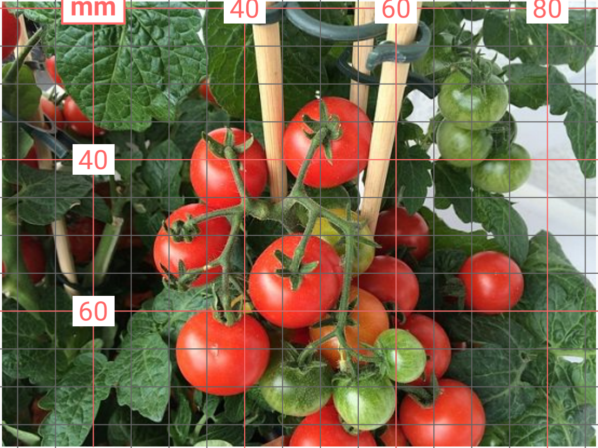
# Modeling optimal control policies

## in stochastic epidemic models

Saúl Díaz Infante Velasco

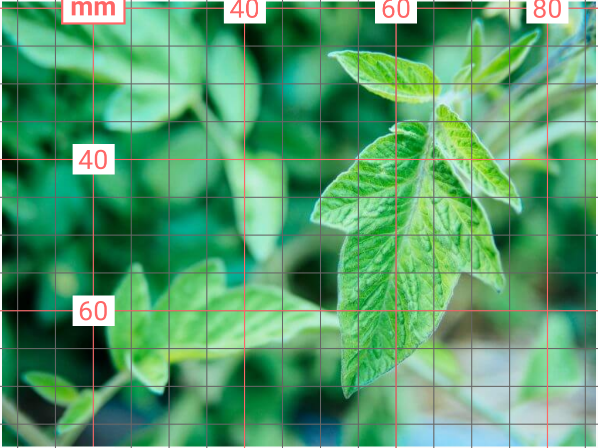
CONACYT-UNIVERSIDAD de SONORA

# Tomato leaf curl virus disease (TYLCVD)



# Tomato leaf curl virus disease (TYLCVD)





# Tomato leaf curl virus disease (TYLCVD)

- Tomato plants **infected early** are severely stunted and will **not produce fruit**
- Leaflets are small and yellowed with edges that curl upwards
- Flowers either do not develop or fall off
- When **older plants** are infected, fruit that is already forming ripens normally, but **no new fruit** is formed after the infection
- TYLCV can be confused with several other conditions such as tomato big bud, herbicide damage and phosphate or magnesium deficiency

# Tomato leaf curl virus disease (TYLCVD)

Society for Industrial and Applied Mathematics

- Tomato plants **infected early** are severely stunted and will **not produce fruit**
- Leaflets are small and yellowed with edges that curl upwards
- Flowers either do not develop or fall off
- When **older plants** are infected, fruit that is already forming ripens normally, but **no new fruit** is formed after the infection
- TYLCV can be confused with several other conditions such as tomato big bud, herbicide damage and phosphate or magnesium deficiency

# Tomato leaf curl virus disease (TYLCVD)

- Tomato plants **infected early** are severely stunted and will **not produce fruit**
- Leaflets are small and yellowed with edges that curl upwards
- Flowers either do not develop or fall off
- When **older plants** are infected, fruit that is already forming ripens normally, but **no new fruit** is formed after the infection
- TYLCV can be confused with several other conditions such as tomato big bud, herbicide damage and phosphate or magnesium deficiency

- Tomato plants **infected early** are severely stunted and will **not produce fruit**
- Leaflets are small and yellowed with edges that curl upwards
- Flowers either do not develop or fall off
- When **older plants** are infected, fruit that is already forming ripens normally, but **no new fruit** is formed after the infection
- TYLCV can be confused with several other conditions such as tomato big bud, herbicide damage and phosphate or magnesium deficiency

# Tomato leaf curl virus disease (TYLCVD)

- Tomato plants **infected early** are severely stunted and will **not produce fruit**
- Leaflets are small and yellowed with edges that curl upwards
- Flowers either do not develop or fall off
- When **older plants** are infected, fruit that is already forming ripens normally, but **no new fruit** is formed after the infection
- TYLCV can be confused with several other conditions such as tomato big bud, herbicide damage and phosphate or magnesium deficiency

siam
Society for Industrial and Applied Mathematics

## Spread

- **TYLCV** is spread by the insect silverleaf whitefly (Bemisia tabaci B biotype)
- Silverleaf whiteflies pick up the virus by feeding on infected host plants. The whiteflies then spread the virus to healthy plants **which show the symptoms 10 to 21 days later**
- Silverleaf whiteflies are **common** in many countries and **feed** on **many** types of **plants**

# Tomato leaf curl virus disease (TYLCVD)



## Spread

- **TYLCV** is spread by the insect silverleaf whitefly (Bemisia tabaci B biotype)
- Silverleaf whiteflies pick up the virus by feeding on infected host plants. The whiteflies then spread the virus to healthy plants **which show the symptoms 10 to 21 days later**
- Silverleaf whiteflies are **common** in many countries and **feed** on **many** types of **plants**

### Spread

- **TYLCV** is spread by the insect silverleaf whitefly (Bemisia tabaci B biotype)
- Silverleaf whiteflies pick up the virus by feeding on infected host plants. The whiteflies then spread the virus to healthy plants **which show the symptoms 10 to 21 days later**
- Silverleaf whiteflies are **common** in many countries and **feed** on **many** types of **plants**

## Control

### Cultural Control

- Physical barriers
- Planting dates
- **Removal of infested plants**
- Host plant resistance

### Biological Control

- Parasitoids
- Predators
- Fungi

## Insecticides

- pymetrozine
- zeta-cypermethrin / bifenthrin

## Control

Cultural Control

- Physical barriers
- Planting dates
- **Removal of infested plants**
- Host plant resistance

Biological Control

- Parasitoids
- Predators
- Fungi

## Insecticides

- pymetrozine
- zeta-cypermethrin / bifenthrin

0

80

## Control

Cultural Control

- Physical barriers
- Planting dates
- **Removal of infested plants**
- Host plant resistance

Biological Control

- Parasitoids
- Predators
- Fungi

## Insecticides

- pymetrozine
- zeta-cypermethrin / bifenthrin

0

80

## Control

Cultural Control

- Physical barriers
- Planting dates
- **Removal of infested plants**
- Host plant resistance

Biological Control

- Parasitoids
- Predators
- Fungi

## Insecticides

- pymetrozine
- zeta-cypermethrin / bifenthrin

0

80

## Control

Cultural Control

- Physical barriers
- Planting dates
- **Removal of infested plants**
- Host plant resistance

Biological Control

- Parasitoids
- Predators
- Fungi

## Insecticides

- pymetrozine
- zeta-cypermethrin / bifenthrin

## Control

Cultural Control

- Physical barriers
- Planting dates
- **Removal of infested plants**
- Host plant resistance

Biological Control

- Parasitoids
- Predators
- Fungi

## Insecticides

- pymetrozine
- zeta-cypermethrin / bifenthrin

0

80

## Control

Cultural Control

- Physical barriers
- Planting dates
- **Removal of infested plants**
- Host plant resistance

Biological Control

- Parasitoids
- Predators
- Fungi

## Insecticides

- pymetrozine
- zeta-cypermethrin / bifenthrin

0

80

# Table of contents



mm 40 60 80 100 120

40

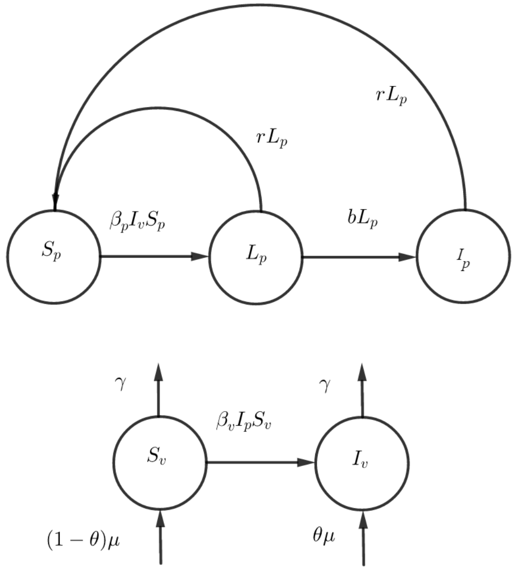
60

80

mm 40

## Hypothesis:

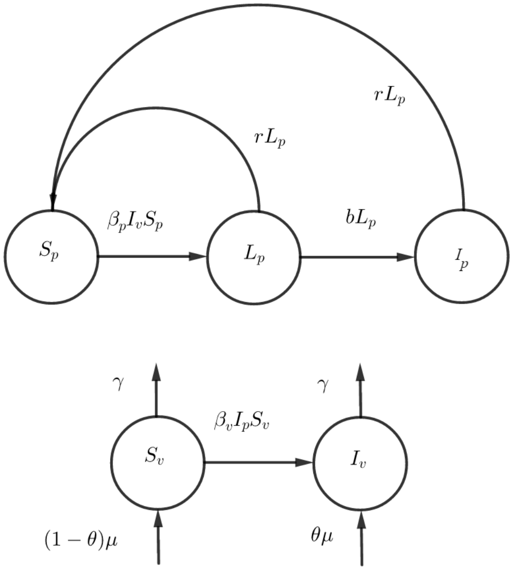
- Remove from latent and infected plants,
- plants become latent plants by infected vectors,
- latent plants become infectious plants,
- vectors become infected vectors by infected plants,
- vectors die per day,
- immigration from alternative hosts.



mm 40

## Hypothesis:

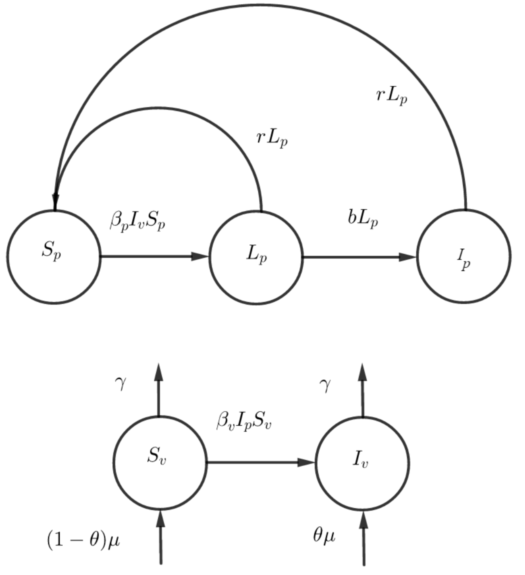
- Remove from latent and infected plants,
- plants become latent plants by infected vectors,
- latent plants become infectious plants,
- vectors become infected vectors by infected plants,
- vectors die per day,
- immigration from alternative hosts.



mm 40

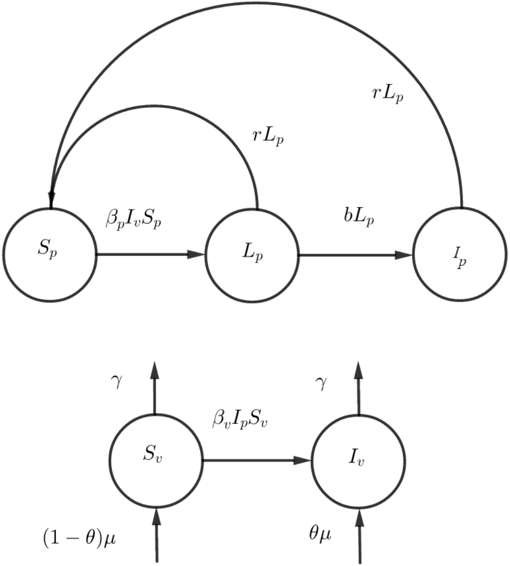
## Hypothesis:

- Remove from latent and infected plants,
- plants become latent plants by infected vectors,
- latent plants become infectious plants,
- vectors become infected vectors by infected plants,
- vectors die per day,
- immigration from alternative hosts.



## Hypothesis:

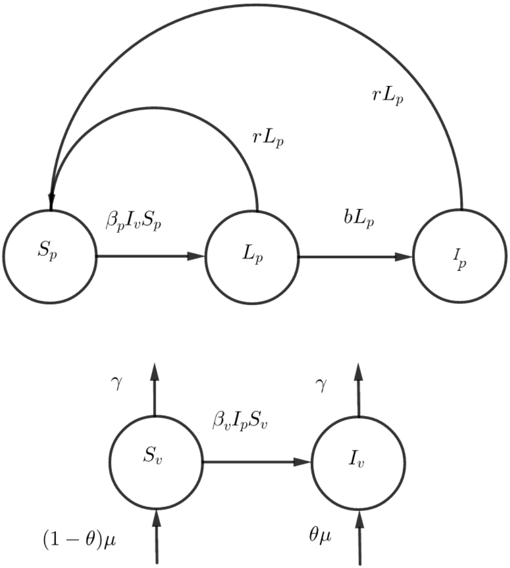
- Remove from latent and infected plants,
- plants become latent plants by infected vectors,
- latent plants become infectious plants,
- vectors become infected vectors by infected plants,
- vectors die per day,
- immigration from alternative hosts.



mm 40

## Hypothesis:

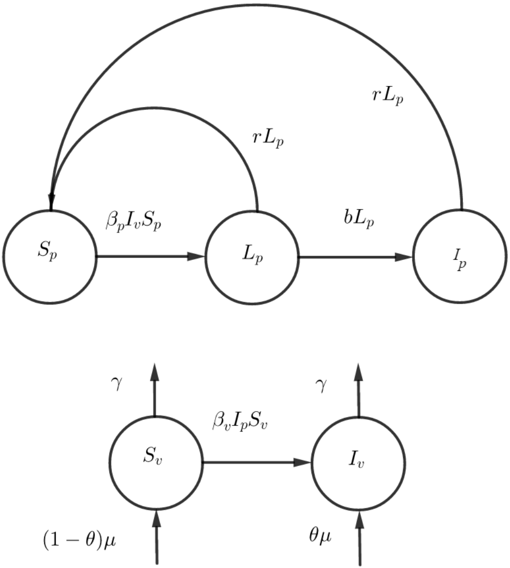
- Remove from latent and infected plants,
- plants become latent plants by infected vectors,
- latent plants become infectious plants,
- vectors become infected vectors by infected plants,
- vectors die per day,
- immigration from alternative hosts.



mm 40

## Hypothesis:

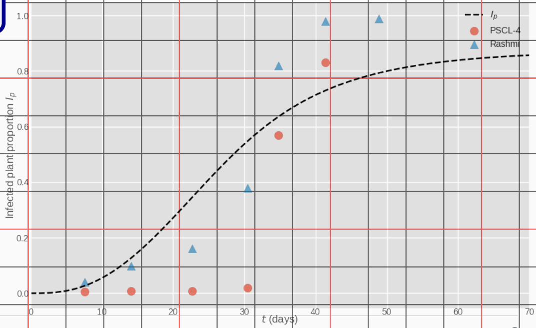
- Remove from latent and infected plants,
- plants become latent plants by infected vectors,
- latent plants become infectious plants,
- vectors become infected vectors by infected plants,
- vectors die per day,
- immigration from alternative hosts.



$$
\begin{aligned}
\frac{dS_p}{dt} &= -\beta_p S_p I_v + r(L_p + I_p) \\
\frac{dL_p}{dt} &= \beta_p S_p I_v - bL_p - rL_p \\
\frac{dI_p}{dt} &= bL_p - rI_p \\
\frac{dS_v}{dt} &= -\beta_v S_v I_p - \gamma S_v + (1-\theta)\mu \\
\frac{dI_v}{dt} &= \beta_v S_v I_p - \gamma I_v + \theta \mu
\end{aligned}
$$

| Par. | Value | Descrip. |
|---|---|---|
| $\beta_p$ | 0.1 | latent rate |
| $r$ | 0.01 | remove rate |
| $1/b$ | 0.075 | time of latency |
| $\gamma$ | 0.06 | vector die or depart rate |
| $\mu$ | 0.3 | immigration rate |
| $\theta$ | 0.2 | infected vectors arrival |
| $\beta_v$ | 0.003 | vector infected rate |

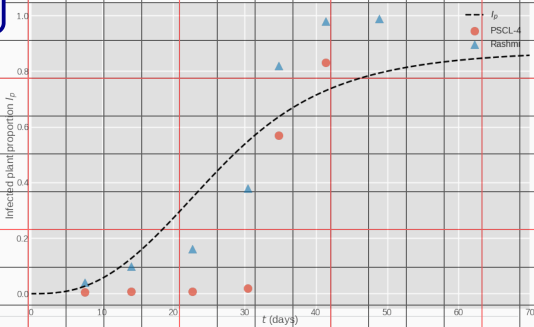
$$
\begin{aligned}
R_0 &= \sqrt{\frac{\beta_v \mu b \beta_p}{r^2 (r+b)\gamma}} \\
DFE &= (N_p, 0, 0, 0, \mu/\gamma)^\top \\
EE &= (S_p^*, L_p^*, I_p^*, S_v^*, I_v^*)^\top
\end{aligned}
$$



$$\frac{dS_p}{dt} = -\beta_p S_p I_v + \mathbf{r}(L_p + I_p)$$
$$\frac{dL_p}{dt} = \beta_p S_p I_v - bL_p - \mathbf{r}L_p$$
$$\frac{dI_p}{dt} = bL_p - \mathbf{r}I_p$$
$$\frac{dS_v}{dt} = -\beta_v S_v I_p - \gamma S_v + (1-\theta)\mu$$
$$\frac{dI_v}{dt} = \beta_v S_v I_p - \gamma I_v + \theta\mu$$

| Par. | Value | Descrip. |
|---|---|---|
| $\beta_p$ | 0.1 | latent rate |
| $r$ | 0.01 | remove rate |
| $1/b$ | 0.075 | time of latency |
| $\gamma$ | 0.06 | vector die or depart rate |
| $\mu$ | 0.3 | immigration rate |
| $\theta$ | 0.2 | infected vectors arrival |
| $\beta_v$ | 0.003 | vector infected rate |

$$R_0 = \sqrt{\frac{\beta_v \mu b \beta_p}{r^2(r+b)\gamma}}$$
$$DFE = (N_p, 0, 0, 0, \mu/\gamma)^\top$$
$$EE = (S_p^*, L_p^*, I_p^*, S_v^*, I_v^*)^\top$$



$$\frac{dS_p}{dt} = -\beta_p S_p I_v + \mathbf{r}(L_p + I_p)$$
$$\frac{dL_p}{dt} = \beta_p S_p I_v - bL_p - \mathbf{r}L_p$$
$$\frac{dI_p}{dt} = bL_p - \mathbf{r}I_p$$
$$\frac{dS_v}{dt} = -\beta_v S_v I_p - \gamma S_v + (1-\theta)\mu$$
$$\frac{dI_v}{dt} = \beta_v S_v I_p - \gamma I_v + \theta\mu$$

**Controls**

$u_1(t)$ : Latent replanting
$u_2(t)$ : Infected replanting
$u_3(t)$ : Fumigation

$u_i(t) \in [0, u_i^{max}]$

80

$$
\begin{aligned}
\frac{dS_p}{dt} &= -\beta_p S_p I_v + \mathbf{r}(L_p + I_p) \\
\frac{dL_p}{dt} &= \beta_p S_p I_v - bL_p - \mathbf{r}L_p \\
\frac{dI_p}{dt} &= bL_p - \mathbf{r}I_p \\
\frac{dS_v}{dt} &= -\beta_v S_v I_p - \gamma S_v + (1-\theta)\mu \\
\frac{dI_v}{dt} &= \beta_v S_v I_p - \gamma I_v + \theta\mu
\end{aligned}
$$

**Controls**

$u_1(t)$ : Latent replanting

$u_2(t)$ : Infected replanting

$u_3(t)$ : Fumigation

$u_i(t) \in [0, u_i^{max}]$

80

$$
\begin{aligned}
\frac{dS_p}{dt} &= -\beta_p S_p I_v + (u_1 + r) L_p + (u_2 + r) I_p \\
\frac{dL_p}{dt} &= \beta_p S_p I_v - b L_p - (u_1 + r) L_p \\
\frac{dI_p}{dt} &= b L_p - (u_2 + r) I_p \\
\frac{dS_v}{dt} &= -\beta_v S_v I_p - (\gamma + u_3) S_v + (1 - \theta)\mu \\
\frac{dI_v}{dt} &= \beta_v S_v I_p - (\gamma + u_3) I_v + \theta\mu
\end{aligned}
$$

**Controls**

$u_1(t)$ : Latent replanting

$u_2(t)$ : Infected replanting

$u_3(t)$ : Fumigation

$$u_i(t) \in [0, u_i^{max}]$$

80

$$\frac{dS_p}{dt} = -\beta_p S_p I_v + (u_1 + r) L_p + (u_2 + r) I_p$$
$$\frac{dL_p}{dt} = \beta_p S_p I_v - b L_p - (u_1 + r) L_p$$
$$\frac{dI_p}{dt} = b L_p - (u_2 + r) I_p$$
$$\frac{dS_v}{dt} = -\beta_v S_v I_p - (\gamma + u_3) S_v + (1 - \theta)\mu$$
$$\frac{dI_v}{dt} = \beta_v S_v I_p - (\gamma + u_3) I_v + \theta\mu$$

$$\int_0^T \left[ A_1 I_p(t) + A_2 L_p(t) + A_3 I_v(t) + \sum_{i=1}^{3} c_i \frac{u_i^2}{2} \right] dt$$

**Controls**

$u_1(t)$ : Latent replanting

$u_2(t)$ : Infected replanting

$u_3(t)$ : Fumigation

$u_i(t) \in [0, u_i^{max}]$

## Optimal Control Problem

$$g(x,u) := A_1 I_p(t) + A_2 L_p(t) + A_3 I_v(t) + c_1 u_1(t)^2 + c_2 u_2(t)^2 + c_3 u_3(t)^2$$

$$\min_{\bar{u}(\cdot)\in\tilde{\mathscr{U}}_{x_0}[0,T]} J(u_1,u_2,u_3) = \int_0^T g(x,u)ds$$

such that:

$$\begin{aligned}
\frac{dS_p}{dt} &= \beta_p S_p I_v + (r+u_1)L_p + (r+u_2)I_p,\\
\frac{dL_p}{dt} &= \beta_p S_p I_v - bL_p - (r+u_1)L_p,\\
\frac{dI_p}{dt} &= bL_p - (r+u_2)I_p,\\
\frac{dS_v}{dt} &= \beta_v S_v I_p - (\gamma+u_3)S_v + (1-\theta)\mu,\\
\frac{dI_v}{dt} &= \beta_v S_v I_p - (\gamma+u_3)I_v + \theta\mu,\\
x(0) &= x_0, \qquad u_i \in [0, u_i^{\max}]
\end{aligned}$$

### Consider the controlled dynamics

$$\begin{cases} \dot{x}(s) = f(s, u(s), x(s)) \; s \in [t_0, T], \\ x(t_0) = x_0, \end{cases}$$

with terminal state constraint

$$x(T; t_0, x_0, u(\cdot)) \in M, M \subseteq \mathbb{R}^n.$$

### Problem $(OC)$

$(t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^n$, find a control policie $\bar{u}(\cdot) \in \tilde{\mathscr{U}}_{x_0}[t_0, T]$ s.t.

$$J(t_0, x_0; \bar{u}(\cdot)) = \inf_{u(\cdot) \in \tilde{\mathscr{U}}_{x_0}[t_0, T]} J(t_0, x_0; u(\cdot)).$$

### Cost functional

$$\tilde{\mathscr{U}}_{x_0}[t_0, T] := \{u : [t_0, T] \to \mathbb{R}^n | \text{ measurable}\}$$

$$J(t_0, x_0; u(\cdot)) = \int_{t_0}^{T} g(s, u(s), x(s)) ds + h(T, x(T))$$

**Hypothesis:**

**(C-1)** $f : \mathbb{R}_+ \times U \times \mathbb{R}^n \to \mathbb{R}^n$ is measurable, satisfies the lipchitz condition in $x$, $|f(t,u,0)| \leq L, \forall (t,u) \in \mathbb{R}_+ \times U$.

**(C-2)** $g : \mathbb{R}_+ \times U \times \mathbb{R}^n \to \mathbb{R}$, $h : \mathbb{R}^n \to \mathbb{R}$ are measurable, and

$$\begin{aligned} &|g(s,u,x_1) - g(s,u,x_2)| + \\ &|h(x_1) - h(x_2)| \\ &\leq \omega(|x_1| \vee |x_2|, |x_1 - x_2|) \end{aligned}$$

$\forall (s,u) \in \mathbb{R}_+ \times U, x_1, x_2 \in \mathbb{R}^n$.

**(C-3)** For a.a. $t \in [0,T]$, Cesari property holds $\forall\, x \in \mathbb{R}^n$.

## Hypothesis:

**(C-1)** $f : \mathbb{R}_+ \times U \times \mathbb{R}^n \to \mathbb{R}^n$ is measurable, satisfies the lipchitz condition in $x$, $|f(t,u,0)| \leq L, \forall (t,u) \in \mathbb{R}_+ \times U$.

**(C-2)** $g : \mathbb{R}_+ \times U \times \mathbb{R}^n \to \mathbb{R}$, $h : \mathbb{R}^n \to \mathbb{R}$ are measurable, and

$$\begin{aligned} &|g(s,u,x_1) - g(s,u,x_2)| + \\ &|h(x_1) - h(x_2)| \\ &\leq \omega(|x_1| \vee |x_2|, |x_1 - x_2|) \end{aligned}$$

$\forall (s,u) \in \mathbb{R}_+ \times U, x_1, x_2 \in \mathbb{R}^n$.

**(C-3)** For a.a. $t \in [0,T]$, Cesari property holds $\forall\, x \in \mathbb{R}^n$.

## Modulus of continuity

$\omega : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$, increasing, $\omega(r,0) = 0\ \forall r \geq 0$.

## Hypothesis:

**(C-1)** $f : \mathbb{R}_+ \times U \times \mathbb{R}^n \to \mathbb{R}^n$ is measurable, satisfies the lipchitz condition in $x$, $|f(t,u,0)| \leq L, \forall (t,u) \in \mathbb{R}_+ \times U$.

**(C-2)** $g : \mathbb{R}_+ \times U \times \mathbb{R}^n \to \mathbb{R}$, $h : \mathbb{R}^n \to \mathbb{R}$ are measurable, and

$$
\begin{aligned}
&|g(s,u,x_1) - g(s,u,x_2)| + \\
&|h(x_1) - h(x_2)| \\
&\leq \omega(|x_1| \vee |x_2|, |x_1 - x_2|)
\end{aligned}
$$

$\forall (s,u) \in \mathbb{R}_+ \times U, x_1, x_2 \in \mathbb{R}^n$.

**(C-3)** For a.a. $t \in [0,T]$, Cesari property holds $\forall\, x \in \mathbb{R}^n$.

## Modulus of continuity

$\omega : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$, increasing, $\omega(r,0) = 0\ \forall r \geq 0$.

## Cesari property

$$
\begin{aligned}
\mathbf{E}(t,x) = \{&(z^0, z) \in \mathbb{R} \times \mathbb{R}^n | \\
&z^0 \geq g(t,u,x), \\
&z = f(t,u,x), u \in U\}.
\end{aligned}
$$

$$
\bigcap_{\delta} \bar{co}\mathbf{E}(t, B_\delta(x)) = \mathbf{E}(t,x)
$$

**Hypothesis:**

**(C-1)** $f:\mathbb{R}_+\times U\times\mathbb{R}^n\to\mathbb{R}^n$ is measurable, satisfies a Lipchitz condition in $x$, $|f(t,u,0)|\le L, \forall (t,u)\in\mathbb{R}_+\times U$.

**(C-2)** $g:\mathbb{R}_+\times U\times\mathbb{R}^n\to\mathbb{R}$, $h:\mathbb{R}^n\to\mathbb{R}$ are measurable, and

$$|g(s,u,x_1)-g(s,u,x_2)|+ \\ |h(x_1)-h(x_2)| \\ \le \omega(|x_1|\vee|x_2|,|x_1-x_2|)$$

$\forall (s,u)\in\mathbb{R}_+\times U, x_1,x_2\in\mathbb{R}^n$.

**(C-3)** For a.a. $t\in[0,T]$, Cesari property holds $\forall\, x\in\mathbb{R}^n$.

**Existence Theorem**

Let (C-1)-(C-3) hold. Then problem $(OC)$ admits at least one optimal pair.

# Optimal Control Characterization



## $(OC)^T$

$$J(t_0,x_0;u(\cdot)) = \int_{t_0}^{T} g(s,u(s),x(s))ds + h(x(T))$$

$$\begin{cases} \dot{x}(s) = f(s,u(s),x(s)) \; s \in [t_0,T] \\ x(t_0) = x_0 \end{cases}$$

## Additional hypothesis:

**(C-4)**

$x \mapsto (f(t,u,x), g(t,u,x), h(x))$ is differentiable,

$$(u,x) \mapsto (f(t,u,x), f_x(t,u,x), g(t,u,x), g_x(t,u,x), h_x(x))$$

is continuous.

## Hamiltonian:

$$H = g(t,x(t),u(t)) + \langle \lambda(t), f(t,x(t),u(t)) \rangle,$$

---

$$\frac{\partial H}{\partial u_i}(t,\bar{x}(\cdot),\bar{u}(\cdot)) = 0.$$

**Pontryagin's Maximum Principle**

If $\bar{u}(t)$ and $\bar{x}(t)$ are optimal for the problem $(OC)$, then there exists a piecewise differentiable adjoint variable $\lambda(t)$ s.t.

$$H(t,\bar{x}(t),u(t),\lambda(t)) \leq H(t,\bar{x}(t),\bar{u}(t),\lambda(t))$$

$\forall$ $u$ at $t$,

$$\begin{aligned}\lambda'(t) &= -\frac{\partial H(t,\bar{x}(t),\bar{u}(t),\lambda(t))}{\partial x},\\ \lambda(T) &= 0.\end{aligned}$$

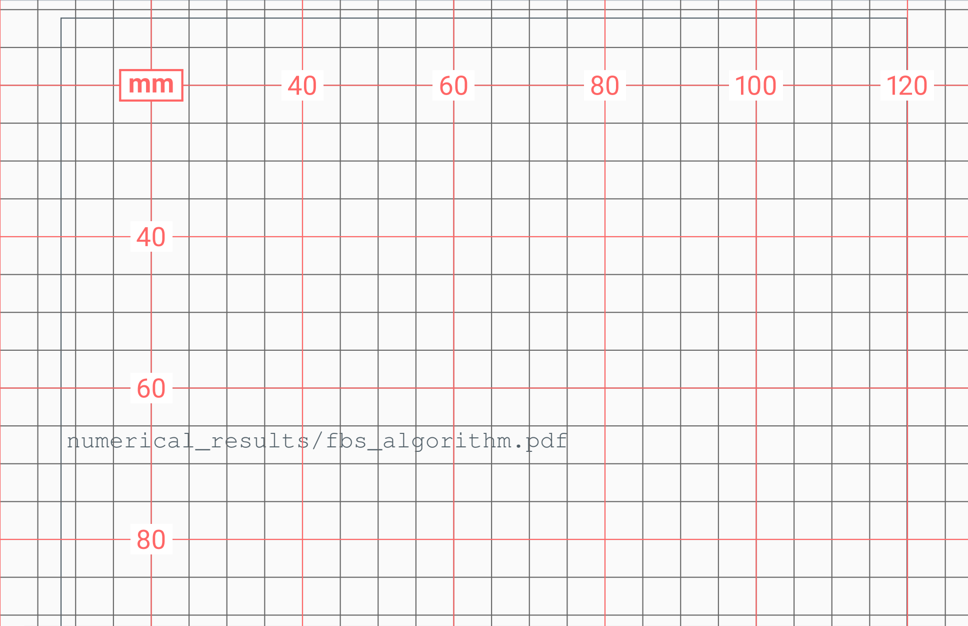
$$H = g(t,x(t),u(t)) + \langle \lambda(t), f(t,x(t),u(t)) \rangle,$$

# Example

$$
\begin{aligned}
\frac{dS_p}{dt} &= -\beta_p S_p I_v + (r + u_1) L_p + (r + u_2) I_p, \\
\frac{dL_p}{dt} &= \beta_p S_p I_v - b L_p - (r + u_1) L_p, \\
\frac{dI_p}{dt} &= b L_p - (r + u_2) I_p, \\
\frac{dS_v}{dt} &= -\beta_v S_v I_p - (\gamma + u_3) S_v + (1 - \theta)\mu, \\
\frac{dI_v}{dt} &= \beta_v S_v I_p - (\gamma + u_3) I_v + \theta\mu,
\end{aligned}
$$

$$
\begin{aligned}
H = {} & A_1 I_v + A_2 L_p + A_3 I_v \\
& + \sum_{i=1}^{3} c_i u_i^2 \\
& + \lambda_1(-\beta_p S_p I_v + (r + u_1) L_p \\
& + (r + u_2) I_p) \\
& + \lambda_2(\beta_p S_p I_v - b L_p \\
& - (r + u_1) L_p) \\
& + \lambda_3(b L_p - (r + u_2) I_p) \\
& + \lambda_4(-\beta_v S_v I_p - (\gamma + u_3) S_v \\
& + (1 - \theta)\mu) \\
& + \lambda_5(\beta_v S_v I_p - (\gamma + u_3) I_v \\
& + \theta\mu).
\end{aligned}
$$

# The most popular



# Case with one controls

mm 40 60 80 100 120

40

./only_infected_replanting-eps-conver

60

80

# Case with one controls

./one_control_simulation_1-eps-conver

## Case with two controls

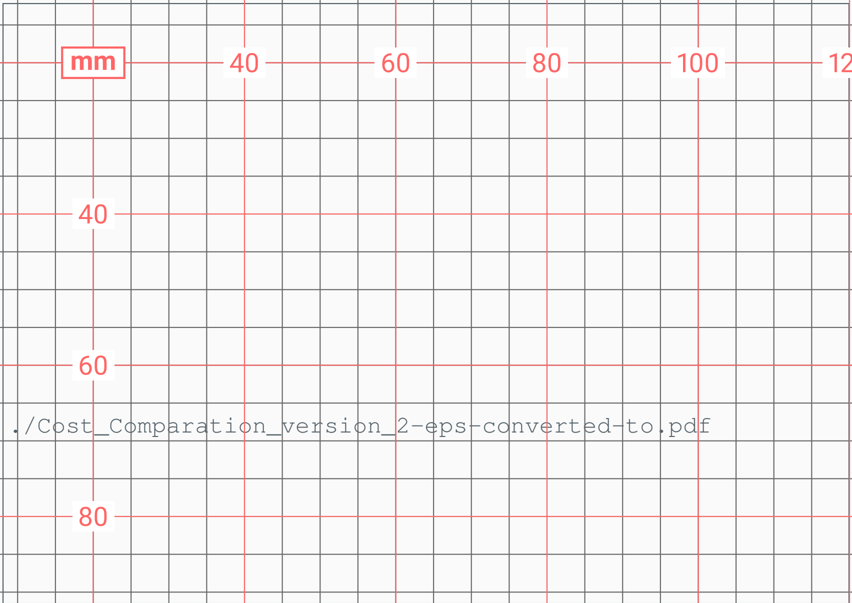
./two_control_simulation_1-eps-conver

# Case with three controls

./three_controls_simulation_1-eps-con

# Cost Function Comparation





mm 40 60 80 100 120

40

60

80