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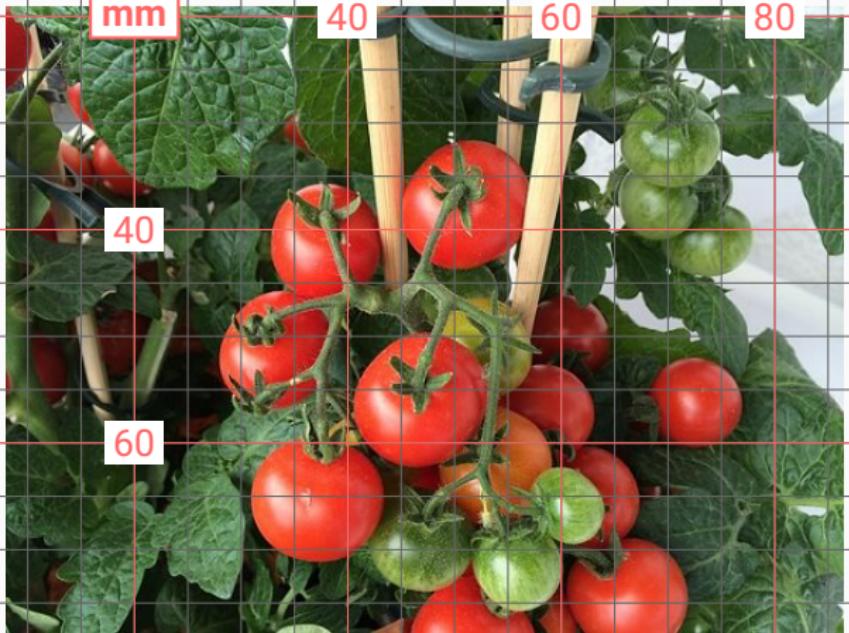
Modeling optimal control policies

in stochastic epidemic models

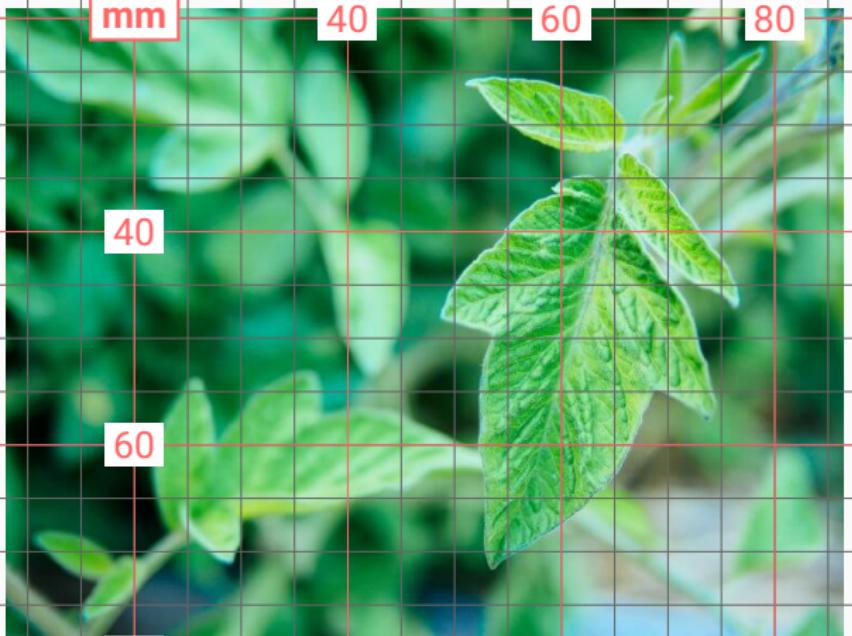
Saúl Díaz Infante Velasco

CONACYT-UNIVERSIDAD de SONORA

Tomato leaf curl virus disease (TYLCVD)



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- Tomato plants **infected early** are severely stunted and will **not produce fruit**
- Leaflets are small and yellowed with edges that curl upwards
- Flowers either do not develop or fall off
- When **older plants** are infected, fruit that is already forming ripens normally, but **no new fruit** is formed after the infection
- TYLCV can be confused with several other conditions such as tomato big bud, herbicide damage and phosphate or magnesium deficiency



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Spread

- **TYLCV** is spread by the insect silverleaf whitefly (Bemisia tabaci B biotype)
- Silverleaf whiteflies pick up the virus by feeding on infected host plants. The whiteflies then spread the virus to healthy plants which show the symptoms 10 to 21 days later
- Silverleaf whiteflies are common in many countries and feed on many types of plants



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Control

Cultural Control

- Physical barriers
- Planting dates
- Removal of infested plants
- Host plant resistance

Biological Control

- Parasitoids
- Predators
- Fungi

Insecticides

- pymetrozine
- zeta-cypermethrin / bifenthrin



Shun-xiang, R., Zhen-zhong, W., Bao-li, Q., and Yuan, X. (2001).

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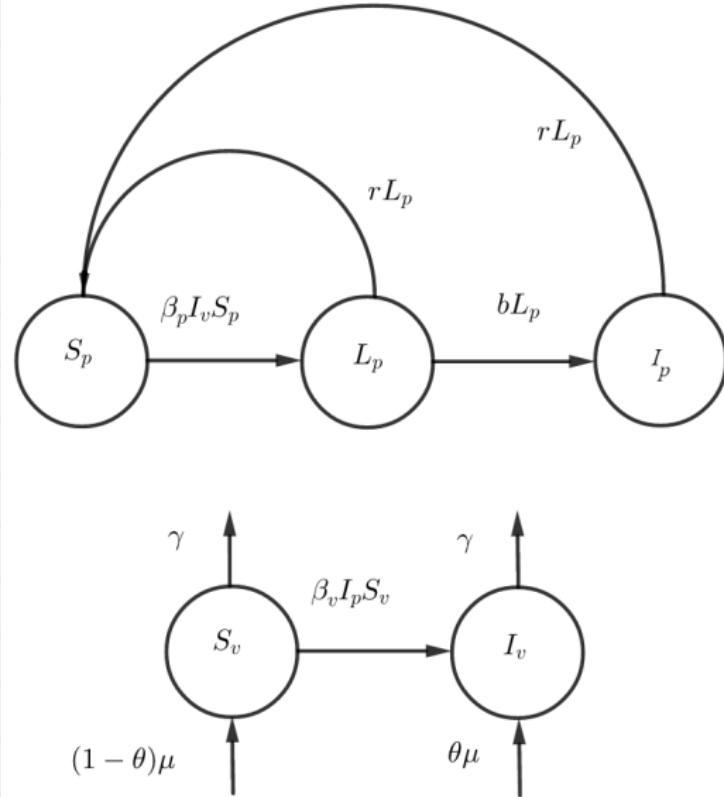


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Journal of Applied Ecology
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- Remove from latent and infected plants,
- plants become latent plants by infected vectors,
- latent plants become infectious plants,
- vectors become infected vectors by infected plants,
- vectors die per day,
- immigration from alternative hosts.



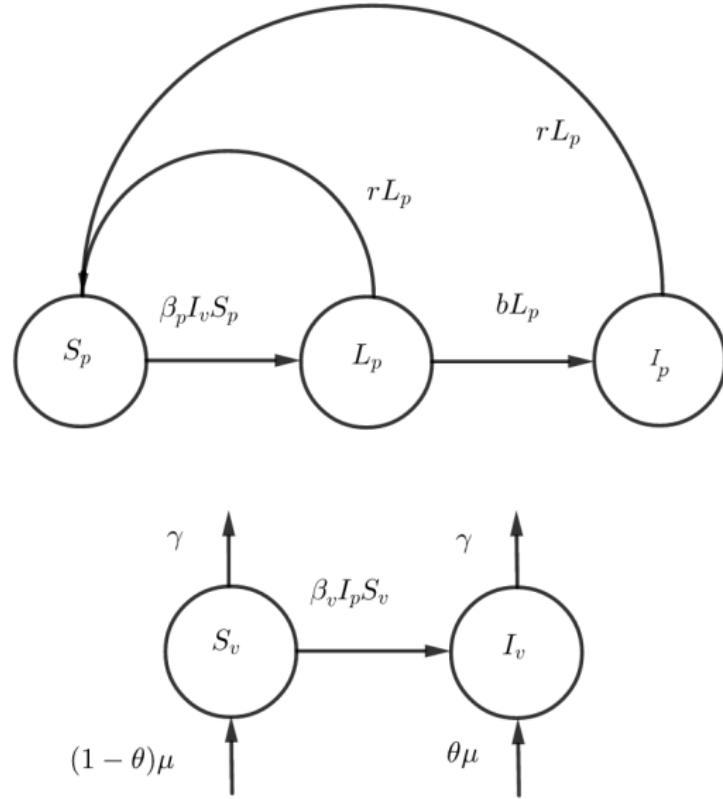


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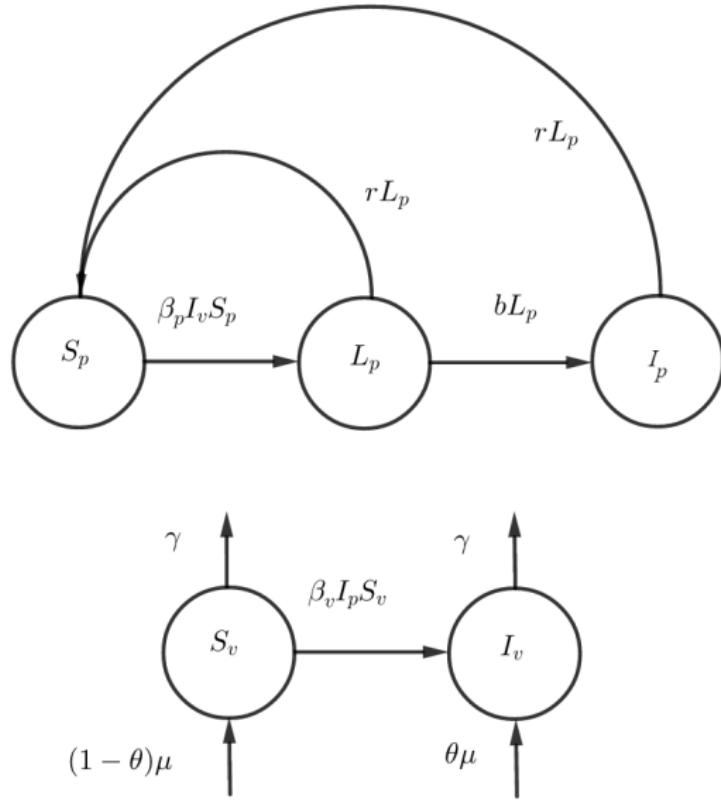
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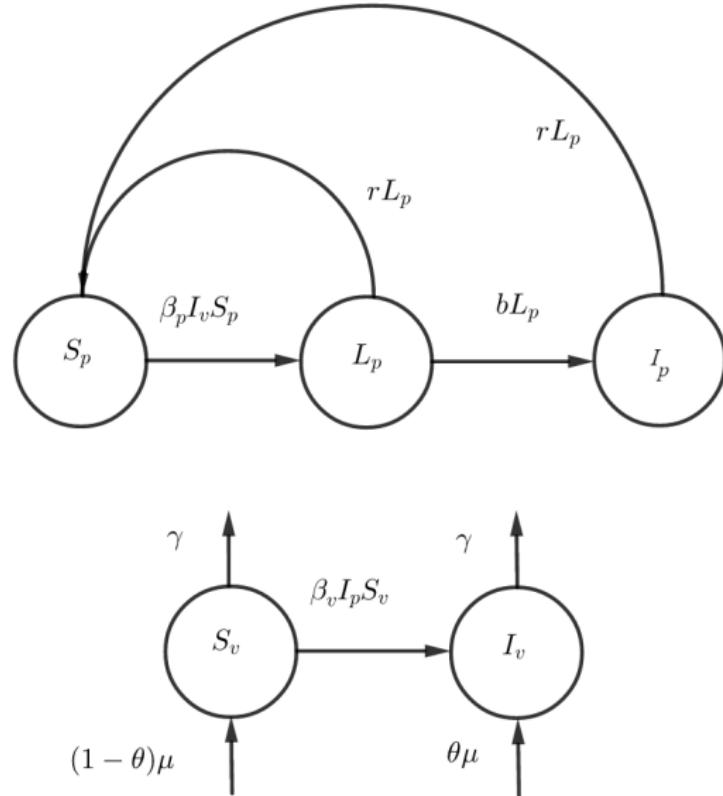
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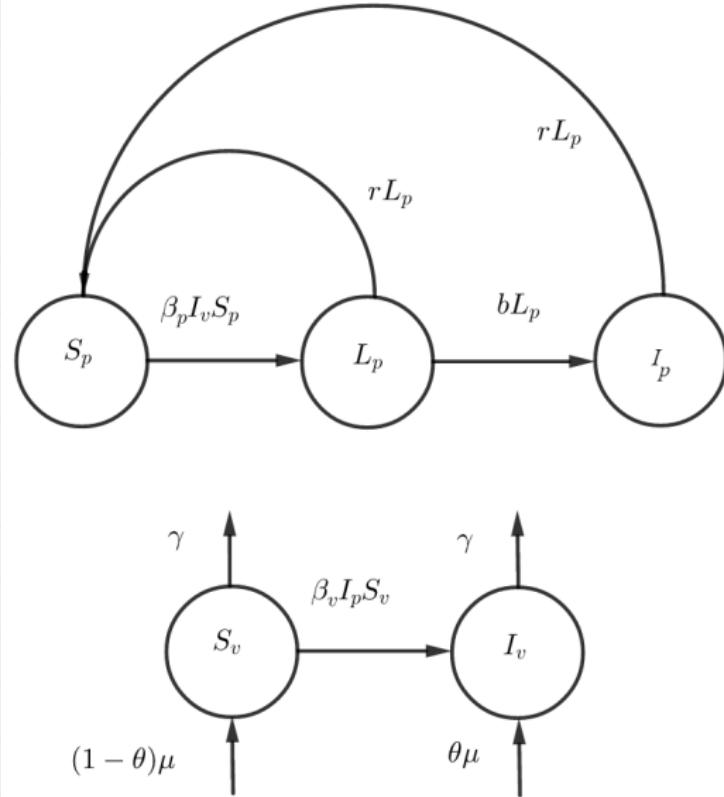


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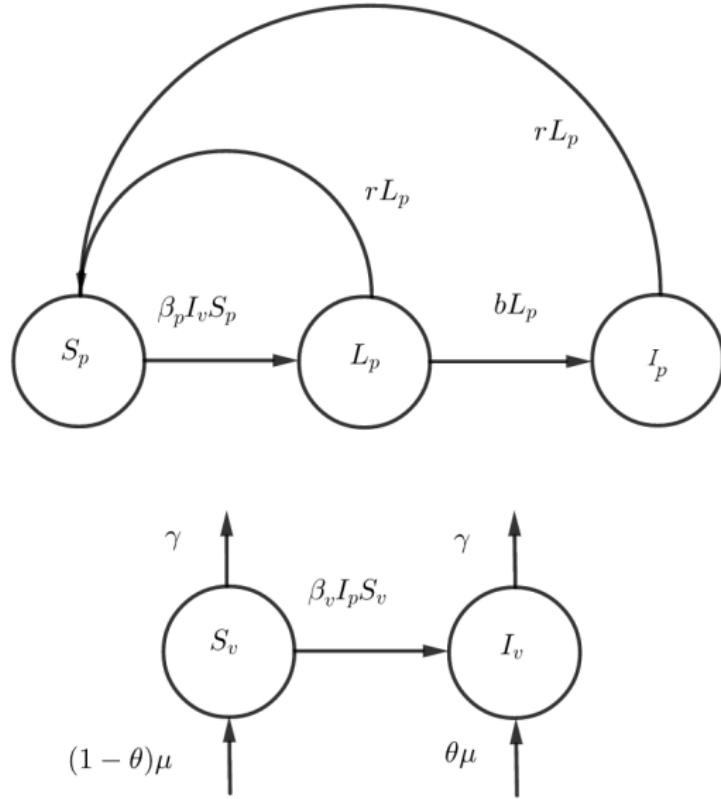


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$$\frac{dI_p}{dt} = b L_p - r I_p$$

$$\frac{dS_v}{dt} = -\beta_v S_v I_p - \gamma S_v + (1 - \theta)\mu$$

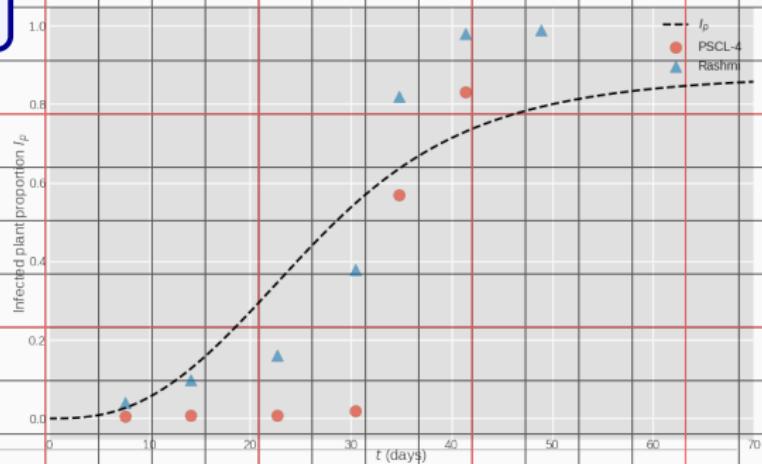
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$$R_0 = \sqrt{\frac{\beta_v \mu b \beta_p}{r^2(r+b)\gamma}}$$

$$DFE = (N_p, 0, 0, 0, \mu/\gamma)^\top$$

$$EE = (S_p^*, L_p^*, I_p^*, S_v^*, I_v^*)^\top$$

Par.	Value	Descrip.
β_p	0.1	latent rate
r	0.01	remove rate
$1/b$	0.075	time of latency
γ	0.06	vector die or depart rate
μ	0.3	immigration rate
θ	0.2	infected vectors arrival
β_v	0.003	vector infected rate



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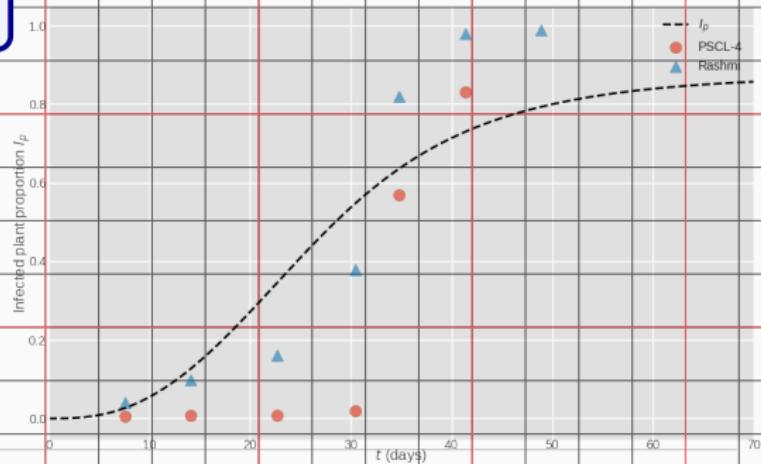
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Controls

$u_1(t)$: Latent replanting

$u_2(t)$: Infected replanting

$u_3(t)$: Fumigation

$$u_i(t) \in [0, u_i^{\max}]$$

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$$\int_0^T \left[A_1 I_p(t) + A_2 L_p(t) + A_3 I_v(t) + \sum_{i=1}^3 c_i \frac{u_i^2}{2} \right] dt$$

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Optimal Control Problem

$$g(x, u) := A_1 I_p(t) + A_2 L_p(t) + A_3 I_v(t) + c_1 u_1(t)^2 + c_2 u_2(t)^2 + c_3 u_3(t)^2$$

$$\min_{\bar{u}(\cdot) \in \tilde{\mathcal{U}}_{x_0}[0, T]} J(u_1, u_2, u_3) = \int_0^T g(x, u) ds$$

such that:

$$\frac{dS_p}{dt} = \beta_p S_p I_v + (r + u_1) L_p + (r + u_2) I_p,$$

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$$x(0) = x_0, \quad u_i \in [0, u_i^{\max}]$$

Consider the controlled dynamics

$$\begin{cases} \dot{x}(s) = f(s, u(s), x(s)) & s \in [t_0, T], \\ x(t_0) = x_0, \end{cases}$$

with terminal state constraint

$$x(T; t_0, x_0, u(\cdot)) \in M, M \subseteq \mathbb{R}^n.$$

Problem (OC)

$(t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^n$, find a control policy
 $\bar{u}(\cdot) \in \tilde{\mathcal{U}}_{x_0}[t_0, T]$ s.t.

$$J(t_0, x_0; \bar{u}(\cdot)) = \inf_{u(\cdot) \in \tilde{\mathcal{U}}_{x_0}[t_0, T]} J(t_0, x_0; u(\cdot)).$$

Cost functional

$$\tilde{\mathcal{U}}_{x_0}[t_0, T] := \{u : [t_0, T] \rightarrow \mathbb{R}^n \mid \text{measurable}\}$$

$$\begin{aligned} J(t_0, x_0; u(\cdot)) = & \int_{t_0}^T g(s, u(s), x(s)) ds \\ & + h(T, x(T)) \end{aligned}$$

Hypothesis:

- (C-1) $f : \mathbb{R}_+ \times U \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is measurable, satisfies the lipchitz condition in x ,
 $|f(t, u, 0)| \leq L, \forall (t, u) \in \mathbb{R}_+ \times U$.
- (C-2) $g : \mathbb{R}_+ \times U \times \mathbb{R}^n \rightarrow \mathbb{R}$,
 $h : \mathbb{R}^n \rightarrow \mathbb{R}$ are measurable, and
- $$\begin{aligned} & |g(s, u, x_1) - g(s, u, x_2)| + \\ & |h(x_1) - h(x_2)| \\ & \leq \omega(|x_1| \vee |x_2|, |x_1 - x_2|) \end{aligned}$$
- $$\forall (s, u) \in \mathbb{R}_+ \times U, x_1, x_2 \in \mathbb{R}^n.$$
- (C-3) For a.a. $t \in [0, T]$, Cesari property holds $\forall x \in \mathbb{R}^n$.

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Modulus of continuity

$\omega : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$, increasing,
 $\omega(r, 0) = 0 \forall r \geq 0.$

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Cesari property

$$\begin{aligned} \mathbf{E}(t, x) = & \{(z^0, z) \in \mathbb{R} \times \mathbb{R}^n | \\ & z^0 \geq g(t, u, x), \\ & z = f(t, u, x), u \in U\}. \end{aligned}$$

$$\bigcap_{\delta} \bar{co} \mathbf{E}(t, B_\delta(x)) = \mathbf{E}(t, x)$$

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- $$\begin{aligned} & |g(s, u, x_1) - g(s, u, x_2)| + \\ & |h(x_1) - h(x_2)| \\ & \leq \omega(|x_1| \vee |x_2|, |x_1 - x_2|) \end{aligned}$$
- $$\forall (s, u) \in \mathbb{R}_+ \times U, x_1, x_2 \in \mathbb{R}^n.$$
- (C-3) For a.a. $t \in [0, T]$, Cesari property holds $\forall x \in \mathbb{R}^n$.

Existence Theorem

Let (C-1)-(C-3) hold. Then problem (OC) admits at least one optimal pair.

$(OC)^T$

$$J(t_0, x_0; u(\cdot)) = \int_{t_0}^T g(s, u(s), x(s)) ds + h(x(T))$$

$$\begin{cases} \dot{x}(s) = f(s, u(s), x(s)) & s \in [t_0, T] \\ x(t_0) = x_0 \end{cases}$$

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Hamiltonian:

$$H = g(t, x(t), u(t)) + \langle \lambda(t), f(t, x(t), u(t)) \rangle,$$

$$\frac{\partial H}{\partial u_i}(t, \bar{x}(\cdot), \bar{u}(\cdot)) = 0.$$

Additional hypothesis:

(C-4)

$x \mapsto (f(t, u, x), g(t, u, x), h(x))$
is differentiable,

$(u, x) \mapsto (f(t, u, x), f_x(t, u, x),$
 $g(t, u, x), g_x(t, u, x),$
 $h_x(x))$

is continuous.

Pontryagin's Maximum Principle

If $\bar{u}(t)$ and $\bar{x}(t)$ are optimal for the problem (OC), then there exists a piecewise differentiable adjoint variable $\lambda(t)$ s.t.

$$H(t, \bar{x}(t), \bar{u}(t), \lambda(t)) = \max_{u \in U} H(t, x(t), u(t), \lambda(t)), \quad \forall t \in [0, T]$$

$$\lambda'(t) = -\frac{\partial H(t, \bar{x}(t), \bar{u}(t), \lambda(t))}{\partial x},$$

$$\lambda(T) = 0.$$

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Hamiltonian

$$H = g(t, x(t), u(t)) + \langle \lambda(t), f(t, x(t), u(t)) \rangle,$$

Example

$$\frac{dS_p}{dt} = -\beta_p S_p I_v + (r + u_1) L_p + (r + u_2) I_p,$$

$$\frac{dL_p}{dt} = \beta_p S_p I_v - b L_p - (r + u_1) L_p,$$

$$\frac{dI_p}{dt} = b L_p - (r + u_2) I_p,$$

$$\frac{dS_v}{dt} = -\beta_v S_v I_p - (\gamma + u_3) S_v + (1 - \theta) \mu,$$

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$$\begin{aligned} H = & A_1 I_v + A_2 L_p + A_3 I_p \\ & + \sum_{i=1}^3 c_i u_i^2 \\ & + \lambda_1 (-\beta_p S_p I_v + (r + u_1) L_p \\ & + (r + u_2) I_p) \\ & + \lambda_2 (\beta_p S_p I_v - b L_p \\ & - (r + u_1) L_p) \\ & + \lambda_3 (b L_p - (r + u_2) I_p) \\ & + \lambda_4 (-\beta_v S_v I_p - (\gamma + u_3) S_v \\ & + (1 - \theta) \mu) \\ & + \lambda_5 (\beta_v S_v I_p - (\gamma + u_3) I_v \\ & + \theta \mu). \end{aligned}$$

$$\frac{d\lambda_1}{dt} = \beta_p(\lambda_1 - \lambda_2),$$

$$\frac{d\lambda_2}{dt} = -A_2 + (r + u_1)(\lambda_2 - \lambda_1) + b(\lambda_2 - \lambda_3),$$

$$\frac{d\lambda_3}{dt} = -A_1 + (r + u_2)(\lambda_3 - \lambda_1) + \beta_v S_v(\lambda_4 - \lambda_5),$$

$$\frac{d\lambda_4}{dt} = \beta_v I_p(\lambda_4 - \lambda_5) + (\gamma + u_3)\lambda_4,$$

$$\frac{d\lambda_5}{dt} = -A_3 + \beta_p S_p(\lambda_1 - \lambda_2) + (\gamma + u_3)\lambda_5,$$

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$$\frac{\partial H}{\partial u}(\bar{u}) = 0 \Rightarrow$$

$$u_i \in [0, u_i^{\max}]$$

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optimal control policies

$$\bar{u}_1 = \min \left(\max \left(0, \frac{L_p(\lambda_2 - \lambda_1)}{2c_1} \right), u_1^{\max} \right)$$

$$\bar{u}_2 = \min \left(\max \left(0, \frac{I_p(\lambda_3 - \lambda_1)}{2c_2} \right), u_2^{\max} \right)$$

$$\bar{u}_3 = \min \left(\max \left(0, \frac{S_v \lambda_4 + I_v \lambda_5}{2c_3} \right), u_3^{\max} \right)$$

The most popular

Algorithm 2 Forward Backward Sweep

```

In mm,  $t_f, x_0, h, \text{tol}$  40 ----- 60 ----- 80 ----- 100 ----- 120
Output:  $x^*, u^*, \lambda$ 

procedure FORWARD_BACKWARD_SWEEP( $g, \lambda_{\text{function}}, u, x_0, \lambda_f, h, n_{\max}$ )
    while  $\epsilon > \text{tol}$  do
         $u_{\text{old}} \leftarrow u$ 
         $x_{\text{old}} \leftarrow x$ 
        40 - RUNGE_KUTTA_FORWARD( $g, u, x_0, h$ )
         $\lambda_{\text{old}} \leftarrow \lambda$ 
         $\lambda \leftarrow \text{RUNGE_KUTTA_BACKWARD}(\lambda_{\text{function}}, x, \lambda_f, h)$ 
         $u_1 \leftarrow \text{OPTIMALITY\_CONDITION}(u, x, \lambda)$ 
         $u \leftarrow \alpha u_1 + (1 - \alpha) u_{\text{old}}, \quad \alpha \in [0, 1]$  ▷ convex combination
         $\epsilon_u \leftarrow \frac{\|u - u_{\text{old}}\|}{\|u\|}$ 
        60 -  $\epsilon_x \leftarrow \frac{\|x - x_{\text{old}}\|}{\|x\|}$  ▷ relative error
         $\epsilon_\lambda \leftarrow \frac{\|\lambda - \lambda_{\text{old}}\|}{\|\lambda\|}$ 
         $\epsilon \leftarrow \max \{\epsilon_u, \epsilon_x, \epsilon_\lambda\}$ 
    end while
    re 80 -  $x^*, u^*, \lambda$  ▷ Optimal pair
end procedure

```

Set up

- 120 -

$$(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$$

$W(t)$: m -dimensional Brownian motion

$$dx(t) = f(t, u(t), x(t))dt + \sigma(t, u(t), x(t))dW(t)$$

$$x(0) = x_0,$$

$$f : [0, T] \times U \times \mathbb{R}^n \rightarrow \mathbb{R}^n,$$

$$\sigma : [0, T] \times U \times \mathbb{R}^n \rightarrow \mathbb{R}^{n+m},$$

U : separable metric space.

$$\mathcal{U}[0, T] := \{u : [0, T] \times \Omega \rightarrow U | u(\cdot) \text{ is } \{\mathcal{F}_t\}_{t \geq 0}\text{-adapted}\}$$

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Weak formulation of optimal control

A 6-tuple $\pi = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}, W(\cdot), u(\cdot))$, is a w-admissible control, if

- $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ is a filtered probability space satisfying the usual conditions,
- $W(t)$ is an m -dimensional standard Brownian motion defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$,
- $u(\cdot)$ is an $\{\mathcal{F}_t\}_{t \geq 0}$ -adapted process on $(\Omega, \mathcal{F}, \mathbb{P})$ taking values in U ,
- $x(\cdot)$ is unique solution,
- some prescribed state constraints are satisfied,
- $g(\cdot, u(\cdot), x(\cdot)) \in L^1_{\mathcal{F}}(0, T; \mathbb{R})$ and
 $h(x(T)) \in L^1_{\mathcal{F}_T}(\Omega; \mathbb{R})$

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Cost functional

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$$J(u(\cdot)) = \mathbb{E} \left\{ \int_0^T g(t, u(t), x(t)) dt + h(x(T)) \right\}$$

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(WS)

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$$J(\bar{\pi}) = \inf_{\pi \in \mathcal{U}_{ad}^w[0, T]} J(\pi) \quad (*)$$

s.t.

$$dx(t) = f(t, u(t), x(t)) dt + \sigma(t, u(t), x(t)) dW(t)$$

$$x(0) = x_0,$$

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problem (WS) is finite, if r.h.s. of (*) is finite.

Hypothesis:

(SE-1) (U, d) is a compact metric space and $T > 0$,

(SE-2) f, σ, g , and h are all continuous, and $\exists L > 0$ s.t.

$$\psi(t, u, x) = \{f(t, u, x), \sigma(t, u, x), g(t, u, x), h(x)\},$$

$$|\psi(t, u, x) - \psi(t, u, \hat{x})| \leq L|x - \hat{x}|,$$

$$\forall t \in [0, T], x, \hat{x} \in \mathbb{R}^n, u \in U,$$

$$|\psi(t, u, 0)| \leq L \forall (t, u) \in [0, T] \times U.$$

(SE-3) $\forall (t, x) \in [0, T] \times \mathbb{R}^n$, the set

$$(f, \sigma\sigma^T, g)(t, x, U) := \{(f_i(t, u, x), (\sigma\sigma^T)^{ij}(t, u, x), g(t, u, x)) | u \in U, i = 1, \dots, n, j = 1, \dots, m\}$$

is convex in \mathbb{R}^{m+nm+1} ,

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is convex in \mathbb{R}^{m+nm+1} ,

(SE-4) $S(t) \equiv \mathbb{R}^n$.

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Existence Theorem (weak formulation)

Under **(SE1)-(SE4)**, if (WS) is finite, then it admits an optimal control.

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