisfactory final formulation.

The other topic is the definition of ultramorphism. This is a purely universal algebraic concept, in the context of structures with underlying *categories* instead of underlying *sets*. In fact, given any "algebra" **S** consisting of a category S, and a family of "operations": functors of the form $S^{\Gamma_1} \to S^{\Gamma_2}$ (with Γ_1, Γ_2 graphs defining the "arity" of the operation), one has an associated concept of "**S**-morphism"; this becomes "ultramorphism" if **S** is the category of sets together with all the ultraproduct functors $[U]$: $\mathrm{SET}^I \to \mathrm{SET}$ as operations. Two remarks should be made: first, an **S** morphism is defined within S, and no ambient class of structures is involved; second, an **S**-morphism is not the same as a "composite" ("polynomial") in terms of the given operations.

The most natural kind of **S**-morphism occurs when the operations of **S** are defined by universal properties such as limit, colimits, etc. If one has a diagram Δ of objects and morphisms in S, with a specification that certain items in Δ are obtained from certain others by one of the operations of **S**, then one has certain new, uniquely defined morphisms between objects of the diagram, existing directly by the universal properties involved. Picking a definite such morphism, and considering the "same" morphism simultaneously for all diagrams "of the same kind as Δ," one has an **S**-morphism that can be called *canonical.*

Here is an example. Suppose $f\colon I \to J$ is a function between the sets I and J, U is an ultrafilter on I, V is the ultrafilter on J for which $B \in V$ iff $f^{-1}(B) \in U$. Then, given any family $\langle A_j \rangle_{j \in J}$ of sets, one has a "canonical" map

$$\delta = \delta_{\langle A_j \rangle}\colon \ \prod_j A_j/V \to \prod_i A_{f(i)}/U$$

defined by

$$\langle a_j \rangle/V \mapsto \langle a_{f(i)} \rangle/U.$$

Moreover, we have the naturality condition depicted as

$$\begin{tikzcd}
A_j \arrow[d, "g_j"'] & \prod A_j/V \arrow[r, "\delta_{\langle A_j \rangle}"] \arrow[d, "\prod g_j/V"'] \arrow[dr, phantom, "\circlearrowleft"] & \prod A_{f(i)}/U \arrow[d, "\prod g_{f(i)}/U"] \\
B_j & \prod B_j/V \arrow[r, "\delta_{\langle B_j \rangle}"'] & \prod B_{f(i)}/U
\end{tikzcd}$$

This is a simple example of an ultramorphism in SET. It is clear that this ultramorphism lifts to the category of models Mod T of a theory: for a family of models $\langle M_j \rangle$, one has $\delta_{\langle M_j \rangle}\colon \prod M_j/V \to \prod M_{f(i)}/U$, with the obvious naturality condition holding.

Since ultraproducts in SET are defined in terms of products and directed colimits, the above ultramorphism in SET can be seen to derive solely from the universal properties of products and colimits. In other words, the ultramorphism is a *canonical* one associated with the algebra $\mathscr{S}$ which is SET together with the small product and directed colimit functors. However, the products no longer exist in Mod T; the ultramorphism in Mod T cannot be "canonically" defined; it remains an "abstract" ultramorphism.

In Section 3, we introduce the general concept of ("abstract") ultramorphism. In the proof of the main result in Section 4, we construct certain ultramorphisms. Much of the verification of their being ultramorphisms is postponed to Section 5, where we show that in fact they are canonical with respect to the full small limit (not just product) and colimit structure of SET.

A summary of the material of the present paper appears in [M3].[1]

PRELIMINARY REMARKS

For unexplained terminology [CWM] (occasionally [CK] and [MR]) should be consulted.

In our underlying set-theory, we use *sets*, *classes*, and *superclasses*. In axiomatic terms, we use Zermelo–Fraenkel set-theory with the axiom of choice, with the additional axiom of the existence of two distinct (strongly) inaccessible cardinals. With θ_0, θ_1 the first. respectively, the second inaccessible, we call elements of V_{θ_0} (sets of rank $<\theta_0$) *sets*, elements of V_{θ_1} *classes*, and arbitrary elements of the universe *superclasses*. A *small category*, a *category*, and a *supercategory* are "categories" whose collection of objects and all hom-"sets" are sets, respectively, classes, respectively, superclasses. The category of sets, SET, is a category, but not a small category; for categories K and S, the functor category Hom(K, S) (denoted S^K in [CWM]) is a category; the "category" of all categories is a supercategory, but not a category.

With any subcategory K of the functor category Hom($\mathscr{C}$, $\mathscr{D}$), one has the well-known evaluation functor

$$\mathrm{ev}\colon\ \mathscr{C} \to \mathrm{Hom}(K, \mathscr{D})$$

defined by the formulas

$$\begin{aligned} &\mathrm{ev}(C)(M) = M(C) && (C \in |\mathscr{C}|, M \in |K|) \\ &\mathrm{ev}(C)(h) = h_C && (h\colon M \to M' \text{ in } K) \\ &(\mathrm{ev}(f))_M = M(f) && (f\colon C \to C' \text{ in } \mathscr{C}). \end{aligned}$$

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[1] *Note added in proof.* The papers [M4] and [M5] develop the subject of this paper in further directions.

The functor ev(C) is also written (C); it is the functor "evaluation at C": $K \to \mathscr{D}$.

1. Basic Notions

A. *Pretoposes*

The notion of *pretopos* was introduced by Grothendieck [SGA4]; see also [MR]. Although the reader will not have to keep in mind the exact definition in order to follow the main part of the paper, for completeness we record the definition. A pretopos is a category T satisfying the following:

(1) T has all finite (left) limits (is finitely complete); or equivalently, T has a terminal object, and all pullbacks.

(2) T has a strict zero-object 0. 0 is an initial object and every morphism with codomain 0 is an isomorphism.

(3) T has a stable disjoint sum of any pair of objects. A disjoint sum $A \sqcup B$ of objects A, B is a coproduct of A and B such that, for the canonical injections i: $A \to A \sqcup B$, j: $B \to A \sqcup B$, i and j are monomorphisms, and in the pullback

$$\begin{tikzcd}
A \arrow[r] & A \sqcup B \\
C \arrow[u] \arrow[r, "\text{p.b.}"] & B \arrow[u]
\end{tikzcd}$$

C is isomorphic to 0; moreover; if in the diagram

$$\begin{tikzcd}
 & A \arrow[rr, "i"] & & \mathrm{A} \sqcup B \\
A' \arrow[ur] \arrow[rr, "\text{p.b.}", "i'"'] & & C' \arrow[ur] & B \arrow[u, "j"'] \\
 & & B' \arrow[u, "j'", "\text{p.b.}"'] \arrow[ur] &
\end{tikzcd}$$

we have two pulbacks as shown, then C' is a disjoint sum of A' and B' with canonical injections i' and j' (stability).

(4) T has stable quotients of equivalence relations. A diagram $E \underset{g}{\overset{f}{\rightrightarrows}} A$ of classes and functions is an equivalence relation if the map $e \mapsto \langle f(e), g(e) \rangle$: $E \to A \times A$ is one-to-one and its image is an equivalence

relation on A in the ordinary sense. A diagram $E \rightrightarrows^{f}_{g} A$ in T is an equivalence relation if for all $B \in |T|$, the induced diagram

$$\mathrm{Hom}(B, E) \underset{\mathrm{Hom}(B,g)}{\overset{\mathrm{Hom}(B,f)}{\rightrightarrows}} \mathrm{Hom}(B, A)$$

is an equivalence relation. A quotient of the equivalence relation $E \rightrightarrows^{f}_{g} A$ is a coequalizer $A \to^{q} Q$ of f and g such that the diagram

$$\begin{tikzcd}
E \arrow[r, "f"] \arrow[d, "g"'] & A \arrow[d, "q"] \\
A \arrow[r, "q"] & Q
\end{tikzcd}$$

is a pullback; it is stable if the following holds: whenever $A \to^{q} Q$ is a quotient of some equivalence relation, and the diagram

$$\begin{tikzcd}
A \arrow[r, "q"] & Q \\
A' \arrow[u] \arrow[r, "q'"'] & Q' \arrow[u]
\end{tikzcd}$$

is a pullback, then $A' \to^{q'} Q'$ is the quotient of some equivalence relation as well.

A functor $T \to T'$ between pretoposes is a *morphism of pretoposes*, or an *elementary functor*, ("logical functor" in [MR]), if it preserves the pretopos structure. More precisely, this means the following. We consider the following five kinds of finite diagrams in any category:

(1.1) a single object, denoted 1; it is a *terminal object diagram* if 1 is a terminal object;

(1.2) a diagram of the form

$$\begin{tikzcd}
A \arrow[r] & C \\
D \arrow[u] \arrow[r] & B;\arrow[u]
\end{tikzcd}$$

it is a *pullback diagram* if the well-known condition holds;

(2) a single object, denoted 0; it is a *zero object diagram*, if 0 is a strict zero object;

(3) a diagram of the form

$$\begin{tikzcd} A \arrow[r, "i"] & C \\ & B; \arrow[u, "j"'] \end{tikzcd}$$

it is a *disjoint sum diagram* if C is a disjoint sum of A and B with canonical injections i and j;

(4) a diagram of the form

$$\begin{tikzcd} E \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & A \arrow[r, "q"] & Q; \end{tikzcd}$$

it is a *quotient diagram*, if (f, g) is an equivalence relation and q is its quotient.

Now, M is elementary if the following holds: for each of the five kinds of diagrams, if a finite diagram in T is of that kind, then M transforms it into a diagram in T' of the same kind. Note in particular that in pretoposes we do not require "distinguished" products, pullbacks, etc., and accordingly, elementary functors are not necessarily "strict," even if such "distinguished" products, etc., are present (as in SET, or in most actual pretoposes).

The most standard example of a pretopos is SET. An elementary functor M: $T \to$ SET is called a *model* of T. In [MR] it is explained in what sense pretoposes correspond to theories, models of pretoposes correspond to models of theories, natural transformations between two models M, N: $T \rightrightarrows$ SET correspond to homomorphisms (in case of a Boolean T, elementary embeddings) of models. In addition, arbitrary elementary functors $T \to T'$ between pretoposes should be regarded as (generalized) interpretations (of T in T'). In particular, note that any functor M: $T \to$ SET *is* a structure of the many-sorted similarity type the underlying graph of T whose sorts are the objects of T and whose only other symbols are unary operations between sorts, the morphisms of T. Therefore, e.g., for a natural transformation h: $M \to M'$ with M, M' models of T makes sense to be called *elementary*: it means that h is an elementary embedding of M (as a many-sorted structure) into M'.

When we want to emphasize that the category T is a pretopos, we write $\mathscr{T}$ for it. T, T_1, T', SET, etc., on the one hand, and $\mathscr{T}$, $\mathscr{T}_1$, $\mathscr{T}'$, $\mathscr{S}et$, etc., on the other, mean respectively the same categories but the script notation emphasizes that we regard them as pretoposes. Accordingly, a notation like

$$M\colon\quad \mathscr{T} \to \mathscr{S}et$$

means that M is a morphism of pretoposes, i.e., an elementary functor. The *category of (all) elementary functors from* $\mathscr{T}$ *to* $\mathscr{T}'$ is denoted $\mathrm{Hom}(\mathscr{T}, \mathscr{T}')$; this is the full subcategory of $\mathrm{Hom}(T, T')$ whose objects are all elementary functors $M: \mathscr{T} \to \mathscr{T}'$. The category $\mathrm{Hom}(\mathscr{T}, \mathscr{Set})$ is written $\mathrm{Mod}\, T$; it is the *category of (all) models* of T.

A functor $M: T \to T'$ between pretoposes T and T' is said to *reflect the pretopos structure*, or is *coelementary*, if the following holds. Whenever $\Gamma_1 \to T$ is a diagram in T, with one of the five finite graphs Γ_1 the five kinds of pretopos diagrams are based on, and the composite $\Gamma_1 \to T \to T'$ is of the corresponding kind (terminal object, zero object, etc.) in T', then $\Gamma_1 \to T$ is of that same kind in T.

B. *Pre-ultracategories*

Next, we elaborate on ultraproducts. Given an ultrafilter U on a (non-empty) set I (cf. [CK]), one has the ultraproduct functor

$$[U]:\quad \mathrm{SET}^I \longrightarrow \mathrm{SET}$$

defined as follows: given any family $\langle A_i: i \in I\rangle = \langle A_i\rangle$ of sets, define the equivalence relation $\sim_U$ on the set of all vectors $\langle a_i: i \in P\rangle$ such that $P \in U$ and $a_i \in A_i$ for $i \in P$ as

$$\langle a_i: i \in P\rangle \sim \langle a_i': i \in P'\rangle \qquad \text{iff} \quad \{i \in P \cap P': a_i = a_i'\} \in U.$$

Denote by $\langle a_i\rangle/U$ the equivalence class of $\langle a_i\rangle$, and define $\prod_{i \in I} A_i/U = \prod A_i/U$ to be the set of all equivalence classes of $\sim$. (Because of the possibility of some A_i being empty, the more usual formulation with vectors indexed by the full set I is not adequate.) This defines the action of the functor $[U]$ on objects: $[U](\langle A_i\rangle) = \prod A_i/U$. Given $\langle f_i: A_i \to B_i | i \in I\rangle = \langle f_i\rangle$, we define

$$[U](\langle f_i\rangle) = \prod f_i/U = f$$

such that

$$f(\langle a_i\rangle/U) = \langle f_i(a_i)\rangle/U;$$

it is easy to check that f is well defined, and that $[U]$ so defined is a functor $\mathrm{SET}^I \to \mathrm{SET}$.

An ultraproduct of sets is a directed colimit of products; in a formula: $\prod A_i/U = \varinjlim_{P \in U^{\mathrm{op}}} \prod_{i \in P} A_i$. The graph of the colimit is U^{op}, the opposite of the partial order on U defined by containment; since U is closed under intersection, U^{op} is directed. The diagram $\Delta: U^{\mathrm{op}} \to \mathrm{SET}$, the colimit of which is $\prod A_i/U$, has $\Delta(P) = \prod_{i \in P} A_i$, and $\Delta(\langle P, Q\rangle) =$ the canonical projection $\prod_{i\in P} A_i \to \prod_{i\in Q} A_i$ for P, Q in U, $P \supseteq Q$. An ultraproduct of morphisms is a canonical map between colimits of products.

Next, we introduce some purely formal concepts.

A *pre-ultracategory* (p.-u.c.) **S** is a category S together with a specified but arbitrary functor $[U]$: $S^I \to S$ formally associated with any ultrafilter U on any set I. **SET** is the p.-u.c. SET with the standard ultraproduct functors. *A strict ultraproduct preserving functor*, or *strict pre-ultrafunctor* between p.-u.c.'s **S** and **S**$'$ is a functor X: $S \to S'$ such that for all ultrafilters U (on I), the diagram

$$\begin{tikzcd}
S^I \arrow[r, "{[U]_{\mathbf{S}}}"] \arrow[d, "X^I"'] & S \arrow[d, "X"] \\
(S')^I \arrow[r, "{[U]_{\mathbf{S}'}}"] & S'
\end{tikzcd}$$

commutes. More generally, a *pre-ultrafunctor* (p.-u.f.) $\mathbf{S} \to \mathbf{S}'$ is a functor X: $S \to S'$, together with a specified *transition isomorphism*

$$[X, U]\colon \quad X \circ [U] \xrightarrow{\cong} [U] \circ X^I$$

for each ultrafilter U. A script p.-u.f. is therefore a p.-u.f. whose transition isomorphisms are all identity natural transformations.

An *ultratransformation* (u.t.) between p.-u.f.'s X and Y: $\mathbf{S} \rightrightarrows \mathbf{S}'$ is a natural transformation σ: $X \to Y$ such that, with the four composite functors and natural transformations from the diagram

$$\begin{tikzcd}
S^I \arrow[r, "{[U]_{\mathbf{S}}}"] \arrow[d, shift right=4, "X^I"'] \arrow[d, shift left=4, "Y^I"] \arrow[d, phantom, "\xrightarrow{\sigma^I}"] & S \arrow[d, shift right=4, "X"'] \arrow[d, shift left=4, "Y"] \arrow[d, phantom, "\xrightarrow{\sigma}"] \\
(S')^I \arrow[r, "{[U]_{\mathbf{S}'}}"] & S'
\end{tikzcd}$$

we have a commutative diagram of functors and natural transformations as follows:

$$\begin{tikzcd}
X \circ [U]_{\mathbf{S}} \arrow[r, "\sigma \circ {[U]_{\mathbf{S}}}"'] \arrow[d, "{[X, U]}"'] \arrow[dr, phantom, "\circlearrowleft"] & Y \circ [U]_{\mathbf{S}} \arrow[d, "{[Y, U]}"] \\
{[U]_{\mathbf{S}'}} \circ X^I \arrow[r, "{[U]_{\mathbf{S}'}} \circ \sigma^I"'] & {[U]_{\mathbf{S}'}} \circ Y^I.
\end{tikzcd}$$

If X and Y are strict, this condition reduces to the equality

$$\sigma \circ [U]_{\mathbf{S}} = [U]_{\mathbf{S}'} \circ \sigma^I.$$

The pre-ultrafunctors between two fixed pre-ultracategories S, S' and ultratransformations between them form a category $\mathrm{Hom}(\mathbf{S}, \mathbf{S}')$. We have the forgetful functor

$$\mathrm{Hom}(\mathbf{S}, \mathbf{S}') \longrightarrow \mathrm{Hom}(S, S')$$

that assigns to each ultrafunctor its functor-part; it is clearly a faithful functor, but it is not full. We will refer to this functor as a *quasi inclusion.*

The strict p.-u.f.'s $\boldsymbol{S} \to \boldsymbol{S}'$ can be identified with p.-u.f.'s with identity transition isomorphisms; the category $\mathrm{Hom}_s(\mathbf{S}, \mathbf{S}')$ of strict p.-u.f.'s with ultratransformations as morphisms then appears as a full subcategory of $\mathrm{Hom}(\mathbf{S}, \mathbf{S}')$.

There is an obvious composition of p.-u.f.'s. Given X: $\mathbf{K} \to \mathbf{K}'$ and Y: $\mathbf{K}' \to \mathbf{K}''$, the composite $Y \circ X$: $\mathbf{K} \to \mathbf{K}'$ has its functor-part the composite of the functor-parts of X and Y; the transition isomorphisms are given as follows. In the diagram

$$\begin{tikzcd}
K^I \arrow[r, "{[U]_{\mathbf{K}}}"] \arrow[d, "X^I"'] & K \arrow[d, "X"] \\
(K')^I \arrow[r, "{[U]_{\mathbf{K}'}}"] \arrow[d, "Y^I"'] & K' \arrow[d, "Y"] \\
(K'')^I \arrow[r, "{[U]_{\mathbf{K}''}}"] & K''
\end{tikzcd}$$

there are three composite functors

$$F, G, H\colon\ \mathbf{K}^I \mathrel{\substack{\to\\ \to\\ \to}} K''\colon$$

$$F = Y \circ X \circ [U]_{\mathbf{K}}$$

$$G = Y \circ [U]_{\mathbf{K}'} \circ X^I$$

$$H = [U]_{\mathbf{K}''} \circ Y^I \circ X^I;$$

we have the composite natural isomorphisms

$$\Phi = Y \circ [X, U]\colon\quad F \xrightarrow{\cong} G$$

$$\Psi = [Y, U] \circ X^I\colon\quad G \xrightarrow{\cong} H;$$

we define $[Y \circ X, U] = \Psi \circ \Phi$.

In this way we have defined the object function of the functor

$$\mathrm{Hom}(X, \mathbf{K}'') = (\ ) \circ X\colon\quad \mathrm{Hom}(\mathbf{K}', \mathbf{K}'') \longrightarrow \mathrm{Hom}(\mathbf{K}, \mathbf{K}'');$$

the action on morphisms is the usual action of composition with X on natural transformations.

2. PRETOPOSES VERSUS PRE-ULTRACATEGORIES

The starting point for this paper is that the category of sets, SET, carries, simultaneously, a pretopos structure ($\mathcal{S}et$) and a pre-ultracategory structure (**SET**); and that, moreover, these two structures "commute" with each other. By this we mean the following. Let U be an ultrafilter on I. The direct power SET^I is a pretopos (as is easily verified; see also below). Now, we have

LOS' THEOREM. *Any ultraproduct functor* $\mathrm{SET}^I \to^{[U]} \mathrm{SET}$ *is elementary.*

Although the theorem is fundamental, it is routine to verify it; we omit the details. In a roundabout way, this theorem follows from a well-known form of Los theorem (the fundamental theorem on ultraproducts [CK]), but since it is more elementary than the latter, this derivation is inappropriate. Rather, using purely formal arguments, we derive the model-theoretical version from the basic Los theorem.

In what follows, we develop a formal theory relating pretoposes and p.-u.c.'s based solely on Los' theorem. To emphasize the abstract character of the context, we take a category S and we assume that S is a pretopos $\mathcal{S}$, that S also carries a p.-u.c. structure **S**, and that $[U]_{\mathcal{S}}: \mathcal{S}^I \to \mathcal{S}$ is elementary for all ultrafilters (I, U). In fact, it will not be important that we talk about pretoposes and p.-u.c.'s specifically, but it would be pointless to introduce greater generality. In what follows, $\mathcal{T}$ $(=T)$, $\mathcal{T}'$ $(=T')$ are arbitrary pretoposes. **K** and **K**$'$ are p.-u.c.'s with respective underlying categories K and K'.

We make a series of statements that are actually definitions of gradually introduced structures. The proofs are all quite formal and easy; we make only occasional remarks concerning them. At this stage there is a perfect symmetry in the roles the two kinds of structures play.

(i)(a) $\mathrm{Hom}(K, S)$ *is a pretopos, denoted* $\mathcal{H}om(K, S)$, *such that we have a factorization*

$$\begin{tikzcd}
K \arrow[r, "\mathrm{ev}_K"] \arrow[dr] & \mathrm{Hom}(\mathrm{Hom}(K, S), S) \\
& \mathrm{Hom}(\mathcal{H}om(K, S), \mathcal{S}) \arrow[u, "\text{inclusion}"']
\end{tikzcd}$$

(*with* ev_K *the evalutation functor*). [*Here K is an arbitrary caregory; the pre-ultrastructure on it does not play any role.*]

The *proof* is well-known, although it does require work.

(i)(b) Hom(T, S) *is a* p.-u.c., *denoted* **Hom**(T, S), *in a unique way such that we have a factorization*

$$\begin{tikzcd}
T \arrow[rr, "\mathrm{ev}_T"] \arrow[d] \arrow[drr, phantom, "\circlearrowleft"] & & \mathrm{Hom}(\mathrm{Hom}(T, S), S) \\
\mathrm{Hom}_s(\mathbf{Hom}(T, S), \mathbf{S}) \arrow[rr, "\text{inclusion}"'] & & \mathrm{Hom}(\mathbf{Hom}(T, S), \mathbf{S}) \arrow[u, "\text{quasi inclusion}"']
\end{tikzcd}$$

[*Here T is an arbitrary caregory.*]

Proof. The factorization condition forces us to define $[U] = [U]_{\mathrm{Hom}(T,S)} = \prod(\ )/U$ by the formulas:

$$\left(\prod M_i/U\right)(A) = \prod M_i(A)/U \qquad (M_i \in |\mathrm{Hom}(T, S)|,\ A \in |T|),$$

$$\left(\prod M_i/U\right)(f) = \prod M_i(f)/U \qquad (f\colon A \to A' \text{ in } T),$$

$$\left(\prod h_i/U\right)_A = \prod (h_i)_A/U \qquad (h_i\colon M_i \to N_i \text{ in } \mathrm{Hom}(T, S)).$$

It is easy to verify that in this way we indeed get what we want. Note that, for the case $S = \mathrm{SET}$, we have just given the usual definition of the ultraproduct of structures of the similarity type the graph of T. ▌

(ii)(a) *Construct the commutative diagram*

$$\begin{tikzcd}
K \arrow[r, "\mathrm{ev}'_K"] \arrow[dr, "\mathrm{ev}_K"'] \arrow[dr, phantom, "\circlearrowleft", near start, shift left=3] & \mathrm{Hom}(\mathrm{Hom}(\mathbf{K}, \mathbf{S}), S) & \mathrm{Hom}(\mathbf{K}, \mathbf{S}) \arrow[d, "q(q.i.)"] \\
 & \mathrm{Hom}(\mathrm{Hom}(K, S), S) \arrow[u, "\mathrm{Hom}(q,S)"'] & \mathrm{Hom}(K, S)
\end{tikzcd}$$

defining ev'_K.*Then* ev'_K *is made into a* p.-u.f.

$$\mathrm{ev}_{\mathbf{K}}\colon\ \mathbf{K} \longrightarrow \mathbf{Hom}(\mathrm{Hom}(\mathbf{K}, \mathbf{S}), S)$$

by the transition isomorphisms $[\mathrm{ev}_{\mathbf{K}}, U]$ *defined by the formula*

$$([\mathrm{ev}_{\mathbf{K}}, U]_{\langle M_i\rangle})_X = [X, U]_{\langle M_i\rangle}$$

(U *an ultrafilter on* I, $M_i \in |K|$ for $i \in I$, $X \in |\mathrm{Hom}(\mathbf{K}, \mathbf{S})|$).

(ii)(b) *Construct the commutative diagram*

$$\begin{tikzcd}
T \arrow[r, "\mathrm{ev}'_T"] \arrow[dr, "\mathrm{ev}_T"'] \arrow[dr, phantom, "\circlearrowleft", shift left=3ex] & \mathrm{Hom}(\mathrm{Hom}(\mathscr{T}, \mathscr{S}), S) & \mathrm{Hom}(\mathscr{T}, \mathscr{S}) \arrow[d, "i"', "\text{(incl.)}"] \\
 & \mathrm{Hom}(\mathrm{Hom}(T, S), S) \arrow[u, "\mathrm{Hom}(i,S)"'] & \mathrm{Hom}(T, S)
\end{tikzcd}$$

defining ev'_T. *Then* ev'_T *is an elementary functor denoted*

$$\mathrm{ev}_{\mathscr{T}}\colon \quad \mathscr{T} \longrightarrow \mathscr{H}\!om(\mathrm{Hom}(\mathscr{T}, \mathscr{S}), S).$$

(iii)(a) (functoriality of $\mathscr{H}\!om(-, S)$). *For any* $X\colon K \to K'$, *the functor* $\mathrm{Hom}(X, S) = (-) \circ X\colon \mathrm{Hom}(K', S) \to \mathrm{Hom}(K, S)$ *is elementary*: $\mathscr{H}\!om(X, S)\colon \mathscr{H}\!om(K', S) \to \mathscr{H}\!om(K, S)$.

(iii)(b) (functoriality of $\mathbf{Hom}(-, S)$). *For any* $M\colon T \to T'$, *the functor*

$$\mathrm{Hom}(M, S) = (-) \circ M\colon \quad \mathrm{Hom}(T', S) \to \mathrm{Hom}(T, S)$$

is a strict pre-ultrafunctor:

$$\mathbf{Hom}(M, S)\colon \quad \mathbf{Hom}(T', S) \to \mathbf{Hom}(T, S).$$

(iv)(a) $\mathrm{Hom}(\mathbf{K}, \mathbf{S})$ *is a pretopos, denoted* $\mathscr{H}\!om(\mathbf{K}, \mathrm{S})$, *such that the quasi-inclusion*

$$\mathscr{H}\!om(\mathbf{K}, \mathrm{S}) \xrightarrow{\text{q.i.}} \mathscr{H}\!om(K, S)$$

is both elementary and coelementary.

Proof. The proof relies on Los' theorem. It also shows why we have to consider general pre-ultrafunctors, not just strict ones. Otherwise, the proof uses only very general features of the definition of pretopos, and in particular, the verifications of each of the four parts of the definition of pretopos for $\mathrm{Hom}(\mathbf{K}, \mathrm{S})$ all follow the same pattern. This pattern could be made explicit but it does not seem worth doing so.

We show how to obtain equalizers in $\mathrm{Hom}(\mathbf{K}, \mathrm{S})$. Let the diagram

$$\begin{tikzcd}
X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y
\end{tikzcd}$$

in $\mathrm{Hom}(\mathbf{K}, \mathrm{S})$ be given; let $E \rightarrow^{e} X$ be a morphism *in* $\mathrm{Hom}(K, S)$ (with X now meaning just the functor part of what X was originally) such that

$$\begin{tikzcd}
E \arrow[r, "e"] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y
\end{tikzcd} \tag{2.1}$$

is an equalizer-diagram in Hom(K, S). We endow E with transition isomorphisms making it an object of Hom(**K**, **S**). Consider the diagram

$$\begin{tikzcd}
E\left(\prod M_i/U\right) \arrow[r, "e_{\prod M_i/U}"] \arrow[d, dashed, "?"'] & X\left(\prod M_i/U\right) \arrow[r, shift left, "f_{\prod M_i/U}"] \arrow[r, shift right, "g_{\prod M_i/U}"'] \arrow[d, "{[X,U]_{\langle M_i\rangle}}"', "\wr"] & Y\left(\prod M_i/U\right) \arrow[d, "{[Y,U]_{\langle M_i\rangle}}", "\wr"'] \\
\prod E(M_i)/U \arrow[r, "\prod e_{M_i/U}"] & \prod X(M_i)/U \arrow[r, shift left, "\prod f_{M_i/U}"] \arrow[r, shift right, "\prod g_{M_i/U}"'] & \prod Y(M_i)/U
\end{tikzcd}$$

The fact that f and g are ultratransformations (morphisms in Hom(**K**, **S**)) implies that the two squares on the right (one with two horizontal arrows involving f's, the other with g's) commute; in other words, we have an isomorphism of the two diagrams, one consisting of the upper pair of morphisms, the other of the lower pair. Los' theorem and the choice of e imply that the lower half of the intire diagram is an equalizer diagram; so is the upper half by the choice of e. By the essential uniqueness of equalizers, therefore, there is a unique vertical arrow ? on the left that, with the two other vertical arrows, makes up an isomorphism of the two equalizer diagrams. We *define* $[E, U]_{\langle M_i\rangle} = ?$. To show that this definition gives us a natural transformation, we consider a family of morphisms

$$h_i\colon \quad M_i \to M'_i;$$

we consider the entire diagram above, with the M_i's replaced by the M'_i's, and we show that the h_i's induce, in a natural way, a natural transformation between the M_i-version and the M'_i-version of the entire diagram. This can be done, using general features of the definition of equalizer.

Because of the commutativity of the square on the left side, we see that e is indeed a morphism in Hom(**K**, **S**). It remains to verify that the diagram (2.1) so obtained is indeed an equalizer diagram; this is easy. The remaining assertions are left to the reader to prove.

Assume in this proof that X and Y were strict ultrafunctors. E still proves to be a general ultrafunctor, unless $e_{\prod M_i/U}$ is literally the same as $\prod e_{M_i}/U$. It seems rather impossible (and pointless) to force the definition of $E \to^e X$ to satisfy this. ▌

(iv)(b) Hom($\mathscr{T}$, $\mathscr{S}$) *is a* p.-u.c., *denoted* **Hom**($\mathscr{T}$, $\mathscr{S}$), *such that the inclusion*

$$\mathbf{Hom}(\mathscr{T}, \mathscr{S}) \longrightarrow \mathrm{Hom}(T, S)$$

is a strict p.-u.f.

Proof. The statement is equivalent to saying that the ultraproduct (according to (i)(b)) of functors $T \to S$ which happen to be elementary is elementary too. This is a direct consequence of Los' theorem, as stated above, and it is identical to the usual version of that theorem, which says that the ultraproduct of models of a theory is a model of the same theory. ▮

(v)(a) (Functoriality of $\mathscr{H}om(-, \mathbf{S})$) *For any ultrafunctor* X: $\mathbf{K} \to \mathbf{K}'$, *the functor*

$$\mathrm{Hom}(X, \mathbf{S})\colon\ \mathrm{Hom}(\mathbf{K}', \mathbf{S}) \to \mathrm{Hom}(\mathbf{K}, \mathbf{S})$$

(*see the end of Sect.* 1) *is elementary,*

$$\mathscr{H}om(X, \mathbf{S})\colon\ \mathscr{H}om(\mathbf{K}',\mathbf{S}) \longrightarrow \mathscr{H}om(\mathbf{K}, \mathbf{S}).$$

Proof. It is clear from the definitions that the diagram

$$\begin{tikzcd}
\mathrm{Hom}(\mathbf{K}', \mathbf{K}) \arrow[r, "\mathrm{Hom}(X{,}\mathbf{S})"] \arrow[d, "\text{q.i.}"'] & \mathrm{Hom}(\mathbf{K}, \mathbf{S}) \arrow[d, "\text{q.i.}"] \\
\mathrm{Hom}(K', S) \arrow[r, "\mathrm{Hom}(X{,}S)"'] & \mathrm{Hom}(K, S)
\end{tikzcd}$$

commutes. By (iv)(a) and (iii)(a), the composite of the left arrow and the bottom arrow is elementary. The right arrow is coelementary by (iv)(a). So the top arrow is elementary. ▮

(v)(b) (Functoriality of $\mathbf{Hom}(-, \mathscr{S})$). *For any elementary* M: $\mathscr{T} \to \mathscr{T}'$, *the functor*

$$\mathrm{Hom}(M, \mathscr{S})\colon\ \mathrm{Hom}(\mathscr{T}', \mathscr{S}) \longrightarrow \mathrm{Hom}(\mathscr{T}, \mathscr{S})$$

is a strict p.-u.f. *denoted*

$$\mathbf{Hom}(M, \mathscr{S})\colon\ \mathbf{Hom}(\mathscr{T}', \mathscr{S}) \longrightarrow \mathbf{Hom}(\mathscr{T}, \mathscr{S}).$$

Proof. Dual to the previous one. ▮

(vi)(a) *Construct the following commutative diagram, defining the functor* e_K: $K \to \mathrm{Hom}(\mathscr{H}om(\mathbf{K}, \mathbf{S}), \mathscr{S})$:

$$\begin{tikzcd}
\mathrm{Hom}(K, S) & \mathrm{Hom}(\mathbf{K}, \mathbf{S}) \arrow[l, "q", "(\mathrm{q.i.})"']
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Hom}(\mathrm{Hom}(K, S), S) \arrow[rr, "\mathrm{Hom}(q,S)"] & & \mathrm{Hom}(\mathrm{Hom}(\mathbf{K}, \mathbf{S}), S) \\
& K \arrow[ul, "\mathrm{ev}_K"'] \arrow[ur, "\mathrm{ev}'_K"] \arrow[dl] \arrow[dr, "e_K"] & \\
\mathrm{Hom}(\mathscr{H}\!\mathit{om}(K, S), \mathscr{S}) \arrow[uu, "\mathrm{incl.}"] \arrow[rr, "\mathrm{Hom}(q,S)"'] & & \mathrm{Hom}(\mathscr{H}\!\mathit{om}(\mathbf{K}, \mathbf{S}), \mathscr{S}) \arrow[uu, "i", "\mathrm{incl.}"']
\end{tikzcd}$$

$$\begin{tikzcd}
\mathscr{H}\!\mathit{om}(K, S) & \mathscr{H}\!\mathit{om}(\mathbf{K}, \mathbf{S}). \arrow[l, "q", "(\mathrm{q.i.})"']
\end{tikzcd}$$

The left triangle is given by (i)(a), the top triangle by (ii)(a), and e_K is defined so that the bottom triangle commutes. The square commutes, as easily seen. As a consequence, the right triangle commutes. Now, by (iv)(b), the right vertical inclusion is a strict p.-u.f.,

$$i\colon\ \mathbf{Hom}(\mathscr{T}.\mathscr{S}) \longrightarrow \mathbf{Hom}(T, S)$$

for $\mathscr{T} = \mathscr{H}\!\mathit{om}(\mathbf{K}, \mathbf{S})$; also by (ii)(a), we have the p.-u.f.

$$\mathrm{ev}_{\mathbf{K}}\colon\ \mathbf{K} \longrightarrow \mathbf{Hom}(T, S),$$

whose functor part is ev'_K. It follows that e_K is made a p.-u.f., denoted

$$\mathrm{e}_{\mathbf{K}}\colon\ \mathbf{K} \longrightarrow \mathbf{Hom}(\mathscr{H}\!\mathit{om}(\mathbf{K}, \mathbf{S}), \mathscr{S}),$$

in a unique way such that $i \circ e_{\mathbf{K}} = \mathrm{ev}_{\mathbf{K}}$.

(vi)(b) *By exactly the dual procedure* (*and using that by* (iv)(a), $q\colon \mathscr{H}\!\mathit{om}(\mathbf{K}, \mathbf{S}) \to \mathscr{H}\!\mathit{om}(K, S)$ *is coelementary*; *here* $\mathbf{K} = \mathbf{Hom}(\mathscr{T}, \mathscr{S})$), *we construct the elementary functor*

$$e_{\mathscr{T}}\colon\ \mathscr{T} \longrightarrow \mathscr{H}\!\mathit{om}(\mathbf{Hom}(\mathscr{T}, \mathscr{S}), \mathbf{S}).$$

It is clear from its definition that $e = e_{\mathscr{T}}$ satisfies the formulas defining evaluation functors:

$$e(A)(M) = M(A) \qquad (A \in |T|,\ M \in |\mathrm{Hom}(\mathscr{T}, \mathscr{S})|)$$

$$e(A)(h) = h_A \qquad (h\colon M \to N \text{ in } \mathrm{Hom}(\mathscr{T}, \mathscr{S}))$$

$$e(f)_M = M(f) \qquad (f\colon A \to B \text{ in } T).$$

3. ULTRAMORPHISMS

The main new concept of this paper is that of ultramorphism, a generalization of the canonical embedding $M \to^{\delta} M^U$ of a structure (or a set) into an ultrapower of it. An *ultramorphism* is a "canonically defined map between ultraproducts." Before giving the general definition, we give an example, itself generalizing the above $\delta\colon M \to M^U$.

Let us give ourselves two sets, I and J, a map $g\colon I \to J$, an ultrafilter U on I, and the induced ultrafilter V on J defined by $Q \in V$ iff $g^{-1}(Q) \in U$. With these fixed data, we associate an ultramorphism in SET as follows. Let $\langle A_j \rangle = \langle A_j\colon j \in J \rangle$ be a J-family of sets. We deduce the I-family $\langle A_{g(i)}\colon i \in I \rangle$ and the two ultraproducts $A = \prod A_j/V$ and $B = \prod A_{g(i)}/U$. We have the following "canonically defined" map $\delta = \delta_{\langle A_j \rangle}\colon A \to B$ given by the formula

$$\delta(\langle a_j \rangle/V) = \langle a_{g(i)} \rangle/U$$

(it is easy to verify that δ is well defined).

The ultramorphism δ is the totality of all maps $\delta_{\langle A_j \rangle}$, for all J-families $\langle A_j \rangle$ of sets. That δ is an ultramorphism amounts, according to our general definition below, to satisfying that whenever $f_j\colon A_j \to B_j$ is a map for every $j \in J$, then the diagram

$$\begin{tikzcd}
\prod A_j/V \arrow[r, "\delta_{\langle A_j \rangle}"] \arrow[d, "\prod f_j/V"'] & \prod A_{g(i)}/U \arrow[d, "\prod f_{g(i)}/U"] \\
\prod B_j/V \arrow[r, "\delta_{\langle B_j \rangle}"'] & \prod B_{g(i)}/U
\end{tikzcd}$$

commutes; this is indeed readily verified. Choosing J to be a singleton, we obtain the first-mentioned example.

We now give the general concept.

An *ultragraph* Γ is given by

(i) two disjoint sets Γ^{f}, Γ^{b}; the elements of Γ^{f} are called *free nodes*, the elements of Γ^{b} are *bound nodes*; $|\Gamma| = \Gamma^{\mathrm{f}} \cup \Gamma^{\mathrm{b}}$ is the set of *nodes*;

(ii) for any pair γ, γ' of nodes, a set $E(\gamma, \gamma')$ of *edges* from γ to γ'; in particular we have a (directed) graph in the sense of [CWM];

(iii) an assignment of a triple $\langle I, U, g \rangle = \langle I_\beta, U_\beta, g_\beta \rangle$ to any bound node $\beta \in \Gamma^{\mathrm{b}}$ such that U is an ultrafilter on I, and $g\colon I \to \Gamma^{\mathrm{f}}$.

An *ultradiagram of type Γ in* a pre-ultracategory **S** is a diagram

$$\mathscr{A}\colon\ \Gamma \to S$$

(i.e., an assignment of an object $\mathscr{A}(\gamma)$ of S to each node γ of Γ, and of a morphism $\mathscr{A}(e)$: $\mathscr{A}(\gamma) \to \mathscr{A}(\gamma')$ to each edge $e \in E(\gamma, \gamma')$) satisfying the condition

$$\mathscr{A}(\beta) = \prod_{i \in I_\beta} \mathscr{A}(g_\beta(i))/U_\beta$$

for all $\beta \in \Gamma^{\mathrm{b}}$. We use the notation

$$\mathscr{A}: \Gamma \longrightarrow \mathrm{S}$$

to indicate that $\mathscr{A}$ is an ultradiagram of type Γ in S.

A *morphism of* ultradiagrams Φ: $\mathscr{A} \to \mathscr{B}$ between ultradiagrams

$$\mathscr{A}, \mathscr{B}: \quad \Gamma \rightrightarrows \mathrm{S}$$

of the same type in the same ultracategory is a natural transformation Φ between $\mathscr{A}$ and $\mathscr{B}$ as diagrams (see [CWM]) satisfying the additional condition

$$\Phi_\beta = \prod_{i \in I_\beta} \Phi_{g_\beta(i)}/U_\beta$$

for all $\beta \in \Gamma^{\mathrm{b}}$.

Let Γ be an ultragraph, and k and l two distinguished nodes (free or bound) in Γ; write Γ^* for the triple $\langle \Gamma, k, l \rangle$. An *ultramorphism* (u.m.) *of type* Γ^* in S (S a p.-u.c.) is given by assigning a morphism

$$\delta_{\mathscr{A}}: \quad \mathscr{A}(k) \longrightarrow \mathscr{A}(l)$$

to each ultradiagram $\mathscr{A}: \Gamma \to S$ such that whenever Φ: $\mathscr{A} \to \mathscr{B}$ is a morphism between ultradiagrams $\mathscr{A}, \mathscr{B}$: $\Gamma \rightrightarrows S$, we have that the diagram

$$\begin{tikzcd}
\mathscr{A}(k) \arrow[r, "\Phi_k"] \arrow[d, "\delta_A"'] & \mathscr{B}(k) \arrow[d, "\delta_B"] \\
\mathscr{A}'(l) \arrow[r, "\Phi_l"'] & \mathscr{B}(l)
\end{tikzcd}$$

commutes.

The same definition more abstractly put looks as follows. We have the category $\mathrm{Hom}(\Gamma, \mathrm{S})$ of all ultradiagrams of type Γ in S as objects and all morphisms between them as explained above; the composition is that of natural transformations. The nodes k and l of Γ define functors

$$\begin{tikzcd}
\mathrm{Hom}(\Gamma, \mathrm{S}) \arrow[r, shift left, "(k)"] \arrow[r, shift right, "(l)"'] & S
\end{tikzcd}$$

"evaluation at k(at$l$)" defined by the formulas

$$(k)(\mathscr{A}) = \mathscr{A}(k) \qquad (\mathscr{A} \in |\mathrm{Hom}(\Gamma, \mathbf{S})|)$$

$$(k)(\Phi) = \Phi_k \qquad (\Phi\colon \mathscr{A} \to \mathscr{B} \text{ in } \mathrm{Hom}(\Gamma, \mathbf{S}))$$

and similarly for (l). An ultramorphism of type Γ^* in $\mathbf{S}$ is a natural transformation δ: $(k) \to (l)$.

To fit our example above into this scheme, we let Γ be the ultragraph specified as

$$\Gamma^{\mathrm{f}} = J$$

$$\Gamma^{\mathrm{b}} = \{k, l\} \qquad \text{(disjoint from } J\text{)}$$

$$E(\gamma, \gamma') = \varnothing \qquad \text{for all } \gamma, \gamma' \in |\Gamma|$$

$$\langle I_k, U_k, g_k \rangle = \langle J, V, id_J \rangle$$

$$\langle I_l, U_l, g_l \rangle = \langle I, U, g \rangle.$$

An ultradiagram $\mathscr{A}$ of type Γ in SET is determined by the family $\langle \mathscr{A}(j) \colon j \in J \rangle = \langle A_j \colon j \in J \rangle$. Defining $\delta_{\mathscr{A}}$ to be $\delta_{\langle A_j \rangle}$ as we did above results in an ultramorphism in **SET**; the requisite commutativity condition is seen to be identical to the one stated above.

We denote the class of all ultramorphisms in **SET** by Δ(SET). An *ultracategory* (u.c.) **K** is a pre-ultracategory together with a specification of an ultramorphism $\delta_{\mathbf{K}}$ associated with any $\delta \in \Delta(\mathrm{SET})$ such that $\delta_{\mathbf{K}}$ is of the same type as δ. **SET** is the "standard" u.c. with $\delta_{\mathrm{SET}} = \delta$ $(\delta \in \Delta(\mathrm{SET}))$. We denote both ultracategories and pre-ultracategories by bold face capitals; when we are forced to, we write pre-**K** for the p.-u.c. part of the u.c. **K**.

Let **K** and **S** be pre-ultracategories, X: $\mathbf{K} \to \mathbf{S}$ a pre-ultrafunctor. Let δ' and δ be ultramorphisms in **K**, respectively, in **S**, of the same type Γ^*. We define what we mean by saying that X *carries* δ' into δ. If X is strict, this means that in the diagram

$$\begin{tikzcd}
\mathrm{Hom}(\Gamma, \mathbf{K}) \arrow[r, shift left=2, "(k)"] \arrow[r, shift right=2, "(l)"'] \arrow[r, phantom, "\delta'\downarrow"] \arrow[d, "\mathrm{Hom}(\Gamma{,}X)"'] & \mathbf{K} \arrow[d, "X"] \\
\mathrm{Hom}(\Gamma, \mathbf{S}) \arrow[r, shift left=2, "(k)"] \arrow[r, shift right=2, "(l)"'] \arrow[r, phantom, "\delta\downarrow"] & S
\end{tikzcd}$$

the two composite natural transformations are equal,

$$\delta \circ \operatorname{Hom}(\Gamma, X) = X \circ \delta'.$$

The point here is that X being strict, the left vertical arrow makes sense. For a general X, we have to slightly modify this definition; this modification could have been avoided by defining the notion of ultradiagram more liberally allowing "nonstrict" ones too.

Let $\mathscr{M}\colon \Gamma \to \mathbf{K}$ be an ultradiagram. There is an ultradiagram, denoted $X\mathscr{M}\colon \Gamma \to \mathbf{S}$, uniuely determined by the following: there is a (natural) diagram isomorphism

$$X \circ \mathscr{M} \xrightarrow[v]{\cong} X\mathscr{M}$$

such that for each free node γ of Γ, $v_\gamma = \mathrm{Id}_{\mathscr{M}(\gamma)}$, and for each bound node β of Γ, $v_\beta = [X, U]_{\bar{M}}$ with $\langle I, U, g\rangle = \langle I_\beta, U_\beta, g_\beta\rangle$ and $\bar{M} = \langle \mathscr{M}(g(i))\colon i \in I\rangle$; this does not use anything except that each $[X, U]_{\bar{M}}$ is an isomorphism. $X\mathscr{M}$ is obtained by replacing

$$X(\mathscr{M}(\beta)) = X\left(\prod \mathscr{M}(g(i))/U\right)$$

by

$$\prod X(\mathscr{M}(g(i))/U$$

so that the basic requirement on ultradiagrams is met. With $X\mathscr{M}$ so defined, we say that X carries δ' into δ if the diagram

$$\begin{tikzcd}
X(\mathscr{M}(k)) \arrow[r, "X(\delta'.\mathscr{M})"] \arrow[d, "v_k"'] & X(\mathscr{M}(l)) \arrow[d, "v_l"] \\
(X\mathscr{M})(k) \arrow[r, "\delta_{X.\mathscr{M}}"'] & (X\mathscr{M})(l)
\end{tikzcd}$$

commutes.

An *ultrafunctor* (u.f.) X: $\mathbf{K} \to \mathbf{S}$ between ultracategories is a pre-ultrafunctor that carries $\delta_{\mathbf{K}}$ into $\delta_{\mathbf{S}}$, for all $\delta \in \Delta(\mathrm{SET})$.

For ultracategories $\mathbf{K}$ and $\mathbf{S}$, Hom($\mathbf{K}$, $\mathbf{S}$) denotes the category whose objects are the ultrafunctors $\mathbf{K} \to \mathbf{S}$ and whose morphisms are all the ultratransformations; Hom($\mathbf{K}$, $\mathbf{S}$) is a full subcategory of Hom(pre-$\mathbf{K}$, pre-$\mathbf{S}$).

LEMMA 3.1. *Suppose* $\Phi\colon Y \to X$ *is a morphism in* Hom(pre-$\mathbf{K}$, pre-$\mathbf{S}$), *with ultracategories* **K**, **S**, *and suppose that* X *is an ultrafunctor and that* Φ_M *is a monomorphism for each* $M \in |\mathbf{K}|$. *Then* Y *is an ultrafunctor.*

Proof. The commutative diagram

$$\begin{tikzcd}
X \circ \mathscr{M} \arrow[r, "\nu^X"] \arrow[dr, phantom, "\circlearrowleft"] & X\mathscr{M} \\
Y \circ \mathscr{M} \arrow[u, "\Phi \circ \mathscr{M}"] \arrow[r, "\nu^Y"'] & Y\mathscr{M} \arrow[u, "\phi"']
\end{tikzcd}$$

defines ϕ as a morphism of the diagrams $Y\mathscr{M}$, $X\mathscr{M}$. The fact that Φ is an ultratransformation easily implies that ϕ is a morphism of the ultradiagrams $Y\mathscr{M}$. $X\mathscr{M}$. Let us write δ for δ_S, δ' for $\delta_{\mathbf{K}}$. In the diagram

$$\begin{tikzcd}
& X(\mathscr{M}(k)) \arrow[rr, "X(\delta'_{.\mathscr{M}})"] & & X(\mathscr{M}(l)) \\
Y(\mathscr{M}(k)) \arrow[ur, "\Phi_{.\mathscr{M}(k)}"] \arrow[rr, "Y(\delta'_{.\mathscr{M}})"] & & Y(\mathscr{M}(l)) \arrow[ur, "\Phi_{.\mathscr{M}(l)}"] & \\
& (X\mathscr{M})(k) \arrow[uu, "(\nu_k^X)^{-1}"'] \arrow[rr, "\delta_{X.\mathscr{M}}"] & & (X\mathscr{M})(l) \arrow[uu, "(\nu_l^X)^{-1}"'] \\
(Y\mathscr{M})(k) \arrow[uu, "(\nu_k^Y)^{-1}"] \arrow[ur, "\varphi_k"'] \arrow[rr, "\delta_{Y.\mathscr{M}}"'] & & (Y\mathscr{M})(l) \arrow[uu, "(\nu_k^Y)^{-1}", near end] \arrow[ur, "\varphi_l"'] &
\end{tikzcd}$$

the left and right faces commute by the definition of ϕ; the bottom face commutes because δ is an u.m. in S; the top face commutes since Φ is natural; and finally, the back face commutes since X is an ultrafunctor. It follows that the two composite diagonals in the front face are coequalized by $\Phi_{\mathscr{M}(l)}$. Since the latter is a monomorphism, the front face commutes. ∎

We have now $\mathbf{S} = \mathbf{SET}$ as an ultracategory. Based on the "dual" structures $\mathscr{S}$ (a pretopos) and **S** (ultracategory) with the same underlying category SET, a "duality" can be set up between pretoposes and ultracategories. Replacing systematically p.-u.c.'s and p.-u.f.'s by u.c.'s and u.f.'s, respectively, and accordingly changing the meaning of Hom(**K**, **S**), we can repeat, without otherwise changing anything, the basic duality theory of Section 2. We refer to the resulting statements–definitions by the notation $(\mathrm{i})_\Delta(\mathrm{a})$–$(\mathrm{vi})_\Delta(\mathrm{b})$ (some of which are identical to their original counterpart). Below, some remarks follow concerning the proofs of statements $(\mathrm{i})_\Delta(\mathrm{a})$–$(\mathrm{vi})_\Delta(\mathrm{b})$.

ad(i)$_\Delta$(b) The additional structure on $\mathbf{K} = \mathrm{Hom}(T, S)$ is defined by the formula

$$((\delta_{\mathbf{K}})_{\mathcal{M}})_A = \delta_{(A)\circ\mathcal{M}};$$

$\mathcal{M}$: $\Gamma \to \mathbf{K}$, $\delta \in \Delta(\mathrm{SET})$ of type Γ^*, $A \in |T|$, (A): $\mathrm{Hom}(T, S) \to S$ is the functor "evaluation at A" (note that (A) is a strict pre-ultrafunctor by (i)(b) and hence $(A)\circ\mathcal{M}$ is an ultradiagram). It is easy to verify that $\delta_{\mathbf{K}}$ so defined is indeed an u.m. in $\mathbf{K}$.

ad(iv)$_\Delta$(a) We show that the full inclusion

$$\mathscr{C} = \mathrm{Hom}(\mathbf{K}, \mathbf{S}) \longrightarrow \mathscr{D} = \mathrm{Hom}(\text{pre-}\mathbf{K}, \text{pre-}\mathbf{S})$$

is elementary; by (iv)(a), the assertion will then follow. In fact, we show that (1.1) any terminal object in $\mathscr{D}$ belongs to $\mathscr{C}$, (1.2) any product object $X \times Y$ in $\mathscr{D}$ of elements X, Y of $|\mathscr{C}|$ belongs to $|\mathscr{C}|$, (1.3) any equalizer object in $\mathscr{D}$ of a pair of morphisms in $\mathscr{C}$ belongs to $\mathscr{C}$, and three more similar conditions corresponding to the other clauses of the definition of pretopos.

The proofs of all these statements are similar, and they are similar to the proof of 3.1; in fact (1.3) is a consequence of 3.1. Some details for (1.2) are as follows.

Let X, $Y \in |\mathscr{C}|$, and

$$\begin{tikzcd} & X \times Y \arrow[dl, "\Pi"'] \arrow[dr, "\Pi'"] & \\ X & & Y \end{tikzcd}$$

a product diagram in $\mathscr{D}$. Let $\mathcal{M}$: $\Gamma \to \mathbf{K}$ be an ultradiagram. We have morphisms of ultradiagrams in S as in

$$\begin{tikzcd} & (X \times Y)\mathcal{M} \arrow[dl, "\Pi.\mathcal{M}"'] \arrow[dr, "\Pi'.\mathcal{M}"] & \\ X\mathcal{M} & & Y\mathcal{M} \end{tikzcd}$$

such that the square

$$\begin{tikzcd} (X \times Y)\circ\mathcal{M} \arrow[r, "\nu^{X \times Y}"] \arrow[d, "\Pi\circ\mathcal{M}"'] & (X \times Y)\mathcal{M} \arrow[d, "\Pi.\mathcal{M}"] \\ X\circ\mathcal{M} \arrow[r, "\nu^X"'] & X\mathcal{M} \end{tikzcd}$$

and the similar other square commute. Writing δ' for $\delta_{\mathbf{K}}$ and δ for $\delta_{\mathbf{S}}$ ($\delta \in \Delta(\mathrm{SET})$ of type Γ^*), we have the diagram

$$\begin{tikzcd}
 & X(\mathscr{M}(k)) \arrow[rr, "X(\delta'_{\mathscr{M}})"] \arrow[dd, "\nu"] & & X(\mathscr{M}(l)) \arrow[dd, "\nu"] \\
(X \times Y)(\mathscr{M}(k)) \arrow[ur, "\Pi_{.\mathscr{M}(k)}"] \arrow[rr, "(X \times Y)(\delta'_{.\mathscr{M}})"'] \arrow[dd, "\nu"'] & & (X \times Y)(\mathscr{M}(l)) \arrow[ur, "\Pi_{.\mathscr{M}(l)}"] \arrow[dd, "\nu"] & \\
 & (X\mathscr{M})(k) \arrow[rr, "\delta_{X.\mathscr{M}}"] & & (X\mathscr{M})(l) \\
((X \times Y)\,\mathscr{M})(k) \arrow[ur, "(\Pi.\mathscr{M})_k"] \arrow[rr, "\delta_{(X \times Y).\mathscr{M}}"] & & ((X \times Y)\,\mathscr{M})(l) \arrow[ur, "(\Pi.\mathscr{M})_l"'] &
\end{tikzcd}$$

and a similar one with X replaced by Y. Similarly to the proof of 3.1, we see that all faces except the front one commute. Let us write α for the horizontal arrow $(X \times Y)(\delta'_{\mathscr{M}})$, and β for the arrow with the same domain and codomain that makes, in place of α, the front face commute. Then putting either α or β in place of the dashed arrow in

$$\begin{tikzcd}
X(\mathscr{M}(k)) \arrow[r, "X(\delta'_{\mathscr{M}})"] & X(\mathscr{M}(l)) \\
(X \times Y)(\mathscr{M}(k)) \arrow[u, "\Pi_{.\mathscr{M}(k)}"] \arrow[r, dashed] \arrow[d, "\Pi_{.\mathscr{M}(k)}"'] & (X \times Y)(\mathscr{M}(l)) \arrow[u, "\Pi_{.\mathscr{M}(l)}"'] \arrow[d, "\Pi'_{.\mathscr{M}(l)}"] \\
Y(\mathscr{M}(k)) \arrow[r, "Y(\delta'_{.\mathscr{M}})"'] & Y(\mathscr{M}(l))
\end{tikzcd}$$

we obtain a commutative diagram. Since the two vertical sides are both product diagrams, it follows that $\alpha = \beta$, as required for the commutativity of the front face.

We emphasize that later we will always be interested in the "ultra" versions as opposed to the "pre-ultra" ones. Accordingly, e.g, $\mathbf{Hom}(\mathscr{T}, \mathscr{S})$ will mean an ultracategory rather than a pre-ultracategory.

Finally, we consider the "canonical map of a set into an ultrapower of it" as an ultramorphism; this turns out to be a slight modification of the special case of the example considered at the beginning of this section with J taken to be a singleton. Given an ultrafilter (I, U), we put $\Gamma^{\mathrm{f}} = \{k\}$, $\Gamma^{\mathrm{b}} = \{l\}$, $E(\gamma, \gamma') = \varnothing$ for all $\gamma, \gamma' \in |\Gamma|$, $\langle I_l, U_l, g_l \rangle = \langle I, U, g \rangle$ with $g(i) = k$ for all $i \in I$; this defines an u.g. Γ (the modification mentioned above consists in taking k to be a free node). The desired u.m. δ of type $\langle \Gamma, k, l \rangle$ is given as follows. An ultradiagram $\mathscr{A}$ of type Γ in $\mathbf{S}$ is determined by a single $A \in |S|$, the value $\mathscr{A}(k) = A$. We write δ_A for $\delta_{\mathscr{A}}$; δ_A: $A \to A^U$ is the map $a \mapsto \langle a \rangle / U$, where $\langle a \rangle$ is the constant function $I \to A$ with value a. For $\mathbf{K} = \mathbf{Hom}(\mathscr{T}, \mathscr{S})$, the induced u.m. $\delta' = \delta_{\mathbf{K}}$ is as follows. Again an ultradiagram $\mathscr{M}$ of type Γ in $\mathbf{K}$ is determined by a single $M \in |\mathrm{Hom}(\mathscr{T}, \mathscr{S})|$. We write δ_M for $\delta'_{\mathscr{M}}$; we have δ_M: $M \to M^U$ such that $(\delta_M)_A = \delta_{M(A)}$. Given a p.-u.f. X: $\mathbf{K} \to \mathbf{S}$, to say that X carries $\delta_{\mathbf{K}}$ into δ means that the diagram

$$\begin{tikzcd}
 & X(M^U) \arrow[dd, "{[X,U]_{\langle M \rangle}}"] \\
X(M) \arrow[ur, "X(\delta_M)"] \arrow[dr, "\delta_{XM}"'] & \\
 & (XM)^U
\end{tikzcd}$$

commutes.

4. RECOVERING A THEORY FROM THE ULTRACATEGORY OF ITS MODELS

Let $T = \mathscr{T}$ be a small pretopos, fixed throughout this section. Let us write **Mod** T for the ultracategory of the models of T, i.e., **Mod** $T = \mathbf{Hom}(\mathscr{T}, \mathscr{S}et)$. The last item in the list of definitions given before, $(\mathrm{vi})_{\Delta}(\mathrm{b})$, gives us the canonical "evaluation" functor e_T: $\mathscr{T} \to \mathscr{H}om(\mathbf{Mod}\ T, \mathbf{SET})$. The main theorem of the paper is

THEOREM 4.1. *For any small pretopos T, e_T is an equivalence of categories.*

The rest of the section is devoted to the proof of the theorem.

Let us denote the pretopos $\mathscr{H}om(\mathbf{Mod}\ T, \mathbf{SET})$ by T'. The essential use of the fact that we are talking about pretoposes, and not just say about "logical" categories in the sense of [MR], is the existence of a criterion usable here for an elementary functor e: $T \to T'$ between pretoposes to be an equivalence. First, we give some definitions.

Let e: $T \to T'$ be an elementary functor between pretoposes. For any $A \in |T|$, e induces a lattice homomorphism

$$e^{(A)}\colon\ \mathrm{Sub}_T(A) \longrightarrow \mathrm{Sub}_{T'}(eA)$$

(where $\mathrm{Sub}_T(A)$ is the lattice of subobjects of A); to the subobject determined by the monomorphism $X \to^{\alpha} A$, $e^{(A)}$ assigns the subobject determined by $e(X) \to^{e(\alpha)} e(A)$. e is called *conservative* if $e^{(A)}$ is injective (one-to-one) for all $A \in |T|$; *subobject-full* if $e^{(A)}$ is surjective for all $A \in |T|$. It is easy to see that e being conservative implies that e is faithful, and e being conservative and subobject-full implies that it is full.

Let $X \in |T'|$. A *partial cover of X (via e)* is a pair (Φ, Ψ) of morphisms in T' of the form

$$\begin{tikzcd}
Y \arrow[r, "\Psi"] \arrow[d, "\Phi"'] & e(A) \\
X &
\end{tikzcd}$$

with $A \in |T|$, and such that Ψ is a monomorphism. With A being specified, we talk about a *partial A-cover of X*. A *finite cover of X via e* is a finite family $\langle (\Phi_i, \Psi_i)\colon i \in I \rangle$ of partial covers of X such that $X = \bigvee_{i \in I} \operatorname{Im}(\Phi_i)$. [$\bigvee$ denotes sup in the lattice $\operatorname{Sub}_{T'}(X)$; $\operatorname{Im}(\Phi)$ is the least subobject of X through which Φ factors.]

LEMMA 4.2. *With e as above, the following conditions are jointly sufficient for e being an equivalence*:

(i) *e is conservative*;

(ii) *e is subobject-full*;

(iii) *every object of T' has a finite cover via e.*

For the proof, see 7.1.7, p. 203 in [MR].

It remains to verify the three conditions in 4.2 for our $e = e_T$. e being conservative is equivalent to saying that ev: $T \to \mathcal{H}om(\text{Mod } T, \text{SET})$ is conservative; ev being conservative is essentially Godel's completeness theorem; see Section 3.5, especially 3.5.5(ii), and also 6.3.5(ii) in [MR].

Next we deal with 4.2(ii).

LEMMA 4.3. *Let $M, N \in |\text{Mod } T|$, $A \in |T|$, $a \in M(A)$ and $b \in N(A)$. Suppose that for all $\Sigma \in \operatorname{Sub}(A)$, $b \in N(\Sigma)$ implies that $a \in M(\Sigma)$. Then there is an ultrafilter (I, U) and a homomorphism $h\colon N \to M^U$ such that $\delta_A(a) = h_A(b)$, where δ is the canonical embedding of M into the ultrapower M^U.*

Proof. The condition is equivalent to saying that every positive existential formula (in the canonical language associated with T) satisfied by a in M is satisfied by b in N. Looking at it this way, we see that the lemma becomes a variant of Tarski's theorem on substructures–extensions [CK]; also compare 7.1.4′ in [MR]. The precise version can be left as an exercise. ▌

Proof of 4.2(ii). Let v: $X \to e(A)$ be a morphism in $T' = \mathcal{H}om(\mathbf{Mod}\ T, \mathbf{SET})$; $A \in |T|$. Let us denote by $X^*(M)$ the subject $\mathrm{Im}(v_M)$ of $e(A)(M) = M(A)$, for $M \in |\mathrm{Mod}\ T)$. Let S denote the collection of all $\Sigma \in \mathrm{Sub}_T(A)$ such that $X^*(N) \subset N(\Sigma)$ for all $N \in |\mathrm{Mod}\ T|$.

Claim 1. $X^*(M) = \bigcap_{\Sigma \in S} M(\Sigma)$, for any $M \in |\mathrm{Mod}\ T|$.

It is obvious that the left-hand side is contained in the right. To show the converse, let a belong to $M(\Sigma)$ for all $\Sigma \in \mathrm{Sub}(A)$ such that $\Sigma \in S$. Let $J = \{\Sigma \in \mathrm{Sub}(A)$: $a \notin M(\Sigma)\}$. J is closed under finite sup(join) in $\mathrm{Sub}(A)$; in particular, $J \neq \varnothing$, since $0_A = \bigvee \varnothing \in J$. Let V be an ultrafilter on J such that for any $\Sigma \in J$, the set $[\Sigma] =_{\mathrm{df}} \{\Phi \in J$: $\Phi \geqslant \Sigma\}$ belongs to V; since $[\Sigma_1] \cap \cdots \cap [\Sigma_n] = [\Sigma_1 \vee \cdots \vee \Sigma_n]$, such V exists.

For each $\Sigma \in J$, there is some $N_\Sigma \in |\mathrm{Mod}\ T|$ and $b_\Sigma \in N_\Sigma(A)$ such that $b_\Sigma \in X^*(N_\Sigma) - N_\Sigma(\Sigma)$; this is a consequence of the fact that $\Sigma \in J$ implies that $\Sigma \notin S$. Consider the ultraproduct $N = \prod_{\Sigma \in J} N_\Sigma / V$ and the element $b = \langle b_\Sigma \rangle / V$ in $N(A)$. Using the fact that $[\Sigma] \in V$ for all $\Sigma \in J$, it is easy to see that for all $\Sigma \in \mathrm{Sub}(A)$, $b \in N(\Sigma)$ implies that $a \in M(\Sigma)$. Therefore, by 4.3, there are an ultrafilter (I, U) and h: $N \to M^U$ such that $\delta_A(a) = h_A(b)$.

Since $b_\Sigma \in X^*(N_\Sigma)$, there is $x_\Sigma \in X(N_\Sigma)$ with $v_N(x_\Sigma) = b_\Sigma$. Since X is a pre-ultrafunctor, and v is an ultratransformation, we have the commutative diagram

$$\begin{tikzcd}
 & & X(N) \arrow[dll, "v_N"'] \arrow[dd, "\wr"', "{[X,Y]_{\langle N_\Sigma \rangle} = \phi}"] \\
\prod N_\Sigma(A)/V & \circlearrowleft & \\
 & & \prod X(N_\Sigma)/V \arrow[ull, "\prod v_{N_\Sigma}/V"]
\end{tikzcd}$$

Let $x = \langle x_\Sigma \rangle / V$, and $y = \phi^{-1}(x)$; it follows that $v_N(y) = b$. From the commutativity of the diagram

$$\begin{tikzcd}
N(A) \arrow[r, "h_A"] \arrow[d, "v_N"'] & M^U(A) \arrow[d, "v_{M^U}"] \\
X(N) \arrow[r, "X(h)"'] & X(M^U)
\end{tikzcd}$$

we infer that $h_A(b) = h_A(v_N(y)) = v_{M^U}(z)$ for $z = (X(h))(y)$. Now consider,

$$\begin{tikzcd}
M^U(A) \arrow[drr, "(v_M)^U"] & & \\
 & \circlearrowleft & \\
X(M^U) \arrow[uu, "v_{M^U}"] \arrow[rr, "\psi =_{\mathrm{df}} {[X,U]_{\langle M \rangle}}"'] & & (XM)^U
\end{tikzcd}$$

and let $\psi(z) = \langle x_i \rangle / U$; it follows that $\langle a \rangle / U = \delta_A(a) = h_A(b) = v_{M^U}(z) = (v_M)^U(\langle x_i \rangle / U) = \langle v_M(x_i) \rangle / U$. Therefore $a = v_M(x_i)$ for U-almost all i, hence for at least one i; in other words, $a \in X^*(M)$, proving the claim.

Claim 2. There is $\Sigma \in S$ such that $X^*(M) = M(\Sigma)$ for all $M \in |\text{Mod } T|$.

Suppose not, i.e., for all $\Sigma \in S$ there are $M_\Sigma \in |\text{Mod } T|$ and $a_\Sigma \in M_\Sigma(\Sigma) - X^*(M_\Sigma)$. Let W be an ultrafilter on S [S is clearly non-empty] such that $[\Sigma] =_{\text{df}} \{\Phi \in S\colon \Phi \leqslant \Sigma\} \in W$ for all $\Sigma \in S$. Since S is closed under finite inf(meet), such W exists. Let $M = \prod M_\Sigma / W$, $a = \langle a_\Sigma \rangle / W \in M(A)$. By the choice of W, we see that $a \in \bigcap_{\Sigma \in S} M(\Sigma)$. Consider the commutative diagram

$$\begin{tikzcd}
\prod X(M_\Sigma)/W \arrow[drr, "\prod v_{M_\Sigma}/W"] & & \\
X(M) \arrow[u, "{[X,W]\langle M_\Sigma \rangle}", "\wr"'] \arrow[rr, "v_M"'] & \circlearrowleft & M(A)
\end{tikzcd}$$

If we had $a \in X^*(M) = \text{Im}(v_M)$, then there would be $\langle x_\Sigma \rangle / W \in \prod X(M_\Sigma)/W$ such that $a_\Sigma = v_{M_\Sigma}(x_\Sigma)$ for W-almost all Σ, hence for at least one Σ; but that would contradict $a_\Sigma \notin X^*(M_\Sigma)$. We have concluded that $a \in \bigcap_{\Sigma \in S} M(\Sigma) - X^*(M)$, contrary to Claim 1. This proves Claim 2.

Let us now assume that v is a monomorphism. Because of the "pointwise" definition of the pretopos structure and in particular, of the finite $\underleftarrow{\lim}$-structure on T', we see that this is equivalent to saying that v_M is a one-to-one function for all $M \in \text{Mod } T$. Let $\Sigma \in S$ satisfy the assertion of Claim 2. Then, we claim, the subobject determined by v equals $e(\Sigma)$; in other words, if Σ is given by the monomorphism $\Sigma \xrightarrow{\sigma} A$, then there is an isomorphism ϕ in T' making the diagram

$$\begin{tikzcd}
X \arrow[rr, "v"] & & e(A) \\
& e(\Sigma) \arrow[ul, "\phi", "\wr"'] \arrow[ur, "e(\sigma)"'] &
\end{tikzcd} \tag{4.1}$$

commute. By $X^*(M) = M(\Sigma)$, clearly there is a unique such isomorphism in Hom(Mod T, SET); it remains to check that it is an ultratransformation. Consider the diagram

$$\begin{tikzcd}
& X\left(\prod M_i/U\right) \arrow[d, "\nu_{\prod M_i/U}"] \arrow[ddr, "{[X,U]_{\langle M_i\rangle}}"] & \\
& \prod M_i(A)/U & \\
\prod M_i(\Sigma)/U \arrow[uur, "\phi_{\prod M_i/U}"] \arrow[ur, "\prod M_i(\sigma)/U"'] \arrow[rr, "\prod \phi_{M_i}/U"'] & & \prod X(M_i)/U \arrow[ul, "\prod \nu_{M_i}/U"]
\end{tikzcd}$$

with an arbitrary ultraproduct $\prod M_i/U$ of models of T. The left upper triangle commutes since it is the instance of (4.1) at $\prod M_i/U$. The right upper triangle commutes because ν is an ultratransformation. The bottom triangle commutes since it is the ultraproduct of the instances of (4.1) at the M_i. Therefore, the bottom side and the composite of the upper sides of the outer triangle are coequalized by $\prod \nu_{M_i}/U$. Since the latter is a monomorphism (in SET), it follows that the outer triangle commutes, proving that ϕ is indeed an ultratransformation. We have proved 4.2(ii). ▌

Next we turn to proving 4.2(iii). Let X be an object of T', fixed until the end of the proof.

Let $M \in |\text{Mod } T|$, $x \in X(M)$, $A \in |T|$ and $a \in M(A)$. We say that (A, a) is a *support* of x if the following holds: for any pair of morphisms h_1, h_2: $M \rightrightarrows N$ in Mod T, $(h_1)_A(a) = (h_2)_A(a)$ implies $(Xh_1)(x) = (Xh_2)(x)$.

The motivation for the use of this concept is as follows. Let T' denote (temporarily) any pretopos, A and X two objects of T'. Let $M \in \text{Mod } T'$, $x \in M(X)$, $a \in M(A)$. We say that a is a "support" of x if for any pair h_1, h_2: $M \rightrightarrows N$ of morphisms in Mod T', $(h_1)_A(a) = (h_2)_A(a)$ implies $(h_1)_X(x) = (h_2)_X(x)$. A "partial A-cover" of X is a pair of morphisms (ψ, ϕ) with ϕ: $\Sigma \to A$ a mono, and ψ: $\Sigma \to X$. Then the following are equivalent:

(i) a is a support of x;

(ii) there are a partial A-cover (ψ, ϕ) of X and $s \in M(\Sigma)$ such that $(M(\phi))(s) = a$ and $(M(\psi))(s) = x$.

The implication (ii)$\Rightarrow$(i) is trivial; the converse is an easy model theoretical argument; it was first observed and used by Bacsich [B].

The preceding observation makes the subsequent work natural enough. In fact, one can see easily that the statement of 4.1 together with the observation imply each of the subsequent lemmas.

LEMMA 4.4. *Every $x \in X(M)$ has some support (A, a).*

Proof. It suffices to show that there is a finite family $a_i \in M(A_i)$ $(i < n)$

such that $(h_1)_{A_i}(a_i) = (h_2)_{A_i}(a_i)$ for all $i < n$ implies $(Xh_1)(x) = (Xh_2)(x)$ for all h_1, h_2 as above. Namely, having the a_i, we can put $A = A_0 \times \cdots \times A_{n-1}$ and $a = \langle a_0, \dots, a_{n-1} \rangle$, the element of $M(A)$ that is mapped to a_i by the ith projection $M(A) \to M(A_i)$ for all $i < n$; it is easy to see that (A, a) is a support of x. Suppose, to the contrary, that no such finite family exists. Let I be the set of all finite sets consisting of pairs (A, a) such that $A \in |T|$ and $a \in M(A)$. For every $i \in I$, there are $N_i \in |\mathrm{Mod}\ T|$ and $h_1^i . h_2^i \colon M \rightrightarrows N_i$ such that $(h_1^i)_A(a) = (h_2^i)_A(a)$ for all $(A, a) \in i$, and $(Xh_1^i)(x) \neq (Xh_2^i)(x)$. For any $i \in I$, let $[i]$ be the subset $\{j \in I \colon j \supseteq i\}$ of I. Clearly, $[i_1] \cap \cdots \cap [i_n] = [i_1 \cup \cdots \cup i_n]$; hence, there is an ultrafilter U on I such that $[i] \in U$ for all $i \in I$. Consider now the diagram

$$\begin{tikzcd}
M \arrow[r, "\delta_M"] & M^U \arrow[r, shift left, "h_1 = \prod h_1^i/U"] \arrow[r, shift right, "h_2 = \prod h_2^i/U"'] & \prod N_i/U
\end{tikzcd} \tag{4.2}$$

with δ_M the canonical embedding of M into its ultrapower M^U. We claim that the two composites are equal,

$$h_1 \circ \delta_M = h_2 \circ \delta_M.$$

Indeed, let $A \in |T|$ and $\mathrm{a} \in M(A)$ be arbitrary. Then for $i = \{(A, a)\}$ we have $[i] \in U$; and, of course, for $j \in [i]$ we have that $h_1^j(a) = h_2^j(a)$. It follows that $(h_1 \circ \delta_M)(a) = \langle h_1^j(a) \rangle / U = \langle h_2^j(a) \rangle / U = (h_2 \circ \delta_M)(a)$, showing the assertion. We now take the X-image of the diagram (4.2), and consider the diagram

$$\begin{tikzcd}
 & X(M^U) \arrow[r, shift left, "X(h_1)"] \arrow[r, shift right, "X(h_2)"'] \arrow[d, "\wr"', "{[X,U]_{\langle M \rangle}}"] & X\left(\prod N_i/U\right) \arrow[d, "\wr"', "{[X,U]_{\langle N_i \rangle}}"] \\
X(M) \arrow[ur, "X(\delta_M)"] \arrow[dr, "\delta_{XM}"'] & & \\
 & (XM)^U \arrow[r, shift left, "\prod Xh_1^i/U"] \arrow[r, shift right, "\prod Xh_2^i/U"'] & \prod XN_i/U
\end{tikzcd}$$

Since X is an ultrafunctor, the left triangle commutes (see the end of Sect. 3). By the naturality of $[X, U]$, the square with h_1's and the square with h_2's commute. Since X is a functor, $X(h_1) \circ X(\delta_M) = X(h_1 \circ \delta_M) = X(h_2 \circ \delta_M) = X(h_2) \circ X(\delta_M)$. It follows that

$$\left(\prod Xh_1^i/U\right) \circ \delta_{XM} = \left(\prod Xh_2^i/U\right) \circ \delta_{XM}.$$

Apply the two sides of this equality to the element $x \in X(M)$. We obtain

$$\langle (Xh_1^i)(x)\rangle/U = \langle (Xh_2^i)(x)\rangle/U.$$

This contradicts the assumption that

$$(Xh_1^i)(x) \neq (Xh_2^i)(x) \qquad \text{for all} \quad i \in I.$$

This proves the lemma. ▮

We are going to prove that, in case (A, a) is a support for $x \in X(M)$, there is a partial A-cover of X *containing* (M, a, x), i.e., $\Phi\colon Y \to X$, $\Psi\colon Y \to e(A)$ (a mono) such that for some $y \in Y(M)$, we have $a = \Psi_M(y)$ and $x = \Phi_M(y)$. To facilitate the proof, we reformulate the notion of subobject in T'. We introduce a terminology which is not strictly necessary but which makes the formalism nicer.

Let * be a new entity called "blank;" it is distinct from all sets, hence it is not an element of any set. The category SET*, the category of pointed sets augmented by blank, is defined as follows. The objects of SET* are pairs (A, a) with A a set a either an element of A, or $a = *$. An object (A, a) with $a \in A$ is called *proper*, $(A, *)$ *improper*. A morphism

$$(A, a) \xrightarrow{f} (B, b)$$

is a function $A \to^{f} B$ such that $f(a) = b$ where we use the convention that $f(*) = *$. Composition and identity are defined in SET* as those in SET. In other words, we have the forgetful functor

$$\begin{tikzcd}
\mathrm{SET}^* \arrow[r, "F"] & \mathrm{SET} \\
(A, a) \arrow[r, mapsto] & A \\
f \arrow[r, mapsto] & f.
\end{tikzcd}$$

SET* is a pre-ultracategory via the following definitions. Define $\prod_{i \in I}(A_i, a_i)/U$ (with U an ultrafilter on I) to be $(\prod A_i/U, \langle a_i\rangle/U)$ with the understanding that

$$\langle a_i\rangle_{i \in I}/U = \langle a_i\rangle_{i \in P}/U$$

in case $P =_{\mathrm{df}} \{i \in I\colon a_i \in A\} \in U$, and

$$\langle a_i\rangle/U = *$$

otherwise. Clearly, we have "ultraproduct" functors

$$(\mathrm{SET}^*)^I \xrightarrow{[U]} \mathrm{SET}^*$$

such that the forgetful functor F: SET* $\to$ SET is a strict pre-ultrafunctor **SET*** $\to$ **SET**.

Let Y be any pre-ultrafunctor **Mod** $T \to$ **SET**. With $K = \text{Mod}\, T$, we define the category $K^* = K^*_Y$ as follows. Its objects are pairs (M, x) with $M \in |K|$, and $x \in Y(M)$ or $x = *$; the first kind is *proper*, the second *improper*. A morphism $(M, x) \to^h (N, y\rangle$ is a morphism $M \to^h N$ such that $(Yh)(x) = y$. With

$$\phi = [Y, U]_{\langle M_i \rangle}\colon\quad Y\Big(\prod M_i/U\Big) \xrightarrow{\cong} \prod YM_i/U$$

the transition isomorphism, we define

$$\prod (M_i, x_i)/U = \Big(\prod M_i/U, \varphi^{-1}(\langle x_i \rangle/U)\Big).$$

Let Y^*: $K^*_Y \to$ SET* be the functor

$$(M, x) \longmapsto (Y(M), x),$$

$$\begin{tikzcd}
(M, x) \arrow[d, "h"] & & (YM, x) \arrow[d, "Yh"'] \\
(N, y) & & (YN, y).
\arrow[from=1-1, to=2-3, phantom, "\longmapsto"]
\end{tikzcd}$$

We observe that Y^* is a pre-ultrafunctor, with essentially the same transition isomorphisms as Y.

We are ready to give our reformulation of the notion of subobject. Let Y be as above. Let Σ^* be a class of objects of $K^* = K^*_Y$ ($K = \text{Mod}\ T$); consider the following conditions (0)–(iii) on Σ^*:

(0) $\langle M, * \rangle \in \Sigma^*$ for all $M \in |K|$;

(i) whenever $p \in \Sigma^*$, $q \in |K^*|$, and there is a morphism $p \to q$ in K^*, then $q \in \Sigma^*$;

(ii) whenever U is an ultrafilter on I, $p_i \in |K^*|$ for $i \in I$, then $\prod p_i/U \in \Sigma^*$;

(iii) whenever (I, U) is an ultrafilter, $p_i \in |K^*|$ for all $i \in I$, and $\prod p_i/U \in \Sigma^*$, then there is $P \in U$ such that $p_i \in \Sigma^*$ for all $i \in P$.

Given any monomorphism μ: $\Sigma \to Y$, define μ^* to be the set

$$\{(M, x)\colon M \in |K|, x \in \text{Im}(\mu_M)\} \cup \{(M, *)\colon M \in |K|\}.$$

LEMMA 4.5. *The subobjects of Y are in one-to-one correspondence with classes Σ^* satisfying* (0)–(iii); *the class corresponding to the subobject determined by μ is μ^*.*

Proof. We leave it to the reader to verify that $\Sigma^* = \mu^*$ satisfies (0)–(iii), and that μ and μ' determine the same subobject of Y if and only if $\mu^* = (\mu')^*$ (the "if" part of this last statement is proved by an argument similar to the one at the end of the proof of 4.2(ii)).

Given any Σ^* satisfying (0)–(iii), define the object function of the functor Σ: $K \to \mathrm{SET}$ by setting

$$\Sigma(M) = \{x \in Y(M) \colon (M, x) \in \Sigma^*\}.$$

For being able to define Σ as a subfunctor of Y, i.e., such that

$$\mu\colon \quad \Sigma \to Y$$

with $\mu_M =$ inclusion: $\Sigma(M) \to Y(M)$ is natural, it is necessary and sufficient to have that for any h: $M \to N$, if $x \in \Sigma(M)$, then we have $(Yh)(x) \in \Sigma(N)$; and then $\Sigma(h) =$ the restriction of h to $\Sigma(M)$. The mentioned condition holds by (i) above.

We want to define transition isomorphisms $v' = [\Sigma, U]_{\langle M_i \rangle}$ such that

$$\begin{tikzcd}
Y\left(\prod M_i/U\right) \arrow[r, "\cong", "v = {[Y,U]_{\langle M_i \rangle}}"'] & \prod YM_i/U \\
\Sigma\left(\prod M_i/U\right) \arrow[u, "\mu_{\prod M_i/U}"] \arrow[r, "\cong", "v'"'] & \prod \Sigma M_i/U \arrow[u, "\prod \mu_{M_i}/U"']
\end{tikzcd}$$

commutes. For this, it is necessary and sufficient that v map the subset $\Sigma(\prod M_i/U)$ of $Y(\prod M_i/U)$ onto the subset $\mathrm{Im}(\prod \mu_{M_i}/U)$ of $\prod YM_i/U$. To show that v "maps into," we use condition (iii), and for v "mapping onto," condition (ii). Indeed, for the second proof, let $x = \langle x_i \rangle_{i \in Q}/U$ be any element of $\mathrm{Im}(\prod \mu_{M_i}/U)$; $Q \in U$. It follows that there is $P \subset Q$, $P \in U$ such that $x_i \in \Sigma(M_i)$ for $i \in P$. Define $p_i = (M_i, x_i)$ for $i \in P$ and $p_i = (M_i, *)$ for $i \in I - P$; then $p_i \in \Sigma^*$ for all $i \in I$ by also (0). We have $\prod p_i/U = (\prod M_i/U, v^{-1}(x))$, and $\prod p_i/U \in \Sigma^*$ by (ii). It follows that $v^{-1}(x) \in \Sigma(\prod M_i/U)$ as desired. It is easy to check that $\mu^* = \Sigma^*$.

We have defined Σ as a pre-ultrafunctor and μ: $\Sigma \to Y$ as an ultratransformation; clearly, μ is a monomorphism. Recall (3.1) that if Y is an ultrafunctor, then Σ is automatically an ultrafunctor, hence we have a subobject of Y in T'. ▌

When considering subobjects of an object Y of T' of the form $Y = Y_1 \times Y_2$, the elements of K^*_Y will look like (M, x_1, x_2) with either both x_1 and x_2 being $*$, or else $x_1 \in Y_1(M)$, $x_2 \in Y_2(M)$.

Let $A \in |T|$. A partial A-cover of X (an object of T') can be construed as a subobject of $Y = e(A) \times X$ with the additional property of being univalent. When translated into the language just introduced, a partial A-cover of X becomes the same as a family Σ^* of objects of K^*_Y with the additional property

(iv) (M, a, x), (M, a, x') are proper points in Σ^* implies that $x = x'$.

Explicitly: given $\Sigma^* \subset (\text{Mod } T)^*_Y$ satisfying (0)–(iv), we have a monomorphism $\Sigma \rightarrow^{\mu} e(A) \times X$ such that $\mu^* = \Sigma^*$ by 4.5. With the canonical projections $\pi_1 : e(A) \times X \rightarrow e(A)$, $\pi_2 : e(A) \times X \rightarrow X$, and with $\Psi = \pi_1 \circ \mu$, $\Phi = \pi_2 \circ \mu$ now it is clear that (Φ, Ψ) is a partial A-cover of X. Let $M_0 \in |\text{Mod } T|$, $a_0 \in M_0(A)$ and $x_0 \in X(M_0)$. With (Φ, Ψ) just constructed, to say that there is $\sigma \in M_0(\Sigma)$ such that $a_0 = \Psi_{M_0}(\sigma)$ and $x_0 = \Phi_{M_0}(\sigma)$, is equivalent to saying that $(M_0, a_0, x_0) \in \Sigma^*$.

Our task restated now looks as follows. We have a "point" $p_0 = (M_0, a_0, x_0)$ in $K^* = K^*_Y$ $(Y = e(A) \times X)$ such that (A, a_0) is a support of x_0; we want to show the existence of $\Sigma^* \subset |K^*|$ satisfying (i)–(iv) and also

(v) $p_0 \in \Sigma^*$.

The fact that $|K^*|$ is a proper class makes it difficult to construct Σ^* directly. Therefore, changing notation we let K^* be an arbitrary small subcategory of what K^* was so far, and we let J^* be a set of triples $\langle I, U, g^* \rangle$ with U an ultrafilter on I, and g^* a function from I to $|K^*|$ such that $\prod g^*(i)/U \in |K^*|$. We write $\mathbf{K}^*$ for $(K^*, \mathbf{J}^*)$, and call $\mathbf{K}^*$ a *small approximation* of $(\text{Mod } T)^* = (\text{Mod } T)^*_Y$. We further make the (trivial) assumptions that $\mathbf{J}^*$ contains all triples of the form $\langle \{0\}, U_0, g^* \rangle$, with U_0 the trivial ultrafilter, g^* any function $\{0\} \rightarrow |K^*|$, and also that if $\langle I, Y, g^* \rangle \in \mathbf{J}^*$ and g^{**}: $I \rightarrow |K^*|$ is such that $g^{**}(i) = (M_i, a'_i, x'_i)$ for some a'_i, x'_i where $g^*(i) = (M_i, a_i, x_i)$, for all $i \in I$, then $\langle I, U, g^{**} \rangle \in \mathbf{J}^*$.

A $\mathbf{K}^*$-*subobject* of Y $(= e(A) \times X)$ is a family $\Sigma^* \subset |K^*|$ satisfying (0)–(iii), with (ii) and (iii) restricted to the case when $\langle I, U, \langle p_i : i \in I \rangle \rangle$ belongs to $\mathbf{J}^*$.

A *partial A-cover of X relative to* $\mathbf{K}^*$ is a $\mathbf{K}^*$-subobject Σ^* of Y satisfying (iv). The main argument of the paper is the proof of

LEMMA 4.6. *Let* $\mathbf{K}^*$ *be as described,* $p_0 = (M_0, a_0, x_0) \in |K^*|$, *and assume that* (A, a_0) *is a support for* x_0. *Then there is a partial A-cover* Σ^* *of* X *relative to* $\mathbf{K}^*$ *such that* $p_0 \in \Sigma^*$.

Proof. We first give an outline of the proof. We try to build the set Σ^*

by successively throwing in more and more points from K^*. We start by throwing in p_0. At any stage, conditions (i) and (ii) are to be honored by simply throwing in necessary points. Satisfying (iii), however, requires a choice of a set $P \in U$. Repeating these steps transfinitely often, including choices "$P \in U$," we end up with a family Σ^* satisfying (0)–(iii) and (v). We might have failed to obain a partial A-cover by having violated (iv). We now make the assumption that, indeed, at all possible series of choices "$P \in U$" we fail. It turns out that this assumption gives rise to two ultramorphisms. Finally, a contradiction arises from the fact that X "preserves" these ultramorphisms.

(1) We start by constructing a specific ultragraph Γ and an ultradiagram $M^*\colon \Gamma \to K^*$ codifying, in a sense, the process of building Σ^* of the outline (although K^* is a pre-ultracategory only partially, namely for ultraproducts given in $\mathbf{J}^*$, $\mathscr{M}^*$ being an u.d. will make sense since we will not refer to ultraproducts outside $\mathbf{J}^*$). We let $\kappa = \text{card}(|K^*|)$, and $\alpha_0 = \kappa^+$. Γ will be specified by defining Γ^{f}, Γ^{b}, $|\Gamma| = \Gamma^{\text{f}} \cup \Gamma^{\text{b}}$, and edges between elements of $|\Gamma|$ by induction on $\alpha < \alpha_0$; in particular, we will define disjoint sets of nodes Γ^{f}_α, Γ^{b}_α and $\Gamma_\alpha = \Gamma^{\text{f}}_\alpha \cup \Gamma^{\text{b}}_\alpha$ for $\alpha < \alpha_0$, and we'll put $\Gamma^{\text{f}} = \bigcup_{\alpha < \alpha_0} \Gamma^{\text{f}}_\alpha$, $\Gamma^{\text{b}} = \bigcup_{\alpha < \alpha_0} \Gamma^{\text{b}}_\alpha$.

With every improper point q of K^*, we associate a free node ϕ_q. With another, distinguished, free node ϕ_0, we put $\Gamma^{\text{f}}_0 = \{\phi_q\colon q \text{ improper in } K^*\} \cup \{\phi_0\}$, $\Gamma^{\text{b}}_0 = \varnothing$; no edges are defined between nodes in $\Gamma_0 = \Gamma^{\text{f}}_0$; put also $\Theta_0 = \varnothing$. The definition of the diagram $\mathscr{M}^*$ on Γ_0 is

$$\mathscr{M}^*(\phi_q) = q, \qquad \mathscr{M}^*(\phi_0) = p_0.$$

Let $\alpha > 0$, $\alpha < \alpha_0$. Assuming that we have made the specifications for ordinals less than α, we let $\Gamma^{\text{f}}_{<\alpha} = \bigcup_{\alpha' < \alpha} \Gamma^{\text{f}}_{\alpha'}$, $\Gamma^{\text{b}}_{<\alpha} = \bigcup_{\alpha' < \alpha} \Gamma^{\text{b}}_{\alpha'}$, $\Theta_{<\alpha} = \bigcup_{\alpha' < \alpha} \Theta_{\alpha'}$. Below, we associate with α an index-set Θ_α, disjoint from $\Theta_{<\alpha}$; we associate with every $t \in \Theta_\alpha$ two ultrafilters (I_t, U_t) and (J_t, V_t), with distinct corresponding bound nodes β_t and γ_t, distinct for distinct t; an edge $e_t\colon \beta_t \to \gamma_t$; and a function $g_t\colon I_t \to \Gamma^{\text{f}}_{<\alpha}$. Once these items are fixed, for all $\alpha < \alpha_0$, the ultradiagram Γ is made up as follows: $\Gamma^{\text{f}}_\alpha = \coprod_{t \in \Theta_\alpha} J_t$ (disjoint sum), $\Gamma^{\text{b}}_\alpha = \{\beta_t\colon\ t \in \Theta_\alpha\} \cup \{\gamma_t\colon\ t \in \Theta_\alpha\}$; $\Gamma^{\text{f}} = \Gamma^{\text{f}}_{<\alpha_0}$, $\Gamma^{\text{b}} = \Gamma^{\text{b}}_{<\alpha_0}$; the set $E(\gamma, \gamma')$ of edges from γ to γ' is empty unless $\gamma = \beta_t$, $\gamma' = \gamma_t$ for some $t \in \Theta_{<\alpha_0}$ in which case $E(\gamma, \gamma') = \{e_t\}$; for the bound node $\beta = \beta_t$, $\langle I_\beta, U_\beta, g_\beta \rangle$ is $\langle I_t, U_t, g_t \rangle$; for the bound node $\beta = \gamma_t$, $\langle I_\beta, U_\beta, g_\beta \rangle$ is $\langle J_t, V_t, i_t \rangle$ with i_t the canonical injection $J_t \to \Gamma^{\text{f}}_\alpha = \coprod_{t' \in \Theta_\alpha} J_{t'}$.

The specifications on the level α (>0) are given as follows. We let Θ_α be the set of all systems

$$t = \langle \alpha; I, U, g; f; J, V, g' \rangle$$

such that: g is a function from I to $\Gamma^{\mathrm{f}}_{<\alpha}$; with $g^* = \langle \mathcal{M}^*(g(i))\colon i \in I\rangle$ (the function $\mathcal{M}^* \circ g$ into $|K^*|$, defined by induction assumption) we have that both $\langle I, U, g^*\rangle$ and (J, V, g') belong to $\mathbf{J}^*$; $\prod g^*(i)/U$ is a proper point of K^*; f is a morphism $f\colon p_1 \to p_2$, with $p_1 = \prod g^*(i)/U$ and $p_2 = \prod g'(j)/V$. For $t \in \Theta_\alpha$ displayed, we put $I_t = I$, $U_t = U$, $g_t = g$, $J_t = J$, $V_t = V$, $\mathcal{M}^*(\beta_t) = p_1$, $\mathcal{M}^*(\gamma_t) = p_2$, $\mathcal{M}^*(e_t) = f$ and $\mathcal{M}^*(i_t(j)) = g'(j)$ for $j \in J$.

This completes the definition of Γ and the diagram $\mathcal{M}^*\colon \Gamma \to K^*$. Clearly, $\mathcal{M}^*$ is an "ultradiagram" $\mathcal{M}^*\colon \Gamma \to \mathbf{K}^*$. By composing $\mathcal{M}^*$ with the forgetful functor $F\colon K^* \to \mathrm{Mod}\, T$ (see above), we obtain the ultradiagram $\mathcal{M}\colon \Gamma \to \mathbf{K}$.

Note the following state of affairs. The nodes ϕ_q (q improper in K^*) are "dummy" nodes in the sense that for an ultradiagram $\mathcal{A}^*\colon \Gamma \to \mathrm{SET}^*$, one can replace each $\mathcal{A}^*(\phi_q) = (A', a)$ by $(A', *)$ without disturbing $\mathcal{A}^*$ being an ultradiagram. This is a consequence of the clause "$\prod g(i)/U$ is a proper point of K^*" in the definition of Θ_α above; as a consequence of this clause, $g_\beta^{-1}(\Gamma_0 - \{\varphi_0\})$ is always a "small" set, i.e., not belonging to U_β, for any $\beta \in \Gamma^{\mathrm{b}}$.

(2) We now formalize the idea of a "possible series of choices $P \in V$" (V replacing U, for notational reasons) of the outline.

Let Θ be any subset of $\Theta_{<\alpha_0}$, and let $P = \langle P_t\colon t \in \Theta\rangle$ be an indexed family with $P_t \in V_t$. We define when an index $t \in \Theta_{<\alpha_0}$ and a node $\gamma \in \Gamma_{<\alpha_0}$ are *accessible* (*from* $\vec{P}$), by induction. $\phi_0 \in \Gamma_0$ is accessible, but each ϕ_q (q improper in K^*) is not. Having defined when $\gamma \in \Gamma_{<\alpha}$ and $t \in \Theta_{<\alpha}$ are accessible, we declare that $t \in \Theta_\alpha$ is accessible iff $\{i \in I_t\colon g_t(i) \text{ is accessible}\}$ belongs to U_t. β_t and γ_t are accessible iff t is. Finally, $\gamma = i_t(j) \in \Gamma^{\mathrm{f}}_\alpha$ ($j \in J_t$) is accessible iff t is accessible, $t \in \Theta$, and $j \in P_t$. Note that for $t \in \Theta_\alpha$, t, β_t and γ_t being accessible from $\vec{P}$ depends only on the initial segment $\vec{P}_{<\alpha} = \langle P_{t'}\colon t' \in \Theta \cap \Theta_{<\alpha}\rangle$.

The vector $\vec{P}$ is *regular* iff for all $t \in \Theta_{<\alpha_0}$, t belongs to Θ iff t is accessible from $\vec{P}$. By the preceding remark, "many regular vectors" can be constructed by induction on α, by honoring the regularity requirement on level α by choosing $\Theta \cap \Theta_\alpha$ depending on $\vec{P}_{<\alpha}$, and then choosing $P_t \in V_t$ for $t \in \Theta \cap \Theta_\alpha$ "arbitrarily." $\Gamma(\vec{P})$ denotes the set of nodes accessible from $\vec{P}$.

The set of all regular vectors is denoted by $\mathcal{P}$. $\mathcal{P}$ is a partially ordered set by the following relation $\leqslant$: $\vec{P} \leqslant \vec{Q}$ iff $\mathrm{dom}(\vec{P}) \subseteq \mathrm{dom}(\vec{Q})$ and for all $t \in \mathrm{dom}(\vec{P})$, $P_t \subseteq Q_t$. In fact, $\mathcal{P}$ with $\leqslant$ is a lower semilattice. Given $\vec{P} = \langle P_t\colon t \in \Theta\rangle$ and $\vec{P}' = \langle P'_t\colon t \in \Theta'\rangle$ in $\mathcal{P}$, we define $\vec{P}'' = \langle P''_t\colon t \in \Theta''\rangle$ as follows. Whenever $t \in \Theta''$, we put $P''_t = P_t \cap P'_t$. Having defined when $t \in \Theta_{<\alpha}$ belongs to Θ'', we declare that $t \in \Theta_\alpha$ belongs to Θ'' exactly if it is accessible from $\langle P''_t\colon t \in \Theta'' \cap \Theta_{<\alpha}\rangle$. It is easy to check that $\vec{P}''$ so defined is the greatest lower bound for $\vec{P}$ and $\vec{P}'$. Also note that $\vec{P} \leqslant \vec{Q}$ implies $\Gamma(\vec{P}) \subseteq \Gamma(\vec{Q})$.

(3) *Until the end of the proof we now make the assumption that the conclusion of the lemma fails.* We *claim* that this implies that for any $\vec{P} \in \mathscr{P}$, there are $\gamma_1, \gamma_2 \in \Gamma^{\mathrm{f}} \cap \Gamma(\vec{P})$ such that $\mathscr{M}^*(\gamma_i) = \langle M_i, a_i, x_i \rangle$ are proper $(i = 1, 2)$ and we have $M_1 = M_2$, $a_1 = a_2$, but $x_1 \neq x_2$.

Indeed, let $\vec{P} \in \mathscr{P}$. Consider the set $\Sigma^* =_{\mathrm{df}} \{q \in |K^*|: q \text{ is improper}\} \cup \{\mathscr{M}^*(\gamma): \gamma \in \Gamma(\vec{P})\}$. We claim that Σ^* satisfies conditions (0)–(iii) and (v). The proof is easy; the details are as follows. (0) is clear. (v) holds since $\phi_0 \in \Gamma(\vec{P})$. Before going on to the rest of the conditions, consider an arbitrary $p \in \Sigma^*$. We have that $p = \mathscr{M}^*(\gamma)$ for some $\gamma \in \Gamma^{\mathrm{f}} \cap \Gamma(\vec{P})$, or else p is improper (in which case $p = \mathscr{M}^*(\phi_p)$). Namely, if $p \in \Sigma^*$ is proper, and $p = \mathscr{M}^*(\beta_t)$ for some $\alpha < \alpha_0$ and $t \in \Theta_\alpha$ accessible from $\vec{P}$, then $\{i \in I_t: g_t(i)$ is accessible from $\vec{P}\} \in U_t$; hence for $t' = \langle \alpha; I_t, U_t, g_t; \mathrm{Id}_p; \{0\}, U_0, g' \rangle$ with U_0 the trivial ultrafilter on $\{0\}$, $g': \{0\} \to |\mathbf{K}^*|$ and $g'(0) = p$ we have $t' \in \Theta_\alpha$ (recall that $\langle \{0\}, U_0, g' \rangle \in \mathbf{J}^*$), and by the definition of accessibility again, t' is accessible from $\vec{P}$ (since $(I_{t'}, U_{t'}, g_{t'}) = (I_t, U_t, g_t)$); and clearly, for $\gamma = i_{t'}(0)$, $\gamma \in \Gamma^{\mathrm{f}}$ and $p = \mathscr{M}^*(\gamma)$. The case $p = \mathscr{M}^*(\gamma_t)$ is similarly handled.

To see (i), let $p = \mathscr{M}^*(\gamma)$ with $\gamma \in \Gamma^{\mathrm{f}} \cap \Gamma(\vec{P})$, and $f: p \to q$ in K^*; let $\alpha < \alpha_0$ be such that $\gamma \in \Gamma^{\mathrm{f}}_{<\alpha}$. Look at the construction of Γ and $\mathscr{M}^*$ at level α, and put $t = \langle \alpha; \{0\}, U_0, g; f; \{0\}, U_0, g' \rangle$ with $g: \{0\} \to \Gamma^{\mathrm{f}}_{<\alpha}$, $g(0) = \gamma$ (by assumption, $\langle \{0\}, U_0, g^* \rangle \in \mathbf{J}^*$), $g' = g^*$. We have $t \in \Theta_\alpha$. Since γ is accessible from $\vec{P}$, so is t; hence, so are β_t and α_t. Of course, $\mathscr{M}^*(\beta_t) = \mathscr{M}^*(\gamma) = p$, and $\mathscr{M}^*(\gamma_t) = q$. It follows that $q \in \Sigma^*$.

To see (ii), let $p_i \in \Sigma^*$ for $i \in I$, and $\langle I, U, \langle p_i: i \in I \rangle \rangle \in \mathbf{J}^*$. If the set $Q = \{i \in I: p_i \text{ is proper}\}\}$ is not in U, the assertion is clear. Assume $Q \in U$. For $i \in Q$, $p_i = \mathscr{M}^*(\gamma_i)$ for some $\alpha_i < \alpha_0$ and $\gamma_i \in \Gamma^{\mathrm{f}}_{\alpha_i} \cap \Gamma(\vec{P})$ such that, in addition, $p_i = p_j$ implies $\gamma_i = \gamma_\gamma$ $\langle i, j \in I \rangle$. Since card $\{\alpha_i: i \in I\} \leqslant \mathrm{card}\{p_i: i \in I\} \leqslant \kappa$, it follows that there is $\alpha < \alpha_0 = \kappa^+$ such that $\alpha_i < \alpha$ for all $i \in I$. Let $p_i^* = p_i$ for $i \in Q$, $p_i^* = (M_i, *, *)$ if $p_i = (M_i, a_i, x_i)$ for $i \in I - Q$. Then clearly, $\prod p_i^*/U = \prod p_i/U \,_{\mathrm{df}}\!= p$. Let

$$t = \langle \alpha; I, U, g; \mathrm{Id}_p; \{0\}, U_0, g' \rangle \in \Theta_\alpha$$

with $g(i) = \gamma_i$ for $i \in Q$, $g(i) = \phi^*_{p_i}$ for $i \in I - Q$, $g'(0) = p$ (note that $\langle I, U, g^* \rangle \in J^*$, by an assumption on $\mathbf{J}^*$). Since each γ_i $(i \in Q)$ is accessible from $\vec{P}$, so are t and β_t. We have $p = \mathscr{M}^*(\beta_t) \in \Sigma^*$ as desired.

To see (iii), assume $p_i \in |K^*|$ $(i \in I)$, $p = \prod p_i/U \in \Sigma^*$, $\langle I, U, \langle p_i: i \in I \rangle \rangle \in \mathbf{J}^*$. If p is improper, the required conclusion is clear. Let p be proper. Then $p = \mathscr{M}^*(\gamma)$ for some $\alpha < \alpha_0$ and $\gamma \in \Gamma^{\mathrm{f}}_{<\alpha} \cap \Gamma(\vec{P})$. Consider

$$t = \langle \alpha; \{0\}, U_0, g; \mathrm{Id}_p; I, U. \langle p_i: i \in I \rangle \rangle \in \Theta_\alpha$$

with $g(0) = \gamma$. Since γ is accessible from $\vec{P}$, so is t. Since $\vec{P} = \langle P_{t'}: t' \in \Theta \rangle$ is regular, $t \in \Theta$. Also, each $\gamma_j = i_t(j)$ with $j \in P_t$ is accessible from $\vec{P}$, and $\mathscr{M}^*(\gamma_j) = p_j$. It follows that for all $j \in P_t \in U$, we have $p_j \in \Sigma^*$, as desired.

We have shown that, indeed, Σ^* satisfies (0)–(iii) and (v). It cannot satisfy (iv) too since otherwise, we would have that, after all, the conclusion of the lemma held. By also using the remark preceding the verification of (i)–(iii), this proves our first-stated claim.

(4) Preliminary to defining the ultramorphisms we will need, we prove

(*) *Given any ultradiagram* $\mathscr{A}: \Gamma \to \mathrm{SET}$, *and an element* $a \in \mathscr{A}(\phi_0)$, *there is a lifting* $\mathscr{A}^* = \mathscr{A}^*_a$ *of* $\mathscr{A}$ *such that* $\mathscr{A}^* = \Gamma \to \mathrm{SET}^*$ *is an ultradiagram and the diagram*

$$\begin{tikzcd}
\Gamma \arrow[r, "A^*"] \arrow[dr, "\mathscr{A}"'] & \mathrm{SET}^* \arrow[d, "\text{forgetful}"] \\
& \mathrm{SET}
\end{tikzcd}$$

commutes.

The *proof* is straightforward; the assertion is a consequence of the "recursive character" of the ultragraph Γ. The definition of $\mathscr{A}^*(\gamma)$ for $\gamma \in \Gamma_\alpha$ is given by induction on α. $\mathscr{A}^*(\phi_0) =_{\mathrm{df}} (\mathscr{A}(\phi_0), a)$, $\mathscr{A}^*(\phi_q) =_{\mathrm{df}} (\mathscr{A}(\phi_q), *)$ (q improper in K^*). Suppose $0 < \alpha < \alpha_0$ and $\mathscr{A}^*(\gamma)$ has been defined for $\gamma \in \Gamma_{<\alpha}$. For $t \in \Theta_\alpha$, define $\mathscr{A}^*(\beta_t) = \prod \mathscr{A}^*(g_t(i)/U_t$. Let $a \in \mathscr{A}(\gamma_t)$ or $a = *$ be the uniquely determined element for which $\mathscr{A}(e_t)$ is a morphism from $\mathscr{A}^*(\beta_t)$ to $(\mathscr{A}(\gamma_t), a)$; put $\mathscr{A}^*(\gamma_t) = (\mathscr{A}(\gamma_t), a)$ and of course, $\mathscr{A}^*(e_t) = \mathscr{A}(e_t)$. Choose (arbitrarily) a set $P \in V_t$ and a vector $\langle a_j : j \in P \rangle$ such that $a = \langle a_j : j \in P \rangle / U$, and put $\mathscr{A}^*(i_t(j)) = (\mathscr{A}(i_t(j)), a_j)$ for $j \in P$, $A^*(i_t(j)) = (\mathscr{A}(i_t(j)), *)$ for $j \in J_t - P$. This completes the inductive definition of $\mathscr{A}^*$; the required properties clearly hold.

Now, consider the set $\mathscr{P}$ of all regular vectors. Given $\vec{P} \in \mathscr{P}$, let $[\vec{P}] = \{\vec{Q} \in \mathscr{P} : \vec{Q} \leqslant \vec{P}\}$. Since $\mathscr{P}$ is a lower semilattice, $[\vec{P}_1] \cap [\vec{P}_2] = [\vec{P}_1 \cap \vec{P}_2]$. It follows that there is an ultrafilter W on $\mathscr{P}$ such that $[\vec{P}] \in W$ for all $\vec{P} \in \mathscr{P}$.

Using part (3), let $\vec{P} \mapsto \gamma_\varepsilon(\vec{P})$ ($\varepsilon = 1, 2$) be two assignments of nodes of Γ to regular vectors such that we have

$$\mathscr{M}^*(\gamma_\varepsilon(\vec{P})) = (M_{\vec{P}}, a_{\vec{P}}, x_{\varepsilon,\vec{P}}) = p_\varepsilon(\vec{P}) \qquad \text{for} \quad \varepsilon = 1, 2;\ x_{1,\vec{P}} \neq x_{2,\vec{P}};$$

also, $p_\varepsilon(\vec{P})$ is proper and $\gamma_\varepsilon(\vec{P}) \in \Gamma^{\mathrm{f}} \cap \Gamma(\vec{P})$.

We define two ultragraphs Γ_ε ($\varepsilon = 1, 2$) as follows. "$\Gamma_\varepsilon = \Gamma \cup \{l\}$"; more precisely, Γ is a sub-ultragraph of Γ_ε; Γ_ε has one additional bound node l

(no other additional nodes or edges), $I_l = \mathscr{P}$, $U_l = W$ and $g_l = \langle \gamma_\varepsilon(\vec{P})\colon \vec{P} \in \mathscr{P} \rangle$.

We define, for $\varepsilon = 1, 2$, an ultramorphism $\delta_\varepsilon \in \Delta(\mathrm{SET})$ of type $(\Gamma_\varepsilon, \phi_0, l)$ as follows. Let $\mathscr{A}^+\colon \Gamma_\varepsilon \to \mathrm{SET}$ be an ultradiagram, $\mathscr{A}\colon \Gamma \to \mathrm{SET}$ its restriction to Γ; we have $\mathscr{A}^+(l) = \prod_{P \in \mathscr{P}} \mathscr{A}(\gamma_\varepsilon(\vec{P}))/W$. $\mathscr{A}$ uniquely determines $\mathscr{A}^+$; we write $(\delta_\varepsilon)_{\mathscr{A}}$ for $(\delta_\varepsilon)_{\mathscr{A}^+}$. To define $(\delta_\varepsilon)_{\mathscr{A}}\colon \mathscr{A}(\phi_0) \to \mathscr{A}^+(l)$, let $a \in \mathscr{A}(\phi_0)$. Choose a lifting $\mathscr{A}^* = \mathscr{A}^*_a$ of $\mathscr{A}$ as in (*) above. Let $(A(\vec{P}), a_\varepsilon(\vec{P}))$ be $\mathscr{A}^*(\gamma_\varepsilon(\vec{P}))$. Define

$$(\delta_\varepsilon)_{\mathscr{A}}(a) = \langle a_\varepsilon(\vec{P})\colon \vec{P} \in \mathscr{P} \rangle / W.$$

It is not hard to verify that $(\delta_\varepsilon)_{\mathscr{A}}(a)$ is indeed in $\mathscr{A}^+(l)$ (it is not *), that it does not depend on the choice of $\mathscr{A}^*$ and that, in fact, $\delta_\varepsilon = \langle (\delta_\varepsilon)_{\mathscr{A}^+} : \mathscr{A}^+$ is an ultradiagram from Γ_ε to SET$\rangle$ so defined is an ultramorphism in SET. We postpone the verification of these facts to the next section where we give a more detailed analysis of these ultramorphisms.

(5) We now complete the proof of the lemma. Let us write $\bar{\delta}_\varepsilon$ for the induced ultramorphism $(\delta_\varepsilon)_{\mathbf{Mod}\,T}$. The u.d. $\mathscr{M}\colon \Gamma \to \mathbf{K}$ defined in part (1) gives rise to the u.d.'s $\mathscr{M}_\varepsilon\colon \Gamma_\varepsilon \to \mathbf{K}$ extending $\mathscr{M}$; $\mathscr{M}_1(l) = \mathscr{M}_2(l) = \prod_{\vec{P} \in \mathscr{P}} M_{\vec{P}}/W =_{\mathrm{df}} M_1$ (for $M_{\vec{P}}$, see above). We have $\bar{\bar{\delta}}_\varepsilon = (\bar{\delta}_\varepsilon)_{\mathscr{M}_\varepsilon}\colon M_0 \to M_1$ defined such that $(\bar{\bar{\delta}}_\varepsilon)_B\colon M_0(B) \to M_1(B)$ is

$$(\bar{\bar{\delta}}_\varepsilon)_B = (\delta_\varepsilon)_{(B)\circ\mathscr{M}_\varepsilon} = (\delta_\varepsilon)_{(B)\circ\mathscr{M}}$$

for all $B \in |T|$ (with $\mathscr{A} = (B) \circ \mathscr{M}$, we have $\mathscr{A}^+ = (B) \circ \mathscr{M}_\varepsilon$).

We now contemplate the pair of morphisms

$$M_0 \overset{\bar{\bar{\delta}}_1}{\underset{\bar{\bar{\delta}}_2}{\rightrightarrows}} M_1 \tag{4.3}$$

and their X-images

$$X(M_0) \overset{X(\bar{\bar{\delta}}_1)}{\underset{X(\bar{\bar{\delta}}_2)}{\rightrightarrows}} X(M_1). \tag{4.4}$$

First, we have that

$$a_0 \xmapsto{(\bar{\bar{\delta}}_\varepsilon)_A} \langle a_{\vec{P}}\colon \vec{P} \in \mathscr{P} \rangle / W;$$

with the $a_{\vec{P}}$ introduced above; in particular, that

$$(\bar{\bar{\delta}}_1)_A(a_0) = (\bar{\bar{\delta}}_2)_A(a_0). \tag{4.5}$$

Namely, denoting by π_1 the "projection" functor

$$\pi_1\colon\quad K^* \longrightarrow \mathrm{SET}^*$$
$$(M, a, x) \longmapsto (M(A), a),$$

we clearly have that $\mathscr{A}^* =_{\mathrm{df}} \pi_1 \circ \mathscr{M}^*\colon \Gamma \to \mathrm{SET}^*$ is an ultradiagram (since π_1 is a strict pre-ultrafunctor); also $\mathscr{A}^*(\phi_0) = (M_0(A), a_0)$ and that

$$\begin{tikzcd}
\Gamma \arrow[r, "\mathscr{A}^*"] \arrow[dr, "\mathscr{A} = (A) \circ \mathscr{M}"'] & \mathrm{SET}^* \arrow[d, "\text{forgetful}"] \\
& \mathrm{SET}
\end{tikzcd}$$

commutes, i.e., $\mathscr{A}^*$ is a "lifting" of $\mathscr{A}$ "at a_0." It follows, by the definition of δ_ε and that of $\mathscr{A}^*$, that we have

$$(\bar{\bar{\delta}}_\varepsilon)_A(a_0) = (\delta_\varepsilon)_{\mathscr{A}}(a_0) = \langle a_{\vec{P}}\colon \vec{P} \in \mathscr{P} \rangle / W \qquad \text{as claimed.}$$

Next, we look at (4.4) and by following through the definitions, we conclude that

$$(X\bar{\bar{\delta}}_1)(x_0) \neq (X\bar{\bar{\delta}}_2)(x_0) \tag{4.6}$$

as follows. Let π_2: (Mod T)$^* \to \mathrm{SET}^*$ be the "second projection" functor defined by

$$\pi_2\colon \begin{cases} (M, a, x) \mapsto (X(M), x) & \\ \quad f \quad \mapsto X(f) & (f\colon p \to q \text{ in } K^*). \end{cases}$$

π_2 is a pre-ultrafunctor, with transition isomorphisms $[\pi_2, U]$ defined by the formula

$$[\pi_2, U]_{\langle p_i : i \in I \rangle} = [X, U]_{\langle M_i : i \in I \rangle}$$

for $p_i = \langle M_i, a_i, x_i \rangle$ $(i \in I)$. Let us write $\mathscr{M}^*$ for the composite $\Gamma \to^{\mathscr{M}^*} K^* \to^{\text{inclusion}} (\mathrm{Mod}\ T)^*$ as well. Then we have the ultradiagram $\pi_2 \mathscr{M}^*\colon \Gamma \to (\mathrm{Mod}\ T)^*$ and an isomorphism

$$\pi_2 \circ \mathscr{M} \xrightarrow[\cong]{} \pi_2 \mathscr{M}^*$$

defined in Section 3. On the other hand, we have the similarly constructed isomorphism

$$X \circ \mathscr{M} \xrightarrow[\cong]{} X\mathscr{M}.$$

It is easy to see that the diagram

$$\begin{tikzcd}
\Gamma \arrow[r, "\pi_2 \mathscr{M}^*"] \arrow[dr, "X\mathscr{M}"'] & \mathrm{SET}^* \arrow[d, "\text{forgetful}"] \\
& \mathrm{SET}
\end{tikzcd}$$

commutes. Also, $(\pi_2\mathscr{M}^*)(\phi_0)=(XM_0, x_0)$. By the definition of δ_ε, we have that $(\delta_\varepsilon)_{X.\mathscr{M}}(X_0)=\langle y_{\varepsilon,\vec{P}}\rangle/W$, where $y_{\varepsilon,\vec{P}}$ is the second component of $(\pi_2\mathscr{M}^*)(\gamma_\varepsilon(\vec{P}))$. But $\gamma=\gamma_\varepsilon(\vec{P})\in\Gamma^{\mathrm{f}}$, hence $(\pi_2\mathscr{M}^*)(\gamma)=(\pi_2\circ\mathscr{M}^*)(\gamma)=\pi_2(\mathscr{M}^*(\gamma))=(M_{\vec{P}}, x_{\varepsilon,\vec{P}})$. We conclude that $(\delta_\varepsilon)_{X.\mathscr{M}}(x_0)=\langle x_{\varepsilon,\vec{P}}\rangle/W$.

Now we use the fact that X, being an ultrafunctor, carries $(\delta_\varepsilon)_{\mathbf{K}}$ into δ_ε ($\mathbf{K}=\mathbf{Mod}\ T$). This implies that the diagram

$$\begin{tikzcd}
X(M_0) \arrow[r, "X(\bar{\bar{\delta}}_\varepsilon)"] \arrow[rd, "(\delta_\varepsilon)_{X.\mathscr{M}}"'] & X(M_1) \arrow[d, "{[X,W]_{\langle M_{\vec{P}}\rangle} \mathrm{df}= \nu}"] \arrow[ld, phantom, "\circlearrowleft"] \\
 & \prod X(M_{\vec{P}})/W
\end{tikzcd}$$

commutes. Therefore, $X(\bar{\bar{\delta}}_\varepsilon)(x_0)=\nu^{-1}(\langle x_{\varepsilon,\vec{P}}\rangle/W)$.

Recall that we have $x_{1,\vec{P}}\neq x_{2,\vec{P}}$ for all $\vec{P}\in\mathscr{P}$. It follows that (4.6) indeed holds.

(4.5) and (4.6) together contradict the assumption that (A, a_0) is a support for x_0. This contradiction shows that our assumption at the beginning of (3) is untenable, i.e., that the lemma is true. ▌

The Keisler-Shelah isomorphism theorem says that if M and N are elementarily equivalent structures ($M\equiv N$), then there is an ultrafilter (I, U) such that $M^U\cong N^U$ (see [CK]).

Let Y be an object of $T'=\mathrm{Hom}(\mathbf{Mod}\ T, \mathbf{SET})$. Let K be a small full subcategory of Mod T, and let $\mathbf{J}$ be a set of triples (I, U, g) such that U is an ultrafilter on I, g: $I\to|K|$. Assume that whenever M, N are in K, and $M\equiv N$, then there is (I, U) such that $M^U\cong N^U$, and for the constant functions g_1, g_2 with domain I with respective values M and N, we have $(I, U, g_i)\in\mathbf{J}$ $(i=1, 2)$. Let κ be the cardinal card $(\amalg\{\mathrm{Hom}(A_1, A_2)$: $A_1, A_2\in|T|\})+\aleph_0$, and assume furthermore that for all $M\in|\mathrm{Mod}\ T|$ with card $M=\mathrm{card}(\amalg\{M(A)$: $A\in|T|\})\leqslant\kappa$ there is $M'\in|K|$ such that $M'\cong M$. Let K^* be the full subcategory of (Mod $T)^*_Y$ whose objects are all $(M, y)\in$ (Mod $T)^*_Y$ with $M\in|K|$. Let J^* be the family of all (I, U, g^*) with g^*: $I\to|K^*|$ such that for g: $I\to|K|$ with $g(i)=$ the first component of $g^*(i)$, we have $(I, U, g)\in\mathbf{J}$. Any $\mathbf{K}^*=(K^*, \mathbf{J}^*)$ obtained this way is called a *small closed approximation* of the pre-ultracategory (Mod $T)^*_Y$. By the Keisler-Shelah isomorphism theorem, any small approximation (see above) of (Mod $T)^*$ can be extended to a closed one.

Recall the notion of a $\mathbf{K}^*$-subobject of Y: it is a family $\Sigma^*\subset|K^*|$ satisfying conditions (0)–(iii), with (ii) and (iii) restricted to ultraproducts put in $\mathbf{J}^*$. For $\Sigma^*\subset|K^*|$, and $M\in|K|$, let $\Sigma^*(M)=\{y\in Y(M)$: $(M, y)\in\Sigma^*\}$.

LEMMA 4.7. *Let* $\mathbf{K}^*$ *be a small closed approximation of* (Mod $T)^*_Y$, *and let* Σ^*_1, Σ^*_2 *be two* $\mathbf{K}^*$*-subobjects of* Y. *If* $\Sigma^*_1(M)=\Sigma^*_2(M)$ *for all* $M\in|K|$ *with card* $M\leqslant\kappa$, *then* $\Sigma^*_1=\Sigma^*_2$.

Proof. Assume the hypotheses. We are going to prove that $\Sigma_1{}^*(N) = \Sigma_2^*(N)$ for all $N \in |K|$.

Let $N \in |K|$. Find $M \in |K|$ with $\mathrm{card}(M) \leqslant \kappa$ such that $N \equiv M$ (by the Lowenheim-Skolem theorem, cf. [CK]). Find an ultrafilter (I, U) "in $\mathbf{K}^*$" such that there is an isomorphism h: $M^U \cong N^U$. Consider the diagram

$$\begin{tikzcd}
YM \arrow[d, "\delta = \delta_{YM}"] & YN \arrow[d, "\delta' = \delta_{YN}"] \arrow[dd, bend left=90, "f"] \\
(YM)^U \arrow[d, "\nu = {[Y, U]}^{-1}_{\langle M \rangle}", "\wr"'] & (YN)^U \arrow[d, "\nu' = {[Y, U]}^{-1}_{\langle N \rangle}", "\wr"'] \\
Y(M^U) \arrow[r, "\sim", "Yh"'] & Y(N^U)
\end{tikzcd}$$

Here δ_B: $B \to B^U$ is the canonical embedding $b \to \langle b \rangle/U$, f is the composite $\nu' \circ \delta'$. Let Σ^* be either of Σ_1^*, Σ_2^*. We *claim* that for any $y \in Y(N)$, $y \in \Sigma^*(N)$ if and only if there are $P \in U$ and $y_i \in \Sigma^*(M)$ for $i \in P$ such that $f(y) = (Yh)(\nu(\langle y_i \rangle/U))$. Suppose first that $y \in \Sigma^*(N)$. Then $(N, y) \in \Sigma^*$, hence by (ii), $(N, y)^U = (N^U, \nu'(\langle y \rangle/U)) = (N^U, f(y))$ belongs to Σ^*. Let $z = (Yh^{-1})(f(y))$. By (i), $(M^U, z) \in \Sigma^*$. Since $z \in Y(M^U)$, there are $y_i \in YM$ for $i \in I$ such that $z = \nu(\langle y_i \rangle/U)$; i.e., $(M^U, z) = \prod (M, y_i)/U$ in the pre-ultracategory $(\mathrm{Mod}\ T)^*$. By (iii), there is $P \in U$ such that $y_i \in \Sigma^*(M)$ for $i \in P$. We have $f(y) = (Yh)(z) = (Yh)(\nu(\langle y_i \rangle/U))$ as desired. Conversely, suppose $P \in U$, $y_i \in \Sigma^*(M)$ for $i \in P$, and $f(y) = (Yh)(\nu(\langle y_i \rangle/U))$. By (ii), we have $z =_{\mathrm{df}} \nu(\langle y_i \rangle\ U) \in \Sigma^*(M^U)$. By (i), $(Yh)(z) \in \Sigma^*(N^U)$. Hence $f(y) \in \Sigma^*(N^U)$, i.e., $(N, y)^U \in \Sigma^*$. By (iii), we must have that $y \in \Sigma^*(N)$, proving the claim.

Inspecting the assertion of the claim, we see that it determines $\Sigma^*(N)$ solely from $\Sigma^*(M)$. The lemma is proved. ▌

LEMMA 4.8. *Let $M \in |\mathrm{Mod}\ T|$, $A \in |T|$, $a \in M(A)$, $X \in |T'|$, $x \in X(M)$, and assume that (A, a) is a support of x. Then there is a partial A-cover of X containing $p = (M, a, x)$.*

Proof. By the remarks following the statement of condition (iv), it suffices to show that there is a family $\Sigma^* \subset (\mathrm{Mod}\ T)_Y^*$ (with $Y = e(A) \times X$) satisfying conditions (0)–(v), with these conditions understood now as referring to $K^* = (\mathrm{Mod}\ T)_Y^*$.

Let $\mathbf{K}_\alpha^* = (K_\alpha^*, \mathbf{J}_\alpha^*)$ $(\alpha \in \mathrm{Ord})$ be a sequence of small closed approximations of $\mathbf{K}_\infty^* = (\mathbf{Mod}\ T)_Y^*$ such that: the sequence is increasing, i.e., $|K_\alpha^*| \subset |K_\beta^*|$ and $\mathbf{J}_\alpha^* \subset \mathbf{J}_\beta^*$ for $\alpha < \beta$, and it is exhaustive, i.e., $\bigcup_{\alpha \in \mathrm{Ord}} |K_\alpha^*| = |K_\infty^*|$, $\bigcup_{\alpha \in \mathrm{Ord}} \mathbf{J}^* =$ the class of all triples (I, U, g^*) with U an ultrafilter on I, and g^*: $I \to |K_\infty^*|$. By the above, such sequence exists.

Let $\langle M_l\colon l\in\Lambda\rangle$ be a small family (Λ a set) of models $M_l\in K_0$ of power $\leqslant\kappa$ (see the definition of "closed") such that for all $M\in|\mathrm{Mod}\ T|$ of power $\leqslant\kappa$, M is isomorphic to some M_l ($l\in\Lambda$). All possible choices $\bar{\Sigma}^*$ of subsets Σ_l^* of $Y(M_l)$ form a *set*: the set $S=_{\mathrm{df}}\{\bar{\Sigma}^*=\langle\Sigma_l^*\colon l\in\Lambda\rangle\colon \Sigma_l^*\subset Y(M_l),\ l\in\Lambda\}$. We claim that there is $\bar{\Sigma}^*\in S$ such that for arbitrarily large α (for all $\alpha'\in\mathrm{Ord}$ there is $\alpha>\alpha'$ such that) there is a partial A-cover $\Sigma^{*(\alpha)}$ of X relative to $\mathbf{K}_\alpha^*$ containing p_0 with the additional property that its trace on the M_l's is the given $\bar{\Sigma}^*$, i.e., $\Sigma^{*(\alpha)}(M_l)=\Sigma_l^*$ for $l\in\Lambda$. By 4.6, for each α, there is one without the additional property; suppose we have chosen one, $\Sigma^{*(\alpha)}$, for each $\alpha\in\mathrm{Ord}$. But since the possible traces form a set, S, there must be at least one element $\bar{\Sigma}^*$ of S that occurs as the trace of $\Sigma^{*(\alpha)}$ for arbitrarily large α's, which is the claim.

Now, with $\bar{\Sigma}^*$ and the $\Sigma^{*(\alpha)}$ of the claim (defined for $\alpha\in C$, an unbounded class of ordinals), notice that for all $\alpha<\beta$, both in C, we must have that $\Sigma^{*(\alpha)}\subset\Sigma^{*(\beta)}$. The reason is that the restriction of $\Sigma^{*(\beta)}$ to $\mathbf{K}_\alpha^*$ is a $\mathbf{K}_\alpha^*$-subobject of Y with the same trace $\bar{\Sigma}^*$ as $\Sigma^{*(\alpha)}$, hence by 4.7, the restriction and $\Sigma^{*(\alpha)}$ coincide. This fact now immediately implies that $\Sigma^*=\bigcup_{\alpha\in C}\Sigma^{*(\alpha)}$ is the family we are looking for. ▌

Proof of Condition 4.2(iii) *for* $T'=\mathrm{Hom}(\mathbf{Mod}\ T,\mathbf{SET})$ It suffices to show the existence of finitely many families $\Sigma^*(0)$, $\Sigma^*(1),\ldots,\ \Sigma^*(n)\subset(\mathrm{Mod}\ T)_Y^*$ satisfying (0)–(iv) and also that for every $M\in|\mathrm{Mod}\ T|$ and every $x\in X(M)$ there is $a\in M(A)$ such that $(M,a,x)\in\Sigma^*(l)$ for at least one $l\leqslant n$; this last condition expresses that the partial covers defined by the $\Sigma^*(l)$ form a finite cover of X.

By 4.4 and 4.8, it is true that for every M and $x\in X(M)$ there are $a\in M(A)$ and Σ^* satisfying (0)–(iv) such that $(M,a,x)\in\Sigma^*$.

Now note that by 4.7, such a Σ^* is determined by its trace on the M_l's (see the proof of 4.8). Hence, there are a *set* Λ and an indexing $\langle\Sigma^*(l)\colon l\in\Lambda\rangle$ such that every Σ^* satisfying (0)–(iv) is one of the $\Sigma^*(l)$ ($l\in\Lambda$).

Assume that the assertion of the first paragraph fails. Let I be the set of all finite subsets of Λ; for $i\in I$, let $[i]=\{j\in I\colon i\subseteq j\}$, and let U be an ultrafilter on I with $[i]\in U$ for all $i\in I$. By assumption, for every $i\in I$ there are M_i and $x_i\in X(M_i)$ such that whenever $a\in M(A)$, then $(M,a,x)\notin\Sigma^*(l)$ for all $l\in i$. Consider $M=\prod M_i/U$, $v=[X,U]_{\langle M_i\rangle}\colon X(M)\xrightarrow{\sim}\prod XM_i/U$, $y=\langle x_i\rangle/U\in\prod XM_i/U$, and $x=v^{-1}(y)$. We *claim* that for all $a\in M(A)$ and all $l\in\Lambda$, we have $(M,a,x)\notin\Sigma^*(l)$. Indeed, let $l\in\Lambda$ and $a\in M(A)$ and suppose that $(M,a,x)\in\Sigma^*(l)$. We have $a=\langle a_i\rangle/U$ for some $P\in U$ and $i\in P$; then $(M,a,x)=\prod(M_i,a_i,x_i)/U$ (in the p.-u.c. $(\mathrm{Mod}\ T)_Y^*$). Condition (iii) applied to $\Sigma^*(l)$ and $(M,a,x)\in\Sigma^*(l)$ says that there is $Q\in U$, $Q\subset P$, such that $(M_i,a_i,x_i)\in\Sigma^*(l)$ for $i\in Q$. Consider the set $[\{l\}]=\{i\in I\colon l\in i\}$; it belongs to U; intersecting it with Q, the intersection still belongs to U, hence it is non-empty. Let i be an element of this intersection.

We have found $i \in I$, $l \in i$, and $a = a_i$, such that $(M_i, a, x_i) \in \Sigma^*(l)$, contrary to the choice of the M_i and x_i. This shows our *claim*.

Now clearly, the assertion of the claim just proved contradicts the facts stated in the second and third paragraphs. This completes the proof. ▌

According to Lemma 4.2, we have finished the proof of Theorem 4.1.

5. CANONICAL ULTRAMORPHISMS

The ultramorphisms used in the last section (the "diagonal" δ_M: $M \to M^U$, and the δ_ε $(\varepsilon = 1, 2)$ in the proof of 4.6) have a stronger canonicity property than the one spelled out in the definition of ultramorphism. Roughly speaking, they are canonical morphisms defined in terms of the universal properties of certain (left) limits and directed colimits in SET. In this section, we make this idea precise by introducing the notion of canonical ultramorphism, and showing that they are all we need for the truth of Theorem 4.1.

A *cone* (see [CWM]) is a graph G with a distinguished node v (*vertex*) such that for every $\gamma \in |G| - \{v\}$, there is exactly one edge (*projection*) $v \to \gamma$ in G, and there are no edges into v; the full subgraph on $|G| - \{v\}$ is the *base* of the cone. A *D-cone* (or directed co-cone) is a graph G with a distinguished node v (*vertex*), such that for every $\gamma \in |G| - \{v\}$ there is exactly one edge (*injection*) $\gamma \to v$, there are no edges from v, and moreover, the full subgraph on $|G| - \{v\}$ (the *base* of the D-cone) is the graph of a directed partial order.

An LD-graph (limit directed colimit) is a small graph Λ, together with a set of data as follows:

(i) a set C_0 of edges of Λ of the form $\lambda \to \lambda$ [intended to be identity morphisms], and a set C_1 of triangles $\begin{tikzcd}\lambda_1 \arrow[r] \arrow[rr, bend right] & \lambda_2 \arrow[r] & \lambda_2\end{tikzcd}$ of edges of Λ [intended to be commutative traingles];

(ii) a family $\mathrm{LD}(\Lambda) = \mathrm{L}(\Lambda) \cup \mathrm{D}(\Lambda)$ of pairs (G, g) such that if $(G, g) \in \mathrm{L}(\Lambda)$, G is a cone, and if $(G, g) \in \mathrm{D}(\Lambda)$, G is a D-cone, and (always) g is a graph-morphism (diagram) g: $G \to \Lambda$.

Let Λ be an LD-graph, and S an arbitrary category. An LD-*diagram of type Λ in S* is a diagram Δ: $\Lambda \to S$ such that Δ carries each element of C_0 into an identity morphism in S, carries each triangle in C_1 into a commutative triangle in S, and "carries each formal limit and colimit in $\mathrm{LD}(\Lambda)$ into a true limit (resp. directed colimit) in $\mathrm{LD}(\Lambda)$": whenever $(G, g) \in \mathrm{L}(\Lambda)$, then the composit $\Delta \circ g$: $G \to S$ is a limit diagram ("limiting cone"), and whenever $(G, g) \in D(\Lambda)$, then $\Delta \circ g$ is a directed colimit diagram (including the condition that the restriction of $\Delta \circ g$ to the base of G is a functor from the base as a category (partial order)).

For an LD-graph Λ, and arbitrary category S, let $\mathrm{Hom}(\Lambda, S)$ stand for the category whose objects are the LD-diagrams $\Lambda \to S$, whose morphisms all the natural transformations between them.

Proposition 5.1. *For any* LD-*graph* Λ, *there are a category* $\hat{\Lambda}$ *and an* LD-*diagram* Δ_Λ: $\Lambda \to \hat{\Lambda}$, *having the following universal property*:

Whenever Δ: $\Lambda \to S$ *is any* LD-*diagram, then there is a unique functor* $\hat{\Delta}$: $\hat{\Lambda} \to S$ *such that*

$$\begin{tikzcd}
\Lambda \arrow[r, "\Delta_\Lambda"] \arrow[dr, "\Delta"'] & \hat{\Lambda} \arrow[d, "\hat{\Delta}"] \\
 & S
\end{tikzcd}$$

commutes.

$\hat{\Lambda}$ *and* Δ_L *are unique up to isomorphism*, $\hat{\Lambda}$ *and* Δ_Λ *can* (*and will*) *be chosen so that the objects of* $\hat{\Lambda}$ *are exactly the nodes of* Λ, *and the object-function of* Δ_Λ *is the identity.*

Moreover, any natural transformation $\Delta \to \Delta'$ ($\Delta, \Delta' \in \mathrm{Hom}(\Lambda, S)$) *is a natural transformation* $\hat{\Delta} \to \hat{\Delta}'$.

Remarks (instead of proof). Intuitively, the morphisms of $\hat{\Lambda}$ are those of Λ, plus all others that necessarily exist because of the prescribed limits and colimits. The universal properties of those limits and colimits also force certain morphisms to be equal. The proof can be given by a transfinite construction through all ordinals.

Let k and 1 be two distinguished nodes of Λ, Λ an LD-graph. A *formal canonical* LD-*morphism* (f.c.LD-m.) *of type* Λ is any morphism $k \to l$ in $\hat{\Lambda}$.

Let Λ_1, Λ_2 be two LD-graphs. Λ_2 is an *extension of* Λ_1 (in symbols: $\Lambda_1 \subset \Lambda_2$) if Λ_1 is a subgraph of Λ_2 (not necessary full) and $\mathrm{LD}(\Lambda_1) \subset \mathrm{LD}(\Lambda_2)$.

Let Z be a subset of Λ, an LD-graph. Let $\hat{Z}$ be the smallest subset Z' of $|\lambda|$ such that $Z \subset Z'$ and whenever $(G, g) \in \mathrm{LD}(\Lambda)$, and $g(i) \in Z'$ for all i in the base of the cone (or D-cone) G, then $g(v) \in Z'$ for the vertex v of G; briefly, $\hat{Z}$ is the closure of Z under the distinguished limits and colimits.

Let $\Lambda_1 \subset \Lambda_2$. We say that Λ_2 is a *generated extension* (g-extension) of Λ_1 if $|\Lambda_1|\hat{}$ taken in Λ_2 is identical to $|\Lambda_2|$: "every node in $|\Lambda_2|$ is obtained by (repeated) distinguished limits and colimits from $|\Lambda_1|$."

Let Λ_2 be an extension of Λ_1, i: $\Lambda_1 \to \Lambda_2$ the graph-inclusion, S a category. A_2 is an *S-conservative* extension of A_1 if the induced functor i^*: $\mathrm{Hom}(A_2, S) \to \mathrm{Hom}(A_1, S)$ is an equivalence of categories.

Consider the following (trivial) example. Let the LD-graphs A_1 and A_2 be defined as follows: $|A_1| = I$ (a set), A_1 has no edges, and $\mathrm{LD}(A_1) = \varnothing$; $|A_2| = I \cup \{p\}$, the edges of A_2 are "projections" $p \to i$, one for each $i \in I$; $L(A) = \{(G, g)\}$, with $G = A_2$ (A_2 is a cone with vertex p) and $g =$ the identity, $D(A) = \varnothing$. Then clearly, A_2 is a g-extension of A_1; in fact, for any category S, we have that i^* above is full and faithful; A_2 is S-conservative over A_1 precisely when S has all I-indexed products.

Let Γ be an ultragraph, in the sense introduced in Section 3. Spelling out the definition of ultraproducts in terms of products and colimits results in a specific LD-graph $\bar{\Gamma}$ as follows. $\bar{\Gamma}$ has Γ as a subgraph; additional nodes and edges are as follows: for each bound node β and for each $P \in U_\beta$ (U_β is the ultrafilter assigned to β in Γ), we have a node $\delta(\beta, P)$ [for the product $\prod_{i \in P} g_\beta(i)$]; for each $i \in P$, an edge $\pi(\beta, P, i)$: $\delta(\beta, P) \to g_\beta(i)$ [projection]; for each pair (P, Q) of elements of U_β with $Q \subseteq P$, an edge $\eta(\beta, P, Q)$: $\delta(\beta, P) \to \delta(\beta, Q)$; for each $P \in U_\beta$, an edge $\eta(\beta, P)$: $\delta(\beta, P) \to \beta$ [injection];

$$\begin{tikzcd}
 & & \delta(\beta, P) \arrow[dll, "\pi(\beta,P,i)"'] \arrow[drr, "\eta(\beta,P)"] \arrow[dd] & & \\
g_\beta(i) & \circlearrowleft \eta(\beta,P,Q) & & \circlearrowleft & \beta \\
 & & \delta(\beta, Q) \arrow[ull, "\pi(\beta,Q,i)"] \arrow[urr, "\eta(\beta,Q)"'] & &
\end{tikzcd}$$

$\mathrm{LD}(\bar{\Gamma})$ is given as follows. The members of $L(\bar{\Gamma})$ are $(G_{\beta,P}, g_{\beta,P})$ for $\beta \in \Gamma^b$ and $P \in U_\beta$, where $G_{\beta,P}$ is the cone with base the set P (no edges in the base), and vertex v; $g_{\beta,P}(i) = g_\beta(i)$ [given in Γ!] for $i \in P$; $g_{\beta,P}(v) = \delta(\beta, P)$; $g_{\beta,P}(v \to^{\mathrm{proj}} i) = \pi(\beta, P, i)$. The members of $D(\bar{\Gamma})$ are $(G_\beta, \bar{g}_\beta)$, for $\beta \in \Gamma^b$, where G_β is the cone whose base is the graph of the partial order U_β^{op}, and whose vertex is v; $\bar{g}_\beta(P) = \delta(\beta, P)$, $\bar{g}_\beta(Q \leqslant P) = \eta(\beta, P, Q)$, $\bar{g}_\beta(v) = \beta$. We also throw in the following commutativity data into the set C_1,

$$\begin{tikzcd}
 & & \delta(\beta, P) \arrow[dll, "\pi(\beta,P,i)"'] \arrow[dd, "\eta(\beta,P,Q)"] \\
g_\beta(i) & \circlearrowleft & \\
 & & \delta(\beta, Q) \arrow[ull, "\pi(\beta,Q,i)"]
\end{tikzcd}$$

for all $\beta \in \Gamma^{\mathrm{b}}$, $Q \subseteq P$ in U_β, $i \in Q_\beta$; also, we add "$\eta(\beta, P, P) = \mathrm{Id}$" to C_0.

The fact that the purpose of the LD-graph $\bar{\Gamma}$ is achieved is expressed by the following statement: the graph- inclusion $\Gamma \to \bar{\Gamma}$ induces an equivalence $\mathrm{Hom}(\bar{\Gamma}, \mathrm{SET}) \to \mathrm{Hom}(\Gamma, \mathrm{SET})$ (the domain category here is a category of LD-diagrams, the second one is a category of ultradiagrams; note that it is not quite literally true that $\mathrm{Hom}(\bar{\Gamma}, \mathrm{SET}) \to \mathrm{Hom}(\mathrm{graph}(\Gamma), \mathrm{SET})$ factors through $\mathrm{Hom}(\Gamma, \mathrm{SET}) \to^{\mathrm{incl.}} \mathrm{Hom}(\mathrm{graph}(\Gamma), \mathrm{SET})$; one gets "non-strict" ultradiagrams too; but still, the above statement is true, by a trivial modification of the functor defined by composition.)

Let Γ be an ultragraph, k and l two nodes in Γ. A *formal canonical ultramorphism* (f.c.u.m.) *of type* $\Gamma^* = (\Gamma, k, l)$ is any morphism $k \to l$ in Λ, for any LD-graph Λ which is a generated extension of the LD-graph $\bar{\Gamma}$ (see above), and which is a conservative extension of $\bar{\Gamma}$ with respect to SET.

It would be much simpler to say that an f.c.u.m. is a morphism in $\bar{\Gamma}$; unfortunately, this kind of special f.c.u.m. is not sufficient for our proof to go through.

An f.c.u.m. δ of type Γ^*, with Λ as above, gives rise to an ultramorphism in $S = \mathrm{SET}$ as follows. The graph-inclusion $\Gamma \to^{i_1} \bar{\Gamma}$ induces an equivalence $\mathrm{Hom}(\bar{\Gamma}, S) \to^{i_1^*} \mathrm{Hom}(\Gamma, S)$, as we said above. The inclusion $\bar{\Gamma} \to^{i_2} \Lambda$ induces an equivalence $\mathrm{Hom}(\Lambda, S) \to^{i_2^*} \mathrm{Hom}(\bar{\Gamma}, S)$ by hypothesis. Let $\mathrm{Hom}_{\mathrm{LD}}(\hat{\Lambda}, S)$ denote the category of all functors $\hat{\Lambda} \to S$ preserving the "distinguished" limits and colimits; $\mathrm{Hom}_{\mathrm{LD}}(\hat{\Lambda}, S)$ has the objects precisely the functors $\hat{\Delta}$ for $\Delta \in |\mathrm{Hom}(\Lambda, S)|$ (see 5.1), and it is a full subcategory of $\mathrm{Hom}(\hat{\Lambda}, S)$. By 5.1, $i_3 = \Delta_\Lambda$: $\Lambda \to \hat{\Lambda}$ induces an equivalence, in fact an isomorphism, $\mathrm{Hom}_{\mathrm{LD}}(\hat{\Lambda}, S) \to^{i_3^*} \mathrm{Hom}(\Lambda, S)$. Composing these three equivalences, we obtain that $i = i_3 \circ i_2 \circ i_1$: $\Gamma \to \Lambda$ induces an equivalence $\mathrm{Hom}_{\mathrm{LD}}(\hat{\Lambda}, S) \to^{i^*} \mathrm{Hom}(\Gamma, S)$. It follows that the induced functor $\mathrm{Hom}(\mathrm{Hom}(\Gamma, S), S) \to \mathrm{Hom}(\mathrm{Hom}_{\mathrm{LD}}(\hat{\Lambda}, S), S)$ is an equivalence as well. In particular, with the functors

$$\begin{tikzcd}\mathrm{Hom}(\Gamma, S) \arrow[r, shift left, "(\mathrm{k})"] \arrow[r, shift right, "(l)"'] & S\end{tikzcd}$$

and

$$\begin{tikzcd}\mathrm{Hom}_{\mathrm{LD}}(\hat{\Lambda}, S) \arrow[r, shift left, "(k)'"] \arrow[r, shift right, "(l)'"'] & S\end{tikzcd}$$

(with $(k)'$ being "evaluation at k" for $k \in |\hat{\Lambda}|$, etc.) we have $(k)' = i^{**}((k))$, $(l)' = i^{**}((l))$, and i^{**} induces a bijection

$$\mathrm{Hom}((k), (l)) \longrightarrow \mathrm{Hom}((k)', (l)').$$

We have our f.c.u.m. δ as a morphism δ: $k \to l$ in $\hat{\Lambda}$. Hence $(\delta) = \mathrm{ev}(\delta)$ is a morphism (δ): $(k)' \to (l)'$ in $\mathrm{Hom}((k)', (l)')$. The unique $\delta_0 \in \mathrm{Hom}((k)), (l))$

such that $i^{**}(\delta_0)=\delta$ is, by definition, the ultramorphism δ_S induced by δ. Any ultramorphism so obtained from an f.c.u.m. is called *canonical.*

In less abstract terms, δ_S is defined as follows. Let $\mathcal{A}\colon \Gamma\to S$ be an ultradiagram. We can find an LD-diagram $\mathcal{A}_1\colon \bar{\Gamma}\to S$ extending Γ, then $\mathcal{A}_2\colon \Lambda\to S$ extending $\mathcal{A}_1$ (by hypothesis), then $\mathcal{A}_3\colon \hat{\Lambda}\to S$ extending $\mathcal{A}_2$ (via Δ_Λ), by 5.1. We have $(\delta_S)_{\mathcal{A}}=\mathcal{A}_3(\delta)$. The above, more abstract, definition has the advantage that from it it is immediate that δ_S is well defined and is an ultramorphism.

A *canonical ultracategory* **K** is a pre-ultracategory together with an ultramorphism $\delta_{\mathbf{K}}$ of type Γ^* assigned (formally) to any f.c.u.m. δ of type Γ^*. Replacing ultracategories by canonical ones, the basic duality theory of Sections 2 and 3 remains valid (of course), and one obtains a canonical functor $e_T^{(c)}$ from T, any pretopos, into $\mathcal{H}om(\mathbf{Hom}^{(c)}(\mathcal{T}, \mathcal{S}et)$, $\mathbf{SET})$, where $\mathbf{Hom}^{(c)}(\mathcal{T}, \mathcal{S}et)$ is the canonical ultracategory of models of T.

THEOREM 5.2. *For any small pretopos T, $e_T^{(c)}$ is an equivalence of categories.*

Of course, 5.2 is a strengthening of 4.1 since $\mathrm{Hom}(\mathbf{Hom}^{(c)}(\mathcal{T}, \mathcal{S}et), \mathbf{SET})$ is a subcategory of $\mathrm{Hom}(\mathbf{Hom}(\mathcal{T}, \mathcal{S}et), \mathbf{SET})$.

Proof. The proof is obtained by inspecting that of 4.1. The proofs of conditions (i) and (ii) in 4.2 did not refer to ultramorphisms at all. In the proof of (iii), specifically in the proof of 4.4, we have used the ultramorphism "$\delta_M\colon M\to M^U$." This is seen to be canonical as follows.

Now $\Gamma^f=\{k\}$, $\Gamma^b=\{l\}$, $(I_l, U_l)=(I, U)$; in $\bar{\Gamma}$ we have the additional nodes $\delta(l, P)=\delta_P$ for $P\in U$, edges $\pi(l, P, i)=\pi_{P,i}\colon \delta_P\to k$ $(i\in P)$, $\eta(l, P, Q)=\eta_{P,Q}\colon \delta_P\to\delta_Q$ $(Q\leqslant P$ in $U)$, and $\eta(l, P)=\eta_P\colon \delta_P\to l$,

$$\begin{tikzcd}
 & \delta_P \arrow[dl, "\pi_{P,i}"'] \arrow[dr, "\eta_P"] \arrow[dd, "\eta_{P,Q}"] & \\
k & & l \\
 & \delta_Q \arrow[ul, "\pi_{Q,i}"] \arrow[ur, "\eta_Q"'] &
\end{tikzcd}$$

Let us consider these nodes and edges objects and morphisms in $\hat{\bar{\Gamma}}$ (by writing γ for $\Delta_{\bar{\Gamma}}(\gamma)$). Let $P\in U$. In the category $\hat{\bar{\Gamma}}$, δ_P is a product of the constant I-indexed family $\langle k\rangle$, with projections $\pi_{P,i}$. Hence, there is a unique $\phi_P\colon k\to\delta_P$ in $\hat{\bar{\Gamma}}$ (diagonal) such that $\pi_{P,i}\circ\phi_P=\mathrm{Id}_k$ for all $i\in I$. Moreover, for $Q\subseteq P$, both in U, we have that

$$\begin{tikzcd}
 & \delta_P \arrow[dd, "\eta_{P,Q}"] \\
k \arrow[ur, "\phi_P"] \arrow[dr, "\phi_Q"'] \arrow[r, phantom, "\circlearrowleft"] & \\
 & \delta_Q
\end{tikzcd}$$

The composite $\eta_P \circ \phi_P$: $k \to l$ will not depend on P,

$$\begin{tikzcd}
 & \delta_P \arrow[dd, "\eta_{P,Q}"] \arrow[dr, "\eta_P"] & \\
k \arrow[ur, "\phi_P"] \arrow[dr, "\phi_Q"'] \arrow[r, phantom, "\circlearrowleft"] & \arrow[r, phantom, "\circlearrowleft"] & l \\
 & \delta_Q \arrow[ur, "\eta_Q"'] &
\end{tikzcd}$$

call it δ: $k \to l$. It is clear that δ_{SET} is our ultramorphism (note that in this case there was no need for a Λ; i.e., $\Lambda = \bar{\Gamma}$).

Next, we turn to the proof of 4.6, specifically part (4) of that proof. Fix ε, either of 1 and 2, and write $\gamma(\vec{P})$ for $\gamma_\varepsilon(\vec{P})$, Γ^+ for Γ_ε; we wish to show that δ_ε is canonical.

$\bar{\Gamma}^+$ now looks as follows. Let $t \in \Theta_{<\alpha}$, $Q' \leqslant P'$ in U_t, $Q \leqslant P$ in V_t, $i \in P'$, $j \in P$. Instead of

$$\begin{tikzcd}
 & \delta(\beta_t, P') \arrow[dl, "\pi(\beta_t, P', i)"'] \arrow[dd, "\eta(\beta_t, P', Q')"] \arrow[dr, "\eta(\beta_t, P')"] & & & \delta(\gamma_t, P) \arrow[dl, "\eta(\gamma_t, P)"'] \arrow[dd, "\eta(\gamma_t, P, Q)"] \arrow[dr, "\pi(\gamma_t, P, j)"] & \\
g_\beta(i) & & \beta_t \arrow[r, "e_t"] & \gamma_t & & i_t(j) \\
 & \delta(\beta_t, Q') & & & \delta(\gamma_t, Q) &
\end{tikzcd}$$

we write

$$\begin{tikzcd}
 & \varepsilon_{t,P'} \arrow[dl, "\rho_{t,P',i}"'] \arrow[dd, "\xi_{t,P',Q'}"] \arrow[dr, "\psi_{t,P'}"] & & & \delta_{t,P} \arrow[dl, "\phi_{t,P}"'] \arrow[dd, "\eta_{t,P,Q}"] \arrow[dr, "\pi_{t,P,j}"] & \\
g_\beta(i) & & \beta_t \arrow[r, "e_t"] & \gamma_t & & i_t(j) \\
 & \varepsilon_{t,Q'} & & & \delta_{t,Q} &
\end{tikzcd}$$

Let $S \subseteq R$ be in W (W is an ultrafilter on $\mathscr{P}$), $\vec{P} \in R$. In place of

$$\begin{tikzcd}
 & \delta(l, R) \arrow[dl, "\pi(l,R,\vec{P})"'] \arrow[dr, "\eta(l,R)"] \arrow[dd, "\eta(l,R,S)"] & \\
\gamma(\vec{P}) & & l \\
 & \delta(l, S) &
\end{tikzcd}$$

we write

$$\begin{tikzcd}
 & \delta_R \arrow[dl, "\pi_{R,\vec{P}}"'] \arrow[dr, "\eta_R"] \arrow[dd, "\eta_{R,S}"] & \\
\gamma(\vec{P}) & & l \\
 & \delta_S &
\end{tikzcd}$$

We now extend the LD-graph $\bar{\Gamma}^{+}$ to an LD-graph Λ. Let $\vec{P} \in \mathscr{P}$, $\vec{P} = \langle P_t \colon t \in \Theta \rangle$. For $t \in \Theta$, let $P'_t = \{ i \in I_t \colon g_t(i) \in \Gamma(\vec{P}) \}$; since t is accessible from $\vec{P}$, $P'_t \in U_t$. Let $\Gamma^* \vec{P}$ be the full subgraph of $\bar{\Gamma}$ whose nodes are the ones in $\Gamma(\vec{P})$ (the ones accessible from $\vec{P}$), plus all $\varepsilon_{\vec{P},t} =_{\mathrm{df}} \varepsilon_{t,P'_t}$ and $\delta_{\vec{P},t} =_{\mathrm{df}} \delta_{t,P_t}$ $(t \in \Theta)$. Λ will have a "formal left limit" of $\Gamma^* \vec{P}$; we throw in a new node $\lambda_{\vec{P}}$, and projections $\bar{\gamma}_{\vec{P}} \colon \lambda_{\vec{P}} \to \gamma$ to every γ in $|\Gamma^* \vec{P}|$; we put in an appropriate pair (G, g) in $L(\Lambda)$. Let $\vec{Q} \leqslant \vec{P}$, both in P. We let $\varepsilon_{\vec{P},\vec{Q},t} =_{\mathrm{df}} \varepsilon_{t,P'_t Q'_t}$ (where P'_t was defined above, from $\vec{P}$; Q'_t is defined similarly, from $\vec{Q}$) and $\eta_{\vec{P},\vec{Q},t} =_{\mathrm{df}} \eta_{t,P_t,Q_t}$, for $t \in \Theta'$. Consider the cone with base the diagram $\Gamma^* \vec{Q}$ and vertex $\lambda_{\vec{P}}$ and whose projections are the projections $\bar{\gamma}_{\vec{P}}$ for $\gamma \in \Gamma(\vec{Q})$, and the (formal) composites $\varepsilon_{\vec{P},\vec{Q},t} \circ (\overline{\varepsilon_{\vec{P},t}})_{\vec{P}} \colon \lambda_{\vec{P}} \to \varepsilon_{\vec{Q},t}$, $\eta_{\vec{P},\vec{Q},t} \circ (\overline{\delta_{\vec{P},t}})_{\vec{P}} \colon \lambda_{\vec{P}} \to \delta_{\vec{Q},t}$, for $t \in \Theta'$. Put in a new arrow $\zeta_{\vec{P},\vec{Q}} \colon \lambda_{\vec{P}} \to \lambda_{\vec{Q}}$ together with commutativity conditions asserting that $\zeta_{\vec{P},\vec{Q}}$ is the (unique) arrow obtained from the universal property of the cone with vertex $\lambda_{\vec{Q}}$ and base $\Gamma^* \vec{Q}$.

Before completing the description of Λ, we want to come to the main point. Let $\bar{\Gamma}$ be the full subgraph of $\bar{\Gamma}^{+}$ as described above obtained by discarding the nodes l and $\delta_R (R \in W)$. Let Λ_1 be the LD-extension of $\bar{\Gamma}$ obtained by the preceding extensions process (this did not refer to l and the δ_R). Let $\nu_{\vec{P}}$ be the projection $\nu_{\vec{P}} = (\bar{\phi}_0)_{\vec{P}} \colon \lambda_{\vec{P}} \to \phi_0$.

Now, let $\mathscr{A} \colon \Gamma \to \mathrm{SET}$ be any ultradiagram. We can regard $\mathscr{A}$ an LD-diagram $\mathscr{A} \colon \bar{\Gamma} \to \mathrm{SET}$. Because SET has small limits, $\mathscr{A}$ can be extended, essentially uniquely, to an LD-diagram $\mathscr{B} \colon \Lambda_1 \to \mathrm{SET}$. We *claim* that $\mathscr{B}$ carries the family of morphisms $\zeta_{\vec{P},\vec{Q}}$ ($\vec{Q} \leqslant \vec{P}$ in $\mathscr{P}$) into a directed system in

SET, and it carries ϕ_0 into a colimit of this directed system, with canonical injections $\mathscr{B}(\nu_{\vec{P}})$ $(\vec{P}\in\mathscr{P})$.

The first part of this claim is clear; actually, $\zeta_{\vec{P},\vec{Q}}$ will form a directed system already in $\hat{A}_1$, because the "canonical definitions" of the $\zeta_{\vec{P},\vec{Q}}$. The second part is a special property of the graph Γ, as well as of SET. To prove the second part, we begin by pointing out a representation of the left limit $\mathscr{B}(\lambda_{\vec{P}})$.

The standard way of representing left limits in SET yields the following. $\mathscr{B}(\lambda_{\vec{P}})$ is the set of all elements (vectors) $\vec{a}\in\prod_{\gamma\in|\Gamma^*\vec{P}|}\mathscr{B}(\gamma)$ such that for any edge $\beta\to^{e}\gamma$ in $\Gamma^*\vec{P}$, $(\mathscr{B}(e))(\vec{a}(\beta))=\vec{a}(\gamma)$. The projections $\mathscr{B}(\lambda_{\vec{P}})\to^{\mathscr{B}(\bar{\gamma})}\mathscr{B}(\gamma)$ are given by restricting the projections from the product. Looking at this representation, we recognize the following connection with part (4) in the proof of 4.6.

Let $\mathscr{A}^*$: $\Gamma\to\mathrm{SET}^*$ be any *proper lifting* of $\mathscr{A}$, i.e., a u.d. such that

$$\begin{tikzcd}
\Gamma \arrow[r, "\mathscr{A}^*"] \arrow[dr, "\mathscr{A}"'] & \mathrm{SET}^* \arrow[d, "\text{forgetful}"] \\
 & \mathrm{SET}
\end{tikzcd}$$

commutes, $\mathscr{A}^*(\phi_0)$ is proper, and $\mathscr{A}^*(\gamma)$ is improper for $\gamma\in\Gamma_0-\{\phi_0\}$. Let Θ be the set of $t\in\Theta_{\alpha_0}$ such that $\mathscr{A}^*(\beta_t)$ is proper, let $P_t=\{j\in J_t\colon\mathscr{A}^*(i_t(j))$ is proper$\}$; for $t\in\Theta$ we have $P_t\in V_t$. It is easy to see that $\vec{P}=\vec{P}_{\mathscr{A}^*}=\langle P_t\colon t\in\Theta\rangle$ is regular, and $\Gamma(\vec{P})$ is identical to the set $\{\phi_0\}\cup\{\gamma\in\Gamma_{<\alpha_0}-\Gamma_0\colon\mathscr{A}^*(\gamma)$ is proper$\}$. Now with a fixed $\mathscr{A}$, and a fixed $\vec{P}\in\mathscr{P}$, let $A_{\vec{P}}$ be the set of all proper liftings $\mathscr{A}^*$ of $\mathscr{A}$ such that $\vec{P}_{\mathscr{A}^*}=\vec{P}$. Inspection shows that the set $\mathscr{B}(\lambda_{\vec{P}})$ described above and $A_{\vec{P}}$ are essentially the same: elements of $\mathscr{B}(\lambda_{\vec{P}})$ are determined by their restrictions to $\Gamma(\vec{P})$, and similarly for elements of $A_{\vec{P}}$; we identify $\vec{a}\in\mathscr{B}(\lambda_{\vec{P}})$ with $\mathscr{A}^*\in A_{\vec{P}}$ such that $\vec{a}\restriction\Gamma(\vec{P})=\mathscr{A}^*\restriction\Gamma(\vec{P})$.

We now choose the representation $\mathscr{B}(\lambda_{\vec{P}})=A_{\vec{P}}$, with the obvious projections; in particular, $\mathscr{B}(\nu_{\vec{P}})$ is the map taking $\mathscr{A}^*$ to a in $(A,a)=\mathscr{A}^*(\phi_0)$. In this representation, the map $\mathscr{B}(\zeta_{\vec{P},\vec{Q}})$: $\mathscr{B}(\lambda_{\vec{P}})\to\mathscr{B}(\lambda_{\vec{Q}})$ becomes the following: it associates with $\mathscr{A}^*\in\mathscr{B}(\lambda_{\vec{P}})$ (such that $\vec{P}_{\mathscr{A}^*}=\vec{P}$) the "restriction" $\mathscr{A}_1^*$ with $\vec{P}_{\mathscr{A}_1^*}=\vec{Q}$ such that $\mathscr{A}_1^*(\gamma)=\mathscr{A}^*(\gamma)$ for $\gamma\in\Gamma(\vec{Q})$ and $\mathscr{A}_1^*(\gamma)$ is the improper point for $\gamma\in\Gamma_{<\alpha_0}-\Gamma(\vec{Q})$.

We remind the reader of the following well-known criterion. Let $(D,\leqslant)$ be a directed partial order, $\langle A_d,\zeta_{d,d'}\colon A_d\to A_{d'},\ \nu_{d,\infty}\colon A_d\to A_\infty\colon d\leqslant d'$ in $D\rangle$ a D-cone in SET. For it to be a colimiting D-cone, it is necessary and sufficient to have the following two conditions fulfilled:

(i) (surjectivity) For any $a\in A_\infty$, there is $d\in D$ and $b\in A_d$ such that $\nu_{d,\infty}(b)=a$;

(ii) (injectivity) whenever $d \in D$, b and b' belong to A_d, and $\nu_{d,\infty}(b) = \nu_{d,\infty}(b')$, then there is $d' \geqslant d$ such that $\zeta_{d,d'}(b) = \zeta_{d,d'}(b')$.

Now, turning to the second part of our claim, we immediately see that in our case, (i) holds, since the statement (*) in (4) in the proof of 4.6 is an equivalent assertion. To prove (ii), let $\mathscr{A}_1^*, \mathscr{A}_2^* \in \mathscr{B}(\lambda_{\vec{P}})$ $(\vec{P} = \langle P_t \colon t \in \Theta \rangle)$ be two proper liftings of $\mathscr{A}$, and suppose that $\mathscr{A}_1^*(\phi_0) = \mathscr{A}_2^*(\phi_0) = (\mathscr{A}(\phi_0), a)$, $a \in \mathscr{A}(\phi_0)$. We define the sets $\Theta' \subset \Theta$ and $P_t' \subset P_t$ for $t \in \Theta'$ as follows. Suppose that $0 < \alpha < \alpha_0$ and that $\Theta' \cap \Theta_\alpha$ and P_t' for $t \in \Theta' \cap \Theta_{<\alpha}$ have been defined such that (*induction hypothesis*) for every $\gamma \in \Gamma_{<\alpha}$ accessible from $\vec{P}'_{<\alpha} = \langle P_t' \colon t \in \Theta' \cap \Theta_{<\alpha} \rangle$ we have $\mathscr{A}_1^*(\gamma) = \mathscr{A}_2^*(\gamma)$. Define $\Theta' \cap \Theta_\alpha$ by declaring that $t \in \Theta_\alpha$ belongs to it if and only if t is accessible from $\vec{P}'_{<\alpha}$. Define $P_t' = \{ j \in J_t \colon \mathscr{A}_1^*(i_t(j)) = \mathscr{A}_2^*(i_t(j)) \}$ for $t \in \Theta' \cap \Theta_\alpha$. Then for $t \in \Theta' \cap \Theta_\alpha$, $\{ i \in I_t \colon g_t(i)$ is accessible from $\vec{P}'_{<\alpha} \}$ belongs to U_t; by the induction hypothesis, therefore $\{ i \in I_t \colon \mathscr{A}_1^*(g_t(i)) = \mathscr{A}_2^*(g_t(i)) \}$ belongs to U_t as well; hence $\mathscr{A}_1^*(\beta_t) = \mathscr{A}_2^*(\beta_t)$ and $\mathscr{A}_1^*(\gamma_t) = \mathscr{A}_2^*(\gamma_t)$. It follows that for $t \in \Theta' \cap \Theta_\alpha$ we have that $P_t' \in V_t$ as required. It also follows that for $\gamma \in \Gamma_\alpha$, γ accessible from $\vec{P}'$ implies that $\mathscr{A}_1^*(\gamma) = \mathscr{A}_2^*(\gamma)$. $\mathscr{A}_1^*(\phi_0) = \mathscr{A}_2^*(\phi_0)$ by assumption. Therefore, for every $\gamma \in \Gamma_{<\alpha+1}$, if γ is accessible from $\vec{P}'$, then $\mathscr{A}_1^*(\gamma) = \mathscr{A}_2^*(\gamma)$ as required for the induction.

By its definition, $\vec{P}' \in \mathscr{P}$ and $\zeta_{\vec{P},\vec{P}'}(A_1^*) = \zeta_{\vec{P},\vec{P}'}(\mathscr{A}_2^*)$. This completes the proof of (ii) ("injectivity"), and thus the proof of the *claim* as well.

The claim proved last asserts that if we add the specification to Λ_1 that ϕ_0 be the colimit of the directed system of the arrows $\zeta_{\vec{P},\vec{Q}}$ with canonical injections $\nu_{\vec{P}}$, then the resulting LD-graph Λ_2 will be a SET-conservative extension of Λ_1 and hence of $\bar{\Gamma}$ as well.

Let Λ be the graph which is the union of $\bar{\Gamma}^+$ and Λ_2 in the obvious sense. Clearly, Λ is an extension of $\bar{\Gamma}$ which is conservative with respect to SET. We point out that a morphism $\delta \colon \phi_0 \to l$ in Λ as follows.

Let $\vec{P} \in \mathscr{P}$. The set $R = \{ \vec{Q} \in \mathscr{P} \colon \vec{Q} \leqslant \vec{P} \}$ belongs to W. We write γ for $\Delta_\Lambda(\gamma)$ again; we put ourselves in the category $\hat{\Lambda}$. δ_R is the product $\prod_{\vec{Q} \leqslant \vec{P}} \gamma(Q)$, with projections $\pi_{R,\vec{Q}}$. Consider the composite

$$\lambda_{\vec{P}} \xrightarrow{\zeta_{\vec{P},\vec{Q}}} \lambda_{\vec{Q}} \xrightarrow{(\overline{\gamma(\vec{Q})})_{\vec{Q}}} \gamma(\vec{Q}),$$

and call it $\chi_{\vec{P},\vec{Q}}$. We have a unique $\chi_{\vec{P}} \colon \lambda_{\vec{P}} \to \delta_R$ such that

$$\begin{tikzcd}
\lambda_{\vec{P}} \arrow[rr, "\chi_{\vec{P},\vec{Q}}"] \arrow[dr, "\chi_P"'] & & \gamma(\vec{Q}) \\
& \delta_R \arrow[ur, "\pi_{R,\vec{Q}}"'] &
\end{tikzcd}$$

commutes for all $\vec{Q} \in R$. Let $\mu_{\vec{P}}\colon \lambda_P \to l$ be the composite $\mu_{\vec{P}} = \eta_R \circ \chi_{\vec{P}}$. It is easy to see that

$$\begin{tikzcd}
\lambda_{\vec{P}} \arrow[rr, "\zeta_{\vec{P},\vec{Q}}"] \arrow[dr, "\mu_{\vec{P}}"'] & & \lambda_{\vec{Q}} \arrow[dl, "\mu_{\vec{Q}}"] \\
& l &
\end{tikzcd}$$

commutes for $\vec{Q} \leqslant \vec{P}$. Since ϕ_0 is the colimit of the $\zeta_{\vec{P},\vec{Q}}$, there is a unique δ: $\phi_0 \to l$ such that

$$\begin{tikzcd}
\lambda_{\vec{P}} \arrow[rr, "\mu_{\vec{P}}"] \arrow[dr, "\nu_{\vec{P}}"'] & & l \\
& \phi_0 \arrow[ur, "\delta"'] &
\end{tikzcd}$$

commutes for all $\vec{P}$. δ is the desired f.c.u.m. Looking at the second description of δ_{SET} given above, it becomes clear that δ_{SET} is the same as what we described as δ_ε in the proof of 4.6.

This completes the proof of Theorem 5.2. ▌

6. EXACT CATEGORIES

The notion of an *exact* category was introduced by M. Barr in [EC]. The definition is obtained from that of pretopos by deleting clauses (2) and (3) (disjoint sums); the exact categories so defined are the same as "exact and finitely complete" categories of [EC]. A functor between exact categories is *exact* if it preserves finite limits and quotients of equivalence relations. For any exact category S, $\mathrm{Ex}(S)$ denotes the category of all exact functors from S to SET, with all natural transformations as morphisms; $\mathrm{Ex}(S)$ is a full subcategory of $\mathrm{Hom}(S, \mathrm{SET})$. It is easy to see that $\mathrm{Ex}(S)$ has arbitrary small products and small directed colimits "inherited from SET": they are preserved by the inclusion $\mathrm{Ex}(S) \to \mathrm{Hom}(S, \mathrm{SET})$, as well as by each $(A)\colon \mathrm{Ex}(S) \to \mathrm{SET}$ ($A \in |S|$; $(A) =$ "evaluation at A"). It follows that $\mathrm{Ex}(S)$ is a pre-ultracategory $\mathbf{Ex}(S)$ such that the inclusion $\mathrm{Ex}(S) \to \mathrm{Hom}(S, \mathrm{SET})$ is a strict p.-u.f. $\mathbf{Ex}(S) \to \mathbf{Hom}(S, \mathrm{SET})$ (for $\mathbf{Hom}(S, \mathrm{SET})$ as a p.-u.c., see (i)(b) in Sect. 2). In fact, it follows that if we define the ultraproduct functor $[U]\colon (\mathrm{Ex}(S))^I \to \mathrm{Ex}(S)$ by the formula $[U](\langle M_i \rangle) = \varinjlim_{P \in U^{\mathrm{op}}} \prod_{i \in P} M_i$, with products and colimits understood in $\mathrm{Ex}(S)$), then we get a functor isomorphic to the ultraproduct functor defined by the formulas in the proof of (i)(b) in Section 2, and if we choose the products and colimits matching (in an obvious sense) the choices defining the ultraproduct functors in SET, this new definition becomes literally the same as the old one.

Ex(S) can be made an ultracategory in a unique way so that the full inclusion **Ex**$(S) \to$ **Hom**$(S$, SET$)$ is a strict ultrafunctor; this is obvious.

Let X: Ex$(S) \to$ SET be a functor preserving all small products and directed colimits; i.e., carrying all small product diagrams (cones) and all small directed colimit diagrams (co-cones) into the corresponding kind of diagram in SET. Then there is a canonical (transition) isomorphism $[X, U]$ for every ultrafilter (I, U), from $X \circ [U]_{\mathbf{K}}$ $(\mathbf{K} = \mathbf{Ex}(S))$ to $[U]_{\mathrm{SET}} \circ X^I$: for any $\langle M_i \rangle$ in K^I, $[X, U]_{\langle M_i \rangle}$ is the natural isomorphism of two versions of $\varinjlim_{P \in U^{\mathrm{op}}} \prod_{i \in P} M_i$, obtained by two sets of choices for the limiting cones and co-cones involved. Hence, X can be regarded a preultrafunctor. Moreover, any natural transformation between two such functors X, Y: Ex$(S) \to$ SET will be automatically an ultratransformation between X and Y as pre-ultrafunctors; this is easily seen.

Let us denote by Hom$_{\prod D}$(**Ex**(S), **SET**) the category whose objects are those X: Ex$(S) \to$ SET which preserve small products and directed colimits and which are also ultrafunctors (as pre-ultrafunctors between the ultracategories **Ex**(S) and **SET**), and whose morphisms are all natural transformations between such functors.

In particular, Hom$_{\prod D}$(**Ex**(S), **SET**) is a full subcategory of Hom(Ex(S), SET). It is clear that ev: $S \to$ Hom(Ex(S), SET) factors through Hom$_{\prod D}$(**Ex**(S), **SET**), giving rise to a functor e'_S: $S \to$ Hom$_{\prod D}$(**Ex**(S), **SET**). The following corollary is a fairly straightforward consequence of Theorem 4.1.

COROLLARY 6.1. *For any small exact category S, e'_s is an equivalence of categories.*

Proof. We first construct the "pretopos completion" of S. Consider the Grothendieck topology on S generated by the families of coverings that are single regular epimorphisms (the regular epimorphism topology). The category of sheaves over S as a site with this topology, $\tilde{S}$, is a coherent topos. Let T be the full subcategory of $\tilde{S}$ consisting of the coherent objects; T is a pretopos. The canonical functor ε: $S \to \tilde{S}$ (Yoneda followed by associated sheaf) factors through T, and gives rise to a functor σ: $S \to T$. Elementary facts of topos theory imply the following three facts.

(i) For any pretopos $\mathscr{T}' = T'$, σ induces an equivalence σ^*: Hom$(\mathscr{T}, \mathscr{T}') \xrightarrow{\sim}$ Ex(S, T') (Hom$(\mathscr{T}, \mathscr{T}')$ is the category of elementary functors from T to T', Ex(S, T') is the category of exact functors from S to T'; $\sigma^*(M) = M \circ \sigma$).

(ii) σ is conservative (hence faithful) and full.

(iii) Every object B in T is covered by finitely many objects coming from S: there are finitely many morphisms $\sigma(A_i) \to^{f_i} B$ $(i<n)$ such that 1_B (the maximal subobject of $B) = \bigvee_{i<n} \mathrm{Im}(f_i)$.

Apply (i) to $T' = \mathrm{SET}$ to obtain an equivalence σ^*: Mod $T \xrightarrow{\sim} \mathrm{Ex}(S)$. σ^* induces an equivalence σ^{**}: $\mathrm{Hom}(\mathrm{Ex}(S), \mathrm{SET}) \xrightarrow{\sim} \mathrm{Hom}(\mathrm{Mod}(T), \mathrm{SET})$.

By inspecting the definitions of **Ex**(S) and **Mod**(T) as ultracategories, we see that σ^* is a strict ultrafunctor **Mod**$(T) \to$ **Ex**(S). Let $S' =$

$$\begin{tikzcd}
M(B) \arrow[rrr, "\nu_M", "\sim"'] \arrow[ddd, "(\pi_i)_B"'] & & & X(\sigma^*M) \arrow[ddd, "X(\rho_i)"] \\
 & a \arrow[r, mapsto] \arrow[d, mapsto] & b \arrow[d, mapsto] & \\
 & a_i \arrow[r, mapsto] & b_i & \\
M_i(B) \arrow[rrr, "\sim", "M_i"'] & & & X(\sigma^*M_i)
\end{tikzcd}$$

for each $i < n$. Let $b_i = \nu_{M_i}(a_i)$. Since X preserves products, the morphisms $X(\rho_i)$ $(i < n)$ form a product diagram in SET. Hence there is $b \in X(\sigma^*M)$ such that $X(\rho_i)(b) = b_i$ for $i < n$. Let $a = \nu_M^{-1}(b)$. By the last mentioned commutativity and ν_{M_i} being an isomorphism, $(\pi_i)_B(a) = a_i$ for $i < n$. Since the f_i: $\sigma A_i \to B$ form a covering, there is $i < n$ and $c \in M(\sigma A_i)$ such that $M(f_i)(c) = a$. But then, for this specific i, and for $c_i = (\pi_i)_B(c)$, we have that $M_i(f_i)(c_i) = a_i$, contradicting $a_i \notin \operatorname{Im}(M_i(f_i))$. This proves the *claim*.

We have proved the existence of A $(= A_i) \in |S|$ and a regular epi $\sigma A \overset{p}{\twoheadrightarrow} B$. Consider the kernel-pair of p,

$$\begin{tikzcd}
 & \sigma A \arrow[dr, "p"] & \\
\sigma A \times_B \sigma A = C \arrow[ur, "\rho"] \arrow[r, "\mu"] \arrow[dr, "\rho'"'] \arrow[ur, phantom, "\circlearrowright" near end, shift right=3] \arrow[dr, phantom, "\circlearrowleft" near end, shift left=3] & \sigma(A \times A) \arrow[u, "\sigma\pi"'] \arrow[d, "\sigma\pi'"] & B; \\
 & \sigma A \arrow[ur, "p"'] &
\end{tikzcd}$$

ρ, ρ', p, p form a pullback, $A \xleftarrow{\pi} A \times A \xrightarrow{\pi'} A$ are canonical projections, μ is a mono; the diagram is in T. It follows that the subobject Σ determined by μ is an equivalence relation on σA, and p is a quotient of it. By (iii), (ii) and the fact that in the exact category S there are images, we see that the subobject Σ is a sup of the form $\Sigma = \bigvee_{i<n} \sigma(\Sigma_i)$, with $\Sigma_i \in \mathrm{Sub}_S(A \times A)$ $(i < n)$. We now want to apply the above argument involving finite products to conclude that $\Sigma = \sigma(\Sigma_i)$ for one of the $i < n$. For this we need that the object C *is preserved by finite products*: whenever $M_j \in |\mathrm{Mod}\ T|$ for $j < m$, $\prod_{j<m} \sigma^*M_j \xrightarrow{\pi_j} \sigma^*M_j$ are canonical projections in $\mathrm{Ex}(S)$, ν: $\sigma^*M \xrightarrow{\cong} \prod_{j<m} \sigma^*M_j$ is an isomorphism, $\sigma^*(\rho_j) = \pi_j \circ \nu$, then the diagram $(M(C) \xrightarrow{(\rho_j)_C} M_j(C): j < m)$ is a product diagram in SET. This can be proved easily starting from the facts that the objects σA and B are preserved by finite products (use that the pullback of product diagrams is a product diagram); in turn, σA is preserved by finite products since $e'(A)$ preserves (finite) products, and B is preserved since $e(B) \cong \sigma^+ X$ and X preserves (finite) products. Using that C is preserved by finite products, we can repeat the above argument to show that its covering given by the fact that $\Sigma = \bigvee_{i<n} \sigma(\Sigma_i)$ should reduce to a single morphism, i.e., that $\Sigma = \sigma(\Sigma_i)$ as claimed; the details are omitted.

We now have that $\Sigma = \sigma(\Sigma')$ for some $\Sigma' \in \mathrm{Sub}_S(A \times A)$. Since Σ is an equivalence relation on σA, by the exactness and conservativeness of σ, it follows that Σ' is an equivalence relation on A. Since S is exact, there is a quotient $A \twoheadrightarrow^{p'} A'$ of Σ'. Since σ is exact, $\sigma A \rightarrow^{\sigma p'} \sigma A'$ is a quotient of Σ. Since $\sigma A \rightarrow^{p} B$ is a quotient of Σ as well, B is isomorphic to $\sigma A'$.

Since $e(B) \cong \sigma^+ X$, it follows that X is isomorphic to $e'(A')$, which was to be proved. ▌

The statement of 6.1 contains the assertion that the evaluation functor ev: $S \rightarrow \mathrm{Hom}(\mathrm{Ex}(S), \mathrm{SET})$ is exact, faithful and full. This was proved before in [M2], sharpening M. Barr's full exact embedding theorem [EC].

7. PRELIMINARIES ON 2-CATEGORIES

Concerning 2-categories, we have the references [CWM] and [FCT]. In order to make the paper selfcontained, and also to emphasize certain points, we include a discussion of 2-categories here.

The definition of category looks as follows: a category has a collection of objects and for each pair A, B of objects, a *class* $\mathrm{Hom}(A, B)$ (of morphisms), etc. If here we replace *class* by *category*, and make all the logical changes in the subsequent parts of the definition, we arrive at the notion of 2-category. [In passing: our 2-categories will usually be super-categories (s-categories): their collections of objects will be superclasses (s-classes).]

We find that it is easier to arrive at an even more general notion first. Let S be a Cartesian s-category: S has distinguished finite products, including a distinguished terminal object $\mathbf{1}$. We wish to define the notion of an S-category so that an S-category C has a s-class of objects, and for each pair of objects A, B, it has an *object-of-morphisms* $\mathrm{Hom}(A, B) = (A, B) = C(A, B)$ which is an object of S. Now note that we do not have *elements* of objects of S available; therefore, we have to express ideas of composition, associativity, etc., by drawing diagrams in the abstract category S. Here is the finished product.

DEFINITION 7.1. Let $\mathscr{S}$ be a Cartesian s-category.

(1) An $\mathscr{S}$-category $\mathscr{C}$ is given by

(i) an s-class $|\mathscr{C}|$ of objects of $\mathscr{C}$;

(ii) for any pair of objects A, B of $\mathscr{C}$, an object $\mathscr{C}(A, B) = (A, B)$ in $\mathscr{S}$ (the *object-of-morphims-of* $\mathscr{C}$ *with-domain* A *and codomain* B);

(iii) for any object A of $\mathscr{C}$, an *identity morphism*: a morphism 1_A: $\mathbf{1} \to (A, A)$ in $\mathscr{S}$, with $\mathbf{1}$ the (distinguished) terminal object of $\mathscr{S}$;

(iv) for any triple of objects A, B, C of $\mathscr{C}$, a *composition-morphism* $\circ = \circ_{A,B,C}$

$$\circ\colon \quad (A, B) \times (B, C) \to (A, C).$$

The data (i)–(iv) have to satisfy:

(a) (identity). The composite (in $\mathscr{S}$),

$$(A, B) \xrightarrow{\cong} \mathbf{1} \times (A, B) \xrightarrow{1_A \times \mathrm{Id}_{(A,B)}} (A, A) \times (A, B) \xrightarrow{\circ_{A,A,B}} (A, B)$$

equals the identity morphism $\mathrm{Id}_{(A,B)}$ (left unit), and a similar condition for 1_A being a right unit.

(b) (associativity). For any quadruple of objects A, B, C, D of $\mathscr{C}$, the diagram (in $\mathscr{S}$),

$$\begin{tikzcd}
(A, B) \times (B, C) \times (C, D) \arrow[r, "\circ_{A,B,C} \times \mathrm{Id}_{(C,D)}"] \arrow[d, "\mathrm{Id}_{(A,B)} \times \circ_{B,C,D}"'] & (A, C) \times (C, D) \arrow[d, "\circ_{A,C,D}"] \\
(A, B) \times (B, D) \arrow[r, "\circ_{A,B,D}"'] & (A, D)
\end{tikzcd}$$

commutes.

(2) For $\mathscr{S}$-categories $\mathscr{C}$ and $\mathscr{D}$, an *$\mathscr{S}$-functor*

$$F\colon \quad \mathscr{C} \to \mathscr{D}$$

is an assignment of an object $F(A)$ of $\mathscr{D}$ to any object A of $\mathscr{C}$, plus an assignment of a morphism in $\mathscr{S}$,

$$\mathscr{C}(A, B) \xrightarrow{F_{A,B}} \mathscr{D}(XA, XB)$$

to any pair of objects A, B of $\mathscr{C}$, such that the following are satisfied:

(a) the composite

$$\mathbf{1} \xrightarrow{1_A} \mathscr{C}(A, A) \xrightarrow{F_{A,A}} \mathscr{D}(XA, XB)$$

equals 1_{FA}, and

(b) the diagram

$$\begin{tikzcd}
\mathscr{C}(A, B)\times\mathscr{C}(B, C) \arrow[r, "c_{A,B,C}"] \arrow[d, "F_{A,B}\times F_{B,C}"'] & \mathscr{C}(A, C) \arrow[d, "F_{A,C}"] \\
\mathscr{D}(FA, FB)\times\mathscr{D}(FB, FC) \arrow[r, "c_{FA,FB,FC}"'] & \mathscr{D}(FA, FC)
\end{tikzcd}$$

commutes.

(3) For $\mathscr{S}$-functors

$$F, G\colon\ \mathscr{C} \rightrightarrows \mathscr{D}$$

an $\mathscr{S}$-transformation μ: $F\to G$ is an assignment of a morphism (in $\mathscr{S}$) μ_A: $1\to\mathscr{D}(FA, GA)$ to any object A of $\mathscr{C}$ such that the diagram

$$\begin{tikzcd}
 & \mathscr{D}(FA, FB) \arrow[r, "\mathrm{Id}_{\mathscr{D}(FA,FB)}\times\mu_B"] & \mathscr{D}(FA, FB)\times\mathscr{D}(FB, GB) \arrow[d] \\
\mathscr{C}(A, B) \arrow[ur, "F_{A,B}"] \arrow[dr, "G_{A,B}"'] & & \mathscr{D}(FA, GB) \\
 & \mathscr{D}(GA, GB) \arrow[r, "\mu_A\times\mathrm{Id}_{\mathscr{D}(GA,GB)}"'] & \mathscr{D}(FA, GA)\times\mathscr{D}(GA, GB) \arrow[u]
\end{tikzcd}$$

commutes.

(4) Given $\mathscr{S}$-functors and $\mathscr{S}$-transformations

$$\begin{tikzcd}
\mathscr{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "F'"'] \arrow[r, phantom, "\downarrow\mu"] & \mathscr{D} \arrow[r, bend left, "G"] \arrow[r, bend right, "G'"'] \arrow[r, phantom, "\downarrow\nu"] & \mathscr{E}
\end{tikzcd}$$

we can define composites $G\circ F$, $G\circ\mu$, $\nu\circ F$, $\nu\circ\mu$ in a natural way; the details of formulating these are left to the reader. Also, note the obvious identity $\mathscr{S}$-functor $\mathrm{Id}_{\mathscr{C}}\colon\mathscr{C}\to\mathscr{C}$; as well as the identity $\mathscr{S}$-transformation Id_F: $F\to F$.

(5) Given $\mathscr{S}$-functors

$$\begin{tikzcd}
\mathscr{C} \arrow[r, shift left, "G"] & \mathscr{D} \arrow[l, shift left, "F"]
\end{tikzcd}$$

an ($\mathscr{S}$-)*adjunction* (making F a *left adjoint* of G, $F\dashv G$) is a pair of $\mathscr{S}$-transformations

$$\eta = \mathrm{Id}_{\mathscr{D}}\to GF, \qquad \varepsilon\colon\ FG\to\mathrm{Id}_{\mathscr{C}}$$

such that the following two composites

$$G \xrightarrow{\eta \circ G} GFG \xrightarrow{G \circ \varepsilon} G$$

$$F \xrightarrow{F \circ \eta} FGF \xrightarrow{\varepsilon \circ F} F$$

are identities (the first one Id_G, the second Id_F). η is called the *unit*, ε the *counit* of the adjunction.

We maintain that the above definitions are the only possible ones one can reasonably give under the restriction that they should work in the full generality of $\mathcal{S}$ being any Cartesian s-category. Also, it is understood that for $\mathcal{S} = \mathrm{CLASS}$, the s-category of all classes and functions, an $\mathcal{S}$-category should be precisely the same as on s-category in which the hom-"sets" are classes; this is indeed so, as can be seen by inspecting the definition; of course, "functor," "natural transformation," etc., will behave similarly well.

Now, let $\mathcal{C}at$ be the s-category of all categories: its objects are the categories, its morphisms are the functors, and composition is the usual one. $\mathcal{C}at$ is Cartesian; in particular $\mathbf{1}$ is the category with one object and one morphism.

DEFINITION 7.2. A *2-category* is a $\mathcal{C}at$-category, a *2-functor* is a $\mathcal{C}at$-functor, and a *2-transformation* is a $\mathcal{C}at$-transformation.

In particular, for a 2-category $\mathcal{C}$, $\mathcal{C}(A, B)$ is a category; usually, an object of a hom-object $\mathcal{C}(A, B)$ is called a 1-cell of $\mathcal{C}$, a morphism of $\mathcal{C}(A, B)$ is called a 2-cell of $\mathcal{C}$. A notation like $f\colon A \to B$, with objects (0-cells) of $\mathcal{C}$, indicates that f is an object of $\mathcal{C}(A, B)$ (a 1-cell); and $\sigma\colon f \to f'$, with $f, f'\colon A \rightrightarrows B$, indicates that σ is a morphism from f and f' in $\mathcal{C}(A, B)$ (σ is a 2-cell). Giving a functor $\mathbf{1} \to K$ (K a category) is equivalent to giving an object of K; hence now 1_A is a 1-cell $1_A\colon A \to A$. For a 2-transformation $\mu\colon F \to G$ (using the notation of 7.1(3)), μ_A is a 1-cell $FA \to GA$.

The prime example for a 2-category is $\mathcal{C}at$ itself; the hom-object $\mathcal{C}at(A, B)$ is $\mathrm{Hom}(A, B)$, the category of all functors and natural transformations from A to B. The fact that $\mathcal{S} = \mathcal{C}at$ can be "enriched" to an $\mathcal{S}$-category structure is related to the fact that $\mathcal{S}$ is *Cartesian closed.* On the other hand, Definition 7.1 does not refer to the Cartesian closed structure of $\mathcal{S}$, in contrast to the usual formulations in the literature.

Let $\mathcal{C}$ be a 2-category, A and B objects in $\mathcal{C}$. A 1-cell, $f\colon A \to B$ is called an *equivalence* if there exists $g\colon B \to A$ and isomorphisms (2-cells) $1_A \to^{\cong} g \circ f$, $1_B \to^{\cong} f \circ g$.

PROPOSITION 7.3. *Suppose* (η, ε) *is an adjunction of the 2-functors* $\mathscr{C} \overset{G}{\underset{F}{\rightleftarrows}} \mathscr{D}$ $(F \dashv G)$. *Let* A *and* B *objects of* $\mathscr{C}$ *and assume that the* 1*-cells* ε_A: $FGA \to A$, ε_B: $FGB \to B$ *are equivalences. Then*

$$G_{A,B}: \quad \mathscr{C}(A, B) \longrightarrow \mathscr{D}(GA, GB) \tag{7.1}$$

is an equivalence of categories.

Proof. By assumption, we have the 1-cells,

$$\varepsilon_A: \quad FGA \longrightarrow A, \qquad \varepsilon'_A: \quad A \longrightarrow FGA,$$

the 2-cells,

$$\zeta_A: \quad 1_A \xrightarrow{\cong} \varepsilon_A \circ \varepsilon'_A, \qquad \xi_A: \quad 1_{FGA} \xrightarrow{\cong} \varepsilon'_A \circ \varepsilon_A,$$

and the same with B replacing A. Let us denote (7.1) by γ: $K_1 \to K_2$. We want to construct a functor δ: $K_2 \to K_1$ and natural isomorphisms

$$\mu: \quad \mathrm{Id}_{K_1} \xrightarrow{\cong} \delta \circ \gamma, \qquad \nu: \quad \mathrm{Id}_{K_2} \xrightarrow{\cong} \gamma \circ \delta.$$

Given a 1-cell g: $GA \to GB$, we put $\delta(g)$ to be the composite in the diagram

$$\begin{tikzcd}
FGA \arrow[r, "Fg"] & FGB \arrow[d, "\varepsilon_B"] \\
A \arrow[u, "\varepsilon'_A"] \arrow[r, dashed, "\delta(g)"'] & B
\end{tikzcd}$$

(of course, Fg means $F_{GA,GB}(g)$). δ is in fact a functor: it is the composite of $F_{GA,GB}$ and the functor

$$\varepsilon_B \circ (-) \circ \varepsilon'_A: \quad \mathscr{C}(FGA, FGB) \longrightarrow \mathscr{C}(A, B)$$

(here $\circ$ refers to the composition law in the 2-category $\mathscr{C}$).

To define μ we have to determine μ_f: $f \to \delta(\gamma(f))$ for any f: $A \to B$. We have $\delta(\gamma(f)) = \varepsilon_B \circ FG(f) \circ \varepsilon'_A = f \circ \varepsilon_A \circ \varepsilon'_A$ by the naturality of ε: $FG \to \mathrm{Id}_{\mathscr{C}}$. Define $\mu_f = f \circ \zeta_A$ [in more detail: recall the composition functor $\circ = \circ_{A,A,B}$: $\mathscr{C}(A, A) \times \mathscr{C}(A, B) \to \mathscr{C}(A, B)$; μ_f is $\circ_{A,A,B}(\langle \zeta_A, \mathrm{Id}_f \rangle)$]. It is easy to see that in this way we have indeed defined a natural isomorphism μ.

Let g: $GA \to GB$ be a 1-cell; we will define ν_g: $g \to^{\cong} \gamma(\delta(g))$. We have $\gamma(\delta(g)) = G\varepsilon_B \circ GFg \circ G\varepsilon'_A$.

First, from the isomorphism $G\xi_A$: $1_{GFGA} \to^{\cong} G\varepsilon'_A \circ G\varepsilon_A$ and from the equality $G\varepsilon_A \circ \eta_{GA} = 1_{GA}$ we obtain the isomorphism

$$\chi \underset{\mathrm{df}}{=} G\xi_A \circ \eta_{GA}: \eta_{GA} \xrightarrow{\cong} G\varepsilon'_A.$$

Using $GFg \circ \eta_{GA} = \eta_{GB} \circ g$ (naturality of η), and $G\varepsilon_B \circ \eta_{GB} = 1_{GB}$ (adjunction), we obtain $G\varepsilon_B \circ GFg \circ \eta_{GA} = G\varepsilon_B \circ \eta_{GB} \circ g = g$. Therefore, we have the isomorphism

$$\nu_g \underset{\text{df}}{=} (G\varepsilon_B \circ GFg) \circ \chi\colon\ g \xrightarrow{\cong} \gamma(\delta(g)).$$

It is easy to check that ν so defined is a natural isomorphism. ▌

8. THE STONE ADJUNCTION

Suppose $\mathscr{L}$ and **U** are two categories of certain kinds of structures. Suppose we have a structure $\mathscr{A}$ in $\mathscr{L}$ and another one, **A** in **U** such that the underlying sets of $\mathscr{A}$ and **A** are identical, say A. Suppose moreover that the $\mathscr{L}$-structure $\mathscr{A}$ and the **U**-structure **A** on the set A "commute" with each other. If the structures in $\mathscr{L}$ and **U** are both algebras defined by (possibly infinitary) operations on an underlying set, then $\mathscr{A}$ commuting with **A** means that we have the identity

$$\alpha(\langle \beta(\langle a_{ij}\colon j \in J\rangle)\colon i \in I\rangle) = \beta(\langle \alpha(\langle a_{ij}\colon i \in I\rangle)\colon j \in J\rangle) \tag{8.1}$$

for a_{ij} in A, where α is any $\mathscr{L}$-operation, β any **U**-operation on A, α is I-ary, and β is J-ary. In case **U** is, e.g., a category of topological spaces and continuous functions, the commutation condition should say that every $\mathscr{L}$-operation on A is continuous with respect to the topology on A.

Under these conditions, and some mild closure conditions on $\mathscr{L}$ and **U** (closure under products and substructures in the purely algebraic case), by a simple formal procedure we can set up a pair of adjoint functors

$$\begin{tikzcd} \mathscr{L}^{\text{op}} \arrow[r, shift left, "G = \mathbf{Hom}(-{,}\mathscr{A})"] & \mathbf{U} \arrow[l, shift left, "F = \mathscr{H}om(-{,}\mathbf{A})"] \end{tikzcd}$$

with F being left adjoint to G, where $G(\mathscr{D}) = \mathbf{Hom}(\mathscr{D}, \mathscr{A})$ is an **U**-structure with underlying set $\mathrm{Hom}(\mathscr{D}, \mathscr{A})$, the set of all $\mathscr{L}$-morphisms from $\mathscr{D}$ to $\mathscr{A}$, and similarly for F. This adjunction is seen to derive naturally from the adjoint pair

$$\begin{tikzcd} \mathrm{SET}^{\text{op}} \arrow[r, shift left, "G_0 = \mathrm{Hom}(-{,}A)"] & \mathrm{SET} \arrow[l, shift left, "F_0 = \mathrm{Hom}(-{,}A)"] \end{tikzcd}$$

with the standard adjunction $\mathrm{Hom}(-, F(=)) \to^{\cong} \mathrm{Hom}(=, G(-))$, given by $f \mapsto (c \mapsto (b \mapsto (f(b))(c)))$ $(f \in \mathrm{Hom}(B, F(C)),\ b \in B,\ c \in C)$.

In the Stone duality between $\mathscr{B}oole$, the category of Boolean algebras, and **Stone**, the category of Stone spaces, the relevant functors

$$\begin{tikzcd} \mathscr{B}oole^{\text{op}} \arrow[r, shift left, "G"] & \mathbf{Stone} \arrow[l, shift left, "F"] \end{tikzcd}$$

arise as above with A the two-element set *2*, $\mathscr{A}$ the two-element B.A., and **A** the two-element discrete space. We emphasize the general nature of this construction, and we call it the *Stone adjunction* associated with the "bi-structure" ($\mathscr{A}$, **A**). In the Stone duality, the Stone adjunction becomes an equivalence: both the unit and the counit of the adjunction are isomorphisms; this is the essential particularity of Stone's theorem.

For precise and general formulations, and illuminating theorems, concerning the above discussion see [LR].

For further motivation, we discuss a variant of H. A. Priestley's duality theory [P] for distributive lattices.

We define the notion of ultraspace (u.s.) as follows. A u.s. **X** is a partially ordered set $(X, \leqslant)$, together with an operation $[U]$: $X^I \to X$ for any ultrafilter U on any set I. [We do not require any relation between $\leqslant$ and the $[U]$ now; but see below.] A morphism of ultraspaces is a function between the underlying sets preserving (but not necessarily reflection) $\leqslant$ and the $[U]$. **U** denotes the category of ultraspaces.

Let $\mathscr{L}$ denote the category of distributive lattices (d.l) (with 0, 1, $\wedge$, $\vee$), with lattice-homomorphisms as morphisms.

We have the following standard d.l.: *2*, with underlying set $2 = \{0, 1\}$, with the well-known lattice operations. On the other hand, we have the u.s. **2** with underlying set 2; $0 \leqslant 1$ but not $1 \leqslant 0$; and

$$[U]\langle\varepsilon_i\rangle = 1 \Leftrightarrow \{i \in I\colon \varepsilon_i = 1\} \in U.$$

It is immediate that *2* and **2** "commute" with each other, in the following sense: on the one hand, we have the commutativity relations (8.1) for any lattice-operation $\alpha = 0, 1, \wedge, \vee$, and any ultraproduct $\beta = [U]$; on the other hand, we have that $\wedge$ and $\vee$ are monotonic with respect $\leqslant$ (the partial order in 2). On the basis of these facts alone, by a simplified version of the work done in Section 2, we can set up a pair of adjoint functors

$$\begin{tikzcd} \mathscr{L}^{\mathrm{op}} \arrow[r, shift left, "G = \mathbf{Hom}(-{,}\, \mathit{2})"] & \mathbf{U} \arrow[l, shift left, "F = \mathscr{H}\!\mathit{om}(-{,}\, \mathbf{2})"] \end{tikzcd};$$

here $G(\mathscr{D}) = \mathbf{Hom}(\mathscr{D}, \mathit{2})$ is the ultraspace of all homomorphisms (prime filters) $\mathscr{D} \to \mathit{2}$, a sub-ultraspace of $\mathbf{Hom}(D, 2)$ defined in the obvious way; similarly for F. The counit ε: $\mathrm{Id}_{\mathscr{L}} \to FG$ of this adjunction has

$$\varepsilon_{\mathscr{D}}\colon \quad \mathscr{D} \to \mathscr{H}\!\mathit{om}(\mathbf{Hom}(\mathscr{D}, \mathit{2}), \mathbf{2})$$

defined by the formula $\varepsilon_{\mathscr{D}}(d)(p) = p(d)$ $(d \in D,\ p \in \mathrm{Hom}(\mathscr{D}, \mathit{2}))$. The unit η: $\mathrm{Id}_{\mathbf{U}} \to GF$ is given by

$$\eta_{\mathbf{X}}(x)(d) = d(x) \qquad (\mathbf{X} \in |\mathbf{U}|,\ x \in X,\ d \in \mathrm{Hom}(\mathbf{X}, \mathbf{2})).$$

By imitating the proof of condition 4.2(ii) in Section 4, we can show that

$\varepsilon_{\mathscr{D}}$ is an isomorphism of d.l.'s for any d.l. $\mathscr{D}$. Here is a sketch of the proof that $\eta_{\mathbf{X}}$ is surjective for any u.s. $\mathbf{X}$.

Let p: $\mathscr{H}om(\mathbf{X}, \mathbf{2}) \to 2$ be given. Consider the set J of pairs $\delta = \langle d_1, d_2 \rangle$ of elements of Hom($\mathbf{X}$, $\mathbf{2}$) such that $p(d_1) = 1$ and $p(d_2) = 0$. Define the partial order $\leqslant$ on J by: $\delta' = \langle d'_1, d'_2 \rangle \leqslant \delta$ iff $d'_1 \leqslant d_1$ and $d'_2 \geqslant d_2$ (the latter $\leqslant$, $\geqslant$ refer to the ordering in $\mathscr{H}om(X, 2)$). It is easy to see that J is a lower semilattice with $\delta \wedge \delta' = \langle d_1 \wedge d'_1,\ d_2 \vee d'_2 \rangle$. Therefore, there is an ultrafilter V on the set J such that $[\delta] =_{\mathrm{df}} [\delta' \in J\colon \delta' \leqslant \delta]$ belongs to V for all $\delta \in J$. For every $\delta = \langle d_1, d_2 \rangle$ in J, there is $x_\delta \in X$ such that $d_1(x_\delta) = 1$ and $d_2(x_\delta) = 0$; otherwise we would have $d_1 \leqslant d_2$, contradicting $p(d_1) = 1$ and $p(d_2) = 0$. Consider $x = [V]\langle x_\delta \rangle = \prod_{\delta \in J} x_\delta / V$. We claim that $\eta_X(x) = p$, i.e., $p(d) = d(x)$ for all $d \in D$. Let, e.g., $p(d) = 1$; let $\delta_0 = \langle d, 0 \rangle \in J$; we have $d(x) = \prod_{\delta \in J} d(x_\delta)/V$. For all $\delta = \langle d_1, d_2 \rangle \in [\delta_0]$, we have $d_1(x_\delta) = 1$, hence by $d_1 \leqslant d$, $d(x_\delta) = 1$; it follows that $d(x) = 1$. We prove similarly that $p(d) = 0$ implies $d(x) = 0$.

η_X is not injective in general. Postulating the injectivity of η_X leads to the following "separation axiom." A u.s. X is called *totally disconnected* (t.d.) if whenever x, $y \in X$ and $x \not\leqslant y$, then there is d: $\mathbf{X} \to 2$ such that $d(x) = 1$ and $d(y) = 0$. It is easy to see that in a t.d. u.s., the ultraproduct operations are monotonic with respect to the ordering given. Let **TU** denote the category of all t.d. u.s.'s; **TU** is a full subcategory of **U**. It is easy to see that **Hom**($\mathscr{D}$, 2) defined above is t.d.; hence we have a pair of adjoint functors ($F \dashv G$),

$$\begin{tikzcd} \mathscr{L}^{\mathrm{op}} \arrow[r, shift left, "G = \mathbf{Hom}(-, 2)"] & \mathbf{TU}, \arrow[l, shift left, "F = \mathscr{H}om(-, \mathbf{2})"] \end{tikzcd}$$

the Stone adjunction for d.l.'s and t.d. u.s.'s. We have shown

PROPOSITION 8.1. *The Stone adjunction for distributive lattices and totally disconnected ultraspaces is an equivalence of categories.*

Although the idea behind the general concept of a Stone adjunction is very simple, we do not have a satisfactory general formulation of it. Such a general formulation should refer to the following situation. We are given a Cartesian closed s-category S; two $\mathscr{S}$-categories $\mathscr{L}$ and $\mathbf{U}$ of two kinds of "structures with underlying $\mathscr{S}$-objects," in particular we have forgetful $\mathscr{S}$-functors F_1: $\mathscr{L} \to \mathscr{S}$, F_2: $\mathbf{U} \to \mathbf{S}$; we have $\mathscr{A} \in |\mathscr{L}|$, $\mathbf{A} \in |\mathbf{U}|$ such that $F_1(\mathscr{A}) = F_2(\mathbf{A}) = A$. It is not clear to this author how to formulate satisfactory the hypothesis that "$\mathbf{A}$ commutes with $\mathscr{A}$." With this hypothesis somehow formulated, plus with certain assumptions (closure conditions?) on $\mathscr{L}$ and $\mathbf{U}$, we should be able to set up a pair of adjoint $\mathscr{S}$-functors $F \dashv G$,

$$\begin{tikzcd} \mathscr{L} \arrow[r, shift left, "G = \mathbf{Hom}(-, \mathscr{A})"] & \mathbf{U} \arrow[l, shift left, "F = \mathscr{H}om(-, \mathbf{A})"] \end{tikzcd}$$

so that the adjunction naturally derives from that in

$$\mathscr{S}^{\mathrm{op}} \underset{\mathrm{Hom}(-,A)}{\overset{\mathrm{Hom}(-,A)}{\rightleftarrows}} \mathscr{S}$$

This general concept successfully formulated should include the special case (with $\mathscr{S} = \mathscr{C}at$) we now turn to: the Stone adjunction of the 2-categories of pretoposes on the one hand and ultracategories on the other.

Note that, just as structures with underlying *sets* (classes) form categories, structures with underlying *categories* form 2-categories. In particular, we have the 2-category $\mathscr{P}\mathscr{T}$ of pretoposes, and the 2-category **UC** of ultracategories.

The objects (0-cells) of $\mathscr{P}\mathscr{T}$ are the pretoposes, the 1-cells are the elementary functors, the 2-cells arbitrary natural transformations between the latter; for $\mathscr{T}, \mathscr{T}' \in |\mathscr{P}\mathscr{T}|$, the category $\mathscr{P}\mathscr{T}(\mathscr{T}, \mathscr{T}')$ is the same as what we denoted by Hom $(\mathscr{T},\mathscr{T}')$ before. The rest of the 2-category structure of $\mathscr{P}\mathscr{T}$ are the obvious data. In particular, we have the forgetful 2-functor

$$F_1\colon\ \mathscr{P}\mathscr{T} \to \mathscr{C}at$$
$$\mathscr{T} \mapsto T$$

with $(F_1)_{\mathscr{T},\mathscr{T}'}\colon\ \mathscr{P}\mathscr{T}(\mathscr{T},\mathscr{T}') \to \mathrm{Hom}(T, T')$ being the full inclusion.

The objects of **UC** are the ultracategories, the 1-cells the ultrafunctors, and the 2-cells the ultratransformations; $\mathbf{UC}(\mathbf{K}, \mathbf{K}')$ is the same thing what we denoted by $\mathrm{Hom}(\mathbf{K}, \mathbf{K}')$ before. The composition structure of **UC** is partly described in the definition given at the end of Section 1; what we did there was to describe the object function of the functor

$$\circ_{\mathbf{K},\mathbf{K}',\mathbf{K}''}\colon\ \mathbf{UC}(\mathbf{K}, \mathbf{K}') \times \mathbf{UC}(\mathbf{K}', \mathbf{K}'') \longrightarrow \mathbf{UC}(\mathbf{K}, \mathbf{K}'').$$

The effect of this functor on ultratransformations is the usual "horizontal" composition of natural transformations. We have a forgetful 2-functor

$$F_2\colon\ \mathbf{UC} \to \mathscr{C}at$$
$$\mathbf{K} \mapsto K$$

with $(F_2)_{\mathbf{K},\mathbf{K}'}\colon\ \mathbf{UC}(\mathbf{K}, \mathbf{K}') \to \mathrm{Hom}(K, K')$ being what we described as a quasi-inclusion before.

We have 2-functors

$$G = \mathbf{Hom}(-, \mathscr{S}et)\colon\ \ \mathscr{P}\mathscr{T}^{\mathrm{op}} \to \mathbf{UC}$$
$$F = \mathscr{H}om(-, \mathbf{SET})\colon\ \ \mathbf{UC} \to \mathscr{P}\mathscr{T}^{\mathrm{op}};$$

the effect of these on objects and on 1-cells is given in (iv)(b), (v)(b), (iv)(a), and (v)(a) in Section 2, more precisely, in their Δ-subscripted versions mentioned in Section 3. The effect of these 2-functors on 2-cells is ordinary "horizontal composition."

We have 2-transformations

$$\eta\colon\quad \mathrm{Id}_{\mathbf{UC}} \to GF$$

$$\varepsilon\colon\quad \mathrm{Id}_{\mathscr{P}\mathscr{T}} \to FG,$$

where $\eta_{\mathbf{K}}$ is $e_{\mathbf{K}}$ defined in $(\mathrm{vi})_\Delta(\mathrm{a})$, and $\varepsilon_{\mathscr{T}}$ is $e_{\mathscr{T}}$ defined in $(\mathrm{vi})_\Delta(\mathrm{b})$ (in Sect. 2). We omit the verification of the fact that these data in fact define 2-transformations.

Finally, we claim that η and ε define an adjunction $F|G$ of the 2-functors F and G. To show this, we have to verify the identities

$$G(e_{\mathscr{T}}) \circ e_{G\mathscr{T}} = \mathrm{Id}_{G\mathscr{T}} \tag{8.2}$$

$$F(e_{\mathbf{K}}) \circ e_{F\mathbf{K}} = \mathrm{Id}_{F\mathbf{K}} \tag{8.3}$$

for $\mathscr{T} \in |\mathscr{P}\mathscr{T}|$, and $\mathbf{K} \in |\mathbf{UC}|$ (note the "opposite sign" on $\mathscr{P}\mathscr{T}$!).

Although the verification of these identities is completely straightforward, we will dwell on them to some extent. Because of a lack of a good general theory of Stone adjunctions, the verification has to be done by tedious computation. Some, but unfortunately not all, of the computation can be done by putting together the diagrams drawn in the various clauses (i)(a)–(vi)(b) in Section 2.

We will use the following abbreviations. (K, S), $(\mathbf{K}, \mathbf{S})$, $(i, \mathscr{S})$, etc, stand for $\mathrm{Hom}(K, S)$, $\mathrm{Hom}(\mathbf{K}, \mathbf{S})$, $\mathrm{Hom}(i, \mathscr{S})$, etc.; $[T, S]$ stands for $\mathbf{Hom}(T, S)$, etc.; $\{\mathbf{K}, \mathbf{S}\}$ stands for $\mathscr{H}\!om(\mathbf{K}, \mathbf{S})$, etc. We consider the diagram to prove (8.3),

$$\begin{tikzcd}
((\mathbf{K}, \mathbf{S}), S\rangle \\
(\{\mathbf{K}, \mathbf{S}\}, \mathscr{S}) \arrow[u, "i"']
\end{tikzcd}
\qquad
\begin{tikzcd}
(\mathbf{K}, \mathbf{S}) \arrow[r, "q_1(\mathrm{q.i.})"] & (K, S)
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=small]
 & & ③ = ((\mathbf{K}, \mathbf{S}), S), S) \arrow[rr, "((q_1, S), S)"] \arrow[dd, "(i, S)"'] \arrow[dr] & & (((K, S), S), S) = ④ \arrow[dl, "(\mathrm{ev}_K, S) = g'"] \\
 & ② = (\mathbf{K}, \mathbf{S}) \arrow[ur, "\mathrm{ev}_{(\mathbf{K}, \mathbf{S})}"] \arrow[rr, "q_2(\mathrm{q.i.})"] \arrow[dl, "e_{(\mathbf{K}, \mathbf{S})} = f'"] \arrow[dr] & & (K, S) = ⑤ \arrow[ur, "g = \mathrm{ev}_{(K, S)}"] & \\
① = ([\{\mathbf{K}, \mathbf{S}\}, \mathscr{S}], \mathbf{S}) \arrow[ur, "f = (e_{\mathbf{K}}, \mathbf{S})"] \arrow[rr, "q_3(\mathrm{q.i})"'] & & ((\{\mathbf{K}, \mathbf{S}\}, \mathscr{S}), S) = ⑥ \arrow[ur, "(e_K, S)"'] & &
\end{tikzcd}$$

The triangles ①, ②, ⑥ (with $f'\colon ② \to ①$) and ②, ③, ⑤ commute; they form (part of) the diagram drawn for (vi)(b) (Sect. 2) defining $f' = e_T$:

$T \to ([\mathcal{T}, \mathcal{S}], \mathbf{S})$ for $T = (\mathbf{K}, \mathbf{S}) = F(\mathbf{K})$. The triangles ③, ⑥, ⑤ and ③, ⑤, ④ (with g': ⑤ → ④) commute; they are obtained by applying the functor $(-, S)$ to (part of) the diagram drawn for (vi)(a) defining e_K: $K \to (\{\mathbf{K}, \mathbf{S}\}, \mathcal{S})$. The quadrangles ①, ②, ⑤, ⑥ (with f: ① → ②) and ②, ③, ④, ⑤ (with g: ⑤ → ④) commute as is seen directly. Finally, we have $g' \circ g = \mathrm{Id}_{(K,S)}$, as seen from the basic adjunction of the functors

$$\begin{tikzcd}\mathscr{C}\!at^{\mathrm{op}} \arrow[r, shift left, "{(-,S)}"] & \mathscr{C}\!at \arrow[l, shift left, "{(-,S)}"]\end{tikzcd}$$

Note that our object is to show that $f \circ f' = \mathrm{Id}_{(\mathbf{K},\mathbf{S})}$.

Using the facts listed, a simple diagram chase shows that $f \circ f' \circ q_2 = q_2$. This means that $f \circ f'$ and $\mathrm{Id}_{(\mathbf{K},\mathbf{S})}$ agree as far as the effects on the functor parts of the objects (ultrafunctors) of $(\mathbf{K}, \mathbf{S})$, as well as the effect on the morphisms in $(\mathbf{K}, \mathbf{S})$ are concerned.

It remains to show that the transition isomorphisms $[X', U]$ of $X' = (f \circ f')(X)$, for any X: $\mathbf{K} \to \mathbf{S}$, agree with $[X, U]$ (and hence $X' = X$).

Referring to the definition at the end of Section 1, we see that

$$[f(H), U] = ([H, U] \circ e') \circ (H \circ [e, U])$$

for $H \in |①|$, $e = e_{\mathbf{K}}$, U an ultrafilter on I. On the other hand, for any $X \in |②|$, $H = f'(X)$ is a strict ultrafunctor (see (i)(b), (ii)(b) and (vi)(b) for $\mathcal{T} = \{\mathbf{K}, \mathbf{S}\}$); therefore for $X' = (f \circ f')(X)$, we have

$$[X', U] = (f'(X))([e, U]).$$

By the definition of f', we have

$$(f'(X))([e, U]_{\langle M_i \rangle}) = ([e, U]_{\langle M_i \rangle})_X \qquad \text{for} \quad M_i \in |K| \quad (i \in I);$$

by (ii)(a) we see that

$$([e, U]_{\langle M_i \rangle})_X = [X, U]_{\langle M_i \rangle};$$

it follows that $[X', U]_{\langle M_i \rangle} = [X, U]_{\langle M_i \rangle}$ as desired.

This completes the verification of (8.3).

When verifying (8.2), we want the composite

$$\phi\colon \quad [\mathcal{T}, \mathcal{S}] \xrightarrow{e[\mathcal{T},\mathcal{S}]} [\{[\mathcal{T}, \mathcal{S}], \mathbf{S}\}, \mathcal{S}] \xrightarrow{[e_{\mathcal{T}},\mathcal{S}]} [\mathcal{T}, \mathcal{S}]$$

to be the identity. Both functors here are ultrafunctors. We leave it to the reader to verify that the composition of their functor parts is the identity functor; we show that the transition isomorphisms of the composite are identities as well.

By (v)(b), the ultrafunctor $[e_{\mathscr{T}}, \mathscr{S}]$ is strict. Hence, by the formula for the transition isomorphism of a composite (used above) we have for $M_i \in |(\mathscr{T}, \mathscr{S})|$ and $A \in |T|$ that

$$\begin{aligned}([\phi, U]_{\langle M_i \rangle})_A &= ((e_{\mathscr{T}}, \mathscr{S})([e_{[\mathscr{T}, \mathscr{S}]}, U]_{\langle M_i \rangle}))_A \\ &= ([e_{[\mathscr{T}, \mathscr{S}]}, U]_{\langle M_i \rangle})_{e_T(A)} \\ &= [e_{\mathscr{T}}(A), U]_{\langle M_i \rangle} = \mathrm{Id}_{\prod M_i(A)/U};\end{aligned}$$

the last equality because $e_{\mathscr{T}}(A)$ is strict.

This completes the verification of the equalities (8.2) and (8.3).

We thus have verified the Stone adjunction

$$\begin{tikzcd} \mathscr{P}\mathscr{T}^{\mathrm{op}} \arrow[r, shift left, "\mathbf{Hom}(-, \mathscr{S}et)"] & \mathbf{UC} \arrow[l, shift left, "\mathscr{H}om(-, \mathbf{SET})"] \end{tikzcd}$$

for pretoposes and ultracategories.

Suppose we have an adjunction (η, ε) of 2-functors

$$\begin{tikzcd} \mathscr{C} \arrow[r, shift left, "G"] & \mathscr{D} \arrow[l, shift left, "F"] \end{tikzcd}$$

and suppose that we have distinguished a class of objects of $\mathscr{C}$, the "small" objects of $\mathscr{C}$; in the application, $\mathscr{C} = \mathscr{P}\mathscr{T}^{\mathrm{op}}$ and its "small" objects are the small pretoposes. We call the given adjunction a *reflection in the small* if ε_A is an equivalence (in the sense of the 2-category $\mathscr{C}$) for all small objects A of $\mathscr{C}$.

In the Stone adjunction for pretoposes and (canonical) ultracategories, $\varepsilon_{\mathscr{T}} (\mathscr{T} \in |\mathscr{P}\mathscr{T}|)$ is the functor e_T of Theorem 4.1 (Theorem 5.2). To say that e_T is an equivalence in the sense of the 2-category $\mathscr{P}\mathscr{T}$ is clearly the same as to say that e_T is an equivalence of categories in the usual sense. Therefore, Theorem 4.1 (Theorem 5.2) implies

THEOREM 8.2. *The Stone adjunction for pretoposes and (canonical) ultracategories is a reflection in the small.*

Proposition 7.3 now implies

COROLLARY 8.3. *The Stone adjunction for pretoposes and ultracategories induces an equivalence between* Hom$(\mathscr{T}, \mathscr{T}')$, *the category of elementary functors* $\mathscr{T} \to \mathscr{T}'$, *and* Hom(Mod T', Mod T), *the category of ultrafunctors and ultratransformations from* Mod T' *to* Mod T, *for any small pretoposes* $T = \mathscr{T}$, *and* $T' = \mathscr{T}'$. ▮

LEMMA 8.4. *An ultrafunctor* X: $\mathbf{K} \to \mathbf{K}'$ *is an equivalence in the sense of* *the 2-category* **UC** *if and only if the functor-part of* X *is an equivalence of categories.*

Proof. The "only if" part is obvious. Assume that the functor-part of X, also denoted by X, is an equivalence of categories. Therefore, there are a functor $Y\colon \mathbf{K}' \to \mathbf{K}$ and isomorphisms $\phi\colon YX \to^{\cong} \mathrm{Id}_K$, $\psi\colon XY \to^{\cong} \mathrm{Id}_{K'}$ such that $X\phi = \psi X$ (see Theorem 1, p. 91 in [CWM]). Let (I, U) be an ultrafilter; for simplicity we abbreviate both $[U]_{\mathbf{K}}$ and $[U]_{\mathbf{K}'}$ by U; we denote the transition isomorphism $[X, U]\colon XU \to^{\cong} UX^I$ by ν. There is a unique isomorphism $\mu_U = \mu\colon YU \to^{\cong} UY^I$ such that the diagram of functors

$$\begin{tikzcd}
XYU \arrow[r, "X\mu"] \arrow[dr, "\psi U"'] & XUY^I \arrow[r, "\nu Y^I"] & UX^IY^I \arrow[dl, "U\psi^I"] \\
 & U &
\end{tikzcd} \tag{8.4}$$

commutes; namely, this diagram defines $X\mu$ uniquely as an isomorphism; since X is full and faithful, μ is uniquely determined too. We make Y into a pre-ultrafunctor by specifying $[Y, U] = \mu_U$ for all ultrafilters U. The commutativity of (8.4) says precisely that ψ is an ultratransformation

$$\psi\colon \quad XY \longrightarrow 1_{\mathbf{K}'},$$

hence an isomorphism in the category Hom(**K**, **K**). We *claim* that ϕ is an ultratransformation

$$\phi\colon \quad YX \longrightarrow 1_{\mathbf{K}},$$

hence an isomorphism in Hom(**K**, **K**). Consider the diagram

$$\begin{tikzcd}
 & XYUX^I \arrow[d, "\psi UX^I"'] \arrow[drr, "X\mu X^I"] & ① & \\
XYXU \arrow[ur, "XY\nu"] \arrow[dr, "X\phi U = \psi XU"'] \arrow[r, phantom, "③"] & UX^I & UX^IY^IX^I \arrow[l, "U\psi^IX^I", "= UX^I\phi^I"'] & XUY^IX^I \arrow[l, "\nu Y^IX^I"'] \arrow[dll, "XU\phi^I"] \\
 & XU \arrow[u, "\nu"] & ② &
\end{tikzcd}$$

Part ① commutes since it is obtained from (8.4) by post-composing with X^I. Part ② commutes by the naturality of ν. Part ③ commutes by the naturality of ψ. It follows that the outer quadrangle commutes. The diagram (similar to (8.4)) whose commutativity expresses that ϕ is an ultratransformation gives rise, upon pre-composing with X, to the mentioned outer quadrangle. Since X is faithful, the assertion of our claim follows.

Since the p-.u.f. XY is isomorphic to the u.f. $l_{K'}$, by Lemma 3.1 it follows that XY is a u.f. Using that X is faithful, now it is easy to show that Y is a u.f.

The isomorphisms ψ and ϕ show that X is an equivalence in the 2-category **UC** with "quasi-inverse" Y. ▮

Lemma 8.5. *Let* G: $\mathscr{C} \to \mathscr{D}$ *be a* 2*-functor, and assume that for any "small" objects* A, B of $\mathscr{C}$, $G_{A,B}$: $\mathscr{C}(A, B) \to \mathscr{D}(GA, GB)$ *is an equivalence of categories. Then, for "small"* A *and* B *in* $\mathscr{C}$, *a* 1*-cell* δ: $GA \to GB$ *is an equivalence* (*in* $\mathscr{D}$) *iff for some* (*equivalently*: *every*) 1*-cell* γ: $A \to B$ *such that* $G_{A,B}(\gamma) \cong \delta$, γ *is an equivalence* (*in* $\mathscr{C}$).

The easy proof is omitted.

Corollary 8.6. *The small pretoposes* T, T' *are equivalent categories if and only if there is an ultrafunctor from* Mod T' *to* Mod T *whose functor part is an equivalence of categories.*

Proof. Immediate by 8.3, 8.4, and 8.5. ▮

We also note that the "comparison theorem," Theorem 7.1.8 in [MR], is an immediate consequence of the preceding; this theorem says that if an elementary functor F: $T \to T'$ between pretoposes induces an equivalence Mod $T' \to$ Mod T, then F itself is an equivalence.

Finally, we comment on the fact that the Stone adjunction for pretoposes and ultracategories fails to be a duality, i.e., an equivalence of categories.

One aspect of the Stone adjunction failing to be an equivalence is the fact that the proof of Theorem 4.1 goes through with various "sub-ultracategories" of **Mod** T replacing **Mod** T. For one thing, one can take a sub-family of the ultraproduct functors so that at each time the ultrafilter existence theorem was used, one can in fact take a required ultrafilter in the given subfamily; it is easy to see (by inspecting the proof) that quite restricted subfamilies of the family of all ultrafilters would suffice. On the other hand, one can restrict the models considered by taking a full subcategory of Mod T closed under ultraproducts and containing at least one model of every complete extension of T.

A "perfect duality theory" requires the replacement of **UC** by the full sub-2-category **UC**$'$ whose objects are (isomorphic) to **Mod** T, for some $T \in |\mathscr{P}\mathscr{T}|$; of course, to make this interesting one would have to find an "abstract" definition of **UC**$'$.

For material concerning adjunctions giving rise to perfect dualities, see [LR].

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