UQ

ODE + noise = Better Model

To fix ideas

Example

$$dN(t) = aN(t)dt$$

To fix ideas

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Perturb in $[t, t + dt)$

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Perturb in $[t, t+dt)$

$$adt \rightsquigarrow adt + \sigma dB(t)$$

To fix ideas

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Get a SDE

$$dN(t) = aN(t)dt + \sigma N(t)dB(t)$$

To fix ideas

Example

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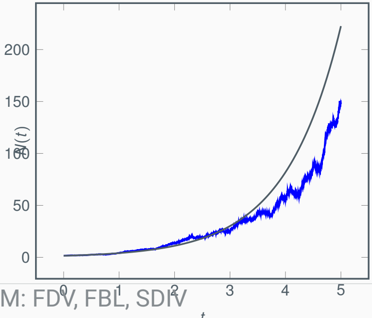

Some Important applications

- Finance
- Physics
- Chemistry
- Biology
- Epidemiology

Henston

$$dS_t = \mu S_t dt + \sqrt{V_t} S_t \left(\sqrt{1-\rho^2} dW_t^{(1)} + \rho dW^{(2)} \right)$$

$$dV_t = \kappa(\lambda - V_t)dt + \theta \sqrt{V_t} dW_t^{(2)}$$

Some Important applications

- Finance
- Physics
- Chemistry
- Biology
- Epidemiology

Langevin

$$dX_t = -(\nabla U)(X_t)dt + \sqrt{2\varepsilon}dW_t$$

Some Important applications

- Finance
- Physics
- Chemistry
- Biology
- Epidemiology

Brusselator

$$dX_t = \left[\delta - (\alpha + 1)X_t + Y_t X_t^2\right] dt + g_1(X_t) dW_t^{(1)}$$
$$dY_t = \left[\alpha X_t + Y_t X_t^2\right] dt + g_2(X_t) dW_t^{(2)}$$

Some Important applications

- Finance
- Physics
- Chemistry
- Biology
- Epidemiology

Lotka Volterra

$$dX_t = (\lambda X_t - kX_t Y_t)dt + \sigma X_t dW_t$$
$$dY_t = (kX_t Y_t - mY_t)dt$$

Some Important applications

- Finance
- Physics
- Chemistry
- Biology
- Epidemiology

SIR

$$
\begin{aligned}
dS_t &= (-\alpha S_t I_t - \delta S_t + \delta)dt - \beta S_t I_t dW_t \\
dI_t &= (\alpha S_t I_t - (\gamma + \delta) I_t)dt + \beta S_t I_t dW_t \\
dR_t &= (\gamma I_t - \delta R_t)dt
\end{aligned}
$$

Why noise?

Environmental effects

- Extinction
- Outbreaks

Why noise?

Environmental effects

Extinction

Outbreaks

Environmental Brownian noise suppresses explosions.

Why noise?

Environmental effects

- Extinction
- Outbreaks

Noise color induces extinction

Why noise?

Environmental effects

- Extinction
- Outbreaks

$\mathscr{R}_0$: Endemic g.a.e. $\rightarrow$ periodic oscillations

Alternativas

In Biology

- **DTMC, CTMC**
- Stochastic perturbation of parameters
- Mean reverting processes

Environmental effects

- Extinction
- Outbreaks

DTMC $+$ CTMC $+$ ME $\rightarrow$ *SDE*

Alternativas

In Biology

- DTMC, CTMC
- **Stochastic perturbation of parameters**
- Mean reverting processes

$$\varphi dt \rightsquigarrow \varphi dt + \sigma dB_t$$

Environmental effects

- Extinction
- Outbreaks

Alternativas

In Biology

- DTMC, CTMC
- **Stochastic perturbation of parameters**
- Mean reverting processes

$$\varphi dt \rightsquigarrow \varphi dt + F(x)dB_t$$

Environmental effects

- Extinction
- Outbreaks

Alternativas

In Biology

- DTMC, CTMC
- Stochastic perturbation of parameters
- Mean reverting processes

$$d\varphi_t = (\varphi_e - \varphi_t)dt + \sigma_\varphi dBt$$

Environmental effects

- Extinction
- Outbreaks

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CDMX Covid-19 data

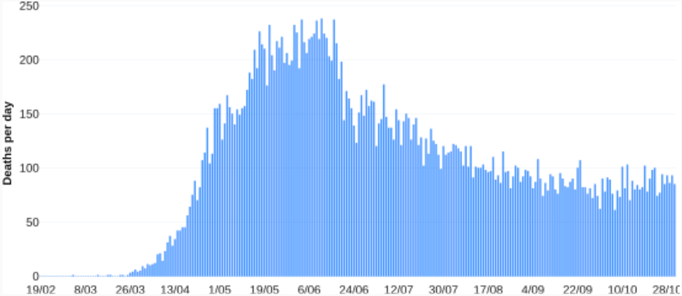


Figure: CDMX data

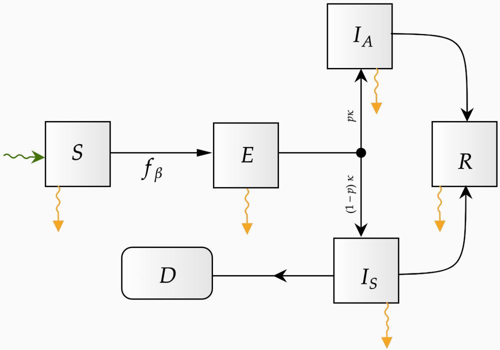
Example: Estimation of the infection rates β_s, β_a, and ratio of asymptomatic cases p.

Argument: Noise could improve the uncertainty quantification.

[1]

[1] Work in progress.
Somited in IJCM. Fernado Baltazar (UNAM-CU) and Francisco Delgado (CONACYT-UNAM-OAXACA)

MCMC with a deterministic SEIRS structure



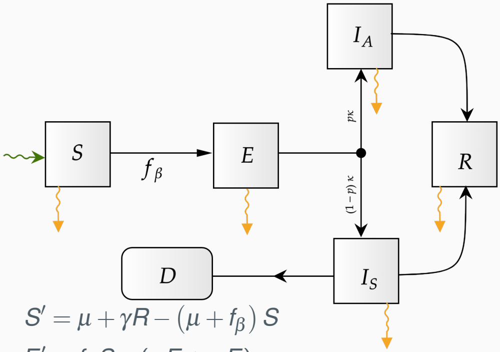
$$f_\beta := \beta_s I_s + \beta_a I_a$$

$$\frac{S + E + I + R + D}{N} = 1$$

$$\beta_\bullet = \frac{\widehat{\beta}_\bullet}{N - D}$$

μ

MCMC with a deterministic SEIRS structure



$$f_\beta := \beta_s I_s + \beta_a I_a$$

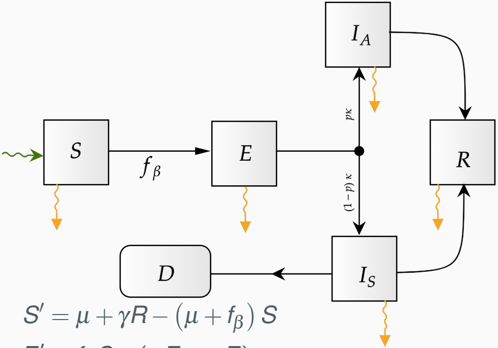
$$\frac{S+E+I+R+D}{N} = 1$$

$$\beta_\bullet = \frac{\widehat{\beta}_\bullet}{N-D}$$

$\rightsquigarrow$ μ

$$S' = \mu + \gamma R - (\mu + f_\beta)\, S$$
$$E' = f_\beta S - (\kappa E + \mu E)$$
$$I_a{}' = p\kappa E - (\alpha_a + \mu)\, I_a$$
$$I_s{}' = (1-p)\kappa E - (\alpha_s + \mu) I_s$$
$$R' = \alpha_a I_a + \alpha_s(1-\theta) I_s - (\mu+\gamma) R$$
$$D' = \theta \alpha_s I_s.$$

MCMC with a deterministic SEIRS structure



$$S' = \mu + \gamma R - (\mu + f_\beta)\, S$$
$$E' = f_\beta S - (\kappa E + \mu E)$$
$$I_a' = p\kappa E - (\alpha_a + \mu)\, I_a$$
$$I_s' = (1-p)\kappa E - (\alpha_s + \mu) I_s$$
$$R' = \alpha_a I_a + \alpha_s (1-\theta) I_s - (\mu + \gamma) R$$
$$D' = \theta \alpha_s I_s.$$

$$f_\beta := \beta_s I_s + \beta_a I_a$$
$$\frac{S + E + I + R + D}{N} = 1$$
$$\beta_\bullet = \frac{\widehat{\beta}_\bullet}{N - D}$$

μ

$$Y_t \sim \mathrm{Poisson}(\lambda_t)$$
$$\lambda_t = \int_0^t (1-p)\kappa E$$
$$p \sim \mathrm{Uniform}(0.3, 08)$$
$$\kappa \sim \mathrm{Gamma}(10, 50)$$
$$\beta_a, \beta_s \sim \mathcal{N}(0.5, 0.1)$$

Overfiting example

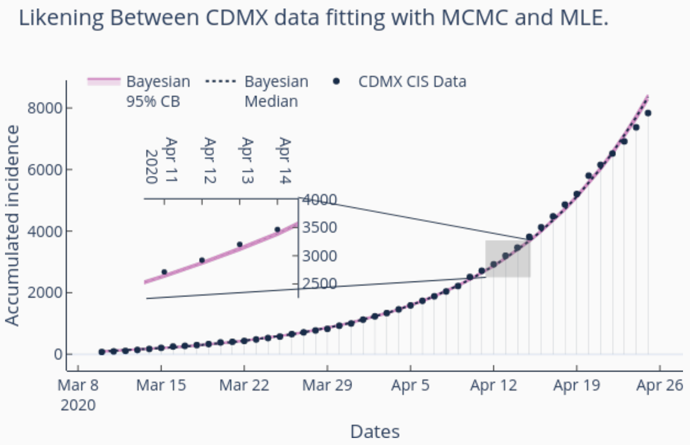


Figure: MCMC Fit of diary new cases of Mexico city during exponential growth. See `https://plotly.com/~sauld/53/` for an electronic version.

Stochastic perturbation

Perturbing the above deterministic base by Brownian Motion

$\mu\, dt \rightsquigarrow \mu\, dt + \sigma dW(t)$ gives our SDE SEIR-Covid-19

$$
\begin{aligned}
dS(t) &= \left[\mu - \mu S(t) - f_\beta S(t) + \gamma R(t)\right] dt + \sigma\left(1 - S(t)\right) dW(t) \\
dE(t) &= \left[f_\beta S(t) - \kappa E(t) - \mu E(t)\right] dt - \sigma E(t) dW(t) \\
dI_a(t) &= \left[p\kappa E(t) - (\alpha_a + \mu) I_a(t)\right] dt - \sigma I_a(t) dW(t) \\
dI_s(t) &= \left[(1-p)\kappa E(t) - (\alpha_s + \mu) I_s(t)\right] dt - \sigma I_s(t) dW(t) \\
dR(t) &= \left[\alpha_a I_a(t) + \alpha_s I_s(t) - (\mu + \gamma) R(t)\right] dt - \sigma R(t) dW(t), \\
& t \in [0, T].
\end{aligned}
$$

Let $\mathbb{P}_{\beta,p}$ the law of solution to SDE. We use the following result [2].

Theorem (Likelihood ratio of Itô processes Särkkä and Solin (2019, Thm. 7.4))

Consider the Itô processes

$$\begin{aligned} dx =& f(x,t) + dB_t, \qquad x(0) = x_0, \\ dy =& g(y,t) + dB_t, \qquad y(0) = x_0. \end{aligned}$$

Then the ratio of probability laws of $\mathscr{X}_t$ *and* $\mathscr{Y}_t$ *is given as*

$$\begin{aligned} \frac{p(\mathscr{X}_t)}{p(\mathscr{Y}_t)} =& Z(t), \\ Z(t) = \exp\Big(-\frac{1}{2}\int_0^t & [f(y,\tau) - g(y,\tau)]^\top \mathbb{Q}^{-1} [f(y,\tau) - g(y,\tau)] d\tau \\ & + \int_0^t [f(y,\tau) - g(y,\tau)]^\top \mathbb{Q}^{-1} dB_\tau \Big) \end{aligned}$$

in the sense that for an arbitrary functional $h(\cdot)$ *of the path from* 0 *to* t*,*

$$\mathbb{E}[h(\mathscr{X}_t)] = \mathbb{E}[Z(t)h(\mathscr{Y}_t)]$$

[2] Särkkä, Simo; Solin, Arno, Applied stochastic differential equations. Institute of Mathematical Statistics Textbooks, 10. Cambridge University Press, Cambridge, 2019. ix+316 pp. ISBN: 978-1-316-64946-6

Suppose we have the SDE

$$dX_t = a(t, X_t)dt + b(X_t)dW_t,$$

where the diffusion coefficient depends only on the state variable. Such SDE can transform into one with unitary diffusion by applying the *Lamperti* transform

$$Y_t := F(X_t) = \int_z^{X_t} \frac{1}{b(u)} du.$$

Here z is an arbitrary value and Y_t solves

$$dY_t = \left(\frac{a(t, X_t)}{b(X_t)} - \frac{1}{2} b_x(X_t) \right) dt + dW_t$$

The results follows form the Itô formula.

Using Itô and Lamperti transformations

$$
\begin{aligned}
dS(t) &= \left[\mu - \mu S(t) - f_\beta S(t) + \gamma R(t)\right] dt + \sigma\left(1 - S(t)\right) dW(t)\\
dE(t) &= \left[f_\beta S(t) - \kappa E(t) - \mu E(t)\right] dt - \sigma E(t) dW(t)\\
dI_a(t) &= \left[p\kappa E(t) - (\alpha_a + \mu) I_a(t)\right] dt - \sigma I_a(t) dW(t)\\
dI_s(t) &= \left[(1-p)\kappa E(t) - (\alpha_s + \mu) I_s(t)\right] dt - \sigma I_s(t) dW(t)\\
dR(t) &= \left[\alpha_a I_a(t) + \alpha_s I_s(t) - (\mu + \gamma) R(t)\right] dt - \sigma R(t) dW(t),\\
&\quad t \in [0, T].
\end{aligned}
$$

$$
-\frac{1}{\sigma} d\mathbf{X}_{\beta,p}(t) = F\left(\mathbf{X}_{\beta,p}(t)\right) dt + d\mathbf{W}(t),
$$

$$
\mathbf{X}_{\beta,p}(t) := \begin{pmatrix} \log\left(1 - S(t)\right) \\ \log\left(E(t)\right) \\ \log\left(I_a(t)\right) \\ \log\left(I_s(t)\right) \\ \log\left(R(t)\right) \end{pmatrix}, \quad
F(\mathbf{X}_{\beta,p}(t)) := \begin{pmatrix} \frac{\mu}{\sigma} - \frac{f_\beta S(t)}{\sigma(1 - S(t))} + \frac{\gamma R(t)}{\sigma(1 - S(t))} + \frac{1}{2}\sigma \\ -\frac{f_\beta S(t)}{\sigma E(t)} + \frac{\kappa}{\sigma} + \frac{\mu}{\sigma} + \frac{1}{2}\sigma \\ -\frac{\kappa p E(t)}{\sigma I_a(t)} + \frac{(\alpha_a + \mu)}{\sigma} + \frac{1}{2}\sigma \\ -\frac{\kappa(1-p)E(t)}{\sigma I_s(t)} + \frac{(\alpha_s + \mu)}{\sigma} + \frac{1}{2}\sigma \\ -\frac{\alpha_a I_a(t) + \alpha_s I_s(t)}{\sigma R(t)} + \frac{\mu + \gamma}{\sigma} + \frac{1}{2}\sigma \end{pmatrix}, \quad
d\mathbf{W}(t) := \begin{pmatrix} dW(t) \\ dW(t) \\ dW(t) \\ dW(t) \\ dW(t) \end{pmatrix}.
$$

MLE estimation

Define $f_\beta := (\beta_s I_s(t) + \beta_a I_a(t))$, thus

$$f_\beta - f_{\beta_0} = (\beta_s - \beta_{s,0}) I_s(t) + (\beta_a - \beta_{a,0}) I_a(t).$$

With this notation we write

$$[F(\mathbf{X}_{\beta,p}(t)) - F(\mathbf{X}_{\beta_0,p_0}(t))] = \begin{pmatrix} -(f_\beta - f_{\beta_0}) \dfrac{S(t)}{\sigma(1 - S(t))} \\ -(f_\beta - f_{\beta_0}) \dfrac{S(t)}{\sigma E(t)} \\ -(p - p_0) \dfrac{\kappa E(t)}{\sigma I_a(t)} \\ (p - p_0) \dfrac{\kappa E(t)}{\sigma I_s(t)} \\ 0 \end{pmatrix},$$

Then we obtain the likelihood (Radon-Nikodyn derivative)

$$
\begin{aligned}
\frac{d\mathbb{P}_\beta}{d\mathbb{P}_{\beta_0}} =& \exp\Bigg[\int_0^T [F(\mathbf{X}_{\beta,p}(t)) - F(\mathbf{X}_{\beta_0,p_0}(t))]^T Q^{-1} d\mathbf{W}(t) \\
&\qquad - \frac{1}{2}\int_0^T [F(\mathbf{X}_{\beta,p}(t)) - F(\mathbf{X}_{\beta_0,p_0}(t))]^T Q^{-1} [F(\mathbf{X}_{\beta,p}(t)) - F(\mathbf{X}_{\beta_0,p_0}(t))] d(t)\Bigg], \\
f_\beta =& (\beta_s I_s(t) + \beta_a I_a(t))
\end{aligned}
$$

$$
[F(\mathbf{X}_{\beta,p}(t)) - F(\mathbf{X}_{\beta_0,p_0}(t))] = \begin{pmatrix} -(f_\beta - f_{\beta_0})\dfrac{S(t)}{\sigma(1-S(t))} \\ -(f_\beta - f_{\beta_0})\dfrac{S(t)}{\sigma E(t)} \\ -(p-p_0)\dfrac{\kappa E(t)}{\sigma I_a(t)} \\ (p-p_0)\dfrac{\kappa E(t)}{\sigma I_s(t)} \\ 0 \end{pmatrix}, \qquad \mathbb{Q} = \mathbb{I}_5 \text{ (identity)}
$$

Therefore we can estimate $\widehat{\varphi} = (\beta_s, \beta_a, p)$ by maximizing $-\log(\text{likelihood})$.

For exmple, to estimte p, we derive the $-\log(\text{likelihood})$ with respect to p and deduce an expresion to find a extrema.

$$(p-p_0)\underbrace{\left(\int_0^T \left[\frac{\kappa^2 E^2(t)}{I_s^2(t)} + \frac{\kappa^2 E^2(t)}{I_a^2(t)}\right] dt\right)}_{:=J_2(T)} - \sigma \int_0^T \left[-\frac{\kappa E(t)}{I_a(t)} + \frac{\kappa E(t)}{I_s(t)}\right] dW(t) = 0$$

$$\hat{p}_{ML} - p_0 = \frac{\sigma}{J_2(T)} \int_0^T \left[-\frac{\kappa E(t)}{I_a(t)} + \frac{\kappa E(t)}{I_s(t)}\right] dW(t),$$

Estimator consistency

Let $X_0^+ := \{(S(t_0), E(t_0), I_a(t_0), I_s(t_0))\}$ initial state where all populations classes are strictly positive. Denote by $\varphi := \{\mu, \beta_s, \beta_a, \kappa, p, \theta, \alpha_s, \alpha_a, \gamma\}$, a model parameter configuration. The reproductive number for the deterministic version $(\sigma = 0)$

$$\mathscr{R}_0^D := \frac{p\kappa\beta_s}{(\mu+\kappa)(\mu+\alpha_s)} + \frac{(1-p)\kappa\beta_a}{(\mu+\kappa)(\mu+\alpha_a)}.$$

Define

$$\Omega^* := \{(S, E, I_a, I_s, R) \times [t_0, T] : S(T) \leq S(t) < S(t_0), \\ E(t) > E(t_0),\ I_a(t) > I_a(t_0),\ I_s(t) > I_s(t_0)\}.$$

Theorem

Let $T_0 > 0$ such that for all $t \in [0, T_0]$

i. *The deterministic threshold $\mathscr{R}_0^D > 1$*

ii. *The initial condition X_0^+ and parameters configuration φ are such that $\Omega^* \neq \emptyset$*

Then, the estimators $(\hat{\beta}_{s,ML}, \hat{\beta}_{a,ML}, \hat{p}_{ML})$ are strongly consistent, that is,

$$\lim_{T \to T_0} \begin{pmatrix} \hat{\beta}_{s,ML} \\ \hat{\beta}_{a,ML} \\ \hat{p}_{ML} \end{pmatrix} = \begin{pmatrix} \beta_{s,0} \\ \beta_{a,0} \\ p_0 \end{pmatrix}, \qquad \textit{w.p.1.}$$

To estimate the parameter of diffusion σ

To estimate the parameter of diffusion σ, we use the quadratic variation over $[0, T]$, $< *, * >_T$, of the solution processes

$$\hat{\sigma}^2 := \sum_{i=1}^{5} \frac{\hat{\sigma}_i^2}{5},$$

where

$$\hat{\sigma}_1^2 = \frac{< S, S >_T}{\int_0^T (1 - S(t))^2 dt}, \qquad \hat{\sigma}_2^2 = \frac{< E, E >_T}{\int_0^T E(t)^2 dt}, \qquad \hat{\sigma}_3^2 = \frac{< I_a, I_a >_T}{\int_0^T I_a(t)^2 dt},$$

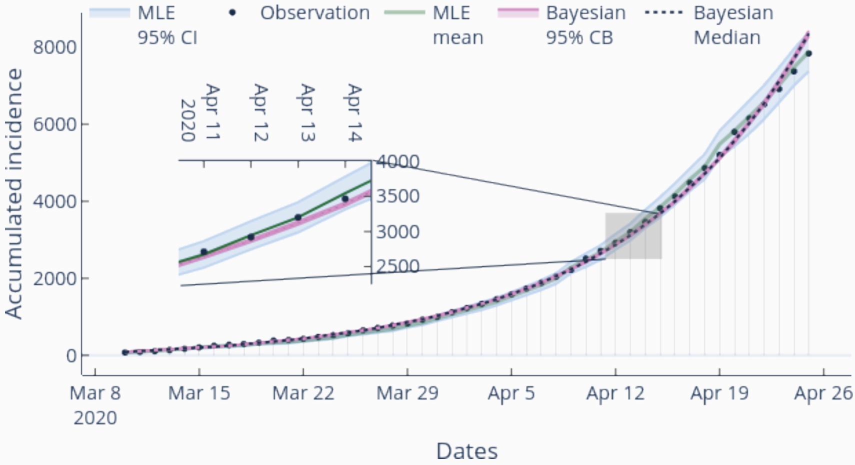
$$\hat{\sigma}_4^2 = \frac{< I_s, I_s >_T}{\int_0^T I_s(t)^2 dt}, \qquad \hat{\sigma}_5^2 = \frac{< R, R >_T}{\int_0^T R(t)^2 dt}.$$

Algorithm 1 Approximation by Euler-Mayurama. $I_s^{mx}(t_n)$ observation data.

1: Fix $E(0), I_a(0)$ and $R(0)$, $S(0) = 1 - E(0) - I_a(0) - R(0) - I_s^{mx}$, and make $n = 0$.
2: Generate $\Delta W \sim N(0, \Delta)$.
3:

$$
\begin{aligned}
S(t_{n+1}) =& S(t_n) + [\mu + \gamma R(t_n) - (\mu + \beta_a I_a(t_n) + \beta_s I_s^{mx}(t_n)) S(t_n)]\Delta \\
&+ \sigma(1 - S(t_n))\Delta W \\
E(t_{n+1}) =& E(t_n) + [(\beta_a I_a(t_n) + \beta_s I_s^{mx}(t_n)) S(t_n) - (\kappa + \mu) E(t_n)]\Delta \\
&- \sigma(E(t_n))\Delta W \\
I_a(t_{n+1}) =& I_a(t_n) + (p\kappa E(t_n) - \alpha_a + \mu) I_a(t_n)\Delta - \sigma I_a(t_n)\Delta w \\
R(t_{n+1}) =& 1 - S(t_{n+1} - E(t_{n+1}) - I_a(t_{n+1}) - I_s^{mx}(t_{n+1})
\end{aligned}
$$

Likening Between CDMX data fitting with MCMC and MLE.



Discrete time

Closed Loop Policies

Games

Git-Hub