

MODELADO DE POLÍTICAS FITOSANITÁRIAS ÓPTIMAS

PARA CULTIVOS DE IMPORTANCIA ECONÓMICA

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Classic epidemic models

Consider the deterministic SI model:

$$\frac{dS}{dt} = -\frac{\beta}{N}SI + (b + \gamma)I$$

$$\frac{dI}{dt} = \frac{\beta}{N}SI - (b + \gamma)I$$

$$N = S(t) + I(t)$$

Where N is constant and

$$S(t) = N - I(t).$$

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$$\mathcal{R}_0 = \frac{\beta}{b + \gamma}$$

$$\mathcal{R}_0 \leq 1$$

$$\Rightarrow \lim_{t \rightarrow \infty} (S(t), N(t)) = (N, 0)$$

$$\mathcal{R}_0 > 1$$

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Kermack-McKendrick Model

$$\frac{dS}{dt} = -\beta SI$$

$$\frac{dI}{dt} = \beta SI$$

$$\frac{dR}{dt} = \gamma IN \quad = S(t) + I(t) + R(t)$$

Where N is constan.

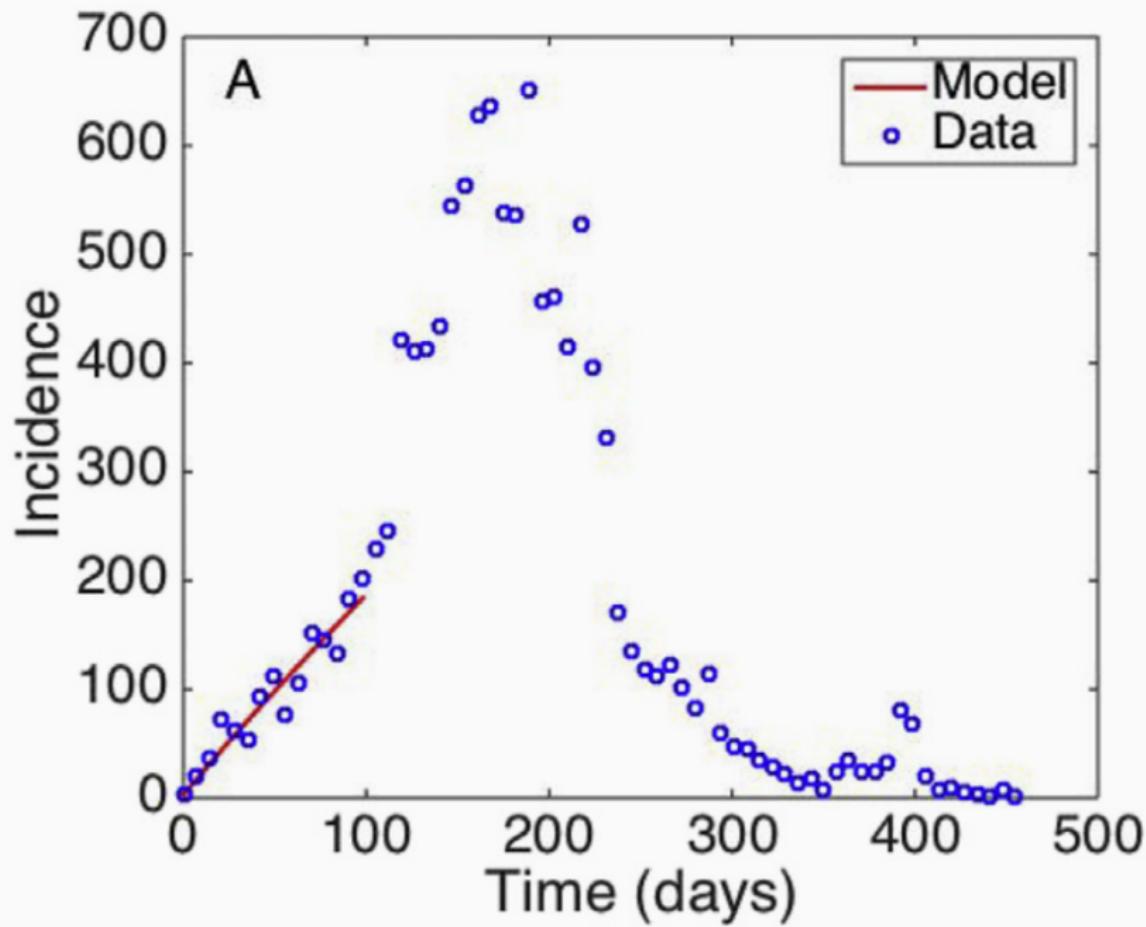
$$\mathcal{R}_0 = \frac{\beta}{\gamma}$$

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$$\mathcal{R}_0 > 1$$

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Important Problems

- Parameter estimation
- Control strategies modeling
- \mathcal{R}_0 definition in exotic dynamics: Stochastic, Fractional, Functional
- Modeling plant and animal diseases
- Forecasting

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Objective:

Illustrate epidemic modeling ideas in the formulation of an optimal controlled model.

Tomato leaf curl virus disease (TYLCVD)



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- Tomato plants **infected early** are severely stunted and will **not produce fruit**
- Leaflets are small and yellowed with edges that curl upwards
- Flowers either do not develop or fall off
- When **older plants** are infected, fruit that is already forming ripens normally, but **no new fruit** is formed after the infection
- TYLCV can be confused with several other conditions such as tomato big bud, herbicide damage and phosphate or magnesium deficiency



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Tomato leaf curl virus disease (TYLCVD)

Spread

- **TYLCV** is spread by the insect silverleaf whitefly (*Bemisia tabaci* B biotype)
- Silverleaf whiteflies pick up the virus by feeding on infected host plants. The whiteflies then spread the virus to healthy plants which show the symptoms 10 to 21 days later
- Silverleaf whiteflies are common in many countries and feed on many types of plants
- More than 22 species, including both annuals and perennials, are also known hosts of TLCV.



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Control

Cultural Control

- Physical barriers
- Planting dates
- Replanting of infected plants
- Host plant resistance

Biological Control

- Parasitoids
- Predators
- Fungi

Insecticides

- pymetrozine
- zeta-cypermethrin / bifenthrin



Shun-xiang, R., Zhen-zhong, W., Bao-li, Q., and Yuan, X. (2001).

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$$\frac{dS_p}{dt} = -\beta_p S_p I_v + r(L_p + I_p)$$

$$\frac{dL_p}{dt} = \beta_p S_p I_v - b L_p - r L_p$$

$$\frac{dI_p}{dt} = b L_p - r I_p$$

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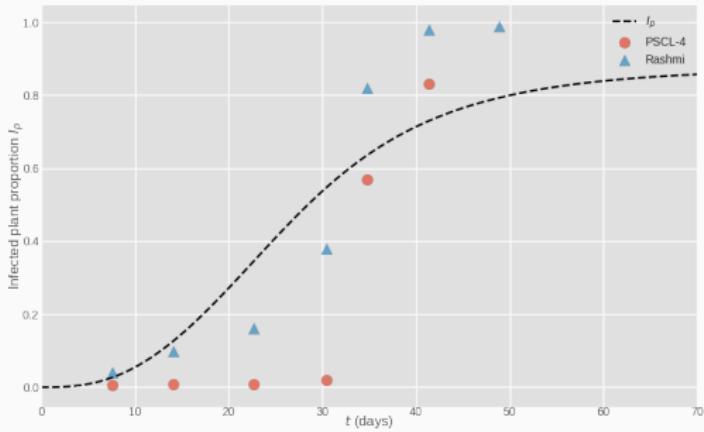
$$\frac{dI_v}{dt} = \beta_v S_v I_p - \gamma I_v + \theta\mu$$

$$R_0 = \sqrt{\frac{\beta_v \mu b \beta_p}{r^2(r+b)\gamma}}$$

$$DFE = (N_p, 0, 0, 0, \mu/\gamma)^\top$$

$$EE = (S_p^*, L_p^*, I_p^*, S_v^*, I_v^*)^\top$$

| Par. | Value | Descrip. |
|-----------|-------|---------------------------|
| β_p | 0.1 | latent rate |
| r | 0.01 | remove rate |
| $1/b$ | 0.075 | time of latency |
| γ | 0.06 | vector die or depart rate |
| μ | 0.3 | immigration rate |
| θ | 0.2 | infected vectors arrival |
| β_v | 0.003 | vector infected rate |



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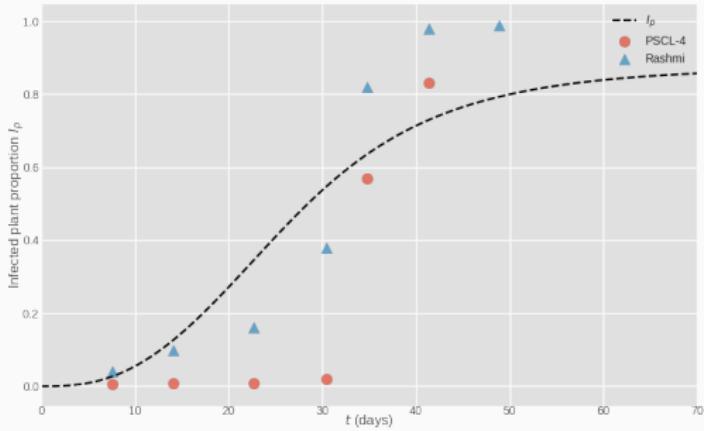
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Controls

$u_1(t)$: Latent replanting

$u_2(t)$: Infected replanting

$u_3(t)$: Fumigation

$$u_i(t) \in [0, u_i^{max}]$$

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$$\int_0^T \left[A_1 I_p(t) + A_2 L_p(t) + A_3 I_v(t) + \sum_{i=1}^3 c_i \frac{u_i^2}{2} \right] dt$$

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Optimal Control Problem

$$g(x, u) := A_1 I_p(t) + A_2 L_p(t) + A_3 I_v(t) + c_1 u_1(t)^2 + c_2 u_2(t)^2 + c_3 u_3(t)^2$$

$$\min_{\bar{u}(\cdot) \in \tilde{\mathcal{U}}_{x_0}[0, T]} J(u_1, u_2, u_3) = \int_0^T g(x, u) ds$$

such that:

$$\frac{dS_p}{dt} = \beta_p S_p I_v + (r + u_1) L_p + (r + u_2) I_p,$$

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$$x(0) = x_0, \quad u_i \in [0, u_i^{\max}]$$

Consider the controlled dynamics

$$\begin{cases} \dot{x}(s) = f(s, u(s), x(s)) & s \in [t_0, T], \\ x(t_0) = x_0, \end{cases}$$

with terminal state constraint

$$x(T; t_0, x_0, u(\cdot)) \in M, M \subseteq \mathbb{R}^n.$$

Problem (OC)

$(t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^n$, find a control policy
 $\bar{u}(\cdot) \in \tilde{\mathcal{U}}_{x_0}[t_0, T]$ s.t.

$$J(t_0, x_0; \bar{u}(\cdot)) = \inf_{u(\cdot) \in \tilde{\mathcal{U}}_{x_0}[t_0, T]} J(t_0, x_0; u(\cdot)).$$

Cost functional

$$\tilde{\mathcal{U}}_{x_0}[t_0, T] := \{u : [t_0, T] \rightarrow \mathbb{R}^n \mid \text{measurable}\}$$

$$\begin{aligned} J(t_0, x_0; u(\cdot)) = & \int_{t_0}^T g(s, u(s), x(s)) ds \\ & + h(T, x(T)) \end{aligned}$$

Hypothesis:

- (C-1) $f : \mathbb{R}_+ \times U \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is measurable, satisfies the lipchitz condition in x ,
 $|f(t, u, 0)| \leq L, \forall (t, u) \in \mathbb{R}_+ \times U$.
- (C-2) $g : \mathbb{R}_+ \times U \times \mathbb{R}^n \rightarrow \mathbb{R}$,
 $h : \mathbb{R}^n \rightarrow \mathbb{R}$ are measurable, and
- $$\begin{aligned} & |g(s, u, x_1) - g(s, u, x_2)| + \\ & |h(x_1) - h(x_2)| \\ & \leq \omega(|x_1| \vee |x_2|, |x_1 - x_2|) \end{aligned}$$
- $$\forall (s, u) \in \mathbb{R}_+ \times U, x_1, x_2 \in \mathbb{R}^n.$$
- (C-3) For a.a. $t \in [0, T]$, Cesari property holds $\forall x \in \mathbb{R}^n$.

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Modulus of continuity

$\omega : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$, increasing,
 $\omega(r, 0) = 0 \forall r \geq 0.$

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Cesari property

$$\begin{aligned} \mathbf{E}(t, x) = & \{(z^0, z) \in \mathbb{R} \times \mathbb{R}^n | \\ & z^0 \geq g(t, u, x), \\ & z = f(t, u, x), u \in U\}. \end{aligned}$$

$$\bigcap_{\delta} \bar{co} \mathbf{E}(t, B_\delta(x)) = \mathbf{E}(t, x)$$

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- (C-1) $f : \mathbb{R}_+ \times U \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is measurable, satisfies a Lipchitz condition in x , $|f(t, u, 0)| \leq L, \forall (t, u) \in \mathbb{R}_+ \times U$.
- (C-2) $g : \mathbb{R}_+ \times U \times \mathbb{R}^n \rightarrow \mathbb{R}$, $h : \mathbb{R}^n \rightarrow \mathbb{R}$ are measurable, and
- $$\begin{aligned} & |g(s, u, x_1) - g(s, u, x_2)| + \\ & |h(x_1) - h(x_2)| \\ & \leq \omega(|x_1| \vee |x_2|, |x_1 - x_2|) \end{aligned}$$
- $$\forall (s, u) \in \mathbb{R}_+ \times U, x_1, x_2 \in \mathbb{R}^n.$$
- (C-3) For a.a. $t \in [0, T]$, Cesari property holds $\forall x \in \mathbb{R}^n$.

Existence Theorem

Let (C-1)-(C-3) hold. Then problem (OC) admits at least one optimal pair.

Optimal Control Characterization

$(OC)^T$

$$J(t_0, x_0; u(\cdot)) = \int_{t_0}^T g(s, u(s), x(s)) ds + h(x(T))$$

$$\begin{cases} \dot{x}(s) = f(s, u(s), x(s)) & s \in [t_0, T] \\ x(t_0) = x_0 \end{cases}$$

Additional hypothesis:

(C-4)

$x \mapsto (f(t, u, x), g(t, u, x), h(x))$
is differentiable,

$(u, x) \mapsto (f(t, u, x), f_x(t, u, x),$
 $g(t, u, x), g_x(t, u, x),$
 $h_x(x))$

is continuous.

Pontryagin's Maximum Principle

If $\bar{u}(t)$ and $\bar{x}(t)$ are optimal for the problem (OC), then there exists a piecewise differentiable adjoint variable $\lambda(t)$ s.t.

$$H(t, \bar{x}(t), \bar{u}(t), \lambda(t)) = \max_{u \in U} H(t, x(t), u(t), \lambda(t)), \quad \forall t \in [0, T]$$

$$\lambda'(t) = -\frac{\partial H(t, \bar{x}(t), \bar{u}(t), \lambda(t))}{\partial x},$$

$$\lambda(T) = 0.$$

Hamiltonian

$$H = g(t, x(t), u(t)) + \langle \lambda(t), f(t, x(t), u(t)) \rangle,$$

Example

$$\frac{dS_p}{dt} = -\beta_p S_p I_v + (\textcolor{red}{r+u_1}) L_p + (\textcolor{red}{r+u_2}) I_p,$$

$$\frac{dL_p}{dt} = \beta_p S_p I_v - b L_p - (\textcolor{red}{r+u_1}) L_p,$$

$$\frac{dI_p}{dt} = b L_p - (\textcolor{red}{r+u_2}) I_p,$$

$$\frac{dS_v}{dt} = -\beta_v S_v I_p - (\textcolor{brown}{\gamma+u_3}) S_v + (1-\theta)\mu,$$

$$\frac{dI_v}{dt} = \beta_v S_v I_p - (\textcolor{brown}{\gamma+u_3}) I_v + \theta\mu,$$

Example

$$\frac{dS_p}{dt} = -\beta_p S_p I_v + (r + u_1) L_p + (r + u_2) I_p,$$

$$\frac{dL_p}{dt} = \beta_p S_p I_v - b L_p - (r + u_1) L_p,$$

$$\frac{dI_p}{dt} = b L_p - (r + u_2) I_p,$$

$$\frac{dS_v}{dt} = -\beta_v S_v I_p - (\gamma + u_3) S_v + (1 - \theta) \mu,$$

$$\frac{dI_v}{dt} = \beta_v S_v I_p - (\gamma + u_3) I_v + \theta \mu,$$

$$\begin{aligned}
 H = & A_1 I_v + A_2 L_p + A_3 I_p \\
 & + \sum_{i=1}^3 c_i u_i^2 \\
 & + \lambda_1 (-\beta_p S_p I_v + (r + u_1) L_p \\
 & + (r + u_2) I_p) \\
 & + \lambda_2 (\beta_p S_p I_v - b L_p \\
 & - (r + u_1) L_p) \\
 & + \lambda_3 (b L_p - (r + u_2) I_p) \\
 & + \lambda_4 (-\beta_v S_v I_p - (\gamma + u_3) S_v \\
 & + (1 - \theta) \mu) \\
 & + \lambda_5 (\beta_v S_v I_p - (\gamma + u_3) I_v \\
 & + \theta \mu).
 \end{aligned}$$

$$\frac{d\lambda_1}{dt} = \beta_p(\lambda_1 - \lambda_2),$$

$$\frac{d\lambda_2}{dt} = -A_2 + (r + u_1)(\lambda_2 - \lambda_1) + b(\lambda_2 - \lambda_3),$$

$$\frac{d\lambda_3}{dt} = -A_1 + (r + u_2)(\lambda_3 - \lambda_1) + \beta_v S_v(\lambda_4 - \lambda_5),$$

$$\frac{d\lambda_4}{dt} = \beta_v I_p(\lambda_4 - \lambda_5) + (\gamma + u_3)\lambda_4,$$

$$\frac{d\lambda_5}{dt} = -A_3 + \beta_p S_p(\lambda_1 - \lambda_2) + (\gamma + u_3)\lambda_5,$$

optimal control policies

$$\bar{u}_1 = \min \left(\max \left(0, \frac{L_p(\lambda_2 - \lambda_1)}{2c_1} \right), u_1^{max} \right)$$

$$\bar{u}_2 = \min \left(\max \left(0, \frac{I_p(\lambda_3 - \lambda_1)}{2c_2} \right), u_2^{max} \right)$$

$$\bar{u}_3 = \min \left(\max \left(0, \frac{S_v \lambda_4 + I_v \lambda_5}{2c_3} \right), u_3^{max} \right)$$

$$\frac{\partial H}{\partial u}(\bar{u}) = 0 \Rightarrow$$

$$u_i \in [0, u_i^{max}]$$

The most popular

Algorithm 2 Forward Backward Sweep

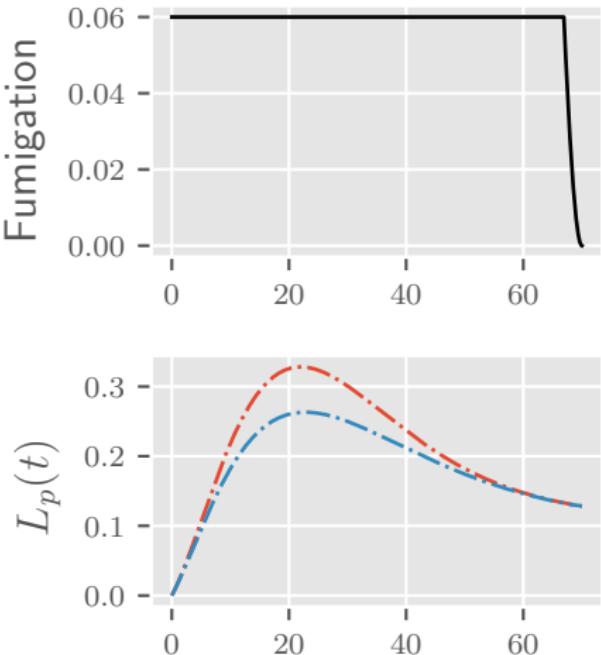
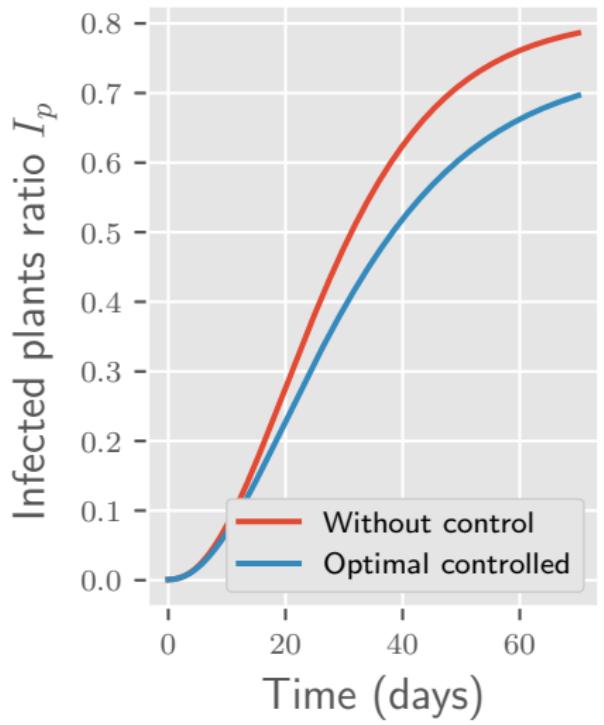
```

Input:  $t_0, t_f, x_0, h, \text{tol}, \lambda_f$ 
Output:  $x^*, u^*, \lambda$ 

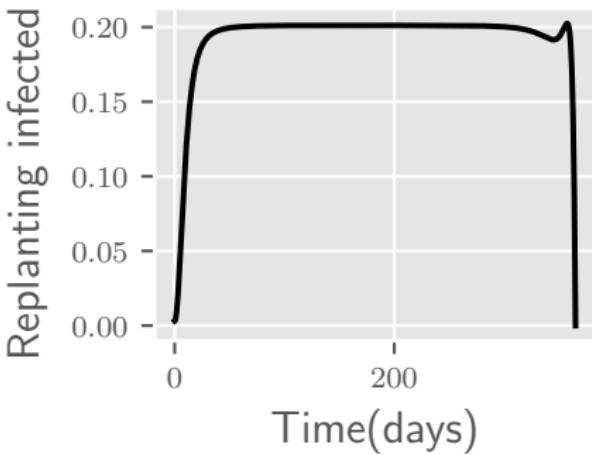
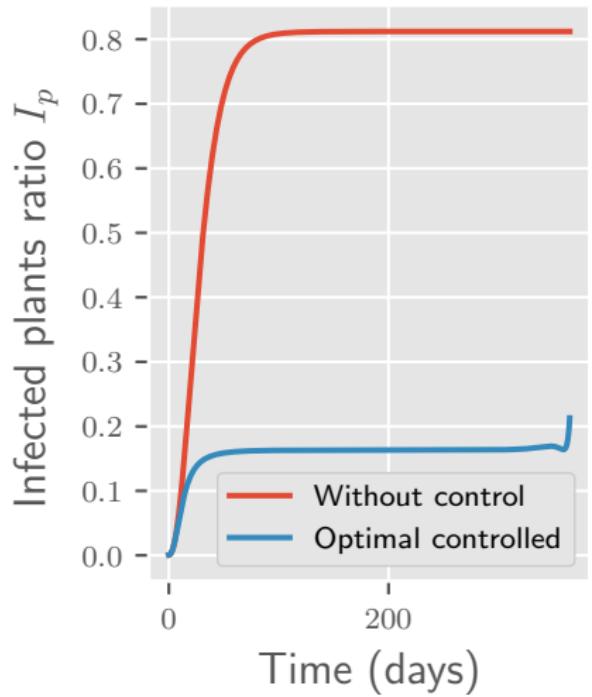
procedure FORWARD _ BACKWARD _ SWEEP( $g, \lambda_{\text{function}}, u, x_0, \lambda_f, h, n_{\max}$ )
    while  $\epsilon > \text{tol}$  do
         $u_{\text{old}} \leftarrow u$ 
         $x_{\text{old}} \leftarrow x$ 
         $x \leftarrow \text{RUNGE\_KUTTA\_FORWARD}(g, u, x_0, h)$ 
         $\lambda_{\text{old}} \leftarrow \lambda$ 
         $\lambda \leftarrow \text{RUNGE\_KUTTA\_BACKWARD}(\lambda_{\text{function}}, x, \lambda_f, h)$ 
         $u_1 \leftarrow \text{OPTIMALITY\_CONDITION}(u, x, \lambda)$ 
         $u \leftarrow \alpha u_1 + (1 - \alpha)u_{\text{old}}, \quad \alpha \in [0, 1]$  ▷ convex combination
         $\epsilon_u \leftarrow \frac{\|u - u_{\text{old}}\|}{\|u\|}$ 
         $\epsilon_x \leftarrow \frac{\|x - x_{\text{old}}\|}{\|x\|}$  ▷ relative error
         $\epsilon_\lambda \leftarrow \frac{\|\lambda - \lambda_{\text{old}}\|}{\|\lambda\|}$ 
         $\epsilon \leftarrow \max \{\epsilon_u, \epsilon_x, \epsilon_\lambda\}$ 
    end while
    return  $x^*, u^*, \lambda$  ▷ Optimal pair
end procedure

```

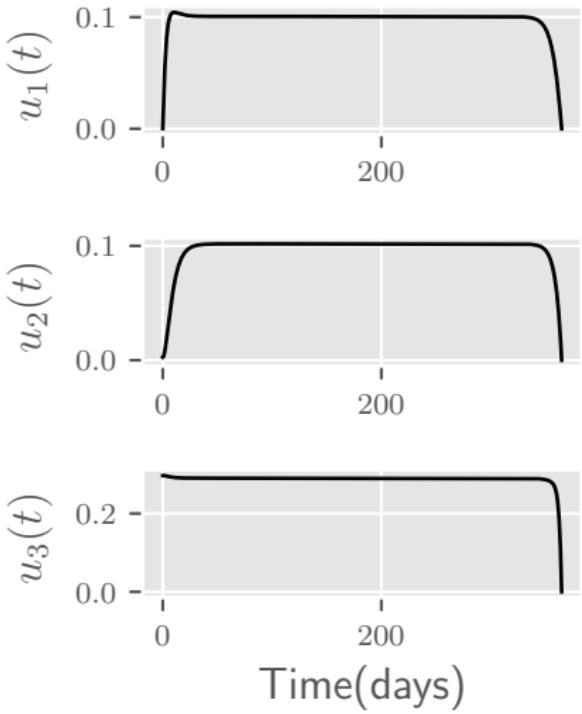
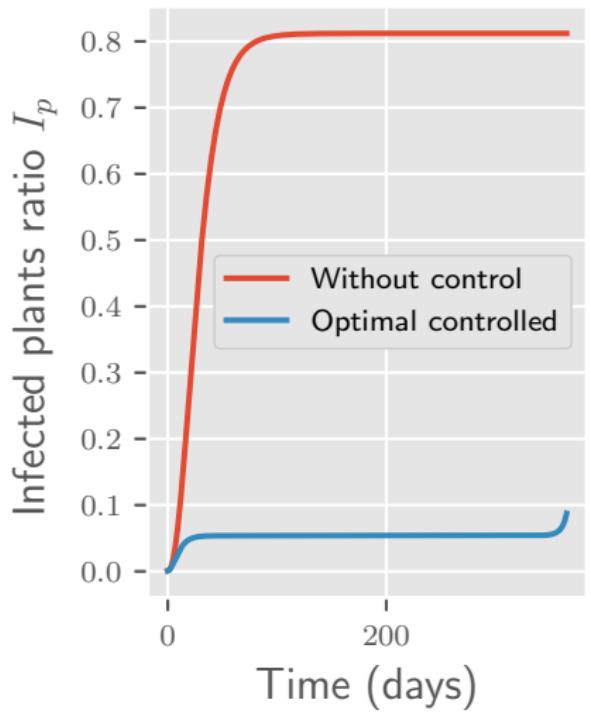
Control by fumigation



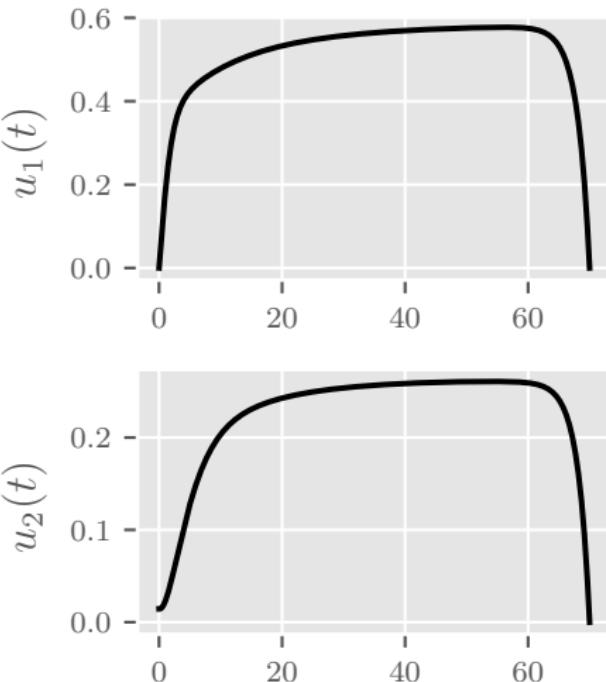
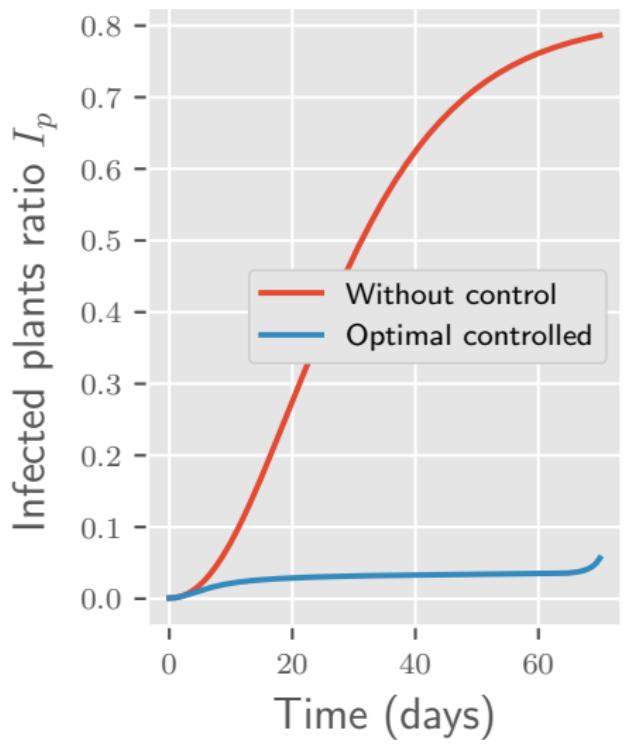
Control by replanting infected plants



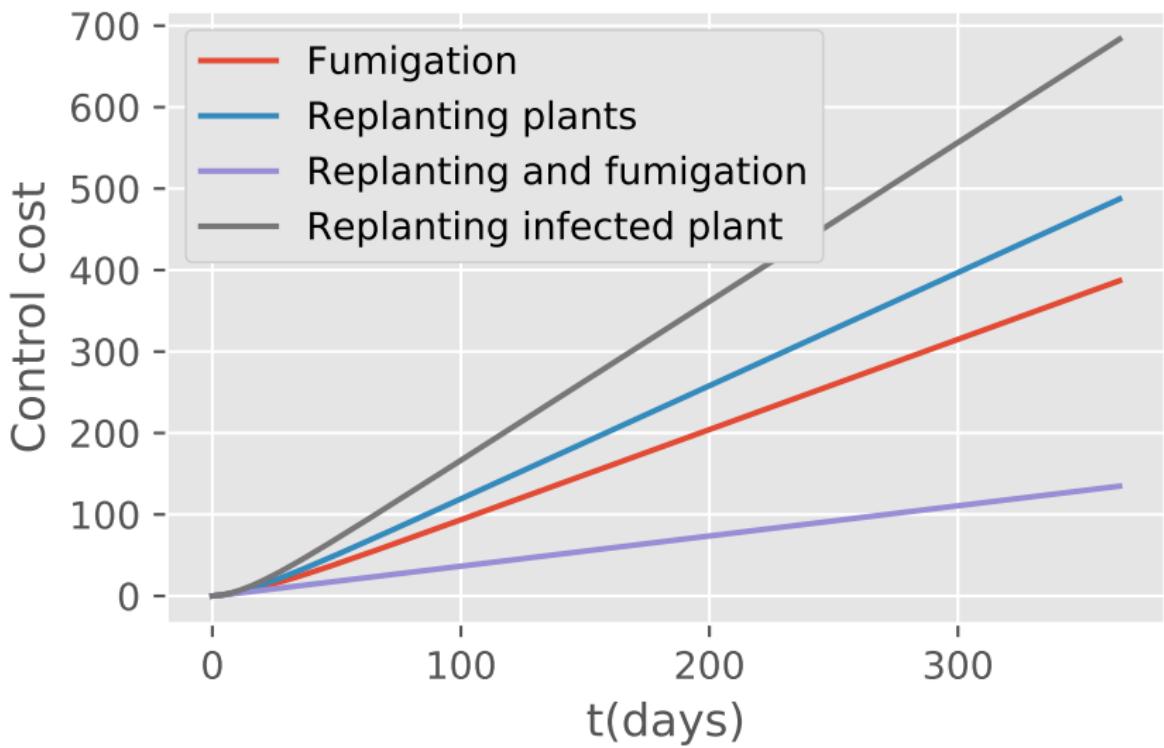
Control by replanting and fumigation



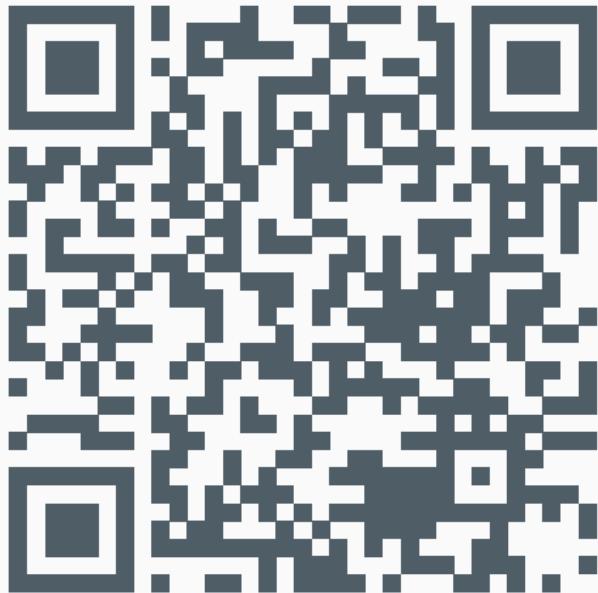
Control by replanting



Control cost likening



Perspectives



Thank You !!!

<https://github.com/SaulDiazInfante/Baemer-SIAM-Section-Mexico>