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Sedenions: algebra and analysis

K. Imaeda [a], M. Imaeda [b,*]

[a] 507-150, Yokoikami, Okayamashi, 701-1145, Japan

[b] IFUP, 24201, Rue De Cezanne, Laguna Niguel, CA 92677, USA

Abstract

A 16-dimensional Cayley–Dickson algebra is presented. Its unique algebraic properties, its zero-divisors, and the solutions to a general linear equation are found. A theory of function is developed in terms of the regularity (monogenicity) conditions and some such functions are constructed.

Keywords: Cayley–Dickson algebra; Sedenions; Monogenicity; Quaternions Octonions

1. Introduction

What we mean here by a sedenion algebra is a non-commutative, non-associative, non-alternative, but power-associative 16-dimensional Cayley–Dickson algebra with a quadratic norm and whose elements are constructed from real numbers, $\mathbb{R}$, by iterations of Cayley–Dickson process [1,2], where -1 is chosen as the field parameter at each step of the construction. In such process 2^N-dimensional algebras of complex numbers, $\mathbb{C}$ $(N=1)$, quaternions, $\mathbb{Q}$ $(N=2)$, octonions, $\mathbb{O}$ $(N=3)$, sedenions, $\mathbb{S}$ $(N=4)$, and other power-associative hypercomplex numbers are successively obtained.

The fact that sedenions do not necessarily constitute a composition nor a division algebra, the reason for which can be traced to the non-associativity of $\mathbb{O}$, and possessing properties that are not obvious within our current physical theories, may have been the motivation for introducing concepts and modifications to achieve or eliminate some specific metric and algebraic properties. Some parts of our work paralells that of other authors [3–5]; but unlike the

* Corresponding author. E-mail: mari_imaeda@bigfoot.com

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sedenions discussed in [3,4] our sedenions are non-alternative, non-modular and pocess real norm. In [5] alternativity is restored by introducing ternary products. Should Hurwitz theorem be one possible indication of non-conformity of sedenions to current mathematical methods in physics? It is clear from Musès [6] and Gürsey [7], and other authors that some new algebraic and analytical structures, such that might be offered by a 16-dimensional algebra, may hold a key to better formulation of physical theories. Furthermore, its relation to the Lie group G_2 remains compelling under any circumstances. We had found that the zero-divisors of sedenions are confined to some hypersurfaces and that the linear equations are solvable under certain conditions. Our first task is to list some basic algebraic properties of sedenions in Section 2. Find the zero-divisors and the conditions for their existence, in Section 3. The non-trivial task of solving a linear equation is treated in Section 4. We also develop a theory of function of a sedenion variable based on Fueter's Morera-type regularity (monogenicity) conditions and construct intrinsic, polynomial, and exponential functions, in Section 5, as the special cases of more general 2^N-dimensional theory studied earlier [8,9]. A brief account of octonions that we refer to in the main text is found in Appendix A.

2. Basic algebra

In this paper we denote sedenions by **S**, **T**, **Z**, in bold capital letters, and octonions by capital letters, A, B, X, Y, etc... We define a sedenion,

$$\mathbf{S} := (A; B) \in \mathbb{S}, \quad A, B \in \mathbb{O},$$

as an ordered pair of two real octonions, A and B. The conjugation of **S** is then defined in terms of $\bar{A}$, the conjugate of A, and of B by

$$\mathbf{S} := (\mathbf{A}; -B).$$

As with octonions this implies that $\bar{\bar{\mathrm{S}}} = \mathbf{S}$, and that the automorphism is improper. The product of two sedenions, $\mathbf{S} := (A; B)$ and $\mathbf{T} := (C; D)$, under Cayley–Dickson process, is

$$\mathbf{ST} = (A; B)(C; D) := (AC + \rho\bar{D}B; B\bar{C} + DA). \tag{1}$$

We choose the field parameter, $\rho{:} = -1$.

In constructing the sedenion basis, $(e_\mu, \mu = 0, \ldots, 15)$, from the octonions basis, $(i_\mu, \mu = 0, \ldots, 7)$, we choose the generator, $e_8 = (0;1)$, and define

$$e_\mu := i_\mu \quad (\mu = 0, 1, \ldots, 7),$$

$$e_{8+\mu} := i_\mu e_8 \quad (i_0 = e_0 = 1).$$

The multiplication of the two basal elements, e_j and e_k (the Greek (Latin) indices take some or all values of $0, 1, \ldots, 15$ $(1, \ldots, 15)$), satisfies

$$e_j e_k = -\delta_{jk} + \varepsilon_{jkm} e_m, \tag{2}$$

where the value of the structure constant, ε_{jkm}, which is totally antisymmetric in its indices, is given by the usual permutation rule for a 3-indexed antisymmetric tensor with values 0 or ± 1, and determined by the appropriate 35 sedenion cycles (j, k, m). In terms of the basis a real sedenion, $\mathbf{S}$, may be written componentwise as

$$\mathbf{S} := s_0 + \sum_{j=1}^{15} s_j e_j, \quad s_\mu \in \mathbb{R},$$

where $\mathbf{s} := \sum_{j=1}^{15} s_j e_j$ is termed as the "pure sedenion". The structure constant, ε_{jkm}, is associated with the non-commutativity of sedenions by

$$(\mathbf{S}, \mathbf{T}) := \mathbf{ST} - \mathbf{TS} = 2 \sum_{(jkm)} s_j t_k \varepsilon_{jkm} e_m, \tag{3}$$

where the summation is over all possible permutations of sedenion cycles (j, k, m). The non-associativity of three sedenions, $\mathbf{S}$, $\mathbf{T}$, and $\mathbf{U}$, is measured by the non-zero (not necessarily zero) associator, $(\mathbf{S}, \mathbf{T}, \mathbf{U}) := (\mathbf{S}\ \mathbf{T})\ \mathbf{U} - \mathbf{S}(\mathbf{T}\ \mathbf{U})$.

Although the multiplication of basal elements is strictly alternative (true for the basis of any 2^N-dimensional Cayley–Dickson algebra with field parameter -1) it is not necessarily true for arbitrary sedenions. For two sedenions, $\mathbf{S}$ and $\mathbf{T}$, the associator

$$(\mathbf{S}, \mathbf{S}, \mathbf{T}) = (-(A, B, D); (A, B, C)) = -(\bar{\mathbf{S}}, \mathbf{S}, \mathbf{T}) \tag{4}$$

is not necessarily zero. Furthermore, unlike octonions,

$$(e_j e_k) e_m + (e_j e_m) e_k \quad (m \neq k)$$

does not always vanish. This translates to the fact that the second structure constants, which deals with triple products of sedenions, are non-antisymmetric in its indices. The norm of a sedenion $\mathbf{S}$, defined as

$$n(\mathbf{S}) := \bar{\mathbf{S}}\mathbf{S} = \mathbf{S}\bar{\mathbf{S}} = n(A) + n(B) = \sum_{\mu=0}^{15} s_\mu^2 \in \mathbb{R},$$

is a non-degenerate quadratic form associated with a symmetric bilinear form

$$\mathbf{S} \cdot \mathbf{T} = \frac{1}{2}(n(\mathbf{S} + \mathbf{T}) - n(\mathbf{S}) - n(\mathbf{T})).$$

Sedenions do not form a composition algebra. Indeed, $n(\mathbf{ST}) - n(\mathbf{S})\, n(\mathbf{T})$ does not vanish in general. Furthermore, sedenions do not constitute a division algebra. For a sedenion $\mathbf{S} = (A;B)$, which satisfies certain norm and "vector-closure" conditions, there exist a sedenion, $\mathbf{Z}$, which is both left and right zero-divisor of $\mathbf{S}$,

$$\mathbf{ZS} = 0 = \mathbf{SZ}.$$

3. Zero-divisors

One cannot solve a general linear equation of sedenion in a usual manner due to the non-alternativity of the algebra. However, we can write a sedenion as an ordered pair of octonions. We first deal with the case of finding the zero-divisors of sedenions. For the right zero-divisor, $\mathbf{Z}$, of $\mathbf{S}$,

$$\mathbf{SZ} = 0. \tag{5}$$

Representing $\mathbf{S}$ and $\mathbf{Z}$ by $(A;B)$ and $(X;Y)$, respectively, where A,B,X,Y are non-zero octonions, for the non-trivial case, we have

$$AX - \bar{Y}B = 0; \quad B\bar{X} + YA = 0, \tag{6}$$

from which we obtain

$$\begin{aligned} A &= \frac{1}{n(\mathbf{Z})}(B,X,Y), \quad B = -\frac{1}{n(\mathbf{Z})}(A,X,Y); \\ X &= \frac{1}{n(\mathbf{S})}(Y,A,B), \quad Y = -\frac{1}{n(\mathbf{S})}(X,A,B). \end{aligned} \tag{7}$$

We can now prove the following.

Proposition (Zero-divisor conditions). *Eq.* (6) *implies the norm condition*:

(I) $n(A) = n(B)$, and $n(X) = n(Y)$.

Eq. (7) *implies the following "vector-closure" conditions*:

(II) A,B,X,Y *are pure octonions*; *thus the left and the right zero-divisors are same for a given sedenion*, $\mathbf{S}$.

(III) *Any product of* A, B, X, Y, *such as* AB, AX, AY, *are pure sedenions*,

(IV) *The triple products arising from* A, B, X, Y, *such as* $X(AB)$, $Y(AB)$, $A(XY)$, *and* $B(XY)$, *are pure octonions*.

(V) *Non-vanishing of a determinant, as discussed below*.

By suitable substitution and by using octonion properties (see Appendix A) we obtain the following coupled equations from (7):

$$\omega X(A,B) + (X,A,B) = 0; \quad \omega Y(A,B) + (Y,A,B) = 0, \tag{8}$$

where

$$\omega := \frac{n(\mathbf{S})^2}{n(A,B)} \in \mathbb{R}.$$

One can show that $\omega = 1$, which implies that Eq. (8) for X reduces to

$$(XA)B - X(BA) = 0. \tag{9}$$

Componentwise this equation is written as

$$\sum_{j=1}^{7} \lambda_{0j} x_j = 0, \tag{10}$$

$$\left(\sum_{j=1}^{7} x_j a_j\right) b_r - \left(\sum_{k=1}^{7} x_k b_k\right) a_r = 0 \quad \forall r = 1, \ldots, 7,$$

where $\lambda_{0j} := \sum_{(km)} a_k b_m \varepsilon_{jkm}$.

The vector-closure conditions for zero-divisors imply

$$\begin{aligned} &x_0 = 0, \\ &(AX)_0 = \sum_{j=1}^{7} a_j x_j = 0, \quad (BX)_0 = \sum_{j=1}^{7} b_j x_j = 0, \\ &(X(AB))_0 = \sum_{j=1}^{7} \lambda_{0j} x_j = 0, \end{aligned} \tag{11}$$

and thus Eq. (9) is automatically satisfied.

We proceed to solve for X in a usual manner: when a_j and b_j $(j = 1, \ldots, 7)$ are given x_j are determined only by the three independent equations (11). By re-indexing the equations (for notational purposes only) so that x_1, x_2, x_3 are the three independent unknown, and x_k $(4 \leqslant k \leqslant 7)$ are the free parameters, such that the determinant,

$$\varDelta := \begin{vmatrix} \lambda_{01} & \lambda_{02} & \lambda_{03} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

does not vanish, we find

$$x_1 = \varDelta^{-1} \begin{vmatrix} -\sum_{f.p} \lambda_{0k} x_k & \lambda_{02} & \lambda_{03} \\ -\sum_{f.p} a_k x_k & a_2 & a_3 \\ -\sum_{f.p} b_k x_k & b_2 & b_3 \end{vmatrix}.$$

Similarly for x_2 and x_3. The corresponding solution for Y can now be obtained using the appropriate equation or simply from

$$Y = \frac{X(A,B)}{n(\mathbf{S})}.$$

We may define a left or a right inverse of $\mathbf{S}$ as

$$\mathbf{S}^{-1} := \frac{\bar{\mathrm{S}}}{n(\mathbf{S})} + \mathbf{Z},$$

however, this does not automatically imply that $(\mathbf{S}^{-1})^{-1} = \mathbf{S}$.

4. Linear equations

We now solve for $\mathbf{Z}$, in terms of $\mathbf{S}$ and $\mathbf{T}$, in a linear equation,

$$\mathbf{SZ} = (A;B)(X;Y) = \mathbf{T}, \quad A \neq 0 \neq B.$$

Using the identity (4) and defining octonions, F and G, by $\bar{\mathbf{S}}\mathbf{T} := (F;G)$, we obtain

$$\begin{aligned} &n(\mathbf{S})X - (A,B,Y) = F, \\ &n(\mathbf{S})Y + (A,B,X) = G. \end{aligned} \tag{12}$$

By substitution we have for X,

$$\omega X(A,B) + (A,B,X) = 2K, \tag{13}$$

where

$$K := \frac{[n(\mathbf{S})F + (A,B,G)]\ (A,B)}{2n(A,B)}, \quad \omega := \frac{n(\mathbf{S})^2}{n(\mathbf{A},\mathbf{B})}.$$

Writing this equation componentwise we have the following simultaneous equations:

$$\begin{aligned} &\omega \sum_{j=1}^{7} \lambda_{0j} x_j = k_0, \\ &\omega \lambda_{0k} x_0 - \sum_{j=1}^{7} \lambda_{jk} x_j + (1-\omega) \sum_{(jm)} \varepsilon_{jkm} \lambda_{0m} x_j = k_k, \end{aligned} \tag{14}$$

where we had used the fact that for each fixed j and k,

$$\sum_{(pq)} a_p b_q \varepsilon_{jkpq} = \sum_m \lambda_{0m} \varepsilon_{jkm} - \lambda_{jk}; \quad \lambda_{jk} := (a_j, b_k).$$

These equations may be written in a matrix form as

$$\alpha_{\mu\nu} x_\nu = k_\mu, \tag{15}$$

where $\alpha_{\mu\nu}$ is a skew-symmetric matrix with

$$\alpha_{\mu\mu} = 0; \quad \alpha_{0j} = -\omega\lambda_{0j} = -\alpha_{j0}; \quad \alpha_{jk} = \lambda_{jk} - (1-\omega)\varepsilon_{jkm}\lambda_{0m},$$

which satisfies

$$\sum_k \alpha_{jk}\alpha_{0k} = 0, \quad j = 1, \ldots, 7.$$

Both x_ν and k_μ are column matrices. Through a tedious but straightforward calculation we obtain the determinant

$$|\alpha_{\mu\nu}| = \omega^4(1-\omega)^4\left(\sum_{j=1}^{7}\lambda_{0j}^2\right)^4. \tag{16}$$

From Eq. (16) we have the following cases:

Case 1: $\omega \neq 1$.

1. $K=0$: then $X=0 \Rightarrow \mathbf{Z}=0$.
2. $K \neq 0$: the solution for x_μ $(\mu = 0, \ldots, 7)$ is uniquely determined by A, B, F and G.

Case 2: $\omega = 1$.

1. $K=0$: the case is that of the zero-divisor discussed in Section 3.
2. $K \neq 0$: from Eq. (14) we have

$$\lambda_{0k}x_0 - b_k v + a_k w = k_k, \tag{17}$$

where

$$v := \sum_{j=1}^{7} a_j x_j, \quad w := \sum_{j=1}^{7} b_j x_j, \quad \sum_{j=1}^{7}\lambda_{0j}x_j = -k_0.$$

The equations are now solved in a conventional manner as follows. Note that the seven equations of (17) are imposed on x_0, v, and w, and that the rank of Δ and M, where

$$L := \begin{pmatrix} \lambda_{01} & a_1 & b_1 \\ \lambda_{02} & a_2 & b_2 \\ \cdot & \cdot & \cdot \\ \lambda_{07} & a_7 & b_7 \end{pmatrix}, \qquad M := \begin{pmatrix} \lambda_{01} & a_1 & b_1 & k_1 \\ \lambda_{02} & a_2 & b_2 & k_2 \\ \cdot & \cdot & \cdot & \cdot \\ \lambda_{07} & a_7 & b_7 & k_7 \end{pmatrix},$$

should be equal to or be less than three.

(i) The rank of L is three: by re-indexing the a_k and b_k (strictly for notational purposes only), so that

$$D = \begin{vmatrix} \lambda_{01} & a_1 & b_1 \\ \lambda_{02} & a_2 & b_2 \\ \lambda_{03} & a_3 & b_3 \end{vmatrix} \neq 0,$$

we find

$$x_0 = \frac{D_1}{D}, \quad v = \frac{D_2}{D}, \quad w = \frac{D_3}{D},$$

where D_j $(j = 1, 2, 3)$ is the determinant in which the three entries of the jth column in D are replaced by k_1, k_2 and k_3, respectively. Now to solve for x_k we have the following system of equations:

$$\sum_{k=1}^{7} \lambda_{0k} x_k = -k_0,$$

$$\sum_{j=1}^{7} a_j x_j = \frac{D_2}{D}, \quad \sum_{j=1}^{7} b_j x_j = \frac{D_3}{D}.$$

The solution is given by,

$$x_j = \frac{\Delta_j}{\Delta} \quad \forall j = 1, 2, 3,$$

where

$$\Delta := \begin{vmatrix} \lambda_{01} & \lambda_{02} & \lambda_{03} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix},$$

and Δ_j is the determinant Δ in which the three entrices in the jth column are replaced by

$$-k_0 - \sum_{j=1}^{7} \lambda_{0j} x_j, \quad \frac{D_2}{D} - \sum_{j=1}^{7} a_j x_j, \quad \frac{D_3}{D} - \sum_{j=1}^{7} b_j x_j,$$

where x_j $(j = 4, \ldots, 7)$ denote the free parameters.

(ii) The rank of L is two: we consider $B = \alpha A$, $\alpha \in \mathbb{R}$, and conclude from Eq. (12) that

$$X = \frac{F}{n(\mathbf{S})}, \quad Y = \frac{G}{n(\mathbf{S})}.$$

Thus, $\mathbf{Z}$ is uniquely determined by $\mathbf{S}$ as

$$\mathbf{Z} = \frac{\bar{\mathbf{S}} T}{n(\mathbf{S})}.$$

.

5. Theory of functions

As found by many authors both past and present the usual Cauchy–Riemann type differentiability conditions, when extended to functions of a hypercomplex variable, are too restrictive to yield interesting, if not useful, functions. Our path is to follow one of the most successful approach in constructing functions, that of regular functions; a theory originally developed for quaternions by Fueter [10], which was then refined, generalized, and extended to the cases of a variable in other Cayley–Dickson algebras [11–16] as well as in Clifford algebras, and which are successfully used in physics by authors too many to list them here. We define the regularity (monogenicity) condition for sedenions in terms of the variation of an integral of a function in a 16-dimensional Euclidean space. For a function, $\mathbf{F}(\mathbf{Z})$, of a sedenion variable, $\mathbf{Z} := \sum_{\mu=0}^{15} z_\mu e_\mu; (z_\mu \in \mathbb{R})$, consider an integral over a closed hypersurface S^{15},

$$J[S^{15}] := \int_{S^{15}} \mathrm{d}\mathbf{Z}\, F(\mathbf{Z}).$$

The surface element $\mathrm{d}\mathbf{Z}$ can be written in terms of the differentials, $\mathrm{d}z_\mu$, or of the directional cosines, γ_μ, and the volume element, $\mathrm{d}\sigma$, of S^{15} as

$$\mathrm{d}\mathbf{Z} = \sum_{\mu=0}^{15} \partial z_\mu e_\mu = \sum_{\mu=0}^{15} \gamma_\mu \,\mathrm{d}\sigma e_\mu.$$

Using Stoke's theorem we can write $J[S^{15}]$ as a volume integral over V^{16} enclosed in S^{15} as

$$J[S^{15}] = \int_{V^{16}} (DF(\mathbf{Z}))\mathrm{d}V,$$

where

$$D := \sum_{\mu=0}^{15} \partial_\mu e_\mu, \quad \mathrm{d}V := \prod_{\mu=0}^{15} \mathrm{d}z_\mu.$$

Now consider an infinitesimal variation of the surface S_A^{15} to S_B^{15} in the neighbourhood of a point $\mathbf{Z}$ on S_A^{15}. The volume enclosed by the two surfaces is

$$\delta V := \int_{S_A - S_B} \mathrm{d}V^{16} \in \mathbb{R}.$$

Define

$$\frac{\delta J[S]}{\delta S} := \lim_{\delta V \to 0} \frac{[J[S_A^{15}] - J[S_B^{15}]]}{\delta V} = \lim_{\delta V \to 0} \frac{\left[\int_{\delta V} (D\mathbf{F})\,\mathrm{d}V\right]}{\delta V} = D\mathbf{F}(\mathbf{Z}).$$

Then the function $F(\mathbf{Z})$ is left regular at $\mathbf{Z}_0$ on S^{15} if $J[S]$ is stationary under the infinitesimal variation of the surface at $\mathbf{Z}_0$,

$$\left.\frac{\delta J[S]}{\delta S}\right|_{\mathbf{Z}=\mathbf{Z}_0} = \left.D\mathbf{F}(\mathbf{Z})\right|_{\mathbf{Z}=\mathbf{Z}_0} = 0.$$

The definition is independent of the choice of S^{15} for $D\mathbf{F}(\mathbf{Z})=0$. The right D-regularity as well as $\bar{D}$-regularity conditions can be defined in a similar manner.

We now outline the construction of some regular functions of a sedenion variable.

(I) *Intrinsic functions*: Starting with an analytic function,

$$g(z) = u(x_0, x) + \mathrm{i}v(x_0, x),$$

of a complex variable, $z = x_0 + \mathrm{i}x$, and replacing $\mathrm{i} = \sqrt{-1}$ by a pure sedenion

$$\mathrm{i} \Rightarrow \frac{\sum_{j=1}^{15} \; x_j e_j}{x},$$

where

$$x := \sqrt{\sum_{j=1}^{15} x_j^2},$$

the function assumes the form

$$\mathbf{G}(\mathbf{X}) = u(x_0, x) + \frac{\sum_{j=1}^{15} v(x_0, x) x_j e_j}{x},$$

where $\mathbf{X}$ is a sedenion variable. Then the function

$$\mathbf{F}(\mathbf{X}) := \square^7 \mathbf{G}(\mathbf{X}); \quad \square := D\bar{D} = \sum_{\mu=0}^{15} \partial_\mu^2,$$

is left-regular.

(II) *Polynomial functions*: Consider a function

$$\mathbf{F}_n(\mathbf{X}, \mathbf{T}) = \frac{[(\mathbf{XT} + \mathbf{TX})]^n}{2} = \left[\sum_{j=1}^{15} (x_0 e_j - x_j) t_j \right]^n,$$

where

$$\mathbf{X} := \sum_{\mu=0}^{15} x_\mu e_\mu, \quad \mathbf{T} := \sum_{j=1}^{15} t_j e_j \in \mathbf{S}.$$

The power associativity ensures without ambiguity that $\mathbf{F}_n(\mathbf{X}, \mathbf{T})$ can be written as a power series in t_k,

$$\mathbf{F}_n(\mathbf{X}, \mathbf{T}) = \sum_{(n_r)} n! \mathbb{P} n_1 n_2 \ldots n_{15} (x_0, x_1, \ldots, x_{15}) t_1^{n_1} t_2^{n_2} \ldots t_{15}^{n_{15}},$$

where $n := \sum_{j=1}^{15} n_j$, and $\mathbb{P}_{n_1 n_2 \ldots n_{15}}$ is a polynomial function of a sedenion variable

$$\mathbb{P}_{n_1 n_2 \ldots n_{15}}(x_0, x_1, \ldots, x_{15}) = \frac{1}{n!} \sum_{(k_r)} (x_0 e_{k_1} - x_{k_1}) \ldots (x_0 e_{k_n} - x_{k_n}).$$

The summation is over all possible permutations of the series $(k_r) = (k_1, \ldots, k_n)$, $\{k_1, \ldots, k_n\} = \{1, 2, \ldots, 15\}$, and the number of times each of $1, 2, \ldots, 15$ appears in the series is $n_1, n_2, \ldots, n_{15}$, respectively. It can be easily shown that $\mathbf{F}_n(\mathbf{X}, \mathbf{T})$ and, henceforth, the polynomials $\mathbb{P}_{n_1 n_2 \ldots n_{15}}$, are both side D-regular.

(III) *Exponential functions*: By again considering the hypercomplex variable $\mathbf{F}(\mathbf{X}) := \sum_{j=1}^{15}(x_0 e_j - x_j)$, we may define exponential function of $\mathbf{S}$ as

$$\mathrm{Exp}[\mathrm{i}\mathbf{F}(\mathbf{X}, \mathbf{T})] := \frac{(\mathrm{i})^n}{n!}\mathbf{F}^n(\mathbf{X}, \mathbf{T}).$$

This function is both side D-regular.

(IV) *Cauchy-type kernel function*: For a function

$$H(\mathbf{X}) := \frac{\mathbf{X}}{n(\mathbf{X})},$$

of a variable $\mathbf{X}$, in a $2^N(=m)$-dimensional Cayley–Dickson algebra, one can readily show that,

$$\Box^n\ \mathbf{H}(\mathbf{X}) = \frac{C_n\ \mathbf{H}(\mathbf{X})}{[\ n(\mathbf{X})]^n}, \quad n \in \mathbb{Z}^+,$$

where

$$C_n = (-1)^n 2^n n! \prod_{k=1}^{n}(m - 2k), \quad \Box := \mathbf{D}\bar{D} = \sum_{\mu=0}^{15} \frac{\partial^2}{\partial x_\mu^2},$$

and that $\Box^n \mathbf{H}(\mathbf{X})$ is a both side regular function of an m-dimensional Cayley–Dickson algebra if $n = m/2 - 1$. Thus, one can construct a Cauchy-type kernel function of a sedenion variable which is both side regular by choosing $n = 7$. However, a *general* integral formula for sedenions based on this Cauchy-type kernel function cannot be constructed due to the non-antisymmetric nature of the second structure constant.

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Appendix A. Octonions

We now list octonion properties and relations that are relevant to our main work. Let A, B, $C \in \mathbb{O}$. Componentwise the octonion A, the conjugate, the norm, and the inverse may be written, respectively, as

$$A := a_0 + \sum_{j=1}^{7} a_j i_j, \quad \bar{A} := a_0 - \sum_{j=1}^{15} a_j i_j,$$
$$n(A) := A\bar{A} = \bar{A}A = \sum_{\mu=0}^{7} a_\mu^2, \quad A^{-1} := \bar{A}/n(A), \tag{A.1}$$

where the multiplication of basal elements, i_j $(j = 1, \ldots 7)$, is governed by the same relations as that of the sedenion basis, e_j $(j = 1, \ldots, 7)$, defined in Eq. (2).

The non-zero (not necessarily zero) associator, which is a pure octonion, is defined by

$$(A, B, C) := (AB)C - A(BC) = 2 \sum_{(jkmn)} a_j b_k c_m \varepsilon_{jkmn} i_n,$$

where the second structure constant, ε_{jkmn}, is antisymmetric in its indices, and is determined from the seven distinct associator cycles, (j, k, m, n) by the usual 4-indexed tensor rule: $\varepsilon_{jkmn} = (+1,\ -1,$ or $0)$, depending on the (even, odd, or otherwise) permutation of the octonion associator cycle.

The following identities hold for octonion associators:

$$\begin{aligned}
&(A, B, C) = -(\bar{A}, B, C) = -(B, A, C), \cdots \\
&(A, B, (A, B, C)) = (A, B)(A, B, C) = -(A, B, C)(A, B); \\
&(A, B) := AB - BA, \\
&(A, AB, C) = (A, B, C)A.
\end{aligned} \tag{A.2}$$

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