PURE DATA FOUNDATIONS OF MATHEMATICS

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Abstract

This is an abstract.

1 Introduction

2 Foundation

Any system of reasoning must necessarily have at least one foundational concept which is understood before a first definition. In our case, the foundational concept is the *finite sequence*. We assume that finite sequences and obvious operations on finite sequences are understood. On the other hand, logic, logical values and predicates (foundational in ZFC) and types (foundational in type theories) will be defined in terms of finite sequences. Taking finite sequences as understood, define 'data' and 'coda' by the following.

- *Data* is a finite sequence of *codas*, and a *coda* is a pair of *data*.

Notationally speaking, if A and B are data, $A\ B$ denotes the concatenation of A and B as finite sequences and $A : B$ denotes the data consisting of the single coda formed by pairing A and B. By the definition, for example, the empty sequence of codas () is data, and, therefore, so is the pairing of two empty sequences '(:)' and so is this sequence of three codas (:)(:)((:) : (:)). We shall refer to this as "pure data" since it is data "made from finite sequences of nothing." We shall see that both data in the ordinary sense (bits, bytes, language expressions) and mathematical concepts such as variables, functions, definitions, logical values, categories, morphisms, types and theorems naturally appear as varieties of pure data with simple definitions expressed in the small algebra of data with the concatenation and the colon operations. Notationally speaking, the colon operation binds from the right first and binds less strongly than concatenation, so that $A : B : C$ means $(A : (B : C))$ and $A : B\ C$ means $(A : (B\ C))$.

In coda, the answers to mathematical questions come from an equivalence relation =, which is defined by a partial function from codas to data called a *context*. Given a context δ, equality is defined by

$$A\ B = \delta(A)\ B = A\ \delta(B) \tag{1}$$

$$A : B = \delta(A) : B = A : \delta(B) \tag{2}$$

for data A and B where the partial function δ has been extended to a function from data to data with identities. Conceptually, relations are assumed until δ is an identity on data. With a given context, data A is *empty* if $A = ()$ and *invariant* if $\delta(A) = A$.

In coda, new definitions are added to a context as partial functions from codas to data. We use a convention to guarantee that each such partial function has it's own disjoint domain. Let the *domain* of a coda $A : B$ be the data consisting of the first coda in the sequence A and the empty sequence if A is empty. A partial function mapping the all of the codas with a particular invariant domain to data is called a *definition.* Definitions can be added to a valid context if they do not clash with previous definitions.

THE AXIOM OF DEFINITION. *The empty context is valid. If δ is a valid context and d is a definition, and if no coda is in the domain of both δ and d, then the union of δ and d is a valid context.*

As an axiomatic system, coda is complete at this point. This is quite a contrast from, for instance, ZFC which has ten axioms even assuming predicate logic. The reason for the simplification will become clear in the sections which follow. In section 4, we show that coda already has an internal logic implicit in the structure as defined. We adopt this logic for reasoning within coda, so we do not need axioms related to propositional or predicate logic. In section 5, we show that coda also contains an internal language which can also be defined just as with any other definition. The language has the unusual property that all byte sequences are valid syntax, so language and language syntax axioms are also not needed. Axioms specifying valid "rules of deduction" are also not needed because deduction, proof, and computation in coda are all determined by the above data equality. Each of these is a merely a data sequence $A_0 = A_1 = A_2 = \cdots = A_n$, deducing A_n from A_0, computing A_n starting with A_0 or proving that $A_0 = A_n$, depending on one's point of view. Unlike type systems with a Curry-Howard correspondence, proof and computation are the same thing in coda.

In the sections that follow, "data" and "pure data" are interchangeable terms and "sequence" will always mean a finite sequence unless otherwise indicated. Since coda only has one axiom, we may refer to the Axiom of Definition as just "the axiom."

3 Definitions

As specified in the Axiom of Definition, a definition is specified by a) choosing invariant data to be the domain of the definition and b) specifying a function from codas with that domain to pure data. For example, we can define that any coda with domain () maps to itself. This establishes that (: B) as an atom for any data B, and if B is invariant, then so is the coda, so (:), for instance, is invariant and can be the domain of more definitions. Continuing in this fashion, we define codas with domain (:) to be "bits", so (:) : is the 0-bit, (:) : (:) is the 1-bit. Codas with the 0-bit as domain are conventionally bit sequences and codas with domain 1-bit are conventionally byte sequences. This makes byte strings available as invariant names for other definitions. For instance, by **(pass A:B)$\rightarrow$ B** we mean a definition so that any coda with domain "pass" maps (pass A:B) to B. If "pass" has not already been used, this definition may be added to context. It can be helpful to think of **pass** as a "command" with "input" B and "argument" A. Typical examples are shown in table X. In addition to the approximately 50, very simple, pre-defined built in definitions, coda includes a definition (def A : B) which lets one add new definitions to a context provided that A is invariant and not already used in the context, according to the Axiom of Definition. A particularly important and typical family of definitions are the *applications* such as **ap** in Table 2. These are simple combinatorial operations on sequences. Taking finite sequences as understood, we presume that descriptions such as those in Table 2 are sufficient for a clear understanding of the meaning of these sorts of definitions. Of course, further description and built in examples are available with the software, and the source code implementation of these is also always short and readable.

Domain	Description	Action
pass	Pass input unchanged	(pass A:B)$\rightarrow$ B
null	Empty data for any input	(null A:B)$\rightarrow$ ()
rev	Reverse the order of data	(rev : B)$\rightarrow$ (), if B is empty (rev : b)$\rightarrow$ b, if b is an atom (rev : B C)$\rightarrow$ (rev:B) (rev:A)
if	Conditional B	(if A:B) $\rightarrow$ B, if B is empty (if A:B) $\rightarrow$ (), if B is atomic.
ap	Apply argument A to each input	(ap A : B)$\rightarrow$ (), if B is empty (ap A : b)$\rightarrow$ A:b, if b is an atom (ap A : B C)$\rightarrow$ (ap A:B) (ap A:C)
nat	The natural numbers	(nat : n)$\rightarrow$ n (nat : $n+1$)
def	Make a new definition	(def A:B)$\rightarrow$ add definition (A A':B')$\rightarrow$ B.

Table 1: *Typical definitions in coda. Each definition is a partial function from codas to data defined on codas with a specified domain. When multiple actions are listed, the total action is defined by the first action where the left hand side pattern applies.*

Schematic Coda	$\rightarrow$	Schematic Result
ap A:b_1 b_2 b_3	$\rightarrow$	(A:b_1) (A:b_2) (A:b_3)
app a_1 a_2 a_3:B	$\rightarrow$	(a_1:B) (a_2:B) (a_3:B)
ap2 a a_1 a_2 a_3:B	$\rightarrow$	(a a_1:b_1) (a a_2:b_2) (a a_3:b_3)
aps A:b_1 b_2 b_3 b_4	$\rightarrow$	(A b_1:(A b_2 : (A b_3:b_4)))
apif A:b_1 b_2...	$\rightarrow$	b_1 (...if (A:b_1) is empty) b_2 (...if (A:b_2) is empty) ...

Table 2: *Typical combinatoric definitions combine data A and B as sequences.*

4 Logic

The previous sections can be simply summarized. We have recursively defined “pure data” to be finite sequences of pairs of pure data called codas. Given such data, any partial function from coda to data qualifies as a “definition.” Under the conditions of the Axiom of Definition, definitions can be added to a “context” which defines an equality on data and meaning within the system. These simple definitions and, for instance, the combinatorial examples in figure 1 are meant to be sufficiently understood based on the foundational concept of the **finite sequence** without the need for logical axioms or logical values. We presume, for instance, that reversing the order of a finite sequence is understood without making a choice between, say, classical and intuitionistic logic.

In the framework of coda, all mathematical objects (values, functions, variables, morphisms, theorems,...) are meant to be merely pure data. This means that “propositions” will appear in the form:

Is data A equal to data B?

Since equality itself is available as a definition, then this question is also the specific data $A = B$ or $(= A : B)$. But if a question like “Is data A equal to data B?” is data, it is natural to expect that the answer to the question should be encoded in the specific data $(= A : B)$. This suggests that “logic” in coda should appear as a natural coarse classification of data in general. The classification should match our intuition and ordinary reasoning and should be stable under adding new definitions to a given context. A suitable classification is indeed available.

Given a particular context, data A is said to be *atomic* if one or more of the codas in it’s sequence is invariant. Note that since being empty and being atomic are disjoint categories which are preserved when adding new definitions to a context, this classification is an interesting candidate:

- data is *true* if it is empty, *false* if it is atomic and *undecided* otherwise.

although true data is “always true” and false data is “always false,” [always], undecided data may become true or false with added definitions. In this sense, undecided data are like “variables” which may receive a true or false value depending on future choices. This is not quite the correct picture, however, because some undecided data actually remain undecided independent of any future added definitions. Such *undecidable* data occurs, for example, with the Godel phenomenon as we shall see in section Y. The fact that coda logic is ”almost” two-valued logic is from the duality between *true* data () and *false* data (:) visible in any display of pure data. Although proposing to change something as fundamental as logic can be disorienting, one soon realizes that the situation is still quite familiar and that the seemingly odd “undecided” category works perfectly. Examples illustrate the flavor of these definitions assuming a context where ”a”, ”b”, ”c” and ”foo” and ”bar” are not the domains of a definition in context:

True	(), (**pass**:), **null** : a b c, (**and**:), (**or** a:), (**xor** : a)
False	a b c, **first** 3 : a b (foo:bar), (**and** a:b), (**or** a:b)
Undecided	foo:bar, **pass**:foo:bar, **last**:a b (foo:bar)

In the above , “and”, “or”, and “xor” are familiar binary operations from classical logic in the cases where their arguments are true or false in the coda sense. For instance, if data A and B are both either true or false then (xor $A : B$) is () for *true* and (:) for *false* according to the standard truth table for the **xor** operation. On the other hand, if either A or B are undecided, the data (xor $A : B$) has no definitions that apply. In effect, (xor $A : B$) “waits” until both A and B are defined enough to have logical values. This is an example where (xor $A : B$) acts as a logical expression with “variables” A and B.

Undefined data may occur even when data does not explicitly contain an undefined coda such as (foo:bar) above. For example the natural numbers in coda is the data (nat:0) because a definition (nat : n) $\rightarrow$ n (nat : $n+1$) means that (nat:0) = 0 1 2 3 4 5 (nat:6), etc. The sum of the first 1000 natural numbers is perfectly well defined and is computed by "sum n : first 1000 : nat : 0". What happens, however, if we decided to sum all natural numbers via "sum n : nat : 0"? The answer is that this data remains undecided no matter how much the expression is evaluated. Here, the undecided logical category is, quite nicely and naturally, the logical value of data that is not defined.

Classical logic can be thought of as establishing at least two things: 1) defining how to reason correctly and 2) assigning logical meaning to "propositions." Within coda, 1) reasoning is correct if and only if it is application of a definition in context and 2) logical meaning is defined by data being either empty, atomic or neither. Rather than propositions being foundational undefined entities, propositions in coda are just data and their logical values are also just data. There is an understandable and perhaps even unpleasant disorientation caused by attempting to re-defining something as fundamental as classical logic. We have found that this disorientation is only temporary once one realizes that the familiar binary logical operations remain, and the seemingly odd class of "undecided" data fits in perfectly as "logic valued variables" and as "the answers to questions that have no answers." Confidence may be gained in section X where we examine Godel phenomena and directly compute some paradoxes.

5 Language

Both predicate logic with ZFC or with dependent type theory are formal languages. They have alphabets with special symbols and axiomatic syntax rules which distinguish meaningful sentences from nonsense sentences such as "$xyxy\forall\exists\exists xx\forall$". The situation with coda is much simpler because we can define the language itself as an ordinary definition, added to a context via the Axiom of Definition just like all other definitions. The basic idea behind the language definition is to give textual control over the concatenation and colon operations. Thus, if x and y are language expressions, partial functions

$$(\{x\ \ y\}\ A : B) \rightarrow (\{x\}\ A : B)\ (\{y\}\ A : B) \tag{3}$$

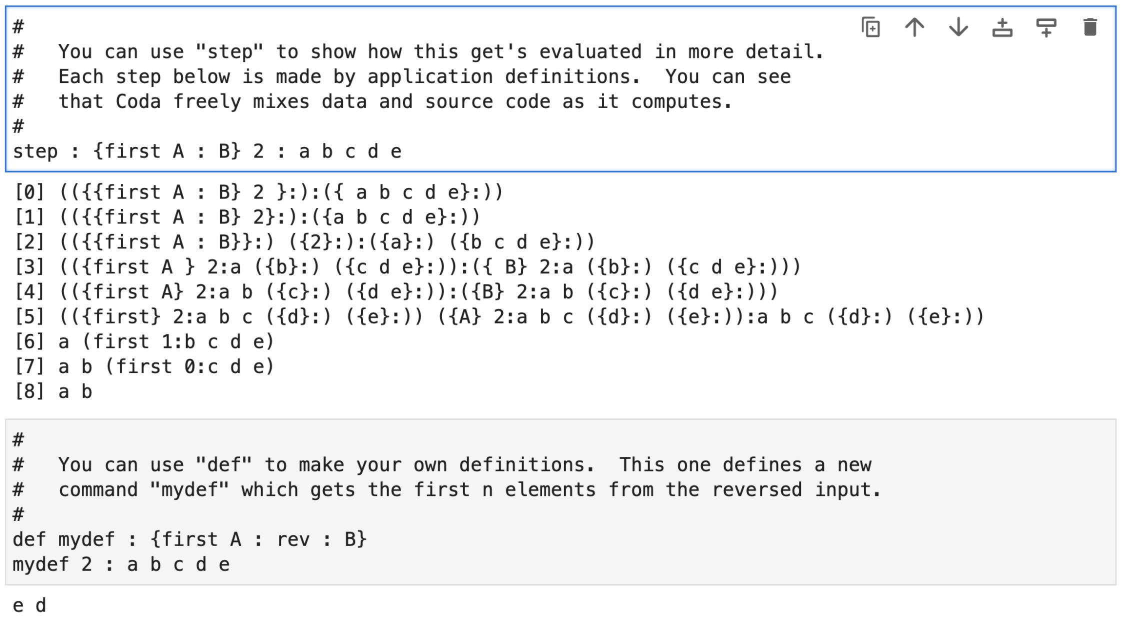
$$(\{x : y\}\ A : B) \rightarrow (\{x\}\ A : B) : (\{y\}\ A : B) \tag{4}$$

allow specification of the two operations. The language literals "A" and "B" allow specification of left and right components of a generic coda $(A : B)$ via

$$(\{A\}A : B) \rightarrow A \tag{5}$$

$$(\{B\}A : B) \rightarrow B \tag{6}$$

allow language specified access to the components of a generic coda $(A : B)$. These are partial functions fused into one language partial function by enforcing an order which also defines the precedence of operations. In the actual language, there are just a few more partial functions defining grouping with parenthesis, string literals with angle brackets, removing extraneous spaces and adding a bit of syntactic sugar so that $A = B$ in the language is interpreted as $(= A : B)$ and $A * B : X$ in the language is interpreted as $A : B : X$ for any data X and $X?$ is interpreted as $(? : X)$. Figure Y shows an evaluation of the language expression "step : {first A : B} 2 : a b c d e". In general, *evaluation* means applying definitions in some unspecified order. The **step** definition causes the evaluation to be displayed at each step, showing language expressions mixing with other



```
#
#   You can use "step" to show how this get's evaluated in more detail.
#   Each step below is made by application definitions.  You can see
#   that Coda freely mixes data and source code as it computes.
#
step : {first A : B} 2 : a b c d e
```

```
[0] (({{first A : B} 2 }:):({ a b c d e}:))
[1] (({{first A : B} 2}:):({a b c d e}:))
[2] (({{first A : B}}:) ({2}:):({a}:) ({b c d e}:))
[3] (({first A } 2:a ({b}:) ({c d e}:)):({ B} 2:a ({b}:) ({c d e}:)))
[4] (({first A} 2:a b ({c}:) ({d e}:)):({B} 2:a b ({c}:) ({d e}:)))
[5] (({first} 2:a b c ({d}:) ({e}:)) ({A} 2:a b c ({d}:) ({e}:)):a b c ({d}:) ({e}:))
[6] a (first 1:b c d e)
[7] a b (first 0:c d e)
[8] a b
```

```
#
#   You can use "def" to make your own definitions.  This one defines a new
#   command "mydef" which gets the first n elements from the reversed input.
#
def mydef : {first A : rev : B}
mydef 2 : a b c d e
```

```
e d
```

Figure 1: This is a caption

data as the evaluation proceeds. As shown, the language includes definitions which allow you to create definitions as in the example “def mydef : first A : rev : B” which takes “input” (B), reverses the order and then selects the first n items in the reversed sequence where n is determined by the “argument” A.

The language is intentionally a minimalist wrap around the basic algebra of data with a colon and a space for concatenation. Only the characters “():{}=*/ ” have language significance. There are no reserved keywords or special syntax for variables, functions, classes or exceptions. The language is unusual in that any byte sequence is a valid language expression. This means that we do not have to specify an alphabet or even define valid and invalid syntax. There is no such thing as invalid syntax. The source code for the compiler and parser is small enough to be easily read and understood by humans [1].

6 Proof and Computation

7 Spaces

8 Mathematical Machine Learning

9 Is Mathematics Consistent?

10 Summary

References

[1] Source code in github.

[2] R.T.Cox, Am.J.Phys. 14, 1 (1946).

Space	Action	Data	Space:()	$F * Space = Space * F$
pass	A→A	All data	()	F distributive
null	A→ ()	() only	()	F such that F:X=()
first	a A→ a	Single atoms	()	F(a) for atom a
bool	A→() *or* (:)	() or (:)	()	F:X which preserves bool:X
type n	A→ (n:2) (n:1) (n:5)	Natural numbers	()	(type n)∗F∗(type n) for any F
sum n	(n:2) (n:1) (n:5)→ (n:8)	Natural numbers	(n:0)	$F(n_1 + n_2) = F(n_1) + F(n_2)$
prod n	(n:2) (n:1) (n:5)→ (n:10)	Natural numbers	(n:1)	$F(n_1 * n_2) = F(n_1) * F(n_2)$
sort n	(n:2) (n:1) (n:5)→ (n:1) (n:2) (n:5)	Natural numbers	()	$n_1 \leq n_2 \implies F(n_1) \leq F(n_2)$
End sum n	$F_1\ F_2 \ldots F_n \to F_1 * F_2 * \cdots * F_n$	Morphisms	pass	Functorial
Hom Sum	$F_1\ F_2 \ldots F_n \to F_1 * F_2 * \cdots * F_n$	Morphisms	pass	Functorial

Table 3: *Spaces.*

[3] S.Youssef, Mod.Phys.Lett A9, 2571 (1994).

[4] S.Youssef, Phys.Lett. A204, 181(1995).