Mollifiers and Smooth Functions

We say a function f from $\mathbb{R} \to \mathbb{C}$ is C^∞ (or simply *smooth*) if all its derivatives to every order exist at every point of $\mathbb{R}$. For $f : \mathbb{R}^k \to \mathbb{C}$, we say f is C^∞ if all partial derivatives to every order exist and are continuous.

Proposition 1. *The function*

$$g(x) = \begin{cases} 0 & \text{if } x \le 0 \\ e^{-\frac{1}{x}} & \text{if } x > 0 \end{cases}$$

is C^∞.

This will follow from several lemmas. Note the only thing we need to prove is that $g^{(n)}(0)$ exists for all n. We will show $g^{(n)}(0) = 0$ for all n.

Lemma 2. *For $x > 0$, $n \ge 0$ the n^{th} derivative $g^{(n)}(x) = P(\frac{1}{x})e^{-\frac{1}{x}}$ for P a polynomial.*

Proof. This is true for $n = 0$. Now inductively, if $g^{(n)}(x) = P(\frac{1}{x})e^{-\frac{1}{x}}$, compute for $x > 0$

$$g^{(n+1)}(x) = P'(\tfrac{1}{x}) \cdot (-\tfrac{1}{x^2}) \cdot e^{-\frac{1}{x}} + P(\tfrac{1}{x}) \cdot e^{-\frac{1}{x}} \cdot (\tfrac{1}{x^2}).$$

Now note that $P'(\frac{1}{x}) \cdot (-\frac{1}{x^2}) + P(\frac{1}{x}) \cdot (\frac{1}{x^2})$ is also a polynomial in $\frac{1}{x}$. □

Lemma 3. *If P is a polynomial, then*

$$\lim_{x \to 0^+} P(\tfrac{1}{x})e^{-\frac{1}{x}} = 0.$$

Proof. Make the substitution $y = \frac{1}{x}$, and note that

$$\lim_{y \to \infty} \frac{P(y)}{e^y} = 0.$$

□

Lemma 4. *For all n, $g^{(n)}(0) = 0$.*

Proof. This is true for $n = 0$. Assume by the inductive hypothesis that $g^{(n)}(0) = 0$ for some n. Compute

$$g^{(n+1)}(0) = \lim_{h \to 0} \frac{g^{(n)}(h) - g^{(n)}(0)}{h} = \lim_{h \to 0} \frac{g^{(n)}(h)}{h}$$

by the inductive hypothesis. For $h < 0$, it is clear that $g^{(n)}(h) = 0$, and thus that

$$\lim_{h \to 0^-} \frac{g^{(n)}(h)}{h} = 0.$$

For $h > 0$, use Lemma 2 to compute

$$\lim_{h\to 0^+} \frac{g^{(n)}(h)}{h} = \lim_{h\to 0^+} \tfrac{1}{h} \cdot P(\tfrac{1}{h}) e^{-\frac{1}{h}} = 0$$

by applying Lemma 3 for the polynomial $\tilde{P}(y) = yP(y)$. Since the left and right limits are equal, we find $g^{(n+1)}(0) = 0$ and by induction this holds for all n. $\square$

The Proposition is proved.

Theorem 1. *There is a function ϕ on $\mathbb{R}^k$ which satisfies*

(1) $\phi(x) \geq 0$ *for all* $x \in \mathbb{R}^k$.
(2) $\phi \in C^\infty(\mathbb{R}^k)$.
(3) *supp* $\phi = \overline{B_1(0)}$.
(4) $\int_{\mathbb{R}^k} \phi \, dm = 1$.

Proof. For g from Proposition 1 above, let

$$\phi(x) = c_k \, g(1 - \|x\|^2) = \begin{cases} 0 & \text{if } \|x\| \geq 1 \\ c_k \exp(-\frac{1}{1-\|x\|^2}) & \text{if } \|x\| < 1 \end{cases}$$

where c_k is defined so that $\int_{\mathbb{R}^k} \phi \, dm = 1$. Then ϕ is C^∞ since $\phi = c_k \, g \circ h$ for $h = 1 - \|x\|^2 = 1 - x_1^2 - \cdots - x_k^2$. h is C^∞ since it's a polynomial. Then we can verify that all the various partial derivatives of $\phi = c_k \, g \circ h$ are continuous by using the usual rules of differentiation (the chain rule in multiple variables, the product rule, etc.) to compute them. $\square$

A *mollifer* is a family of functions ϕ_δ based on ϕ given by $\phi_\delta(x) = \delta^{-k} \phi(\frac{x}{\delta})$ for all $\delta > 0$. Then it is easy to check that

Proposition 5. *For all $\delta > 0$,*

(1) $\phi_\delta(x) \geq 0$ *for all* $x \in \mathbb{R}^k$.
(2) $\phi_\delta \in C^\infty(\mathbb{R}^k)$.
(3) *supp* $\phi_\delta = \overline{B_\delta(0)}$.
(4) $\int_{\mathbb{R}^k} \phi_\delta \, dm = 1$.

Proof. All of these properties are obvious except the last one. For the last one, we use the change of variables formula in multiple integrals to prove

$$\int_{\mathbb{R}^k} \phi_\delta(x) \, dm(x) = \int_{\mathbb{R}^k} \delta^{-k} \phi(\tfrac{x}{\delta}) \, dm(x) = \int_{\mathbb{R}^k} \phi(y) \, dm(y) = 1,$$

by using using the substitution $y = \frac{x}{\delta}$, which implies $dm(y) = \delta^{-k} dm(x)$ for δ^{-k} the Jacobian determinant. $\square$

A function f is mollified by convolution with ϕ_δ. Define

$$f_\delta(x) = (f*\phi_\delta)(x) = \int_{\mathbb{R}^k} f(x-y)\phi_\delta(y)\,dm(y) = \int_{\mathbb{R}^k} f(y)\phi_\delta(x-y)\,dm(y),$$

as long as the integrals converge. In this case, note that the integrals in the line above are equal by making the substitution $z = x-y$, which implies $dm(z) = dm(y)$ (the multiplicative factor is the absolute value of the Jacobian determinant, which is 1).

The idea is that $f_\delta(x)$ is a weighted, smoothed average all the values of f in the ball $B_\delta(x)$ of radius δ and center x. To see this, we make an analogous construction using the function

$$\alpha(x) = m(B_1(0))^{-1}\,\chi_{B_1(0)}(x).$$

Then α satisfies the same properties as ϕ in Theorem 1 except it is not smooth. Then if we define $\alpha_\delta(x) = \delta^{-k}\alpha(\frac{x}{\delta})$, the convolution

$$(f*\alpha_\delta)(x) = \int_{\mathbb{R}^k} f(y)\alpha_\delta(x-y)\,dm(y) = m(B_\delta(x))^{-1}\int_{B_\delta(x)} f(y)\,dm(y)$$

is the average value of f over the ball $B_\delta(x)$ of radius δ and center x. Since $\phi_\delta \geq 0$ and has integral 1 over its support $\overline{B_\delta(0)}$, we can consider $(f*\phi_\delta)(x)$ to be a weighted average of f over $B_\delta(x)$.

We say a complex function f on $\mathbb{R}^k$ is *locally* L^1 if $\chi_K f \in L^1$ for every compact subset K of $\mathbb{R}^k$. It is an easy consequence of Hölder's inequality that every $f \in L^p(\mathbb{R}^k)$ is locally L^1 for $1 \leq p \leq \infty$. Compute for q the conjugate exponent of p:

$$\int_{\mathbb{R}^k} |f\chi_K|\,dm = \int_K |f|\,dm \leq \|f\|_{L^p(K)}\cdot\|1\|_{L^q(K)} \leq \|f\|_{L^p(\mathbb{R}^k)}\cdot m(K)^{\frac{1}{q}} < \infty$$

Theorem 2. *If $f : \mathbb{R}^k \to \mathbb{C}$ is locally L^1 and $\psi : \mathbb{R}^k \to \mathbb{C}$ is C^∞ with compact support, then $f*\psi$ is C^∞.*

In order to prove this theorem, we need a few lemmas:

Lemma 6. *If $f \in L^1_{\text{loc}}(\mathbb{R}^k)$ and $\psi \in C_c(\mathbb{R}^k)$, then $f*\psi$ is continuous.*

Proof. Let $x_n \to x$ be a convergent sequence. The lemma is proved if we can show

$$(f*\psi)(x_n) = \int_{\mathbb{R}^k} f(y)\psi(x_n-y)\,dm(y) \to \int_{\mathbb{R}^k} f(y)\psi(x-y)\,dm(y) = (f*\psi)(x).$$

Note that the definition of $f*\psi$ is unchanged if f is redefined on a set of measure 0. Thus, without loss of generality, assume f is defined everywhere.

We may assume $x_n \in \overline{B_1(x)}$. Choose $r > 0$ so that supp $\psi \subset \overline{B_r(0)}$ (since ψ has compact support). Since ψ is continuous, we have for all y,

$$f(y)\psi(x_n - y) \to f(y)\psi(x - y).$$

Moreover, if $C = \sup |\psi| < \infty$, then for all x_n, we have

$$|f(y)\psi(x_n - y)| \le C|f(y)|\chi_{\overline{B_{r+1}(x)}}(y).$$

(To show the last term is justified, note that if $\|y\| > r + 1$, then $\|x_n - y\| \ge \|y\| - \|x_n\| > (r + 1) - 1 = r$ and so $\psi(x_n - y) = 0$.) By assumption, the function on the right-hand side is integrable. So the Dominated Convergence Theorem applies to show that

$$\begin{aligned} \lim_{n\to\infty} (f * \psi)(x_n) &= \lim_{n\to\infty} \int_{\mathbb{R}^k} f(y)\psi(x_n - y)\, dm(y) \\ &= \int_{\mathbb{R}^k} \lim_{n\to\infty} f(y)\psi(x_n - y)\, dm(y) \\ &= \int_{\mathbb{R}^k} f(y)\psi(x - y)\, dm(y) \\ &= (f * \psi)(x). \end{aligned}$$

□

Lemma 7. *If $f \in L^1_{\mathrm{loc}}(\mathbb{R}^k)$ and $\psi \in C_c^\infty(\mathbb{R}^k)$, then for all $i = 1, \ldots, k$,*

$$\frac{\partial}{\partial x^i}(f * \psi) = f * \frac{\partial \psi}{\partial x^i}.$$

Proof. We may assume ψ is real, since we may otherwise consider the real and imaginary parts. Compute at x, for e_i the i^{th} coordinate vector

$$\begin{aligned} \frac{\partial}{\partial x^i}(f * \psi)(x) &= \lim_{h\to 0} \tfrac{1}{h}[(f * \psi)(x + he_i) - (f * \psi)(x)] \\ &= \lim_{h\to 0} \frac{1}{h}\left[\int_{\mathbb{R}^k} f(y)\psi(x + he_i - y)\, dm(y)\right. \\ &\qquad \left. - \int_{\mathbb{R}^k} f(y)\psi(x - y)\, dm(y)\right] \\ &= \lim_{h\to 0} \int_{\mathbb{R}^k} f(y)\left(\frac{\psi(x + he_i - y) - \psi(x - y)}{h}\right) dm(y) \\ &= \lim_{h\to 0} \int_{\mathbb{R}^k} f(y)\frac{\partial \psi}{\partial x^i}(x - y + c(h)e_i)\, dm(y), \end{aligned}$$

where we use the Mean Value Theorem to find a value $c(h)$ so that $|c(h)| \leq |h|$ (here we use that ψ is real). Continue to compute at x

$$\begin{aligned}\frac{\partial}{\partial x^i}(f * \psi) &= \lim_{h\to 0}\left(f * \frac{\partial \psi}{\partial x^i}\right)(x + c(h)e_i) \\ &= \left(f * \frac{\partial \psi}{\partial x^i}\right)(x)\end{aligned}$$

where we apply Lemma 6 to take the last limit. □

Proof of Theorem 2. Apply Lemma 7 to compute the first partial derivative

$$\frac{\partial}{\partial x^i}(f * \psi) = f * \frac{\partial \psi}{\partial x^i}.$$

This function is continuous by Lemma 6 and since $\frac{\partial \psi}{\partial x^i} \in C_c(\mathbb{R}^k)$. The higher-order partial derivatives can be handled by induction. □

Lemma 8. *If f is locally L^1 and $f_\delta = f * \phi_\delta$ is the standard mollifier, then*

$$\mathrm{supp} f_\delta \subset \mathrm{supp} f + \overline{B_{\delta(0)}} = \{x + y \,\big|\, x \in \mathrm{supp} f,\ y \in \overline{B_{\delta(0)}}\}.$$

Proof. If

$$0 \neq f_\delta(x) = \int_{\mathbb{R}^k} f(x-y)\phi_\delta(y)\,dm(y) = \int_{B_\delta(0)} f(x-y)\phi_\delta(y)\,dm(y),$$

then $f(x-y)$ cannot be identically zero for all $y \in B_\delta(0)$. So for some such y, $x - y \in \mathrm{supp}\ f$, and thus $x = (x-y)+y \in \mathrm{supp}\ f + B_\delta(0)$. □

Theorem 3. *If $f : \mathbb{R}^k \to \mathbb{C}$ is continuous, and ϕ_δ is the standard mollifier, then $f_\delta = f * \phi_\delta \to f$ uniformly on compact subsets of $\mathbb{R}^k$.*

Proof. Let $K \subset \mathbb{R}^k$ be compact. Then there is an $r > 0$ so that $K \subset \overline{B_r(0)}$. On the compact set $\overline{B_{r+1}(0)}$, f is uniformly continuous. Choose $\epsilon > 0$. There is an $\eta > 0$ so that if $z, w \in \overline{B_{r+1}(0)}$, then $\|z - w\| < \eta$ implies $|f(z) - f(w)| < \epsilon$. We may additionally assume $\eta < 1$. Let $\delta \in (0, \eta)$.

For $x \in K \subset \overline{B_r(0)}$, compute

$$\begin{aligned}
|f_\delta(x) - f(x)| &= \left| \int_{\mathbb{R}^k} f(x-y)\phi_\delta(y)\, dm(y) - f(x) \right| \\
&= \left| \int_{\mathbb{R}^k} f(x-y)\phi_\delta(y)\, dm(y) - \int_{\mathbb{R}^k} f(x)\phi_\delta(y)\, dm(y) \right| \\
&\le \int_{\mathbb{R}^k} |f(x-y) - f(x)|\phi_\delta(y)\, dm(y) \\
&= \int_{\overline{B_\delta(0)}} |f(x-y) - f(x)|\phi_\delta(y)\, dm(y) \\
&< \int_{\overline{B_\delta(0)}} \epsilon\, \phi_\delta(y)\, dm(y) = \epsilon
\end{aligned}$$

The last line is justified since $x \in \overline{B_r(0)} \subset \overline{B_{r+1}(0)}$ and $x - y \in \overline{B_r(0)} + \overline{B_\delta(0)} \subset \overline{B_{r+1}(0)}$. Since this estimate applies to all x in the compact set K, we have that $f_\delta \to f$ uniformly on K. □

Corollary 9. *If $f \in C_c(\mathbb{R}^k)$, then $f_\delta \to f$ uniformly on $\mathbb{R}^k$.*

Theorem 4. *For $1 \le p < \infty$, $C_c^\infty(\mathbb{R}^k)$ is dense in $L^p(\mathbb{R}^k)$.*

Proof. Let $f \in L^p(\mathbb{R}^k)$ and $\epsilon > 0$. Then Theorem 3.14 of Rudin implies there is a $g \in C_c(\mathbb{R}^k)$ so that $\|f - g\|_p < \frac{\epsilon}{2}$.

Now let $g_\delta = g * \phi_\delta$ for ϕ the standard mollifier. Then Corollary 9 implies $g_\delta \to g$ uniformly as $\delta \to 0$. Assume supp $g \subset B_r(0)$ for some $r > 0$. Then supp $g_\delta \subset B_r(0) + \overline{B_\delta(0)} = B_{r+\delta}(0)$. Compute

$$\|g_\delta - g\|_p^p = \int_{\mathbb{R}^k} |g_\delta - g|^p\, dm = \int_{B_{r+\delta}(0)} |g_\delta - g|^p\, dm \le \sup |g_\delta - g|^p \cdot m(B_{r+\delta}(0)).$$

Now $\sup |g_\delta - g| \to 0$ since $g_\delta \to g$ uniformly. Thus $\|g_\delta - g\|_p \to 0$ and there is a $\delta > 0$ so that $\|g_\delta - g\|_p < \frac{\epsilon}{2}$.

Therefore, $\|f - g_\delta\|_p \le \|f - g\|_p + \|g - g_\delta\|_p < \epsilon$. $g_\delta \in C_c^\infty(\mathbb{R}^k)$ since supp $g_\delta \subset$ supp $g + \overline{B_\delta(0)}$ is compact. g_δ is C^∞ by Theorem 2. □

A similar but more precise result is

Theorem 5. *For any $1 \le p < \infty$ and $f \in L^p(\mathbb{R}^k)$, then $\|f * \phi_\delta - f\|_p \to 0$ as $\delta \to 0$, where ϕ is any nonnegative measurable function on $\mathbb{R}^k$ with total integral one.*

The proof is essentially contained in Theorems 9.5, 9.9 and 9.10 in Rudin.

Proposition 10. *Let $f \in L^p(\mathbb{R}^k)$ for $1 \le p < \infty$, and let $y \in \mathbb{R}^k$. Let $f^y(x) = f(x - y)$. Then the map $y \mapsto f^y$ is a continuous map from $\mathbb{R}^k$ to $L^p(\mathbb{R}^k)$.*

Proof. Let $\epsilon > 0$. By Rudin, Theorem 3.14, choose a continuous function g with compact support so that $\|f - g\|_p < \epsilon$. Let $R > 0$ be so that supp $g \subset B_R(0)$. Since g is continuous with compact support, it is uniformly continuous. Thus there is a $\delta \in (0, R)$ so that $|s - t| < \delta$ implies

$$|g(s) - g(t)| < m(B_0(2R))^{-\frac{1}{p}}\epsilon,$$

for m Lebesgue measure on $\mathbb{R}^k$. Then compute for $|s - t| < \delta$

$$\|g^s - g^t\|_p^p = \int_{\mathbb{R}^k} |g(x-s) - g(x-t)|^p \, dx < m(B_{2R}(0))^{-1} \epsilon^p \, m(B_{R+\delta}(s)) < \epsilon^p.$$

Whenever $|s - t| < \delta$,

$$\begin{aligned}
\|f^s - f^t\|_p &\leq \|f^s - g^s\|_p + \|g^s - g^t\|_p + \|g^t - f^t\|_p \\
&= \|(f - g)^s\|_p + \|g^s - g^t\|_p + \|(g - f)^t\|_p \\
&= \|f - g\|_p + \|g^s - g^t\|_p + \|g - f\|_p \\
&< 3\epsilon.
\end{aligned}$$

Here we have used the change of variables $z = x - s$ for $\alpha = f - g$ to compute

$$\|\alpha^s\|_p^p = \int_{\mathbb{R}^k} \alpha(x - s) \, dx = \int_{\mathbb{R}^k} \alpha(z) \, dz = \|\alpha\|_p^p.$$

□

Proof of Theorem 5. We need to prove that

$$\lim_{\delta \to 0} \|f * \phi_\delta - f\|_p = 0.$$

As in the proof of Theorem 3 above, we have

$$|(f * \phi_\delta)(x) - f(x)| \leq \int_{\mathbb{R}^k} |f(x - y) - f(x)| \phi_\delta(y) \, dm(y).$$

Since ϕ_δ is a positive function with integral one, we may apply Jensen's Inequality for the convex function $t \mapsto t^p$ to find

$$\begin{aligned}
|(f * \phi_\delta)(x) - f(x)|^p &\leq \left(\int_{\mathbb{R}^k} |f(x - y) - f(x)| \phi_\delta(y) \, dm(y) \right)^p \\
&\leq \int_{\mathbb{R}^k} |f(x - y) - f(x)|^p \phi_\delta(y) \, dm(y).
\end{aligned}$$

Now we may integrate this inequality over $\mathbb{R}^k$ in x and use Fubini's Theorem to find

$$\begin{aligned}\|f * \phi_\delta - f\|_p^p &\le \int_{\mathbb{R}^k}\int_{\mathbb{R}^k} |f(x-y) - f(x)|^p \phi_\delta(y)\, dm(y)\, dm(x) \\ &= \int_{\mathbb{R}^k}\int_{\mathbb{R}^k} |f(x-y) - f(x)|^p \, dm(x)\, \phi_\delta(y)\, dm(y) \\ &= \int_{\mathbb{R}^k} \|f^y - f\|_p^p \, \phi_\delta(y)\, dm(y).\end{aligned}$$

Let $g(y) = \|f^y - f\|_p^p$. Proposition 10 above shows g is a continuous function, and it is clear that $g(0) = 0$. Moreover, g is bounded since

$$g(y) = \|f^y - f\|_p^p \le (\|f^y\|_p + \|f\|_p)^p = (2\|f\|_p)^p.$$

We continue computing to find

$$\begin{aligned}\|f * \phi_\delta - f\|_p^p &\le \int_{\mathbb{R}^k} g(y)\phi_\delta(y)\, dm(y) \\ &= \int_{\mathbb{R}^k} g(y)\delta^{-k}\phi(y\delta^{-1})\, dm(y), \\ &= \int_{\mathbb{R}^k} g(\delta s)\phi(s)\, dm(s)\end{aligned}$$

for the change of variables $s = y\delta^{-1}$. As $\delta \to 0$, $g(\delta s)\phi(s) \to g(0)\phi(s) = 0$ pointwise on $\mathbb{R}^k$. Moreover, the integrand $g(\delta s)\phi(s) \le \|g\|_\infty \phi(s)$ for all δ. Since g is bounded and ϕ is integrable, the Dominated Convergence Theorem applies to show that

$$\lim_{\delta \to 0} \int_{\mathbb{R}^k} g(\delta s)\phi(s)\, dm(s) = 0.$$

This implies $\|f * \phi_\delta - f\|_p \to 0$ as $\delta \to 0$. $\square$