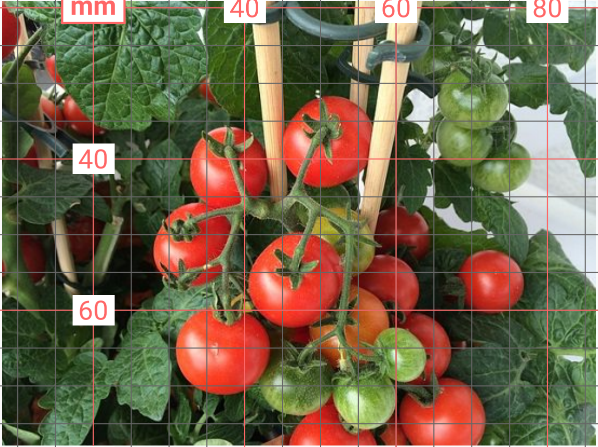
Modeling optimal control policies

in stochastic epidemic models

Saúl Díaz Infante Velasco

CONACYT-UNIVERSIDAD de SONORA

Tomato leaf curl virus disease (TYLCVD)



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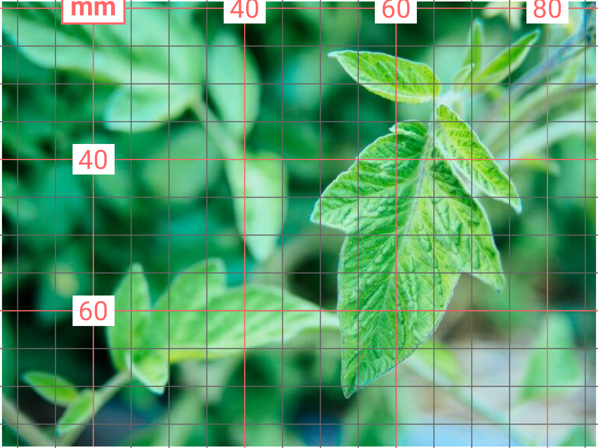

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- Tomato plants **infected early** are severely stunted and will **not produce fruit**
- Leaflets are small and yellowed with edges that curl upwards
- Flowers either do not develop or fall off
- When **older plants** are infected, fruit that is already forming ripens normally, but **no new fruit** is formed after the infection
- TYLCV can be confused with several other conditions such as tomato big bud, herbicide damage and phosphate or magnesium deficiency

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Tomato leaf curl virus disease (TYLCVD)

Spread

- **TYLCV** is spread by the insect silverleaf whitefly (Bemisia tabaci B biotype)
- Silverleaf whiteflies pick up the virus by feeding on infected host plants. The whiteflies then spread the virus to healthy plants **which show the symptoms 10 to 21 days later**
- Silverleaf whiteflies are **common** in many countries and **feed** on **many** types of **plants**

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Society for Industrial and Applied Mathematics

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Control

Cultural Control

- Physical barriers
- Planting dates
- **Removal of infested plants**
- Host plant resistance

Biological Control

- Parasitoids
- Predators
- Fungi

Insecticides

- pymetrozine
- zeta-cypermethrin / bifenthrin

Shun-xiang, R., Zhen-zhong, W., Bao-li, Q., and Yuan, X. (2001). **THE PEST STATUS OF BEMISIA TABACI IN CHINA AND NON-CHEMICAL CONTROL STRATEGIES.** *Insect Science*, 8(3):279–288.

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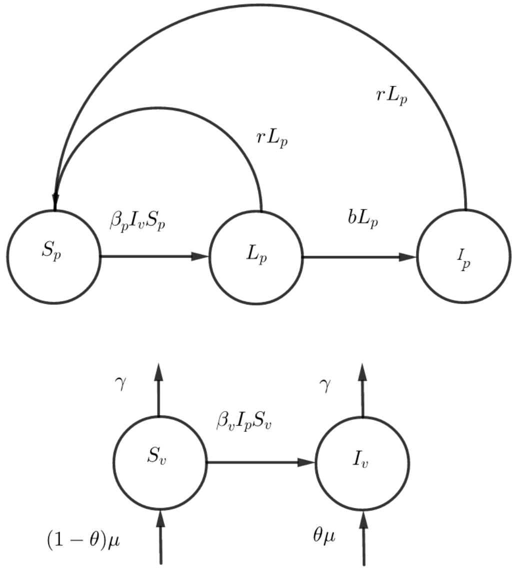
- plants become latent plants by infected vectors,
- latent plants become infectious plants,
- vectors become infected vectors by infected plants,
- vectors die per day,
- immigration from alternative hosts.



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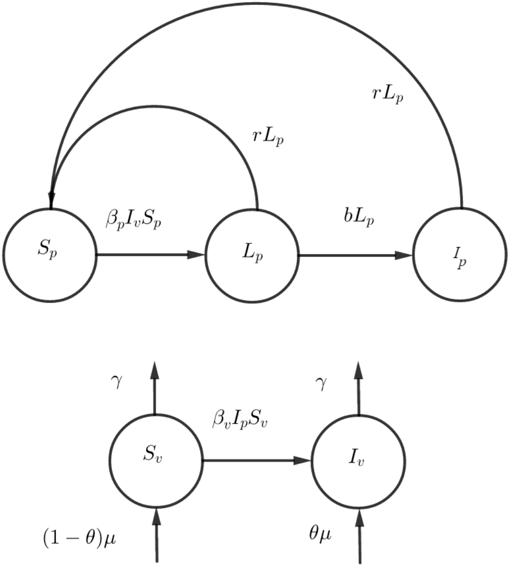

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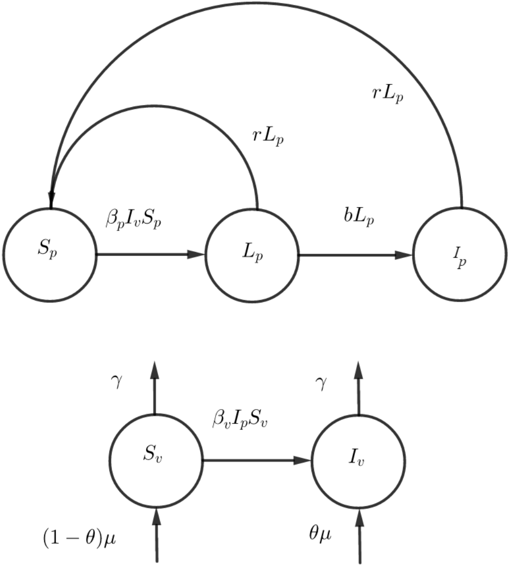

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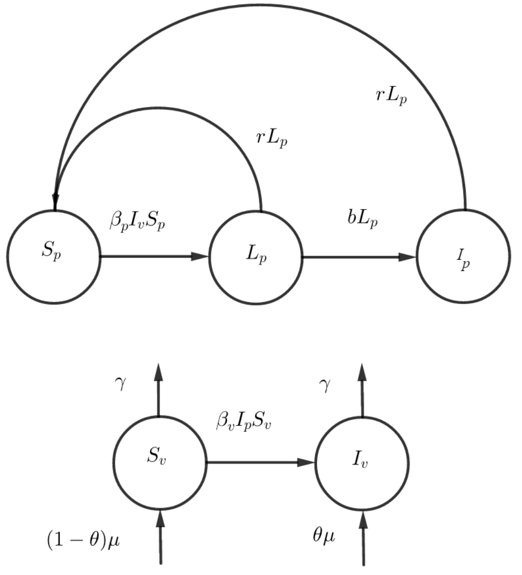

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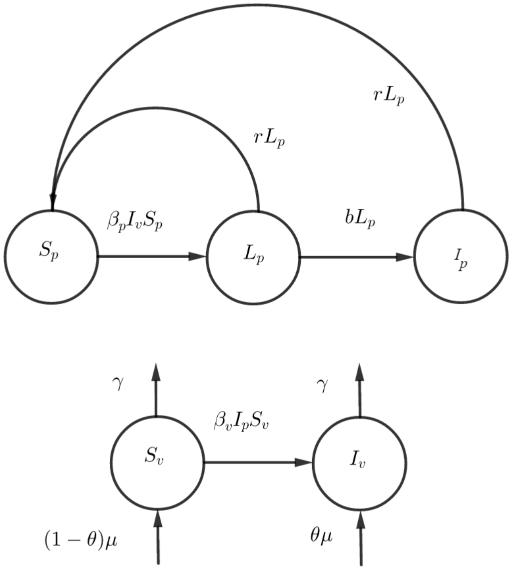

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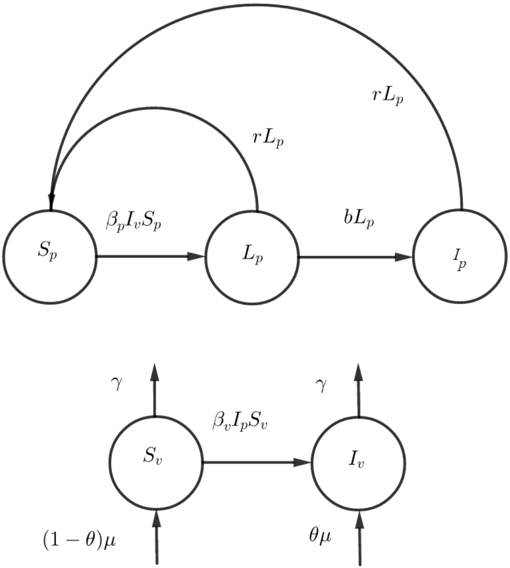

$$\frac{dS_p}{dt} = -\beta_p S_p I_v + r(L_p + I_p)$$
$$\frac{dL_p}{dt} = \beta_p S_p I_v - bL_p - rL_p$$
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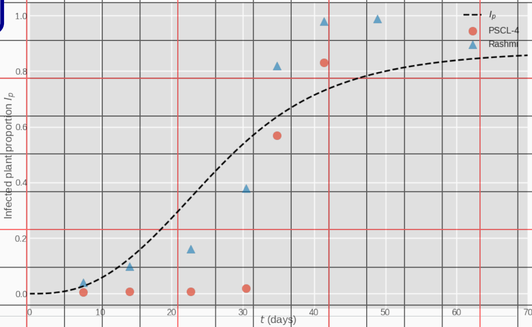
Par.	Value	Descrip.
β_p	0.1	latent rate
r	0.01	remove rate
$1/b$	0.075	time of latency
γ	0.06	vector die or depart rate
μ	0.3	immigration rate
θ	0.2	infected vectors arrival
β_v	0.003	vector infected rate

$$R_0 = \sqrt{\frac{\beta_v \mu b \beta_p}{r^2(r+b)\gamma}}$$
$$DFE = (N_p, 0, 0, 0, \mu/\gamma)^\top$$
$$EE = (S_p^*, L_p^*, I_p^*, S_v^*, I_v^*)^\top$$



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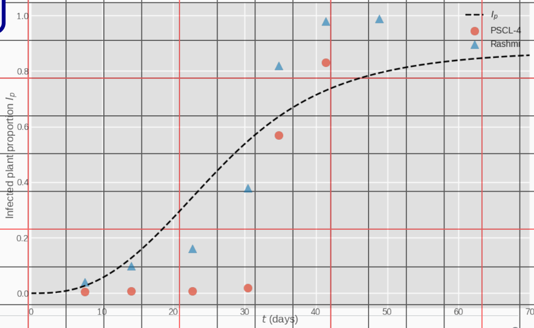

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Controls

$u_1(t)$: Latent replanting
$u_2(t)$: Infected replanting
$u_3(t)$: Fumigation

$$u_i(t) \in [0, u_i^{max}]$$

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$$\int_0^T \left[A_1 I_p(t) + A_2 L_p(t) + A_3 I_v(t) + \sum_{i=1}^{3} c_i \frac{u_i^2}{2} \right] dt$$

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$u_i(t) \in [0, u_i^{max}]$

Optimal Control Problem

$$g(x,u) := A_1 I_p(t) + A_2 L_p(t) + A_3 I_v(t) + c_1 u_1(t)^2 + c_2 u_2(t)^2 + c_3 u_3(t)^2$$

$$\min_{\bar{u}(\cdot)\in\tilde{\mathscr{U}}_{x_0}[0,T]} J(u_1,u_2,u_3) = \int_0^T g(x,u)ds$$

such that:

$$\begin{aligned}
\frac{dS_p}{dt} &= \beta_p S_p I_v + (r+u_1)L_p + (r+u_2)I_p,\\
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x(0) &= x_0, \qquad u_i \in [0, u_i^{\max}]
\end{aligned}$$

Consider the controlled dynamics

$$\begin{cases} \dot{x}(s) = f(s, u(s), x(s)) \; s \in [t_0, T], \\ x(t_0) = x_0, \end{cases}$$

with terminal state constraint

$$x(T; t_0, x_0, u(\cdot)) \in M, M \subseteq \mathbb{R}^n.$$

Problem (OC)

$(t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^n$, find a control policie $\bar{u}(\cdot) \in \tilde{\mathscr{U}}_{x_0}[t_0, T]$ s.t.

$$J(t_0, x_0; \bar{u}(\cdot)) = \inf_{u(\cdot) \in \tilde{\mathscr{U}}_{x_0}[t_0, T]} J(t_0, x_0; u(\cdot)).$$

Cost functional

$$\tilde{\mathscr{U}}_{x_0}[t_0, T] := \{u : [t_0, T] \to \mathbb{R}^n | \text{ measurable}\}$$

$$J(t_0, x_0; u(\cdot)) = \int_{t_0}^{T} g(s, u(s), x(s)) ds + h(T, x(T))$$

Hypothesis:

(C-1) $f : \mathbb{R}_+ \times U \times \mathbb{R}^n \to \mathbb{R}^n$ is measurable, satisfies the lipchitz condition in x, $|f(t,u,0)| \leq L, \forall (t,u) \in \mathbb{R}_+ \times U$.

(C-2) $g : \mathbb{R}_+ \times U \times \mathbb{R}^n \to \mathbb{R}$, $h : \mathbb{R}^n \to \mathbb{R}$ are measurable, and

$$|g(s,u,x_1) - g(s,u,x_2)| + |h(x_1) - h(x_2)| \leq \omega(|x_1| \vee |x_2|, |x_1 - x_2|)$$

$\forall (s,u) \in \mathbb{R}_+ \times U, x_1, x_2 \in \mathbb{R}^n$.

(C-3) For a.a. $t \in [0,T]$, Cesari property holds $\forall\, x \in \mathbb{R}^n$.

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Modulus of continuity

$\omega : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$, increasing, $\omega(r,0) = 0 \; \forall r \ge 0$.

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\begin{aligned}
&|g(s,u,x_1) - g(s,u,x_2)| + \\
&|h(x_1) - h(x_2)| \\
&\leq \omega(|x_1| \vee |x_2|, |x_1 - x_2|)
\end{aligned}
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$\forall (s,u) \in \mathbb{R}_+ \times U, x_1, x_2 \in \mathbb{R}^n$.

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$\omega : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$, increasing, $\omega(r,0) = 0 \; \forall r \geq 0$.

Cesari property

$$
\begin{aligned}
\mathbf{E}(t,x) = \{(z^0, z) \in \mathbb{R} \times \mathbb{R}^n |\\
z^0 \geq g(t,u,x),\\
z = f(t,u,x), u \in U\}.
\end{aligned}
$$

$$
\bigcap_{\delta} \bar{co}\mathbf{E}(t, B_\delta(x)) = \mathbf{E}(t,x)
$$

Hypothesis:

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(C-2) $g : \mathbb{R}_+ \times U \times \mathbb{R}^n \to \mathbb{R}$, $h : \mathbb{R}^n \to \mathbb{R}$ are measurable, and

$$|g(s,u,x_1) - g(s,u,x_2)| + |h(x_1) - h(x_2)| \leq \omega(|x_1| \vee |x_2|, |x_1 - x_2|)$$

$\forall (s,u) \in \mathbb{R}_+ \times U, x_1, x_2 \in \mathbb{R}^n$.

(C-3) For a.a. $t \in [0,T]$, Cesari property holds $\forall\, x \in \mathbb{R}^n$.

Existence Theorem

Let (C-1)-(C-3) hold. Then problem (OC) admits at least one optimal pair.

Optimal Control Characterization

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$(OC)^T$

$$J(t_0,x_0;u(\cdot)) = \int_{t_0}^{T} g(s,u(s),x(s))ds + h(x(T))$$

$$\begin{cases} \dot{x}(s) = f(s,u(s),x(s)) \; s \in [t_0,T] \\ x(t_0) = x_0 \end{cases}$$

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Hamiltonian:

$$H = g(t,x(t),u(t)) + \langle \lambda(t), f(t,x(t),u(t)) \rangle,$$

$$\frac{\partial H}{\partial u_i}(t,\bar{x}(\cdot),\bar{u}(\cdot)) = 0.$$

Additional hypothesis:

(C-4)

$x \mapsto (f(t,u,x), g(t,u,x), h(x))$ is differentiable,

$(u,x) \mapsto (f(t,u,x), f_x(t,u,x), g(t,u,x), g_x(t,u,x), h_x(x))$

is continuous.

Pontryagin's Maximum Principle

If $\bar{u}(t)$ and $\bar{x}(t)$ are optimal for the problem (OC), then there exists a piecewise differentiable adjoint variable $\lambda(t)$ s.t.

$$H(t,\bar{x}(t),u(\bar{t}),\lambda(t)) = \max_{u\in U} H(t,x(t),u(t),\lambda(t)), \qquad \forall t \in [0,T]$$

$$\lambda'(t) = -\frac{\partial H(t,\bar{x}(t),\bar{u}(t),\lambda(t))}{\partial x},$$
$$\lambda(T) = 0.$$

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Hamiltonian

$$H = g(t,x(t),u(t)) + \langle \lambda(t), f(t,x(t),u(t)) \rangle,$$

Example

$$
\begin{aligned}
\frac{dS_p}{dt} &= -\beta_p S_p I_v + (r + u_1) L_p + (r + u_2) I_p, \\
\frac{dL_p}{dt} &= \beta_p S_p I_v - b L_p - (r + u_1) L_p, \\
\frac{dI_p}{dt} &= b L_p - (r + u_2) I_p, \\
\frac{dS_v}{dt} &= -\beta_v S_v I_p - (\gamma + u_3) S_v + (1 - \theta)\mu, \\
\frac{dI_v}{dt} &= \beta_v S_v I_p - (\gamma + u_3) I_v + \theta \mu,
\end{aligned}
$$

Example

$$
\begin{aligned}
\frac{dS_p}{dt} &= -\beta_p S_p I_v + (r + u_1)L_p + (r + u_2)I_p,\\
\frac{dL_p}{dt} &= \beta_p S_p I_v - bL_p - (r + u_1)L_p,\\
\frac{dI_p}{dt} &= bL_p - (r + u_2)I_p,\\
\frac{dS_v}{dt} &= -\beta_v S_v I_p - (\gamma + u_3)S_v + (1-\theta)\mu,\\
\frac{dI_v}{dt} &= \beta_v S_v I_p - (\gamma + u_3)I_v + \theta\mu,
\end{aligned}
$$

$$
\begin{aligned}
H = {} & A_1 I_v + A_2 L_p + A_3 I_v\\
& + \sum_{i=1}^{3} c_i u_i^2\\
& + \lambda_1(-\beta_p S_p I_v + (r + u_1)L_p\\
& + (r + u_2)I_p)\\
& + \lambda_2(\beta_p S_p I_v - bL_p\\
& - (r + u_1)L_p)\\
& + \lambda_3(bL_p - (r + u_2)I_p)\\
& + \lambda_4(-\beta_v S_v I_p - (\gamma + u_3)S_v\\
& + (1-\theta)\mu)\\
& + \lambda_5(\beta_v S_v I_p - (\gamma + u_3)I_v\\
& + \theta\mu).
\end{aligned}
$$

$$
\begin{aligned}
\frac{d\lambda_1}{dt} &= \beta_p(\lambda_1 - \lambda_2), \\
\frac{d\lambda_2}{dt} &= -A_2 + (r + u_1)(\lambda_2 - \lambda_1) + b(\lambda_2 - \lambda_3), \\
\frac{d\lambda_3}{dt} &= -A_1 + (r + u_2)(\lambda_3 - \lambda_1) + \beta_v S_v(\lambda_4 - \lambda_5), \\
\frac{d\lambda_4}{dt} &= \beta_v I_p(\lambda_4 - \lambda_5) + (\gamma + u_3)\lambda_4, \\
\frac{d\lambda_5}{dt} &= -A_3 + \beta_p S_p(\lambda_1 - \lambda_2) + (\gamma + u_3)\lambda_5,
\end{aligned}
$$

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$$\frac{\partial H}{\partial u}(\bar{u}) = 0 \Rightarrow$$

$$u_i \in [0, u_i^{max}]$$

optimal control policies

$$
\begin{aligned}
\bar{u}_1 &= \min\left(\max\left(0, \frac{L_p(\lambda_2 - \lambda_1)}{2c_1}\right), u_1^{max}\right) \\
\bar{u}_2 &= \min\left(\max\left(0, \frac{I_p(\lambda_3 - \lambda_1)}{2c_2}\right), u_2^{max}\right) \\
\bar{u}_3 &= \min\left(\max\left(0, \frac{S_v\lambda_4 + I_v\lambda_5}{2c_3}\right), u_3^{max}\right)
\end{aligned}
$$

Algorithm 2 Forward Backward Sweep

Input: [illegible], t_f, x_0, h, tol
Output: x^*, u^*, λ

procedure FORWARD_BACKWARD_SWEEP$(g, \lambda_{\text{function}}, u, x_0, \lambda_f, h, n_{max})$
  while $\epsilon >$ tol **do**
    $u_{\text{old}} \leftarrow u$
    $x_{\text{old}} \leftarrow x$
    $x \leftarrow$ RUNGE_KUTTA_FORWARD(g, u, x_0, h)
    $\lambda_{\text{old}} \leftarrow \lambda$
    $\lambda \leftarrow$ RUNGE_KUTTA_BACKWARD$(\lambda_{\text{function}}, x, \lambda_f, h)$
    $u_1 \leftarrow$ OPTIMALITY_CONDITION(u, x, λ)
    $u \leftarrow \alpha u_1 + (1-\alpha) u_{old}, \qquad \alpha \in [0,1]$ ▷ convex combination
    $\epsilon_u \leftarrow \dfrac{||u - u_{\text{old}}||}{||u||}$
    $\epsilon_x \leftarrow \dfrac{||x - x_{\text{old}}||}{||x||}$ ▷ relative error
    $\epsilon_\lambda \leftarrow \dfrac{||\lambda - \lambda_{\text{old}}||}{||\lambda||}$
    $\epsilon \leftarrow \max\{\epsilon_u, \epsilon_x, \epsilon_\lambda\}$
  end while
  return x^*, u^*, λ ▷ Optimal pair
end procedure

Set up

$$
\begin{aligned}
&(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P}) \\
W(t) &: m\text{-dimensional Brownian motion} \\
dx(t) &= f(t, u(t), x(t))dt + \sigma(t, u(t), x(t))dW(t) \\
x(0) &= x_0, \\
f &: [0, T] \times U \times \mathbb{R}^n \to \mathbb{R}^n, \\
\sigma &: [0, T] \times U \times \mathbb{R}^n \to \mathbb{R}^{n+m}, \\
U &: \text{separable metric space.}
\end{aligned}
$$

$$\mathscr{U}[0, T] := \{u : [0, T] \times \Omega \to U | u(\cdot) \text{ is } \{\mathscr{F}_t\}_{t\geq 0}\text{-adapted}\}$$

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Weak formulation of optimal control

A 6-tuple $\pi = (\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P}, W(\cdot), u(\cdot))$, is a w-admissible control, if

- $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$ is a filtered probability space satisfying the usual conditions,
- $W(t)$ is an m-dimensional standard Brownian motion defined on $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$,
- $u(\cdot)$ is an $\{\mathscr{F}_t\}_{t\geq 0}$-adapted process on $(\Omega, \mathscr{F}, \mathbb{P})$ taking values in U,
- $x(\cdot)$ is unique solution,
- some prescribed state constraints are satisfied,
- $g(\cdot, u(\cdot), x(\cdot)) \in L^1_{\mathscr{F}}(0, T; \mathbb{R})$ and $h(x(T)) \in L^1_{\mathscr{F}_T}(\Omega; \mathbb{R})$

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- $x(\cdot)$ is unique solution,
- some prescribed state constraints are satisfied,
- $g(\cdot, u(\cdot), x(\cdot)) \in L^1_{\mathscr{F}}(0, T; \mathbb{R})$ and $h(x(T)) \in L^1_{\mathscr{F}_T}(\Omega; \mathbb{R})$

Cost functional

$$J(u(\cdot)) = \mathbb{E}\left\{ \int_0^T g(t,u(t),x(t))dt + h(x(T)) \right\}$$

(WS)

$$J(\bar{\pi}) = \inf_{\pi \in \mathscr{U}_{ad}^{w}[0,T]} J(\pi) \qquad (*)$$

s.t.

$$dx(t) = f(t,u(t),x(t))dt + \sigma(t,u(t),x(t))dW(t)$$
$$x(0) = x_0,$$

---

problem (WS) is finite, if r.h.s. of (*) is finite.

Hypothesis:

(SE-1) (U,d) is a compact metric space and $T>0$,

(SE-2) $f,\sigma,g,$ and h are all continuous, and $\exists L>0$ s.t. $\psi(t,u,x)=\{f(t,u,x),\sigma(t,u,x),g(t,u,x),h(x)\}$,

$$|\psi(t,u,x)-\psi(t,u,\hat{x})|\leq L|x-\hat{x}|,\quad \forall t\in[0,T],x,\hat{x}\in\mathbb{R}^n,u\in U,$$
$$|\psi(t,u,0)|\leq L\forall(t,u)\in[0,T]\times U.$$

(SE-3) $\forall(t,x)\in[0,T]\times\mathbb{R}^n$, the set

$$(f,\sigma\sigma^T,g)(t,x,U):=\{(f_i(t,u,x),(\sigma\sigma^T)^{ij}(t,u,x),g(t,u,x))|\ u\in U,i=1,\dots,n,j=1,\dots,m\}$$

is convex in $\mathbb{R}^{m+nm+1}$,

(SE-4) $S(t)\equiv\mathbb{R}^n$.

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$$
\begin{aligned}
|\psi(t,u,x) - \psi(t,u,\hat{x})| &\leq L|x - \hat{x}|, \\
&\qquad \forall t \in [0,T], x, \hat{x} \in \mathbb{R}^n, u \in U, \\
|\psi(t,u,0)| &\leq L \forall (t,u) \in [0,T] \times U.
\end{aligned}
$$

(SE-3) $\forall (t,x) \in [0,T] \times \mathbb{R}^n$, the set

$$
\begin{aligned}
(f, \sigma\sigma^T, g)(t,x,U) := \{&(f_i(t,u,x), (\sigma\sigma^T)^{ij}(t,u,x), g(t,u,x)) \mid \\
&u \in U, i = 1, \ldots, n, j = 1, \ldots, m\}
\end{aligned}
$$

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$$\begin{aligned}|\psi(t,u,x)-\psi(t,u,\hat{x})| &\leq L|x-\hat{x}|,\\ &\qquad \forall t\in[0,T],x,\hat{x}\in\mathbb{R}^n,u\in U,\\ |\psi(t,u,0)| &\leq L\forall(t,u)\in[0,T]\times U.\end{aligned}$$

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$$\forall t \in [0,T], x, \hat{x} \in \mathbb{R}^n, u \in U,$$
$$|\psi(t,u,0)| \leq L \forall (t,u) \in [0,T] \times U.$$

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is convex in $\mathbb{R}^{m+nm+1}$,

(SE-4) $S(t) \equiv \mathbb{R}^n$.

Existence Theorem (weak formulation)

Under (**SE**1)-(**SE**4), if (WS) is finite, then it admits an optimal control.

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mm 40 60 80 100 120

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