A skeptical history of numbers

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Number theory

Fruit Counting

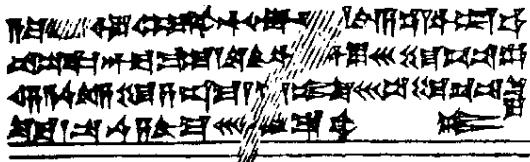| 4 | 3 | 10 | 6 |

Whole numbers and so on

$$\mathbb{N} = \{0, 1, 2, 3, \ldots\}$$

$$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, 3, \ldots\}$$

$$\mathbb{Q} = \{22/7, 94/100, -2/3, 47/50, \ldots\}$$

Linear equations: ax + b = 0

Algebra

Solve   a x^2 + b x + c = 0.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

820 AD
Muḥammad ibn Mūsā al-Khwārizmī   *(algoritmi)*

Diophantus 210 AD

Al-kitāb al-mukhtaṣar fī ḥisāb al-ǧabr wa'l-muqābala

DIOPHANTI
ALEXANDRINI
ARITHMETICORVM
LIBRI SEX.

Solving the quadratic, circa 2000 BC

Solving the cubic, circa 1500 AD

REGVLA.

Deducito tertiam partem numeri rerum ad cubum, cui addes quadratum dimidij numeri æquationis, & totius accipe radicem, scili cet quadratam, quam seminabis, unicaq; dimidium numeri quod iam in se duxeras, adijcies, ab altera dimidium idem minus, habebisq; Bi nomium cum sua Apotome, inde detracta ℞ cubica Apotomæ ex ℞ cubica sui Binomij, residuũ quod ex hoc relinquitur, est rei æstimatio. Exemplum. cubus & 6 positiones, æquantur 20, ducito 2, tertiam partem 6, ad cubum, fit 8, duc 10 dimidium numeri in se, fit 100, iunge 100 & 8, fit 108, accipe radi cem quæ est ℞ 108, & eam geminabis, alte ri'addes 10, dimidium numeri, ab altero mi nues tantundem, habebis Binomiũ ℞ 108 p:10, & Apotomen ℞ 108 m:10, horum accipe ℞ᵘ cub᷑ & minue illam quæ est Apo tomæ, ab ea quæ est Binomij, habebis rei æstimationem, ℞ v: cub: ℞ 108 p:10 m:℞ v: cubica ℞ 108 m:10.

cub᷑ p:6 reb᷑ æqlis 20
2
8 ———— 20
8 ———— 10
108
℞:108 p:10
℞:108 m:10
℞v: cu.℞:108 p:10
m:℞ v:cu.℞:108 m:10

Solving the quartic, circa 1500 AD

DEMONSTRATIO.

Sit quadratum A F, divisum in duo quadrata A D & D F, & duo supplementa D C & D E, & uelim addere gnomonem K P G circũcirca, ut remaneat quadratum totum A H, dico quod talis gnomo, constabit ex duplo G C additæ lineæ, in C A, cum quadrato G C, nam F G constat æqualis G C in C F, ex diffinitione data in initio secundi elementorum, et c F est æqualis C A, ex diffinitione quadrati, & per 44ᵘᵐ primi elemen torum, K F est æqualis F G, igitur duæ superficies G F & F K, constant ex G C, in duplum C A, & quadratũ G C est F E, ex corrolario quartæ se cundi elementorũ, igitur patet pro positum, si igitur A D sit 1 q̃d᷑q̃d᷑ & C D A C D E, 3 quadrata, & D F 9, erunt B A 1 quadratum, & B C 3 nes cessario, cum igitur uoluerimus ad dere q̃drata aliqua, ad D C & D E, & fuerint C L & K M, erit ad cõplen dum quadratum totum necessaria superficies L N M, quæ ut demon stratum est, constat ex quadrato G C numeri quadratorum dimidiati,

Irrational numbers

$$\overline{\mathbb{Q}} = \{\sqrt{2}, 5^{2/17}, \sqrt[3]{5} + \sqrt[5]{3}, \ldots\}$$

$$x^2 = 2 \qquad x = \sqrt{2}$$

$$x^3 = x + 1$$

$$x = \frac{\sqrt[3]{9 - \sqrt{69}} + \sqrt[3]{9 + \sqrt{69}}}{\sqrt[3]{18}}$$

Quintic polynomials

$$x^5 = x + 1?$$

Abel: Cannot be expressed in terms of nth roots and whole numbers.

Solving the quintic, circa 2000 AD

(Doyle-M)
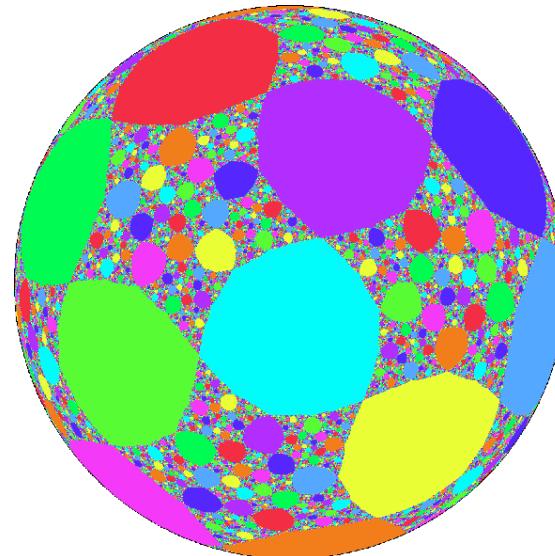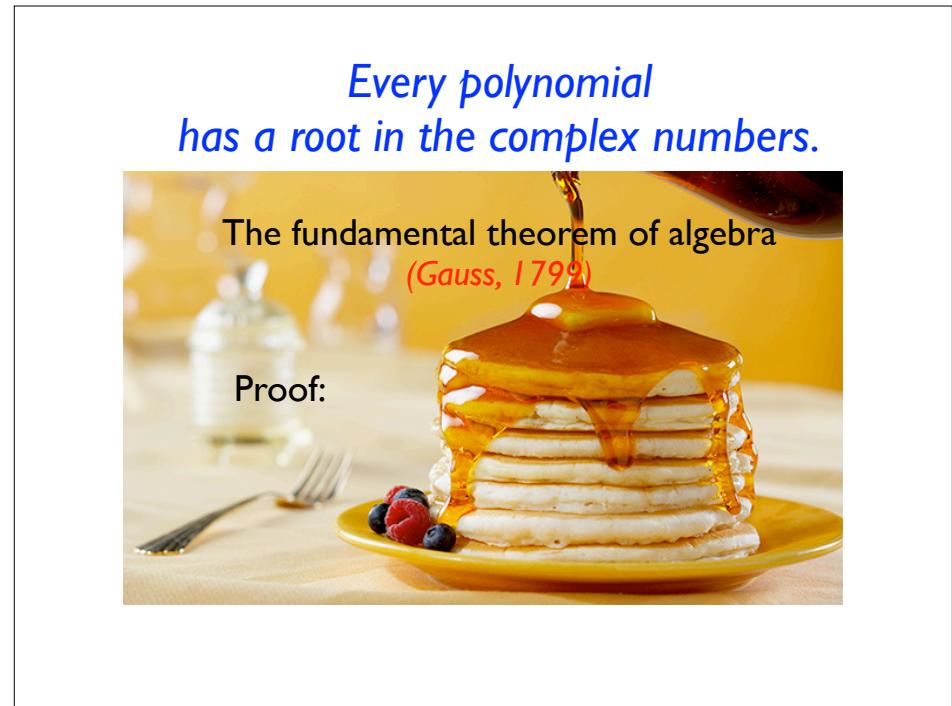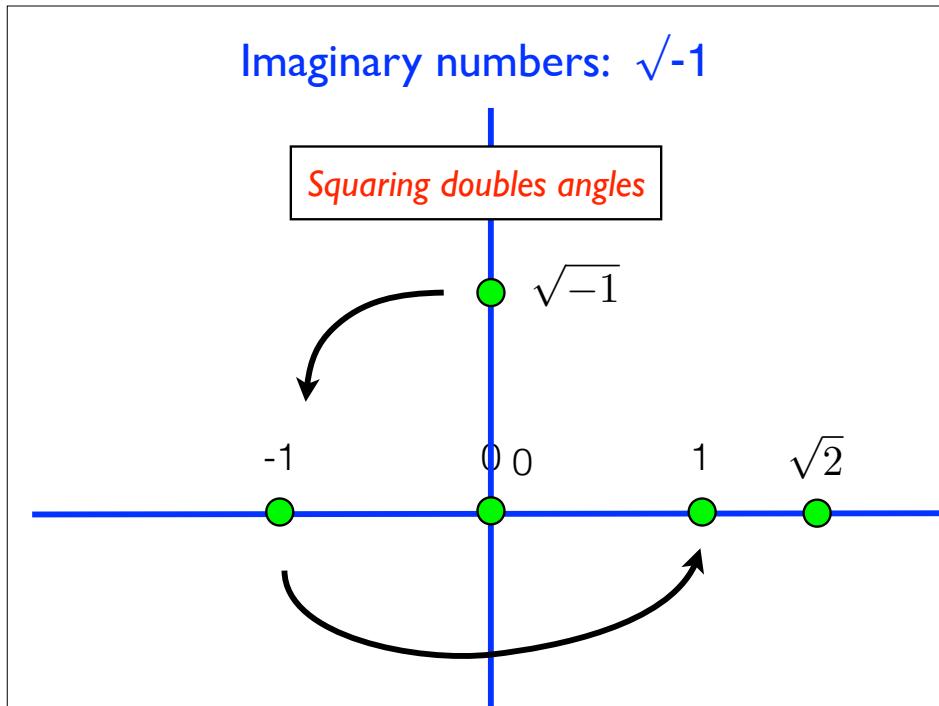

Quintic polynomials

$$x^5 = x + 1?$$

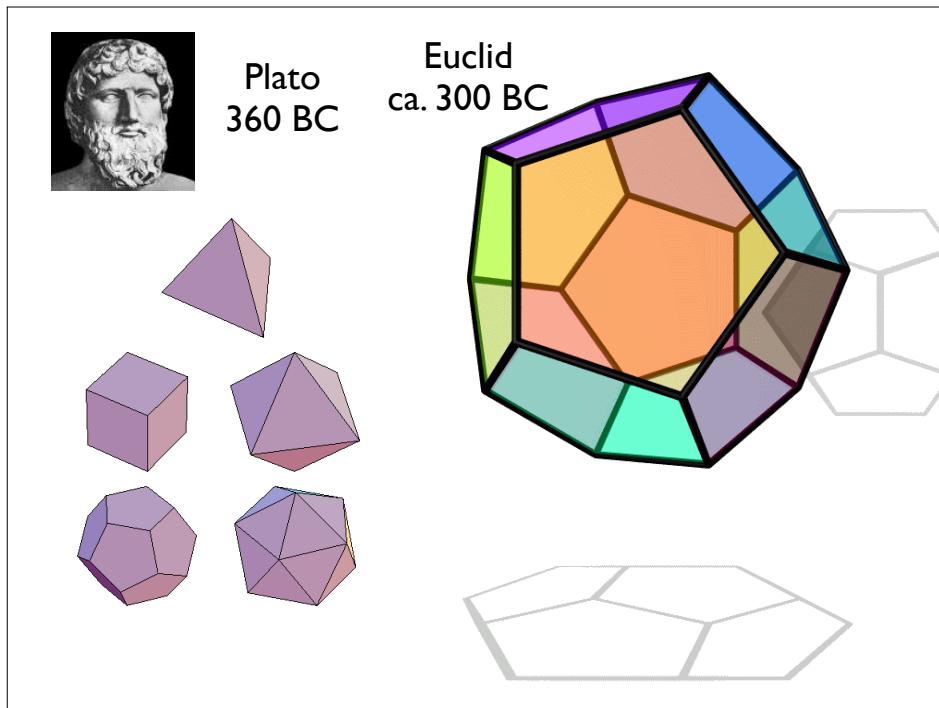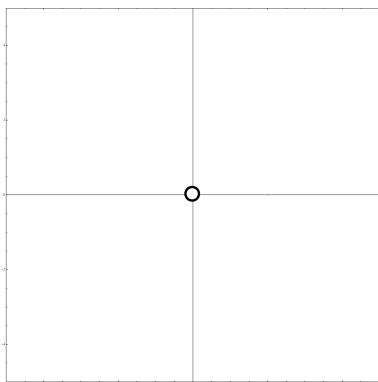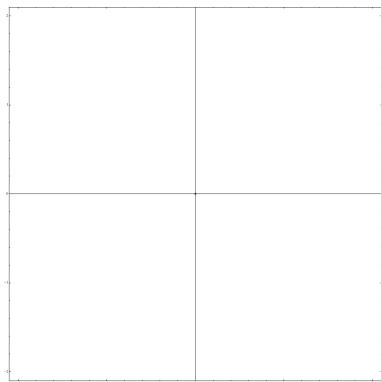$$x = 1.1673039782614186843....$$

What kind of number is this?

Geometry

Plato
360 BC

Euclid
ca. 300 BC

$\sqrt{2}$ Real numbers π

$\mathbb{R}$

$\pi = 3.1415926535897....$

the continuum

-3 -2 -1 0 1 2 3 4 5 6 7 8 9 10 11 12 13 14

Imaginary numbers: $\sqrt{-1}$

Squaring doubles angles

$\sqrt{-1}$

-1   0 0   1 $\sqrt{2}$

*Every polynomial
has a root in the complex numbers.*

The fundamental theorem of algebra
(Gauss, 1799)

Proof:

Slide 1 (top-left)

To solve: $P(z) = 0$

Look at: $z \mapsto P(z)$

Slide 2 (top-right)

Whole number equations

$$X^2 + Y^2 = Z^2 \qquad 5^2 + 12^2 = 13^2$$

$$X^n + Y^n = Z^n \qquad 0^n + 1^n = 1^n$$

$$Y^2 = X^3 - 2 \qquad 5^2 = 3^3 - 2$$

$$\left(\frac{113259286337279}{449455096000}\right)^2 = \left(\frac{2340922881}{58675600}\right)^3 - 2$$

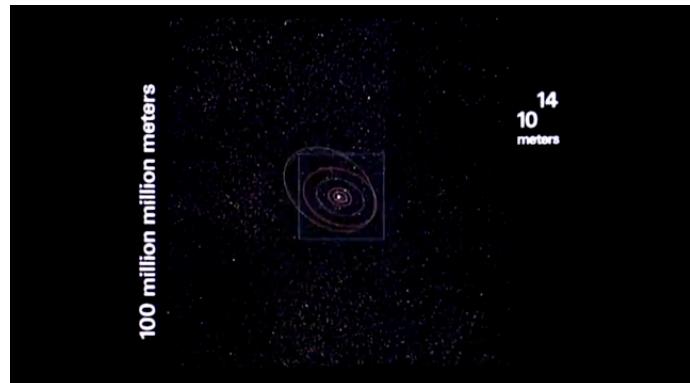## Slide 3 (bottom-left)

Large Numbers

MMMDCCCLXXXVIII = 3,888

250 BC: Archimedes: *The Sand Reckoner*

myriad = 10,000
myriads of myriads of ...
estimated 10^{63} grains of sand to fill the universe.

Slide 4 (bottom-right)

Powers of 10

100 million million meters

10^{14} meters

Charles and Ray Eames, 1968 / 1977

Towers

$T(1) = 10$

$T(2) = 10^{10}$ = 10 billion

Googol = 10^{100} = 10,[...100 zeros]...000

             >> atoms in observable Universe

$T(3) = 10^{(10^{10})}$ = 10,000,...[10 billion zeros]...000

$T(4) = 10^{10^{10^{10}}}$ << Skewes' number = $10^{10^{10^{34}}}$

     = bound for when first π(x) > li(x)   *1933*

$\ll T(5), \ldots$

Wowsers

the untamed power of induction!
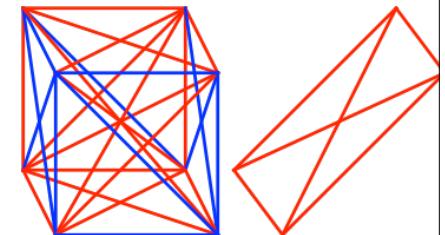
W(1) = 10

W(2) = T(W(1)) = tower of height 10

W(3) = tower of height W(2)

<<< Graham's number G

    12 < N < G

       1977

= size of our ignorance

Busy beaver function

B(n) =

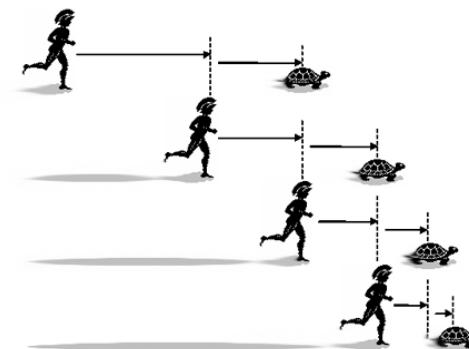largest possible output of a rogue but mortal computer program of length n

CONTROL DATA 166 Line Printer

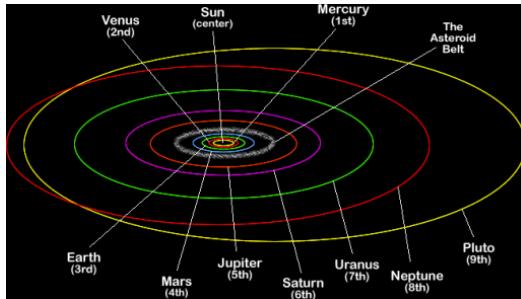*Is this number defined?*

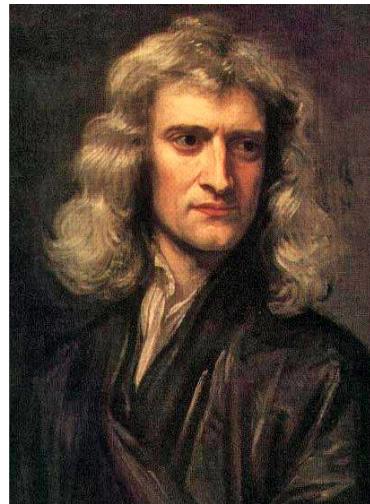## Paradox of Infinity

Zeno
430 BC

Infinitesimals

$$\dot{P} \;=\; \frac{dP}{dt} \;=\; \frac{P(t+\epsilon) - P(t)}{\epsilon}$$
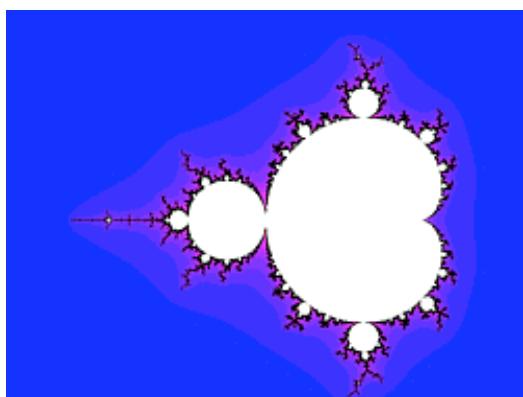
All Calculus, Physical laws

Newton 1689

And what are these same evanescent increments? They are neither finite quantities, nor quantities infinitely small, nor yet nothing. May we not call them the ghosts of departed quantities?

—George Berkeley

for every $\epsilon > 0$ there exists a δ....

Infinity

$$\mathbb{N} = \{0,1,2,3,4,....\}$$

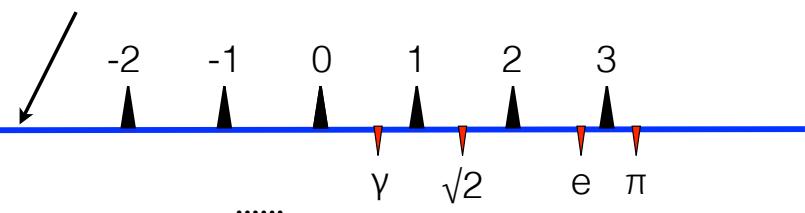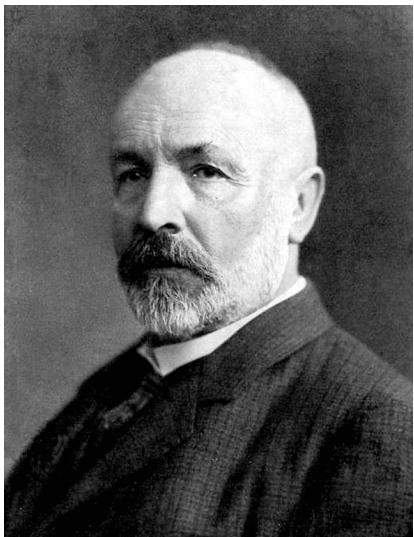## Many infinities

$|\mathbb{N}|$ $\{0,1,2,3,...\}$ the number of possible books

is smaller than

$|\mathbb{R}|$ {all real numbers}   the number of points in a line (or cube or...)

the silent majority

-2   -1   0   1   2   3

γ   √2   e   π

......

Set theory

Georg Cantor
1845-1918

"No one shall expel us from the Paradise that Cantor has created."
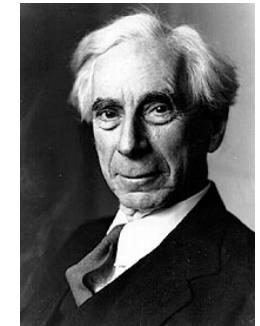
David Hilbert

Frege and Russell   *1903*

BEGRIFFSSCHRIFT,

EINE DER ARITHMETISCHEN NACHGEBILDETE

FORMELSPRACHE

DES REINEN DENKENS.

VON

D^r. GOTTLOB FREGE.

PRIVATDOCENTEN DER MATHEMATIK AN DER UNIVERSITÄT JENA.

"Hardly anything more unfortunate can befall a scientific writer than to have one of the foundations of his edifice shaken after the work is finished."
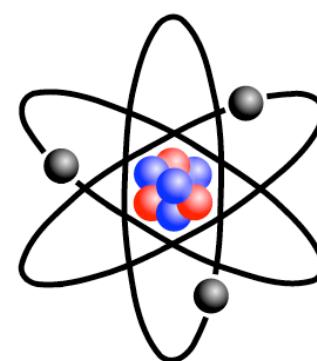

Crisis!

Berry's number

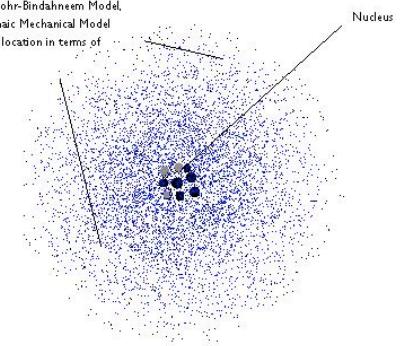N =   [the smallest positive integer not definable
          in fewer than twelve words]

Russell's paradox

Let A = {all sets which are not members of themselves}.
          Is A a member of A?

Picture of the atom

Electrons –
Contrary to the Bohr-Bindahneem Model, the Quantum-Lemaic Mechanical Model describes electron location in terms of probabilty

Nucleus

20th century revolutions

Absolute space	Relativity
Solar system atom	Quantum atom
Determinism	Uncertainty
Positivism	Existentialism

Foundational Crisis: Solutions(?)

(1) Be careful not to define A in terms of A. (Type theory)

(2) Only deal with things you can construct. (Intuitionism)

(3) Agree on Axioms, and only admit conclusions from them.

The Dust Settles

Hilbert 1930

For us there is no *ignorabimus*, and in my opinion none whatever in natural science.

Wir müssen wissen — wir werden wissen!

Gödel 1931

Mathematics is, and will always be, incomplete.

A: Refused to accept the uncertainties of quantum mechanics (God playing dice)

B: Established that mathematics will never be complete.

Incompleteness:
Some questions have no answers

Are there infinitely many Mersenne primes $p = 2^n - 1$?

Is there a set A with $|\mathbb{N}| < |A| < |\mathbb{R}|$?

Is the dynamical system $x \Rightarrow x^2 - c$ chaotic
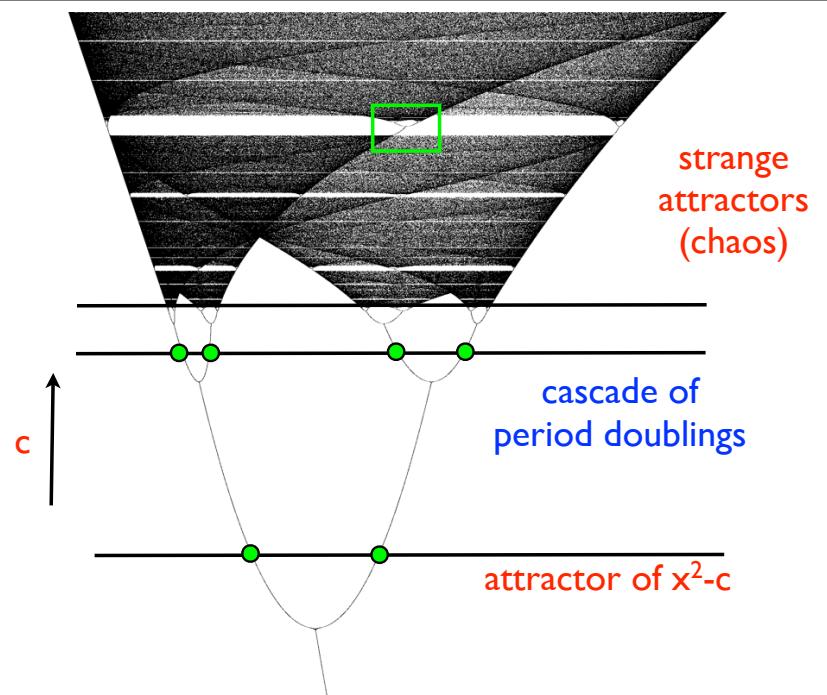for $c = 1.5$?

What is chaos?
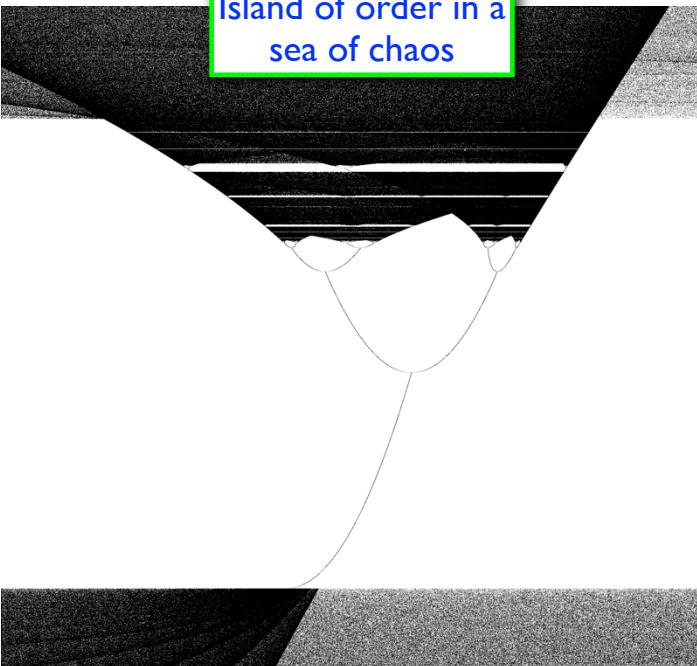

Quadratic dynamics

$$x_{n+1} = x_n^2 - c$$

$$x_0 = 0$$

$0 \Rightarrow 0 \Rightarrow 0 \Rightarrow 0...$ (c=0)

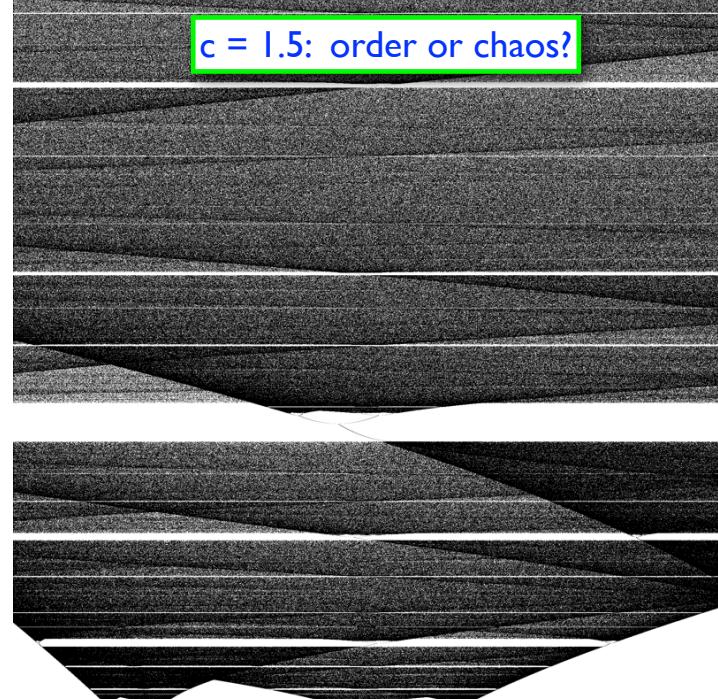$0 \Rightarrow -1 \Rightarrow 0 \Rightarrow -1....$ (c=1)

$0 \Rightarrow -3 \Rightarrow 6 \Rightarrow 33 ....$ (c=3)

---

strange attractors (chaos)

c

cascade of period doublings

attractor of $x^2 - c$

Island of order in a sea of chaos

c = 1.5: order or chaos?

← c = 1.5

Is mathematics consistent?

0=1?!

Deductions

Axioms

But the integers exist! ⇒ Arithmetic is consistent

Kronecker, 1865

Die ganzen Zahlen hat der liebe Gott gemacht, alles andere ist Menschenwerk

"God made the integers, all else is the work of man."

Nelson, 2010

The notion of the actual infinity of all numbers is a product of human imagination; the story is simply made up.

Consistency radius

Contradictions: at what scale?

N = Length of the shortest proof that 0=1.

Gödel:
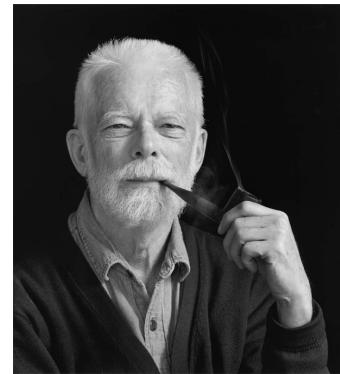*We can assume N is finite without danger!*

``Healthy skepticism''

Non-standard numbers

[0, 1, 2, ..., n, n+1, ..., N-1, N, N+1 ....]

standard   *non-standard*

A. Everything that used to be true is still true.

B. 0 is standard

C. n standard ⇒ n+1 standard

D. There exists a nonstandard N

Edward Nelson, 1932-2014

Virtues of non-standard numbers

Newton rehabilitated   Cantor deprecated

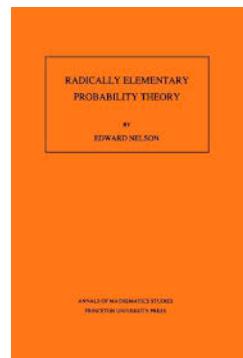Analysis simplified

N non-standard

ε replaced by 1/N

⇒ working theory
of infinitesimals
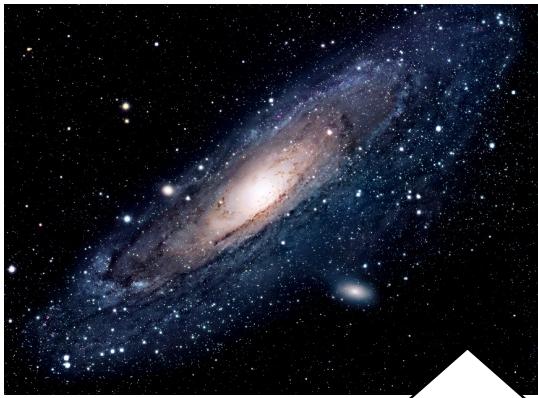
∞ replaced by N
⇒
avoids measure theory

RADICALLY ELEMENTARY
PROBABILITY THEORY

BY
EDWARD NELSON

ANNALS OF MATHEMATICS STUDIES
PRINCETON UNIVERSITY PRESS

The vacuum is not empty

nanoparticle

$\Delta E \Delta t = \hbar$

vacuum fluctuations of virtual particle-antiparticle pairs
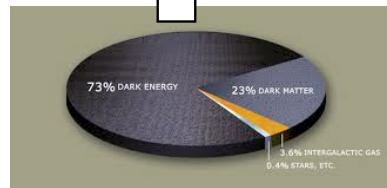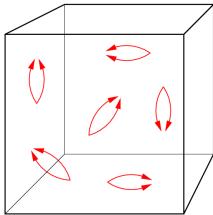
70% of the Universe is made up of inconsistencies

Dark Energy

Mathematics is a model

What image of mathematics fits best
with the world as we now know it?

$$10^{10^{10^{10^{10^{10^{10^{10}}}}}}}$$