*Journal of Mathematical Sciences, Vol. 149, No. 6, 2008*

# ALMOST SURE POLYNOMIAL ASYMPTOTIC STABILITY OF STOCHASTIC DIFFERENCE EQUATIONS

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Abstract. In this paper, we establish the almost sure asymptotic stability and decay results for solutions of an autonomous scalar difference equation with a nonhyperbolic equilibrium at the origin, which is perturbed by a random term with a fading state-independent intensity. In particular, we show that if the unbounded noise has tails that fade more quickly than polynomially, then the state-independent perturbation dies away at a sufficiently fast polynomial rate in time, and if the autonomous difference equation has a polynomial nonlinearity at the origin, then the almost sure polynomial rate of decay of solutions can be determined exactly.

## Introduction

In this paper, we study the almost sure convergence to the equilibrium of a stochastic difference equation. Without loss of generality, this equilibrium can be taken to be zero. In this difference equation, linearization of the equation close to the equilibrium does not determine the asymptotic behavior because the terms that depend on the state of the system are $o(x)$ as $x \to 0$. The equations studied may be viewed as stochastically perturbed versions of autonomous difference equations, where the random perturbation is independent of the state.

As a consequence of the fact that the equation admits a trivial linearization at the equilibrium, which can yield only a weak restoring force towards the equilibrium, it must be suspected that the convergence of solutions of the difference equation to its equilibrium state cannot take place at an exponentially fast rate. The nonexponential convergence of solutions of stochastic differential and functional differential equations with bounded delay has been studied in a number of papers. Stability of polynomial systems, without consideration of the rate of decay, has been studied in [24, 27]. The nonexponential rate of convergence of equations that are quasilinear but have many nonautonomous features has been considered in [7, 16–18, 20, 21]. Slower- and faster-than-exponential pathwise convergence to the equilibria of autonomous stochastic differential equations is considered in [3, 5], and some interesting earlier work in this direction may be found in [32, 33]. The work in [3, 5] was inspired by an idea of Gikhman and Skorohod in [8].

In [3, 5, 32, 33], the stochastic perturbation has an intensity that depends on the state. In this paper, we concentrate on difference equations with state-independent perturbations of fading intensity. Papers in which nonlinear equations with fading state-independent perturbations have been considered include [1, 2]. In these papers, the almost sure exact polynomial rate of convergence of solutions is studied, and this paper is an attempt to extend some of these results to stochastic difference equations.

There is also literature on the asymptotic behavior of nonlinear stochastic difference equations. Some of the relevant papers are quoted here. In [13, 14], the coefficients are assumed to be linear or having sublinear growth. In [23, 25, 26], some nonlinearity was allowed in the coefficients. In [28–30], the coefficients were permitted to have polynomial growth. Stability and stabilization of general nonlinear scalar difference equations was considered in [4].

The novelty of the present paper comes from the fact that we are attempting to determine the exact nonlinear rate of decay for a stochastic difference equation where the terms depending on the state are $o(x)$ as $x \to 0$. The nonautonomous features are assumed to be responsible for the polynomial decay of solutions; very rapid decay of the external perturbation is permitted. In the process, we prove some

Translated from Sovremennaya Matematika. Fundamental'nye Napravleniya (Contemporary Mathematics. Fundamental Directions), Vol. 17, Differential and Functional Differential Equations. Part 3, 2006.

1072–3374/08/1496–1629  

new results for the asymptotic decay of perturbed nonlinear deterministic difference equations. These results are quoted separately because they may be of independent interest. Furthermore, we attempt to connect our results with those seen in [1] because the difference equation studied here can be viewed as a discretization of the stochastic differential equation considered there. In doing so, we show that the Euler–Maruyama method is dynamically consistent with the stochastic differential equation in that, for a sufficiently small step size, solutions of the difference equation converge almost surely (and do so at exactly the same rate) under conditions that imply the almost sure convergence of the stochastic differential equation.

The paper is organized as follows. In Sec. 1, we give some notation, terminology, and auxiliary results on stochastic processes. In Sec. 2, which is subdivided into several subsections, the main results of the paper are stated and discussed. The first subsection gives a result on the asymptotic stability of general scalar stochastic difference equations with state-independent perturbations. The second subsection considers the exact polynomial rate of convergence of solutions to scalar deterministic difference equations. These results are applied in the next subsection to give the main result of the paper, which concerns the almost sure polynomial rate of decay of solutions of stochastic difference equations. The last subsection of the discussion considers the connections between the convergence of the stochastic difference equation studied here and a related stochastic differential equation; mention is also made of the connection between the stochastic difference equation and a discretization of the stochastic differential equation. The penultimate section of the paper, Sec. 3, shows some numerical simulations, which demonstrate the results. The final section of the paper, Sec. 4, contains the proofs of all results: a separate subsection is devoted to each proof.

## 1. Mathematical Preliminaries

In this section, we introduce the notation and give a number of necessary definitions and a lemma that we use to prove our results. A detailed exposition of the definitions and facts of the theory of random processes can be found in, e.g., [15].

The signum function will be denoted by sgn, where $\operatorname{sgn}(x) = 1$ for $x > 0$, $\operatorname{sgn}(x) = -1$ for $x < 0$, and $\operatorname{sgn}(x) = 0$ for $x = 0$. As usual, let $x \vee y$ denote the maximum of $x$, $y \in \mathbb{R}$ and $x \wedge y$ denote the minimum. We denote by $C(I; J)$ the space of continuous functions from $I$ to $J$.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}_{n\in\mathbb{N}}, \mathbb{P})$ be a complete filtered probability space. Let $\{\xi_n\}_{n\in\mathbb{N}}$ be a sequence of independent random variables with $\mathbb{E}\xi_n = 0$. We assume that the filtration $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ is naturally generated:

$$\mathcal{F}_{n+1} = \sigma\{\xi_{i+1} : i = 0, 1, \dots, n\}.$$

Among all the sequences $\{X_n\}_{n\in\mathbb{N}}$ of random variables we distinguish those for which $X_n$ are $\mathcal{F}_n$-measurable for all $n \in \mathbb{N}$. A stochastic sequence $\{X_n\}_{n\in\mathbb{N}}$ is said to be an *$\mathcal{F}_n$-martingale* if $\mathbb{E}|X_n| < \infty$ and $\mathbb{E}\big[X_n \big| \mathcal{F}_{n-1}\big] = X_{n-1}$ for all $n \in \mathbb{N}$ a.s. A stochastic sequence $\{\xi_n\}_{n\in\mathbb{N}}$ is said to be an *$\mathcal{F}_n$-martingale-difference* if $\mathbb{E}|\xi_n| < \infty$ and $\mathbb{E}\big(\xi_n \big| \mathcal{F}_{n-1}\big) = 0$ a.s. for all $n \in \mathbb{N}$. We use the standard abbreviation "a.s." for the wordings "almost sure" or "almost surely" throughout the text.

In what follows, $\{X_n \to\}$ denotes the set of all $\omega \in \Omega$ for which $\lim_{n\to\infty} X_n$ exists and is finite.

**Lemma 1.** *Let $\{Z_n\}_{n\in\mathbb{N}}$ be a nonnegative $\mathcal{F}_n$-measurable process, $\mathbb{E}|Z_n| < \infty$ for all $n \in \mathbb{N}$, and*

$$Z_{n+1} \le Z_n + u_n - v_n + \nu_{n+1}, \quad n = 0, 1, 2, \dots,$$

*where $\{\nu_n\}_{n\in\mathbb{N}}$ is an $\mathcal{F}_n$-martingale-difference, while $\{u_n\}_{n\in\mathbb{N}}$, $\{v_n\}_{n\in\mathbb{N}}$ are nonnegative $\mathcal{F}_n$-measurable processes with $\mathbb{E}|u_n|$, $\mathbb{E}|v_n| < \infty$ for all $n \in \mathbb{N}$.*

*Then*

$$\left\{\omega : \sum_{n=1}^{\infty} u_n < \infty\right\} \subseteq \left\{\omega : \sum_{n=1}^{\infty} v_n < \infty\right\} \bigcap \{Z \to\}.$$

## 2. Discussion of Results

**2.1. Asymptotic stability.** Let $\zeta$ be a nonrandom number, $h > 0$ be fixed, and $\{\xi_n\}_{n\in\mathbb{N}}$ be a sequence of independent random variables with

$$\mathbb{E}\xi_n = 0, \quad \mathbb{E}[\xi_n^2] = 1, \quad n \in \mathbb{N}. \tag{2.1}$$

We consider the equation

$$X_{n+1} = X_n - hf(X_n) + \sqrt{h}\sigma_n\xi_{n+1}, \quad n \in \mathbb{N}; \qquad X_0 = \zeta. \tag{2.2}$$

We suppose that $f \in C(\mathbb{R};\mathbb{R})$ is a nonrandom function and $(\sigma_n)_{n\in\mathbb{N}}$ is a nonrandom sequence. In order that zero be an equilibrium of the unperturbed equation, we ask that

$$f(0) = 0. \tag{2.3}$$

We now impose some sufficient conditions on $f$ and $\sigma$ to ensure that all solutions of (2.2) tend to zero almost surely. We do not claim these conditions to be optimal for this purpose; rather we wish to show that solutions of (2.2) can be almost surely asymptotically stable. In other results, we take the asymptotic stability as a fact and then determine the rate at which solutions tend to the limit, given information on the asymptotic behavior of $\sigma_n$ as $n \to \infty$ and $f(x)$ as $x \to 0$.

Turning to the asymptotic stability result, we suppose that zero is a stable solution of the unperturbed equation in the sense that

$$xf(x) > 0, \quad x \neq 0. \tag{2.4}$$

**Theorem 1.** *Suppose that $f \in C(\mathbb{R};\mathbb{R})$ obeys (2.3) and (2.4) and suppose that there exists $K > 0$ such that*

$$|f(x)| \le K|x|, \quad x \in \mathbb{R}, \tag{2.5}$$

*so that $Kh < 2$. Let the sequence $(\sigma_n)_{n\in\mathbb{N}}$ satisfy*

$$\sum_{n=1}^{\infty} \sigma_n^2 < \infty. \tag{2.6}$$

*If $(X_n)$ is the unique solution to (2.2), then*

$$\lim_{n\to\infty} X_n = 0 \quad a.s. \tag{2.7}$$

**Remark 1.** We note that (2.1) together with (2.6) implies that the stochastic perturbation in (2.2) tends to zero almost surely as $n \to \infty$:

$$\lim_{n\to\infty} \sigma_n\xi_{n+1} = 0, \quad \text{a.s.} \tag{2.8}$$

This fact is well known and follows by the standard method of applying Chebyshev's inequality and then using the first Borel–Cantelli lemma.

**2.2. Decay rate for perturbed deterministic difference equations.** Here we are interested in determining the rate of decay for solutions of (2.2) under some hypotheses about the rate of decay of $\sigma_n$ and the asymptotic behavior of $f(x)$ as $x \to 0$. In order to prove such results, we first prove a theorem for the rate of decay for perturbed difference equations, which need not be stochastic. Although the results are used to establish the rate of decay of stochastic equations, we present them here in more detail, as we do not know of such results appearing in the deterministic literature and believe that they may be of independent interest.

We consider the deterministic equation

$$x_{n+1} = x_n - hf(x_n) + g_n, \quad n \in \mathbb{N}; \qquad x_0 = \zeta. \tag{2.9}$$

Since we are interested in studying polynomial decay of solutions, we impose the following condition on $f$:

$$\text{there exists } a > 0 \text{ and } \beta > 1 \text{ such that } \lim_{x\to 0} \frac{\operatorname{sgn}(x)f(x)}{|x|^\beta} = a. \tag{2.10}$$

We also require that $g$ decay at a polynomial rate related to the asymptotic behavior of $f$ at $x = 0$:

$$\limsup_{n\to\infty} \frac{\log|g_n|}{\log n} < -\frac{\beta}{\beta - 1}. \tag{2.11}$$

Condition (2.10) ensures that $f(x)$ behaves like $x^\beta$ as $x \downarrow 0$. However, when $\beta$ is not a rational number, $x^\beta$ is not well defined for $x < 0$. Therefore, in order to maintain symmetry, we extend $f$ to behave like $-|x|^\beta$ as $x \uparrow 0$.

The preservation of symmetry is a crucial hypothesis in the existence of a well-defined decay rate. If the exponent $\beta$ in (2.10) has different values for $x < 0$ and $x > 0$, then the decay rate of the solution to zero depends on whether the solution approaches zero from above or below. However, in the presence of a stochastic perturbation, it is not clear whether the solution would necessarily be nonoscillatory (i.e., whether it ultimately approaches the equilibrium from one side).

This condition on $g_n$ is chosen so that the rate of decay of the perturbed difference equation (2.9) is the same as the one for the unperturbed equation $x_{n+1} = x_n - hf(x_n)$. If a slower rate of polynomial decay is imposed on $g_n$, it should still be possible to determine a polynomial rate of decay of $x_n$, as similar results can be proven for analogous ordinary differential equations. However, we prefer to defer such a treatment of slowly fading perturbations to a later work, in which the asymptotic behavior of a larger class of difference equations is considered.

In the following results, we take the convergence of $x_n \to 0$ as $n \to \infty$ as a hypothesis. We do this so that we may concentrate on the pathwise rate of decay of solutions to zero when convergence takes place and also separate hypotheses that simply ensure convergence from those that determine the convergence rate.

**Theorem 2.** *Suppose that $f$ is a continuous function that obeys* (2.3) *and* (2.10) *and $(g_n)_{n\in\mathbb{N}}$ is a sequence that obeys* (2.11)*, where $\beta > 1$ is the exponent in* (2.10)*. Let $x_n$ be the unique solution of* (2.9)*. If*

$$\lim_{n\to\infty} x_n = 0 \tag{2.12}$$

*and $a$ is the constant defined in* (2.10)*, then there is a constant $L$ that assumes either the values* 0 *or* $[a(\beta-1)]^{-1/(\beta-1)}$ *such that*

$$\lim_{n\to\infty} (nh)^{\frac{1}{\beta-1}}|x_n| = L. \tag{2.13}$$

This result is a consequence of the following three lemmas.

**Lemma 2.** *Suppose that $f$ is a continuous function that obeys* (2.3) *and* (2.10) *and $(g_n)_{n\in\mathbb{N}}$ is a sequence that obeys* (2.11)*, where $\beta > 1$ is the exponent in* (2.10)*. Let $x_n$ be the unique solution of* (2.9)*. If $x_n$ obeys* (2.12) *and $a$ is the constant defined in* (2.10)*, then*

$$\limsup_{n\to\infty} (nh)^{\frac{1}{\beta-1}}|x_n| \le \left(\frac{1}{a(\beta-1)}\right)^{\frac{1}{\beta-1}}. \tag{2.14}$$

Once we have proven this result, we can show that either the inequality in (2.14) is an equality or the solutions decay to zero more quickly than $n^{-1/(\beta-1)}$.

**Lemma 3.** *Suppose that $f$ is a continuous function that obeys* (2.3) *and* (2.10) *and $(g_n)_{n\in\mathbb{N}}$ is a sequence that obeys* (2.11)*, where $\beta > 1$ is the exponent in* (2.10)*. Let $x_n$ be the unique solution of* (2.9)*. If $x_n$ obeys* (2.12) *and $a$ is the constant defined in* (2.10)*, then*

$$\lim_{n\to\infty} (nh)^{\frac{1}{\beta-1}}|x_n| = 0 \ \textit{or} \ \limsup_{n\to\infty} (nh)^{\frac{1}{\beta-1}}|x_n| = [a(\beta-1)]^{-1/(\beta-1)}. \tag{2.15}$$

Finally, we show that if the polynomial upper bound on the decay rate of solutions is achieved, then solutions attain exactly this rate of decay as $n \to \infty$.

**Lemma 4.** *Suppose that $f$ is a continuous function which obeys* (2.3) *and* (2.10) *and $(g_n)_{n\in\mathbb{N}}$ is a sequence which obeys* (2.11)*, where $\beta > 1$ is the exponent in* (2.10)*. Let $x_n$ be the unique solution of* (2.9)*. If $x_n$ obeys* (2.12)*, $a$ is the constant defined in* (2.10)*, and*

$$\limsup_{n\to\infty}(nh)^{\frac{1}{\beta-1}}|x_n| = [a(\beta-1)]^{-1/(\beta-1)}\,, \tag{2.16}$$

*then*

$$\lim_{n\to\infty}(nh)^{\frac{1}{\beta-1}}|x_n| = [a(\beta-1)]^{-1/(\beta-1)}\,. \tag{2.17}$$

**Remark 2.** In the theorem above, both the cases where $L = 0$ and $L = (a(\beta-1))^{-1/(\beta-1)}$ can be realized even if the order of magnitude of the perturbation remains the same as $n\to\infty$. Indeed, the initial-value problem

$$x_{n+1} = x_n - \operatorname{sgn}(x_n)\,x_n^2 + \left(-\frac{n+\frac{1}{\sqrt{2}}+\frac{1}{2}}{\left(n+\frac{1}{\sqrt{2}}\right)^2\left(n+1+\frac{1}{\sqrt{2}}\right)^2} + \frac{1}{4}\frac{1}{\left(n+\frac{1}{\sqrt{2}}\right)^4}\right), \quad n\in\mathbb{N}, \quad x_0 = 1,$$

obeys all the hypotheses of the theorem and has the unique solution

$$x_n = \frac{1}{2}\left(n+\frac{1}{\sqrt{2}}\right)^{-2}$$

for all $n\in\mathbb{N}$. Clearly, this solution satisfies

$$\lim_{n\to\infty} n x_n = 0,$$

hence, $L = 0$ for this problem. On the other hand, the unique solution of the initial-value problem

$$x_{n+1} = x_n - x_n^2\operatorname{sgn}(x_n) + \frac{1}{(n+1)^2(n+2)}, \quad n\in\mathbb{N}, \qquad x_0 = 1,$$

is

$$x_n = \frac{1}{n+1},$$

and so satisfies

$$\lim_{n\to\infty} n x_n = 1.$$

Thus, we have obtained an example where $L = (a(\beta-1))^{-1/(\beta-1)}$, whereas for this problem $a = 1$ and $\beta = 2$. Note that the initial-value condition is the same and the perturbation has the same decay rate in both examples, i.e.,

$$\lim_{n\to\infty} n^3|g_n| = 1.$$

**Remark 3.** It is always possible to get an arbitrarily fast rate of decay for the solution of

$$x_{n+1} = x_n - hf(x_n) + g_n, \quad n\in\mathbb{N}; \quad x_0 = \zeta,$$

provided that the perturbation has the appropriate form and rate of decay. Indeed, let the rate of decay of $d_n$ be such that $d_0 = 1$ and

$$\lim_{n\to\infty}\frac{d_{n+1}}{d_n} = 0.$$

The latter condition implies that $d_n$ decays to zero faster than any geometric progression. If the perturbation is

$$g_n = \zeta(d_{n+1} - d_n) + hf(\zeta d_n)$$

and $\lim_{x\to 0} f(x)/x = 0$, then the perturbation $g_n$ tends to zero faster than any geometric progression,

$$\lim_{n\to\infty}\frac{g_n}{d_n} = -\zeta,$$

and the solution of the initial-value problem is $x_n = \zeta d_n$.

**2.3. Decay rate for stochastic difference equations.** If we define the stochastic process $\{G_n\}_{n\in\mathbb{N}}$ by the formula

$$G_n = \sqrt{h}\sigma_n\xi_{n+1}, \quad n \in \mathbb{N}, \tag{2.18}$$

then the realization $X(\omega) = (X_n(\omega))_{n\in\mathbb{N}}$ of the solution of Eq. (2.2) can be identified with the solution of (2.9), where $x_n = X_n(\omega)$ and $g_n = G_n(\omega)$. Due to the results of the last section, if we can show that $g_n = G_n(\omega)$ exhibits the decay property (2.11) for almost all $\omega \in \Omega$, then we can apply the deterministic results of the last section to each path associated with the outcome $\omega$. Therefore, if we can show that

$$\limsup_{n\to\infty} \frac{\log|G_n|}{\log n} < -\frac{\beta}{\beta-1}, \quad \text{a.s.}, \tag{2.19}$$

then all the results in the last section are applied to the stochastic equation (2.2).

In order to prove (2.19), we require that $\Phi$, which is the distribution function of each $\xi_n$, have tails, which decay faster than any polynomial function, namely,

$$\lim_{x\to\infty}(1-\Phi(x))x^\gamma = 0, \quad \lim_{x\to-\infty}\Phi(x)x^\gamma = 0 \quad \text{for every } \gamma > 0. \tag{2.20}$$

This condition is true for bounded random variables, two-sided exponential random variables (i.e., a random variable with density $\phi$ given by $\phi(x) = \frac{1}{2}\alpha e^{-\alpha|x|}$ for all $x \in \mathbb{R}$ and some $\alpha > 0$), and lognormally distributed random variables. It is also true for normally distributed random variables; therefore, it applies in the case where we view (2.2) as a strong Euler–Maruyama approximation of a stochastic differential equation with uniform mesh size $h > 0$.

Condition (2.20) ensures that the large fluctuations of $(\xi_n)$ grow more slowly than any polynomially growing function.

**Lemma 5.** *Suppose that $\Phi$, which is the distribution function of $\xi_n$, obeys* (2.20)*. Then*

$$\lim_{n\to\infty} n^\varepsilon|\xi_n| = 0 \quad a.s.$$

*for every fixed $\varepsilon > 0$.*

According to (2.11), if (2.20) is satisfied, then (2.19) is satisfied for the stochastic equation provided that $\sigma_n$ decays polynomially.

**Lemma 6.** *Suppose that $\Phi$, which is the distribution function of $\xi_n$, obeys* (2.20) *and for some $\beta > 1$, we have*

$$\limsup_{n\to\infty} \frac{\log|\sigma_n|}{\log n} < -\frac{\beta}{\beta-1}. \tag{2.21}$$

*If we define $G_n$ by* (2.18)*, then $G_n$ obeys* (2.19)*.*

**Remark 4.** We note that if $\sigma_n$ obeys (2.21) for some $\beta > 1$, then $\sigma_n$ obeys (2.6).

By Theorem 2, Lemma 6, and the discussion in this section, we see immediately that the following result, which is the main theorem of the paper, must hold.

**Theorem 3.** *Suppose that $f$ is a continuous function that obeys* (2.3) *and* (2.10)*. Let $(\sigma_n)_{n\in\mathbb{N}}$ be a sequence obeying* (2.21)*.*

*If $X_n$ is a solution of* (2.2) *that obeys* (2.7)*, $\beta > 1$ is the exponent in* (2.10) *and* (2.21)*, and $a$ is the constant defined in* (2.10)*, then there is a random variable $L$ which assumes either the values* 0 *or* $[a(\beta-1)]^{-1/(\beta-1)}$ *such that*

$$\lim_{n\to\infty}(nh)^{\frac{1}{\beta-1}}|X_n| = L \quad a.s. \tag{2.22}$$

**2.4. Numerical methods and stochastic differential equations.** In this section, we show how the asymptotic stability of the solution of the stochastic difference equation (2.2) compares with that of the solution of the stochastic differential equation (SDE)

$$dX(t) = -f(X(t))\,dt + \sigma(t)\,dB(t). \tag{2.23}$$

Here $(\xi_n)_{n\geq 0}$ is a sequence of standardized, normally distributed random variables (which satisfy (2.1)), while $B$ is a scalar standard Brownian motion.

To explain the relationship between Eqs. (2.2) and (2.23), let us recall the Euler–Maruyama (EM) numerical method that computes approximations $X_n(h) \approx X(nh)$ by

$$X_{n+1}(h) = X_n(h) - hf(X_n(h)) + \sigma(nh)\Delta B_{n+1}, \tag{2.24}$$

where $h > 0$ is the constant step size and $\Delta B_{n+1} = B((n+1)h) - B(nh)$. We see that if

$$\sigma_n = \sigma(nh), \quad \xi_{n+1} = \frac{B((n+1)h) - B(nh)}{\sqrt{h}},$$

then (2.24) coincides with (2.2). Moreover, we note that condition (2.20) is satisfied for the common distribution function of the $\xi$s, which is that of a standardized normal random variable.

There is an intensive literature on the finite-time strong convergence of the EM approximation (2.24) to the true solution of (2.23), e.g., [10, 12, 19, 22]. However, our aim here is not to show the finite-time strong convergence of $x_n$ to $X(nh)$ but to analyze the stability of the numerical method. Stability analysis of numerical methods for SDEs is motivated by the question "for what choice of step size does the numerical method reproduce the characteristics of the test equation?" The mean-square asymptotic stability of numerical methods when the test equation is a linear scalar SDE has been studied by several authors (see, e.g., [9, 31]). More recently, in [11], the exponential mean-square stability of numerical methods to general nonlinear $n$-dimensional SDEs has been investigated. But the almost sure asymptotic stability of numerical methods to SDEs has been less studied than the mean-square stability. Only recently, in [31], was the almost sure asymptotic stability of the weak EM method to a linear scalar SDE discussed. Our aim here is to discuss the almost sure asymptotic stability of the strong EM method (2.24) to the nonlinear scalar SDE (2.23).

It transpires that (2.2) is a good discrete model of (2.23) because, under the conditions stipulated for Theorem 3, solutions of the continuous problem have the same asymptotic behavior. The following theorem shows that the correspondence with Theorem 3 is exact.

**Theorem 4.** *Suppose that $f$ is a locally Lipschitz continuous function that obeys* (2.3) *and* (2.10) *and $\sigma$ obeys*

$$\limsup_{t\to\infty} \frac{\log|\sigma(t)|}{\log t} < -\frac{\beta}{\beta-1}, \tag{2.25}$$

*where $\beta > 1$ is the exponent in* (2.10).

*If $X$ is the unique continuous $\mathcal{F}^B(t)$–adapted process that obeys* (2.23) *and satisfies $\lim_{t\to\infty} X(t) = 0$ a.s. and a is the constant defined in* (2.10)*, then there is a random variable $L$, which assumes either the values $0$ or $[a(\beta-1)]^{-1/(\beta-1)}$, such that*

$$\lim_{t\to\infty} t^{\frac{1}{\beta-1}}|X(t)| = L \quad a.s. \tag{2.26}$$

The condition that $X(t) \to 0$ as $t \to \infty$ a.s. can be satisfied under a variety of conditions on $f$ and $\sigma$ consistent with those required to prove Theorem 4. For example, an analogue of Theorem 1 can be proven by means of a continuous analogue of Lemma 1. Alternatively, we can employ a result of Chan and Williams in [6] to show that $X(t) \to 0$ as $t \to \infty$. The continuous analogue, whose proof we do not supply, is as follows.

**Theorem 5.** *Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz continuous function, which obeys* (2.3), (2.5), *and* (2.4), *and that $\sigma \in C([0,\infty);\mathbb{R})$ obeys*

$$\int_0^\infty \sigma^2(s)\,ds < \infty.$$

*If $X$ is the unique continuous $\mathcal{F}^B(t)$-adapted process that obeys* (2.23), *then*

$$\lim_{t\to\infty} X(t) = 0 \quad a.s.$$

Since Theorem 3 faithfully reproduces the rate of decay of the continuous problem stated in Theorem 4 when convergence takes place, it is reasonable to ask whether any discretization (2.2) of (2.23) with $\sigma_n = \sigma(nh)$ will suffice. It is at this point that the condition that solutions converge becomes important and that a restriction on the step size $h$ of the Euler–Maruyama scheme should be imposed in order to ensure this consistency. Such a condition ($h < 2/K$) is required to guarantee convergence in Theorem 1.

The proof of Theorem 4 can be inferred easily from results in [1] and [2]. The reasoning is as follows. By (2.25) and [2, Lemma 5.1], the solution of

$$dY(t) = -Y(t)\,dt + \sigma(t)\,dB(t)$$

obeys

$$\lim_{t\to\infty} \frac{\log|Y(t)|}{\log t} < -\frac{\beta}{\beta-1} \quad \text{a.s.}$$

Then $Z(t) = X(t) - Y(t)$ obeys $\lim_{t\to\infty} Z(t) = 0$ a.s. and, if $g(t) := f(Z(t)) - f(Z(t)+Y(t)) + Y(t)$, then

$$Z'(t) = -f(Z(t)) + g(t), \quad t > 0.$$

Since $f$ is locally Lipschitz continuous, while $Z(t) \to 0$ and $Z(t)+Y(t) \to 0$ as $t \to \infty$, it follows that for each $\omega$ in an almost sure set, there is a finite $K(\omega) > 0$ such that $|g(t,\omega)| \le K(\omega)|Y(t,\omega)|$ for all $t \ge 0$, and so

$$\limsup_{t\to\infty} \frac{\log|g(t)|}{\log t} < -\frac{\beta}{\beta-1} \quad \text{a.s.}$$

Using this estimate, (2.10), and the fact that $Z(t) \to 0$ as $t \to \infty$, by [1, Lemmas 4.1–4.3] we have that

$$\lim_{t\to\infty} t^{\frac{1}{\beta-1}} Z(t) = L \quad \text{a.s.},$$

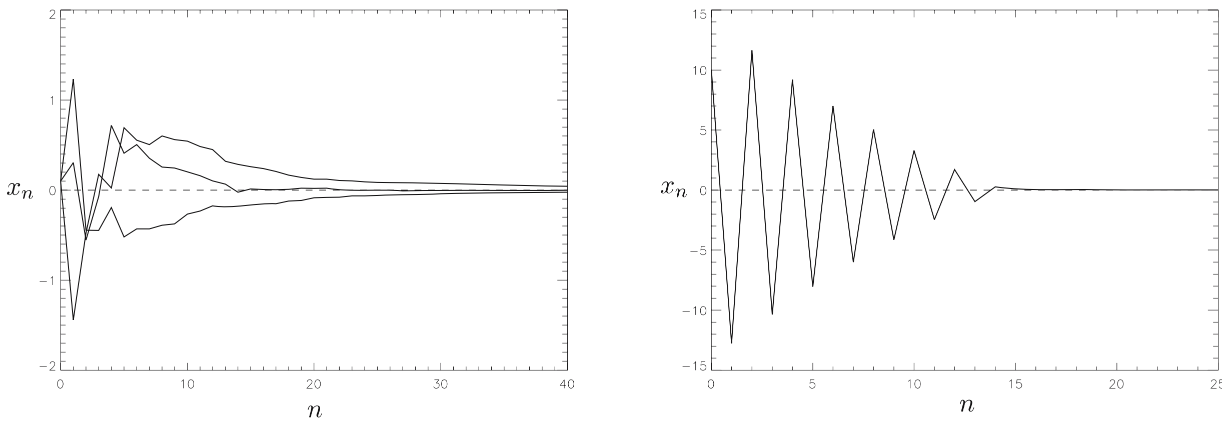
where $L$ assumes either the value 0 or $(a(\beta-1))^{-1/(\beta-1)}$. The claim now follows for $X = Z + Y$.

## 3. Simulations

In this section, we present some computer simulations of the stochastic difference equation (2.2). Our methodology for this simulation is standard, but we record it here for the sake of completeness. The code was implemented in the C programming language. Uniform pseudorandom numbers in $[0,1]$ were generated via the in-built integer random number generator `rand()`. Standardized normal random variables $\xi$ were then generated according to $\xi = \sqrt{-2\log U_1}\cos(2\pi U_2)$, where $U_1$ and $U_2$ are uniform random numbers in $[0,1]$.

In the simulations below, the function $f$ and noise intensity $(\sigma_n)$ always take the same form. We have chosen

$$f(x) = \begin{cases} \frac{3}{2}x + \frac{1}{2}, & x < -1, \\ \operatorname{sgn}(x)|x|^{3/2}, & x \in [-1,1], \\ \frac{3}{2}x - \frac{1}{2}, & x > 1, \end{cases}$$




$h = 0.2$, $X_0 = 0.1$ $\qquad$ $h = 1.3$, $X_0 = 10$

Fig. 1. $X_n \to 0$ as $n \to \infty$, as predicted by Theorem 1

and

$$\sigma_n = \frac{5}{(1+nh)^{7/2}}, \quad n \geq 0.$$

In each case, we have taken $(\xi_n)_{n\geq 1}$ to be a sequence of independent normal random variables with zero mean and variance one. In the nomenclature of the problem, we see that $\beta = 3/2$, $a = 1$, and we may define $K > 0$ according to

$$K = K_1 \vee K_2, \qquad \text{where } K_1 = \sup_{x>0} \frac{f(x)}{x}, \quad K_2 = \sup_{x<0} \frac{f(x)}{x}.$$

Therefore, $K = 3/2$. By dint of this choice of $(\sigma_n)$, we may view Eq. (2.2) as a discretization of the stochastic differential equation

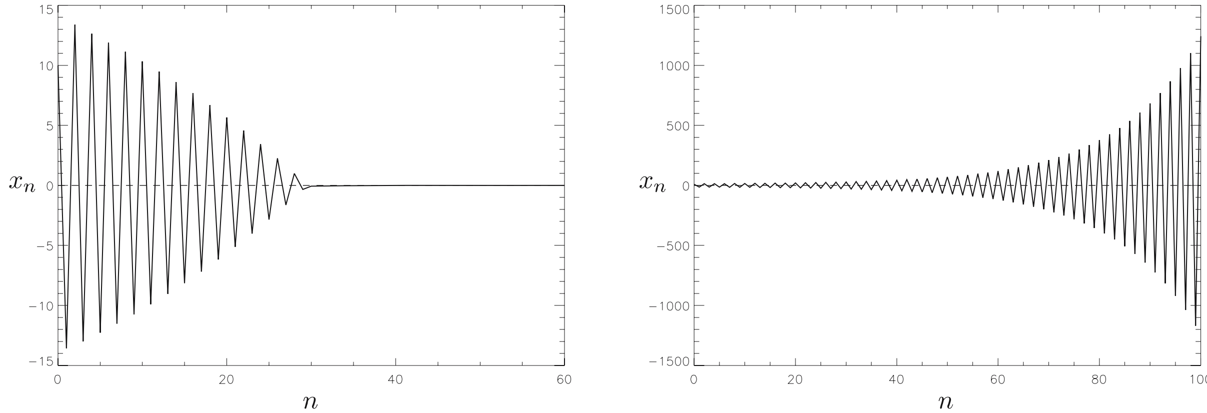
$$dX(t) = -f(X(t))\,dt + \sigma(t)\,dB(t),$$

where $B$ is a standard one-dimensional Brownian motion and $\sigma(t) = 5/(1+t)^{7/2}$ for all $t \geq 0$.

By Theorem 1, it follows that $\lim_{n\to\infty} X_n = 0$ a.s. provided that $0 < h < 2/K = 4/3 = 1.333\ldots = h_{\text{crit}}$. In Fig. 1 below, we show that the simulations reflect this result for two values of $h$. In the first instance, when $h = 0.2$, we show the asymptotic behavior of three different sample paths $n \mapsto X_n(\omega)$. The randomness is introduced by selecting a different integer seed (and, therefore, a different outcome $\omega$) for the random number generator in each case.

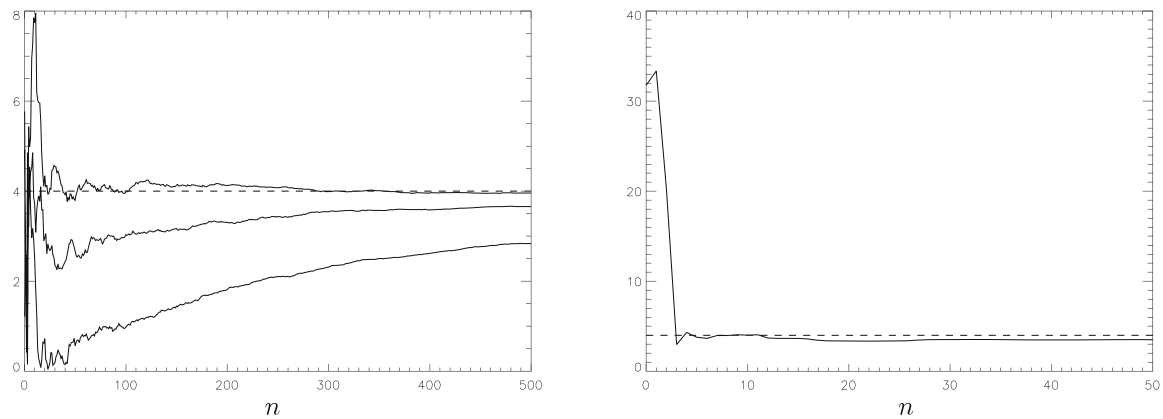
We observe that the sample path in the case where $h = 1.3$ tends to zero only after a number of oscillations around the equilibrium position. It may be suspected, as is often the case, that these oscillations are a precursor to the instability in the difference equation. That this happens for $h$ close to $h_{\text{crit}}$ suggests that the analysis in Theorem 1 may be quite sharp in the restriction it places on $h$ if the almost sure global stability of the solution is to be assured. In the second case (when $h = 1.3$), note that the initial-value condition $X_0 = 10$ was selected far from the equilibrium, while the initial-value condition is relatively close to the equilibrium $X_0 = 0.1$ in the first case (when $h = 0.2$). If the second case is viewed as a simulated sample path of the related stochastic differential equation, we see that the asymptotic stability is still assured, even when the initial-value condition is far from equilibrium and the step size is quite large.

In Fig. 2, we consider sample paths of $X$ in cases where Theorem 1 is not applicable, i.e., for which $h > h_{\text{crit}}$. From the first simulation (for $h = 1.35$), it appears that solutions of (2.2) can still be asymptotically stable with at least positive probability for $h > h_{\text{crit}}$. However, for a slightly larger value of $h$ ($h = 1.375$) and a sufficiently large initial-value condition, the second graph in Fig. 2 shows that




$h = 0.2,\ X_0 = 0.1$ $\qquad$ $h = 1.3,\ X_0 = 10$

Fig. 2. Cases not covered by Theorem 1




$h = 0.2,\ X_0 = 0.1$ $\qquad$ $h = 1,\ X_0 = 10$

Fig. 3. Graph of $n$ vs. $(nh)^2|X_n|$, suggesting that $(nh)^2|X_n| \to 4$ as $n \to \infty$, as predicted by Theorem 3. In each case, the dashed horizontal line is 4 units above the horizontal axis

solutions become unbounded with at least positive probability. Interpreting this result for numerical simulations of stochastic differential equations, this suggests that the analysis in Theorem 1 may place quite stringent estimates on the maximum allowable step size of the Euler scheme. Moreover, since Theorem 5 holds, Fig. 2 also suggests that the Euler scheme will not provide qualitatively accurate information about the pathwise stability of solutions of the underlying nonlinear stochastic differential equation when the step size is too large.

In each of Figs. 1 and 2, we plot $X_n$ versus $n$. However, in Fig. 3 we try to determine whether the theoretical asymptotic rate of decay given by Theorem 3 is exhibited by the simulations. Given that $h < 4/3$, Theorem 3 predicts that

$$\lim_{n\to\infty} (nh)^2 |X_n| = L \quad \text{a.s.},$$

where the random variable $L$ assumes either of the values 0 or 4. Therefore, we plot $n$ versus $(nh)^2|X_n|$ and examine whether the graph tends to 0 or 4 as $n \to \infty$. The simulations are consistent with the result of Theorem 3; indeed, based on simulations such as those shown in Fig. 3, we are led to suspect that $L = 4$ a.s.

Finally, we have not proven results for the local asymptotic stability of the stochastic difference equation (2.2) in this paper, nor have we proven a result on the rate of decay of solutions when the noise intensity $\sigma_n$ decays to zero more slowly than allowed by Theorem 3. Based upon the consistency exhibited between theoretical predictions and the simulations that we present here, we anticipate that for these open questions, such simulations will provide useful guidance in formulating conjectures concerning the asymptotic behavior.

## 4. Proofs

**4.1. Proof of Theorem 1.** Define $\eta_{n+1} = \xi_{n+1}^2 - \mathbb{E}[\xi_{n+1}^2]$ for each $n \in \mathbb{N}$. Then $(\eta_{n+1})_{n\in\mathbb{N}}$ is an $\mathcal{F}_n$-martingale difference. Noting that $\mathbb{E}[\xi_{n+1}^2] = 1$ and squaring both sides of (2.2) yields

$$X_{n+1}^2 = X_n^2 + hf(X_n)[-2X_n + hf(X_n)] + h\sigma_n^2 + h\sigma_n^2\eta_{n+1} + 2\sigma_n\sqrt{h}(X_n - hf(X_n))\xi_{n+1}.$$

Identify

$$u_n := h\sigma_n^2,$$
$$\nu_{n+1} := h\sigma_n^2\eta_{n+1} + 2\sigma_n\sqrt{h}(X_n - hf(X_n))\xi_{n+1},$$
$$v_n := hf(X_n)[2X_n - hf(X_n)],$$

and

$$Z_n = X_n^2.$$

Then $(\nu_n)_{n\in\mathbb{N}}$ is an $\mathcal{F}_n$-martingale difference. Clearly, $u_n \geq 0$. As to $v_n$, we note that when $x_n = 0$, we have $v_n = 0$. When $X_n \neq 0$, we use (2.4), (2.5) and the fact that $Kh < 2$ to obtain

$$v_n = hX_nf(X_n)\left(2 - h\frac{f(X_n)}{X_n}\right) > 0.$$

Therefore, $v_n \geq 0$, and $v_n$ is $\mathcal{F}_n$-measurable. Finally, as

$$|v_n| \leq h|f(X_n)|[2|X_n| + h|f(X_n)|] \leq |X_n|^2hK(2 + hK),$$

we have that $\mathbb{E}|v_n| < \infty$ for all $n \in \mathbb{N}$ because $\mathbb{E}[X_n^2] < \infty$ for all $n \in \mathbb{N}$. Therefore, all the hypotheses of Lemma 1 hold, and it follows from (2.6) that

$$\lim_{n\to\infty} X_n^2 \quad \text{exists a.s. and is a.s. finite}$$

and

$$\sum_{n=1}^{\infty} v_n < \infty \quad \text{a.s.}$$

Now there exists $l > 0$ such that $2 - hf(x)/x \geq l$ for all $x \neq 0$. Thus, for $X_n \neq 0$, we have

$$v_n = hX_nf(X_n)\left(2 - h\frac{f(X_n)}{X_n}\right) \geq hlX_nf(X_n),$$

and this estimate holds in the case where $X_n = 0$ as well. Therefore,

$$\sum_{n=1}^{\infty} X_nf(X_n) < \infty.$$

This implies that $X_nf(X_n) \to 0$ as $n \to \infty$. Define

$$\Omega_+ = \{\omega : \lim_{n\to\infty} X_n^2(\omega) = L(\omega) > 0\}$$

and suppose that $\mathbb{P}[\Omega_+] > 0$. Then, for $\omega \in \Omega_+$, we have $\lim_{n\to\infty} f(X_n(\omega)) \to 0$.

If $X_n(\omega)$ tends to a limit (which must be $\pm\sqrt{L(\omega)} \neq 0$), then the continuity of $f$ implies that $f(\pm\sqrt{L(\omega)}) = 0$, which contradicts the assumption that $L(\omega) > 0$. So, in order to avoid a contradiction, it must be that there exist sequences $(n_j(\omega))_{j\mathbb{N}}$ and $(m_j(\omega))_{j\in\mathbb{N}}$ such that $n_j \nearrow \infty$ and $m_j \nearrow \infty$ as $j \to \infty$ and we have

$$X_{n_j(\omega)}(\omega) \to -\sqrt{L(\omega)}, \quad X_{m_j(\omega)}(\omega) \to \sqrt{L(\omega)} \quad \text{as } j \to \infty.$$

Then, by the continuity of $f$, we have

$$0 = \lim_{j\to\infty} f(X_{n_j(\omega)}(\omega)) = f(-\sqrt{L(\omega)}),$$

and so, we must have $L(\omega) = 0$ once more. Therefore, the set $\Omega_+$ must have zero probability, in contradiction to the earlier supposition, and so the theorem is true.

**4.2. Proof of Lemma 2.** By (2.10), there exists $\delta > 0$ such that

$$x - hf(x) > 0, \quad x \in (0,\delta), \qquad -x + hf(x) > 0, \quad x \in (-\delta, 0).$$

Since $x_n \to 0$ as $n \to \infty$, it is possible to choose $N \in \mathbb{N}$ so that $|x_n| < \delta$ for all $n > N$. Also, as $x_n \to 0$ as $n \to \infty$, for every $\varepsilon \in (0, a/2)$, there is an $N_1(\varepsilon) \in \mathbb{N}$ such that, for all $n > N_1$, we have

$$\left|\frac{\operatorname{sgn}(x_n)f(x_n)}{|x_n|^\beta} - a\right| < \varepsilon.$$

Suppose now that $n > N \vee N_1$. We consider the cases $x_n > 0$ (a), $x_n < 0$ (b), and $x_n = 0$ (c). In each case we show that

$$|x_{n+1}| \le |x_n| - h(a-\varepsilon)|x_n|^\beta + |g_n|, \quad n > N \vee N_1. \tag{4.1}$$

(a) If $x_n > 0$ as $|x_n| < \delta$, then we have $x_n - hf(x_n) > 0$, and so

$$|x_{n+1}| \le |x_n - hf(x_n)| + |g_n| = x_n - hf(x_n) + |g_n| = |x_n| - hf(x_n) + |g_n|.$$

Since $n > N_1$, it follows that $-f(x_n) < -(a-\varepsilon)|x_n|^\beta$, and (4.1) holds.

(b) If $x_n < 0$ as $|x_n| < \delta$, then we have $-x_n + hf(x_n) > 0$, and so

$$|x_{n+1}| \le |x_n - hf(x_n)| + |g_n| = |-x_n + hf(x_n)| + |g_n| = -x_n + hf(x_n) + |g_n| = |x_n| + hf(x_n) + |g_n|.$$

Since $n > N_1$, we have $f(x_n) < -(a-\varepsilon)|x_n|^\beta$ and (4.1) holds.

(c) If $x_n = 0$, then $f(x_n) = 0$ and we have

$$|x_{n+1}| = |x_n - hf(x_n) + g_n| = |g_n| = |x_n| - h(a-\varepsilon)|x_n|^\beta + |g_n|,$$

which is (4.1).

Define $x_* > 0$ by $x_*^{\beta-1} = (ha\beta/2)^{-1}$. Since $x_n \to 0$ as $n \to \infty$, there is $N_2 \in \mathbb{N}$ such that $|x_n| < x_*$ for all $n > N_2$. Suppose that there exists $N_3 \ge 1 + (N \vee N_1 \vee N_2)$ such that $|x_{N_3}| > 0$; if such an $N_3$ does not exist, then $x_n = 0$ for all $n \ge 1 + (N \vee N_1 \vee N_2)$ and (2.13) holds with $L = 0$. Now, let $f_2$ be an increasing and continuous function on $[x_*, \infty)$ such that

$$f_2(x_*) = x_* - h(a-\varepsilon)x_*^\beta < x_*$$

and $\lim_{x\to\infty} f_2(x) < x_*$. Further, define

$$F_\varepsilon(x) = \begin{cases} x - h(a-\varepsilon)x^\beta, & x \in [0, x_*), \\ f_2(x), & x \ge x_*. \end{cases}$$

Then $x - F_\varepsilon(x) > 0$ for all $x > 0$ and $F_\varepsilon$ is increasing on $(0,\infty)$. The last fact follows from the definition by showing that $F_\varepsilon'(x) > 0$ for $x \in (0, x_*)$. Since $|x_n| < x_*$ for all $n \ge N_3$, we have (by (4.1))

$$|x_{n+1}| \le F_\varepsilon(|x_n|) + |g_n|, \quad n \ge N_3; \qquad |x_n| < x_*, \quad n \ge N_3. \tag{4.2}$$

Define $(\phi_n)_{n\ge N_3}$ by

$$\phi_{n+1} = F_\varepsilon(\phi_n), \quad n \ge N_3; \qquad \phi_{N_3} = \frac{1}{2}|x_{N_3}| < x_*. \tag{4.3}$$

It is routine to show that $\phi_n \in (0, x_*)$ for all $n \geq N_3$ and $(\phi_n)$ is decreasing. Therefore, there is $\phi_* \in [0, x_*)$ such that $\phi_n \to \phi_*$ as $n \to \infty$. Since $F_\varepsilon$ is continuous, we have $\phi_* = F_\varepsilon(\phi_*)$. However, $F_\varepsilon(x) < x$ for $x \in (0, x_*)$, so we must have $\phi_* = 0$. Define $(w_n)_{n \geq N_3}$ by

$$w_{n+1} = F_\varepsilon(w_n) + |g_n|, \quad n \geq N_3; \qquad w_{N_3} = 2|x_{N_3}| > |x_{N_3}|. \tag{4.4}$$

We now prove that

$$w_n > |x_n|, \quad n \geq N_3. \tag{4.5}$$

This is certainly true for $n = N_3$. Suppose that $w_n > |x_n|$ for $n > N_3$. Then, since $F_\varepsilon$ is increasing on $[0, \infty)$, $(x_n)_{n \geq N_3}$ obeys (4.2), and $(w_n)_{n \geq N_3}$ obeys (4.4), we have

$$w_{n+1} = F_\varepsilon(w_n) + |g_n| > F_\varepsilon(|x_n|) + |g_n| \geq |x_{n+1}|,$$

hence, (4.5) holds by induction.

Similarly, we have

$$w_n > \phi_n, \quad n \geq N_3. \tag{4.6}$$

This holds valid for $n = N_3$. Suppose that $w_n > \phi_n$ for $n > N_3$. Then, since $F_\varepsilon$ is increasing on $[0, \infty)$, $(\phi_n)_{n \geq N_3}$ obeys (4.3), and $(w_n)_{n \geq N_3}$ obeys (4.4), we have

$$w_{n+1} = F_\varepsilon(w_n) + |g_n| \geq F_\varepsilon(w_n) > F_\varepsilon(\phi_n) = \phi_{n+1};$$

hence, (4.6) holds by induction.

Since $g_n \to 0$ as $n \to \infty$, there is a $c > 0$ such that $|g_n| < c$ for all $n \geq N_3$. Also, we have $F_\varepsilon(x) < x_*$ for all $x \geq 0$. It then follows that $w_n < x_* + c$ for all $n \geq N_3$. Suppose, to the contrary, that there is $n \geq N_3$ such that $w_{n+1} \geq x_* + c$. Then

$$x_* + c \leq w_{n+1} = F_\varepsilon(w_n) + |g_n| < x_* + c,$$

a contradiction. Therefore, $w_* := \limsup_{n \to \infty} w_n \geq 0$ exists and is finite. Suppose that $w_* > 0$. Then, since $g_n \to 0$ as $n \to \infty$ and $F_\varepsilon$ is increasing, we have

$$w_* = \limsup_{n \to \infty} \bigl[F_\varepsilon(w_n) + g_n\bigr] = \limsup_{n \to \infty} F_\varepsilon(w_n) \leq F_\varepsilon(w_*).$$

But $x > F_\varepsilon(x)$ for $x > 0$, so this is impossible, which forces $w_* = 0$. Hence, $w_n \to 0$ as $n \to \infty$.

Since $\phi_n \in (0, x_*)$ for all $n \geq N_3$, it follows that

$$\phi_{n+1} = \phi_n - h(a - \varepsilon)\phi_n^\beta, \quad n \geq N_3.$$

We also know that $\phi_n \to 0$ as $n \to \infty$. We now prove that

$$\lim_{n \to \infty} (nh)^{\frac{1}{\beta - 1}} \phi_n = \left( \frac{1}{(a - \varepsilon)(\beta - 1)} \right)^{\frac{1}{\beta - 1}}. \tag{4.7}$$

Define

$$G(x) = -\frac{1}{h(a - \varepsilon)} \frac{x^{1 - \beta}}{\beta - 1}, \quad x > 0.$$

Thus, $G \in C^2((0, \infty); \mathbb{R})$. Then, there exists $\eta_n \in (\phi_n - h(a - \varepsilon)\phi_n^\beta, \phi_n)$ such that

$$G(\phi_{n+1}) = G(\phi_n - h(a - \varepsilon)\phi_n^\beta) = G(\phi_n) - G'(\phi_n)h(a - \varepsilon)\phi_n^\beta + \frac{1}{2}G''(\eta_n)h^2(a - \varepsilon)^2\phi_n^{2\beta},$$

so for $n \geq N_3$, by defining $y_n = G(\phi_n)$, we have

$$y_{n+1} = y_n - 1 - \frac{1}{2}\beta h(a - \varepsilon)\eta_n^{-\beta - 1}\phi_n^{2\beta}.$$

Since $\phi_n \to 0$ as $n \to \infty$, we have $\eta_n/\phi_n \to 1$ as $n \to \infty$, and, therefore,

$$\lim_{n \to \infty} \bigl[y_{n+1} - y_n\bigr] = -1.$$

By writing

$$\frac{y_n}{n} = \frac{1}{n} y_{N_3} + \frac{1}{n} \sum_{j=N_3}^{n-1} [y_{j+1} - y_j], \quad n \geq N_3,$$

and letting $n \to \infty$, we see that $\lim_{n\to\infty} y_n/n = -1$, and so $\lim_{n\to\infty} G(\phi_n)/n = -1$. Manipulation of this limit now gives (4.7).

Since $(g_n)$ obeys (2.11), it follows that

$$\lim_{n\to\infty} n^{\frac{\beta}{\beta-1}} g_n = 0,$$

and so, by (4.6) and (4.7), we have

$$\frac{|g_n|}{w_n^\beta} < \frac{|g_n|}{\phi_n^\beta} = \frac{n^{\beta/(\beta-1)}|g_n|}{(n^{1/(\beta-1)}\phi_n)^\beta} \to 0 \quad \text{as } n \to \infty.$$

Next, as $w_n \to 0$ as $n \to \infty$, we have $w_n < x_*$ for all $n > N_4$, and so, for $n > N_4 \vee N_3$, we have

$$\frac{w_{n+1} - w_n}{w_n^\beta} = -h(a-\varepsilon) + \frac{g_n}{w_n^\beta} := -a_n(\varepsilon).$$

Since $g_n/w_n^\beta \to 0$ as $n \to \infty$, we have $a_n(\varepsilon) \to h(a-\varepsilon) > 0$ as $n \to \infty$. Therefore, there is $N_5 \in \mathbb{N}$ such that $a_n(\varepsilon) > 0$ for all $n > N_5$. Take $N_6 = N_3 \vee N_4 \vee N_5$. We now proceed as in the proof of (4.7) above to show that

$$\lim_{n\to\infty} (nh)^{\frac{1}{\beta-1}} w_n = \left(\frac{1}{(a-\varepsilon)(\beta-1)}\right)^{\frac{1}{\beta-1}}. \tag{4.8}$$

To do this, observe that for all $n \geq N_6$, there exists $\mu_n \in (w_n - a_n(\varepsilon)w_n^\beta, w_n)$ such that

$$G(w_{n+1}) = G(w_n - a_n(\varepsilon)w_n^\beta) = G(w_n) - G'(w_n)a_n(\varepsilon)w_n^\beta + \frac{1}{2}G''(\mu_n)a_n^2(\varepsilon)w_n^{2\beta},$$

so for $n \geq N_6$, by defining $u_n = G(w_n)$, we have

$$u_{n+1} = u_n - \frac{a_n(\varepsilon)}{h(a-\varepsilon)} - \frac{1}{2}\frac{\beta}{h(a-\varepsilon)} a_n^2(\varepsilon)\mu_n^{-\beta-1}w_n^{2\beta}.$$

Since $w_n \to 0$ as $n \to \infty$, we have $\mu_n/w_n \to 1$ as $n \to \infty$, and since $a_n(\varepsilon) \to h(a-\varepsilon)$, it follows that

$$\lim_{n\to\infty} \left[u_{n+1} - u_n\right] = -1.$$

This implies that $\lim_{n\to\infty} G(w_n)/n = \lim_{n\to\infty} u_n/n = -1$, which rearranges to give (4.8).

The result now follows easily: since $|x_n| < w_n$ for all $n \geq N_3$, we have

$$\limsup_{n\to\infty} (nh)^{\frac{1}{\beta-1}} |x_n| \leq \left(\frac{1}{(a-\varepsilon)(\beta-1)}\right)^{\frac{1}{\beta-1}},$$

so, by letting $\varepsilon \to 0^+$, we get (2.14).

**4.3. Proof of Lemma 3.** Let $L = (a(\beta-1))^{-1/(\beta-1)}$. According to (2.14), there exists $0 \leq L_0 \leq L$ such that

$$\limsup_{n\to\infty} (nh)^{\frac{1}{\beta-1}} |x_n| = L_0. \tag{4.9}$$

If $L_0 = 0$, then we have the first part of (2.15). Suppose now that $L_0 \in (0, L)$. Then for every $\varepsilon \in (0, L - L_0)$, there is $N_1(\varepsilon) > 0$ such that

$$(nh)^{\frac{1}{\beta-1}} |x_n| \leq L_0 + \varepsilon, \quad n \geq N_1(\varepsilon).$$

Since $g$ obeys (2.11), it follows that $\sum_{j=n}^{\infty} |g_j|$ is well defined for all $n \geq 0$ and, moreover, as

$$(nh)^{\frac{\beta}{\beta-1}}|x_n|^{\beta} \leq (L_0 + \varepsilon)^{\beta},$$

we have that $\sum_{j=n}^{\infty} |f(x_j)|$ is well defined for every $n \geq 0$. By (2.12), we have

$$-x_n = -h \sum_{j=n}^{\infty} f(x_j) + \sum_{j=n}^{\infty} g_j,$$

and so

$$(nh)^{\frac{1}{\beta-1}}|x_n| \leq h(nh)^{\frac{1}{\beta-1}} \sum_{j=n}^{\infty} \frac{|f(x_j)|}{|x_j|^{\beta}} \left((jh)^{\frac{1}{\beta-1}}|x_j|\right)^{\beta} (jh)^{-\frac{\beta}{\beta-1}} + (nh)^{\frac{1}{\beta-1}} \sum_{j=n}^{\infty} |g_j|. \tag{4.10}$$

Next, for every $\varepsilon \in (0, a)$, there is $N_2(\varepsilon) > 0$ such that $|f(x_n)|/|x_n|^{\beta} < a + \varepsilon$ for all $n > N_2(\varepsilon)$ and $(nh)^{\beta/(\beta-1)}|g_n| < \varepsilon$. Now, let $\varepsilon \in (0, a \wedge (L - L_0))$ and $N(\varepsilon) = N_1(\varepsilon) \vee N_2(\varepsilon)$. Then for $n > N(\varepsilon)$, by (4.10) we have

$$\begin{aligned}(nh)^{\frac{1}{\beta-1}}|x_n| &\leq (a+\varepsilon)(L_0+\varepsilon)^{\beta} n^{\frac{1}{\beta-1}} \sum_{j=n}^{\infty} j^{-\frac{\beta}{\beta-1}} + (nh)^{\frac{1}{\beta-1}} \sum_{j=n}^{\infty} (jh)^{\beta/(\beta-1)}|g_j|(jh)^{-\beta/(\beta-1)} \\ &\leq \left((a+\varepsilon)(L_0+\varepsilon)^{\beta} + \varepsilon h^{-1}\right) n^{\frac{1}{\beta-1}} \sum_{j=n}^{\infty} j^{-\frac{\beta}{\beta-1}}.\end{aligned}$$

Therefore,

$$L_0 = \limsup_{n\to\infty} (nh)^{\frac{1}{\beta-1}}|x_n| \leq \left((a+\varepsilon)(L_0+\varepsilon)^{\beta} + \varepsilon h^{-1}\right)(\beta - 1) \tag{4.11}$$

because

$$\lim_{n\to\infty} n^{\frac{1}{\beta-1}} \sum_{j=n}^{\infty} j^{-\frac{\beta}{\beta-1}} = \beta - 1.$$

The validity of this limit follows readily from the fact that $\int_n^{\infty} s^{-\beta/(\beta-1)}\, ds = (\beta-1)n^{-1/(\beta-1)}$ and

$$\sum_{j=n+1}^{\infty} j^{-\frac{\beta}{\beta-1}} \leq \int_n^{\infty} s^{-\beta/(\beta-1)}\, ds \leq \sum_{j=n}^{\infty} j^{-\frac{\beta}{\beta-1}}.$$

Returning to (4.11) and letting $\varepsilon \downarrow 0$ yields $L_0 \leq aL_0^{\beta}(\beta-1)$, so $L_0 \geq L$. But this contradicts $L_0 < L$. Thus, either $L_0 = 0$ or $L_0 = L$ in (4.9), as needed in (2.15).

**4.4. Proof of Lemma 4.** Let $L = (ah(\beta-1))^{-1/(\beta-1)}$. Fix $C \in (0, L)$ and choose $\varepsilon > 0$ so small that $\varepsilon < 1 \wedge (a/2)$ and

$$-\frac{C}{\beta-1} + h(a+\varepsilon)C^{\beta}(1+\varepsilon)^{\frac{\beta}{\beta-1}} + \varepsilon(1+2\varepsilon)^{\frac{\beta}{\beta-1}} < 0, \qquad (1+\varepsilon)^{1/(\beta-1)} < \frac{L}{C}. \tag{4.12}$$

By (2.10), there exists $\delta > 0$ such that

$$x - hf(x) > 0, \quad x \in (0, \delta), \qquad -x + hf(x) > 0, \quad x \in (-\delta, 0),$$

and $F^{\varepsilon} : [0, \delta) \to [0, \infty)$ defined by

$$F^{\varepsilon}(x) = x - h(a+\varepsilon)x^{\beta}, \quad x \in [0, \delta), \tag{4.13}$$

is increasing on $(0,\delta)$. Since $x_n \to 0$ as $n \to \infty$, it is possible to choose $N \in \mathbb{N}$ so that $|x_n| < \delta$ for all $n > N$. Also, since $x_n \to 0$ as $n \to \infty$ and $g_n$ obeys (2.11), for every $\varepsilon \in (0, a/2)$ there is an $N'(\varepsilon) \in \mathbb{N}$ such that

$$\left|\frac{\operatorname{sgn}(x_n) f(x_n)}{|x_n|^\beta} - a\right| < \varepsilon, \quad n^{\beta/(\beta-1)}|g_n| < \varepsilon.$$

Suppose now that $n > 1 + (N \vee N'(\varepsilon)) =: N_0(\varepsilon) > 1$. We consider the cases $x_n > 0$, $x_n < 0$, and $x_n = 0$. In each case we show that

$$|x_{n+1}| > F^\varepsilon(|x_n|) - |g_n|, \quad n > N_0(\varepsilon). \tag{4.14}$$

(a) If $x_n > 0$ and $|x_n| < \delta$, we have $x_n - hf(x_n) > 0$, so

$$|x_{n+1}| \geq |x_n - hf(x_n)| - |g_n| = x_n - hf(x_n) - |g_n| = |x_n| - hf(x_n) - |g_n|.$$

Since $n > N_0(\varepsilon)$, we have $-f(x_n) > -(a+\varepsilon)|x_n|^\beta$ and (4.14) holds.

(b) If $x_n < 0$ and $|x_n| < \delta$, we have $-x_n + hf(x_n) > 0$, so

$$|x_{n+1}| \geq |x_n - hf(x_n)| - |g_n| = |-x_n + hf(x_n)| - |g_n| = -x_n + hf(x_n) - |g_n| = |x_n| + hf(x_n) - |g_n|.$$

Since $n > N_0(\varepsilon)$, we have $f(x_n) > -(a+\varepsilon)|x_n|^\beta$ and (4.14) holds.

(c) If $x_n = 0$, then $f(x_n) = 0$ and we have

$$|x_{n+1}| = |x_n - hf(x_n) + g_n| = |g_n| \geq -|g_n| = |x_n| - h(a+\varepsilon)|x_n|^\beta - |g_n|,$$

which is (4.14).

Since $\limsup_{n\to\infty} n^{1/(\beta-1)}|x_n| = L$, there exists $N_3(\varepsilon) > 0$ and a sequence $(n_j)_{j\in\mathbb{N}}$ with $\lim_{j\to\infty} n_j = \infty$ such that

$$n_j^{\frac{1}{\beta-1}}|x_{n_j}| > C(1+\varepsilon)^{-\frac{1}{\beta-1}}$$

for all $n_j > N_3(\varepsilon)$. This is made possible by the fact that $L > C(1+\varepsilon)^{-\frac{1}{\beta-1}}$.

Now choose $N_1(\varepsilon)$ to be the smallest member of the sequence $(n_j)_{j\in\mathbb{N}}$ that is greater than $N_0(\varepsilon)/\varepsilon$. Also define $N_2(\varepsilon) = \varepsilon N_1(\varepsilon)$. Note that $|x_{N_1(\varepsilon)}| > 0$ and $N_1(\varepsilon) > N_0(\varepsilon)$. Define

$$y_n = C(n + N_2(\varepsilon))^{-\frac{1}{\beta-1}}, \quad n \geq N_1(\varepsilon). \tag{4.15}$$

Clearly, $(y_n)_{n\in\mathbb{N}}$ is a decreasing sequence. Also, by the definition of $N_2$, we have

$$y_{N_1(\varepsilon)} N_1(\varepsilon)^{\frac{1}{\beta-1}} = C(N_1(\varepsilon) + N_2(\varepsilon))^{-\frac{1}{\beta-1}} N_1(\varepsilon)^{\frac{1}{\beta-1}} = C(1+\varepsilon)^{-\frac{1}{\beta-1}}.$$

The construction of $N_1(\varepsilon)$ implies that

$$N_1(\varepsilon)^{\frac{1}{\beta-1}}|x_{N_1(\varepsilon)}| > C(1+\varepsilon)^{-\frac{1}{\beta-1}},$$

hence, we have

$$y_n \leq y_{N_1(\varepsilon)} < |x_{N_1(\varepsilon)}| < \delta, \qquad n \geq N_1(\varepsilon).$$

Let $n \geq N_1(\varepsilon)$. Since $y_n \in (0,\delta)$, we have $F^\varepsilon(y_n) = y_n - h(a+\varepsilon)y_n^\beta$, and so, defining the function $H$ by $H(x) = (x + N_2(\varepsilon))^{-1/(\beta-1)}$ for $x \geq 0$, we have

$$\begin{aligned}
&y_{n+1} - F^\varepsilon(y_n) + |g_n| \\
&= y_{n+1} - y_n + h(a+\varepsilon)y_n^\beta + |g_n| \\
&< C\left[H(n+1) - H(n)\right] + h(a+\varepsilon)C^\beta(n + N_2(\varepsilon))^{-\frac{\beta}{\beta-1}} + \varepsilon n^{-\frac{\beta}{\beta-1}}.
\end{aligned}$$

By the mean-value theorem, there exists $\mu_n \in [n, n+1]$ such that $H(n+1) - H(n) = H'(\mu_n)$. Moreover, $H'$ is an increasing function, so $H(n+1)-H(n) < H'(n+1)$. For notational compactness, we temporarily suppress the $\varepsilon$-dependence on $N_1$ and $N_2$ and thus obtain

$$\begin{aligned}
&y_{n+1} - F^\varepsilon(y_n) + |g_n| \\
&< \frac{-1}{\beta-1} C(n+N_2+1)^{-\frac{\beta}{\beta-1}} + h(a+\varepsilon)C^\beta (n+N_2)^{-\frac{\beta}{\beta-1}} + \varepsilon n^{-\frac{\beta}{\beta-1}} \\
&= (n+N_2+1)^{-\frac{\beta}{\beta-1}} \left( -\frac{C}{\beta-1} + h(a+\varepsilon)C^\beta \left(\frac{n+N_2+1}{n+N_2}\right)^{\frac{\beta}{\beta-1}} + \varepsilon \left(\frac{n+N_2+1}{n}\right)^{\frac{\beta}{\beta-1}} \right) \\
&\le (n+N_2+1)^{-\frac{\beta}{\beta-1}} \left( -\frac{C}{\beta-1} + h(a+\varepsilon)C^\beta \left(\frac{N_1+N_2+1}{N_1+N_2}\right)^{\frac{\beta}{\beta-1}} + \varepsilon \left(\frac{N_1+N_2+1}{N_1}\right)^{\frac{\beta}{\beta-1}} \right).
\end{aligned}$$

Using the fact that $N_2(\varepsilon) = \varepsilon N_1(\varepsilon)$ and $N_1(\varepsilon) > N_0(\varepsilon)/\varepsilon > 1/\varepsilon$ and the implication

$$N_1(\varepsilon)(1+\varepsilon) > N_0(\varepsilon)(1+\varepsilon)/\varepsilon > \frac{1+\varepsilon}{\varepsilon}$$

in turn, we have

$$\begin{aligned}
&y_{n+1} - F^\varepsilon(y_n) + |g_n| \\
&< (n+N_2(\varepsilon)+1)^{-\frac{\beta}{\beta-1}} \left( -\frac{C}{\beta-1} + h(a+\varepsilon)C^\beta \left(1 + \frac{1}{N_1(\varepsilon)(1+\varepsilon)}\right)^{\frac{\beta}{\beta-1}} + \varepsilon \left(1+\varepsilon+\frac{1}{N_1(\varepsilon)}\right)^{\frac{\beta}{\beta-1}} \right) \\
&< (n+N_2(\varepsilon)+1)^{-\frac{\beta}{\beta-1}} \left( -\frac{C}{\beta-1} + h(a+\varepsilon)C^\beta \left(1 + \frac{\varepsilon}{1+\varepsilon}\right)^{\frac{\beta}{\beta-1}} + \varepsilon \left(1+2\varepsilon\right)^{\frac{\beta}{\beta-1}} \right) \\
&< (n+N_2(\varepsilon)+1)^{-\frac{\beta}{\beta-1}} \left( -\frac{C}{\beta-1} + h(a+\varepsilon)C^\beta (1+\varepsilon)^{\frac{\beta}{\beta-1}} + \varepsilon(1+2\varepsilon)^{\frac{\beta}{\beta-1}} \right) \\
&< 0,
\end{aligned}$$

where we used (4.12) at the last step.

We now summarize what has been proven so far in this lemma: we have shown that

$$\begin{gathered}
y_{n+1} < F^\varepsilon(y_n) - |g_n|, \quad n \ge N_1(\varepsilon); \qquad 0 < y_{N_1(\varepsilon)} < |x_{N_1(\varepsilon)}|, \qquad 0 < y_n < \delta, \quad n \ge N_1(\varepsilon); \\
|x_{n+1}| > F^\varepsilon(|x_n|) - |g_n|, \quad n \ge N_1(\varepsilon); \qquad |x_n| < \delta, \quad n \ge N_1(\varepsilon).
\end{gathered}$$

We now prove that $|x_n| > y_n$ for all $n \ge N_1(\varepsilon)$. This is clearly true for $n = N_1(\varepsilon)$. Suppose that $|x_n| > y_n$ for some $n > N_1(\varepsilon)$. Then, as $y_n < |x_n| < \delta$, and $F^\varepsilon$ is increasing on $[0, \delta)$, we have

$$|x_{n+1}| > F^\varepsilon(|x_n|) - |g_n| > F^\varepsilon(y_n) - |g_n| > y_{n+1}.$$

Therefore, $|x_n| > y_n$ for all $n \ge N_1(\varepsilon)$ by induction. Therefore, it follows that

$$\liminf_{n\to\infty} n^{\frac{1}{\beta-1}} |x_n| \ge C,$$

and so, by letting $C \uparrow L$ and using (2.16), we see that (2.17) must hold.

### 4.5. Proof of Lemma 5.

Let $x > 0$. Then

$$\mathbb{P}[|\xi_{n+1}| > x] = \mathbb{P}[\{\xi_{n+1} > x\} \cup \{\xi_{n+1} < -x\}] \le \mathbb{P}[\xi_{n+1} > x] + \mathbb{P}[\xi_{n+1} < -x] = 1 - \Phi(x) + \Phi(-x).$$

Now, for every $\gamma > 0$, there exists $C_\gamma > 0$ such that

$$1 - \Phi(x) \le \frac{C_\gamma}{x^\gamma}, \quad \Phi(-x) \le \frac{C_\gamma}{x^\gamma}, \quad x > 0.$$

Now suppose that $(a_n)$ is an increasing sequence such that $a_n \to \infty$. Then

$$\sum_{n=0}^{\infty} \mathbb{P}[|\xi_{n+1}| > a_n] \leq \sum_{n=0}^{\infty} 1 - \Phi(a_n) + \Phi(-a_n) \leq \sum_{n=0}^{\infty} \frac{2C_\gamma}{a_n^\gamma}.$$

Choose $a_n = (1+n)^{2/\gamma}$. Then, by the first Borel–Cantelli lemma, for each $\gamma > 0$,

$$\limsup_{n\to\infty} (1+n)^{2/\gamma} |\xi_n| \leq 1 \quad \text{a.s.}$$

The claim follows.

**Acknowledgment.** The results in this paper were presented at the International Conference on Differential and Functional Differential Equations, held at the Steklov Mathematical Institute, Moscow, August 2005. The first author was supported by an Albert College Fellowship awarded by the Dublin City University Research Advisory Panel. The first and third authors also gratefully acknowledge the support of the Boole Centre for Research in Informatics, University College Cork, where the research was partly conducted. The second author wishes to thank the CosmoGrid project funded under the Programme for Research in Third Level Institutions (PRTLI) administered by the Irish Higher Education Authority for the use of simulation software. Finally, the authors are pleased to acknowledge Prof. X. Mao, University of Strathclyde, for his insight into the numerical methods of stochastic differential equations.

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