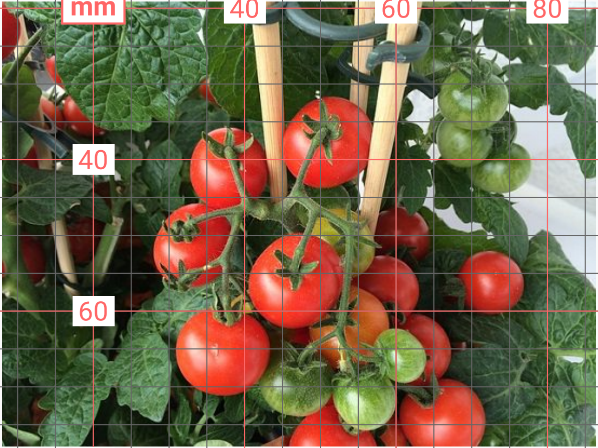
Modeling optimal control policies

in stochastic epidemic models

Saúl Díaz Infante Velasco

CONACYT-UNIVERSIDAD de SONORA

Tomato leaf curl virus disease (TYLCVD)



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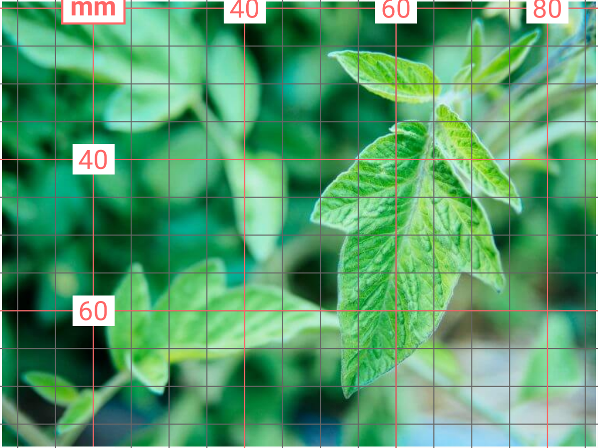

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- Tomato plants **infected early** are severely stunted and will **not produce fruit**
- Leaflets are small and yellowed with edges that curl upwards
- Flowers either do not develop or fall off
- When **older plants** are infected, fruit that is already forming ripens normally, but **no new fruit** is formed after the infection
- TYLCV can be confused with several other conditions such as tomato big bud, herbicide damage and phosphate or magnesium deficiency

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Spread

- **TYLCV** is spread by the insect silverleaf whitefly (Bemisia tabaci B biotype)
- Silverleaf whiteflies pick up the virus by feeding on infected host plants. The whiteflies then spread the virus to healthy plants **which show the symptoms 10 to 21 days later**
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Control

Cultural Control

- Physical barriers
- Planting dates
- **Removal of infested plants**
- Host plant resistance

Biological Control

- Parasitoids
- Predators
- Fungi

Insecticides

- pymetrozine
- zeta-cypermethrin / bifenthrin

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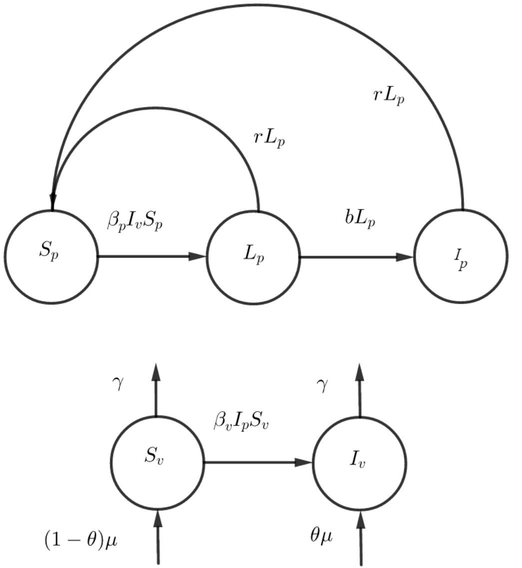
Hypothesis:

- Remove from latent and infected plants,
- plants become latent plants by infected vectors,
- latent plants become infectious plants,
- vectors become infected vectors by infected plants,
- vectors die per day,
- immigration from alternative hosts.



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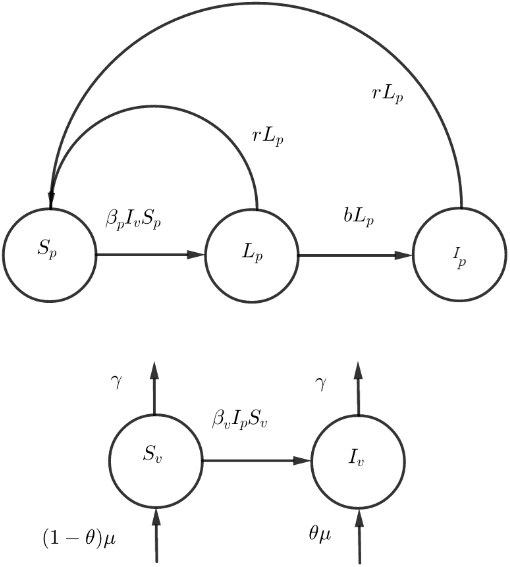

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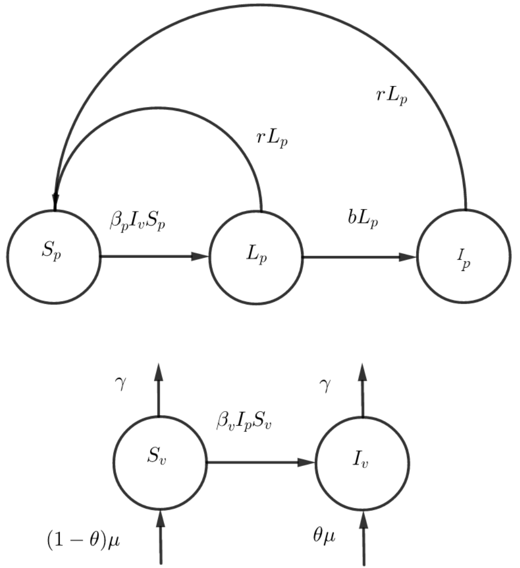

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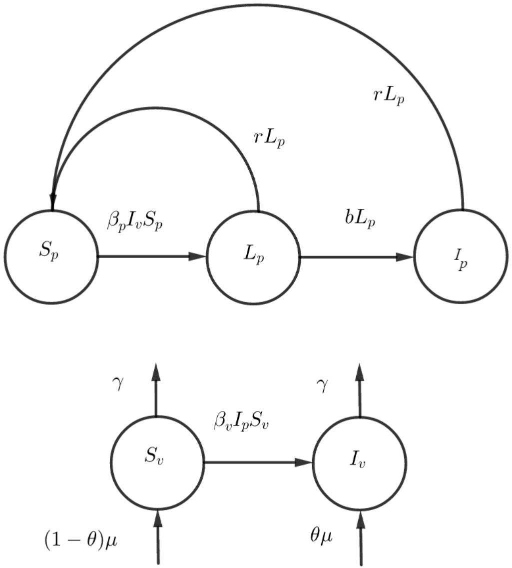

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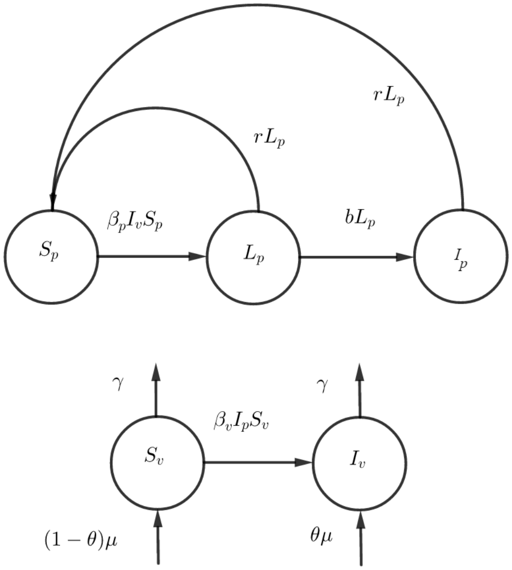

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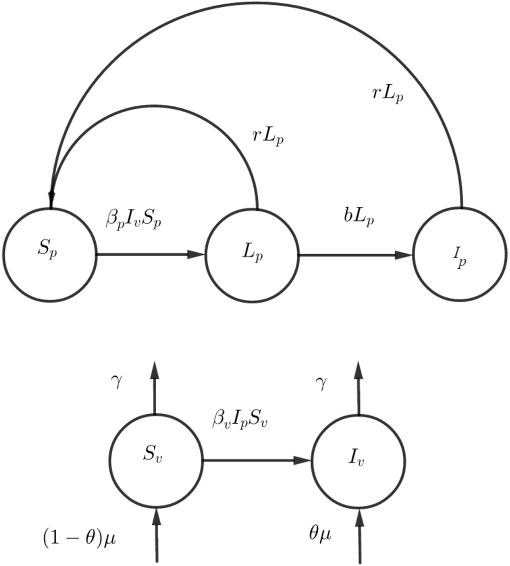

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\begin{aligned}
\frac{dS_p}{dt} &= -\beta_p S_p I_v + r(L_p + I_p) \\
\frac{dL_p}{dt} &= \beta_p S_p I_v - bL_p - rL_p \\
\frac{dI_p}{dt} &= bL_p - rI_p \\
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Par.	Value	Descrip.
β_p	0.1	latent rate
r	0.01	remove rate
$1/b$	0.075	time of latency
γ	0.06	vector die or depart rate
μ	0.3	immigration rate
θ	0.2	infected vectors arrival
β_v	0.003	vector infected rate

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\begin{aligned}
R_0 &= \sqrt{\frac{\beta_v \mu b \beta_p}{r^2(r+b)\gamma}} \\
DFE &= (N_p, 0, 0, 0, \mu/\gamma)^\top \\
EE &= (S_p^*, L_p^*, I_p^*, S_v^*, I_v^*)^\top
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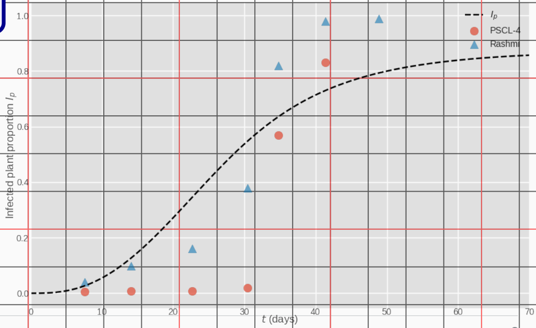

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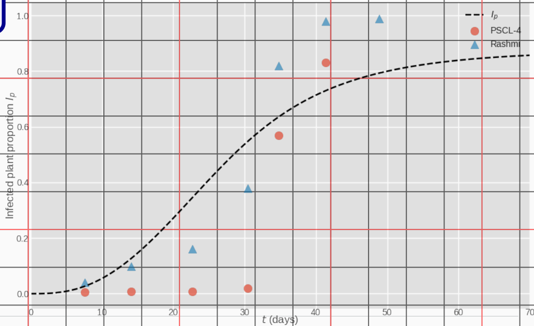

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Controls

$u_1(t)$: Latent replanting

$u_2(t)$: Infected replanting

$u_3(t)$: Fumigation

$u_i(t) \in [0, u_i^{max}]$

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$$
\int_0^T \left[A_1 I_p(t) + A_2 L_p(t) + A_3 I_v(t) + \sum_{i=1}^{3} c_i \frac{u_i^2}{2} \right] dt
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Optimal Control Problem

$$g(x,u) := A_1 I_p(t) + A_2 L_p(t) + A_3 I_v(t) + c_1 u_1(t)^2 + c_2 u_2(t)^2 + c_3 u_3(t)^2$$

$$\min_{\bar{u}(\cdot)\in\tilde{\mathscr{U}}_{x_0}[0,T]} J(u_1,u_2,u_3) = \int_0^T g(x,u)ds$$

such that:

$$\begin{aligned}
\frac{dS_p}{dt} &= \beta_p S_p I_v + (r+u_1)L_p + (r+u_2)I_p,\\
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x(0) &= x_0, \qquad u_i \in [0, u_i^{\max}]
\end{aligned}$$

Consider the controlled dynamics

$$\begin{cases} \dot{x}(s) = f(s, u(s), x(s)) \; s \in [t_0, T], \\ x(t_0) = x_0, \end{cases}$$

with terminal state constraint

$$x(T; t_0, x_0, u(\cdot)) \in M, M \subseteq \mathbb{R}^n.$$

Problem (OC)

$(t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^n$, find a control policie $\bar{u}(\cdot) \in \tilde{\mathscr{U}}_{x_0}[t_0, T]$ s.t.

$$J(t_0, x_0; \bar{u}(\cdot)) = \inf_{u(\cdot) \in \tilde{\mathscr{U}}_{x_0}[t_0, T]} J(t_0, x_0; u(\cdot)).$$

Cost functional

$$\tilde{\mathscr{U}}_{x_0}[t_0, T] := \{u : [t_0, T] \to \mathbb{R}^n | \text{ measurable}\}$$

$$J(t_0, x_0; u(\cdot)) = \int_{t_0}^{T} g(s, u(s), x(s)) ds + h(T, x(T))$$

Hypothesis:

(C-1) $f : \mathbb{R}_+ \times U \times \mathbb{R}^n \to \mathbb{R}^n$ is measurable, satisfies the lipchitz condition in x, $|f(t,u,0)| \leq L, \forall (t,u) \in \mathbb{R}_+ \times U$.

(C-2) $g : \mathbb{R}_+ \times U \times \mathbb{R}^n \to \mathbb{R}$, $h : \mathbb{R}^n \to \mathbb{R}$ are measurable, and

$$|g(s,u,x_1) - g(s,u,x_2)| + |h(x_1) - h(x_2)| \leq \omega(|x_1| \vee |x_2|, |x_1 - x_2|)$$

$\forall (s,u) \in \mathbb{R}_+ \times U, x_1, x_2 \in \mathbb{R}^n$.

(C-3) For a.a. $t \in [0,T]$, Cesari property holds $\forall\, x \in \mathbb{R}^n$.

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Modulus of continuity

$\omega : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$, increasing, $\omega(r,0) = 0\ \forall r \ge 0$.

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Cesari property

$$\begin{aligned} \mathbf{E}(t,x) = \{&(z^0, z) \in \mathbb{R} \times \mathbb{R}^n | \\ &z^0 \geq g(t,u,x), \\ &z = f(t,u,x), u \in U\}. \end{aligned}$$

$$\bigcap_{\delta} \bar{co}\mathbf{E}(t, B_\delta(x)) = \mathbf{E}(t,x)$$

Hypothesis:

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$\forall (s,u) \in \mathbb{R}_+ \times U, x_1, x_2 \in \mathbb{R}^n$.

(C-3) For a.a. $t \in [0,T]$, Cesari property holds $\forall\, x \in \mathbb{R}^n$.

Existence Theorem

Let (C-1)-(C-3) hold. Then problem (OC) admits at least one optimal pair.

Optimal Control Characterization

$(OC)^T$

$$J(t_0,x_0;u(\cdot)) = \int_{t_0}^{T} g(s,u(s),x(s))ds + h(x(T))$$

$$\begin{cases} \dot{x}(s) = f(s,u(s),x(s)) \; s \in [t_0,T] \\ x(t_0) = x_0 \end{cases}$$

Hamiltonian:

$$H = g(t,x(t),u(t)) + \langle \lambda(t), f(t,x(t),u(t)) \rangle,$$

$$\frac{\partial H}{\partial u_i}(t,\bar{x}(\cdot),\bar{u}(\cdot)) = 0.$$

Additional hypothesis:

(C-4)

$x \mapsto (f(t,u,x), g(t,u,x), h(x))$ is differentiable,

$$(u,x) \mapsto (f(t,u,x), f_x(t,u,x), g(t,u,x), g_x(t,u,x), h_x(x))$$

is continuous.

Pontryagin's Maximum Principle

If $\bar{u}(t)$ and $\bar{x}(t)$ are optimal for the problem (OC), then there exists a piecewise differentiable adjoint variable $\lambda(t)$ s.t.

$$H(t,\bar{x}(t),u(t),\lambda(t)) \leq H(t,\bar{x}(t),\bar{u}(t),\lambda(t))$$

$\forall$ u at t,

$$\begin{aligned} \lambda'(t) &= -\frac{\partial H(t,\bar{x}(t),\bar{u}(t),\lambda(t))}{\partial x}, \\ \lambda(T) &= 0. \end{aligned}$$

$$H = g(t,x(t),u(t)) + \langle \lambda(t), f(t,x(t),u(t)) \rangle,$$

Example

$$
\begin{aligned}
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\frac{dI_p}{dt} &= bL_p - (r+u_2)I_p,\\
\frac{dS_v}{dt} &= -\beta_v S_v I_p - (\gamma+u_3)S_v + (1-\theta)\mu,\\
\frac{dI_v}{dt} &= \beta_v S_v I_p - (\gamma+u_3)I_v + \theta\mu,
\end{aligned}
$$

$$
\begin{aligned}
H = {} & A_1 I_v + A_2 L_p + A_3 I_v\\
& + \sum_{i=1}^{3} c_i u_i^2\\
& + \lambda_1(-\beta_p S_p I_v + (r+u_1)L_p\\
& + (r+u_2)I_p)\\
& + \lambda_2(\beta_p S_p I_v - bL_p\\
& - (r+u_1)L_p)\\
& + \lambda_3(bL_p - (r+u_2)I_p)\\
& + \lambda_4(-\beta_v S_v I_p - (\gamma+u_3)S_v\\
& + (1-\theta)\mu)\\
& + \lambda_5(\beta_v S_v I_p - (\gamma+u_3)I_v\\
& + \theta\mu).
\end{aligned}
$$

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Algorithm 2 Forward Backward Sweep

In[illegible] $, t_f, x_0, h,$ to[illegible]
Output: x^*, u^*, λ

procedure FORWARD_BACKWARD_SWEEP$(g, \lambda_{\text{function}}, u, x_0, \lambda_f, h, n_{max})$
  while $\epsilon >$ tol **do**
    $u_{\text{old}} \leftarrow u$
    $x_{\text{old}} \leftarrow x$
    [illegible] RUNGE_KUTTA_FORWARD(g, u, x_0, h)
    $\lambda_{\text{old}} \leftarrow \lambda$
    $\lambda \leftarrow$ RUNGE_KUTTA_BACKWARD$(\lambda_{\text{function}}, x, \lambda_f, h)$
    $u_1 \leftarrow$ OPTIMALITY_CONDITION(u, x, λ)
    $u \leftarrow \alpha u_1 + (1 - \alpha) u_{old}, \qquad \alpha \in [0, 1]$ ▷ convex combination
    $\epsilon_u \leftarrow \dfrac{||u - u_{\text{old}}||}{||u||}$
    $\epsilon_x \leftarrow \dfrac{||x - x_{\text{old}}||}{||x||}$ ▷ relative error
    $\epsilon_\lambda \leftarrow \dfrac{||\lambda - \lambda_{\text{old}}||}{||\lambda||}$
    $\epsilon \leftarrow \max \{\epsilon_u, \epsilon_x, \epsilon_\lambda\}$
  end while
  re[illegible] x^*, u^*, λ ▷ Optimal pair
end procedure

mm 40 60 80 100 120

40

60

80