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New Second-Order Schemes for Forward Backward Stochastic Differential Equations

Yabing Sun[1] and Weidong Zhao[2,*]

[1]*School of Mathematics, Shandong University, Jinan, Shandong 250100, China.*
[2]*School of Mathematics & Finance Institute, Shandong University, Jinan, Shandong 250100, China.*

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Abstract. The Feynman-Kac formulas are used to develop new second-order numerical schemes for the forward-backward stochastic differential equations (FBSDEs) of the first and second order. The methods are simple and allow an easy implementation. Numerous numerical tests for FBSDEs, fully nonlinear second-order parabolic partial differential equations and the Hamilton-Jacobi-Bellman equations show the stability and a high accuracy of the methods.

AMS subject classifications: 65C20, 65C30, 60H30, 60H35

Key words: Forward backward stochastic differential equations, Feynman-Kac formula, difference approximation, second-order scheme.

1. Introduction

Let $T>0$ denote a deterministic terminal time and $(\Omega,\mathscr{F},\mathbb{F},P)$ be a filtered complete probability space with the natural filtration $\mathbb{F}=\{\mathscr{F}_t\}_{0\le t\le T}$ of an m-dimensional Brownian motion $W=(W_t)_{0\le t\le T}$. The decoupled forward-backward stochastic differential equation (FBSDE) on $(\Omega,\mathscr{F},\mathbb{F},P)$ has the form

$$
\begin{aligned}
X_t &= X_0+\int_0^t b(s,X_s)ds+\int_0^t \sigma(s,X_s)dW_s,\\
Y_t &= \varphi(X_T)+\int_t^T f(s,X_s,Y_s,Z_s)ds-\int_t^T Z_s dW_s,
\end{aligned}
\tag{1.1}
$$

where $t\in[0,T]$, $X_0\in\mathscr{F}_0$ is an initial condition, $b:[0,T]\times\mathbb{R}^d\to\mathbb{R}^d$ and $\sigma:[0,T]\times\mathbb{R}^d\to\mathbb{R}^{d\times m}$ are, respectively, drift and diffusion coefficients, $\varphi:\mathbb{R}^d\to\mathbb{R}^n$ and $f:[0,T]\times\mathbb{R}^d\times\mathbb{R}^n\times\mathbb{R}^{n\times m}\to\mathbb{R}^n$ is a driver function. A triple (X_t,Y_t,Z_t) is called the L^2-adapted solution of the FBSDE (1.1) if it is $\mathscr{F}_t$-adapted, square integrable and satisfies the Eq. (1.1).

*Corresponding author. *Email addresses:* `sunybly@163.com` (Y. Sun), `wdzhao@sdu.edu.cn` (W. Zhao).

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We note that Pardoux and Peng [30] established the existence of a unique adapted solution for nonlinear backward stochastic differential equations (BSDEs). Using the results of Ref. [30], Peng [27] proposed a probabilistic interpretation of quasilinear parabolic partial differential equations (PDEs). Cheridito *et al.* [6] extended this interpretation to fully nonlinear parabolic PDEs by introducing a second-order forward-backward SDEs (2FBSDEs) and observing that the solutions of fully nonlinear parabolic PDE allow to solve the corresponding 2FBSDE. Since then FBSDEs are vigorously studied and applied in various fields, including mathematical finance [5,8,21,23], stochastic control [28], risk measure [15,28], mean-field BSDEs [2,3,32,33], and stochastic differential games [18].

Nevertheless, explicit closed-form solutions of FBSDEs can be rarely found, so that various numerical approaches to BSDEs, FBSDEs and 2FBSDEs have been proposed recently. In particular, Refs. [5, 7, 20, 23, 25, 44] use relationships between the solutions of BSDEs, FBSDEs, 2FBSDEs and the related PDEs [6, 21, 23, 27]. On the other hand, there are many approximation methods applied directly to BSDEs [4,16,36,39–41], FBSDEs [11–14, 22, 34, 35, 37, 38, 42, 43, 45] and 2FBSDEs [20, 44]. For FBSDEs, most of the methods determine both Y_t and Z_t directly, which is a complicated and time consuming procedure, especially for Z_t in high dimensional situations.

In this paper, we propose new approximation methods for FBSDEs and 2FBSDEs. These methods are based, respectively, on the first-order [21,27] and the second-order Feynman-Kac formulas [20, 44] and on the difference approximations of derivatives. They can be applied to nonlinear parabolic PDEs and Hamilton-Jacobi-Bellman (HJB) equations, arising in stochastic optimal control problems. The main features of the methods are:

- As soon as an approximation of Y_t is found, it is used in the approximation of Z_t.
- If the parameters of the methods are properly chosen, the accuracy of approximation of Y_t and Z_t can be of order 2 in time.
- The methods are simple and can be easily coded.
- The methods are applicable to fully nonlinear second-order parabolic PDEs.

Moreover, numerical experiments show that the methods are stable, efficient, and provide very accurate solutions of FBSDEs, 2FBSDEs, second-order nonlinear parabolic PDEs, and stochastic optimal control problems.

The paper is organised as follows. In Section 2 we recall the Feynman-Kac formulas and finite difference approximations. Section 3 deals with the discretisation of FBSDEs and 2FBSDEs. Novel fully discrete numerical methods for FBSDEs, 2FBSDEs and fully nonlinear parabolic PDEs are considered in Section 4. Here we also mention new backward stochastic differential equations and backward stochastic difference differential equations (BSDDEs). Numerical experiments reported in Section 5, demonstrate the efficiency and the accuracy of the methods, and our conclusions are in Section 6.

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2. Preliminaries

2.1. Feynman-Kac formula

Assume that the decoupled FBSDE (1.1) has a unique solution (X_s, Y_s, Z_s). We denote by $(X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x})$ the solution starting at the time-space point (t,x) — i.e. the triple $(X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x})$, $s \in [t,T]$ satisfies the following FBSDEs:

$$\begin{aligned} X_s^{t,x} &= x + \int_t^s b(r, X_r^{t,x})dr + \int_t^s \sigma(r, X_r^{t,x})dW_r, \\ Y_s^{t,x} &= \varphi(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})dr - \int_s^T Z_r^{t,x} dW_r, \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} Y_t^{t,x} &= \mathbb{E}[\varphi(X_T^{t,x}) + \int_t^T f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})dr | \mathscr{F}_t] \\ &= \mathbb{E}_t^x[\varphi(X_T) + \int_t^T f(r, X_r, Y_r, Z_r)dr], \end{aligned}$$

where $\mathbb{E}_t^x[\cdot] = \mathbb{E}[\cdot | X_t = x, \mathscr{F}_t]$ and the superscript t,x shows that forward stochastic differential equation starts at (t,x). If no confusion is possible, this superscript is omitted.

Let us denote by $u = u(t,x)$ the function $Y_t^{t,x}$.

Lemma 2.1 (cf. Refs. [8,16,21,27]). *If $u \in C_b^{1,2}([0,T] \times \mathbb{R}^d)$, then u is the classical solution of the second-order semilinear parabolic PDE problem*

$$\begin{aligned} & u_t + Du \cdot b + \frac{1}{2}\sigma\sigma^\top : D^2 u + f\big(t, x, u, Du\sigma\big) = 0, \quad (t,x) \in [0,T) \times \mathbb{R}^d, \\ & u(T,x) = \varphi(x), \quad x \in \mathbb{R}^d, \end{aligned} \tag{2.2}$$

where $Du(x)$ and $D^2u(x)$ are, respectively, the gradient and the Hessian matrix of u with respect to x, and if B and C are $d \times d$ matrices, then $B : C$ is defined by $B : C := \sum_{i,j=1}^d B_{ij} C_{ij}$. Conversely, if $u \in C_b^{1,2}([0,T] \times \mathbb{R}^d)$ is a classical solution of the problem (2.2) *and (X_s, Y_s, Z_s) is the solution of the FBSDEs* (1.1)*, then*

$$Y_s = u(s, X_s), \quad Z_s = Du(s, X_s)\sigma(s, X_s). \tag{2.3}$$

Representations (2.3) are called the Feynman-Kac formulas.

Let $C_b^{l,k,k}$, $k \in \{0\} \cup \mathbb{N}$ denote the set of functions $\phi : [0,T] \times \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}$, which have uniformly bounded partial derivatives $\partial_t^{l_1}\phi$, $\partial_y^{k_1}\partial_z^{k_2}\phi$ for all $l_1 \le l$ and $k_1 + k_2 \le k$. It is known that if $b, \sigma \in C_b^{1+k,2+2k}$, $f \in C_b^{1+k,2+2k,2+2k}$ and $\varphi \in C_b^{2+2k+\alpha}$ for an $\alpha \in (0,1)$, then $u \in C_b^{1+k,2+2k}$ — cf. Ref. [16].

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2.2. Second-order Feynman-Kac formula

Consider PDE

$$\begin{aligned} &u_t + F(t,x,u,Du,D^2u) = 0, \quad (t,x) \in [0,T) \times \mathbb{R}^d, \\ &u(T,x) = g(x), \end{aligned} \tag{2.4}$$

where $F: [0,T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times S^d \to \mathbb{R}$ and $g: \mathbb{R}^d \to \mathbb{R}$ are given functions, S^d is the set of $d \times d$ symmetric matrices, and $u: [0,T) \times \mathbb{R}^d \to \mathbb{R}$ is unknown. The Eq. (2.4) is called fully nonlinear if the function F is nonlinear with respect to the second-order derivatives D^2u.

Let γ_1, γ_2 be $d \times d$ matrices. We write $\gamma_1 \geq \gamma_2$ if $\gamma_1 - \gamma_2$ is a positive semidefinite matrix.

Definition 2.1 (cf. Refs. [20, 44]). An operator F is called elliptic, if for all $(t,x,\lambda,p) \in [0,T] \times \mathbb{R}^k \times \mathbb{R} \times \mathbb{R}^k$ and for all $\gamma_1, \gamma_2 \in S^d$ such that $\gamma_1 \geq \gamma_2$ the inequality

$$F(t,x,\lambda,p,\gamma_1) \geq F(t,x,\lambda,p,\gamma_2),$$

holds.

A fully non-linear PDE (2.4) is called fully nonlinear parabolic if F is an elliptic operator. The probabilistic representation for the solution u of the problem (2.4) has been studied in [6] by an FBSDEs, called the second-order FBSDEs. It has the form

$$\begin{aligned} X_s &= x + \int_0^s b(r,X_r)ds + \int_0^s \sigma(r,X_r)dW_r, \\ Y_s &= g(X_T) + \int_s^T \hat{f}(r,X_r,Y_r,Z_r,\Gamma_r)dr - \int_s^T Z_r dW_r, \qquad s \in [0,T], \\ Z_s &= Z_0 + \int_0^s A_r dr + \int_0^s \Gamma_r dW_r, \end{aligned} \tag{2.5}$$

where X_s, Y_s, Z_s, Γ_s and A_s are unknowns. Note that highly accurate approximation methods for such equations and for fully nonlinear parabolic PDEs have been studied in Refs. [20, 44].

Let us consider the operator

$$\mathscr{L}\varphi := \varphi_t + D\varphi\, b + \frac{1}{2}\sigma\sigma^\top : D^2\varphi, \quad \varphi \in C_b^{1,2}, \tag{2.6}$$

and the function

$$\hat{f}_t := \hat{f}(t,X_t,Y_t,Z_t,\Gamma_t) = f(t,X_t,u(t,X_t),Du(t,X_t),D^2u(t,X_t)), \tag{2.7}$$

where

$$f(t,x,u,Du,D^2u) = F(t,x,u,Du,D^2u) - \left(Du \cdot b + \frac{1}{2}\sigma\sigma^\top : D^2u\right). \tag{2.8}$$

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Lemma 2.2 (cf. Refs. [6,20,44]). *Assume that $(X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x}, \Gamma_s^{t,x}, A_s^{t,x})$ is the unique solution of 2FBSDE* (2.5) *and let $u(t,x) := Y_t^{t,x}$. If $u \in C_b^{1,2}([0,T]\times\mathbb{R}^d)$, then u is the viscosity solution of the problem* (2.4)*, and if $u \in C_b^{1,3}([0,T]\times\mathbb{R}^d)$ is the classical solution of* (2.4)*, then*

$$\begin{aligned} Y_s^{t,x} &= u(s, X_s^{t,x}), & Z_s^{t,x} &= (Du\sigma)(s, X_s^{t,x}), \\ \Gamma_s^{t,x} &= (D(Du\sigma)\sigma)(s, X_s^{t,x}), & A_s^{t,x} &= (\mathscr{L}(Du\sigma))(s, X_s^{t,x}). \end{aligned} \tag{2.9}$$

Representations (2.9) are called the second-order Feynman-Kac formulas.

2.3. Finite difference approximations

For a given function $g = g(x)$, the difference quotient operators are defined by

$$\begin{aligned} D_h^{+1} g(x) &= \frac{g(x+h) - g(x)}{h}, \\ D_h^{-1} g(x) &= \frac{g(x) - g(x-h)}{h}, \\ D_h^{0} g(x) &= \frac{g(x+h) - g(x-h)}{2h}, \\ D_h^{2} g(x) &= \frac{g(x+h) - 2g(x) + g(x-h)}{h^2}, \end{aligned} \tag{2.10}$$

with a positive real number h. Using Taylor expansion, we can write

$$\begin{aligned} D_h^{i} g(x) - g'(x) &= \mathscr{O}(h), & g &\in C_b^2, \\ D_h^{0} g(x) - g'(x) &= \mathscr{O}(h^2), & g &\in C_b^3, \\ D_h^{2} g(x) - g''(x) &= \mathscr{O}(h^2), & g &\in C_b^4, \end{aligned} \tag{2.11}$$

where $i = \pm 1$ and $\mathscr{O}(h)$ means that there is a constant $M > 0$ such that

$$|\mathscr{O}(h)| \le Mh.$$

For example, if $g \in C_b^4$, then

$$\begin{aligned} g(x+h) &= g(x) + g'(x)h + \frac{h^2}{2} g''(x) + \frac{h^3}{3!} g'''(x) + \frac{h^4}{4!} f^{(4)}(\xi_1), \\ g(x-h) &= g(x) - g'(x)h + \frac{h^2}{2} g''(x) - \frac{h^3}{3!} g'''(x) + \frac{h^4}{4!} g^{(4)}(\xi_2), \end{aligned}$$

where $\xi_1 \in [x, x+h]$, $\xi_2 \in [x-h, x]$, and it follows that

$$D_h^2 g(x) - g''(x) = \frac{h^2}{4!}\big(g^{(4)}(\xi_1) + g^{(4)}(\xi_2)\big) = \mathscr{O}(h^2).$$

The Eqs. (2.11) imply that $D_h^i g(x)$, $i = -1, +1$ and $D_h^0 g(x)$, respectively, approximate the first-order derivative $g'(x)$ with errors $\mathscr{O}(h)$ and $\mathscr{O}(h^2)$, and $D_h^2 g(x)$ approximates the second-order derivative $g''(x)$ with an error $\mathscr{O}(h^2)$. Note that in numerical tests below we use the approximation $D_h^0 g(x)$.

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3. Discretisation

Let $0 = t_0 < t_1 < \cdots < t_N = T$ be a regular partition of the time interval $[0,T]$ and $\Delta t_n := t_{n+1} - t_n$, $\Delta t := \max\limits_{0\le n\le N-1} \Delta t_n$. Moreover, let $\Delta W_s := W_s - W_{t_n}$, $t_n \le s \le t_{n+1}$. Note that ΔW_s is a standard Brownian motion, and $\Delta W_s \backsim N(0, s-t_n)$ in distributions. Without loss of generality, we can consider a one-dimensional process $X_t = X_0 + W_t \in \mathbb{R}$, since the approach below is applicable to general diffusion processes $X_t \in \mathbb{R}^d$.

For given t and x, let $\mathbb{E}_t^x[\cdot]$ denote the conditional expectation operator on random variables under the condition $X_t = x$ — i.e. $\mathbb{E}_t^x[\cdot] = \mathbb{E}[\cdot|X_t = x] = \mathbb{E}[\cdot|\mathscr{F}_t, X_t = x]$.

3.1. Discretisation of FBSDEs

If (X_t, Y_t, Z_t) is a solution of the FBSDE (1.1), then for $n = 0, 1, \cdots, N-1$ we have

$$Y_{t_n} = Y_{t_{n+1}} + \int_{t_n}^{t_{n+1}} f_s ds - \int_{t_n}^{t_{n+1}} Z_s dW_s, \tag{3.1}$$

where $f_s = f\big(s, X_s, Y_s(X_s), Z_s(X_s)\big)$ and we write $Y_t(X_t)$ and $Z_t(X_t)$ for Y_t and Z_t, respectively. Taking the conditional mathematical expectation $\mathbb{E}_{t_n}^x[\cdot]$ of both sides of (3.1) leads to the relation

$$\begin{aligned} Y_{t_n} &= \mathbb{E}_{t_n}^x[Y_{t_{n+1}}] + \int_{t_n}^{t_{n+1}} \mathbb{E}_{t_n}^x[f_s] ds \\ &= \mathbb{E}_{t_n}^x[Y_{t_{n+1}}] + \theta \Delta t_n f_{t_n} + (1-\theta)\Delta t_n \mathbb{E}_{t_n}^x[f_{t_{n+1}}] + R_{y1}^n, \end{aligned} \tag{3.2}$$

where $\theta \in [0,1]$ and

$$R_{y1}^n = \int_{t_n}^{t_{n+1}} \Big(\mathbb{E}_{t_n}^x[f_s] - \theta f_{t_n} - (1-\theta)\mathbb{E}_{t_n}^x[f_{t_{n+1}}]\Big) ds.$$

According to Lemma 2.1, any sufficiently smooth solution $u(t,x)$ of the PDE (2.2) yields

$$Y_t = u(t, X_t), \quad Z_t = Du(t, X_t).$$

Moreover, the solution Z_t can be represented as

$$Z_t = u_x'(t, X_t) = D_h^i Y_t(X_t) + \mathscr{O}(h^{2-|i|}), \tag{3.3}$$

where D_h^i, $i = -1, 0, +1$, are difference operators defined by (2.10). It follows from (3.2) and (3.3) that

$$Y_{t_n} = \mathbb{E}_{t_n}^x[Y_{t_{n+1}}] + \Delta t_n \mathbb{E}_{t_n}^x[f_{t_{n+1}}^{h,i}] + R_{yr}^n = \bar{Y}_{t_n}(x) + R_{yr}^n, \tag{3.4}$$

$$Y_{t_n} = \mathbb{E}_{t_n}^x[Y_{t_{n+1}}] + \theta \Delta t_n f_{t_n}^{h,i} + (1-\theta)\Delta t_n \mathbb{E}_{t_n}^x[f_{t_{n+1}}^{h,i}] + R_{y1}^n + R_{y2}^n, \tag{3.5}$$

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where $f_{t_n}^{h,i}=f\big(t_n,X_{t_n},Y_{t_n}(X_{t_n}),D_h^iY_{t_n}(X_{t_n})\big)$ and

$$\begin{aligned}R_{yr}^n&=\int_{t_n}^{t_{n+1}}\big(\mathbb{E}_{t_n}^x[f_s]-\mathbb{E}_{t_n}^x[f_{t_{n+1}}]\big)ds+\Delta t_n\big(\mathbb{E}_{t_n}^x[f_{t_{n+1}}-f_{t_{n+1}}^{h,i}]\big),\\ R_{y2}^n&=\theta\Delta t_n\big(f_{t_n}-f_{t_n}^{h,i}\big)+(1-\theta)\Delta t_n\big(\mathbb{E}_{t_n}^x[f_{t_{n+1}}-f_{t_{n+1}}^{h,i}]\big).\end{aligned}$$

Thus Eqs. (3.4) and (3.5) show that $Y_{t_n}=Y_{t_n}(x)$ satisfies the following equation

$$Y_{t_n}=\mathbb{E}_{t_n}^x[Y_{t_{n+1}}]+\theta\Delta t_n\bar{f}_{t_n}^{h,i}+(1-\theta)\Delta t_n\mathbb{E}_{t_n}^x[f_{t_{n+1}}^{h,i}]+R_y^n, \tag{3.6}$$

where

$$\bar{f}_{t_n}^{h,i}=f(t_n,x,\bar{Y}_{t_n}(x),D_h^i\bar{Y}_{t_n}(x)),\quad R_y^n=R_{y1}^n+R_{y2}^n+R_{y3}^n,\quad R_{y3}^n=\theta\Delta t_n\big(f_{t_n}^{h,i}-\bar{f}_{t_n}^{h,i}\big).$$

Remark 3.1. If generator f and terminal condition φ are sufficiently smooth, then for $\theta=1/2$ and $\theta\neq 1/2$ the truncation errors R_{y1}^n are, respectively, $\mathscr{O}\big((\Delta t)^3\big)$ and $\mathscr{O}\big((\Delta t)^2\big)$ — cf. Ref. [40]. It follows from (3.3) that

$$R_{y2}^n=\mathscr{O}\big(h^{2-|i|}\Delta t\big),\quad R_{yr}^n=\mathscr{O}\big((\Delta t)^2+h^{2-|i|}\Delta t\big).$$

Since $R_{yr}^n(x)=Y_{t_n}(x)-\bar{Y}_{t_n}(x)$ and Y_t is sufficiently smooth, we obtain

$$|f_{t_n}^{h,i}-\bar{f}_{t_n}^{h,i}|\leq C\big(|Y_{t_n}(x)-\bar{Y}_{t_n}(x)|+\big|D_h^iY_{t_n}(x)-D_h^i\bar{Y}_{t_n}(x)\big|\big)=\mathscr{O}\big((\Delta t)^2+h^{2-|i|}\big), \tag{3.7}$$

that leads to the relation

$$R_{y3}^n=\theta\Delta t_n\big(f_{t_n}^{h,i}-\bar{f}_{t_n}^{h,i}\big)=\mathscr{O}\big((\Delta t)^3+h^{2-|i|}\Delta t\big).$$

Note that constant C in (3.7) depends on the derivative f'.

Thus we have

$$R_y^n=\begin{cases}\mathscr{O}\Big((\Delta t)^3+h^{2-|i|}\Delta t\Big), & \text{if}\quad \theta=\dfrac{1}{2},\\ \mathscr{O}\Big((\Delta t)^2+h^{2-|i|}\Delta t\Big), & \text{if}\quad \theta\neq\dfrac{1}{2},\end{cases} \tag{3.8}$$

or, more precisely,

1. If $i=0$, then

$$R_y^n=\begin{cases}\mathscr{O}\big((\Delta t)^3\big), & \text{if}\quad h=\mathscr{O}(\Delta t),\quad \theta=\dfrac{1}{2},\\ \mathscr{O}\big((\Delta t)^2\big), & \text{if}\quad h=\mathscr{O}(\Delta t)^{\frac{1}{2}},\quad \theta\neq\dfrac{1}{2}.\end{cases}$$

2. If $i=\pm1$, then

$$R_y^n=\begin{cases}\mathscr{O}\big((\Delta t)^3\big), & \text{if}\quad h=\mathscr{O}((\Delta t)^2),\quad \theta=\dfrac{1}{2},\\ \mathscr{O}\big((\Delta t)^2\big), & \text{if}\quad h=\mathscr{O}(\Delta t),\quad \theta\neq\dfrac{1}{2}.\end{cases}$$

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3.2. Discretisation of 2FBSDEs

Let $(X_t, Y_t, Z_t, \Gamma_t, A_t)$ be the solution of the 2FBSDE (2.5) and $\hat{f}_t := \hat{f}(t, X_t, Y_t, Z_t, \Gamma_t)$. Analogously to (3.1), we have

$$Y_{t_n} = Y_{t_{n+1}} + \int_{t_n}^{t_{n+1}} \hat{f}_s ds - \int_{t_n}^{t_{n+1}} Z_s dW_s \tag{3.9}$$

for any $n = 0, 1, \cdots, N-1$. Taking the conditional mathematical expectation $\mathbb{E}_{t_n}^x[\cdot]$ of (3.9), we deduce that

$$\begin{aligned} Y_{t_n} &= \mathbb{E}_{t_n}^x[Y_{t_{n+1}}] + \int_{t_n}^{t_{n+1}} \mathbb{E}_{t_n}^x[\hat{f}_s] ds \\ &= \mathbb{E}_{t_n}^x[Y_{t_{n+1}}] + \theta \Delta t_n \hat{f}_{t_n} + (1-\theta)\Delta t_n \mathbb{E}_{t_n}^x[\hat{f}_{t_{n+1}}] + R_{Y1}^n, \end{aligned} \tag{3.10}$$

where

$$R_{Y1}^n = \int_{t_n}^{t_{n+1}} \Big(\mathbb{E}_{t_n}^x[\hat{f}_s] - \theta \hat{f}_{t_n} - (1-\theta)\mathbb{E}_{t_n}^x[\hat{f}_{t_{n+1}}] \Big) ds.$$

Taking into account the Eqs. (2.9), we can approximate $Z_t = Z_t(X_t)$ and $\Gamma_t = \Gamma_t(X_t)$ as

$$Z_t \approx D_h^i Y_t(X_t), \quad i = -1, 0, +1, \quad \Gamma_t \approx D_h^2 Y_t(X_t),$$

where the operators D_h^i and D_h^2 are defined by (2.10). It follows that

$$Y_{t_n} = \mathbb{E}_{t_n}^x[Y_{t_{n+1}}] + \theta \Delta t_n \hat{f}_{t_n}^{h,i} + (1-\theta)\Delta t_n \mathbb{E}_{t_n}^x[\hat{f}_{t_{n+1}}^{h,i}] + R_{Y1}^n + R_{Y2}^n,$$

with

$$\begin{aligned} &\hat{f}_s^{h,i} = \hat{f}\big(s, X_s, Y_s(X_s), D_h^i Y_s(X_s), D_h^2 Y_s(X_s)\big), \\ &R_{Y2}^n = \theta \Delta t_n \big(\hat{f}_{t_n} - \hat{f}_{t_n}^{h,i}\big) + (1-\theta)\Delta t_n \big(\mathbb{E}_{t_n}^x[\hat{f}_{t_{n+1}} - \hat{f}_{t_{n+1}}^{h,i}]\big), \end{aligned}$$

and, consequently,

$$Y_{t_n} = \mathbb{E}_{t_n}^x[Y_{t_{n+1}}] + \theta \Delta t_n \tilde{f}_{t_n}^{h,i} + (1-\theta)\Delta t_n \mathbb{E}_{t_n}^x[\hat{f}_{t_{n+1}}^{h,i}] + R_Y^n, \tag{3.11}$$

with

$$\begin{aligned} &\tilde{f}_{t_n}^{h,i} = f\big(t_n, x, \bar{Y}_{t_n}(x), D_h^i \bar{Y}_{t_n}(x), D_h^2 \bar{Y}_{t_n}(x)\big), \\ &\bar{Y}_{t_n}(x) = \mathbb{E}_{t_n}^x[Y_{t_{n+1}}] + \Delta t_n \mathbb{E}_{t_n}^x[\hat{f}_{t_{n+1}}^{h,i}], \\ &R_Y^n = R_{Y1}^n + R_{Y2}^n + R_{Y3}^n, \\ &R_{Y3}^n = \theta \Delta t_n (\hat{f}_{t_n}^{h,i} - \tilde{f}_{t_n}^{h,i}). \end{aligned}$$

Remark 3.2. The truncation error term R_Y^n in (3.11) can be evaluated analogously to Remark 3.1 and the corresponding estimate is similar to (3.8).

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4. Numerical Schemes

Let us approximate the first-order derivatives by the central differences — i.e. we set $i=0$ in $f_s^{h,i}$, $\bar{f}_s^{h,i}$, $\hat{f}_s^{h,i}$ and $\tilde{f}_s^{h,i}$.

4.1. Numerical scheme for FBSDEs

By (Y^n, Z^n) we denote the approximation of the analytic solution (Y_t, Z_t) of the BSDE (1.1) at the time-space point (t_n, X_{t_n}), $n=N, N-1, \cdots, 0$ and let

$$f^{n,h} = f\big(t_n, X_{t_n}, Y^n(X_{t_n}), D_h^0 Y^n(X_{t_n})\big), \quad n = N, \cdots, 1.$$

Removing the truncation error term R_y^n in (3.6), we arrive at the following explicit scheme for solving the BSDE in (1.1).

Scheme 4.1. Given random variable Y^N we determine the random variables $Y^n = Y^n(X_{t_n})$ and $Z^n = Z^n(X_{t_n})$ from the equations

$$\begin{aligned}
&\bar{Y}^n = \mathbb{E}_{t_n}^{X_{t_n}}[Y^{n+1}] + \Delta t_n \mathbb{E}_{t_n}^{X_{t_n}}[f^{n+1,h}],\\
&Y^n = \mathbb{E}_{t_n}^{X_{t_n}}[Y^{n+1}] + \theta \Delta t_n \bar{f}^{n,h} + (1-\theta)\Delta t_n \mathbb{E}_{t_n}^{X_{t_n}}[f^{n+1,h}],\\
&Z^n = D_h^0 Y^n,\\
&n = N-1, \cdots, 0,
\end{aligned}$$

where $\bar{f}^{n,h} = f\big(t_n, X_{t_n}, \bar{Y}^n(X_{t_n}), D_h^0 \bar{Y}^n(X_{t_n})\big)$.

Remark 4.1. The following θ-scheme [36]:

$$\begin{aligned}
&(1-\theta_1)\Delta t_n Z^n = \mathbb{E}_{t_n}^{X_{t_n}}[Y^{n+1}\Delta W_{t_{n+1}}]\\
&\qquad + \theta_1 \Delta t_n \mathbb{E}_{t_n}^{X_{t_n}}[Z^{n+1}] + \theta_1 \Delta t_n \mathbb{E}_{t_n}^{X_{t_n}}[f^{n+1}\Delta W_{t_{n+1}}],\\
&Y^n = \mathbb{E}_{t_n}^{X_{t_n}}[Y^{n+1}] + \theta_1 \Delta t_n f^n + (1-\theta_1)\Delta t_n \mathbb{E}_{t_n}^{X_{t_n}}[f^{n+1}],\\
&f^n = f\big(t_n, X_{t_n}, Y^n(X_{t_n}), Z^n(X_{t_n})\big),\\
&n = N-1, \cdots, 0,
\end{aligned}$$

which is based on random variables Y^N and Z^N, is more complicated and more difficult to implement, especially in high-dimensional cases.

4.2. Numerical scheme for 2FBSDEs

Let (Y^n, Z^n, Γ^n) denote the approximation of the solution (Y_t, Z_t, Γ_t) of (2.5) at the time level $t = t_n$, $n = N, N-1, \cdots, 0$, and let

$$\hat{f}^{n,h} := \hat{f}\big(t_n, X_{t_n}, Y^n(X_{t_n}), D_h^0 Y^n(X_{t_n}), D_h^2 Y^n(X_{t_n})\big), \quad 0 \le n \le N.$$

The next explicit scheme for solving (2.5) is obtained from (3.11) by removing the truncation error term R_Y^n.

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Scheme 4.2. Given random variable Y^N we determine $Y^n := Y^n(X_{t_n})$, $Z^n := Z^n(X_{t_n})$ and $\Gamma^n := \Gamma^n(X_{t_n})$ from the equations

$$\begin{aligned}
&\bar{Y}^n = \mathbb{E}_{t_n}^{X_{t_n}}[Y^{n+1}] + \Delta t_n \mathbb{E}_{t_n}^{X_{t_n}}[\hat{f}^{n+1,h}],\\
&Y^n = \mathbb{E}_{t_n}^{X_{t_n}}[Y^{n+1}] + \theta\Delta t_n \tilde{f}^{n,h} + (1-\theta)\Delta t_n \mathbb{E}_{t_n}^{X_{t_n}}[\hat{f}^{n+1,h}],\\
&Z^n = D_h^0 Y^n,\\
&\Gamma^n = D_h^2 Y^n,\\
&n = N-1, \cdots, 0,
\end{aligned}$$

where $\tilde{f}^{n,h} := \hat{f}\big(t_n, X_{t_n}, \bar{Y}^n(X_{t_n}), D_h^0\bar{Y}^n(X_{t_n}), D_h^2\bar{Y}^n(X_{t_n})\big)$.

Remark 4.2. In order to extend the Schemes 4.1 and 4.2 to general diffusion processes X_t, we have to approximate them. In numerical tests below, the second order weak Taylor scheme

$$\begin{aligned}
X^{n+1} = X^n &+ b\Delta t_n + \sigma\Delta W_{t_{n+1}} + \frac{1}{2}\sigma\sigma'_x((\Delta W_{t_{n+1}})^2 - \Delta t_n) + b'_x\sigma\Delta Z_n\\
&+ \frac{1}{2}(b'_t + bb'_x + \frac{1}{2}b''_{xx}\sigma^2)\Delta t_n^2 + (\sigma'_t + b\sigma'_x + \frac{1}{2}\sigma''_{xx}\sigma^2)(\Delta W_{t_{n+1}}\Delta t_n - \Delta Z_n),
\end{aligned} \tag{4.1}$$

where

$$b = b(t_n, X^n), \quad \sigma = \sigma(t_n, X^n), \quad \Delta Z_n = \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} dW_r ds,$$

is used to solve forward stochastic differential equation and to respectively replace $X_{t_{n+1}}$ and the conditional expectation $\mathbb{E}_{t_n}^{X_{t_n}}[\cdot]$ by X^{n+1} and $\mathbb{E}_{t_n}^{X^n}[\cdot]$ in Schemes 4.1 and 4.2. Correspondingly, the approximations Z^n and Γ^n in Schemes 4.1 and 4.2 are calculated by the formulas

$$Z^n = D_h^0 Y^n\sigma, \quad \Gamma^n = D_h^2 Y^n\sigma^2 + D_h^0 Y^n\sigma'_x\sigma.$$

Remark 4.3. Lemma 2.2 also shows that Scheme 4.2 can be used for solving fully nonlinear second-order parabolic PDE (2.4). For this, one can take $X_t := X_0 + bt + \sigma W_t$ in (2.5) and define $\hat{f}$ by (2.7)-(2.8) — i.e.

$$\begin{aligned}
\hat{f}^{n,h} &= \hat{f}\big(t_n, X_{t_n}, Y^n(X_{t_n}), D_h^0 Y^n(X_{t_n})\sigma, D_h^2 Y^n(X_{t_n})\sigma^2\big)\\
&= f\big(t_n, X_{t_n}, Y^n(X_{t_n}), D_h^0 Y^n(X_{t_n}), D_h^2 Y^n(X_{t_n})\big).
\end{aligned}$$

4.3. Fully discrete schemes for FBSDEs and 2FBSDEs

Schemes 4.1 and 4.2 are time semi-discrete, but in applications the space discretisation of the conditional expectation $\mathbb{E}_{t_n}^x[\cdot]$ is needed. Therefore, at each time level t_n we consider a partition S_h^n of $\mathbb{R}^d$ by a set of points depending on a parameter $h_n > 0$. Let $\operatorname{dist}(x, S_h^n)$

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denote the distance from x to S_h^n and $\mathcal{N}_e$ be a positive integer. For each $x \in \mathbb{R}^d$, we define a subset $S_{h,x}^n$ of S_h^n such that every set $S_{h,x}^n$ contains at most $\mathcal{N}_e$ elements and

$$\operatorname{dist}(x, S_h^n) < \operatorname{dist}(x, S_h^n \backslash S_{h,x}^n).$$

The sets $S_{h,x}^n$ are called the neighbor grid sets of x in S_h^n.

In order to determine Y^n, we have to employ the values of Y^{n+1} at the points $X_{t_{n+1}}$ and $X_{t_{n+1}} \pm h$, which can be outside of the set S_h^{n+1} since $X_{t_n} = x \in S_h^n$. This can be done by using an interpolation procedure. Let I_h denote the local interpolation operator such that $I_h g(x) = g(x)$ for $x \in S_{h,x}^n$ for a fixed n, and let $D_h^{0,I}$ and $D_h^{2,I}$ be the operators defined by

$$\begin{aligned}
D_h^{0,I} g(x) &= \frac{I_h g(x+h) - I_h g(x-h)}{2h}, \\
D_h^{2,I} g(x) &= \frac{I_h g(x+h) + I_h g(x-h) - 2 I_h g(x)}{h^2}.
\end{aligned}$$

We also consider an approximation $\hat{\mathbb{E}}_{t_n}^x[\cdot]$ of the conditional expectation $\mathbb{E}_{t_n}^x[\cdot]$, and if

$$\begin{aligned}
\bar{f}_s^{h,I} &= f\big(s, X_s, \bar{Y}_s(X_s), D_h^{0,I} \bar{Y}_s(X_s)\big), \\
f_s^{h,I} &= f\big(s, X_s, I_h Y_s(X_s), D_h^{0,I} Y_s(X_s)\big),
\end{aligned}$$

then the Eq. (3.6), which discretises FBSDEs (1.1) with $i = 0$ can be written as

$$Y_{t_n} = \hat{\mathbb{E}}_{t_n}^x [I_h Y_{t_{n+1}}] + \theta \Delta t_n \bar{f}_{t_n}^{h,I} + (1-\theta) \Delta t_n \hat{\mathbb{E}}_{t_n}^x [f_{t_{n+1}}^{h,I}] + \bar{R}_y^n, \tag{4.2}$$

where

$$\begin{aligned}
R_y^n &= R_y^n + R_y^{n,\mathbb{E}} + R_y^{n,I} + R_f^{n,D} + R_f^{n,\mathbb{E}} + R_f^{n,I}, \\
R_y^{n,\mathbb{E}} &= (\mathbb{E}_{t_n}^x - \hat{\mathbb{E}}_{t_n}^x)[Y_{t_{n+1}}], \quad R_y^{n,I} = \hat{\mathbb{E}}_{t_n}^x \big[Y_{t_{n+1}} - I_h Y_{t_{n+1}}\big], \\
R_f^{n,\mathbb{E}} &= (1-\theta)\Delta t_n (\mathbb{E}_{t_n}^x - \hat{\mathbb{E}}_{t_n}^x)\big[f_{t_{n+1}}^{h,0}\big], \quad R_f^{n,D} = \theta \Delta t_n \big(\bar{f}_{t_n}^{h,0} - \bar{f}_{t_n}^{h,I}\big), \\
R_f^{n,I} &= (1-\theta)\Delta t_n \hat{\mathbb{E}}_{t_n}^x \big[f_{t_{n+1}}^{h,0} - f_{t_{n+1}}^{h,I}\big].
\end{aligned}$$

Removing the truncation error term $\bar{R}_y^n$ in (4.2), we arrive to the following fully discrete scheme for the FBSDE (1.1).

Scheme 4.3 (Scheme for FBSDEs)**.** If Y^N is a function defined on the set S_h^N, then for any $x \in S_h^n$, $n = N-1, \cdots, 0$, the random variables $Y^n = Y^n(x)$ and $Z^n = Z^n(x)$ can be determined from the equations

$$\begin{aligned}
\bar{Y}^n &= \hat{\mathbb{E}}_{t_n}^x [I_h Y^{n+1}] + \Delta t_n \hat{\mathbb{E}}_{t_n}^x [f^{n+1,h,I}], \\
Y^n &= \hat{\mathbb{E}}_{t_n}^x [I_h Y^{n+1}] + \theta \Delta t_n \bar{f}^{n,h,I} + (1-\theta) \Delta t_n \hat{\mathbb{E}}_{t_n}^x [f^{n+1,h,I}], \\
Z^n &= D_h^{0,I} Y^n,
\end{aligned}$$

where

$$\begin{aligned}
\bar{f}^{n,h,I} &= f\big(t_n, x, \bar{Y}^n(x), D_h^{0,I} \bar{Y}^n(x)\big), \\
f^{n,h,I} &= f\big(t_n, X_{t_n}, I_h Y^n(X_{t_n}), D_h^{0,I} Y^n(X_{t_n})\big).
\end{aligned}$$

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A fully discrete scheme for the 2FBSDE (2.5) can be constructed analogously.

Scheme 4.4 (Scheme for 2FBSDEs)**.** If Y^N is a function defined on the set S_h^N, then for any $x \in S_h^n$, $n = N-1, \cdots, 0$, the random variables $Y^n = Y^n(x)$, $Z^n = Z^n(x)$ and $\Gamma^n = \Gamma^n(x)$ can be determined from the equations

$$\begin{aligned}
&\bar{Y}^n = \hat{\mathbb{E}}_{t_n}^x[I_h Y^{n+1}] + \Delta t_n \hat{\mathbb{E}}_{t_n}^x[\hat{f}^{n+1,h,I}],\\
&Y^n = \hat{\mathbb{E}}_{t_n}^x[I_h Y^{n+1}] + \theta \Delta t_n \tilde{f}^{n,h,I} + (1-\theta)\Delta t_n \hat{\mathbb{E}}_{t_n}^x[\hat{f}^{n+1,h,I}],\\
&Z^n = D_h^{0,I} Y^n, \quad \Gamma^n = D_h^{2,I} Y^n,
\end{aligned}$$

where

$$\begin{aligned}
\tilde{f}^{n,h,I} &= \hat{f}\big(t_n, x, \bar{Y}^n(x), D_h^{0,I}\bar{Y}^n(x), D_h^{2,I}\bar{Y}^n(x)\big),\\
\hat{f}^{n,h,I} &= \hat{f}\big(t_n, X_{t_n}, I_h Y^n(X_{t_n}), D_h^{0,I} Y^n(X_{t_n}), D_h^{2,I} Y^n(X_{t_n})\big).
\end{aligned}$$

Remark 4.4. In numerical tests in Section 5 we will compare our approach with the results obtained by the following θ-Scheme.

Scheme 4.5 (cf. Zhao *et al.* [36])**.** Given random variables Y^N, Z^N on S_h^N, for any $x \in S_h^n$, $n = N-1, \cdots, 0$, find the random variables $Y^n = Y^n(x)$ and $Z^n = Z^n(x)$ from the equations

$$\begin{aligned}
(1-\theta_1)\Delta t_n Z_n &= \hat{\mathbb{E}}_{t_n}^x[I_h Y^{n+1}\Delta W_{t_{n+1}}] + \theta_1 \Delta t_n \hat{\mathbb{E}}_{t_n}^x[I_h Z^{n+1}]\\
&\quad + \theta_2 \Delta t_n \hat{\mathbb{E}}_{t_n}^x[f^{n+1,I}\Delta W_{t_{n+1}}],\\
Y^n &= \hat{\mathbb{E}}_{t_n}^x[I_h Y^{n+1}] + \theta_3 \Delta t_n f^n + (1-\theta_3)\Delta t_n \hat{\mathbb{E}}_{t_n}^x[f^{n+1,I}],
\end{aligned}$$

where

$$\begin{aligned}
f^n &= f\big(t_n, x, Y^n(x), Z^n(x)\big),\\
f^{n,I} &= f\big(t_n, X_{t_n}, I_h Y^n(X_{t_n}), I_h Z^n(X_{t_n})\big).
\end{aligned}$$

Remark 4.5. If x is fixed and $X_{t_n} = x$, then $X_{t_{n+1}} = X_{t_n} + b\Delta t + \sigma \Delta W_{t_{n+1}}$ is a Gaussian random variable and the operator $\hat{\mathbb{E}}_{t_n}^x[\cdot]$ can be constructed by using the Gauss-Hermite quadratures. Correspondingly, one can make the error terms $R_y^{n,\mathbb{E}}$ and $R_f^{n,\mathbb{E}}$ sufficiently small — cf. Refs. [36,37]. If Δx and r denote, respectively, the space discretisation step and the degree of the interpolation method, then the local errors of the fully discrete schemes contain the following items:

- the time discrete error $(\Delta t)^{k+1}$,
- the difference approximation error $h^2 \Delta t$,
- the interpolation error $(\Delta x)^{r+1}$,
- the difference and interpolation errors $(\Delta x)^{r+1} h^{-1}\Delta t$ and $(\Delta x)^{r+1} h^{-2}\Delta t$,

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where $k=1$ for $\theta \neq 0.5$ and $k=2$ for $\theta = 0.5$.

Remark 4.6. Note that Schemes 4.1 and 4.2 can be considered as standard θ-schemes for the following BSDE:

$$\bar{Y}_t = \varphi(X_T) + \int_t^T f^h(s, X_s, \bar{Y}_s^h) ds - \int_t^T \bar{Z}_s dW_s, \tag{4.3}$$

where h is a positive number, and

$$\begin{aligned}\bar{Y}_s^h &= (Y_{s,-i_0 h}, Y_{s,-(i_0-1)h}, \cdots, Y_s, \cdots, Y_{s,(j_0-1)h}, Y_{s,j_0 h}),\\ Y_{s,ih} &= \bar{Y}_s(X_s + ih) \ (i = -i_0, -i_0+1, \cdots, j_0-1, j_0).\end{aligned}$$

In particular, Scheme 4.1 is an explicit θ-scheme for the BSDE (4.3) with

$$\begin{aligned}f^h =& f^h\big(s, X_s, \bar{Y}_s(X_s - h), \bar{Y}_s(X_s), \bar{Y}_s(X_s + h)\big)\\ =& f\Big(s, X_s, \bar{Y}_s(X_s), \frac{\bar{Y}_s(X_s+h) - \bar{Y}_s(X_s-h)}{2h}\Big).\end{aligned}$$

We observe that (4.3) is a new type of BSDEs. It will be called backward stochastic difference differential equation (BSDDE). Such equations can be helpful in the development of numerical methods for BSDEs and FBSDEs. Nevertheless, at the moment there are only scarce results available in the literature.

5. Numerical Tests

Let us now consider a few numerical examples testing the efficiency of the discrete schemes 4.3 and 4.4. For simplicity, here we use only uniform partitions. The time interval $[0, T]$ is divided into N equal subintervals of the length $\Delta t = T/N$ by the points $t_n = n\Delta t,\ n = 0, 1, \cdots, N$, whereas for $\mathbb{R}^d$ we have

$$S_h^n = S_h = S_{1,h} \times S_{2,h} \times \cdots \times S_{d,h}, \tag{5.1}$$

where

$$S_{j,h} = \{x_i^j : x_i^j = ih, i = 0, \pm 1, \cdots, \pm\infty\}, \quad j = 1, 2, \cdots, d,$$

and $S_{h,x} \subset S_h$ is a neighbor grids at the point x.

We also set $T = 1$ and choose the number of Gauss-Hermite quadrature points sufficiently large, so that the error of the conditional expectation approximations does not affect the convergence rate of Schemes 4.3 and 4.4 with respect to the time step Δt. For simplicity, we also chose $\Delta x = h$ with the same h as in (5.1) and approximate the derivative $f'(x)$ by the central difference $D_h^0 f(x)$. In this case, the local errors are equivalent to $(\Delta t)^{k+1}$, $h^2\Delta t$, h^{r+1} and $h^r \Delta t$ for Scheme 4.3, and $(\Delta t)^{k+1}$, $h^2 \Delta t$, h^{r+1} and $h^{r-1}\Delta t$ for Scheme 4.4. In order to adjust the errors from time and space discretisations in Scheme 4.3, the space step h and the time discrete step Δt should be chosen such that all the terms $h^2\Delta t$, h^{r+1}

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and $h^r\Delta t$ are dominated by $(\Delta t)^{k+1}$. Analogously, for Scheme 4.4 all terms $h^2\Delta t$, h^{r+1} and $h^{r-1}\Delta t$ should be dominated by $(\Delta t)^{k+1}$, where k is the expected order of the truncation error with respect to Δt. We also use the cubic spline interpolation, so that $r=3$. Moreover, in both schemes $h=\sqrt{\Delta t}$ for $\theta\neq 0.5$ ($k=1$), and $h=\Delta t$ for $\theta=0.5$ ($k=2$). For more details, the reader is referred to Refs. [36,37].

Let us also note that in all numerical examples, the convergence rate (denoted by CR) with respect to Δt is obtained by linear least square fitting of the errors.

Example 5.1. We consider the BSDE

$$\begin{aligned}-dY_t &= \frac{Y_t/2-Z_t}{Y_t^2+Z_t^2}dt - Z_t dW_t,\\ Y_T &= \sin(T+W_T),\end{aligned}$$

which has analytic solution

$$\begin{aligned}Y_t &= \sin(t+W_t),\\ Z_t &= \cos(t+W_t),\end{aligned}$$

where W_t is standard Brownian motion.

To solve this problem we use Scheme 4.3. The errors $|Y_0-Y^0|$ and $|Z_0-Z^0|$ and convergence rate are reported in Table 1. The corresponding results for Scheme 4.5 are presented in Table 2. Tables 1-2 show that for $\theta=0$ and $\theta=0.5$, Scheme 4.3 has linear

Table 1: Errors and convergence rate of Scheme 4.3.

	$\theta=0$		$\theta=0.5$	
Δt	$\lvert Y_0-Y^0\rvert$	$\lvert Z_0-Z^0\rvert$	$\lvert Y_0-Y^0\rvert$	$\lvert Z_0-Z^0\rvert$
1/32	2.423E-02	2.014e-02	3.511E-04	3.093e-04
1/64	1.224E-02	9.950e-03	8.830E-05	7.722e-05
1/128	6.200E-03	4.885e-03	2.194E-05	1.928e-05
1/256	3.093E-03	2.419e-03	4.797E-06	4.769e-06
1/512	1.551E-03	1.198e-03	7.614E-07	1.469e-06
CR	0.992	1.018	2.190	1.945

Table 2: Errors and convergence rate of Scheme 4.5.

	$\theta_1=0$		$\theta_1=0.5$	
Δt	$\lvert Y_0-Y^0\rvert$	$\lvert Z_0-Z^0\rvert$	$\lvert Y_0-Y^0\rvert$	$\lvert Z_0-Z^0\rvert$
1/32	4.319E-02	5.346e-02	3.521E-04	5.716e-04
1/64	2.213E-02	2.843e-02	8.839E-05	1.433e-04
1/128	1.119E-02	1.472e-02	2.195E-05	3.564e-05
1/256	5.626E-03	7.491e-03	4.798E-06	8.080e-06
1/512	2.820E-03	3.780e-03	7.645E-07	9.891e-07
CR	0.985	0.957	2.190	2.250

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and quadratic order of convergence, respectively. Moreover, the accuracy of Scheme 4.3 is almost the same as of Scheme 4.5 in [36].

Example 5.2. Consider the nonlinear decoupled 2FBSDE

$$\begin{aligned} dX_t &= \sin(t+X_t)dt + \frac{1}{2}\cos(t+X_t)dW_t, \\ -dY_t &= \left(-\cos(t+X_t)(Y_t^2+Y_t+2Z_t) - \frac{1}{4}\Gamma_t\right)dt - Z_t dW_t, \\ dZ_t &= A_t dt + \Gamma_t dW_t, \end{aligned} \tag{5.2}$$

with the terminal condition $Y_T = \sin(T+X_T)$. The exact solution of problem (5.2) is

$$Y_t = \sin(t+X_t), \quad Z_t = \frac{1}{2}\cos(t+X_t), \quad \Gamma_t = -\frac{1}{2}\sin(t+X_t)\cos^2(t+X_t).$$

Now we use Scheme 4.4 and determine the process X_t by the weak second order Taylor scheme (4.1). The errors $|Y_0 - Y^0|$, $|Z_0 - Z^0|$, $|\Gamma_0 - \Gamma^0|$ and convergence rate are reported in Table 3. It shows the efficiency of Scheme 4.4 for decoupled second-order FBSDEs. The accuracy is of order 1 if $\theta = 0$ and of order 2 if $\theta = 0.5$, which is consistent with theoretical estimates in Remark 3.2.

Table 3: Errors and convergence rate in Example 5.2.

	$\theta=0$			$\theta=0.5$		
Δt	$\lvert Y_0-Y^0\rvert$	$\lvert Z_0-Z^0\rvert$	$\lvert\Gamma_0-\Gamma^0\rvert$	$\lvert Y_0-Y^0\rvert$	$\lvert Z_0-Z^0\rvert$	$\lvert\Gamma_0-\Gamma^0\rvert$
1/32	3.537E-03	4.990e-03	3.808e-03	1.312E-04	3.136e-04	3.792e-04
1/64	1.800E-03	2.557e-03	1.943e-03	2.714E-05	7.239e-05	9.032e-05
1/128	8.905E-04	1.285e-03	9.869e-04	5.560E-06	1.629e-05	2.199e-05
1/256	4.558E-04	6.575e-04	4.871e-04	1.157E-06	3.734e-06	6.426e-06
1/512	2.169E-04	3.121e-04	2.566e-04	4.265E-07	7.669e-07	1.460e-06
CR	1.004	0.996	0.978	2.108	2.163	1.986

Example 5.3. Let us demonstrate the efficiency of Scheme 4.4 for two-dimensional 2FBSDEs. Consider the model

$$\begin{aligned} \begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix} &= \begin{pmatrix} X_1(0) \\ X_2(0) \end{pmatrix} + \int_0^t \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} ds + \int_0^t \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} dW_s, \\ \begin{pmatrix} Y_1(t) \\ Y_2(t) \end{pmatrix} &= \begin{pmatrix} \sin(T+X_1(T)) + \sin(T+X_2(T)) \\ \cos(T+X_1(T)) + \cos(T+X_2(T)) \end{pmatrix} \\ &\quad + \int_t^T \begin{pmatrix} \hat{f}_s^1 \\ \hat{f}_s^2 \end{pmatrix} ds - \int_t^T \begin{pmatrix} Z_1(s) \\ Z_2(s) \end{pmatrix} dW_s, \end{aligned} \tag{5.3}$$

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where $b_1,\ b_2,\ c_1$ and c_2 are constants,

$$\begin{aligned}\hat{f}_s^1 =& -\frac{1}{2}\Big((\sigma_1+\sigma_2)Z_2+\sigma_1\sigma_2 Y_1\Big)-b_1\cos(t+x_1)-b_2\cos(t+x_2)-Y_2\\&+\alpha_1\Big((\Gamma_1+\sigma_2^2\sin(t+x_2))^2+(\Gamma_2+\sigma_2^2\cos(t+x_2))^2-\sigma_1^4\Big),\\\hat{f}_s^2 =& \frac{1}{2}\Big((\sigma_1+\sigma_2)Z_1-\sigma_1\sigma_2 Y_2\Big)+b_1\sin(t+x_1)+b_2\sin(t+x_2)+Y_1\\&+\alpha_2\Big((\Gamma_1+\sigma_1^2\sin(t+x_1))^2+(\Gamma_2+\sigma_1^2\cos(t+x_1))^2-\sigma_2^4\Big),\end{aligned}$$

and $\alpha_1=\alpha_2=0.5$. The problem (5.3) has exact solution

$$\begin{aligned}&Y_1(t)=\sin(t+X_1(t))+\sin(t+X_2(t)),\\&Y_2(t)=\cos(t+X_1(t))+\sin(t+X_2(t)),\\&Z_1(t)=\sigma_1\cos(t+X_1(t))+\sigma_2\cos(t+X_2(t)),\\&Z_2(t)=-\sigma_1\sin(t+X_1(t))-\sigma_2\sin(t+X_2(t)),\\&\Gamma_1(t)=-\sigma_1^2\sin(t+X_1(t))-\sigma_2^2\sin(t+X_2(t)),\\&\Gamma_2(t)=-\sigma_1^2\cos(t+X_1(t))-\sigma_2^2\cos(t+X_2(t)).\end{aligned}$$

We set $b_1=b_2=0.1$ and $\sigma_1=\sigma_2=0.25$. The errors $|Y_i(0)-Y_i^0|$, $|Z_i(0)-Z_i^0|$ and $|\Gamma_i(0)-\Gamma_i^0|$, $i=1,2$ with the time step $\Delta t=1/N$, and the convergence rate with respect to Δt are reported in Tables 4 and 5. Tables 4 and 5 show that Scheme 4.4 can be effectively

Table 4: Errors and convergence rate in Example 5.3 with $\theta=0$.

	$\theta=0$					
N	$\lvert Y_1(0)-Y_1^0\rvert$	$\lvert Z_1(0)-Z_1^0\rvert$	$\lvert\Gamma_1(0)-\Gamma_1^0\rvert$	$\lvert Y_2(0)-Y_2^0\rvert$	$\lvert Z_2(0)-Z_2^0\rvert$	$\lvert\Gamma_2(0)-\Gamma_2^0\rvert$
5	2.315E-01	4.319e-02	2.203e-02	2.432E-01	5.018e-02	1.945e-02
10	1.117E-01	2.156e-02	1.122e-02	1.206E-01	2.347e-02	9.404e-03
15	7.298E-02	1.425e-02	8.743e-03	7.945E-02	1.576e-02	7.510e-03
20	5.329E-02	1.038e-02	5.949e-03	5.844E-02	1.091e-02	4.932e-03
25	4.160E-02	8.064e-03	5.197e-03	4.572E-02	8.579e-03	4.386e-03
CR	1.064	1.039	0.901	1.036	1.095	0.930

Table 5: Errors and convergence rate in Example 5.3 with $\theta=0.5$.

	$\theta=0.5$					
N	$\lvert Y_1(0)-Y_1^0\rvert$	$\lvert Z_1(0)-Z_1^0\rvert$	$\lvert\Gamma_1(0)-\Gamma_1^0\rvert$	$\lvert Y_2(0)-Y_2^0\rvert$	$\lvert Z_2(0)-Z_2^0\rvert$	$\lvert\Gamma_2(0)-\Gamma_2^0\rvert$
5	1.501E-02	5.593e-03	3.288e-03	1.428E-02	4.402e-03	3.300e-03
10	3.781E-03	1.324e-03	8.805e-04	3.501E-03	1.159e-03	7.405e-04
15	1.686E-03	5.740e-04	4.030e-04	1.543E-03	5.233e-04	3.058e-04
20	9.497E-04	3.186e-04	2.295e-04	8.646E-04	2.968e-04	1.632e-04
25	6.084E-04	2.026e-04	1.555e-04	5.520E-04	1.907e-04	1.013e-04
CR	1.992	2.063	1.906	2.021	1.950	2.167

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used for solving multidimensional decoupled second-order FBSDEs. The accuracy is of order 1 for $\theta = 0$ and of order 2 for $\theta = 1/2$, consistent with theoretical estimates in Remark 3.2.

Example 5.4 (Tracking a particle under microscope)**.** Consider the system

$$dX_t = \beta \alpha_t dt + \sigma dW_t,$$

where X_t denotes the distance from a particle to the focus of the microscope, $\beta \in \mathbb{R}$ the gain in our servo loop and $\sigma > 0$ the diffusion constant of the particle. We would like to hold the particle in focus — i.e. we expect the particle to be as close as possible to the origin. Since servo motor cannot be driven with arbitrarily large input power, there is also a power constraint

$$J(\alpha) = \mathbb{E}\left[p \int_0^T X_t^2 dt + q \int_0^T \alpha^2 dt \right],$$

where $p, q > 0$ allow to select tradeoff between good tracking and low feedback power. We consider the associated Bellman equation

$$\begin{aligned} &u_t + \inf_{\alpha \in \mathbb{R}} \left\{ \frac{\sigma^2}{2} u_{xx} + \beta \alpha u_x + p x^2 + q \alpha^2 \right\} = 0, \\ &u(T, x) = 0, \end{aligned} \tag{5.4}$$

with the constants $\sigma = 0.5$, $\beta = 0.5$, $p = 0.5$, $q = 1.5$, $T = 1.0$. The optimal control parameter α^* is $-(\beta/2q)u_x$. Substituting it into the Eq. (5.4), we obtain

$$\begin{aligned} &u_t + F(t, x, u, Du, D^2 u) = 0, \quad (t, x) \in [0, T] \times \mathbb{R}, \\ &u(T, x) = 0, \quad x \in \mathbb{R}, \end{aligned} \tag{5.5}$$

where

$$F(t, x, u, Du, D^2 u) = \frac{\sigma^2}{2} u_{xx} - \frac{\beta^2}{4q} u_x^2 + p x^2.$$

The true solution of the Eq. (5.5) is

$$\begin{aligned} u(t, x) &= a(t) x^2 + b(t), \\ a(t) &= \frac{\sqrt{pq}}{\beta} \tanh\left(\beta \sqrt{\frac{p}{q}} (T - t) \right), \\ b(t) &= \frac{q \sigma^2}{\beta^2} \log\left(\cosh\left(\beta \sqrt{\frac{p}{q}} (T - t) \right) \right). \end{aligned}$$

Let us point out, that in addition to the applicability of Scheme 4.4 to 2FBSDEs, we also want to study its efficiency in solving second-order quasi-linear parabolic HJB equations arising in stochastic optimal control problems. Since the corresponding 2FBSDEs are not

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unique, we can also test the effects of forward stochastic differential equation in 2FBSDEs. Thus we choose the following forward stochastic differential equation

$$dX_t = c_1\beta dt + c_2\sigma dW_t, \quad t\in[0,1],$$

with constants c_1, c_2. The corresponding generator has the form

$$\begin{aligned} f(t,x,u,Du,D^2u) &= F(t,x,u,Du,D^2u) - b^\top Du - \frac{1}{2}\sigma\sigma^\top : D^2u \\ &= \frac{\sigma^2}{2}(1-c_2^2)u_{xx} - \frac{\beta^2}{4q}u_x^2 + px^2 - c_1\beta u_x, \end{aligned}$$

and the corresponding 2FBSDEs are

$$\begin{aligned} dX_t &= c_1\beta dt + c_2\sigma dW_t, \\ -dY_t &= \left(\frac{1}{2}\Gamma_t\frac{1-c_2^2}{c_2^2} - \frac{\beta^2}{4q\sigma^2c_2^2}Z_t^2 - \frac{c_1\beta}{c_2\sigma}Z_t + pX_t^2\right)dt - Z_t dW_t, \\ dZ_t &= A_t dt + \Gamma_t dW_t. \end{aligned}$$

The errors $|Y_0 - Y^0|$, $|Z_0 - Z^0|$, $|\Gamma_0 - \Gamma^0|$ and the convergence rate for various values of c_1 and c_2 are reported in Tables 6-10.

These numerical experiments show that Scheme 4.4 is very efficient in solving the stochastic optimal control problems and produces highly accurate approximate solutions.

Table 6: Errors and convergence rate in Example 5.4: $c_1 = 1.5$, $c_2 = 1.0$.

	$\theta=0$			$\theta=0.5$		
Δt	$\lvert Y_0-Y^0\rvert$	$\lvert Z_0-Z^0\rvert$	$\lvert\Gamma_0-\Gamma^0\rvert$	$\lvert Y_0-Y^0\rvert$	$\lvert Z_0-Z^0\rvert$	$\lvert\Gamma_0-\Gamma^0\rvert$
1/32	2.366E-02	1.171e-02	2.936e-04	1.545E-04	1.409e-05	3.388e-06
1/64	1.191E-02	5.848e-03	1.473e-04	3.853E-05	3.490e-06	8.453e-07
1/128	5.975E-03	2.923e-03	7.380e-05	9.622E-06	8.683e-07	2.111e-07
1/256	2.992E-03	1.461e-03	3.694e-05	2.404E-06	2.166e-07	5.276e-08
1/512	1.497E-03	7.304e-04	1.848e-05	6.009E-07	5.407e-08	1.319e-08
CR	0.996	1.001	0.998	2.001	2.006	2.001

Table 7: Errors and convergence rate in Example 5.4: $c_1 = 1.5$, $c_2 = 0.05$.

	$\theta=0$			$\theta=0.5$		
Δt	$\lvert Y_0-Y^0\rvert$	$\lvert Z_0-Z^0\rvert$	$\lvert\Gamma_0-\Gamma^0\rvert$	$\lvert Y_0-Y^0\rvert$	$\lvert Z_0-Z^0\rvert$	$\lvert\Gamma_0-\Gamma^0\rvert$
1/32	2.746E-02	5.854e-04	7.340e-07	1.545E-04	7.045e-07	8.469e-09
1/64	1.381E-02	2.924e-04	3.683e-07	3.853E-05	1.745e-07	2.113e-09
1/128	6.923E-03	1.461e-04	1.845e-07	9.622E-06	4.341e-08	5.279e-10
1/256	3.466E-03	7.305e-05	9.234e-08	2.404E-06	1.083e-08	1.319e-10
1/512	1.734E-03	3.652e-05	4.619e-08	6.009E-07	2.704e-09	3.298e-11
CR	0.996	1.001	0.998	2.001	2.006	2.001

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Table 8: Errors and convergence rate in Example 5.4: $c_1 = 1.5$, $c_2 = 2.0$.

	$\theta = 0$			$\theta = 0.5$		
Δt	$\lvert Y_0 - Y^0\rvert$	$\lvert Z_0 - Z^0\rvert$	$\lvert\Gamma_0 - \Gamma^0\rvert$	$\lvert Y_0 - Y^0\rvert$	$\lvert Z_0 - Z^0\rvert$	$\lvert\Gamma_0 - \Gamma^0\rvert$
1/32	3.224E-02	2.342e-02	1.174e-03	1.545E-04	2.818e-05	1.355e-05
1/64	1.601E-02	1.170e-02	5.894e-04	3.853E-05	6.979e-06	3.381e-06
1/128	7.978E-03	5.846e-03	2.952e-04	9.622E-06	1.737e-06	8.446e-07
1/256	3.982E-03	2.922e-03	1.477e-04	2.404E-06	4.331e-07	2.111e-07
1/512	1.989E-03	1.461e-03	7.391e-05	6.009E-07	1.081e-07	5.277e-08
CR	1.005	1.001	0.998	2.001	2.006	2.001

Table 9: Errors and convergence rate in Example 5.4: $c_1 = 1.5$, $c_2 = 12$.

	$\theta = 0$			$\theta = 0.5$		
Δt	$\lvert Y_0 - Y^0\rvert$	$\lvert Z_0 - Z^0\rvert$	$\lvert\Gamma_0 - \Gamma^0\rvert$	$\lvert Y_0 - Y^0\rvert$	$\lvert Z_0 - Z^0\rvert$	$\lvert\Gamma_0 - \Gamma^0\rvert$
1/32	5.651E-01	1.405e-01	4.232e-02	1.546E-04	1.689e-04	5.366e-04
1/64	2.823E-01	7.018e-02	2.122e-02	3.853E-05	4.187e-05	1.217e-04
1/128	1.411E-01	3.507e-02	1.063e-02	9.612E-06	1.024e-05	3.315e-05
1/256	7.051E-02	1.753e-02	5.319e-03	2.404E-06	2.601e-06	7.710e-06
1/512	3.525E-02	8.765e-03	2.661e-03	6.009E-07	6.488e-07	1.902e-06
CR	1.001	1.001	0.998	2.002	2.006	2.026

Table 10: Errors and convergence rate in Example 5.4: $c_1 = 0$, $c_2 = \frac{1}{\sigma} = 2.0$.

	$\theta = 0$			$\theta = 0.5$		
Δt	$\lvert Y_0 - Y^0\rvert$	$\lvert Z_0 - Z^0\rvert$	$\lvert\Gamma_0 - \Gamma^0\rvert$	$\lvert Y_0 - Y^0\rvert$	$\lvert Z_0 - Z^0\rvert$	$\lvert\Gamma_0 - \Gamma^0\rvert$
1/32	2.703E-02	1.600e-02	1.174e-03	6.014E-06	1.355e-05	1.355e-05
1/64	1.346E-03	7.994e-03	5.894e-04	1.501E-06	3.381e-06	3.381e-06
1/128	6.716E-03	3.995e-03	2.952e-04	3.748E-07	8.446e-07	8.446e-07
1/256	3.355E-03	1.997e-03	1.477e-04	9.366E-08	2.110e-07	2.110e-07
1/512	1.677E-04	9.985e-04	7.391e-05	2.341E-08	5.275e-08	5.277e-08
CR	1.003	1.001	0.998	2.001	2.001	2.001

Example 5.5. Consider the fully nonlinear HJB equation

$$\begin{aligned} &u_t + F(t,x,u,Du,D^2u) = 0, \quad (t,x) \in [0,T] \times \mathbb{R}, \\ &u(T,x) = \frac{e^{T+x}}{1+e^{T+x}}, \quad x \in \mathbb{R}, \end{aligned} \tag{5.6}$$

where

$$F(t,x,u,Du,D^2u) = \frac{u_{xx}}{2+u_{xx}^2} - u_x - \frac{e^{t+x}(1+e^{t+x})^3(1-e^{t+x})}{2(1+e^{t+x})^6 + e^{2t+2x}(1-e^{t+x})^2}.$$

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The true solution of the Eq. (5.6) is $u(t,x)=e^{t+x}/(1+e^{t+x})$.

Now we consider the forward stochastic differential equation

$$dX_t = b\,dt + \sigma dW_t, \quad t\in[0,1],$$

with constants b and σ. For $s\in[t,T]$, let

$$\begin{aligned}
Y_s^{t,x} &= u(s,X_s^{t,x}) = \frac{e^{t+X_s^{t,x}}}{1+e^{t+X_s^{t,x}}}, \quad Z_s^{t,x} = (Du\sigma)(s,X_s^{t,x}) = \frac{\sigma e^{t+X_s^{t,x}}}{(1+e^{t+X_s^{t,x}})^2},\\
\Gamma_s^{t,x} &= (D(Du\sigma)\sigma)(s,X_s^{t,x}) = \frac{\sigma^2 e^{t+X_s^{t,x}}(1-e^{t+X_s^{t,x}})}{(1+e^{t+X_s^{t,x}})^3},\\
A_s^{t,x} &= (\mathscr{L}(Du\sigma))(s,X_s^{t,x})\\
&= \frac{\sigma e^{t+X_s^{t,x}}\Big(2(1+b)(1-e^{2t+2X_s^{t,x}})+\sigma^2(1-4e^{t+X_s^{t,x}}+e^{2t+2X_s^{t,x}})\Big)}{2(1+e^{t+X_s^{t,x}})^4},
\end{aligned}$$

where operator $\mathscr{L}$ is defined by (2.6). According to Lemma 2.2, vector $(Y_s^{t,x}, Z_s^{t,x}, \Gamma_s^{t,x}, A_s^{t,x})$ is the solution of the 2FBSDE

$$\begin{aligned}
dX_t &= b\,dt + \sigma dW_t,\\
-dY_t &= \Big(\frac{\sigma^2\Gamma_t}{2\sigma^4+\Gamma_t^2} - \frac{1}{2}\Gamma_t - \frac{b+1}{\sigma}Z_t\\
&\qquad - \frac{Y_t(1+e^{t+x})^3(1-e^{2t+2x})}{2(1+e^{t+x})^6+e^{2t+2x}(1-e^{t+x})^2}\Big)dt - Z_t dW_t,\\
dZ_t &= A_t dt + \Gamma_t dW_t.
\end{aligned} \tag{5.7}$$

To find an approximate solution of the problem (5.7) we use Scheme 4.4. The errors $|Y_0-Y^0|$, $|Z_0-Z^0|$, $|\Gamma_0-\Gamma^0|$, and convergence rate for different values of b and σ are reported in Tables 11-13.

Thus Scheme 4.4 produces highly accurate approximate solutions of stochastic optimal control problems and fully nonlinear second-order PDEs. It is stable and not sensitive to the choice of the associated forward stochastic differential equation. This allows to increase the performance of the method by selecting suitable forward stochastic differential equation similar to Example 5.4.

Table 11: Errors and convergence rate in Example 5.5: $b=2.5$, $\sigma=1.0$.

	$\theta=0$			$\theta=0.5$		
Δt	$\lvert Y_0-Y^0\rvert$	$\lvert Z_0-Z^0\rvert$	$\lvert\Gamma_0-\Gamma^0\rvert$	$\lvert Y_0-Y^0\rvert$	$\lvert Z_0-Z^0\rvert$	$\lvert\Gamma_0-\Gamma^0\rvert$
1/16	3.418E-02	4.212e-02	3.977e-02	2.283E-03	2.016e-03	4.668e-03
1/32	1.620E-02	1.907e-02	1.776e-02	5.734E-04	5.040e-04	1.208e-03
1/64	7.794E-03	9.034e-03	8.276e-03	1.439E-04	1.267e-04	3.340e-04
1/128	3.814E-03	4.398e-03	3.986e-03	3.609E-05	3.183e-05	1.025e-04
1/256	1.886E-03	2.170e-03	1.956e-03	9.026E-06	7.976e-06	3.265e-05
CR	1.045	1.067	1.085	1.996	1.995	1.788

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Table 12: Errors and convergence rate in Example 5.5: $b = 3.0$, $\sigma = 1.0$.

	$\theta = 0$			$\theta = 0.5$		
Δt	$\lvert Y_0 - Y^0\rvert$	$\lvert Z_0 - Z^0\rvert$	$\lvert \Gamma_0 - \Gamma^0\rvert$	$\lvert Y_0 - Y^0\rvert$	$\lvert Z_0 - Z^0\rvert$	$\lvert \Gamma_0 - \Gamma^0\rvert$
1/16	4.603E-02	5.854e-02	5.565e-02	3.453E-03	3.075e-03	5.696e-03
1/32	2.148E-02	2.566e-02	2.404e-02	8.657E-04	7.641e-04	1.370e-03
1/64	1.021E-02	1.193e-02	1.095e-02	2.171E-04	1.918e-04	3.820e-04
1/128	4.958E-03	5.754e-03	5.213e-03	5.443E-05	4.812e-05	1.138e-04
1/256	2.442E-03	2.827e-03	2.543e-03	1.361E-05	1.205e-05	3.589e-05
CR	1.059	1.090	1.111	1.996	1.998	1.821

Table 13: Errors and convergence rate in Example 5.5: $b = 3.5$, $\sigma = 1.0$.

	$\theta = 0$			$\theta = 0.5$		
Δt	$\lvert Y_0 - Y^0\rvert$	$\lvert Z_0 - Z^0\rvert$	$\lvert \Gamma_0 - \Gamma^0\rvert$	$\lvert Y_0 - Y^0\rvert$	$\lvert Z_0 - Z^0\rvert$	$\lvert \Gamma_0 - \Gamma^0\rvert$
1/16	6.029E-02	7.971e-02	7.573e-02	4.977E-03	4.471e-03	7.353e-03
1/32	2.774E-02	3.374e-02	3.187e-02	1.242E-03	1.100e-03	1.816e-03
1/64	1.301E-02	1.534e-02	1.414e-02	3.115E-04	2.756e-04	4.623e-04
1/128	6.265E-03	7.320e-03	6.640e-03	7.806E-05	6.908e-05	1.256e-04
1/256	3.073E-03	3.578e-03	3.217e-03	1.953E-05	1.729e-05	3.920e-05
CR	1.073	1.116	1.138	1.998	2.002	1.896

6. Conclusion

Using the Feynman-Kac formulas, we developed new second-order numerical schemes for the forward-backward stochastic differential equations of the first and second order. The methods presented here have a simple structure and allow an easy implementation. Numerous numerical tests carried out for FBSDEs, fully nonlinear second-order parabolic PDEs, and HJB equations show the stability and a high accuracy of the methods.

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