12
Lipschitz Continuity

> Calculus required continuity, and continuity was supposed to require the infinitely little, but nobody could discover what the infinitely little might be. (Russell)

12.1 Introduction

When we graph a function $f(x)$ of a rational variable x, we make a leap of faith and assume that the function values $f(x)$ vary "smoothly" or "continuously" between the sample points x, so that we can draw the graph of the function without lifting the pen. In particular, we assume that the function value $f(x)$ does not make unknown sudden jumps for some values of x. We thus assume that the function value $f(x)$ changes by a small amount if we change x by a small amount. A basic problem in Calculus is to measure how much the function values $f(x)$ may change when x changes, that is, to measure the "degree of continuity" of a function. In this chapter, we approach this basic problem using the concept of *Lipschitz continuity*, which plays a basic role in the version of Calculus presented in this book.

There will be a lot of inequalities ($<$ and $\leq$) and absolute values ($|\cdot|$) in this chapter, so it might be a good idea before you start to review the rules for operating with these symbols from Chapter *Rational numbers*.

Fig. 12.1. Rudolph Lipschitz (1832–1903), Inventor of Lipschitz continuity: "Indeed, I have found a very nice way of expressing continuity..."

12.2 The Lipschitz Continuity of a Linear Function

To start with we consider the behavior of a linear polynomial. The value of a constant polynomial doesn't change when we change the input, so the linear polynomial is the first interesting example to consider. Suppose the linear function is $f(x) = mx + b$, with $m \in \mathbb{Q}$ and $b \in \mathbb{Q}$ given, and let $f(x_1) = mx_1 + b$ and $f(x_2) = mx_2 + b$ to be the function values values for $x = x_1$ and $x = x_2$. The change in the input is $|x_2 - x_1|$ and for the corresponding change in the output $|f(x_1) - f(x_2)|$, we have

$$|f(x_2) - f(x_1)| = |(mx_2 + b) - (mx_1 + b)| = |m(x_2 - x_1)| = |m||x_2 - x_1|. \tag{12.1}$$

In other words, the absolute value of the change in the function values $|f(x_2) - f(x_1)|$ is proportional to the absolute value of the change in the input values $|x_2 - x_1|$ with constant of proportionality equal to the slope $|m|$. In particular, this means that we can make the change in the output arbitrarily small by making the change in the input small, which certainly fits our intuition that a linear function varies continuously.

Example 12.1. Let $f(x) = 2x$ give the total number of miles for an "out and back" bicycle ride that is x miles one way. To increase a given ride by a total of 4 miles, we increase the one way distance x by $4/2 = 2$ miles while to increase a ride by a total of .01 miles, we increase the one way distance x by .005 miles.

We now make an important observation: the slope m of the linear function $f(x) = mx + b$ determines how much the function values change as the input value x changes. The larger $|m|$ is, the steeper the line is, and the more the function changes for a given change in input. We illustrate in Fig. 12.2.

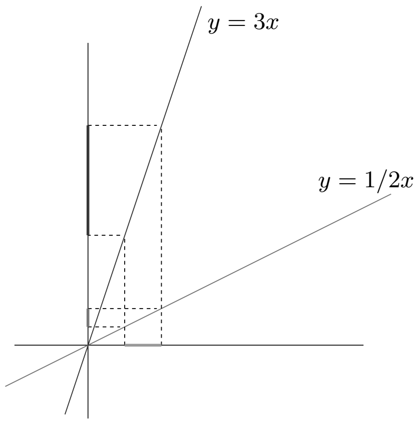

Fig. 12.2. These two linear functions which change a different amount for a given change in input

Example 12.2. Suppose that $f_1(x) = 4x + 1$ while $f_2(x) = 100x - 5$. To increase the value of $f_1(x)$ at x by an amount of .01, we change the value of x by $.01/4 = .0025$. On the other hand, to change the value of $f_2(x)$ at x by an amount of .01, we change the value of x by $.01/100 = .0001$.

12.3 The Definition of Lipschitz Continuity

We are now prepared to introduce the concept of Lipschitz continuity, designed to measure change of function values versus change in the independent variable for a general function $f : I \to \mathbb{Q}$ where I is a set of rational numbers. Typically, I may be an interval of rational numbers $\{x \in \mathbb{Q} : a \leq x \leq b\}$ for some rational numbers a and b. If x_1 and x_2 are two numbers in I, then $|x_2 - x_1|$ is the change in the input and $|f(x_2) - f(x_1)|$ is the corresponding change in the output. We say that f is *Lipschitz continuous* with *Lipschitz constant* L_f on I, if there is a (necessarily nonnegative) constant L_f such that

$$|f(x_1) - f(x_2)| \leq L_f|x_1 - x_2| \quad \text{for all } x_1, x_2 \in I. \tag{12.2}$$

As indicated by the notation, the Lipschitz constant L_f depends on the function f, and thus may vary from being small for one function to be large for another function. If L_f is small, then $f(x)$ may change only a little with a small change of x, while if L_f is large, then $f(x)$ may change a lot under only a small change of x. Again: L_f may vary from small to large depending on the function f.

Example 12.3. A linear function $f(x) = mx + b$ is Lipschitz continuous with Lipschitz constant $L_f = |m|$ on the entire set of rational numbers $\mathbb{Q}$.

Example 12.4. We show that $f(x) = x^2$ is Lipschitz continuous on the interval $I = [-2, 2]$ with Lipschitz constant $L_f = 4$. We choose two rational numbers x_1 and x_2 in $[-2, 2]$. The corresponding change in the function values is

$$|f(x_2) - f(x_1)| = |x_2^2 - x_1^2|.$$

The goal is to estimate this in terms of the difference in the input values $|x_2 - x_1|$. Using the identity for products of polynomials derived in Section 10.6, we get

$$|f(x_2) - f(x_1)| = |x_2 + x_1|\,|x_2 - x_1|. \tag{12.3}$$

We have the desired difference on the right, but it is multiplied by a factor that depends on x_1 and x_2. In contrast, the analogous relationship (12.1) for the linear function has a factor that is constant, namely $|m|$. At this point, we have to use the fact that x_1 and x_2 are in the interval $[-2, 2]$, which means that

$$|x_2 + x_1| \le |x_2| + |x_1| \le 2 + 2 = 4,$$

by the triangle inequality. We conclude that

$$|f(x_2) - f(x_1)| \le 4|x_2 - x_1|$$

for all x_1 and x_2 in $[-2, 2]$.

Lipschitz continuity quantifies the idea of continuous behavior of a function $f(x)$ using the Lipschitz constant L_f. We repeat: If L_f is moderately sized then small changes in input x yield small changes in the function's output $f(x)$, but a large Lipschitz constant means that the function's values $f(x)$ may make a large change when the input values x change by only a small amount.

However it is important to note that there is a certain amount of imprecision inherit to the definition of Lipschitz continuity (12.2) and we have to be circumspect about drawing conclusions when the Lipschitz constant is large. The reason is that (12.2) is only an **upper estimate** on how much the function changes and the actual change might be much smaller than indicated by the constant.

Example 12.5. From Example 12.4, we know that $f(x) = x^2$ is Lipschitz continuous on $I = [-2, 2]$ with Lipschitz constant $L_f = 4$. It is also Lipschitz constant on I with Lipschitz constant $L_f = 121$ since

$$|f(x_2) - f(x_1)| \le 4|x_2 - x_1| \le 121|x_2 - x_1|.$$

But the second value of L_f greatly overestimates the change in f, whereas the value $L_f = 4$ is just about right when x_1 and x_2 are near 2 since $2^2 - 1.9^2 = .39 = 3.9 \times (2 - 1.9)$ and $3.9 \approx 4$.

To determine the Lipschitz constant, we have to make some estimates and the result can vary greatly depending on how difficult the estimates are to compute and our skill at making estimates.

It is also important to note that the size and location of the interval in the definition is important and if we change the interval then we expect to get a different Lipschitz constant L_f.

Example 12.6. We show that $f(x) = x^2$ is Lipschitz continuous on the interval $I = [2, 4]$, with Lipschitz constant $L_f = 8$. Starting with (12.3), for x_1 and x_2 in $[2, 4]$ we have

$$|x_2 + x_1| \le |x_2| + |x_1| \le 4 + 4 = 8$$

so

$$|f(x_2) - f(x_1)| \le 8|x_2 - x_1|$$

for all x_1 and x_2 in $[2, 4]$.

The reason that the Lipschitz constant is bigger in the second example is clear from the graph, see Fig. 12.3, where we show the change in f corresponding to equal changes in x near $x = 2$ and $x = 4$. Because $f(x) = x^2$ is steeper near $x = 4$, f changes more near $x = 4$ for a given change in input.

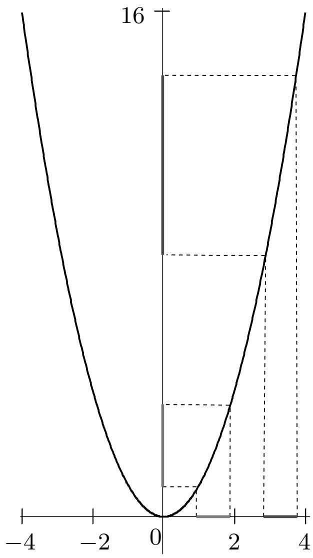

Fig. 12.3. The change in $f(x) = x^2$ for equal changes in x near $x = 2$ and $x = 4$

Example 12.7. $f(x) = x^2$ is Lipschitz continuous on $I = [-8, 8]$ with Lipschitz constant $L_f = 16$ and on $I = [-400, 200]$ with $L_f = 800$.

In all of the examples involving $f(x) = x^2$, we use the fact that the interval under consideration is of finite size. A set of rational numbers I is *bounded* with size a if $|x| \leq a$ for all x in I, for some (finite) rational number a.

Example 12.8. The set of rational numbers $I = [-1, 500]$ is bounded but the set of even integers is not bounded.

While linear functions are Lipschitz continuous on the unbounded set $\mathbb{Q}$, functions that are not linear are usually only Lipschitz continuous on bounded sets.

Example 12.9. The function $f(x) = x^2$ is **not** Lipschitz continuous on the set $\mathbb{Q}$ of rational numbers. This follows from (12.3) because $|x_1 + x_2|$ can be made arbitrarily large by choosing x_1 and x_2 freely in $\mathbb{Q}$, so it is not possible to find a constant L_f such that

$$|f(x_2) - f(x_1)| = |x_2 + x_1||x_2 - x_1| \leq L_f|x_2 - x_1|$$

for all x_1 and x_2 in $\mathbb{Q}$.

The definition of Lipschitz continuity is due to the German mathematician Rudolph Lipschitz (1832–1903), who used his concept of continuity to prove existence of solutions to some important differential equations. This is not the usual definition of continuity used in Calculus courses, which is purely qualitative, while Lipschitz continuity is quantitative. Of course there is a strong connection, and a function which is Lipschitz continuous is also continuous according to the usual definition of continuity, while the opposite may not be true: Lipschitz continuity is a somewhat more demanding property. However, quantifying continuous behavior in terms of Lipschitz continuity simplifies many aspects of mathematical analysis and the use of Lipschitz continuity has become ubiquitous in engineering and applied mathematics. It also has the benefit of eliminating some rather technical issues in defining continuity that are tricky yet unimportant in practice.

12.4 Monomials

Continuing the investigation of continuous functions, we next show that the monomials are Lipschitz continuous on bounded intervals, as we expect based on their graphs.

Example 12.10. We show that the function $f(x) = x^4$ is Lipschitz continuous on $I = [-2, 2]$ with Lipschitz constant $L_f = 32$. We choose x_1 and x_2 in I and we want to estimate

$$|f(x_2) - f(x_1)| = |x_2^4 - x_1^4|$$

in terms of $|x_2 - x_1|$.

To do this we first show that

$$x_2^4 - x_1^4 = (x_2 - x_1)(x_2^3 + x_2^2 x_1 + x_2 x_1^2 + x_1^3)$$

by multiplying out

$$\begin{aligned}(x_2 - x_1)&(x_2^3 + x_2^2 x_1 + x_2 x_1^2 + x_1^3) \\ &= x_2^4 + x_2^3 x_1 + x_2^2 x_1^2 + x_2 x_1^3 - x_2^3 x_1 - x_2^2 x_1^2 - x_2 x_1^3 - x_1^4\end{aligned}$$

and then cancelling the terms in the middle to get $x_2^4 - x_1^4$.

This means that

$$|f(x_2) - f(x_1)| = |x_2^3 + x_2^2 x_1 + x_2 x_1^2 + x_1^3|\,|x_2 - x_1|.$$

We have the desired difference $|x_2 - x_1|$ on the right and we just have to bound the factor $|x_2^3 + x_2^2 x_1 + x_2 x_1^2 + x_1^3|$. By the triangle inequality

$$|x_2^3 + x_2^2 x_1 + x_2 x_1^2 + x_1^3| \le |x_2|^3 + |x_2|^2|x_1| + |x_2||x_1|^2 + |x_1|^3.$$

Now because x_1 and x_2 are in I, $|x_1| \le 2$ and $|x_2| \le 2$, so

$$|x_2^3 + x_2^2 x_1 + x_2 x_1^2 + x_1^3| \le 2^3 + 2^2\,2 + 2\,2^2 + 2^3 = 32$$

and

$$|f(x_2) - f(x_1)| \le 32|x_2 - x_1|.$$

Recall that the Lipschitz constant of $f(x) = x^2$ on I is $L_f = 4$. The fact that the Lipschitz constant of x^4 is larger than the constant for x^2 on $[-2, 2]$ is not surprising considering the plots of the two functions, see Fig. 10.12.

We can use the same technique to show that the function $f(x) = x^m$ is Lipschitz continuous where m is any natural number.

Example 12.11. The function $f(x) = x^m$ is Lipschitz continuous on any interval $I = [-a, a]$, where a is a positive rational number, with Lipschitz constant $L_f = ma^{m-1}$. Given x_1 and x_2 in I, we want to estimate

$$|f(x_2) - f(x_1)| = |x_2^m - x_1^m|$$

in terms of $|x_2 - x_1|$. We can do this using the fact that

$$\begin{aligned}x_2^m - x_1^m &= (x_2 - x_1)(x_2^{m-1} + x_2^{m-2} x_1 + \cdots + x_2 x_1^{m-2} + x_1^{m-1}) \\ &= (x_2 - x_1) \sum_{i=0}^{m-1} x_2^{m-1-i} x_1^i.\end{aligned}$$

We show this by first multiplying out

$$(x_2 - x_1) \sum_{i=0}^{m-1} x_2^{m-1-i} x_1^i = \sum_{i=0}^{m-1} x_2^{m-i} x_1^i - \sum_{i=0}^{m-1} x_2^{m-1-i} x_1^{i+1}$$

To see that there is a lot of cancellation among the terms in the middle in the two sums on the right, we separate the first term out of the first sum and the last term in the second sum

$$(x_2 - x_1) \sum_{i=0}^{m-1} x_2^{m-1-i} x_1^i = x_2^m + \sum_{i=1}^{m-1} x_2^{m-i} x_1^i - \sum_{i=0}^{m-2} x_2^{m-1-i} x_1^{i+1} - x_1^m$$

and then changing the index in the second sum to get

$$\begin{aligned}(x_2 - x_1) &\sum_{i=0}^{m-1} x_2^{m-1-i} x_1^i \\ &= x_2^m + \sum_{i=1}^{m-1} x_2^{m-i} x_1^i - \sum_{i=1}^{m-1} x_2^{m-i} x_1^i - x_1^m = x_2^m - x_1^m.\end{aligned}$$

This is tedious, but it is good practice to go through the details and make sure this argument is correct.

This means that

$$|f(x_2) - f(x_1)| = \left| \sum_{i=0}^{m-1} x_2^{m-1-i} x_1^i \right| |x_2 - x_1|.$$

We have the desired difference $|x_2 - x_1|$ on the right and we just have to bound the factor

$$\left| \sum_{i=0}^{m-1} x_2^{m-1-i} x_1^i \right|.$$

By the triangle inequality

$$\left| \sum_{i=0}^{m-1} x_2^{m-1-i} x_1^i \right| \le \sum_{i=0}^{m-1} |x_2|^{m-1-i} |x_1|^i.$$

Now because x_1 and x_2 are in $[-a, a]$, $|x_1| \le a$ and $|x_2| \le a$. So

$$\left| \sum_{i=0}^{m-1} x_2^{m-1-i} x_1^i \right| \le \sum_{i=0}^{m-1} a^{m-1-i} a^i = \sum_{i=0}^{m-1} a^{m-1} = m a^{m-1}.$$

and

$$|f(x_2) - f(x_1)| \le m a^{m-1} |x_2 - x_1|.$$

12.5 Linear Combinations of Functions

Now that we have seen that the monomials are Lipschitz continuous on a given bounded interval, it is a short step to show that any polynomial is Lipschitz continuous on a given bounded interval. But rather than just do this for polynomials, we show that a linear combination of arbitrary Lipschitz continuous functions is Lipschitz continuous

Suppose that f_1 is Lipschitz continuous with constant L_1 and f_2 is Lipschitz continuous with constant L_2 on the interval I. Note that here (and below) we condense the notation and write e.g. L_1 instead of L_{f_1}. Then $f_1 + f_2$ is Lipschitz continuous with constant $L_1 + L_2$ on I, because if we choose two points x and y in I, then

$$\begin{aligned}
|(f_1 + f_2)(y) - (f_1 + f_2)(x)| &= |(f_1(y) - f_1(x)) + (f_2(y) - f_2(x))| \\
&\leq |f_1(y) - f_1(x)| + |f_2(y) - f_2(x)| \\
&\leq L_1|y - x| + L_2|y - x| \\
&= (L_1 + L_2)|y - x|
\end{aligned}$$

by the triangle inequality. The same argument shows that $f_2 - f_1$ is Lipschitz continuous with constant $L_1 + L_2$ as well (not $L_1 - L_2$ of course!). It is even easier to show that if $f(x)$ is Lipschitz continuous on an interval I with Lipschitz constant L then $cf(x)$ is Lipschitz continuous on I with Lipschitz constant $|c|L$.

From these two facts, it is a short step to extend the result to any linear combination of Lipschitz continuous functions. Suppose that $f_1, \cdots, f_n$ are Lipschitz continuous on I with Lipschitz constants $L_1, \cdots, L_n$ respectively. We use induction, so we begin by considering the linear combination of two functions. From the remarks above, it follows that $c_1 f_1 + c_2 f_2$ is Lipschitz continuous with constant $|c_1|L_1 + |c_2|L_2$. Next given $i \leq n$, we assume that $c_1 f_1 + \cdots + c_{i-1} f_{i-1}$ is Lipschitz continuous with constant $|c_1|L_1 + \cdots + |c_{i-1}|L_{i-1}$. To prove the result for i, we write

$$c_1 f_1 + \cdots + c_i f_i = \big(c_1 f_1 + \cdots + c_{i-1} f_{i-1}\big) + c_n f_n.$$

But the assumption on $\big(c_1 f_1 + \cdots + c_{i-1} f_{i-1}\big)$ means that we have written $c_1 f_1 + \cdots + c_i f_i$ as the sum of two Lipschitz continuous functions, namely $\big(c_1 f_1 + \cdots + c_{i-1} f_{i-1}\big)$ and $c_n f_n$. The result follows by the result for the linear combination of two functions. By induction, we have proved

Theorem 12.1 *Suppose that $f_1, \cdots, f_n$ are Lipschitz continuous on I with Lipschitz constants $L_1, \cdots, L_n$ respectively. Then the linear combination $c_1 f_1 + \cdots + c_n f_n$ is Lipschitz continuous on I with Lipschitz constant $|c_1|L_1 + \cdots + |c_n|L_n$.*

Corollary 12.2 *A polynomial is Lipschitz continuous on any bounded interval.*

Example 12.12. We show that the function $f(x) = x^4 - 3x^2$ is Lipschitz continuous on $[-2, 2]$, with constant $L_f = 44$. For x_1 and x_2 in $[-2, 2]$, we have to estimate

$$\begin{aligned} |f(x_2) - f(x_1)| &= \left|\left(x_2^4 - 3x_2^2\right) - \left(x_1^4 - 3x_1^2\right)\right| \\ &= \left|\left(x_2^4 - x_1^4\right) - \left(3x_2^2 - 3x_1^2\right)\right| \\ &\leq \left|x_2^4 - x_1^4\right| + 3\left|x_2^2 - x_1^2\right|. \end{aligned}$$

From Example 12.11, we know that x^4 is Lipschitz continuous on $[-2, 2]$ with constant 32 while x^2 is Lipschitz continuous on $[-2, 2]$ with Lipschitz constant 4. Therefore

$$|f(x_2) - f(x_1)| \leq 32|x_2 - x_1| + 3 \times 4|x_2 - x_1| = 44|x_2 - x_1|.$$

12.6 Bounded Functions

Lipschitz continuity is related to another important property of a function called boundedness. A function f is *bounded* on a set of rational numbers I if there is a constant M such that, see Fig. 12.4

$$|f(x)| \leq M \text{ for all } x \text{ in } I.$$

In fact if we think about the estimates we have made to verify the definition of Lipschitz continuity (12.2), we see that in every case these involved showing that some function is bounded on the given interval.

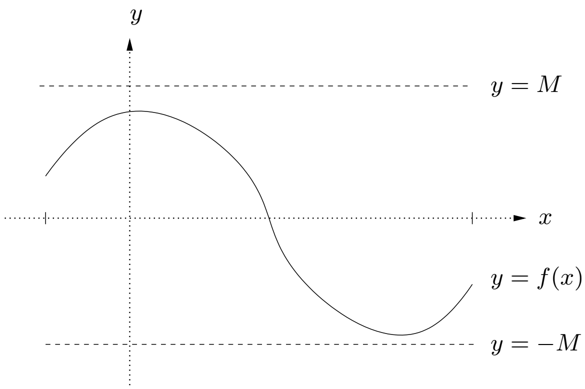

Fig. 12.4. A bounded function on I

Example 12.13. To show that $f(x) = x^2$ is Lipschitz continuous on $[-2, 2]$ in Example 12.4, we proved that $|x_1 + x_2| \leq 4$ for x_1 and x_2 in $[-2, 2]$.

It turns out that a function that is Lipschitz continuous on a bounded domain is automatically bounded on that domain. To be more precise, suppose that a function f is Lipschitz continuous with Lipschitz constant L_f on a bounded set I with size a and choose a point y in I. Then for any other point x in I

$$|f(x) - f(y)| \leq L_f|x - y|.$$

First we know that $|x - y| \leq |x| + |y| \leq 2a$. Also, since $|b + c| \leq |d|$ means that $|b| \leq |d| + |c|$ for any numbers a, b, c, we get

$$|f(x)| \leq |f(y)| + L_f|x - y| \leq |f(y)| + 2L_f a.$$

Even though we don't know $|f(y)|$, we do know that it is finite. This shows that $|f(x)|$ is bounded by the constant $M = |f(y)| + 2L_f a$ for any x in Q. We express this by saying that $f(x)$ is *bounded on* I. We have thus proved

Theorem 12.3 *A Lipschitz continuous function on a bounded set I is bounded on I.*

Example 12.14. In Example 12.12, we showed that $f(x) = x^4 + 3x^2$ is Lipschitz continuous on $[-2, 2]$ with Lipschitz constant $L_f = 44$. Using this argument, we find that

$$|f(x)| \leq |f(0)| + 44|x - 0| \leq 0 + 44 \times 2 = 88$$

for any x in $[-2, 2]$. Since x^4 is increasing for $0 \leq x$, in fact we know that $|f(x)| \leq |f(2)| = 16$ for any x in $[-2, 2]$. So the estimate on the size of $|f|$ using the Lipschitz constant is not very accurate.

12.7 The Product of Functions

The next step in investigating which functions are Lipschitz continuous is to consider the product of two Lipschitz continuous functions on a bounded interval I. We show that the product is also Lipschitz continuous on I. More precisely, if f_1 is Lipschitz continuous with constant L_1 and f_2 is Lipschitz continuous with constant L_2 on a bounded interval I then $f_1 f_2$ is Lipschitz continuous on I. We choose two points x and y in I and estimate by using the old trick of adding and subtracting the same quantity

$$\begin{aligned}
&|f_1(y)f_2(y) - f_1(x)f_2(x)| \\
&\qquad = |f_1(y)f_2(y) - f_1(y)f_2(x) + f_1(y)f_2(x) - f_1(x)f_2(x)| \\
&\qquad \leq |f_1(y)f_2(y) - f_1(y)f_2(x)| + |f_1(y)f_2(x) - f_1(x)f_2(x)| \\
&\qquad = |f_1(y)|\,|f_2(y) - ff_2(x)| + |f_2(x)|\,|f_1(y) - f_1(x)|
\end{aligned}$$

Now Theorem 12.3, which says that Lipschitz continuous functions are bounded, implies there is some constant M such that $|f_1(y)| \leq M$ and

$|f_2(x)| \leq M$ for $x, y \in I$. Using the Lipschitz continuity of f_1 and f_2 in I, we find

$$\begin{aligned}|f_1(y)f_2(y) - f_1(x)f_2(x)| &\leq ML_1|y-x| + ML_2|y-x| \\ &= M(L_1 + L_2)|y-x|.\end{aligned}$$

We summarize

Theorem 12.4 *If f_1 and f_2 are Lipschitz continuous on a bounded interval I then $f_1 f_2$ is Lipschitz continuous on I.*

Example 12.15. The function $f(x) = (x^2+5)^{10}$ is Lipschitz continuous on the set $I = [-10, 10]$ because $x^2 + 5$ is Lipschitz continuous on I and therefore $(x^2+5)^{10} = (x^2+5)(x^2+5)\cdots(x^2+5)$ is as well by Theorem 12.4.

12.8 The Quotient of Functions

Continuing our investigation, we now consider the ratio of two Lipschitz continuous functions. In this case however, we require more information about the function in the denominator than just that it is Lipschitz continuous. We also have to know that it does not become too small. To understand this, we first consider an example.

Example 12.16. We show that $f(x) = 1/x^2$ is Lipschitz continuous on the interval $[1/2, 2]$, with Lipschitz constant $L = 64$. We choose two points x_1 and x_2 in Q and we estimate the change

$$|f(x_2) - f(x_1)| = \left|\frac{1}{x_2^2} - \frac{1}{x_1^2}\right|$$

by first doing some algebra

$$\frac{1}{x_2^2} - \frac{1}{x_1^2} = \frac{x_1^2}{x_1^2 x_2^2} - \frac{x_2^2}{x_1^2 x_2^2} = \frac{x_1^2 - x_2^2}{x_1^2 x_2^2} = \frac{(x_1+x_2)(x_1-x_2)}{x_1^2 x_2^2}.$$

This means that

$$|f(x_2) - f(x_1)| = \left|\frac{x_1+x_2}{x_1^2 x_2^2}\right| |x_2 - x_1|.$$

Now we have the good difference on the right, we just have to bound the factor. The numerator of the factor is the same as in Example 12.4, and we know that

$$|x_1 + x_2| \leq 4.$$

We also know that

$$x_1 \geq \frac{1}{2} \text{ implies } \frac{1}{x_1} \leq 2 \text{ implies } \frac{1}{x_1^2} \leq 4$$

and likewise $\frac{1}{x_2^2} \leq 4$. So we get

$$|f(x_2) - f(x_1)| \leq 4 \times 4 \times 4\,|x_2 - x_1| = 64|x_2 - x_1|.$$

In this example, we have to use the fact that the left-hand endpoint of the interval I is $1/2$. The closer the left-hand endpoint is to zero, the larger the Lipschitz constant will be. In fact, $1/x^2$ is **not** Lipschitz continuous on $[0, 2]$.

We mimic this example in the general case f_1/f_2 by assuming that the denominator f_2 is *bounded below* by a positive constant. We give the proof of the following theorem as an exercise.

Theorem 12.5 *Assume that f_1 and f_2 are Lipschitz continuous functions on a bounded set I with constants L_1 and L_2 and moreover assume there is a constant $m > 0$ such that $|f_2(x)| \geq m$ for all x in I. Then f_1/f_2 is Lipschitz continuous on I.*

Example 12.17. The function $1/x^2$ does not satisfy the assumptions of Theorem 12.5 on the interval $[0, 2]$ and we know that it is not Lipschitz continuous on that interval.

12.9 The Composition of Functions

We conclude the investigation into Lipschitz continuity by considering the composition of Lipschitz continuous functions. This is actually easier than either products or ratios of functions. The only complication is that we have to be careful about the domains and ranges of the functions. Consider the composition $f_2(f_1(x))$. Presumably, we have to restrict x to an interval on which f_1 is Lipschitz continuous and we also have to make sure that the values of f_1 are in a set on which f_2 is Lipschitz continuous.

So we assume that f_1 is Lipschitz continuous on I_1 with constant L_1 and that f_2 is Lipschitz continuous on I_2 with constant L_2. If x and y are points in I_1 then as long as $f_1(x)$ and $f_1(y)$ are in I_2 then

$$|f_2(f_1(y)) - f_2(f_1(x))| \leq L_2|f_1(y) - f_1(x)| \leq L_1L_2|y - x|.$$

We summarize as a theorem.

Theorem 12.6 *Let f_1 be Lipschitz continuous on a set I_1 with Lipschitz constant L_1 and f_2 be Lipschitz continuous on I_2 with Lipschitz constant L_2 such that $f_1(I_1) \subset I_2$. Then the composite function $= f_2 \circ f_1$ is Lipschitz continuous on I_1 with Lipschitz constant L_1L_2.*

Example 12.18. The function $f(x) = (2x - 1)^4$ is Lipschitz continuous on any bounded interval since $f_1(x) = 2x - 1$ and $f_2(x) = x^4$ are Lipschitz continuous on any bounded interval. If we consider the interval $[-.5, 1.5]$ then $f_1(I) \subset [-2, 2]$. From Example 12.10, we know that x^4 is Lipschitz continuous on $[-2, 2]$ with Lipschitz constant 32 while the Lipschitz constant of $2x - 1$ is 2. Therefore, f is Lipschitz continuous on $[-.5, 1.5]$ with constant 64.

Example 12.19. The function $1/(x^2 - 4)$ is Lipschitz continuous on any closed interval that does not contain either 2 or -2. This follows because $f_1(x) = x^2 - 4$ is Lipschitz continuous on any bounded interval while $f_2(x) = 1/x$ is Lipschitz continuous on any closed interval that does not contain 0. To avoid zero, we must avoid $x^2 = 4$ or $x = \pm 2$.

12.10 Functions of Two Rational Variables

Until now, we have considered functions $f(x)$ of one rational variable x. But of course, there are functions that depend on more than one input. Consider for example the function

$$f(x_1, x_2) = x_1 + x_2,$$

which to each pair of rational numbers x_1 and x_2 associates the sum x_1+x_2. We may write this as $f : \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$, meaning that to each $x_1 \in Q$ and $x_2 \in \mathbb{Q}$ we associate a value $f(x_1, x_2) \in \mathbb{Q}$. For example, $f(x_1, x_2) = x_1+x_2$. We say that $f(x_1, x_2)$ is a *function of two independent rational variables* x_1 *and* x_2. Here, we think of $\mathbb{Q} \times \mathbb{Q}$ as the set of all pairs (x_1, x_2) with $x_1 \in \mathbb{Q}$ and $x_2 \in \mathbb{Q}$.

We shall write $\mathbb{Q}^2 = \mathbb{Q}\times\mathbb{Q}$ and consider $f(x_1, x_2) = x_1+x_2$ as a function $f : \mathbb{Q}^2 \to \mathbb{Q}$. We will also consider functions $f : I \times J \to Q$, where I and J are subsets such as intervals, of Q. This just means that for each $x_1 \in I$ and $x_2 \in J$, we associate a value $f(x_1, x_2) \in \mathbb{Q}$.

We may naturally extend the concept of Lipschitz continuity to functions of two rational variables. We say that $f : I \times J \to \mathbb{Q}$ is Lipschitz continuous with Lipschitz constant L_f if

$$|f(x_1, y_1) - f(x_2, y_2)| \leq L_f(|x_1 - x_2| + |y_1 - y_2|)$$

for $x_1, x_2 \in I$ and $y_1, y_2 \in J$.

Example 12.20. The function $f : \mathbb{Q}^2 \to \mathbb{Q}$ defined by $f(x_1, x_2) = x_1 + x_2$ is Lipschitz continuous with Lipschitz constant $L_f = 1$.

Example 12.21. The function $f : [0, 2] \times [0, 2] \to \mathbb{Q}$ defined by $f(x_1, x_2) = x_1x_2$ is Lipschitz continuous with Lipschitz constant $L_f = 2$, since for $x_1, x_2 \in [0, 1]$

$$\begin{aligned} |x_1x_2 - y_1y_2| &= |x_1x_2 - y_1x_2 + y_1x_2 - y_1y_2| \\ &\leq |x_1 - y_1|x_2 + y_1|x_2 - y_2| \leq 2(|x_1 - y_1| + |x_2 - y_2|). \end{aligned}$$

12.11 Functions of Several Rational Variables

The concept of a function also extends to several variables, i.e. we consider functions $f(x_1, \dots, x_d)$ of d rational variables. We write $f : \mathbb{R}^d \to \mathbb{Q}$ if for given rational numbers $x_1, \cdots, x_d$, a rational number denoted by $f(x_1, \dots, x_d)$ is given.

The definition of Lipschitz continuity also directly extends. We say that $f : \mathbb{Q}^d \to \mathbb{Q}$ is Lipschitz continuous with Lipschitz constant L_f if for all $x_1, \cdots, x_d \in \mathbb{Q}$ and $y_1, \cdots, y_d \in \mathbb{Q}$,

$$|f(x_1, \dots, x_d) - f(y_1, \dots, y_d)| \le L_f(|x_1 - y_1| + \cdots + |x_d - y_d|).$$

Example 12.22. The function $f : \mathbb{R}^d \to \mathbb{Q}$ defined by $f(x_1, \dots, x_d) = x_1 + x_2 + \cdots x_d$ is Lipschitz continuous with Lipschitz constant $L_f = 1$.

Chapter 12 Problems

12.1. Verify the claims in Example 12.7.

12.2. Show that $f(x) = x^2$ is Lipschitz continuous on $[10, 13]$ directly and compute a Lipschitz constant.

12.3. Show that $f(x) = 4x - 2x^2$ is Lipschitz continuous on $[-2, 2]$ directly and compute a Lipschitz constant.

12.4. Show that $f(x) = x^3$ is Lipschitz continuous on $[-2, 2]$ directly and compute a Lipschitz constant.

12.5. Show that $f(x) = |x|$ is Lipschitz continuous on $\mathbb{Q}$ directly and compute a Lipschitz constant.

12.6. In Example 12.10, we show that x^4 is Lipschitz continuous on $[-2, 2]$ with Lipschitz constant $L = 32$. Explain why this is a reasonable value for the Lipschitz constant.

12.7. Show that $f(x) = 1/x^2$ is Lipschitz continuous on $[1, 2]$ directly and compute the Lipschitz constant.

12.8. Show that $f(x) = 1/(x^2 + 1)$ is Lipschitz continuous on $[-2, 2]$ directly and compute a Lipschitz constant.

12.9. Compute the Lipschitz constant of $f(x) = 1/x$ on the intervals (a) $[.1, 1]$, (b) $[.01, 1]$, and $[.001, 1]$.

12.10. Find the Lipschitz constant of the function $f(x) = \sqrt{x}$ with $D(f) = (\delta, \infty)$ for given $\delta > 0$.

12.11. Explain why $f(x) = 1/x$ is not Lipschitz continuous on $(0, 1]$.

12.12. (a) Explain why the function

$$f(x) = \begin{cases} 1, & x < 0 \\ x^2, & x \geq 0 \end{cases}$$

is **not** Lipschitz continuous on $[-1, 1]$. (b) Is f Lipschitz continuous on $[1, 4]$?

12.13. Suppose the Lipschitz constant L of a function f is equal to $L = 10^{100}$. Discuss the continuity properties of $f(x)$ and in particular decide if f continuous from a practical point of view.

12.14. Assume that f_1 is Lipschitz continuous with constant L_1, f_2 is Lipschitz continuous with constant L_2 on a set I, and c is a number. Show that $f_1 - f_2$ is Lipschitz continuous with constant $L_1 + L_2$ on I and cf_1 is Lipschitz continuous with constant cL_1 on I.

12.15. Show that the Lipschitz constant of a polynomial $f(x) = \sum_{i=0}^{n} a_i x^i$ on the interval $[-c, c]$ is

$$L = \sum_{i=1}^{n} |a_i| i c^{i-1} = |a_1| + 2c|a_2| + \cdots + nc^{n-1}|a_n|.$$

12.16. Explain why $f(x) = 1/x$ is not bounded on $[-1, 0]$.

12.17. Prove Theorem 12.5.

12.18. Use the theorems in this chapter to show that the following functions are Lipschitz continuous on the given intervals and try to estimate a Lipschitz constant or prove they are not Lipschitz continuous.

(a) $f(x) = 2x^4 - 16x^2 + 5x$ on $[-2, 2]$

(b) $\dfrac{1}{x^2 - 1}$ on $\left[-\dfrac{1}{2}, \dfrac{1}{2}\right]$

(c) $\dfrac{1}{x^2 - 2x - 3}$ on $[2, 3)$

(d) $\left(1 + \dfrac{1}{x}\right)^4$ on $[1, 2]$

12.19. Show the function

$$f(x) = \frac{1}{c_1 x + c_2(1 - x)}$$

where $c_1 > 0$ and $c_2 > 0$ is Lipschitz continuous on $[0, 1]$.

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