Lecture 8

Chebyshev collocation method for differential equations

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Two point boundary value problem
Time <u>independent</u> – linear/non-linear

- Time independent boundary value problem in a general form

$$\varepsilon u_{xx}(x) + p(x)u_x(x) + q(x)u_x(x) = f(x,u), \qquad -1 \le x \le 1$$

ε >0 is a fixed parameter and $p(x)$, $q(x)$ and $f(x)$ are given functions

- Boundary conditons on general (mixed) form:

$$\alpha_- u(-1) + \beta_- u_x(-1) = g_-$$

$$\alpha_+ u(1) + \beta_+ u_x(1) = g_+$$

- Dirichlet boundary conditions: $u(-1) = g_-$ $\quad u(1) = g_+$
- Neumann boundary conditions: $u_x(-1) = g_-$ $\quad u_x(1) = g_+$

The coefficients $\alpha_\pm$ $\beta_\pm$ $g_\pm$ are known

Time dependent problems - linear/non-linear

- Heat equation

  $u_t - \varepsilon u_{xx} = 0$ with boundary conditions given

  on $u(\text{-}1,t) = g_-$ and $u(1,t) = g_+$ and initial condition $u(x,0) = f(x)$
- Linear wave equation

  $u_t + u_x = 0$ boundary conditions given

  on $u(\text{-}1,t) = g_-$ and $u_x(1,t) = g_+$ and initial condition $u(x,0) = f(x)$
- Burgers equation

  $u_t + uu_x - \varepsilon u_{xx} = 0$ with boundary conditions given

  on $u(\text{-}1,t) = g_-$ and $u(1,t) = g_+$ and initial condition $u(x,0) = f(x)$

Review on the Chebyshev transform

- A function $u(x)$ can be expanded in Chebyshev series

$$u(x)=\sum_{k=0}^{\infty}\hat{u}_k T_k(x) \qquad \hat{u}_k=\frac{2}{\pi c_k}\int_{-1}^{1}u(x)T_k(x)\omega(x)\,dx$$

$\omega(x)=(1-x^2)^{-1/2}$ and $c_k=2$ for $k=0$ and $c_k=1$ for $k\geq 1$

- The Chebyshev polynomials are given by $T_k(x)=\cos(k\cos^{-1}(x))$
- In the discrete Chebyshev-Gauss-Lobatto case:

$$x_j=\cos\frac{\pi j}{N},\ j=0,1,2\ldots,N \text{ and } \omega_j=\frac{\pi}{2\tilde{c}_k} \text{ where}$$

$$\tilde{c}_k=2 \text{ for } j=0,N \text{ and } \tilde{c}_k=1 \text{ for } j=1,2,\ldots,N-1$$

- The discrete Chebyshev coefficients are given by

$$\tilde{u}_k=\frac{2}{\tilde{c}_k N}\sum_{j=0}^{N}\frac{1}{\tilde{c}_j}u(x_j)\cos\left(\frac{\pi kj}{N}\right)=\frac{2}{\tilde{c}_k N}\sum_{j=0}^{N}\frac{1}{\tilde{c}_j}u(x_j)\Re\left(e^{i\frac{kj\pi}{N}}\right),\ k=0,1,\ldots N$$

Review on the Chebyshev transform, cont.

- A function $u(x)$ can be expanded in discrete Chebyshev series

$$u(x_j) = \sum_{k=0}^{N} \tilde{u}_k T_k(x_j), \quad j = 0,1,\ldots,N$$

- Chebyshev interpolant $I_N u = \sum_{k=0}^{N} \tilde{u}_k T_k(x)$ (1)

- Or in terms of Chebyshev lagrangian polynomials $I_N u = \sum_{j=0}^{N} u(x_j) \Upsilon_j(x)$ (2)

$$\Upsilon_j(x) = \frac{(-1)^{j+1}(1-x^2)T_N'(x)}{\tilde{c}_j N^2 (x-x_j)}$$

- Note that $I_N u(x_j) = u(x_j)$ and that (1)$\Leftrightarrow$ (2)

Review on the Chebyshev transform - derivatives

- Derivative in transform (polynomial) space (discrete)

$$\left(I_N u(x_j)\right)' := \left(D_N u\right)_j = \sum_{k=0}^{N} \tilde{u}_k^{(1)} T_k(x_j)$$

$$\left(I_N u(x_j)\right)'' := \left(D_N^2 u\right)_j = \sum_{k=0}^{N} \tilde{u}_k^{(2)} T_k(x_j)$$

- The coefficients can be found by a recursive relation

Typo!!

$$c_k \tilde{u}_k^{(1)} = \tilde{u}_{k+2}^{(1)} + 2(k+1)\tilde{u}_{k+1}, \quad k = N-1, N-2, \ldots 0$$

$$c_k \tilde{u}_k^{(2)} = \tilde{u}_{k+2}^{(2)} + 2(k+1)\tilde{u}_{k+1}^{(1)} \quad k = N-1, N-2, \ldots 0$$

$$\tilde{u}_{N+1}^{(1)} = \tilde{u}_N^{(1)} = 0 \quad \text{and} \quad \tilde{u}_{N+1}^{(2)} = \tilde{u}_N^{(2)} = 0$$

- Cost of $O(N \log N)$ if FFT is used

Review on the Chebyshev transform - derivatives

- Derivative in "physical" space $$\boxed{\left(D_N u\right)_j = \sum_{l=0}^{N} u(x_l)\Upsilon_l'(x_j) = \sum_{l=0}^{N} D_{jl} u(x_l)}$$
- D_{jl} are the entries in the Chebyshev derivative matrix, **D**
- The matrix entries for the second order derivative matrix, $\mathbf{D}^2$, are given by $(\mathbf{DD})_{jl}$ where **DD** is a matrix-multiplication
- The Chebyshev derivative matrix at quadrature points is given by

$$D_{jl} = \begin{cases} \dfrac{\tilde{c}_j (-1)^{j+l}}{\tilde{c}_l (x_j - x_l)} & l \neq j \\[2ex] -\dfrac{x_l}{2(1-x_l^2)} & 1 \leq l = j \leq N-1 \end{cases} \qquad D_{jl} = \begin{cases} \dfrac{2N^2+1}{6} & l = j = 0 \\[2ex] -\dfrac{2N^2+1}{6} & l = j = N \end{cases}$$

- The matrix approach costs $\mathrm{O}(N^2)$

Example of a Dirichlet problem

- We wish to solve, by a Chebyshev collocation method,

$$u_{xx} + xu_x - u = \underbrace{(24+5x^2)e^{5x} + (2+2x^2)\cos(x^2) - (4x^2+1)\sin(x^2)}_{f(x)},$$

on $-1 \le x \le 1$ (1)

- With boundary conditions

$$u(-1) = e^{-5} + \sin(1) = g_- \quad \text{and} \quad u(1) = e^{5} + \sin(1) = g_+$$

- Let $\mathbf{x} = (x_0, x_1, \ldots, x_N)^T$ where $x_j = \cos\left(\frac{j\pi}{N}\right)$, $j = 0,1,2,\ldots N$

  are the Chebyshev-Gauss-Lobatto points

- Let $\mathbf{f} = (f(x_0), f(x_1), \ldots, f(x_N))^T$
- Let $\mathbf{u} = (g_+, u(x_1), u(x_2), \ldots, u(x_{N-1}), g_-)^T$ and

  $\mathbf{u}_M = (u(x_1), u(x_2), \ldots, u(x_{N-1}))^T$ be the vector of unknows to be determined

- We will work with the Chebyshev derivative matrices $\mathbf{D}$ and $\mathbf{D}^2$
- The entries in $\mathbf{D}$ are given by D_{ij}, $0 \le i, j \le N$ and $\mathbf{D}^2 = \mathbf{DD}$

Example of a Dirichlet problem, cont.

- The approximation to (1) is given by

$$\mathbf{D}_M^2\mathbf{u}_M - \mathbf{x}_M \otimes \mathbf{D}_M\mathbf{u}_M - \mathbf{u}_M = \mathbf{F}$$

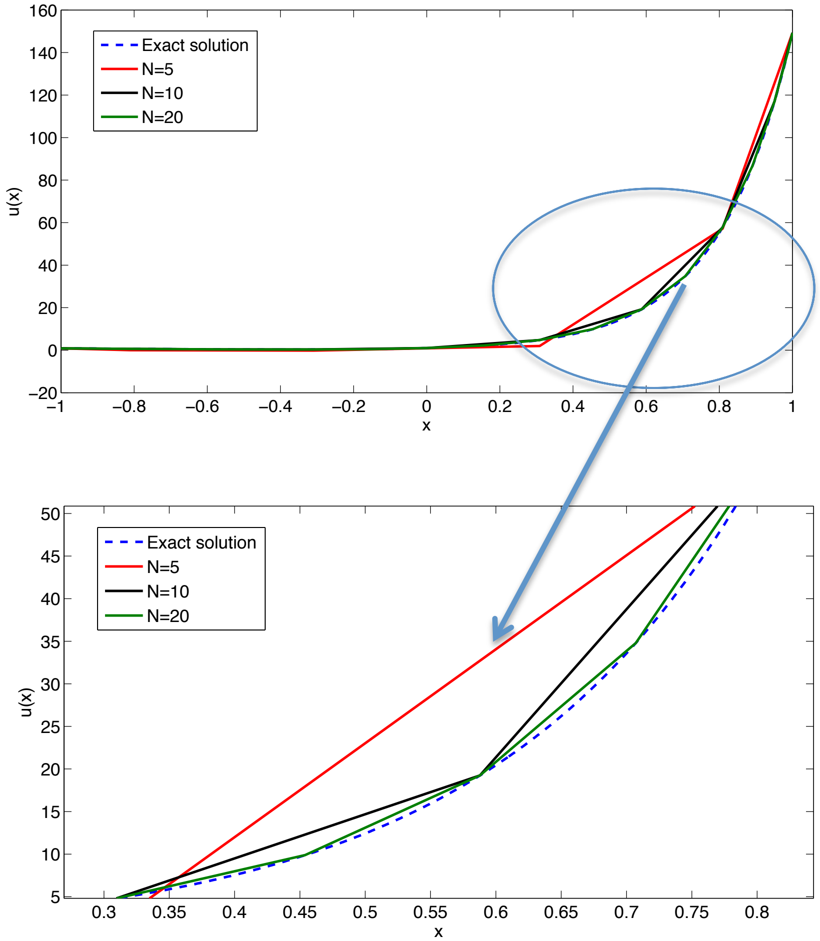
- $\mathbf{x}_M = \mathbf{x}(1:N-1), \quad \mathbf{D}_M = \mathbf{D}(1:N-1,1:N-1), \quad \mathbf{D}_M^2 = \mathbf{D}^2(1:N-1,1:N-1)$

$$\mathbf{F} = \mathbf{f}(1:N-1) - \left[\mathbf{D}^2(1:N-1,0) + \mathbf{x}_M \otimes \mathbf{D}(1:N-1,0)\right]g_+ -$$
$$\left[\mathbf{D}^2(1:N-1,N) + \mathbf{x}_M \otimes \mathbf{D}(1:N-1,N)\right]g_-$$

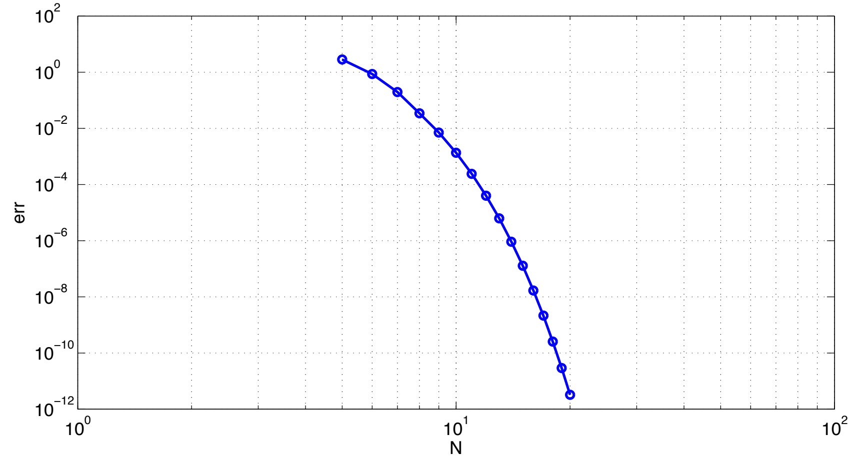
- The approximation can also be written on the form $\mathbf{A}\mathbf{u}_M = \mathbf{F}$ where $\mathbf{A}$ is a matrix of size $(N-1)\times(N-1)$
- The matrix $\mathbf{A}$ is given by $\mathbf{A} = \mathbf{D}^2 + \Lambda\mathbf{D} - \mathbf{I}$ where Λ is a diagonal matrix with the values of $\mathbf{x}_M$ on the diagonal and $\mathbf{I}$ is the identity matrix
- The approximate solution is given as the solution to the linear system of equations $\mathbf{u}_M = \mathbf{A}^{-1}\mathbf{F}$

Solution

- The exact solution is $u(x) = e^{5x} + \sin(x^2)$
- The error is defined by $err = \underbrace{\max}_{1 \le j \le N-1} \left| u_j - u(x_j) \right|$



Convergence



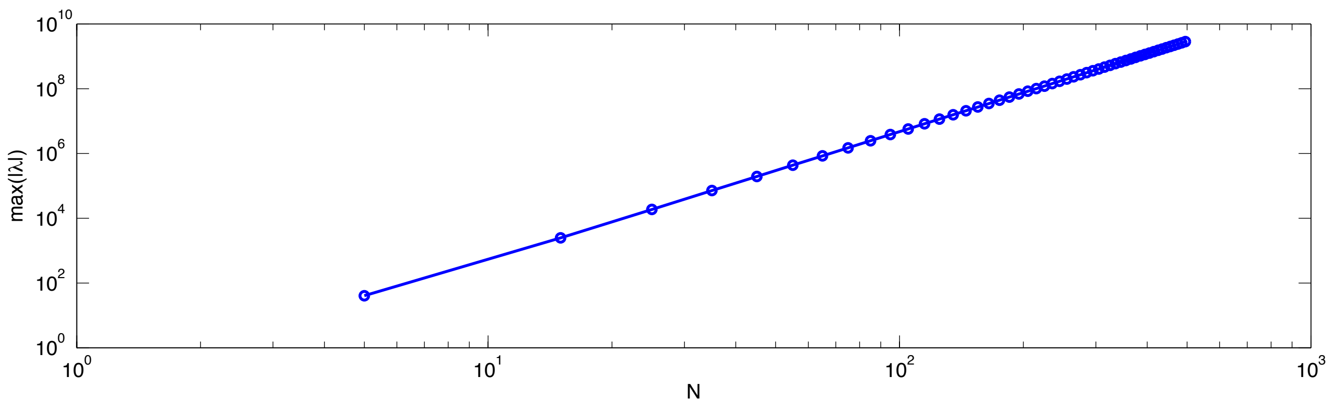
- The error obtained by a second order finite difference approximation with $N = 512$ is approximately the same as with $N = 10$ in the spectral method

Code example - MATLAB

```
N=10
[D, x] = cheb(N);           Chebyshev derivation matrix, D
D2=D*D;                     Chebyshev derivation matrix, D^2

DM=D(2:N,2:N);              For the inner points only
D2M=D2(2:N,2:N);            Note that the numbering of the
L=diag(x(2:N),0);           elements in a vector/matrix is
I=diag(ones(N-1,1),0);      from 1 to N+1 (not 0 to N)

A=D2M+L*DM-I;               A as in Au=F

f=(24+5*x).*exp(5*x)+(2+2*x.^2).*cos(x.^2);
f=f-(4*x.^2+1).*sin(x.^2);
gminus=exp(-5)+sin(1);
gplus=exp(5)+sin(1);

F=f(2:N)-(D2(2:N,1)+x(2:N).*D(2:N,1))*gplus
F=F-(D2(2:N,end)+x(2:N).*D(2:N,end))*gminus;

sol=A\F                     u=A^-1F

u=[gplus;sol;gminus];       Add the known boundary conditions
```

Remarks

- The collocation method will lead to a full and ill-conditioned linear system
- Gaussian elimination to solve **Au=F** is only feasible for problems with a small number of unknowns (one-dimensional problems)
- For multi-dimensional problems an iterative method together with an appropriate pre-conditioner should be used
- We will get back to this when we speak about multi-dimensional problems



Eigenvalues of **A** are growing as N^4