Formuling a SIS-SDE

Now consider $\{$ $\}_{\geq 0}$. Defin

$$p(y, t+\Delta t; x, t)$$

$$y = I_{t+\Delta t}, \quad x = I_t$$

With this notation. Set $\Delta i = 1$, then

$$\begin{aligned}
\frac{dp_i}{dt} &= p_{i-1} b(i_1) + p_{i+1} d(i+1) - p_i[b(i) + d(i)] \\
&= -\frac{p_{i+1}[d(i+1) - d(i+1)] - p_{i-1}[d(i-1) - d(i-1)]}{2\Delta i} \\
&+ \frac{1}{2} \frac{p_{i+1}[d(i+1) + d(i+1)] - 2p_i[b(i) + d(i)] + p_{i-1}[d(i-1) + d(}{(\Delta_i)^2}
\end{aligned}$$

Let $i = x$, $\Delta i = \Delta x$ and $pi(t) = p(x,t)$. Thus, letting $\Delta x to 0$, we obtain the FKE

$$
\begin{aligned}
\frac{\partial p(x,t)}{\partial t} &= \frac{\partial}{\partial x}\{[b(x) - d(x)]p(x,t)\} + \frac{1}{2}\frac{\partial^2}{\partial t^2}\{b(x) + d(x)p(x,t)\} \\
&= \frac{\partial}{\partial x}\left\{\left[\frac{beta}{N}x(N-x) - (b+\gamma)x\right]p(x,t)\right\} \\
&\quad + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left\{\left[\frac{\beta}{N}x(N-x) + (\beta+\gamma)x\right]p(x,t)\right\}
\end{aligned}
$$

Using the SIS-CTMC probablity transition kernel

$$p_{ji}(\Delta t) := \begin{cases} \frac{\beta i(N-i)}{N}\Delta t + o(\Delta t), & j = i+1 \\ (b+\gamma)i\Delta t + o(\Delta t), & j = i-1 \\ 1 - \left[\frac{\beta i(N-i)}{N} + (b+\gamma)i\right]\Delta t + o(\Delta t), & j = i \\ o(\Delta t) & \text{otherwise} \end{cases}$$

If we assume that increment Δt of transition follows a exponential distribution and is suficiently small. Results that increment

$$\Delta I = It + \Delta t - I_t$$

has normal distribuition, with following expectation and variance. Fix time t such taht $I_t = i$

$$\begin{aligned} \mathbb{E}\Delta I &= b(I_t)\Delta t - d(I_t)\Delta t + o(\Delta t) \\ &= \underbrace{[b(I)_t - d(I_t)]}_{:=\mu(I_t)} + o(t) \end{aligned}$$

Thus

$$\begin{aligned}
\mathrm{Var}\left[\Delta I_t\right] &= \mathbb{E}\left[\Delta I_t^2\right] - \left[\mathbb{E}\left[\Delta I_t\right]\right]^2 \\
&= \underbrace{\left[b(I_t) + d(I_t)\right]}_{:=\sigma^2(I_t)} \Delta t + o(\Delta t)
\end{aligned}$$

Since $\Delta I_t \sim \mathcal{N}(\mu(I_t)\Delta t, \sigma^2(I_t)\Delta t)$, we see that

$$\begin{aligned}
I_{t+\Delta t} &= I_t + \Delta I_t \\
&\approx I_t + \mu(I_t)\Delta t + \sigma(I_t)\sqrt{\Delta t}\eta \\
\eta &\sim \mathcal{N}(0,1)
\end{aligned}$$

The Euler-Maruyama’s recurrence equation.

Further, becouse under this setting, the Euler-Maruyama converge. Letting $\Delta t \to 0$, we deduce our SIS-SDE:

$$dI_t = \mu(I_t) + \sigma(I_t) dW_t$$

Sustituting, the notation for birth and death processes

$$dI_t = \frac{\beta}{N} I_t (N - I_t) - (b + \gamma) I_t + \sqrt{\frac{\beta}{N} I_t (N - I_t) + (b + \gamma) I_t} dW_t$$