

# A Summary of Complex Analysis

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## Abstract

This is a summary of complex analysis. The main references are *Complex Analysis* by Elias M. Stein and Rami Shakarchi and the lecture notes of École Polytechnique.

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# 1 Holomorphic Functions

## 1.1 Basic Definitions

What is a holomorphic function? In fact, it is a complex function that is complex differentiable in a neighborhood of every point. And complex differentiability is equivalent to satisfying the Cauchy-Riemann equations.

**Theorem 1.1** (Cauchy-Riemann Equations). *Let  $f: \Omega \rightarrow \mathbb{C}$  be a complex function defined on an open set  $\Omega \subset \mathbb{C}$ . Write  $f(z) = u(x, y) + iv(x, y)$  where  $z = x + iy$ , and  $u, v: \Omega \rightarrow \mathbb{R}$  are real-valued functions. Then  $f$  is holomorphic at a point  $z_0 = x_0 + iy_0 \in \Omega$  if and only if the partial derivatives of  $u$  and  $v$  exist and are continuous in a neighborhood of  $(x_0, y_0)$ , and satisfy the Cauchy-Riemann equations:*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

at the point  $(x_0, y_0)$ .

Using the Cauchy-Riemann equations, we can easily verify that  $\operatorname{Re}(f)$  determines  $\operatorname{Im}(f)$  up to a constant, and vice versa. In addition, we can also consider

$$\frac{\partial}{\partial z} := \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right), \quad \frac{\partial}{\partial \bar{z}} := \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).$$

Then the Cauchy-Riemann equations are equivalent to  $\frac{\partial f}{\partial \bar{z}} = 0$ . Here are some properties about  $\frac{\partial}{\partial z}$  and  $\frac{\partial}{\partial \bar{z}}$ .

**Proposition 1.2.** *If  $f$  and  $g$  are complex valued functions, then*

- $\frac{\partial \bar{f}}{\partial \bar{z}} = \overline{\frac{\partial f}{\partial z}}$ .
- $\frac{\partial (f \circ g)}{\partial z}(z) = \left( \frac{\partial f}{\partial z} \circ g \right)(z) \cdot \frac{\partial g}{\partial z}(z) + \left( \frac{\partial f}{\partial \bar{z}} \circ g \right)(z) \cdot \frac{\partial \bar{g}}{\partial z}(z)$

Using these operators, we can easily verify that  $\overline{f(\bar{z})}$  is holomorphic if  $f(z)$  is holomorphic.

## 1.2 Integration along Curves

Now we are going to an important property of holomorphic functions. A holomorphic function is infinitely complex differentiable. Before we prove it, we need to introduce the concept of integration along curves.

Given a smooth curve  $\gamma: [a, b] \rightarrow \mathbb{C}$  and a complex function  $f: \Omega \rightarrow \mathbb{C}$  defined on an open set  $\Omega \subset \mathbb{C}$  containing the image of  $\gamma$ , we define the complex integral of  $f$  along  $\gamma$  by

$$\int_{\gamma} f(z) dz = \int_a^b f(\gamma(t)) \gamma'(t) dt.$$

**Proposition 1.3.** *Suppose  $\Omega$  is a connected open set. If a continuous function  $f: \Omega \rightarrow \mathbb{C}$  has a primitive  $F$  in  $\Omega$ , and  $\gamma: [a, b] \rightarrow \Omega$  is a smooth curve in  $\Omega$ , then*

$$\int_{\gamma} f(z) dz = F(\gamma(b)) - F(\gamma(a)).$$

With the definition above, we can state Cauchy's integral formulas.

**Theorem 1.4** (Cauchy's Integral Formulas). *Let  $\Omega \subset \mathbb{C}$  be open set containing  $\bar{D}(z_0, r)$ , and let  $f: \Omega \rightarrow \mathbb{C}$  be a holomorphic function. Then*

$$f(z) = \frac{1}{2\pi i} \int_{C(z_0, r)} \frac{f(\omega)}{\omega - z} d\omega,$$

for all  $z \in D(z_0, r)$ .

The key to proving Cauchy's integral formulas is to consider

$$\frac{f(\omega)}{\omega - z} = \frac{f(\omega) - f(z)}{\omega - z} + \frac{f(z)}{\omega - z}$$

and the keyhole contour.

Using Cauchy's integral formulas, we can easily prove that holomorphic functions are infinitely complex differentiable.

**Corollary 1.5.** *Let  $\Omega \subset \mathbb{C}$  be open set containing  $\bar{D}(z_0, r)$ , and let  $f: \Omega \rightarrow \mathbb{C}$  be a holomorphic function. Then*

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{C(z_0, r)} \frac{f(\omega)}{(\omega - z)^{n+1}} d\omega,$$

for all  $z \in D(z_0, r)$ .

Hence holomorphic functions are analytic, i.e., they can be represented by power series. Actually, the converse is also true, which is simple by term-by-term differentiation of power series. We can write  $f(z) = \sum_{n \geq 0} a_n(z - z_0)^n$  in  $D(z_0, r) \subseteq \Omega$  with

$$\begin{aligned} a_n &= \frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i} \int_{C(z_0, r)} \frac{f(\omega)}{(\omega - z_0)^{n+1}} d\omega \\ &= \frac{1}{2\pi r^n} \int_0^{2\pi} f(z_0 + re^{i\theta}) e^{-in\theta} d\theta. \end{aligned}$$

Denote by  $\text{Hol}(\Omega)$  the set of holomorphic functions on  $\Omega$  and  $\mathcal{A}(\Omega)$  the set of analytic functions on  $\Omega$ . Since  $\text{Hol}(\Omega) = \mathcal{A}(\Omega)$ , we can conclude the existence of a primitive of holomorphic functions on a disc.

**Proposition 1.6.** *Let  $\Omega \subset \mathbb{C}$  be an open set containing two piecewise smooth curves  $\gamma_1, \gamma_2$  and their interiors, and let  $f: \Omega \rightarrow \mathbb{C}$  be a holomorphic function. If  $\gamma_1$  and  $\gamma_2$  are homotopic in  $\Omega$ , then*

$$\int_{\gamma_1} f(z) dz = \int_{\gamma_2} f(z) dz.$$

The proof is nontrivial. There are many details. The key point is covering two nearby curves with discs since holomorphic functions have primitives on discs.

As a corollary of Theorem 1.4, we have the Liouville theorem by using Cauchy's estimates.

**Theorem 1.7** (Liouville's Theorem). *Let  $f: \mathbb{C} \rightarrow \mathbb{C}$  be a bounded entire function. Then  $f$  is constant.*

**Remark.** *By Liouville's theorem, we can easily prove the fundamental theorem of algebra.*

### 1.3 Analytic Continuation and Uniqueness Theorem

Now we consider the analytic continuation. To prove the uniqueness of analytic continuation, we may use the fact that the zeros of a holomorphic function are isolated unless the function is identically zero.

**Theorem 1.8** (Isolation of Zeros). *Let  $\Omega \subset \mathbb{C}$  be a connected open set, and let  $f: \Omega \rightarrow \mathbb{C}$  be a holomorphic function. If there exists a point  $z_0 \in \Omega$  such that  $f(z_0) = 0$  and  $f$  is not identically zero in any neighborhood of  $z_0$ , then there exists a neighborhood  $D(z_0, r) \subseteq \Omega$  such that  $f(z) \neq 0$  for all  $z \in D(z_0, r) \setminus \{z_0\}$ .*

The key point of the proof is considering the set  $U := \{z \in \Omega: f^{(n)}(z) = 0, \forall n \geq 0\}$ , which is closed (since it is an intersection of closed sets) and open (since the Taylor series shows that it is identically zero in a neighborhood).

Then we obtain the identity theorem which is proved by the isolation of zeros of holomorphic functions.

**Theorem 1.9** (Identity Theorem). *Let  $\Omega \subset \mathbb{C}$  be a connected open set, and let  $f, g: \Omega \rightarrow \mathbb{C}$  be holomorphic functions. If there exists a subset  $S \subset \Omega$  with an accumulation point in  $\Omega$  such that  $f(z) = g(z)$  for all  $z \in S$ , then  $f(z) = g(z)$  for all  $z \in \Omega$ .*

This theorem induces the uniqueness of analytic continuation.

### 1.4 Sequences of Holomorphic Functions

Denote by  $\|f\|_{\infty, U} = \sup_{z \in U} |f(z)|$  for a function  $f: U \rightarrow \mathbb{C}$  and a set  $U \subseteq \mathbb{C}$ .

**Definition 1.10.** Let  $\Omega \subset \mathbb{C}$  be an open set, and let  $(f_n)$  be a sequence of complex valued functions defined on  $\Omega$ . We say that  $(f_n)$  converges locally uniformly if there exists a function  $f: \Omega \rightarrow \mathbb{C}$  such that for all  $x \in \Omega$ , there exists a neighborhood  $U_x \subseteq \Omega$  such that

$$\lim_{n \rightarrow \infty} \|f_n - f\|_{\infty, U_x} = 0.$$

We say that  $(f_n)$  is a local Cauchy sequence if for all  $x \in \Omega$ , there exists a neighborhood  $U_x \subseteq \Omega$  such that  $(f_n)$  is a Cauchy sequence with respect to the norm  $\|\cdot\|_{\infty, U_x}$ , i.e., for all  $\varepsilon > 0$ , there exists  $N > 0$  such that for all  $m, n > N$ ,

$$\|f_n - f_m\|_{\infty, U_x} < \varepsilon.$$

**Proposition 1.11.** *Let  $\Omega \subset \mathbb{C}$  be an open set, and let  $(f_n)$  be a sequence of complex valued functions defined on  $\Omega$ . Then  $(f_n)$  converges locally uniformly to a function  $f: \Omega \rightarrow \mathbb{C}$  if and only if for all compact subset  $K \subseteq \Omega$ ,  $(f_n)$  converges uniformly to  $f$  on  $K$ . Moreover,  $(f_n)$  is a local Cauchy sequence if and only if for all compact subset  $K \subseteq \Omega$ ,  $(f_n)$  is a Cauchy sequence with respect to the norm  $\|\cdot\|_{\infty, K}$ .*

The proof is simple by using the compactness of  $K$  and the fact that  $\mathbb{C}$  is locally compact.

**Proposition 1.12.** *Let  $\Omega \subset \mathbb{C}$  be an open set, and let  $(f_n)$  be a sequence of complex valued functions defined on  $\Omega$ . Then TFAE:*

- $(f_n)$  converges locally uniformly.
- $(f_n)$  is a local Cauchy sequence.

By applying the previous proposition, it is trivial.

**Theorem 1.13** (Weierstrass Convergence Theorem). *Let  $\Omega \subset \mathbb{C}$  be an open set, and let  $(f_n)$  be a sequence of holomorphic functions defined on  $\Omega$ . If  $(f_n)$  converges locally uniformly to a function  $f: \Omega \rightarrow \mathbb{C}$ , then*

- $f$  is holomorphic on  $\Omega$ .
- For all  $k \geq 0$ , the sequence  $(f_n^{(k)})$  converges locally uniformly to  $f^{(k)}$  on  $\Omega$ .

*Proof.* Given a point  $z_0 \in \Omega$ , choose  $r > 0$  such that  $\bar{D}(z_0, r) \subseteq \Omega$ . Since  $f_n$  is continuous,  $f$  is continuous. For all  $z \in D(z_0, r)$ , we can define

$$F(z) = \frac{1}{2\pi i} \int_{C(z_0, r)} \frac{f(\omega)}{\omega - z} d\omega.$$

$F$  is holomorphic on  $D(z_0, r)$  since  $f$  is continuous. Since  $(f_n)$  converges locally uniformly to  $f$  on  $\Omega$ , by Proposition 1.11 it converges uniformly to  $f$  on  $\bar{D}(z_0, r)$ . Hence for all  $\varepsilon > 0$ , there exists  $N > 0$  such that for all  $n > N$  and all  $\omega \in C(z_0, r)$ ,  $|f_n(\omega) - f(\omega)| < \varepsilon$ . Therefore, for all  $z \in D(z_0, r/2)$  and all  $n > N$ ,

$$\begin{aligned} \left| f_n(z) - \frac{1}{2\pi i} \int_{C(z_0, r)} \frac{f(\omega)}{\omega - z} d\omega \right| &= \left| \frac{1}{2\pi i} \int_{C(z_0, r)} \frac{f_n(\omega) - f(\omega)}{\omega - z} d\omega \right| \\ &\leq \frac{1}{2\pi} \int_{C(z_0, r)} \frac{|f_n(\omega) - f(\omega)|}{|\omega - z|} d\omega \\ &< \frac{\varepsilon}{2\pi} \int_{C(z_0, r)} \frac{1}{|\omega - z|} |d\omega| \\ &= \frac{\varepsilon \cdot r}{r - |z - z_0|}. \\ &\leq 2\varepsilon. \end{aligned}$$

Hence,  $f(z) = F(z)$  for all  $z \in D(z_0, r/2)$ . We conclude that  $f$  is holomorphic on  $\Omega$ . The second part can be proved similarly by using the Cauchy Integral Formulas for derivatives.  $\square$

**Corollary 1.14.** *If  $f = \sum f_n$  is a series of holomorphic functions converging locally absolutely on an open set  $\Omega \subset \mathbb{C}$ , then  $f$  is holomorphic on  $\Omega$  and for all  $k \geq 0$ ,*

$$f^{(k)} = \sum f_n^{(k)},$$

where the series on the right converges locally absolutely on  $\Omega$ .

## 1.5 Holomorphic Functions Defined in Terms of Integrals

In the end, we talk about the holomorphic functions defined in terms of integrals. Suppose  $\Omega$  is an open set and  $X$  is a measurable subset of  $\mathbb{R}^N$ . If  $f: \Omega \times X \rightarrow \mathbb{C}$  is a continuous function, then we consider the function defined by

$$f(z) = \int_X F(z, x) dx.$$

**Theorem 1.15.** *Suppose that  $f$  is well-defined for all  $z \in \Omega$ .*

1. *If for each  $x \in X$ , the function  $F(\cdot, x): \Omega \rightarrow \mathbb{C}$  is holomorphic, then  $f: \Omega \rightarrow \mathbb{C}$  is holomorphic.*
2. *If in addition, for each  $z_0 \in \Omega$ , there exists a neighborhood  $D(z_0, r_0) \subseteq \Omega$  and an integrable function  $g_{z_0}: X \rightarrow [0, \infty)$  such that the function  $|F(z, x)| \leq g_{z_0}(x)$  for all  $z \in D(z_0, r_0)$  and almost all  $x \in X$ , then for all  $k \geq 0$  and all  $z \in \Omega$ ,*

$$f^{(k)}(z) = \int_X \frac{\partial^k F}{\partial z^k}(z, x) dx,$$

**Remark.** *Actually, the second condition makes sure that  $f$  is well-defined.*

## 2 Meromorphic Functions and Residue Theorem

### 2.1 Basic Definitions

**Definition 2.1.**  $P$  is a subset of  $\mathbb{C}$ .

- $P$  is called discrete if all points in  $P$  are isolated, i.e. for all  $z \in P$ , there exists  $r > 0$  such that  $D(z, r) \cap P = \{z\}$ .
- Suppose  $\Omega$  is an open set in  $\mathbb{C}$  containing  $P$ .  $P$  is called locally finite in  $\Omega$  if for all  $a \in \Omega$ , there exists a neighborhood  $U_a$  such that  $P \cap U_a$  is finite, or equivalently, for all compact subset  $K \subseteq \Omega$ ,  $P \cap K$  is finite.

**Proposition 2.2.** Suppose  $\Omega$  is an open set in  $\mathbb{C}$  and  $P$  is closed subset of  $\Omega$ . TFAE:

- $P$  is discrete.
- $P$  is locally finite in  $\Omega$ .
- $P$  doesn't have accumulation points in  $\Omega$ . ( $a$  is an accumulation point of  $P$  if for all  $r > 0$ ,  $(D(a, r) \cap P) \setminus \{a\} \neq \emptyset$ .)

**Definition 2.3.** Let  $\Omega \subset \mathbb{C}$  be an open set, and let  $z_0$  be a point in  $\Omega$ . A function  $f: \Omega \setminus \{z_0\} \rightarrow \mathbb{C}$  is holomorphic.

- If there exists a holomorphic function  $g: \Omega \rightarrow \mathbb{C}$  such that  $g(z) = f(z)$  for all  $z \in \Omega \setminus \{z_0\}$ , then we say that  $f$  has a removable singularity at  $z_0$ .
- If there exists an integer  $m \geq 1$  such that  $(z - z_0)^m f(z)$  is holomorphic at  $z_0$  and nonzero at  $z_0$ , then we say that  $f$  has a pole of order  $m$  at  $z_0$ .
- If  $f$  has neither a removable singularity nor a pole at  $z_0$ , then we say that  $f$  has an essential singularity at  $z_0$ .

**Remark.** 1. We can easily distinguish by verifying the value of  $f$  in the neighborhood of  $z_0$ . If  $\lim_{z \rightarrow z_0} f(z)$  exists, then it is a removable singularity. If  $\lim_{z \rightarrow z_0} |f(z)| = +\infty$ , then it is a pole. Otherwise, it is an essential singularity.

2. Here is another definition of pole :  $f$  has a pole of order  $m$  at  $z_0$  if and only if  $1/f$  is a holomorphic function in a neighborhood of  $z_0$  and has a zero of order  $m$  at  $z_0$ .

**Definition 2.4.** Let  $\Omega \subset \mathbb{C}$  be an open set, and let  $P$  be a locally finite subset of  $\Omega$ . A function  $f: \Omega \setminus P \rightarrow \mathbb{C}$  is called meromorphic on  $\Omega$  if  $f$  is holomorphic on  $\Omega \setminus P$  and each point of  $P$  is either a pole or a removable singularity.

**Remark.** Here is another definition of meromorphic functions : A function  $f$  is called meromorphic on  $\Omega$  if for every point  $z$  of  $\Omega$  there is a neighborhood of  $z$  in which at least one of  $f$  and  $1/f$  is holomorphic. (Actually, it is not a good definition since we don't know whether  $f$  together with  $1/f$  are well defined on  $\Omega$ .)

Denote by  $Z(f)$  the set of zeros of  $f$  and  $P(f)$  the set of poles of  $f$ . By Definition 2.4, we can easily verify that if  $f$  is meromorphic on  $\Omega$ , then  $Z(f)$  and  $P(f)$  are both locally finite in  $\Omega$ .

### 2.2 Residue Theorem

**Definition 2.5** (Laurent Series and Residue). Let  $f: \Omega \rightarrow \mathbb{C}$  be a meromorphic function defined on an open set  $\Omega \subset \mathbb{C}$  and let  $z_0$  be a point in  $\Omega$ . Then there exists  $m \in \mathbb{N}$  and  $r > 0$  such that

$$f(z) = \sum_{n=-m}^{+\infty} a_n (z - z_0)^n,$$

for all  $z \in D(z_0, r) \setminus \{z_0\}$ . We called this series the Laurent series of  $f$  centered at  $z_0$ . This series is unique and absolutely convergent on an annulus  $\{r_1 < |z - z_0| < r_2\}$  with  $0 \leq r_1 < r_2 < r$ . And  $f$  has a pole of order  $m$  at  $z_0$  if and only if  $m \geq 1$  and  $a_{-m} \neq 0$ . In particular, the coefficient  $a_{-1}$  is called the residue of  $f$  at  $z_0$ , denoted by  $\text{Res}(f, z_0)$ . Moreover, the coefficients  $a_n$  can be computed by

$$a_n = \frac{1}{2\pi i} \int_{C(z_0, \rho)} \frac{f(\omega)}{(\omega - z_0)^{n+1}} d\omega,$$

for all  $n \geq -m$  and all  $\rho \in (r_1, r_2)$ .

**Example 2.6.** We give some examples about computing residues.

1. If  $f$  has a removable singularity at  $z_0$ , then  $\text{Res}(f, z_0) = 0$ .
2. If  $f$  has a pole of order  $n$  at  $z_0$ , then

$$\text{Res}(f, z_0) = \frac{1}{(n-1)!} \lim_{z \rightarrow z_0} \frac{d^{n-1}}{dz^{n-1}} ((z - z_0)^n f(z)).$$

3. If we can write  $f(z) = \sum_{n=-m}^{+\infty} a_n(z - z_0)^n$  in a neighborhood of  $z_0$ , then  $\text{Res}(f, z_0) = a_{-1}$ , for example,  $f(z) = 1/(z^2 + 1)$ , we can write

$$f(z) = \frac{1}{(z-i)(z+i)} = \frac{1}{2i} \cdot \frac{1}{z-i} - \frac{1}{2i} \cdot \frac{1}{z+i},$$

so we have  $\text{Res}(f, i) = \frac{1}{2i}$  and  $\text{Res}(f, -i) = -\frac{1}{2i}$ .

4. If  $f(z) = g(z)/h(z)$  where  $g$  and  $h$  are holomorphic functions in a neighborhood of  $z_0$ ,  $g(z_0) \neq 0$  and  $h$  has a simple zero at  $z_0$ , then  $f$  has a pole of order 1 at  $z_0$  and

$$\text{Res}(f, z_0) = \frac{g(z_0)}{h'(z_0)}.$$

What is a pole/residue at infinity? We can consider  $g(w) = f(1/w)$ . If  $g$  has a zero of order  $m$  at 0, then we say that  $f$  has a pole of order  $m$  at infinity. And we define the residue of  $f$  at infinity by

$$\text{Res}(f, \infty) = -\text{Res}\left(\frac{1}{w^2} \cdot f\left(\frac{1}{w}\right), 0\right).$$

If  $g$  has an expansion in Laurent series at 0 like  $g(w) = \sum_{n=-m}^{+\infty} a_n w^n$ , then we have

$$\text{Res}(f, \infty) = -a_1.$$

Now we state the residue theorem.

**Theorem 2.7** (Residue Theorem). *Let  $\Omega \subset \mathbb{C}$  be a simply connected open set. Suppose  $f$  is a meromorphic function on  $\Omega$ . If  $\gamma$  is a closed smooth curve in  $\Omega \setminus P(f)$ , then*

$$\int_{\gamma} f(z) dz = 2\pi i \sum_{z_0 \in S} \text{Res}(f, z_0) \cdot \text{Ind}(\gamma, z_0),$$

where  $S \subseteq P(f)$  is the set of poles in the interior of  $\gamma$  and  $\text{Ind}(\gamma, z_0)$  is the winding number of  $\gamma$  around  $z_0$ , i.e.

$$\text{Ind}(\gamma, z_0) = \frac{1}{2\pi i} \int_{\gamma} \frac{1}{z - z_0} dz = \frac{1}{2\pi i} \int_a^b \frac{\gamma'(t)}{\gamma(t) - z_0} dt.$$

**Remark.** The definition of winding number shows that it is always an integer. And if  $\gamma$  is a simple closed curve, then  $\text{Ind}(\gamma, z_0) = 1$  if  $z_0$  is in the interior of  $\gamma$  and  $\text{Ind}(\gamma, z_0) = 0$  otherwise.

**Example 2.8.** We give some examples about applying the residue theorem to compute real integrals.

1. Compute  $\int_{C(0,1)} \frac{1}{z} \left(z + \frac{1}{z}\right)^n dz$  for  $n \in \mathbb{N}$ . And deduce the value of  $\int_0^{2\pi} \cos^n \theta d\theta$ .

$$\int_{C(0,1)} \frac{1}{z} \left(z + \frac{1}{z}\right)^n dz = \begin{cases} 2\pi i \cdot \binom{n}{n/2}, & n \text{ is even,} \\ 0, & n \text{ is odd.} \end{cases}$$

$$\int_0^{2\pi} \cos^n \theta d\theta = \begin{cases} \frac{\pi}{2^{n-1}} \cdot \binom{n}{n/2}, & n \text{ is even,} \\ 0, & n \text{ is odd.} \end{cases}$$

2. Compute  $I_{m,n} = \int_0^{+\infty} \frac{x^n}{1+x^m} dx$  for  $m, n \in \mathbb{N}$  and  $m \geq n+2$ .

We consider the contour integral of  $f(z) = \frac{z^n}{1+z^m}$  on a sector with angle  $\frac{2\pi}{m}$ .

$$I_{m,n} = \frac{\pi}{m \sin\left(\frac{(n+1)\pi}{m}\right)}.$$

3. Compute  $\int_0^{2\pi} \frac{dt}{a^2 \cos^2 t + b^2 \sin^2 t}$  for  $a > b > 0$ .

We consider the change of variables  $z = e^{it}$ .

$$\int_0^{2\pi} \frac{dt}{a^2 \cos^2 t + b^2 \sin^2 t} = \frac{2\pi}{ab}.$$

4. Compute  $I(s) = \int_0^{+\infty} \frac{x^{s-1}}{1+x} dx$  for  $0 < \text{Re}(s) < 1$ .

We consider the change of variables  $x = e^t$  and the contour integral of  $f(z) = \frac{e^{sz}}{1+e^z}$  on the rectangle  $(-R, R, R + 2\pi i, -R + 2\pi i)$ .

$$I(s) = \frac{\pi}{\sin(\pi s)}.$$

**Theorem 2.9** (Argument Principle). *Let  $\Omega \subset \mathbb{C}$  be a simply connected open set containing a closed piecewise smooth curve  $C$  and its interior. Suppose  $f$  is a meromorphic function on  $\Omega$  which has no zeros or poles on  $C$ . Then*

$$\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} dz = N - P,$$

where  $N$  is the number of zeros of  $f$  inside  $C$  counting multiplicities, and  $P$  is the number of poles of  $f$  inside  $C$  counting multiplicities.

Note that for all  $z_0 \in \Omega$ ,  $f(z) = (z - z_0)^m g(z)$  with  $m \in \mathbb{Z}$  and  $g(z)$  a holomorphic function which is nonzero in a neighborhood of  $z_0$ . The key point is applying the residue theorem to  $\frac{f'(z)}{f(z)}$ .

### 2.3 Some Important Theorems about Holomorphic Functions

Now, we can give some important theorems about holomorphic functions.

**Theorem 2.10** (Rouché's Theorem). *Let  $\Omega \subset \mathbb{C}$  be a connected open set containing a circle  $C$  and its interior, and let  $f, g: \Omega \rightarrow \mathbb{C}$  be holomorphic functions. If  $|f(z)| > |g(z)|$  for all  $z \in C$ , then  $f$  and  $f + g$  have the same number of zeros (counting multiplicities) inside  $C$ .*

We can prove Rouché's Theorem by applying the argument principle to  $f_t(z) = f(z) + tg(z)$ . We can conclude that

$$n_t := \frac{1}{2\pi i} \int_C \frac{f'_t(z)}{f_t(z)} dz$$

is constant for  $t \in [0, 1]$ .

**Theorem 2.11** (Open Mapping Theorem). *If  $f: \Omega \rightarrow \mathbb{C}$  is a non-constant holomorphic function defined on a connected open set  $\Omega \subset \mathbb{C}$ , then  $f$  is an open map.*

By Rouché's Theorem, we can prove the open mapping theorem. Fix  $z_0 \in \Omega$ , there exists  $\varepsilon > 0$  such that for all  $w \in D(f(z_0), \varepsilon) \cap \text{Im}(f)$ , we can write  $f(z) - w = g(z) + h(z)$  where  $g(z) = f(z) - f(z_0)$  and  $h(z) = f(z_0) - w$  and applying Rouché's Theorem on a small circle centered at  $z_0$ . Then we can conclude that  $f(\Omega)$  is open. By the restriction, we can also conclude that  $f$  is open.

**Theorem 2.12** (Maximum Modulus Principle). *Let  $\Omega \subset \mathbb{C}$  be a connected open set, and let  $f: \Omega \rightarrow \mathbb{C}$  be a holomorphic function. If there exists a point  $z_0 \in \Omega$  such that  $|f(z_0)| \geq |f(z)|$  for all  $z \in \Omega$ , then  $f$  is constant.*

By the open mapping theorem, the proof is trivial.

**Corollary 2.13.** *Let  $\Omega \subset \mathbb{C}$  be a connected open set, and let  $f: \overline{\Omega} \rightarrow \mathbb{C}$  be a restriction of holomorphic function on  $\overline{\Omega}$ . Then  $\sup_{z \in \overline{\Omega}} |f(z)| = \sup_{z \in \partial\Omega} |f(z)|$ .*

And now we give an important lemma about holomorphic functions on the unit disc.

**Lemma 2.14** (Schwarz Lemma). *Let  $f: \mathbb{D} \rightarrow \mathbb{D}$  be a holomorphic function with  $f(0) = 0$ , where  $\mathbb{D} = \{z \in \mathbb{C}: |z| < 1\}$  is the unit disc. Then  $|f(z)| \leq |z|$  for all  $z \in \mathbb{D}$ , and  $|f'(0)| \leq 1$ . Moreover, if there exists a point  $z_0 \in \mathbb{D} \setminus \{0\}$  such that  $|f(z_0)| = |z_0|$  or if  $|f'(0)| = 1$ , then  $f(z) = e^{i\theta} z$  for some  $\theta$ .*

**Remark.** *By Schwarz lemma, we can compute the automorphism group of the unit disc, which is*

$$\left\{ e^{i\theta} \frac{a - z}{1 - \bar{a}z} : a \in \mathbb{D}, \theta \in \mathbb{R} \right\}.$$

Suppose  $\psi_a(z) = \frac{a-z}{1-\bar{a}z}$ , we have  $\psi_a \circ \psi_a = \text{id}$ .

The key point of the proof is considering the function  $g(z) = \frac{f(z)}{z}$  and applying the maximum modulus principle.

**Theorem 2.15** (Schwarz-Pick Theorem). *Let  $f: \mathbb{D} \rightarrow \mathbb{D}$  be a holomorphic function. Then for all  $z_1, z_2 \in \mathbb{D}$ ,*

$$\left| \frac{f(z_1) - f(z_2)}{1 - \overline{f(z_2)}f(z_1)} \right| \leq \left| \frac{z_1 - z_2}{1 - \bar{z}_2 z_1} \right|,$$

and for all  $z \in \mathbb{D}$ ,

$$\frac{|f'(z)|}{1 - |f(z)|^2} \leq \frac{1}{1 - |z|^2}.$$

The key point of the proof is considering the function  $\psi_{f(z_2)} \circ f \circ \psi_{z_2}$  and applying Schwarz lemma.

And at last, we give the existence of holomorphic primitive and holomorphic logarithm. In Section 1.2, we have proved that holomorphic functions on a disc have holomorphic primitives. Now we consider the general case.

**Theorem 2.16** (Existence of Holomorphic Primitive). *Let  $\Omega \subset \mathbb{C}$  be a simply connected open set. Then all holomorphic functions  $f: \Omega \rightarrow \mathbb{C}$  have holomorphic primitives on  $\Omega$ .*

Fix  $z_0 \in \Omega$ , we can define

$$F(z) = \int_{\gamma} f(\omega) d\omega,$$

where  $\gamma$  is a piecewise smooth curve in  $\Omega$  from  $z_0$  to  $z$ . By the simply connectivity of  $\Omega$  and Proposition 1.3,  $F(z)$  is well-defined. Then we can easily verify that  $F$  is a holomorphic primitive of  $f$ .

**Remark.** *The simply connectivity of  $\Omega$  contains the connectivity of  $\Omega$ . And for the open set  $\Omega \subset \mathbb{C}$ , the connectivity is equivalent to the path connectivity. So we can define the holomorphic primitive by the integral along a path from a fixed point to  $z$ .*

**Corollary 2.17.** *Let  $\Omega \subset \mathbb{C}$  be a simply connected open set containing a closed piecewise smooth curve  $\gamma$  and its interior. Then for all holomorphic functions  $f: \Omega \rightarrow \mathbb{C}$ , we have*

$$\int_{\gamma} f(z) dz = 0.$$

**Theorem 2.18** (Existence of Holomorphic Logarithm). *Let  $\Omega \subset \mathbb{C}$  be a simply connected open set. Then for all  $f \in \text{Hol}(\Omega)$  which don't have zeros, there exists a holomorphic logarithm  $g := \log(f(z))$  such that  $g'(z) = f'(z)/f(z)$  for all  $z \in \Omega$ .*

It is trivial by Theorem 2.16 since  $f'(z)/f(z)$  is holomorphic on  $\Omega$ .

**Remark.** *We can also write  $f(z) = e^{g(z)}$ .*

**Example 2.19.** 1. The simply connectivity is necessary. We give a counterexample. Consider the function  $f(z) = z$  defined on  $\Omega = \mathbb{C} \setminus \{0\}$ . If there exists a holomorphic logarithm  $g: \Omega \rightarrow \mathbb{C}$  such that  $g'(z) = 1/z$  for all  $z \in \Omega$ , then by the residue theorem, we have

$$\int_{C(0,1)} \frac{1}{z} dz = 2\pi i \neq 0,$$

which contradicts the existence of holomorphic primitive of  $1/z$  on  $\Omega$ .

2. In the slit plane  $\mathbb{C} \setminus (-\infty, 0]$ , we have the principal branch of logarithm.

$$\log z = \log |z| + i \arg z,$$

where  $\arg z \in (-\pi, \pi)$ .

3. For all  $\alpha \in \mathbb{C}$ , we can define the principal branch of power function in the slit plane  $\mathbb{C} \setminus (-\infty, 0]$  by

$$z^\alpha = e^{\alpha \log z}.$$

## 3 Conformal Mappings

Question : Given two open sets  $\Omega_1, \Omega_2 \subset \mathbb{C}$ , does there exist a biholomorphic bijection  $f: \Omega_1 \rightarrow \Omega_2$ ? If such a mapping exists, can we find it?

### 3.1 Examples

Here are some examples of biholomorphic bijections.

**Example 3.1.** 1. The map  $f(z) = az + b$  with  $a \neq 0$  is a biholomorphic bijection from  $\mathbb{C}$  to  $\mathbb{C}$ .

2. The map  $f(z) = e^z$  is a biholomorphic bijection from the strip  $S = \{z \in \mathbb{C}: -\pi < \text{Im}(z) < \pi\}$  to  $\mathbb{C} \setminus \{0\}$ .

3. The map  $f(z) = -\frac{1}{2}(z + 1/z)$  is a biholomorphic bijection from the half-disc  $\{z = x + iy \in \mathbb{C}: y > 0, |z| < 1\}$  to the upper half-plane  $\{z \in \mathbb{C}: \text{Im}(z) > 0\}$ .

4. The map  $e^{i\theta} \psi_a(z) = e^{i\theta} \frac{a-z}{1-\bar{a}z}$  with  $a \in \mathbb{D}$  and  $\theta \in \mathbb{R}$  is a biholomorphic bijection from the unit disc  $\mathbb{D}$  to  $\mathbb{D}$ .

5. The map  $f(z) = \frac{z-z_0}{z-z_1}$  with  $z_0, z_1 \in \mathbb{C}, z_0 \neq z_1$  is a biholomorphic bijection from  $\mathbb{C} \setminus \{z_1\}$  to  $\mathbb{C} \setminus \{1\}$ .

6. The map  $f(z) = \frac{az+b}{cz+d}$  with  $ad-bc \neq 0$  is a biholomorphic bijection from  $\mathbb{C}_\infty$  to  $\mathbb{C}_\infty$ , where  $\mathbb{C}_\infty = \mathbb{C} \sqcup \{\infty\}$  is the Riemann sphere.



Actually,  $\text{Aut}_{\text{hol}}(\mathbb{C}) = \{az + b \mid a \neq 0\}$ ,  $\text{Aut}_{\text{hol}}(\mathbb{D}) = \{e^{i\theta}\psi_a \mid a \in \mathbb{D}, \theta \in \mathbb{R}\}$ ,  $\text{Aut}_{\text{hol}}(\mathbb{C}_\infty) = \left\{\frac{az+b}{cz+d} \mid ad-bc \neq 0\right\}$ .

Consider the upper half-plane  $\mathbb{H} = \{z \in \mathbb{C} : \text{Im}(z) > 0\}$ . The map  $F(z) = \frac{z-i}{z+i}$  is a biholomorphic bijection from  $\mathbb{H}$  to  $\mathbb{D}$  and the map  $G(z) = i\frac{1+z}{1-z} = F^{-1}(z)$  is a biholomorphic bijection from  $\mathbb{D}$  to  $\mathbb{H}$ . Hence  $\mathbb{H}$  and  $\mathbb{D}$  are biholomorphically equivalent. In particular, we can conclude that  $\text{Aut}_{\text{hol}}(\mathbb{H}) = \{G \circ f \circ F \mid f \in \text{Aut}_{\text{hol}}(\mathbb{D})\}$ . And we have the following theorem.

**Theorem 3.2.** *Every automorphism of  $\mathbb{H}$  is a fractional linear transformation taking the form  $f_M$ , where*

$$f_M(z) = \frac{az+b}{cz+d}, \quad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{R}).$$

Moreover,  $\text{Aut}_{\text{hol}}(\mathbb{H}) \simeq \text{PSL}_2(\mathbb{R})$ .

**Proposition 3.3.** *If  $f: \Omega_1 \rightarrow \Omega_2$  is holomorphic and injective, then  $f'(z) \neq 0$  for all  $z \in \Omega_1$ . In particular, the inverse of  $f$  defined on its range is holomorphic, and thus the inverse of a conformal map is also holomorphic.*

The key point of the proof is considering the Taylor series of  $f$  at a point  $z_0 \in \Omega_1$ . If  $f'(z_0) = 0$ , then the Taylor series is

$$f(z) = f(z_0) + a_n(z - z_0)^n + a_{n+1}(z - z_0)^{n+1} + \dots,$$

with  $n \geq 2$  and  $a_n \neq 0$ . Then we can easily find two different points  $z_1, z_2$  in a small neighborhood of  $z_0$  by Rouché's theorem such that  $f(z_1) = f(z_2)$ , which contradicts the injectivity of  $f$ .

## 3.2 Riemann Mapping Theorem

Actually, I never really understand the Riemann mapping theorem until I write down its proof carefully.

**Theorem 3.4** (Riemann Mapping Theorem). *Let  $\Omega \subset \mathbb{C}$  be a simply connected open set which is not equal to  $\mathbb{C}$ . Then for all  $z_0 \in \Omega$ , there exists a unique conformal map  $f: \Omega \rightarrow \mathbb{D}$  such that  $f(z_0) = 0$  and  $f'(z_0) > 0$ .*

Before proving the Riemann mapping theorem, we need some preparations.

**Theorem 3.5** (Hurwitz's Theorem). *Suppose  $(f_n)$  is a sequence of holomorphic functions defined on a connected open set  $\Omega \subseteq \mathbb{C}$ . And  $(f_n)$  converges locally uniformly to a holomorphic function  $f: \Omega \rightarrow \mathbb{C}$ . Then*

1. *If for all  $n \in \mathbb{N}$ ,  $f_n$  is not identically zero, then either  $f$  is identically zero or  $f$  doesn't have zeros in  $\Omega$ .*
2. *If for all  $n \in \mathbb{N}$ ,  $f_n$  is injective, then either  $f$  is constant or  $f$  is injective.*

For the first part, the key point of the proof is proving that if  $f$  is not identically zero, then  $f'_n/f_n$  converges uniformly to  $f'/f$  on a small circle where  $f(z)$  does not vanish. Then we can apply the argument principle to conclude. The second part can be proved similarly by considering  $g_n(z) = f_n(z) - f_n(w)$  for  $w \in \Omega$ .

**Theorem 3.6** (Montel's Theorem). *Let  $\Omega \subset \mathbb{C}$  be an open set, and let  $\mathcal{F}$  be a family of holomorphic functions  $f: \Omega \rightarrow \mathbb{C}$ . Then TFAE:*

- $\mathcal{F}$  is normal, i.e. every sequence in  $\mathcal{F}$  has a subsequence which converges uniformly on every compact subset of  $\Omega$  to a holomorphic function (this limit function may be not in  $\mathcal{F}$ ).
- $\mathcal{F}$  is uniformly bounded on every compact subset of  $\Omega$ , i.e. for every compact subset  $K \subset \Omega$ , there exists a constant  $M_K > 0$  such that  $|f(z)| \leq M_K$  for all  $z \in K$  and all  $f \in \mathcal{F}$ .
- $\mathcal{F}$  is locally uniformly bounded, i.e. for every point  $z_0 \in \Omega$ , there exists a neighborhood  $U \subset \Omega$  of  $z_0$  and a constant  $M_U > 0$  such that  $|f(z)| \leq M_U$  for all  $z \in U$  and all  $f \in \mathcal{F}$ .

*Proof.* The equivalence between the second and third statements is trivial. The implication from the first to the second statement is also trivial by the definition of normal family. Now we prove the implication from the third to the first statement.

Suppose  $(f_n)$  is a sequence in  $\mathcal{F}$ . We need to show that it has a subsequence converging uniformly on compact subsets of  $\Omega$  to a holomorphic function.

### • Step 1 : Local uniform boundedness implies equicontinuity on compact sets.

Let  $K \subset \Omega$  be compact. Since  $\mathcal{F}$  is locally uniformly bounded, for each  $z \in K$ , there exists  $r_z > 0$  and  $M_z > 0$  such that  $D(z, 2r_z) \subset \Omega$  and  $|f(w)| \leq M_z$  for all  $w \in D(z, 2r_z)$  and all  $f \in \mathcal{F}$ . The open discs  $\{D(z, r_z) : z \in K\}$  cover  $K$ , so by compactness, there exists a finite subcover  $\{D(z_i, r_{z_i})\}_{i=1}^N$ . Let  $M = \max\{M_{z_1}, \dots, M_{z_N}\}$  and  $r = \min\{r_{z_1}, \dots, r_{z_N}\}$ .

Now take any  $z, w \in K$  with  $|z - w| < r/2$ . Then  $z \in D(z_i, r_{z_i})$  for some  $i$ , and  $w \in D(z_i, 2r_{z_i})$  since  $|w - z_i| \leq |w - z| + |z - z_i| < r/2 + r_{z_i} \leq 2r_{z_i}$ . By Cauchy's integral formula for derivatives, for any  $f \in \mathcal{F}$ :

$$f'(z) = \frac{1}{2\pi i} \int_{C(z_i, 2r_{z_i})} \frac{f(\zeta)}{(\zeta - z)^2} d\zeta.$$

On the circle  $|\zeta - z_i| = 2r_{z_i}$ , we have  $|\zeta - z| \geq |\zeta - z_i| - |z_i - z| > 2r_{z_i} - r_{z_i} = r_{z_i}$ , so

$$|f'(z)| \leq \frac{2M}{r_{z_i}} \leq \frac{2M}{r}.$$

Thus,  $\mathcal{F}$  has uniformly bounded derivatives on  $K$ , and by the mean value theorem, for  $|z - w| < r/2$ :

$$|f(z) - f(w)| \leq \sup_{\xi \in [z, w]} |f'(\xi)| \cdot |z - w| \leq \frac{2M}{r} |z - w|.$$

This shows  $\mathcal{F}$  is equicontinuous on  $K$ .

• **Step 2 : Application of Arzelà-Ascoli theorem.**

Let  $\{K_j\}_{j=1}^\infty$  be an exhaustion of  $\Omega$  by compact sets (i.e.,  $K_j \subset \text{int}(K_{j+1})$  and  $\bigcup_{j=1}^\infty K_j = \Omega$ ). The existence of such an exhaustion is trivial. We consider the set  $K_j = \{z \in \Omega \mid d(z, \partial\Omega) \geq 1/j, |z| \leq j\}$ .

On  $K_1$ , the family  $\mathcal{F}$  is uniformly bounded and equicontinuous, so by Arzelà-Ascoli,  $(f_n)$  has a subsequence  $(f_{n,1})$  converging uniformly on  $K_1$ . On  $K_2$ , the sequence  $(f_{n,1})$  has a subsequence  $(f_{n,2})$  converging uniformly on  $K_2$ . Continue this process to get subsequences  $(f_{n,j})$  converging uniformly on  $K_j$ . Now consider the diagonal sequence  $g_n = f_{n,n}$ . This sequence converges uniformly on every  $K_j$ , hence on every compact subset of  $\Omega$ .

• **Step 3 : The limit is holomorphic.**

Let  $g$  be the limit function. Since each  $g_n$  is holomorphic and the convergence is uniform on compact sets, by Weierstrass convergence theorem,  $g$  is holomorphic on  $\Omega$ . This completes the proof that  $\mathcal{F}$  is normal. □

*Proof of Riemann Mapping Theorem.* Once we have established the technical results above, the rest of the proof is very elegant. It consists of four steps, which we isolate.

- Step 1 : Uniqueness. Suppose  $f_1, f_2: \Omega \rightarrow \mathbb{D}$  are two conformal maps such that  $f_1(z_0) = f_2(z_0) = 0$  and  $f_1'(z_0), f_2'(z_0) > 0$ . Consider the map

$$g(z) = f_2 \circ f_1^{-1}(z).$$

Then  $g: \mathbb{D} \rightarrow \mathbb{D}$  is a conformal map with  $g(0) = 0$ . By the Schwarz lemma, we have  $g(z) = e^{i\theta}z$ . Since  $g'(0) = f_2'(z_0)/f_1'(z_0) > 0$ , we have  $\theta = 0$ . Hence  $g(z) = z$  and  $f_1 = f_2$ .

- Step 2 : We claim that  $\Omega$  is conformally equivalent to an open subset of the unit disc that contains the origin. Choose a point  $a \in \mathbb{C} \setminus \Omega$  (possible since  $\Omega \neq \mathbb{C}$ ). Since  $z - a \neq 0$  on  $\Omega$ , by Theorem 2.18, there exists a holomorphic logarithm  $h: \Omega \rightarrow \mathbb{C}$  such that  $e^{h(z)} = z - a$  for all  $z \in \Omega$ , which proves in particular that  $h$  is injective. Pick a point  $z_0 \in \Omega$ , by contradiction, we have

$$h(z) \neq h(z_0) + 2\pi i, \forall z \in \Omega.$$

And by contradiction, there exists a disc centered at  $h(z_0) + 2\pi i$  that contains no points of the image  $h(\Omega)$ . So we can define a holomorphic function  $H$  on  $\Omega$  by

$$H(z) = \frac{1}{h(z) - (h(z_0) + 2\pi i)}.$$

Since  $h$  is injective, by Proposition 3.3,  $H: \Omega \rightarrow H(\Omega)$  is a conformal map. Moreover,  $H(\Omega)$  is bounded. We may therefore translate and rescale the function  $H$  in order to obtain a conformal map from  $\Omega$  to an open subset of  $\mathbb{D}$  that contains the origin.

- Step 3 : By the second step, we may assume that  $\Omega$  is an open subset of  $\mathbb{D}$  with  $0 \in \Omega$ . Consider the family

$$\mathcal{F} = \{f: \Omega \rightarrow \mathbb{D} \mid f \text{ is holomorphic, injective, } f(0) = 0\}.$$

$\mathcal{F}$  is non-empty since the inclusion map is in  $\mathcal{F}$ . Moreover,  $\mathcal{F}$  is uniformly bounded by definition.

**Goal:** find  $f \in \mathcal{F}$  such that  $|f'(0)|$  is maximal.

By Cauchy's estimates,  $|f'(0)|$  is bounded. Suppose  $s = \sup_{f \in \mathcal{F}} |f'(0)|$ . Then we can choose a sequence  $(f_n)$  in  $\mathcal{F}$  such that  $\lim_{n \rightarrow \infty} |f_n'(0)| = s$ . By Montel's theorem, there exists a subsequence  $(f_{n_k})$  which converges uniformly on every compact subset of  $\Omega$  to a holomorphic function  $f: \Omega \rightarrow \mathbb{D}$ . Since  $s \geq 1$  (because the identity map is in  $\mathcal{F}$ ),  $f$  is non-constant, hence injective by Hurwitz's theorem. Since we clearly have  $f(0) = 0$ , we have  $f \in \mathcal{F}$  with  $|f'(0)| = s$ .

- Step 4 : We claim that  $f$  is a conformal map from  $\Omega$  to  $\mathbb{D}$ . Since  $f$  is already injective, it suffices to prove  $f(\Omega) = \mathbb{D}$ . If this were not true, we could construct a function in  $\mathcal{F}$  with derivative at 0 greater than  $s$ . Suppose there exists a point  $\alpha \in \mathbb{D} \setminus f(\Omega)$ . Consider the map

$$\psi_\alpha = \frac{\alpha - z}{1 - \bar{\alpha}z},$$

which is an automorphism of  $\mathbb{D}$  that interchanges  $\alpha$  and 0. Since  $\Omega$  is simply connected, so is  $U = (\psi_\alpha \circ f)(\Omega)$ , and moreover,  $0 \notin U$ . It is therefore possible to define a square root function on  $U$  by

$$g(w) = e^{\frac{1}{2} \log(w)}, w \in U.$$

Next, consider the function

$$F = \psi_{g(\alpha)} \circ g \circ \psi_\alpha \circ f.$$

We claim that  $F \in \mathcal{F}$ . Clearly  $F$  is holomorphic and it maps 0 to 0. Also  $F$  maps into the unit disc since this is true of each of the functions in the composition. Finally,  $F$  is injective.

Denote by  $\varphi$  the square function  $\varphi(z) = z^2$ . Then we have

$$f = \psi_\alpha^{-1} \circ \varphi \circ \psi_{g(\alpha)}^{-1} \circ F =: \Phi \circ F.$$

But  $\Phi$  maps  $\mathbb{D}$  into  $\mathbb{D}$  with  $\Phi(0) = 0$ , and is not injective because  $F$  is and  $h$  is not. By the last part of the Schwarz lemma, we conclude that  $|\Phi'(0)| < 1$ . The proof is complete once we observe that

$$f'(0) = \Phi'(0)F'(0),$$

and thus

$$|f'(0)| < |F'(0)|,$$

contradicting the maximality of  $|f'(0)|$  in  $\mathcal{F}$ . Finally, we multiply  $f$  by a complex number of absolute value 1 so that  $f'(0) > 0$ , which ends the proof. □

This proof is copied from *Complex Analysis* by Elias M. Stein and Rami Shakarchi. Furthermore, in this book, there is another proof from Koebe who applies ideas of Carathéodory. We will introduce it in the future.

## 4 Some Interesting Problems

**Problem 1.** Although  $\Omega = \mathbb{C} \setminus \{0\}$  is not simply connected, prove that  $e^{\frac{1}{z}} - \frac{1}{z}$  has a holomorphic primitive on  $\Omega$ .

**Solution 1.** Consider the Laurent series of  $e^{\frac{1}{z}} - \frac{1}{z}$  at 0, we have

$$e^{\frac{1}{z}} - \frac{1}{z} = \sum_{n=0}^{+\infty} \frac{1}{n!} \cdot \frac{1}{z^n} - \frac{1}{z} = 1 + \frac{1}{2!} \cdot \frac{1}{z^2} + \frac{1}{3!} \cdot \frac{1}{z^3} + \cdots.$$

So we can see that  $e^{\frac{1}{z}} - \frac{1}{z}$  has a primitive

$$F(z) = z - \sum_{n=1}^{+\infty} \frac{1}{n!} \cdot \frac{1}{(n-1)z^{n-1}}.$$

Since the series is convergent absolutely on every compact subset of  $\Omega$ , by Weierstrass convergence theorem,  $F$  is holomorphic on  $\Omega$ .

**Problem 2.** Suppose  $\Omega = \mathbb{C} \setminus [0, 1]$ . Prove that there exist two different holomorphic functions well defined on  $\Omega$  such that their squares are  $z \mapsto z(z-1)$ .

**Solution 2.** We can prove it in 3 steps.

- Step 1 : if we can find two different holomorphic functions  $h_1$  and  $h_2$  satisfying  $h_1^2(z) = h_2^2(z) = z(z-1)$ , then  $h_1 = -h_2$ .
- Step 2 : For  $\mathbb{C} \setminus (-\infty, 1]$ , we can define

$$h_1(z) = \exp\left(\frac{1}{2} \log z + \frac{1}{2} \log(z-1)\right).$$

- Step 3 : We can extend  $h_1$  to  $\Omega$  by defining  $h(x) = -\sqrt{x(x-1)}$  for  $x \in \mathbb{R}_{<0}$ . Then we need to verify that the extension is holomorphic at every point in  $(-\infty, 0)$ .

**Problem 3.** For which  $\alpha \in \mathbb{C}$ ,  $f_\alpha: \mathbb{C} \setminus (-\infty, 0] \rightarrow \mathbb{C}$  defined by  $f_\alpha(z) = z^\alpha$  has an analytic continuation on  $\mathbb{C}^*$  or  $\mathbb{C}$ ?

**Solution 3.** If  $f_\alpha$  has an analytic continuation on  $\mathbb{C}^*$ , we can see that

$$\lim_{\theta \rightarrow \pi} f_\alpha(e^{i\theta}) = \lim_{\theta \rightarrow \pi} f_\alpha(e^{-i\theta}),$$

which gives us  $e^{2\pi i\alpha} = 1$ . Hence  $\alpha \in \mathbb{Z}$ . Then we can check for all  $\alpha \in \mathbb{Z}$ ,  $f_\alpha$  can be extended to  $\mathbb{C}^*$  by defining  $f_\alpha(x) = x^\alpha$  for all  $x \in (-\infty, 0]$ .

If  $f_\alpha$  has an analytic continuation on  $\mathbb{C}$ , we obtain  $\alpha \in \mathbb{N}$  since 0 is a pole if  $\alpha < 0$ . And we can check that  $f_\alpha$  can be extended for all  $\alpha \in \mathbb{N}$ .

**Problem 4.** Suppose  $X = \{z \in \mathbb{C} \mid \operatorname{Re}(z) = 0, |\operatorname{Im}(z)| \geq 1\}$  and  $\Omega = \mathbb{C} \setminus X$ .

(a) Prove that there exists a unique holomorphic function  $f: \Omega \rightarrow \mathbb{C}$  such that  $f'(z) = \frac{1}{1+z^2}$  and  $f(0) = 0$ .

(b) Does  $z \mapsto \frac{1}{1+z^2}$  have a holomorphic primitive on  $\mathbb{C} \setminus \{\pm i\}$ ?

(c) Does  $z \mapsto \frac{1}{1+z^2}$  have a holomorphic primitive on  $\mathbb{C} \setminus [-i, i]$ ?

**Solution 4.** (a) Since  $\Omega$  is simply connected as a star-shaped domain, by Theorem 2.16, there exists a holomorphic primitive  $f$  of  $z \mapsto \frac{1}{1+z^2}$  on  $\Omega$ . And the uniqueness is trivial.

(b) No. If there exists a holomorphic primitive  $F$  of  $z \mapsto \frac{1}{1+z^2}$  on  $\mathbb{C} \setminus \{\pm i\}$ , then by the residue theorem, we have

$$\int_{C(i,1)} \frac{1}{1+z^2} dz = \pi \neq 0,$$

which contradicts the existence of holomorphic primitive.

(c) Yes. At first, we can define a primitive  $F$  of  $z \mapsto \frac{1}{1+z^2}$  on  $\mathbb{C} \setminus [-i\infty, i]$  by

$$F(z) = \int_{[2i, z]} \frac{1}{1+\omega^2} d\omega,$$

Then we can extend  $F$  to  $\mathbb{C} \setminus [-i, i]$  by defining

$$F(x) = \int_{\gamma_x} \frac{1}{1+\omega^2} d\omega, \quad x \in (-i\infty, i),$$

where  $\gamma_x$  is a piecewise smooth path in  $\mathbb{C} \setminus [-i, i]$  from  $2i$  to  $x$  which is contained in either the right half-plane or the left half-plane. Note that by the residue theorem, for all simple closed piecewise smooth curve  $\gamma$  in  $\mathbb{C} \setminus [-i, i]$  around the segment  $[-i, i]$ , the integral along  $\gamma$  is 0. So the extension is well-defined. Finally, we only need to verify that the extension is holomorphic at every point in  $(-i\infty, -i)$ .

**Problem 5.** Suppose  $f$  is an entire function such that  $|f(z)| = 1$  for all  $z$  with  $|z| = 1$ . Prove that  $f(z) = az^n$  with  $|a| = 1$  and  $n \in \mathbb{N}$ .

**Solution 5.** Consider the function  $g(z) = \overline{f(1/\bar{z})}^{-1}$ , which is holomorphic on  $\mathbb{C} \setminus \{0\}$ . Since  $|f(z)| = 1$  for all  $z$  with  $|z| = 1$ , we have  $f(z) = g(z)$  for all  $z$  with  $|z| = 1$ . By the identity theorem, we have  $f(z) = g(z)$  for all  $z \in \mathbb{C} \setminus \{0\}$ .

If  $f(0) \neq 0$ , we can regard  $f$  as the analytic continuation of  $g$  on  $\mathbb{C}$ . By defining  $g(0) = f(0)$ . Moreover, we can prove that  $g$  is a bounded entire function. By Liouville's theorem,  $g$  is constant.

If  $f(0) = 0$ , we can write  $f(z) = z^n h(z)$  with  $n \in \mathbb{N}$  and  $h$  an entire function with  $h(0) \neq 0$ . Then we can see that  $|h(z)| = 1$  for all  $z$  with  $|z| = 1$ . By the above argument, we can conclude that  $h$  is constant.

**Problem 6.** Suppose  $(f_n)$  is a sequence of holomorphic functions defined on an open set  $\Omega \subseteq \mathbb{C}$ . If  $(f_n)$  converges pointwise to a function  $f: \Omega \rightarrow \mathbb{C}$ . And for all compact  $K \subset \Omega$ , there exists a constant  $C(K)$  such that

$$\sup_{z \in K} |f_n(z)| \leq C(K), \quad \forall n \in \mathbb{N}.$$

Prove that  $f$  is holomorphic on  $\Omega$  and for all  $k \in \mathbb{N}$  and for all compact  $K \subset \Omega$ , there exists a constant  $C(K, k)$  such that

$$\sup_{z \in K} |f_n^{(k)}(z)| \leq C(K, k), \quad \forall n \in \mathbb{N}.$$

**Solution 6.** We can see that  $(f_n)$  is locally uniformly bounded. By Montel's theorem, there exists a subsequence  $(f_{n_k})$  which converges uniformly on every compact subset of  $\Omega$  to a holomorphic function  $g: \Omega \rightarrow \mathbb{C}$ . Since  $(f_n)$  converges pointwise to  $f$ , we have  $f = g$  on  $\Omega$ . Hence  $f$  is holomorphic on  $\Omega$  and

Let  $K \subset \Omega$  be compact. Since  $(f_n)$  is locally uniformly bounded, for each  $z \in K$ , there exists  $r_z > 0$  and  $M_z > 0$  such that  $D(z, 2r_z) \subset \Omega$  and  $|f_n(w)| \leq M_z$  for all  $w \in D(z, 2r_z)$  and all  $n \in \mathbb{N}$ . The open discs  $\{D(z, r_z): z \in K\}$  cover  $K$ , so by compactness, there exists a finite subcover  $\{D(z_i, r_{z_i})\}_{i=1}^N$ . Let  $M = \max\{M_{z_1}, \dots, M_{z_N}\}$  and  $r = \min\{r_{z_1}, \dots, r_{z_N}\}$ .

By Cauchy's integral formula for derivatives, for any  $n \in \mathbb{N}$  and  $z \in K$ :

$$f^{(k)}(z) = \frac{k!}{2\pi i} \int_{C(z_i, 2r_{z_i})} \frac{f(\omega)}{(\omega - z)^{k+1}} d\omega.$$

On the circle  $|\omega - z_i| = 2r_{z_i}$ , we have  $|\omega - z| \geq |\omega - z_i| - |z_i - z| > 2r_{z_i} - r_{z_i} = r_{z_i}$ , so

$$\left| f^{(k)}(z) \right| \leq \frac{k! \cdot 2M}{r_{z_i}^k} \leq \frac{k! \cdot 2M}{r^k}.$$

Hence we can take  $C(K, k) = k! \cdot 2M/r^k$ .

**Problem 7.** If  $\Omega$  is a connected open bounded set in  $\mathbb{C}$  with  $C^1$  boundary. Suppose that there exist a sequence of automorphisms  $f_n: \Omega \rightarrow \Omega$  with a point  $a \in \Omega$  such that  $f_n(a) \rightarrow b \in \partial\Omega$  as  $n \rightarrow +\infty$ . Prove that  $\Omega$  is conformally equivalent to the unit disc.

**Solution 7.** By Riemann mapping theorem, it suffices to prove that  $\Omega$  is simply connected. We conclude it by three steps.

- Step 1 : Since  $\Omega$  is bounded, by applying Montel's theorem to  $g_n(z) = f_n(z) - b$ , there exists a subsequence  $(g_{n_k})$  which converges uniformly on every compact subset of  $\Omega$  to a holomorphic function  $g: \Omega \rightarrow \mathbb{C}$ . Clearly, we have  $g(a) = 0$ . Since  $g_{n_k}$  does not have zeros, by Hurwitz's theorem,  $g(z) = 0$  for all  $z \in \Omega$ . Hence  $(f_{n_k})$  converge locally uniformly to the constant function  $b$ .
- Step 2 : Suppose  $\gamma$  is a path in  $\Omega$ . We claim that for all  $\varepsilon > 0$ , there exists  $N \in \mathbb{N}$  such that for all  $n > N$ ,  $f_n(\text{Im}(\gamma)) \subset \Omega \cap D(b, \varepsilon)$ . It is trivial by the local uniform convergence of  $(f_{n_k})$  to  $b$ .
- Step 3 : By contradiction, suppose  $\Omega$  is not simply connected. Then there exists a non-contractible closed path  $\gamma$  in  $\Omega$ . By the above step, for all  $\varepsilon > 0$ , there exists  $N \in \mathbb{N}$  such that for all  $n > N$ ,  $f_n(\text{Im}(\gamma)) \subset \Omega \cap D(b, \varepsilon)$ . Since  $\partial\Omega$  is  $C^1$ , we can choose  $\varepsilon$  small enough such that  $\Omega \cap D(b, \varepsilon)$  is simply connected (This is because we can  $\varepsilon$  small enough such that  $\Omega \cap D(b, \varepsilon)$  is homeomorphic to a half disc). Hence  $f_n(\text{Im}(\gamma))$  is contractible in  $\Omega \cap D(b, \varepsilon)$ , which contradicts the fact that  $f_n$  is an automorphism of  $\Omega$  since  $\gamma$  is non-contractible.