

mm

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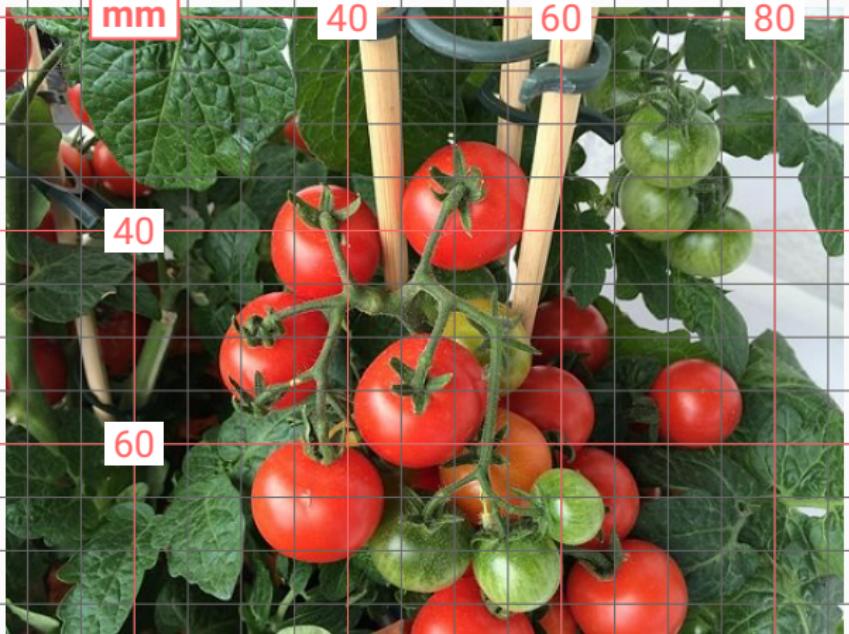
# Modeling optimal control policies

in stochastic epidemic models

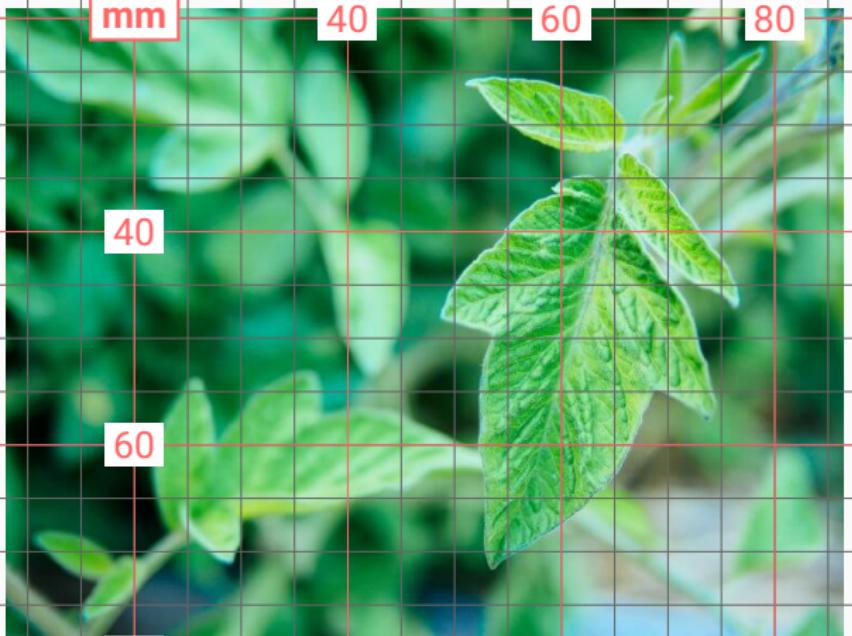
Saúl Díaz Infante Velasco

CONACYT-UNIVERSIDAD de SONORA

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- Leaflets are small and yellowed with edges that curl upwards
- Flowers either do not develop or fall off
- When **older plants** are infected, fruit that is already forming ripens normally, but **no new fruit** is formed after the infection
- TYLCV can be confused with several other conditions such as tomato big bud, herbicide damage and phosphate or magnesium deficiency



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## Spread

- **TYLCV** is spread by the insect silverleaf whitefly (Bemisia tabaci B biotype)
- Silverleaf whiteflies pick up the virus by feeding on infected host plants. The whiteflies then spread the virus to healthy plants which show the symptoms 10 to 21 days later
- Silverleaf whiteflies are common in many countries and feed on many types of plants



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## Control

### Cultural Control

- Physical barriers
- Planting dates
- Removal of infested plants
- Host plant resistance

### Biological Control

- Parasitoids
- Predators
- Fungi

## Insecticides

- pymetrozine
- zeta-cypermethrin / bifenthrin



Shun-xiang, R., Zhen-zhong, W., Bao-li, Q., and Yuan, X. (2001).

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4	Characterization of optimal policies	
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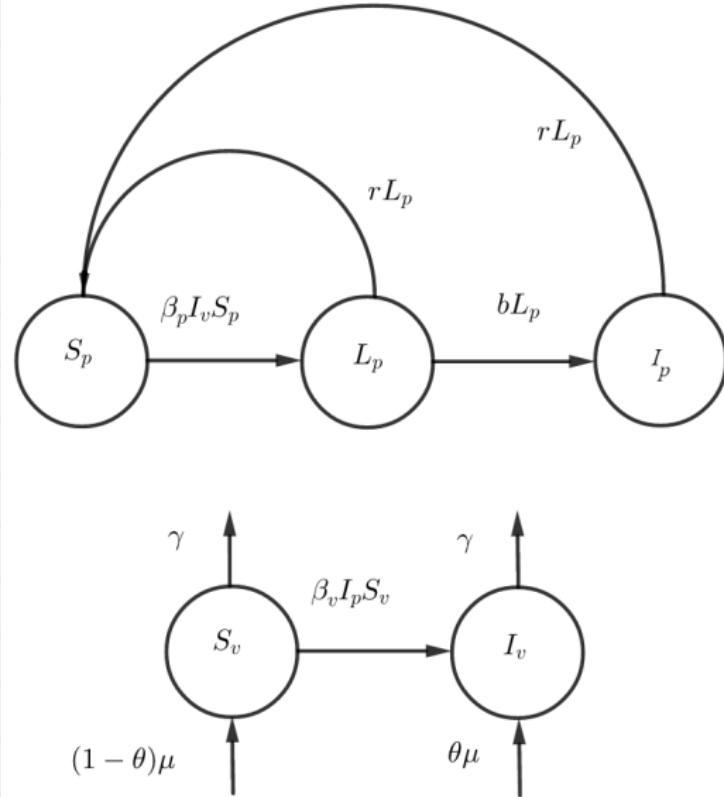


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*Journal of Applied Ecology*  
36(5):625–638, oct 1999. 40

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- Remove from latent and infected plants,
- plants become latent plants by infected vectors,
- latent plants become infectious plants,
- vectors become infected vectors by infected plants,
- vectors die per day,
- immigration from alternative hosts.



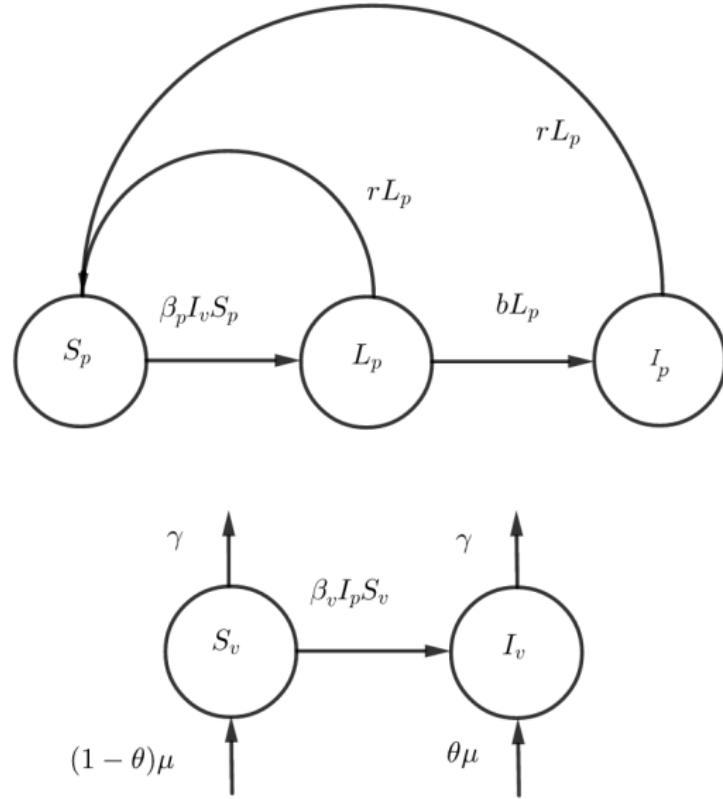


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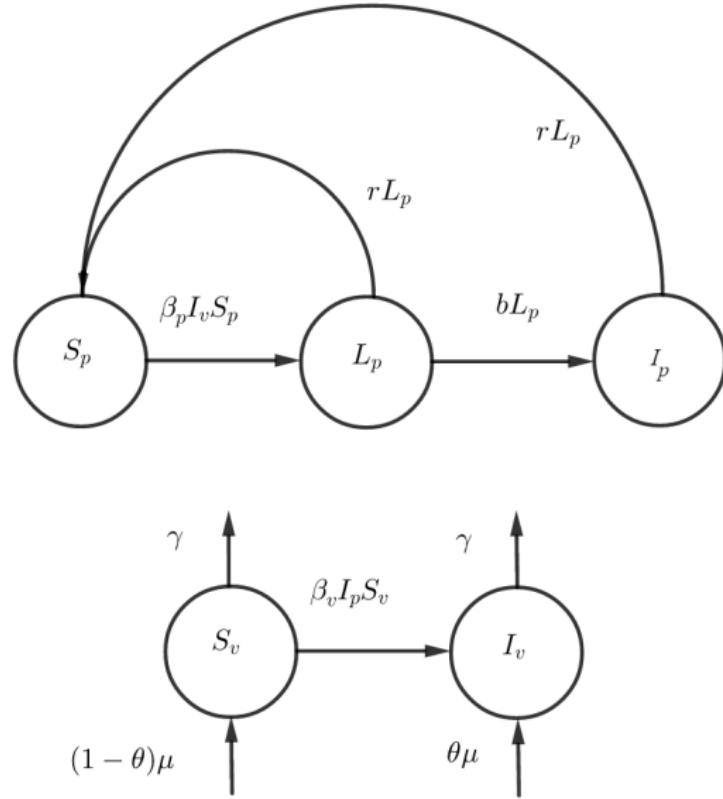


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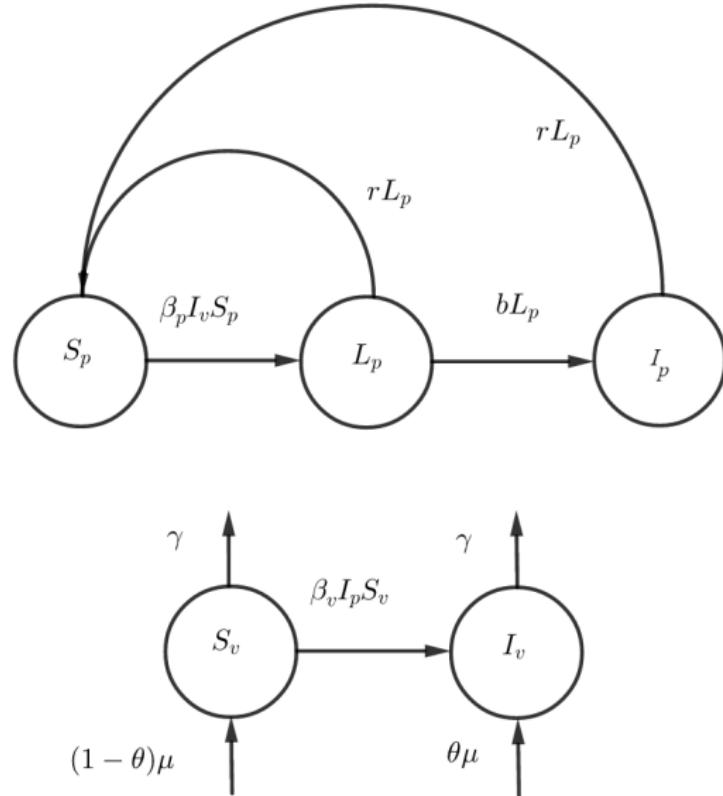
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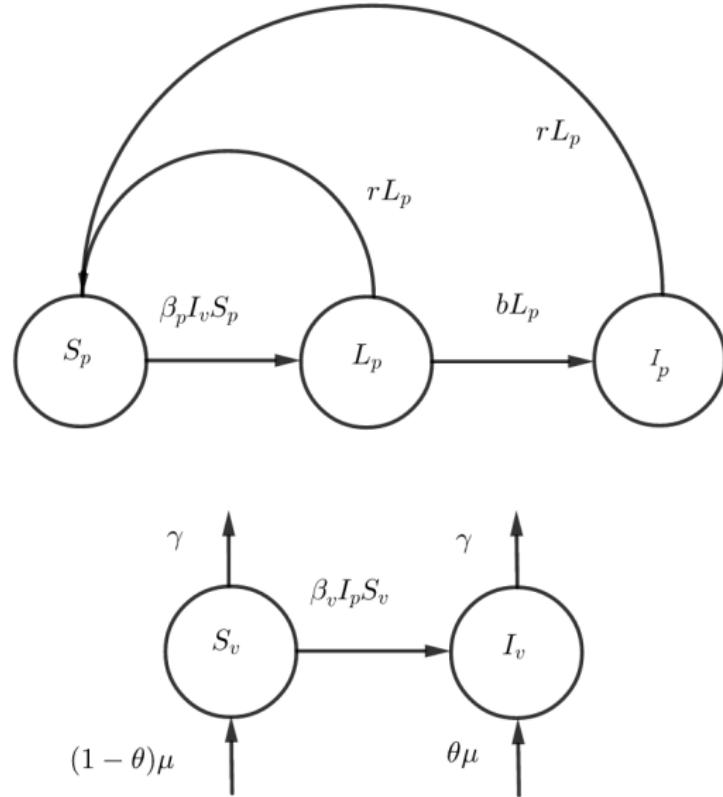


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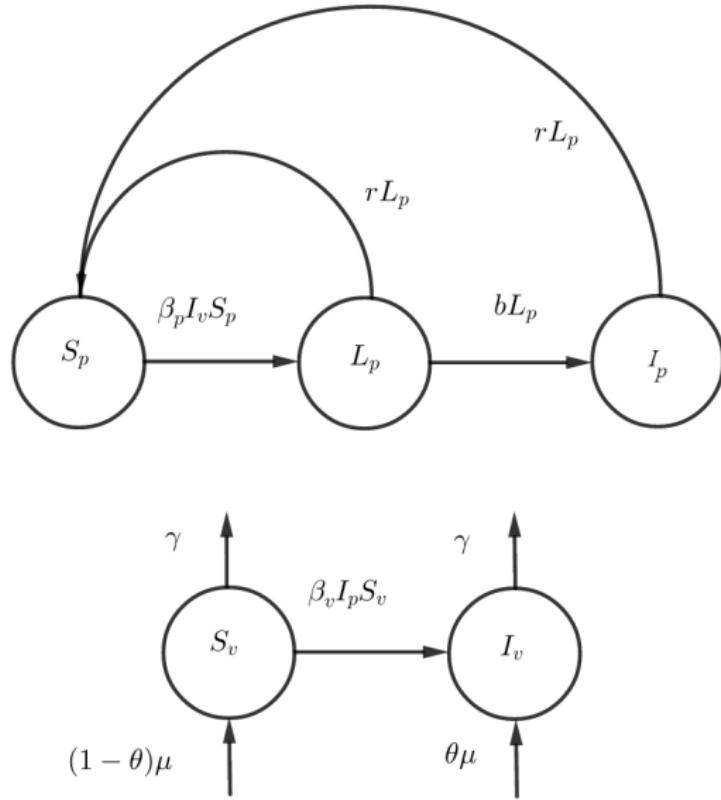


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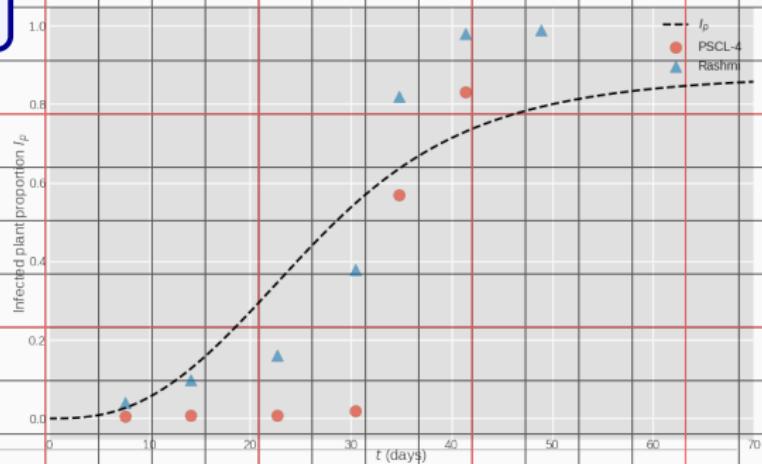
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$$R_0 = \sqrt{\frac{\beta_v \mu b \beta_p}{r^2(r+b)\gamma}}$$

$$DFE = (N_p, 0, 0, 0, \mu/\gamma)^\top$$

$$EE = (S_p^*, L_p^*, I_p^*, S_v^*, I_v^*)^\top$$

Par.	Value	Descrip.
$\beta_p$	0.1	latent rate
$r$	0.01	remove rate
$1/b$	0.075	time of latency
$\gamma$	0.06	vector die or depart rate
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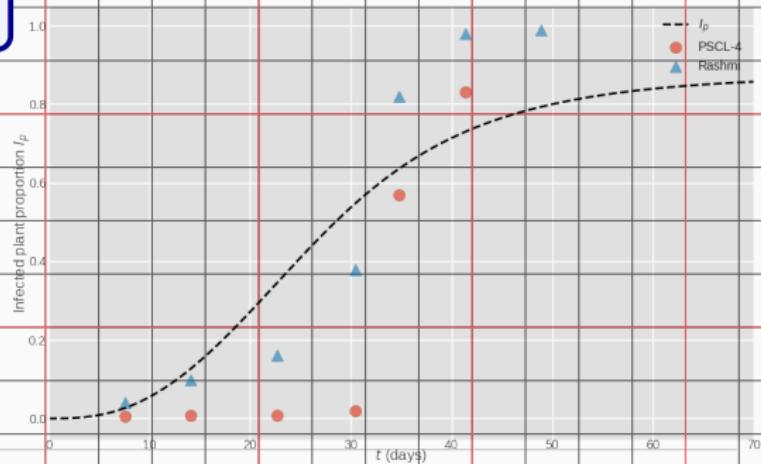
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## Controls

$u_1(t)$  : Latent replanting

$u_2(t)$  : Infected replanting

$u_3(t)$  : Fumigation

$$u_i(t) \in [0, u_i^{\max}]$$

80

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$$\int_0^T \left[ A_1 I_p(t) + A_2 L_p(t) + A_3 I_v(t) + \sum_{i=1}^3 c_i \frac{u_i^2}{2} \right] dt$$

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## Optimal Control Problem

$$g(x, u) := A_1 I_p(t) + A_2 L_p(t) + A_3 I_v(t) + c_1 u_1(t)^2 + c_2 u_2(t)^2 + c_3 u_3(t)^2$$

$$\min_{\bar{u}(\cdot) \in \tilde{\mathcal{U}}_{x_0}[0, T]} J(u_1, u_2, u_3) = \int_0^T g(x, u) ds$$

such that:

$$\frac{dS_p}{dt} = \beta_p S_p I_v + (r + u_1) L_p + (r + u_2) I_p,$$

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$$x(0) = x_0, \quad u_i \in [0, u_i^{\max}]$$

## Consider the controlled dynamics

$$\begin{cases} \dot{x}(s) = f(s, u(s), x(s)) & s \in [t_0, T], \\ x(t_0) = x_0, \end{cases}$$

with terminal state constraint

$$x(T; t_0, x_0, u(\cdot)) \in M, M \subseteq \mathbb{R}^n.$$

## Problem (OC)

$(t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^n$ , find a control policy  
 $\bar{u}(\cdot) \in \tilde{\mathcal{U}}_{x_0}[t_0, T]$  s.t.

$$J(t_0, x_0; \bar{u}(\cdot)) = \inf_{u(\cdot) \in \tilde{\mathcal{U}}_{x_0}[t_0, T]} J(t_0, x_0; u(\cdot)).$$

## Cost functional

$$\tilde{\mathcal{U}}_{x_0}[t_0, T] := \{u : [t_0, T] \rightarrow \mathbb{R}^n \mid \text{measurable}\}$$

$$\begin{aligned} J(t_0, x_0; u(\cdot)) = & \int_{t_0}^T g(s, u(s), x(s)) ds \\ & + h(T, x(T)) \end{aligned}$$

## Hypothesis:

- (C-1)  $f : \mathbb{R}_+ \times U \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  is measurable, satisfies the lipchitz condition in  $x$ ,  
 $|f(t, u, 0)| \leq L, \forall (t, u) \in \mathbb{R}_+ \times U$ .
- (C-2)  $g : \mathbb{R}_+ \times U \times \mathbb{R}^n \rightarrow \mathbb{R}$ ,  
 $h : \mathbb{R}^n \rightarrow \mathbb{R}$  are measurable, and
- $$|g(s, u, x_1) - g(s, u, x_2)| + |h(x_1) - h(x_2)| \leq \omega(|x_1| \vee |x_2|, |x_1 - x_2|)$$
- $$\forall (s, u) \in \mathbb{R}_+ \times U, x_1, x_2 \in \mathbb{R}^n.$$
- (C-3) For a.a.  $t \in [0, T]$ , Cesari property holds  $\forall x \in \mathbb{R}^n$ .

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$\omega : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ , increasing,  
 $\omega(r, 0) = 0 \forall r \geq 0$ .

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## Cesari property

$$\begin{aligned} \mathbf{E}(t, x) = \{ & (z^0, z) \in \mathbb{R} \times \mathbb{R}^n | \\ & z^0 \geq g(t, u, x), \\ & z = f(t, u, x), u \in U \}. \end{aligned}$$

$$\bigcap_{\delta} \bar{co} \mathbf{E}(t, B_\delta(x)) = \mathbf{E}(t, x)$$

## Hypothesis:

- (C-1)  $f : \mathbb{R}_+ \times U \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  is measurable, satisfies a Lipschitz condition in  $x$ ,  $|f(t, u, 0)| \leq L, \forall (t, u) \in \mathbb{R}_+ \times U$ .
- (C-2)  $g : \mathbb{R}_+ \times U \times \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $h : \mathbb{R}^n \rightarrow \mathbb{R}$  are measurable, and
- $$\begin{aligned} & |g(s, u, x_1) - g(s, u, x_2)| + \\ & |h(x_1) - h(x_2)| \\ & \leq \omega(|x_1| \vee |x_2|, |x_1 - x_2|) \end{aligned}$$
- $$\forall (s, u) \in \mathbb{R}_+ \times U, x_1, x_2 \in \mathbb{R}^n.$$
- (C-3) For a.a.  $t \in [0, T]$ , Cesari property holds  $\forall x \in \mathbb{R}^n$ .

## Existence Theorem

Let (C-1)-(C-3) hold. Then problem (OC) admits at least one optimal pair.

$(OC)^T$ 

$$J(t_0, x_0; u(\cdot)) = \int_{t_0}^T g(s, u(s), x(s)) ds + h(x(T))$$

$$\begin{cases} \dot{x}(s) = f(s, u(s), x(s)) & s \in [t_0, T] \\ x(t_0) = x_0 \end{cases}$$

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Hamiltonian:

$$H = g(t, x(t), u(t)) + \langle \lambda(t), f(t, x(t), u(t)) \rangle,$$

$$\frac{\partial H}{\partial u_i}(t, \bar{x}(\cdot), \bar{u}(\cdot)) = 0.$$

Additional hypothesis:

(C-4)

$x \mapsto (f(t, u, x), g(t, u, x), h(x))$   
is differentiable,

$(u, x) \mapsto (f(t, u, x), f_x(t, u, x),$   
 $g(t, u, x), g_x(t, u, x),$   
 $h_x(x))$

is continuous.

## Pontryagin's Maximum Principle

If  $\bar{u}(t)$  and  $\bar{x}(t)$  are optimal for the problem (OC), then there exists a piecewise differentiable adjoint variable  $\lambda(t)$  s.t.

$$H(t, \bar{x}(t), u(t), \lambda(t)) \leq H(t, \bar{x}(t), \bar{u}(t), \lambda(t))$$

$\forall u$  at  $t$ ,

$$\lambda'(t) = -\frac{\partial H(t, \bar{x}(t), \bar{u}(t), \lambda(t))}{\partial x},$$

$$\lambda(T) = 0.$$

$$H = g(t, x(t), u(t)) + \langle \lambda(t), f(t, x(t), u(t)) \rangle,$$

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## Example

$$\frac{dS_p}{dt} = -\beta_p S_p I_v + (r + u_1) L_p + (r + u_2) I_p,$$

$$\frac{dL_p}{dt} = \beta_p S_p I_v - b L_p - (r + u_1) L_p,$$

$$\frac{dI_p}{dt} = b L_p - (r + u_2) I_p,$$

$$\frac{dS_v}{dt} = -\beta_v S_v I_p - (\gamma + u_3) S_v + (1 - \theta) \mu,$$

$$\frac{dI_v}{dt} = \beta_v S_v I_p - (\gamma + u_3) I_v + \theta \mu,$$

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$$\begin{aligned} H = & A_1 I_v + A_2 L_p + A_3 I_p \\ & + \sum_{i=1}^3 c_i u_i^2 \\ & + \lambda_1 (-\beta_p S_p I_v + (r + u_1) L_p \\ & + (r + u_2) I_p) \\ & + \lambda_2 (\beta_p S_p I_v - b L_p \\ & - (r + u_1) L_p) \\ & + \lambda_3 (b L_p - (r + u_2) I_p) \\ & + \lambda_4 (-\beta_v S_v I_p - (\gamma + u_3) S_v \\ & + (1 - \theta) \mu) \\ & + \lambda_5 (\beta_v S_v I_p - (\gamma + u_3) I_v \\ & + \theta \mu). \end{aligned}$$

