apartness relation

Context

Relations

Apartness relations

1. Idea

An *apartness relation* is a binary relation that, instead of saying when two things are the same (as an equivalence relation), states when two things are different – an inequality relation.

Apartness relations are most used in constructive mathematics; in classical mathematics, equivalence relations can take their place (mediated by negation).

The apartness relations that we discuss here are sometimes called ***point–point apartness***, to distinguish this from the related concepts of *set–set* or *point–set* apartness relations; see proximity space and apartness space (respectively) for these.

2. Definitions

Abstract

A set S equipped with an ***apartness relation*** is a groupoid (with S as the set of objects) enriched over the cartesian monoidal category TV^{op}, that is the opposite of the poset of truth values, made into a monoidal category using disjunction. By the law of excluded middle (which says that TV is self-dual under negation), this is equivalent to equipping S with an equivalence relation (which makes S a groupoid enriched over the cartesian category TV *itself*). But in constructive mathematics (or interpreted internally), it is a richer concept with a topological flavour, as TV^{op} is a co-Heyting algebra.

Concrete

Of course, nobody but a category-theorist would use the above as a *definition* of an apartness relation. Normally, one defines an apartness relation on S as a binary relation $\#$ satisfying these three properties:

- irreflexivity: for all $x: S$, $x \# x$ is false;
- symmetry: for all $x, y: S$, if $y \# x$, then $x \# y$;
- comparison: for all $x, y, z: S$, if $x \# z$, then $x \# y$ or $y \# z$.

(Notice that these are dual to the axioms for an equivalence relation; like those axioms, these correspond to identity morphisms, inverses, and composition in a groupoid.)

Related notions

The negation of an apartness relation is an equivalence relation. (On the other hand, the statement that every equivalence relation is the negation of some apartness relation is equivalent to excluded middle, and the statement that the negation of an equivalence relation is always an apartness relation is equivalent to the nonconstructive de Morgan law.) An apartness relation is a tight apartness relation if this equivalence relation is equality; any apartness relation defines a tight apartness relation on the quotient set.

If S and T are both sets equipped with apartness relations, then a function $f: S \to T$ is ***strongly extensional*** if $x \# y$ whenever $f(x) \# f(y)$; that is, f reflects apartness. The strongly extensional functions are precisely the enriched functors between TV^{op}-enriched groupoids, so they are the correct morphisms. (Note that there is no nontrivial notion of enriched natural isomorphism, at least not when the apartness in T is tight.)

3. (2,1)-category of sets with apartness relations

Sets with apartness relations, strongly extensional functions, and equivalences of strongly extensional functions, which serve to identify unequal but equivalent (that is, not apart) elements of a set, form a locally thin (2,1)-category; i.e. a bicategory enriched in thin groupoids. This bicategory is locally small and a univalent bicategory.

4. Topological aspects

The apartness topology

Let S be a set equipped with an apartness relation $\neq$. Using $\neq$, many topological notions may be defined on S. (Often one assumes that the apartness is tight; this corresponds to the T_0 separation axiom in topology.)

If U is a subset of S and x is an element, then U is a $\neq$-***neighbourhood*** (or $\neq$-***neighborhood***) of x if, given any $y: S$, $x \neq y$ or $y \in U$; note that $x \in U$ by irreflexivity. The neighbourhoods of x form a filter: a superset of a neighbourhood is a neighbourhood, and the intersection of 0 or 2 (hence of any finite number) of neighbourhoods is a neighbourhood.

A subset G is $\neq$-***open*** if it's a neighbourhood of all of its members. The open subsets form a topology (in the sense of Bourbaki): any union of open subsets is open, and the intersection of 0 or 2 (hence of any finite number) of open subsets is open.

The $\neq$-***complement*** of x is the subset $\{y: S \mid x \neq y\}$; this is open by comparison. More generally, the $\neq$-***complement*** of any subset A is the set $\tilde{A}$, defined as:

$$\tilde{A} \coloneqq \{y \mid \forall x \in A, x \neq y\}.$$

This is not in general open, but you would use it where you would classically use the set-theoretic complement. However, if A is open to begin with, then $\tilde{A}$ equals the set-theoretic complement.

If $x \neq y$, then $x \in \tilde{y}$ and $y \in \tilde{x}$. Thus, if $\neq$ is tight, then $(S, \neq)$ satisfies the T_1 separation axiom. Symmetry is important here; if we removed symmetry from the axioms of apartness (obtaining a quasi-apartness?) but retained tightness, then we would still get a T_0 topology, but it would not be T_1. This is a version of the fact that failure of T_1 is given by a partial order (or a preorder if T_0 might also fail).

The $\neq$-***closure*** $\bar{A}$ of a subset A is the complement of its complement. This closure is a closure operator: $A \subset \bar{A}$, $\bar{\bar{A}} = \bar{A}$ (in fact, $\tilde{\bar{A}} = \tilde{A}$), $\bar{A} \subset \bar{B}$ whenever $A \subset B$, $\bar{S} = S$ (in fact, $\bar{\emptyset} = \emptyset$ too), and $\overline{A \cap B} = \bar{A} \cap \bar{B}$ (but *not* $\overline{A \cup B} = \bar{A} \cup \bar{B}$).

The ***antigraph*** of a function $f: S \to T$ is

$$\{(x, y) \mid x: S, y: T \mid f(x) \neq y\}.$$

Recall that in ordinary topology, a function between Hausdorff spaces is continuous iff its graph is closed. Similarly, a function $f: S \to T$ is strongly extensional iff its antigraph is open. (Then the graph of f is the complement of the antigraph.)

One important topological concept that doesn't appear classically is locatedness; in an inequality space, a subset A is ***located*** if, given any point x and any neighbourhood U of x, either $U \cap A$ is inhabited (that is, it has a point) or some neighbourhood of x (not necessarily U) is contained in $\tilde{A}$. Note that every point is located. (For an example of a set that need not be located, consider $\{x: S \mid p\}$, where p is an arbitrary truth value. In an inhabited space, this set is located iff p is true or false.)

Relation to metric spaces

Recall that, as Bill Lawvere taught us, a metric space is a groupoid (or $\dagger$ -category) enriched over the category $([0, \infty[^{op}, +)$ of nonnegative real numbers, ordered in reverse, and made monoidal under addition. (Actually, you get a metric only if you impose a tightness condition, although again you can recover this up to equivalence from the 2-morphisms. Furthermore, Lawvere advocated using $[0, \infty]$ instead of $[0, \infty[$, and also dropping the symmetry requirement to get enriched categories instead of groupoids. Thus, he dealt with extended quasipseudometric spaces. These details are not really important here.)

There is a monoidal functor from $([0, \infty[^{op}, +)$ to TV^{op} that maps a nonnegative real number x to the truth value of the statement that $x > 0$. Accordingly, any (symmetric) metric space becomes an inequality space, and any function satisfying $d(f(x), f(y)) \leq d(x, y))$ is strongly extensional.

The topological properties of metric spaces fit well with those of inequality spaces if you always work in this direction. For example, a set which is d-open will also be $\neq$ -open, but not necessarily the other way around. Similarly, a (merely) continuous function between metric spaces is (still) strongly extensional.

Relation to gauge spaces and uniform spaces

In analysis, many spaces are given as gauge spaces, that is by families of pseudometrics; these also become inequality spaces by declaring that $x \neq y$ iff $d(x, y) > 0$ for some pseudometric d in the family. (This will actually be a *tight* apartness iff the family of pseudometrics is separating.)

Classically, any uniform space may be given by a family of pseudometrics, but this doesn't hold constructively. In particular, a topological group may not be an inequality group (as in the next section). However, we can generalize a bit beyond gauge spaces: any uniformly regular uniform space becomes an inequality space by declaring that $x \neq y$ iff there is an entourage U with $(x, y) \notin U$. (If the uniform space is not uniformly regular, the result is merely an inequality relation, not an apartness.)

Relation to proximity spaces

The constructive theory of proximity spaces is based on a generalisation of apartness relations (which here go between *points*) to an apartness relation between *sets*. These are called *apartness spaces*; just as apartness relations (between points) are classically equivalent to equivalence relations, so apartness spaces are classically equivalent to proximity spaces, with two sets being proximate if and only if they are not apart.

Of course, any apartness space has an apartness relation between points: x and y are apart iff $\{x\}$ and $\{y\}$ are apart.

Relation to locales

Let X be a set, regarded as a discrete locale, whose frame of opens is $O(X) = P(X)$, the power set of X. That is, the opens in the locale X are precisely the subsets of the set X. Since discrete locales are locally compact (every set is the union of its K-finite subsets), the locale product $X \times X$ agrees with the spatial product, so that $X \times X$ is also discrete and every subset of $X \times X$ is open. Thus, the opens in the locale $X \times X$ are precisely the subsets of $X \times X$. In particular, an equivalence relation on the set X can be identified with an *open* equivalence relation (in Loc) on the discrete locale X.

Thus, the following theorem gives a different precise sense in which apartness relations are dual to equivalence relations.

Theorem 4.1. *An apartness relation on a set X is the same as a (strongly) closed equivalence relation on the discrete locale X. Moreover, the apartness topology defined above is, as a locale, the quotient of this equivalence relation.*

Proof. By definition, a (strongly) closed sublocale of a locale Y is one of the form CU, for some open $U \in O(Y)$. Thus, when X is a discrete locale, a closed sublocale of $X \times X$ is of the form CU for some subset U of $X \times X$. This subset is the extension of the apartness relation, i.e. $U = \{(x, y) \mid x \# y\}$.

For the first claim, therefore, it remains to show that the three axioms of an equivalence relation for CU correspond to the apartness axioms for $\#$. Note that pullback along locale maps respects closed complements, i.e. $f^*(CU) = C(f^* U)$. Thus, the pullback of CU along the twist map $X \times X \to X \times X$ is the closed sublocale corresponding to the twist of U, i.e. the set $\{(x, y) \mid y \# x\}$. Since C is a contravariant order-isomorphism between the posets of open and closed sublocales, symmetry for CU is equivalent to symmetry for $\#$. Similarly, pulling CU

back to $X \times X \times X$ along one of the three canonical projections gives the closed sublocale dual to the corresponding pullback of U itself, and C transforms unions to intersections; thus transitivity for CU is equivalent to comparison for $\#$. Finally, the pullback of CU along the diagonal is the closed sublocale dual to the similar pullback of U, so to say that the former is all of X is equivalent to saying that the latter is $\varnothing$; thus reflexivity for CU is equivalent to irreflexivity for $\#$.

Now, the quotient in Loc of such an an equivalence relation in particular comes equipped with a surjective locale map from X. Thus, it is a spatial locale and can be regarded as a topology on the set X. Moreover, quotients in Loc are constructed as equalizers in Frm, so we have to compute the equalizer of the two maps $O(X) = P(X) \to O(CU)$, where $O(CU)$ is the frame of opens of CU regarded as a locale in its own right. Equivalently, this means the equalizer of the two maps $P(X) \xrightarrow{\pi_i} P(X \times X) \xrightarrow{j_{CU}} P(X \times X)$, where j_{CU} is the nucleus corresponding to CU.

Now by definition, $j_{CU}(V) = V \cup U$. Thus, the elements of this equalizer — which is to say, the opens in the locale quotient — are subsets V of X such that $(V \times X) \cup U = (X \times V) \cup U$. Reexpressed in terms of $\#$, that means that for any $x, y \in X$ we have $(x \in V \vee x \# y) \Leftrightarrow (y \in V \vee x \# y)$. But since $\#$ is symmetric, this is equivalent to the unidirectional implication $(x \in V \vee x \# y) \to (y \in V \vee x \# y)$, and since $x \# y$ always implies itself, this is equivalent to $x \in V \to (y \in V \vee x \# y)$, which is precisely the condition defining the open sets in the apartness topology above. ▮

Recall that the negation of an apartness relation on X is an equivalence relation on the *set* X. This is the spatial part of the above closed localic equivalence relation, which in general (constructively) need not be itself spatial. The apartness relation is tight just when this spatial part is the diagonal. (By contrast, to say that the closed localic equivalence relation is *itself* the diagonal is to say that the discrete locale X is Hausdorff, which is only true if X has decidable equality.)

Another characterization of the $\#$-open sets is that U is $\#$-open if $U \times X \subseteq (X \times U) \cup W_\#$, where $W_\#$ is $\#$ regarded as a subset of $X \times X$. Rephrased in terms of complementary closed sublocales, this says that CU is "closed under the equivalence relation" dual to $\#$. Thus, the closed sublocales of X with its $\#$-topology (i.e. the formal complements of $\#$-open sets) correspond precisely to the closed sublocales of X (the formal complements of arbitrary subsets of X) that respect this equivalence relation.

As a partial converse to the above theorem, if X is a locally strongly Hausdorff topological space, meaning that its diagonal is a strongly closed sublocale, then the pullback of this diagonal to the discrete locale on the set of points of X is a closed localic equivalence relation, hence an apartness, whose $\neq$-topology refines the given topology. See this theorem. If we are given an apartness relation $\neq$, it is unclear whether the $\neq$-topology is locally strongly

Hausdorff; but if it is, then the apartness relation resulting from this topology is stronger than the given $\neq$.

5. Related concepts

- inequality relation
  - denial inequality
- antisubalgebra
- inequality space

symbol	in logic
$\in$	element relation
$:$	typing relation
$=$	equality
$\vdash$	entailment / sequent
$\top$	true / top
$\bot$	false / bottom
$\Rightarrow$	implication
$\Leftrightarrow$	logical equivalence
$\neg$	negation
$\neq$	negation of equality / apartness
$\notin$	negation of element relation
$\neg\neg$	negation of negation
$\exists$	existential quantification
$\forall$	universal quantification
$\wedge$	logical conjunction
$\vee$	logical disjunction

symbol	in logic
symbol	*in type theory (propositions as types)*
$\to$	function type (implication)
$\times$	product type (conjunction)
$+$	sum type (disjunction)
0	empty type (false)
1	unit type (true)
$=$	identity type (equality)
$\simeq$	equivalence of types (logical equivalence)
$\sum$	dependent sum type (existential quantifier)
$\prod$	dependent product type (universal quantifier)
symbol	*in linear logic*
$\multimap$	linear implication
$\otimes$	multiplicative conjunction
$\oplus$	additive disjunction
$\&$	additive conjunction
$⅋$	multiplicative disjunction
$!$	exponential conjunction

6. References

According to Troelstra and van Dalen:

> Brouwer (1919) introduced the notion of apartness (örtlich verschieden, Entfernung)…. The axioms of the theory of apartness were formulated by Heyting (1925).

- L.E.J. Brouwer, *Begründung der Mengenlehre unabhängig vom logischen Satz vom ausgeschlossenen Dritten II: Theorie der Punktmengen. 1919*
- Arend Heyting, *Intuïtionistische Axiomatiek der Projectieve Meetkunde* (Dutch), Ph.D. Thesis, 1925
- Errett Bishop's *Foundations of Constructive Analysis* (1967) uses apartness for the real numbers and more general metric spaces.
- The 1985 edition with Douglas Bridges, *Constructive Analysis*, includes the general definition of apartness relation, there called an "inequality relation" (though in many other sources, as here, an inequality relation need not satisfy comparison).
- Anne Troelstra's and Dirk van Dalen's *Constructivism in Mathematics* (1988) uses apartness for the reals (volume 1), and general apartness relations in algebra (volume 2, chapter 8). They say "preapartness" and "apartness" instead of "apartness" and "tight apartness" respectively.
- Apartness plays a minimal role in *A Course in Constructive Algebra* (also 1988), by Ray Mines, Fred Richman, and Wim Ruitenburg.
- A great reference for point-set topology in constructive mathematics is the Ph.D. thesis of Frank Waaldijk, *Modern Intuitionist Topology* (1996).

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