Initial conditions stability of a numerical approximation for Kolmogorov equations in infinite dimension.

Francisco Delgado-Vences [*], Alan Matzumiya [†]
and Saul Diaz-Infante [‡]

May 16, 2020

Abstract

Accoring to the numerical scheme reported in [Infin. Dimens. Anal. Quantum Probab. Relat. Top. **19** (2016), no. 3, 1650020, 37 pp.], we characterize numerical stability respect to initial conditions. Our approach consists in solving the associated Kolmogorov equation for a familiy of parabolic SPDE whit a spectral method and deduce mild conditions. We illustrate our results with stochastic versions of the Burgers and Fisher equations.

[*]CONACYT-UNAM, Instituto de Matemáticas, Sede Oaxaca, Antonio de León #2, altos, Col. Centro, Oaxaca de Juárez, CP. 68000 Phone: +52(951) 5160541 delgado@im.unam.mx

[†]Universidad de Sonora Departamento de Matemáticas, Departamento de Matemáticas. Blvd Luis Encinas y Rosales S/N, Colonia Centro. Edificio 3K-1. C.P. 83000. Hermosillo, Sonora, México, alan.matzumiya@gmail.com

[‡]CONACYT-Universidad de Sonora, Departamento de Matemáticas. Blvd Luis Encinas y Rosales S/N, Colonia Centro. Edificio 3K-1. C.P. 83000. Hermosillo, Sonora, México, Phone: +52(662) 2592219 ext. 2430 saul.diazinfante@unison.mx. (CORRESPONDING AUTHOR).

Keywords: Stability, spectral method, Kolmogorov equation, parabolic stochastic partial differential equations.

1 Kolmogorov equations for SPDEs in Hilbert spaces

Let $\mathcal{H}$ be a separable infinite-dimensional Hilbert space with inner product $(,)_{\mathcal{H}}$ and norm $\|\cdot\|_{\mathcal{H}}$. Consider the stochastic differential equation in $\mathcal{H}$

$$dX_t = AX_t dt + B(X_t)dt + \sqrt{Q}dW_t, \tag{1}$$

where the operator $A : \mathcal{D}(A) \subset \mathcal{H} \to \mathcal{H}$ is the infinitesimal generator of a strongly continuous semigroup e^{tA} in $\mathcal{H}$, Q is a bounded operator from another Hilbert space $\mathcal{U}$ to $\mathcal{H}$ and $B : \mathcal{D}(B) \subset \mathcal{H} \to \mathcal{H}$ is a nonlinear mapping.

We define

$$u(t,x) = \mathbb{E}\Big[\varphi(X_t^x)\Big], \tag{2}$$

where $\varphi : \mathcal{H} \to \mathbb{R}$ and X_t^x is the solution to (41) with initial conditions $X_0 = x$ with $x \in \mathcal{H}$. Then u satisfies the Kolmogorov equation:

$$\frac{\partial u}{\partial t} = \frac{1}{2}Tr(QD^2 u) + \langle Ax, Du \rangle_{\mathcal{H}} + \langle B(x), Du \rangle_{\mathcal{H}}, \qquad x \in D(A). \tag{3}$$

Several authors have proved results on existence and uniqueness of the solution of the Kolmogorov equations, see for instance Da Prato [5] for a survey, Da Prato-Debussche [6] for the Burgers equation, Barbu-Da Prato [1] for the 2D Navier-Stokes stochastic flow in a channel.

1.1 On the numerical scheme for the Kolmogorov equation

Following [3], in $\mathcal{H}$ we define a Gaussian measure μ with mean zero and nuclear covariance operator Λ with $Tr(\Lambda) < +\infty$ and since $\Lambda : \mathcal{H} \mapsto \mathcal{H}$ is a positive definite, self-adjoint operator then its square-root operator $\Lambda^{1/2}$ is a positive definite, self-adjoint Hilbert-Schmidt operator on $\mathcal{H}$.

Define the inner product $(g,h)_0 := \left(\Lambda^{-1/2}g, \Lambda^{-1/2}h\right)_{\mathcal{H}}$, for $g, h \in \Lambda^{1/2}\mathcal{H}$. Let $\mathcal{H}_0$ denote the Hilbert subspace of $\mathcal{H}$, which is the completion of $\Lambda^{1/2}\mathcal{H}$

1

with respect to the norm $\|g\|_0 := (g, g)_0^{1/2}$. Then $\mathcal{H}_0$ is dense in $\mathcal{H}$ and the inclusion map $i : \mathcal{H}_0 \hookrightarrow \mathcal{H}$ is compact. The triple $(i, \mathcal{H}_0, \mathcal{H})$ forms an abstract Wiener space.

Let $\mathbb{H} = L^2(\mathcal{H}, \mu)$ denote the Hilbert space of Borel measurable functionals on the probability space with inner product

$$\big[\Phi, \Psi\big]_{\mathbb{H}} := \int_{\mathcal{H}} \Phi(v)\Psi(v)\mu(dv), \quad \text{for } \Phi, \Psi \in \mathbb{H},$$

and norm $\|\Phi\|_{\mathbb{H}} := \big[\Phi, \Phi\big]_{\mathbb{H}}^{1/2}$. We choose a basis system $\{\varphi_k\}$ for $\mathcal{H}$.

A functional $\Phi : \mathcal{H} \mapsto \mathbb{R}$, is said to be a smooth simple functional (or a cylinder functional) if there exists a C^∞-function ϕ on $\mathbb{R}^n$ and n-continuous linear functional $l_1, \ldots, l_n$ on $\mathcal{H}$ such that for $h \in \mathcal{H}$

$$\Phi(h) = \phi(h_1, \ldots, h_n) \quad \text{where} \quad h_i = l_i(h), \quad i = 1, \ldots, n. \tag{4}$$

The set of all such functionals will be denoted by $\mathcal{S}(\mathbb{H})$. Denote by $P_k(x)$ the Hermite polynomial of degree k taking values in $\mathbb{R}$. Then, $P_k(x)$ is given by the following formula

$$P_k(x) = \frac{(-1)^k}{(k!)^{1/2}} e^{\frac{x^2}{2}} \frac{d^k}{dx^k} e^{-\frac{x^2}{2}}$$

with $P_0 = 1$. It is well-known that $\{P_k(\cdot)\}_{k \in \mathbb{N}}$ is a complete orthonormal system for $L^2(\mathbb{R}, \mu_1(dx))$ with $\mu_1(dx) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$. Define the set of infinite multi-index as

$$\mathcal{J} = \left\{ \boldsymbol{\alpha} = (\alpha_i, i \geq 1) \;\Big|\; \alpha_i \in \mathbb{N} \cup \{0\}, \quad |\boldsymbol{\alpha}| := \sum_{i=1}^{\infty} \alpha_i < +\infty \right\}.$$

For $\boldsymbol{n} \in \mathcal{J}$ define the *Hermite polynomial functionals* on $\mathcal{H}$ by

$$H_{\boldsymbol{n}}(h) = \prod_{i=1}^{\infty} P_{n_i}(l_i(h)), \quad h \in \mathcal{H}_0, \quad \boldsymbol{n} \in \mathcal{J}, \tag{5}$$

and where $l_i(h) = \langle h, \Lambda^{-1/2}\varphi_i \rangle_{\mathcal{H}}, \quad i = 1, 2, \ldots$ where $P_n(\xi)$ is the usual Hermite polynomial for $\xi \in \mathbb{R}$ and $n \in \mathbb{N}$.

Remark 1. Notice that $l_i(h)$ is defined only for $h \in \mathcal{H}_0$. However, regarding h as a μ-random variable in $\mathcal{H}$, we have $\mathbb{E}\big(l_i(h)\big) = \|\varphi_i\|^2 = 1$ and then $l_k(h)$ can be defined μ-a.e. $h \in \mathcal{H}$, similar to defining a stochastic integral.

It is possible to identify the Hermite polynomial functionals defined in (5), for $h \in \mathcal{H}_0$, as a deterministic version of the Wick polynomials defined on the canonical Wiener space (for further details see [13] for instance).

2

We have the following result (See Theorems 9.1.5 and 9.1.7 in Da Prato-Zabczyk [8] or Lemma 3.1 in Chapter 9 from Chow [3]).

Lemma 1.1. *For $h \in \mathcal{H}$ let $l_i(h) = \langle h, \Lambda^{-1/2}\varphi_i \rangle_{\mathcal{H}}$, $i = 1, 2, \ldots$. The set $\{H_{\boldsymbol{n}}\}$ of all Hermite polynomials on $\mathcal{H}$ forms a complete orthonormal system for $\mathbb{H}$. Hence the set of all functionals are dense in $\mathbb{H}$. Moreover, we have the direct sum decomposition: $\mathbb{H} = \bigoplus_{j=0}^{\infty} K_j$, where K_j is the subspace of $\mathbb{H}$ spanned by $\{H_{\boldsymbol{n}} : |\boldsymbol{n}| = j\}$.*

Remark 2. Using this lemma, we write the solution of the Kolmogorov equation (3) as

$$u(t, x) = \sum_{\boldsymbol{n} \in \mathcal{J}} u_{\boldsymbol{n}}(t) H_{\boldsymbol{n}}(x), \qquad x \in \mathcal{H}, \quad t \in [0, T], \tag{6}$$

where $u_{\boldsymbol{n}} : [0, T] \mapsto \mathbb{R}$ and $H_{\boldsymbol{n}}(x)$ are the Hermite functionals. This spectral decomposition is used in [9] to construct a numerical scheme for approximate the solution of the Kolmogorov equation. The aim of this paper is to prove the stability with respect to the initial condition of this scheme.

2   Some preliminary results

Let Φ be a smooth simple functional given by (4). Then the Fréchet derivatives, $D\Phi = \Phi'$ and $D_2\Phi = \Phi''$ in $\mathcal{H}$ can be computed as follows:

$$(D\Phi(h), v) = \sum_{k=1}^{n} \left[\partial_k \phi(h_1, \ldots, h_n) \right] l_k(v)$$

$$(D^2\Phi(h), v) = \sum_{j,k=1}^{n} \left[\partial_j \partial_k \phi(h_1, \ldots, h_n) \right] l_j(v) l_k(v),$$

for any $u, v \in \mathcal{H}$, where $\partial_k \phi = \frac{\partial}{\partial h_k}\phi$. Similarly, for $m > 2$, $D^m\Phi(h)$ is a $m-$linear form on $\mathcal{H}^m$ with inner product $(\cdot, \cdot)_m$. We have $[D^m\Phi(h)](v_1, \cdots, v_m) = (D^m\Phi(h), v_1 \otimes \cdots \otimes v_m)_m$, for $h, v_1, \ldots, v_m \in \mathcal{H}$. Consider the following linear stochastic equation

$$du_t = Au_t dt + dW_t, \quad u_0 = h \in \mathcal{H}. \tag{7}$$

Where $A : \mathcal{D}(A) \subset \mathcal{H} \to \mathcal{H}$ is the infinitesimal generator of a strongly continuous semigroup e^{tA} in $\mathcal{H}$. W_t is a Q-Wiener process in $\mathcal{H}$. Chow in [3, Lemma 9.4.1] has shown the following result.

3

Lemma 2.1. *Suppose that A and Q satisfy the following:*

1. $A : \mathcal{D}(A) \subset \mathcal{H} \to \mathcal{H}$ *is self-adjoint and there is $\beta > 0$ such that*

$$\langle Av, v \rangle_{\mathcal{H}} \leq -\beta \|v\|_{\mathcal{H}} \quad \forall v \in \mathcal{H}.$$

2. *A commutes with Q in $\mathcal{D}(A) \subset \mathcal{H}$.*

Then (7) has a unique invariant measure μ which is a Gaussian measure on $\mathcal{H}$ with zero mean and covariance operator $\Lambda = \frac{1}{2}Q(-A)^{-1} = \frac{1}{2}(-A)^{-1}Q$.

Suppose that A and Q have the same eigenfunctions e_k with eigenvalues λ_k and ρ_k respectively. It is well-know (See for instance Da Prato and Zabczyk [8]) that the solution of (7) is a time-homogeneous Markov process with transition operator P_t defined for $\Phi \in \mathbb{H}$ given by

$$(P_t \Phi)(h) = \int_{\mathcal{H}} \Phi(v) \mu_t^h(dv) = \mathbb{E}\Big[\Phi(u_t^h)\Big]. \tag{8}$$

Let $\Phi \in \mathcal{S}(\mathbb{H})$ be a smooth simple functional. By setting $\varphi_k = e_k$ in (4), it takes the form $\Phi(h) = \phi(l_1(h), \cdots, l_n(h))$, where $l_k(h) = (h, \Lambda^{-1/2}e_k)$. Define a differential operator $\mathcal{A}_0$ on $\mathcal{S}(\mathbb{H})$ by

$$\mathcal{A}_0 \Phi(v) = \tfrac{1}{2}Tr[RD^2\Phi(v)] + \langle Av, D\Phi(v) \rangle, \qquad v \in H, \tag{9}$$

which is well defined, since $D\Phi \in D(A)$ and $\langle Av, D\Phi(v) \rangle = (v, AD\Phi(v))_{\mathcal{H}}$.

The following results have been proved in [3].

Lemma 2.2. *Let P_t be the transition operator as defined by (7). Then the following properties hold:*

1. *$P_t : \mathcal{S}(\mathbb{H}) \to \mathcal{S}(\mathbb{H})$ for $t \geq 0$.*

2. *$\{P_t, t \geq 0\}$ is a strongly continuous semigroup on $\mathcal{S}(\mathbb{H})$ so that, for any $\Phi \in \mathcal{S}(\mathbb{H})$, we have $P_0 = I$, $P_{t+s}\Phi = P_t P_s\Phi$, for all $t, s \geq 0$, and $\lim_{t\downarrow 0} P_t\Phi = \Phi$.*

3. *$\mathcal{A}_0$ is the infinitesimal generator of P_t so that, for each $\Phi \in \mathcal{S}(\mathbb{H})$,*

$$\lim_{t\downarrow 0} \tfrac{1}{t}\Big(P_t - I\Big)\Phi = \mathcal{A}_0\Phi.$$

$\square$

Lemma 2.3. *Let $H_n(h)$ be a Hermite polynomial functional given by (5). Then the following hold:*

$$\mathcal{A}_0 H_{\mathbf{n}}(h) = -\lambda_{\mathbf{n}} H_{\mathbf{n}}(h), \tag{10}$$

$$P_t H_{\mathbf{n}}(h) = \exp\{-\lambda_{\mathbf{n}} t\} H_{\mathbf{n}}(h), \tag{11}$$

for any $\mathbf{n} \in \mathcal{J}$ and $h \in H$, where $\lambda_{\mathbf{n}} = \sum_{i=1}^{\infty} n_i \lambda_i$.

The following Theorem is a Green formula that we will need forward. Its proof can be seen, for instance, in [3, Thm. 3.3, Ch. 9].

Theorem 2.4. *Let $\Phi \in \mathcal{S}(\mathbb{H})$ be a smooth simple functional and let $\mu \sim N(0, \Lambda)$ be a Gaussian measure in $\mathcal{H}$. Then, for any $g, h \in \mathcal{H}$ the following formula holds*

$$\int_{\mathcal{H}} (\Lambda h, D\Phi(v))_{\mathcal{H}} \mu(dv) = \int_{\mathcal{H}} (v, h)_{\mathcal{H}} \Phi(v) \mu(dv) . \tag{12}$$

Lemma 2.5. *Assume the conditions for Lemma 2.3 hold. Then, for any $\Phi, \Psi \in \mathcal{S}(\mathbb{H})$, the following Green's formula holds:*

$$\int_{\mathcal{H}} (\mathcal{A}_0 \Phi) \Psi d\mu = \int_{\mathcal{H}} \Phi(\mathcal{A}_0 \Psi) d\mu = -\frac{1}{2} \int_{\mathcal{H}} (QD\Phi, D\Psi) d\mu . \tag{13}$$

By Lemma 1.1, for $\Phi \in \mathbb{H}$, it can be represented as

$$\Phi(v) = \sum_{n=0}^{\infty} \phi_{\mathbf{n}} H_{\mathbf{n}}(v), \tag{14}$$

where $n = |\mathbf{n}|$ and $\mathbf{n} \in \mathcal{J}$. Notice that we can think in $\mathbf{n}$ as a vector of r dimension, i.e. $\mathbf{n} = (n_1, \ldots, n_r)$. Let $\alpha_{\mathbf{n}} = \alpha_{n_1} \cdots \alpha_{n_r}$ be a sequence of positive numbers with $\alpha_{\mathbf{n}} > 0$, such that $\alpha_{\mathbf{n}} \to \infty$ as $n \to \infty$. Define

$$|||\Phi|||_{k,\alpha} = \left[\sum_{\mathbf{n}} \left(1 + \alpha_{\mathbf{n}}\right)^k |\phi_n|^2 \right]^{1/2},$$

$$|||\Phi|||_{0,\alpha} = |||\Phi||| = \left[\sum_{\mathbf{n}} |\phi_n|^2 \right]^{1/2},$$

which is $L^2(\mu)$-norm of Φ. For the given sequence $\alpha = \{\alpha_n\}$, let $\mathbb{H}_{k,\alpha}$ denote the completion of $\mathcal{S}(\mathbb{H})$ with respect to the norm $||| \cdot |||_{k,\alpha}$. Then $\mathbb{H}_{k,\alpha}$ is

called a Gauss–Sobolev space of order k with parameter α. The dual space of $\mathbb{H}_{k,\alpha}$ is $\mathbb{H}_{-k,\alpha}$. From now on, we will fix the sequence $\alpha_{\mathbf{n}} = \lambda_{\mathbf{n}}$, where $\lambda_{\mathbf{n}}$ is given in Lemma 2.3. We shall simply denote $\mathbb{H}_{k,\alpha}$ by $\mathbb{H}_k$ and $|||\Phi|||_{k,\alpha}$ by $|||\Phi|||_k$.

The following results ensure the existence of an extension for the operator $\mathcal{A}_0$ to a domain containing $\mathbb{H}_2$. Their proofs can be found in [3] for instance.

Theorem 2.6. *Let the conditions on A and Q in Lemma 2.1 hold. Then $P_t : \mathbb{H} \to \mathbb{H}$, for $t \geq 0$, is a contraction semigroup with the infinitesimal generator $\tilde{A}$. The domain of $\tilde{A}$ contains $\mathbb{H}_2$ and we have $\tilde{A} = \mathcal{A}_0$ in $\mathcal{S}(\mathbb{H})$.*

Theorem 2.7. *Let the conditions of Theorem 2.6 hold, then the differential operator $\mathcal{A}_0$ defined by (9) in $\mathcal{S}(\mathbb{H})$ can be extended to be a self-adjoint linear operator A in $\mathbb{H}$ with domain $\mathbb{H}_2$.*

Since both $\tilde{A}$ and A are extensions of $\mathcal{A}_0$ to a domain containing $\mathbb{H}_2$, they must coincide there.

Given the Gauss-Sobolev space $\mathbb{H}_k$ with norm $||| \cdot |||_k$ we denote its dual space by $\mathbb{H}_{-k}$ with norm $|||\cdot|||_{-k}$. Thus, we have the inclusions, $\mathbb{H}_k \subset \mathbb{H} \subset \mathbb{H}_k$. We denote the duality between $\mathbb{H}_k$ and $\mathbb{H}_{-k}$ by $\langle\langle \Psi, \Phi \rangle\rangle_k$, $\Phi \in \mathbb{H}_k$, $\Psi \in \mathbb{H}_{-k}$. We also set $\mathbb{H}_0 = \mathbb{H}$, with $||| \cdot |||_0 = ||| \cdot |||$ and $\langle\langle \cdot, \cdot \rangle\rangle_1 = \langle\langle \cdot, \cdot \rangle\rangle$, $\langle\langle \cdot, \cdot \rangle\rangle_0 = [\cdot, \cdot]$. Consider the following Kolmogorov equation,

$$\frac{\partial}{\partial t}\Psi(v,t) = \mathcal{A}\Psi(v,t) + \langle B(v), D\Psi(v,t)\rangle_{\mathcal{H}}, \quad \text{a.e. } v \in \mathbb{H}_2,$$
$$\Psi(v,0) = \phi(v) \,,$$

where, as defined in Theorem 2.6, $\mathcal{A} : \mathbb{H}_2 \to \mathbb{H}$ is given by

$$\mathcal{A}\Phi = \tfrac{1}{2}Tr[RD^2\Phi(v)] + \langle Av, D\Phi(v)\rangle \,. \tag{15}$$

Hypothesis on B will be specified latter. For now, we will consider that it is a locally Lipschitz function. The additional term $\langle B(v), D\Psi(v,t)\rangle_{\mathcal{H}}$ is defined μ-a.e. $v \in \mathbb{H}_2$. We will allow the initial datum ϕ will be in $\mathbb{H}$.

We will study a mild solution of the equation (15). Let $\lambda > 0$ be a parameter. By changing Ψ to $e^{\lambda t}\Psi$ in (15) we get the following equation:

$$\frac{\partial}{\partial t}\Psi(v,t) = \mathcal{A}_\lambda\Psi(v,t) + \langle B(v), D\Psi(v,t)\rangle_{\mathcal{H}}, \quad \text{a.e. } v \in \mathbb{H}_2,$$
$$\Psi(v,0) = \phi(v) \,, \tag{16}$$

where $\mathcal{A}_\lambda = \mathcal{A} - \lambda I$, with I the identity operator in $\mathbb{H}$. Clearly, the problems (15) and (16) are equivalent, as far for the existence and uniqueness questions are concerned. We will work on the problem (16).

Denote by P_t the semigroup with infinitesimal generator $\mathcal{A}_\lambda$. The existence of P_t is ensured by the Theorem 2.6. Then, we can rewrite the equation (16) in an integral form by using the semigroup P_t

$$\Psi(v,t) = e^{-\lambda t}(P_t\phi)(v) + \int_0^t e^{-\lambda(t-s)}[P_{t-s}(B, D\Psi_s)](v)ds, \qquad (17)$$

where we denote $\phi = \phi(\cdot)$ and $\Psi_s = \Psi(\cdot, s)$. Chow [3] had proved the following lemma.

Lemma 2.8. *Let $\Psi \in L^2((0,T); \mathbb{H})$ for some $T > 0$. Then, for any $\lambda > 0$ there exists $C_\lambda > 0$ such that*

$$\left|\left|\left| \int_0^t e^{-\lambda(t-s)} P_{t-s}\Psi_s ds \right|\right|\right|^2 \leq C_\lambda \int_0^T |||\Psi_s|||_{-1}^2 ds, \qquad 0 < t \leq T . \qquad (18)$$

We now prove the following theorem on existence and uniqueness of a mild solution to (16).

Theorem 2.9. *Suppose that $B : \mathcal{H} \to \mathcal{H}_0$ satisfies $(B, D\Phi) \in L^2((0,T); \mathbb{H})$ for any $\Phi \in \mathbb{H}$ and*

$$\sup_{v \in \mathcal{H}} ||\Lambda^{-1/2} B(v)||_{\mathcal{H}} < +\infty.$$

Then, B satisfies

$$\left|\left|\left|\left(B(v), D\Phi(v)\right)\right|\right|\right|_{-1}^2 \leq C|||\Phi(v)|||^2 \quad \text{for any } \Phi \in \mathbb{H}, \quad v \in \mathbb{H}_2, \qquad (19)$$

for some $C > 0$. Moreover, for $\Phi \in \mathbb{H}$, the initial-value problem (16) has a unique mild solution $\Psi \in C((0,T); \mathbb{H})$.

For the part of the existence and uniqueness of the solution we will adapt the proof of the Theorem 5.2 in Chapter 9 from [3].

Proof. First we will prove (19). We have

$$\left|\left|\left|\left(B(v), D\Phi(v)\right)\right|\right|\right|_{-1}^2 = \sum_{\mathbf{n}}(1 + \lambda_{\mathbf{n}})^{-1}|\phi_n|^2,$$

with

$$\phi_n = \left((B(v), D\Phi(V))_{\mathcal{H}}, H_{\mathbf{n}}(v)\right)_{\mathbb{H}} = \int_{\mathcal{H}}(B(v), D\Phi(v))_{\mathcal{H}} H_{\mathbf{n}}(v)\mu(dv). \qquad (20)$$

7

By the Theorem 2.4, in particular (12), we have

$$\int_{\mathcal{H}} (\Lambda h, D\Phi(v))_{\mathcal{H}} \mu(dv) = \int_{\mathcal{H}} (v, h)_{\mathcal{H}} \Phi(v) \mu(dv),$$

for all $\Phi \in \mathcal{S}(\mathbb{H})$, $g, h \in \mathcal{H}$ and $\mu \sim N(0, \Lambda)$. Then, in particular, in each direction $H_{\mathbf{n}}$ this formula is still true, so we have

$$\int_{\mathcal{H}} (\Lambda h, D\Phi(v))_{\mathcal{H}} H_{\mathbf{n}}(v) \mu(dv) = \int_{\mathcal{H}} (v, h)_{\mathcal{H}} \Phi(v) H_{\mathbf{n}}(v) \mu(dv) \ .$$

Then, applying this last equality to (20) we get

$$\begin{aligned}
\phi_n &= \int_{\mathcal{H}} \left(\Lambda[\Lambda^{-1} B(v)], D\Phi(v) \right)_{\mathcal{H}} H_{\mathbf{n}}(v) \mu(dv) \\
&= \int_{\mathcal{H}} \left(\Lambda^{-1} B(v), v \right)_{\mathcal{H}} \Phi(v) H_{\mathbf{n}}(v) \mu(dv) \\
&= \int_{\mathcal{H}} \left(\Lambda^{-1/2} B(v), \Lambda^{1/2} v \right)_{\mathcal{H}} \Phi(v) H_{\mathbf{n}}(v) \mu(dv) \ .
\end{aligned}$$

Thus,

$$\begin{aligned}
|\phi_n|^2 &= \left| \int_{\mathcal{H}} \left(\Lambda^{-1/2} B(v), \Lambda^{1/2} v \right)_{\mathcal{H}} \Phi(v) H_{\mathbf{n}}(v) \mu(dv) \right|^2 \\
&\le \int_{\mathcal{H}} \left| \left(\Lambda^{-1/2} B(v), \Lambda^{1/2} v \right)_{\mathcal{H}} \right|^2 \left| H_{\mathbf{n}}(v) \right|^2 \mu(dv) \int_{\mathcal{H}} \left| \Phi(v) \right|^2 \mu(dv) \ .
\end{aligned} \tag{21}$$

We now focus on the first integral. Let I_1 be the first integral of (21). Then,

$$\begin{aligned}
I_1 &\le \int_{\mathcal{H}} \left\| \Lambda^{-1/2} B(v) \right\|_{\mathcal{H}}^2 \left\| \Lambda^{1/2} v \right\|_{\mathcal{H}}^2 \left| H_{\mathbf{n}}(v) \right|^2 \mu(dv) \\
&\le \sup_{v \in \mathcal{H}} \left\| \Lambda^{-1/2} B(v) \right\|_{\mathcal{H}}^2 \int_{\mathcal{H}} \left\| \Lambda^{1/2} v \right\|_{\mathcal{H}}^2 \left| H_{\mathbf{n}}(v) \right|^2 \mu(dv) \\
&\le C \int_{\mathcal{H}} \left\| v \right\|_{\mathcal{H}}^2 \left| H_{\mathbf{n}}(v) \right|^2 \mu(dv) \\
&\le C \ .
\end{aligned}$$

The last inequality follows by using proposition 9.2.10 in page 198 from [7]. Then, by using this bound on (21) we have.

$$\begin{aligned}
|\phi_n|^2 &\le C \int_{\mathcal{H}} \left| \Phi(v) \right|^2 \mu(dv) \\
&\le C ||| \Phi(v) |||^2.
\end{aligned}$$

8

Thus,

$$|||\big(B(v), D\Phi(v)\big)|||^2_{-1} \le C|||\Phi(v)|||^2 \sum_{\mathbf{n}}(1+\lambda_{\mathbf{n}})^{-1} \le C|||\Phi(v)|||^2,$$

which proves (19).

We now prove the existence and uniqueness of a solution to the initial-value problem (16). Let $\mathbb{X}_T$ denote the Banach space $\mathcal{C}([0,T];\mathbb{H})$ with the sup-norm

$$|||\Psi|||_T := \sup_{0 \le t \le T}|||\Psi||| \ .$$

In $\mathbb{X}_T$ define the linear operator $\mathbb{Q}$ as

$$\mathbb{Q}\Psi = e^{-\lambda t}P_t\Phi + \int_0^t e^{-\lambda(t-s)}P_{t-s}(B, D\Psi_s)ds, \quad \text{for any } \Psi \in \mathbb{X}_T.$$

By Theorem 2.6 P_t is a contraction semigroup, then using this fact and Lemma 2.8 we have

$$|||\mathbb{Q}\Psi|||^2 \le 2\left[|||e^{-\lambda t}P_t\Phi|||^2 + |||\int_0^t e^{-\lambda(t-s)}P_{t-s}(B, D\Psi_s)ds|||^2\right]$$

$$\le 2\left[|||\Phi|||^2 + C_\lambda \int_0^t |||(B, D\Psi_s)|||^2_{-1}ds\right]$$

$$\le 2|||\Phi|||^2 + C_1 \int_0^t |||\Psi_s|||^2 ds,$$

for some $C_1 > 0$. Hence, $|||\mathbb{Q}\Psi|||_T \le C(1 + |||\Psi|||_T)$, with $C = C(\Phi, \lambda, T)$. Then, the map $\mathbb{Q} : \mathbb{X}_T \to \mathbb{X}_T$ is well defined. We now show that is a contraction for a small t. Let $\Psi, \Psi' \in \mathbb{X}_T$. Then

$$|||\mathbb{Q}\Psi - \mathbb{Q}\Psi'|||^2 = |||\int_0^t e^{-\lambda(t-s)}P_{t-s}\big[(B, D\Psi_s) - (B, D\Psi'_s)\big]ds|||^2$$

$$\le C_\lambda \int_0^t |||(B, D\Psi_s - D\Psi')|||^2_{-1}ds$$

$$\le C_2 \int_0^t |||\Psi_s - \Psi'|||^2 ds,$$

for some $C_2 > 0$.

It follows that $|||\mathbb{Q}\Psi - \mathbb{Q}\Psi'|||_T \le \sqrt{C_2 T}|||\Psi_s - \Psi'|||_T$. Then, for small T, $\mathbb{Q}$ is a contraction on $\mathbb{X}_T$. Hence the Cauchy problem (16) has a unique mild solution. $\qquad\square$

3 Numerical stability respect to initial conditions

In this section, we prove the continuity with respect to the initial conditions for a numerical approximation of the Kolmogorov equation associated with an SPDE. Here we understand that a numerical scheme is stable respect to initial conditions if this method reproduces the same behavior when the continuous problem satisfies continuity respect initial conditions.

We first prove the dependence on initial conditions for the mild solution of (16).

Theorem 3.1. *Suppose that $B : \mathcal{H} \to \mathcal{H}_0$ satisfies $(B, D\Phi) \in L^2((0,T); \mathbb{H})$ for any $\Phi \in \mathbb{H}$ and*

$$\sup_{v \in \mathcal{H}} ||\Lambda^{-1/2} B(v)||_{\mathcal{H}} < +\infty. \tag{22}$$

Then, the unique mild solution $\Psi \in C((0,T); \mathbb{H})$ for (16) depends continuously on the initial conditions.

Proof. We know, with the assumption (22), that the existence of a unique mild solution for (16) is guaranteed by Theorem 2.9. We will denote by Ψ_t^φ its mild solution at time t with initial condition φ:

$$\Psi_t^\varphi = e^{-\lambda t} P_t \varphi + \int_0^t e^{-\lambda(t-s)} P_{t-s}(B, D\Psi_s^\varphi) ds .$$

Then,

$$\Psi_t^\varphi - \Phi_t^\psi = e^{-\lambda t} P_t \varphi - e^{-\lambda t} P_t \psi + \int_0^t e^{-\lambda(t-s)} P_{t-s}(B, D\Psi_s^\varphi - D\Phi_s^\psi) ds$$

$$= e^{-\lambda t} P_t(\varphi - \psi) + \int_0^t e^{-\lambda(t-s)} P_{t-s}(B, D\Psi_s^\varphi - D\Phi_s^\psi) ds.$$

From this expression we get

$$|||\Psi_t^\varphi - \Phi_t^\psi|||^2 \le |||e^{-\lambda t} P_t(\varphi - \psi)|||^2 + ||| \int_0^t e^{-\lambda(t-s)} P_{t-s}(B, D\Psi_s^\varphi - D\Phi_s^\psi)|||^2 ds$$

$$\le |||\varphi - \psi|||^2 + C_\lambda \int_0^t |||(B, D\Psi_s^\varphi - D\Phi_s^\psi)|||_{-1}^2 ds$$

$$\le |||\varphi - \psi|||^2 + C_2 \int_0^t |||\Psi_s^\varphi - \Phi_s^\psi|||^2 ds.$$

Thus, by Gronwall's inequality we obtain

$$\||\Psi_t^\varphi - \Phi_t^\psi\||^2 \le \exp(C_2 t)\||\varphi - \psi\||^2, \tag{23}$$

which implies, $\||\Psi_t^\varphi - \Phi_t^\psi\|| \le \exp(Ct)\||\varphi - \psi\||$. This completes the proof. $\quad\square$

We now will study the stability of the numerical scheme. As before, we use Lemma 1.1 to write the solution Ψ_t^φ as in a Fourier-Hermite decomposition:

$$\Psi_t^\varphi = \sum_{\boldsymbol{n} \in \mathcal{J}} u_{\boldsymbol{n}}(t) H_{\boldsymbol{n}}(x), \qquad x \in \mathcal{H}, \quad t \in [0, T] \ . \tag{24}$$

Note that coefficients $u_{\boldsymbol{n}}(t)$ on the numerical approximations depend on the functional and on the initial condition and not the initial condition. First we prove an auxiliary result.

Lemma 3.2. *Set $\{P_k(\xi)\}_{k \in \mathbb{N}}$ the family of normalized Hermite polynomials in $\mathbb{R}$. For every $k \in \mathbb{N}$ and $\xi, \eta \in \mathbb{R}$ such that $\eta < \xi$ we have that*

$$P_k(\xi) - P_k(\eta) = C(k) Pe_{k+1}(\gamma) \cdot (\xi - \eta), \tag{25}$$

where $\gamma \in (\eta, \xi)$ and $C(k) = \frac{(-1)^k}{(k+1)(k!)^{1/2}}$. Moreover, $Pe_k(x)$ is the unnormalized Hermite polynomial of k degree.

Proof. We know that $P_k(\xi) = \frac{(-1)^k}{(k!)^{1/2}} e^{\xi^2/2} \frac{d}{d\xi^k} e^{-\xi^2/2}$. Set $c(k) = (-1)^k (k!)^{-1/2}$, then

$$P_k(\xi) - P_k(\eta) = c(k)\left[e^{\xi^2/2}\frac{d}{d\xi^k}e^{-\xi^2/2} - e^{\eta^2/2}\frac{d}{d\eta^k}e^{-\eta^2/2}\right]$$

$$= c(k)\left[e^{x^2/2}\frac{d}{dx^k}e^{-x^2/2}\Big|_{x=\eta}^{\xi}\right]$$

$$= c(k)\int_\eta^\xi F_k(x)dx,$$

where F_k is a continuous function such that $F_k'(x) = e^{x^2/2}\frac{d}{dx^k}e^{-x^2/2}$. In fact, denoting by $Pe_k(x)$ the unnormalized Hermite polynomial of k degree, results

$$F_k'(x) = e^{x^2/2}\frac{d}{dx^k}e^{-x^2/2} = Pe_k(x),$$

11

and since the Hermite polynomials constitute an Appell sequence we have that

$$F_k'(x) = Pe_k(x) = \frac{1}{k+1}Pe_{k+1}'(x),$$

which implies that $F_k(x) = \frac{1}{k+1}Pe_{k+1}(x)$. Now, since $F_k(x)$ is a continuous function, then there exists $\gamma \in (\eta, \xi)$ such that

$$\int_\eta^\xi F_k(x)dx = F_k(\gamma) \cdot (\xi - \eta).$$

All these implies that $P_k(\xi) - P_k(\eta) = c(k)F_k(\gamma) \cdot (\xi - \eta)$. From this expression the lemma follows immediately. $\qquad\square$

We will use some technical results on the SPDE to prove the following result—the main result of this work.

Theorem 3.3. *Assume that the eigenvalues of Λ, satisfies that for every $k \in \mathbb{N}$, $\lambda_k < \lambda_{k+1} \to \infty$. Assume that the functional φ is Lipschitz. Then, the numeric approximation Ψ_t^φ (given by (43)) to the solution of the Kolmogorov equation $\Psi \in C((0,T); \mathbb{H})$ depends continuously on the initial conditions.*

Proof. Let $x, y \in H$ be two different initial values. We want to estimate $\Psi_t^x - \Psi_t^y$. By definition,

$$\Psi_t^x = \sum_{\bar{n} \in \mathcal{J}} u_{\bar{n}}^x(t) H_{\bar{n}}(x) . \tag{26}$$

Thus,

$$\begin{aligned}
\Psi_t^x - \Psi_t^y &= \sum_{\bar{n} \in \mathcal{J}} u_{\bar{n}}^x(t) H_{\bar{n}}(x) - \sum_{\bar{n} \in \mathcal{J}} u_{\bar{n}}^y(t) H_{\bar{n}}(y) \\
&= \sum_{\bar{n} \in \mathcal{J}} \Big[u_{\bar{n}}^x(t) - u_{\bar{n}}^y(t)\Big] H_{\bar{n}}(x) + \sum_{\bar{n} \in \mathcal{J}} u_{\bar{n}}^y(t)\Big[H_{\bar{n}}(x) - H_{\bar{n}}(y)\Big].
\end{aligned} \tag{27}$$

We focus on the first term in (46). From the definition of the initial condition we obtain the following expression for the time-dependent coefficient

$$u_{\bar{n}}^x(t) = \int_{\mathcal{H}} H_{\bar{n}}(x)\mathbb{E}\Big[\varphi(X_t^x)\Big]\mu(dx) .$$

From this we get

$$
\begin{aligned}
u_{\bar{n}}^{x}(t) - u_{\bar{n}}^{y}(t) &= \int_{\mathcal{H}} H_{\bar{n}}(x)\mathbb{E}\Big[\varphi(X_t^x)\Big]\mu(dx) - \int_{\mathcal{H}} H_{\bar{n}}(y)\mathbb{E}\Big[\varphi(X_t^y)\Big]\mu(dy) \\
&= \int_{\mathcal{H}\times\mathcal{H}} H_{\bar{n}}(x)\mathbb{E}\Big[\varphi(X_t^x)\Big]\mu(dx)\mu(dy) \\
&\quad - \int_{\mathcal{H}\times\mathcal{H}} H_{\bar{n}}(y)\mathbb{E}\Big[\varphi(X_t^y)\Big]\mu(dx)\mu(dy) \\
&= \int_{\mathcal{H}\times\mathcal{H}} H_{\bar{n}}(x)\Big(\mathbb{E}\Big[\varphi(X_t^x)\Big] - \mathbb{E}\Big[\varphi(X_t^y)\Big]\Big)\mu(dx)\mu(dy) \\
&\quad + \int_{\mathcal{H}\times\mathcal{H}}\Big(H_{\bar{n}}(x) - H_{\bar{n}}(y)\Big)\mathbb{E}\Big[\varphi(X_t^y)\Big]\mu(dx)\mu(dy) \ .
\end{aligned}
$$

Then, by the Cauchy-Schwartz inequality, we obtain

$$
\begin{aligned}
|u_{\bar{n}}^{x}(t) - u_{\bar{n}}^{y}(t)|^2 &\leq \left|\int_{\mathcal{H}\times\mathcal{H}} H_{\bar{n}}(x)\left(\mathbb{E}\left[\varphi(X_t^x)\right] - \mathbb{E}\left[\varphi(X_t^y)\right]\right)\mu(dx)\mu(dy)\right|^2 \\
&\quad + \left|\int_{\mathcal{H}\times\mathcal{H}}\left(H_{\bar{n}}(x) - H_{\bar{n}}(y)\right)\mathbb{E}\left[\varphi(X_t^y)\right]\mu(dx)\mu(dy)\right|^2 \\
&\leq \int_{\mathcal{H}\times\mathcal{H}} H_{\bar{n}}^2(x))\mu(dx)\mu(dy) \\
&\quad \times \int_{\mathcal{H}\times\mathcal{H}}\left|\mathbb{E}\left[\varphi(X_t^x)\right] - \mathbb{E}\left[\varphi(X_t^y)\right]\right|^2\mu(dx)\mu(dy) \\
&\quad + \int_{\mathcal{H}\times\mathcal{H}}\mathbb{E}^2\left[\varphi(X_t^y)\right]\mu(dx)\mu(dy) \\
&\quad \times \int_{\mathcal{H}\times\mathcal{H}}\left|H_{\bar{n}}(x) - H_{\bar{n}}(y)\right|^2\mu(dx)\mu(dy) \\
&= \int_{\mathcal{H}\times\mathcal{H}}\left|\mathbb{E}\left[\varphi(X_t^x)\right] - \mathbb{E}\left[\varphi(X_t^y)\right]\right|^2\mu(dx)\mu(dy) \\
&\quad + \int_{\mathcal{H}\times\mathcal{H}}\mathbb{E}^2\left[\varphi(X_t^y)\right]\mu(dx)\mu(dy) \\
&\quad \times \int_{\mathcal{H}\times\mathcal{H}}\left|H_{\bar{n}}(x) - H_{\bar{n}}(y)\right|^2\mu(dx)\mu(dy) \ .
\end{aligned}
\tag{28}
$$

We now estimate the norm of the expression (46) with the help of (47).

$$
\begin{aligned}
\|\Psi_t^x - \Psi_t^y\|^2_{\left(L^2(\mathcal{H},\mu)\right)^2} &= \int_{\mathcal{H}\times\mathcal{H}} |\Psi_t^x - \Psi_t^y|^2 \mu(dx)\mu(dy) \\
&\leq \int_{\mathcal{H}\times\mathcal{H}} \left| \sum_{\bar{n}\in\mathcal{J}} \left[u_{\bar{n}}^x(t) - u_{\bar{n}}^y(t)\right] H_{\bar{n}}(x) \right|^2 \mu(dx)\mu(dy) \\
&\quad + \int_{\mathcal{H}\times\mathcal{H}} \left| \sum_{\bar{n}\in\mathcal{J}} u_{\bar{n}}^y(t)\left[H_{\bar{n}}(x) - H_{\bar{n}}(y)\right] \right|^2 \mu(dx)\mu(dy) \\
&\leq \int_{\mathcal{H}\times\mathcal{H}} \sum_{\bar{n}\in\mathcal{J}} \left| u_{\bar{n}}^x(t) - u_{\bar{n}}^y(t)\right|^2 H_{\bar{n}}^2(x)\mu(dx)\mu(dy) \\
&\quad + \int_{\mathcal{H}\times\mathcal{H}} \sum_{\bar{n}\in\mathcal{J}} \left[u_{\bar{n}}^y(t)\right]^2 \sum_{\bar{n}\in\mathcal{J}} \left|H_{\bar{n}}(x) - H_{\bar{n}}(y)\right|^2 \mu(dx)\mu(dy) \\
&= \sum_{\bar{n}\in\mathcal{J}} \left| u_{\bar{n}}^x(t) - u_{\bar{n}}^y(t)\right|^2 \int_{\mathcal{H}\times\mathcal{H}} H_{\bar{n}}^2(x)\mu(dx)\mu(dy) \\
&\quad + \sum_{\bar{n}\in\mathcal{J}} \left[u_{\bar{n}}^y(t)\right]^2 \sum_{\bar{n}\in\mathcal{J}} \int_{\mathcal{H}\times\mathcal{H}} \left|H_{\bar{n}}(x) - H_{\bar{n}}(y)\right|^2 \mu(dx)\mu(dy) \qquad (29) \\
&= \sum_{\bar{n}\in\mathcal{J}} \left[u_{\bar{n}}^y(t)\right]^2 \sum_{\bar{n}\in\mathcal{J}} \int_{\mathcal{H}\times\mathcal{H}} \left|H_{\bar{n}}(x) - H_{\bar{n}}(y)\right|^2 \mu(dx)\mu(dy) \\
&\quad + \sum_{\bar{n}\in\mathcal{J}} \left| u_{\bar{n}}^x(t) - u_{\bar{n}}^y(t)\right|^2 \\
&= \int_{\mathcal{H}\times\mathcal{H}} \left| \mathbb{E}\left[\varphi(X_t^x)\right] - \mathbb{E}\left[\varphi(X_t^y)\right] \right|^2 \mu(dx)\mu(dy) \\
&\quad + \int_{\mathcal{H}\times\mathcal{H}} \mathbb{E}^2\left[\varphi(X_t^y)\right]\mu(dx)\mu(dy) \\
&\quad \times \sum_{\bar{n}\in\mathcal{J}} \int_{\mathcal{H}\times\mathcal{H}} \left|H_{\bar{n}}(x) - H_{\bar{n}}(y)\right|^2 \mu(dx)\mu(dy) \\
&\quad + \sum_{\bar{n}\in\mathcal{J}} \left[u_{\bar{n}}^y(t)\right]^2 \sum_{\bar{n}\in\mathcal{J}} \int_{\mathcal{H}\times\mathcal{H}} \left|H_{\bar{n}}(x) - H_{\bar{n}}(y)\right|^2 \mu(dx)\mu(dy) .
\end{aligned}
$$

Note that $\mathbb{E}^2\left[\varphi(X_t^y)\right] = u^2(t,x) \in L^2(\mathcal{H},\mu)$, therefore the first integral in the second term is a continuous bounded function of t. Moreover, $\sum_{\bar{n}\in\mathcal{J}} \left[u_{\bar{n}}^y(t)\right]^2$ is the $L^2(\mathcal{H},\mu)$-norm of the function $u(t,x)$, then the series converges and it is also a continuous bounded function of t. Thus, from (48)

14

we get

$$\|\Psi_t^x - \Psi_t^y\|^2_{\left(L^2(\mathcal{H},\mu)\right)^2} \leq \int_{\mathcal{H}\times\mathcal{H}} \left|\mathbb{E}\left[\varphi(X_t^x)\right] - \mathbb{E}\left[\varphi(X_t^y)\right]\right|^2 \mu(dx)\mu(dy)$$

$$+ f(t) \sum_{\bar{n}\in\mathcal{J}} \int_{\mathcal{H}\times\mathcal{H}} \left|H_{\bar{n}}(x) - H_{\bar{n}}(y)\right|^2 \mu(dx)\mu(dy) ,\tag{30}$$

where $f(t) = \sum_{\bar{n}\in\mathcal{J}} \left[u_{\bar{n}}^y(t)\right]^2 + \int_{\mathcal{H}} \mathbb{E}^2\left[\varphi(X_t^y)\right]\mu(dy)$. From the proof of Theorem 3.1 (see (23)) we know that

$$\||\Psi_t^\varphi - \Phi_t^\psi\||^2 = \int_{\mathcal{H}\times\mathcal{H}} \left|\mathbb{E}\left[\varphi(X_t^x)\right] - \mathbb{E}\left[\varphi(X_t^y)\right]\right|^2 \mu(dx)\mu(dy)$$

$$\leq \exp(Ct) \int_{\mathcal{H}\times\mathcal{H}} \|x - y\|_{\mathcal{H}}^2 \mu(dx)\mu(dy)\tag{31}$$

$$= \exp(Ct)\||x - y\||^2.$$

Therefore the first term in the right side of (49) is bounded by (50). We now focus on the second term in the last inequality. Notice that for every $\bar{n} \in \mathcal{J}$ we have

$$H_{\bar{n}}(x) - H_{\bar{n}}(y) = \prod_{i=1}^{\infty} \left[P_{n_i}(\xi_i) - P_{n_i}(\eta_i)\right],\tag{32}$$

where $\xi_i = \langle x, \Lambda^{-1/2}e_i\rangle_{\mathcal{H}}$ and $\eta_i = \langle y, \Lambda^{-1/2}e_i\rangle_{\mathcal{H}}$ (see (5) and lines after that for the definition). Hence, applying Lemma 4.1 to equation (51) we have that

$$H_{\bar{n}}(x) - H_{\bar{n}}(y) = \prod_{i=1}^{\infty} C(i)Pe_{i+1}(\gamma_i) \cdot (\xi_i - \eta_i)$$

$$= \prod_{i=1}^{\infty} C(i)Pe_{i+1}(\gamma_i)\langle x - y, \Lambda^{-1/2}e_i\rangle_{\mathcal{H}},\tag{33}$$

15

here $\gamma_i \in \left(\xi_i \wedge \eta_i, \xi_i \vee \eta_i \right)$ for every $i \in \mathbb{N}$. Then

$$\sum_{\bar{n} \in \mathcal{J}} \int_{\mathcal{H} \times \mathcal{H}} \left| H_{\bar{n}}(x) - H_{\bar{n}}(y) \right|^2 \mu(dx) \mu(dy)$$

$$= \sum_{\bar{n} \in \mathcal{J}} \int_{\mathcal{H} \times \mathcal{H}} \left| \prod_{i=1}^{\infty} C(i) Pe_{i+1}(\gamma_i) \langle x - y, \Lambda^{-1/2} e_i \rangle_{\mathcal{H}} \right|^2 \mu(dx) \mu(dy)$$

$$= \sum_{\bar{n} \in \mathcal{J}} \int_{\mathcal{H} \times \mathcal{H}} \prod_{i=1}^{\infty} \left[C(i) Pe_{i+1}(\gamma_i) \right]^2 \left| \langle x - y, \Lambda^{-1/2} e_i \rangle_{\mathcal{H}} \right|^2 \mu(dx) \mu(dy)$$

$$\leq \sum_{\bar{n} \in \mathcal{J}} \int_{\mathcal{H} \times \mathcal{H}} \prod_{i=1}^{\infty} \left[C(i) Pe_{i+1}(\gamma_i) \right]^2 \|x - y\|_{\mathcal{H}}^2 \|\Lambda^{-1/2} e_i\|_{\mathcal{H}}^2 \mu(dx) \mu(dy) \tag{34}$$

$$= \sum_{\bar{n} \in \mathcal{J}} \int_{\mathcal{H} \times \mathcal{H}} \prod_{i=1}^{\infty} \left[C(i) Pe_{i+1}(\gamma_i) \right]^2 \|x - y\|_{\mathcal{H}}^2 \lambda_i^{-1} \|e_i\|_{\mathcal{H}}^2 \mu(dx) \mu(dy)$$

$$= \|x - y\|_{\mathcal{H}}^2 \sum_{\bar{n} \in \mathcal{J}} \prod_{i=1}^{\infty} \left[C(i) \right]^2 \lambda_i^{-1} \int_{\mathcal{H} \times \mathcal{H}} \left[Pe_{i+1}(\gamma_i) \right]^2 \mu(dx) \mu(dy).$$

Recall that for every $i \in \mathbb{N}$ we have that $\gamma_i \in \left(\xi_i \wedge \eta_i, \xi_i \vee \eta_i \right)$, set $\hat{\gamma}_i \in \left(\xi_i \wedge \eta_i, \xi_i \vee \eta_i \right)$ such that $Pe_i^2(\gamma_i) \leq Pe_{i+1}^2(\hat{\gamma}_i)$ for every $\gamma_i \in \left(\xi_i \wedge \eta_i, \xi_i \vee \eta_i \right)$, notice that the existence of $\hat{\gamma}_i$ is guaranteed since $Pe_{i+1}^2(\cdot)$ is a continuous function. Then, from (53) we get

$$\sum_{\bar{n} \in \mathcal{J}} \int_{\mathcal{H} \times \mathcal{H}} \left| H_{\bar{n}}(x) - H_{\bar{n}}(y) \right|^2 \mu(dx) \mu(dy)$$

$$\leq \|x - y\|_{\mathcal{H}}^2 \sum_{\bar{n} \in \mathcal{J}} \prod_{i=1}^{\infty} \left[C(i) \right]^2 \lambda_i^{-1} \left[Pe_{i+1}(\hat{\gamma}_i) \right]^2 \int_{\mathcal{H}} \int_{\mathcal{H}} \mu(dx) \mu(dy)$$

$$= \|x - y\|_{\mathcal{H}}^2 \sum_{\bar{n} \in \mathcal{J}} \prod_{i=1}^{\infty} \left[C(i) \right]^2 \lambda_i^{-1} \left[Pe_{i+1}(\hat{\gamma}_i) \right]^2. \tag{35}$$

Here, we recall that $C(i) = \frac{(-1)^i}{(i+1)(i!)^{1/2}}$ then $\frac{(-1)^i}{[(i+1)!]^{1/2}} Pe_{i+1}(\hat{\gamma}_i)$ is the normalized Hermite polynomial of $i + 1$ degree evaluated on $\hat{\gamma}_i$ which is bounded by a constant C for every $i \in \mathbb{N}$. Moreover, since $\lambda_k < \lambda_{k+1} \to \infty$ then this implies that

$$\sum_{\bar{n} \in \mathcal{J}} \prod_{i=1}^{\infty} \left[C(i) \right]^2 \lambda_i^{-1} \left[Pe_{i+1}(\hat{\gamma}_i) \right]^2 \leq C \sum_{\bar{n} \in \mathcal{J}} \prod_{i=1}^{\infty} \lambda_i^{-1} (i+1)^{-1} \leq C, \tag{36}$$

16

where C is a finite constant. Putting together (53) and (55) we get that

$$\sum_{\bar{n} \in \mathcal{J}} \int_{\mathcal{H} \times \mathcal{H}} \left| H_{\bar{n}}(x) - H_{\bar{n}}(y) \right|^2 \mu(dx)\mu(dy) \leq C\|x - y\|_{\mathcal{H}} . \tag{37}$$

Putting together inequalities (49), (50) and (56) we obtain

$$\|\Psi_t^x - \Psi_t^y\|_{\left(L^2(\mathcal{H},\mu)\right)^2}^2 \leq \exp(Ct) \int_{\mathcal{H} \times \mathcal{H}} \|x - y\|_{\mathcal{H}}^2 \mu(dx)\mu(dy)$$
$$+ f(t)\|x - y\|_{\mathcal{H}}. \tag{38}$$

Now, if $\|x-y\|_{\mathcal{H}} \leq \delta$, then from (57) we get $\|\Psi_t^x - \Psi_t^y\|_{\left(L^2(\mathcal{H},\mu)\right)^2}^2 \leq G(t)\delta.$ $\square$

Remark 3. If we consider in addition the supremum norm on t, then from (57) we get

$$\sup_{0 \leq t \leq T} \|\Psi_t^x - \Psi_t^y\|_{\left(L^2(\mathcal{H},\mu)\right)^2}^2 \leq C\|x - y\|_{\mathcal{H}}^2 \sup_{0 \leq t \leq T} f(t)$$
$$+ \exp(CT) \int_{\mathcal{H} \times \mathcal{H}} \|x - y\|_{\mathcal{H}}^2 \mu(dx)\mu(dy) . \tag{39}$$

Notice that $f(t)$ is differentiable and continuous, then $\sup_{0 \leq t \leq T} f(t) \leq C$, then from (58) we obtain

$$\sup_{0 \leq t \leq T} \|\Psi_t^x - \Psi_t^y\|_{\left(L^2(\mathcal{H},\mu)\right)^2}^2 \leq C\|x - y\|_{\mathcal{H}}$$
$$+ \exp(CT) \int_{\mathcal{H} \times \mathcal{H}} \|x - y\|_{\mathcal{H}}^2 \mu(dx)\mu(dy) . \tag{40}$$

From this inequality it is possible to show the continuously dependence on the initial conditions for this norm.

4 Numerical stability respect to initial conditions

In this section, we prove the continuity with respect to the initial conditions for a numerical approximation of the Kolmogorov equation associated with an SPDE. Here we understand that a numerical scheme is stable respect

to initial conditions if this method reproduces the same behavior when the continuous problem satisfies continuity respect initial conditions.

Consider the stochastic differential equation in $\mathcal{H}$

$$dX_t = AX_t dt + B(X_t)dt + \sqrt{Q}dW_t, \tag{41}$$

where the operator $A : \mathcal{D}(A) \subset \mathcal{H} \to \mathcal{H}$ is the infinitesimal generator of a strongly continuous semigroup e^{tA} in $\mathcal{H}$, Q is a bounded operator from another Hilbert space $\mathcal{U}$ to $\mathcal{H}$ and $B : \mathcal{D}(B) \subset \mathcal{H} \to \mathcal{H}$ is a nonlinear mapping.

The equation (41) can be associated to a Kolmogorov equation in the next way, we define

$$u(t, x) = \mathbb{E}\Big[\varphi(X_t^x)\Big], \tag{42}$$

where $\varphi : \mathcal{H} \to \mathbb{R}$ and X_t^x is the solution to (41) with initial conditions $X_0 = x$ where $x \in \mathcal{H}$. Then u satisfies the Kolmogorov equation (3).

We use Lemma 1.1 to write the solution Ψ_t^φ as in a Fourier-Hermite decomposition:

$$\Psi_t^\varphi = \sum_{\boldsymbol{n} \in \mathcal{J}} u_{\boldsymbol{n}}(t) H_{\boldsymbol{n}}(x), \qquad x \in \mathcal{H}, \quad t \in [0, T] . \tag{43}$$

Note that coefficients $u_{\boldsymbol{n}}(t)$ only depends on time. First we prove an auxiliary result.

Lemma 4.1. *Set $\{P_k(\xi)\}_{k \in \mathbb{N}}$ the family of normalized Hermite polynomials in $\mathbb{R}$. For every $k \in \mathbb{N}$ and $\xi, \eta \in \mathbb{R}$ such that $\eta < \xi$ we have that*

$$P_k(\xi) - P_k(\eta) = C(k)Pe_{k+1}(\gamma) \cdot (\xi - \eta), \tag{44}$$

where $\gamma \in (\eta, \xi)$ and $C(k) = \frac{(-1)^k}{(k+1)(k!)^{1/2}}$. Moreover, $Pe_k(x)$ is the unnormalized Hermite polynomial of k degree.

Proof. We know that $P_k(\xi) = \frac{(-1)^k}{(k!)^{1/2}}e^{\xi^2/2}\frac{d}{d\xi^k}e^{-\xi^2/2}$. Set $c(k) = (-1)^k(k!)^{-1/2}$, then

$$P_k(\xi) - P_k(\eta) = c(k)\left[e^{\xi^2/2}\frac{d}{d\xi^k}e^{-\xi^2/2} - e^{\eta^2/2}\frac{d}{d\eta^k}e^{-\eta^2/2}\right]$$

$$= c(k)\left[e^{x^2/2}\frac{d}{dx^k}e^{-x^2/2}\Big|_{x=\eta}^{\xi}\right]$$

$$= c(k)\int_\eta^\xi F_k(x)dx,$$

18

where F_k is a continuous function such that $F'_k(x) = e^{x^2/2} \frac{d}{dx^k} e^{-x^2/2}$. In fact, denoting by $Pe_k(x)$ the unnormalized Hermite polynomial of k degree, results

$$F'_k(x) = e^{x^2/2} \frac{d}{dx^k} e^{-x^2/2} = Pe_k(x),$$

and since the Hermite polynomials constitute an Appell sequence we have that

$$F'_k(x) = Pe_k(x) = \frac{1}{k+1} Pe'_{k+1}(x),$$

which implies that $F_k(x) = \frac{1}{k+1} Pe_{k+1}(x)$. Now, since $F_k(x)$ is a continuous function, then there exists $\gamma \in (\eta, \xi)$ such that

$$\int_\eta^\xi F_k(x) dx = F_k(\gamma) \cdot (\xi - \eta).$$

All these implies that $P_k(\xi) - P_k(\eta) = c(k) F_k(\gamma) \cdot (\xi - \eta)$. From this expression the lemma follows immediately. □

We will use some technical results on the SPDE to prove the following result—the main result of this section.

Theorem 4.2. *Assume that the eigenvalues of Λ, satisfies that for every $k \in \mathbb{N}$, $\lambda_k < \lambda_{k+1} \to \infty$. Assume that the functional φ is Lipschitz. Then, the numeric approximation Ψ_t^φ (given by (43)) to the solution of the Kolmogorov equation $\Psi \in C((0,T); \mathbb{H})$ depends continuously on the initial conditions.*

Proof. Let $x, y \in H$ be two different initial values. We want to estimate $\Psi_t^x - \Psi_t^y$. By definition,

$$\Psi_t^x = \sum_{\bar{n} \in \mathcal{J}} u_{\bar{n}}^x(t) H_{\bar{n}}(x) . \tag{45}$$

Thus,

$$\begin{aligned}
\Psi_t^x - \Psi_t^y &= \sum_{\bar{n} \in \mathcal{J}} u_{\bar{n}}^x(t) H_{\bar{n}}(x) - \sum_{\bar{n} \in \mathcal{J}} u_{\bar{n}}^y(t) H_{\bar{n}}(y) \\
&= \sum_{\bar{n} \in \mathcal{J}} \left[u_{\bar{n}}^x(t) - u_{\bar{n}}^y(t) \right] H_{\bar{n}}(x) + \sum_{\bar{n} \in \mathcal{J}} u_{\bar{n}}^y(t) \left[H_{\bar{n}}(x) - H_{\bar{n}}(y) \right].
\end{aligned} \tag{46}$$

We focus on the first term in (46). From the definition of the initial condition we obtain the following expression for the time-dependent coefficient

$$u_{\bar{n}}^x(t) = \int_{\mathcal{H}} H_{\bar{n}}(x) \mathbb{E}\left[\varphi(X_t^x) \right] \mu(dx) .$$

19

From this we get

$$
\begin{aligned}
u_{\bar{n}}^x(t) - u_{\bar{n}}^y(t) &= \int_{\mathcal{H}} H_{\bar{n}}(x) \mathbb{E}\big[\varphi(X_t^x)\big] \mu(dx) - \int_{\mathcal{H}} H_{\bar{n}}(y) \mathbb{E}\big[\varphi(X_t^y)\big] \mu(dy) \\
&= \int_{\mathcal{H} \times \mathcal{H}} H_{\bar{n}}(x) \mathbb{E}\big[\varphi(X_t^x)\big] \mu(dx) \mu(dy) \\
&\quad - \int_{\mathcal{H} \times \mathcal{H}} H_{\bar{n}}(y) \mathbb{E}\big[\varphi(X_t^y)\big] \mu(dx) \mu(dy) \\
&= \int_{\mathcal{H} \times \mathcal{H}} H_{\bar{n}}(x) \Big(\mathbb{E}\big[\varphi(X_t^x)\big] - \mathbb{E}\big[\varphi(X_t^y)\big] \Big) \mu(dx) \mu(dy) \\
&\quad + \int_{\mathcal{H} \times \mathcal{H}} \Big(H_{\bar{n}}(x) - H_{\bar{n}}(y) \Big) \mathbb{E}\big[\varphi(X_t^y)\big] \mu(dx) \mu(dy) .
\end{aligned}
$$

Then, by the Cauchy-Schwartz inequality, we obtain

$$
\begin{aligned}
|u_{\bar{n}}^x(t) - u_{\bar{n}}^y(t)|^2 &\leq \left| \int_{\mathcal{H} \times \mathcal{H}} H_{\bar{n}}(x) \left(\mathbb{E}\left[\varphi(X_t^x)\right] - \mathbb{E}\left[\varphi(X_t^y)\right] \right) \mu(dx) \mu(dy) \right|^2 \\
&\quad + \left| \int_{\mathcal{H} \times \mathcal{H}} \left(H_{\bar{n}}(x) - H_{\bar{n}}(y) \right) \mathbb{E}\left[\varphi(X_t^y)\right] \mu(dx) \mu(dy) \right|^2 \\
&\leq \int_{\mathcal{H} \times \mathcal{H}} H_{\bar{n}}^2(x)) \mu(dx) \mu(dy) \\
&\quad \times \int_{\mathcal{H} \times \mathcal{H}} \left| \mathbb{E}\left[\varphi(X_t^x)\right] - \mathbb{E}\left[\varphi(X_t^y)\right] \right|^2 \mu(dx) \mu(dy) \\
&\quad + \int_{\mathcal{H} \times \mathcal{H}} \mathbb{E}^2\left[\varphi(X_t^y)\right] \mu(dx) \mu(dy) \\
&\quad \times \int_{\mathcal{H} \times \mathcal{H}} |H_{\bar{n}}(x) - H_{\bar{n}}(y)|^2 \mu(dx) \mu(dy) \\
&= \int_{\mathcal{H} \times \mathcal{H}} \left| \mathbb{E}\left[\varphi(X_t^x)\right] - \mathbb{E}\left[\varphi(X_t^y)\right] \right|^2 \mu(dx) \mu(dy) \\
&\quad + \int_{\mathcal{H} \times \mathcal{H}} \mathbb{E}^2\left[\varphi(X_t^y)\right] \mu(dx) \mu(dy) \\
&\quad \times \int_{\mathcal{H} \times \mathcal{H}} |H_{\bar{n}}(x) - H_{\bar{n}}(y)|^2 \mu(dx) \mu(dy) .
\end{aligned}
\tag{47}
$$

We now estimate the norm of the expression (46) with the help of (47).

$$
\|\Psi_t^x - \Psi_t^y\|^2_{\left(L^2(\mathcal{H},\mu)\right)^2} = \int_{\mathcal{H}\times\mathcal{H}} |\Psi_t^x - \Psi_t^y|^2 \mu(dx)\mu(dy)
$$

$$
\leq \int_{\mathcal{H}\times\mathcal{H}} \left| \sum_{\bar{n}\in\mathcal{J}} \left[u_{\bar{n}}^x(t) - u_{\bar{n}}^y(t) \right] H_{\bar{n}}(x) \right|^2 \mu(dx)\mu(dy)
$$

$$
+ \int_{\mathcal{H}\times\mathcal{H}} \left| \sum_{\bar{n}\in\mathcal{J}} u_{\bar{n}}^y(t) \left[H_{\bar{n}}(x) - H_{\bar{n}}(y) \right] \right|^2 \mu(dx)\mu(dy)
$$

$$
\leq \int_{\mathcal{H}\times\mathcal{H}} \sum_{\bar{n}\in\mathcal{J}} \left| u_{\bar{n}}^x(t) - u_{\bar{n}}^y(t) \right|^2 H_{\bar{n}}^2(x)\mu(dx)\mu(dy)
$$

$$
+ \int_{\mathcal{H}\times\mathcal{H}} \sum_{\bar{n}\in\mathcal{J}} \left[u_{\bar{n}}^y(t) \right]^2 \sum_{\bar{n}\in\mathcal{J}} \left| H_{\bar{n}}(x) - H_{\bar{n}}(y) \right|^2 \mu(dx)\mu(dy)
$$

$$
= \sum_{\bar{n}\in\mathcal{J}} \left| u_{\bar{n}}^x(t) - u_{\bar{n}}^y(t) \right|^2 \int_{\mathcal{H}\times\mathcal{H}} H_{\bar{n}}^2(x)\mu(dx)\mu(dy)
$$

$$
+ \sum_{\bar{n}\in\mathcal{J}} \left[u_{\bar{n}}^y(t) \right]^2 \sum_{\bar{n}\in\mathcal{J}} \int_{\mathcal{H}\times\mathcal{H}} \left| H_{\bar{n}}(x) - H_{\bar{n}}(y) \right|^2 \mu(dx)\mu(dy) \qquad (48)
$$

$$
= \sum_{\bar{n}\in\mathcal{J}} \left[u_{\bar{n}}^y(t) \right]^2 \sum_{\bar{n}\in\mathcal{J}} \int_{\mathcal{H}\times\mathcal{H}} \left| H_{\bar{n}}(x) - H_{\bar{n}}(y) \right|^2 \mu(dx)\mu(dy)
$$

$$
+ \sum_{\bar{n}\in\mathcal{J}} \left| u_{\bar{n}}^x(t) - u_{\bar{n}}^y(t) \right|^2
$$

$$
= \int_{\mathcal{H}\times\mathcal{H}} \left| \mathbb{E}\left[\varphi(X_t^x) \right] - \mathbb{E}\left[\varphi(X_t^y) \right] \right|^2 \mu(dx)\mu(dy)
$$

$$
+ \int_{\mathcal{H}\times\mathcal{H}} \mathbb{E}^2\left[\varphi(X_t^y) \right] \mu(dx)\mu(dy)
$$

$$
\times \sum_{\bar{n}\in\mathcal{J}} \int_{\mathcal{H}\times\mathcal{H}} \left| H_{\bar{n}}(x) - H_{\bar{n}}(y) \right|^2 \mu(dx)\mu(dy)
$$

$$
+ \sum_{\bar{n}\in\mathcal{J}} \left[u_{\bar{n}}^y(t) \right]^2 \sum_{\bar{n}\in\mathcal{J}} \int_{\mathcal{H}\times\mathcal{H}} \left| H_{\bar{n}}(x) - H_{\bar{n}}(y) \right|^2 \mu(dx)\mu(dy) .
$$

Note that $\mathbb{E}^2\left[\varphi(X_t^y) \right] = u^2(t,x) \in L^2(\mathcal{H},\mu)$, therefore the first integral in the second term is a continuous bounded function of t. Moreover, $\sum_{\bar{n}\in\mathcal{J}} \left[u_{\bar{n}}^y(t) \right]^2$ is the $L^2(\mathcal{H},\mu)$-norm of the function $u(t,x)$, then the series converges and it is also a continuous bounded function of t. Thus, from (48)

we get

$$\|\Psi_t^x - \Psi_t^y\|^2_{\left(L^2(\mathcal{H},\mu)\right)^2} \leq \int_{\mathcal{H}\times\mathcal{H}} \left|\mathbb{E}\Big[\varphi(X_t^x)\Big] - \mathbb{E}\Big[\varphi(X_t^y)\Big]\right|^2 \mu(dx)\mu(dy)$$
$$+ f(t) \sum_{\bar{n}\in\mathcal{J}} \int_{\mathcal{H}\times\mathcal{H}} \left|H_{\bar{n}}(x) - H_{\bar{n}}(y)\right|^2 \mu(dx)\mu(dy) , \tag{49}$$

where $f(t) = \sum_{\bar{n}\in\mathcal{J}} \left[u_{\bar{n}}^y(t)\right]^2 + \int_{\mathcal{H}} \mathbb{E}^2\Big[\varphi(X_t^y)\Big]\mu(dy)$. From the proof of Theorem 3.1 (see (23)) we know that

$$\||\Psi_t^\varphi - \Phi_t^\psi\||^2 = \int_{\mathcal{H}\times\mathcal{H}} \left|\mathbb{E}\Big[\varphi(X_t^x)\Big] - \mathbb{E}\Big[\varphi(X_t^y)\Big]\right|^2 \mu(dx)\mu(dy)$$
$$\leq \exp(Ct) \int_{\mathcal{H}\times\mathcal{H}} \|x - y\|_{\mathcal{H}}^2 \mu(dx)\mu(dy) \tag{50}$$
$$= \exp(Ct)\||x - y\||^2.$$

Therefore the first term in the right side of (49) is bounded by (50).

We now focus on the second term in the last inequality. Notice that for every $\bar{n} \in \mathcal{J}$ we have

$$H_{\bar{n}}(x) - H_{\bar{n}}(y) = \prod_{i=1}^{\infty} \Big[P_{n_i}(\xi_i) - P_{n_i}(\eta_i)\Big], \tag{51}$$

where $\xi_i = \langle x, \Lambda^{-1/2}e_i\rangle_{\mathcal{H}}$ and $\eta_i = \langle y, \Lambda^{-1/2}e_i\rangle_{\mathcal{H}}$ (see (5) and lines after that for the definition). Hence, applying Lemma 4.1 to equation (51) we have that

$$H_{\bar{n}}(x) - H_{\bar{n}}(y) = \prod_{i=1}^{\infty} C(i) P e_{i+1}(\gamma_i) \cdot (\xi_i - \eta_i)$$
$$= \prod_{i=1}^{\infty} C(i) P e_{i+1}(\gamma_i)\langle x - y, \Lambda^{-1/2}e_i\rangle_{\mathcal{H}}, \tag{52}$$

here $\gamma_i \in \left(\xi_i \wedge \eta_i, \xi_i \vee \eta_i \right)$ for every $i \in \mathbb{N}$. Then

$$\sum_{\bar{n} \in \mathcal{J}} \int_{\mathcal{H} \times \mathcal{H}} \left| H_{\bar{n}}(x) - H_{\bar{n}}(y) \right|^2 \mu(dx)\mu(dy)$$

$$= \sum_{\bar{n} \in \mathcal{J}} \int_{\mathcal{H} \times \mathcal{H}} \left| \prod_{i=1}^{\infty} C(i) Pe_{i+1}(\gamma_i) \langle x - y, \Lambda^{-1/2} e_i \rangle_{\mathcal{H}} \right|^2 \mu(dx)\mu(dy)$$

$$= \sum_{\bar{n} \in \mathcal{J}} \int_{\mathcal{H} \times \mathcal{H}} \prod_{i=1}^{\infty} \left[C(i) Pe_{i+1}(\gamma_i) \right]^2 \left| \langle x - y, \Lambda^{-1/2} e_i \rangle_{\mathcal{H}} \right|^2 \mu(dx)\mu(dy)$$

$$\leq \sum_{\bar{n} \in \mathcal{J}} \int_{\mathcal{H} \times \mathcal{H}} \prod_{i=1}^{\infty} \left[C(i) Pe_{i+1}(\gamma_i) \right]^2 \|x - y\|_{\mathcal{H}}^2 \|\Lambda^{-1/2} e_i\|_{\mathcal{H}}^2 \mu(dx)\mu(dy) \qquad (53)$$

$$= \sum_{\bar{n} \in \mathcal{J}} \int_{\mathcal{H} \times \mathcal{H}} \prod_{i=1}^{\infty} \left[C(i) Pe_{i+1}(\gamma_i) \right]^2 \|x - y\|_{\mathcal{H}}^2 \lambda_i^{-1} \|e_i\|_{\mathcal{H}}^2 \mu(dx)\mu(dy)$$

$$= \|x - y\|_{\mathcal{H}}^2 \sum_{\bar{n} \in \mathcal{J}} \prod_{i=1}^{\infty} \left[C(i) \right]^2 \lambda_i^{-1} \int_{\mathcal{H} \times \mathcal{H}} \left[Pe_{i+1}(\gamma_i) \right]^2 \mu(dx)\mu(dy).$$

Recall that for every $i \in \mathbb{N}$ we have that $\gamma_i \in \left(\xi_i \wedge \eta_i, \xi_i \vee \eta_i \right)$, set $\hat{\gamma}_i \in \left(\xi_i \wedge \eta_i, \xi_i \vee \eta_i \right)$ such that $Pe_i^2(\gamma_i) \leq Pe_{i+1}^2(\hat{\gamma}_i)$ for every $\gamma_i \in \left(\xi_i \wedge \eta_i, \xi_i \vee \eta_i \right)$, notice that the existence of $\hat{\gamma}_i$ is guaranteed since $Pe_{i+1}^2(\cdot)$ is a continuous function. Then, from (53) we get

$$\sum_{\bar{n} \in \mathcal{J}} \int_{\mathcal{H} \times \mathcal{H}} \left| H_{\bar{n}}(x) - H_{\bar{n}}(y) \right|^2 \mu(dx)\mu(dy)$$

$$\leq \|x - y\|_{\mathcal{H}}^2 \sum_{\bar{n} \in \mathcal{J}} \prod_{i=1}^{\infty} \left[C(i) \right]^2 \lambda_i^{-1} \left[Pe_{i+1}(\hat{\gamma}_i) \right]^2 \int_{\mathcal{H}} \int_{\mathcal{H}} \mu(dx)\mu(dy) \qquad (54)$$

$$= \|x - y\|_{\mathcal{H}}^2 \sum_{\bar{n} \in \mathcal{J}} \prod_{i=1}^{\infty} \left[C(i) \right]^2 \lambda_i^{-1} \left[Pe_{i+1}(\hat{\gamma}_i) \right]^2.$$

Here, we recall that $C(i) = \frac{(-1)^i}{(i+1)(i!)^{1/2}}$ then $\frac{(-1)^i}{[(i+1)!]^{1/2}} Pe_{i+1}(\hat{\gamma}_i)$ is the normalized Hermite polynomial of $i + 1$ degree evaluated on $\hat{\gamma}_i$ which is bounded by a constant C for every $i \in \mathbb{N}$. Moreover, since $\lambda_k < \lambda_{k+1} \to \infty$ then this implies that

$$\sum_{\bar{n} \in \mathcal{J}} \prod_{i=1}^{\infty} \left[C(i) \right]^2 \lambda_i^{-1} \left[Pe_{i+1}(\hat{\gamma}_i) \right]^2 \leq C \sum_{\bar{n} \in \mathcal{J}} \prod_{i=1}^{\infty} \lambda_i^{-1} (i+1)^{-1} \leq C, \qquad (55)$$

23

where C is a finite constant. Putting together (53) and (55) we get that

$$\sum_{\bar{n} \in \mathcal{J}} \int_{\mathcal{H} \times \mathcal{H}} \left| H_{\bar{n}}(x) - H_{\bar{n}}(y) \right|^2 \mu(dx)\mu(dy) \leq C\|x - y\|_{\mathcal{H}} \ . \tag{56}$$

Putting together inequalities (49), (50) and (56) we obtain

$$\|\Psi_t^x - \Psi_t^y\|_{\left(L^2(\mathcal{H},\mu)\right)^2}^2 \leq \exp(Ct) \int_{\mathcal{H} \times \mathcal{H}} \|x - y\|_{\mathcal{H}}^2 \mu(dx)\mu(dy) \tag{57}$$
$$+ f(t)\|x - y\|_{\mathcal{H}}.$$

Now, if $\|x-y\|_{\mathcal{H}} \leq \delta$, then from (57) we get $\|\Psi_t^x - \Psi_t^y\|_{\left(L^2(\mathcal{H},\mu)\right)^2} \leq G(t)\delta.$ $\quad\square$

Remark 4. If we consider in addition the supremum norm on t, then from (57) we get

$$\sup_{0 \leq t \leq T} \|\Psi_t^x - \Psi_t^y\|_{\left(L^2(\mathcal{H},\mu)\right)^2}^2 \leq C\|x - y\|_{\mathcal{H}}^2 \sup_{0 \leq t \leq T} f(t) \tag{58}$$
$$+ \exp(CT) \int_{\mathcal{H} \times \mathcal{H}} \|x - y\|_{\mathcal{H}}^2 \mu(dx)\mu(dy) \ .$$

Notice that $f(t)$ is differentiable and continuous, then $\sup_{0 \leq t \leq T} f(t) \leq C$, then from (58) we obtain

$$\sup_{0 \leq t \leq T} \|\Psi_t^x - \Psi_t^y\|_{\left(L^2(\mathcal{H},\mu)\right)^2} \leq C\|x - y\|_{\mathcal{H}} \tag{59}$$
$$+ \exp(CT) \int_{\mathcal{H} \times \mathcal{H}} \|x - y\|_{\mathcal{H}}^2 \mu(dx)\mu(dy) \ .$$

From this inequality it is possible to show the continuously dependence on the initial conditions for this norm.

5   Numerical experiments

In this section we run numerical experiments to illustrate that our scheme preserves the underlying initial condition continuity. To this end, we solve a stochastic version of the Fisher and Burgers PDEs with two near initial function conditions $x(\xi)$, $\hat{x}(\xi)$. In [16] we provide a GitHub repository with a Python implementation to reproduce the following figures. We also provide in [10, 11], the 3D color on-line plotly versions of the regarding solutions.

Stochastic Fisher-KPP equation in an interval

Let $\mathcal{H} = L^2(0,1)$. We consider the stochastic Fisher-KPP equation

$$
\begin{aligned}
dX(t,\xi) &= \left[\nu\partial_\xi^2 X(t,\xi) + X(t,\xi)(1 - X(t,\xi))\right] dt + dW(t,\xi), \\
X(t,0) &= X(t,1) = 0, \quad t > 0, \\
X(0,\xi) &\in \mathcal{H}, \; \xi \in [0,1],
\end{aligned}
\tag{60}
$$

in the interval $[0,1]$ and with initial function conditions $x(\xi))$ and $\widehat{x}(\xi)$. In order to fix this initial function conditions close, we use for our experiments

$$
x(\xi) := \operatorname{sech}^2(5(\xi - 0.5)), \qquad \widehat{x}(\xi) := \sum_{k=0}^{N} T_k(x(\xi)),
\tag{61}
$$

where $T_k(\cdot)$ denotes the Chebyshev polynomial of the first kind. That is, $\widehat{x}(\cdot)$ is the Chebyshev truncated expansion of $x(\cdot)$.

Figure 1 suggest the conclusion of Theorem 4.2, that is, the solutions of equation (60) are continuous respect to initial conditions and satisfies the estimation (50).

Stochastic Burgers equation

Let $\mathcal{H} = L^2(0,1)$, consider the stochastic Burgers equation in the interval $[0,1]$

$$
\begin{aligned}
dX(t,\xi) &= \left[\nu\partial_\xi^2 X(t,\xi) + \frac{1}{2}\partial_\xi X^2(t,\xi)\right] dt + dW(t,\xi), \\
X(t,0) &= X(t,1) = 0, \quad t > 0, \\
X(0,\xi) &= x(\xi), \quad x \in \mathcal{H} .
\end{aligned}
\tag{62}
$$

We use the initial conditions $x(\xi)$ and its truncated Chebyshev expansion

$$
x(\xi) := \sin(\pi\xi), \qquad \widehat{x}(\xi) := \sum_{k=0}^{N} T_k x(\xi).
\tag{63}
$$

According to the $\mathcal{L}(\mathcal{H}, \mu)$-distance between the two underlying solutions, Figure 2 confirms the conclusion of Theorem 4.2.
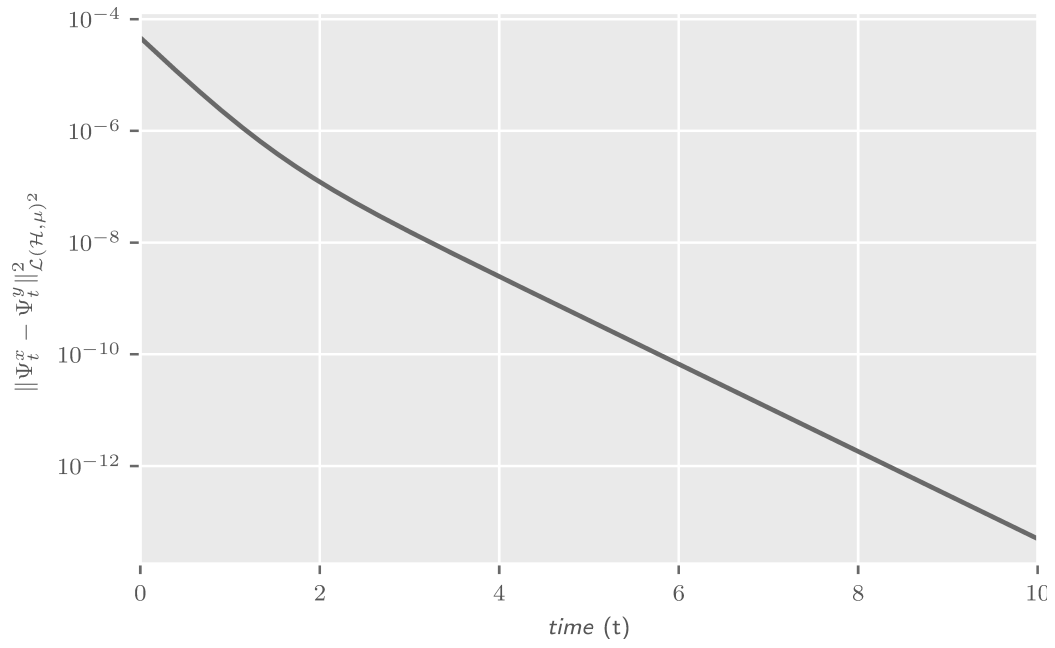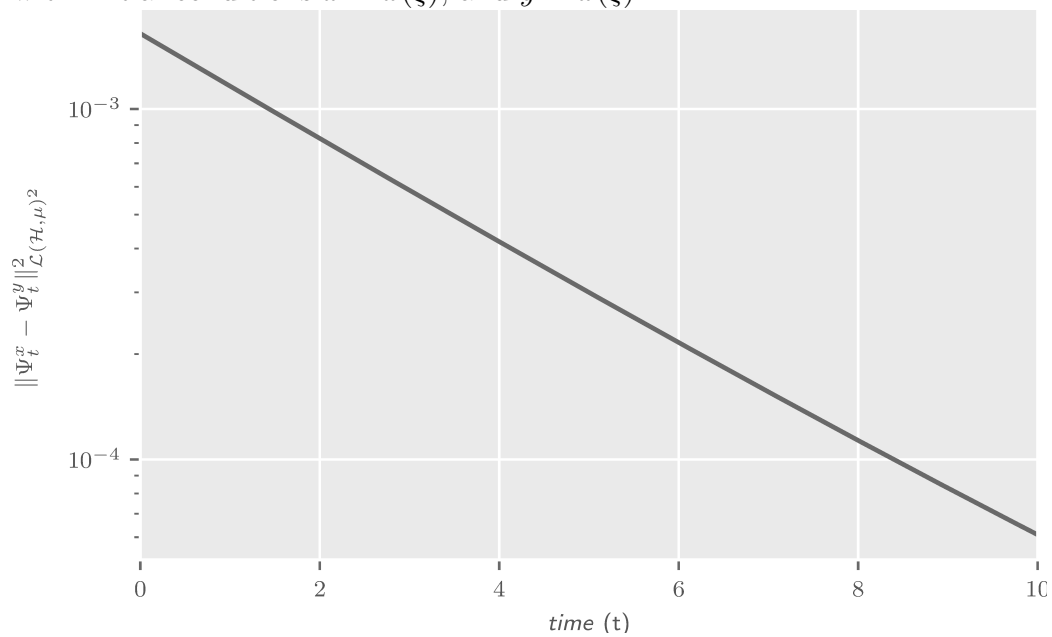
Figure 1: $\mathcal{L}^2(\mathcal{H}, \mu)$ distance between two solutions of the stochastic Fisher PDE with initial conditions $x = x(\xi)$, and $y = \widehat{x}(\xi)$.

Figure 2: Distance between two solutions of the stochastic Burgers eq. (62) with initial conditions $x = x(\xi)$, and $y = \widehat{x}(\xi)$.

6 Conclusions

To the best of our knowledge, our results represent the first contribution to the numeric stability respect to initial conditions of weak approximations of Kolmogorov equations in infinite dimensions. This kind of stability, combining with the weak approximation approach, would save computation time. That is, since our scheme asks specific conditions to obtain a weak numerical solution of an underlying SPDE, we convert the stochastic problem into a deterministic ODE for the first moment. This procedure overcome Montecarlo type simulations to approximate moments or distributions—simulate many realization of the numerical stochastic process to approximate distributions or moments. Further, under our setting, the regarding spectral approximation assures high precision and order of convergence. Thus we guess that our method would improve the time and save resources of computation. We are preparing another article to confirm this conjectures.

References

[1] Viorel Barbu and Giuseppe Da Prato. The Kolmogorov equation for a 2D-Navier–Stokes stochastic flow in a channel. *Nonlinear Analysis: Theory, Methods & Applications*, 69(3):940–949, 2008. ISSN 0362-546X. doi: 10.1016/j.na.2008.02.072. URL http://www.sciencedirect.com/science/article/pii/S0362546X08001569. Trends in Nonlinear Analysis: in Honour of Professor V.Lakshmikantham.

[2] Vladimir Bogachev, Giuseppe Da Prato, and Michael Röckner. Existence results for fokker–planck equations in hilbert spaces. In Robert Dalang, Marco Dozzi, and Francesco Russo, editors, *Seminar on Stochastic Analysis, Random Fields and Applications VI*, page 23–35, Basel, 2011. Springer Basel. ISBN 978-3-0348-0021-1.

[3] Pao-Liu Chow. *Stochastic partial differential equations*. Chapman & Hall/CRC Applied Mathematics and Nonlinear Science Series. Chapman & Hall/CRC, Boca Raton, FL, 2007. ISBN 978-1-58488-443-9; 1-58488-443-6.

[4] G. Da Prato, F. Flandoli, and M. Röckner. Fokker-planck equations for spde with non-trace-class noise. *Commun. Math. Stat.*, 1(3):281–304, 2013. ISSN 2194-6701. doi: 10.1007/s40304-013-0015-5. URL https://doi.org/10.1007/s40304-013-0015-5.

[5] Giuseppe Da Prato. *Kolmogorov equations for stochastic PDEs*. Advanced Courses in Mathematics. CRM Barcelona. Birkhäuser Verlag, Basel, 2004. ISBN 3-7643-7216-8. doi: 10.1007/978-3-0348-7909-5. URL https://doi.org/10.1007/978-3-0348-7909-5.

[6] Giuseppe Da Prato and Arnaud Debussche. m-Dissipativity of Kolmogorov Operators Corresponding to Burgers Equations with Space-time White Noise. *Potential Analysis*, 26(1):31–55, Feb 2007. ISSN 1572-929X. doi: 10.1007/s11118-006-9021-5. URL https://doi.org/10.1007/s11118-006-9021-5.

[7] Giuseppe Da Prato and Jerzy Zabczyk. *Stochastic equations in infinite dimensions*, volume 44 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1992. ISBN 0-521-38529-6. doi: 10.1017/CBO9780511666223. URL https://doi.org/10.1017/CBO9780511666223.

[8] Giuseppe Da Prato and Jerzy Zabczyk. *Second order partial differential equations in Hilbert spaces*, volume 293 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2002. ISBN 0-521-77729-1. doi: 10.1017/CBO9780511543210. URL https://doi.org/10.1017/CBO9780511543210.

[9] Francisco Delgado-Vences and Franco Flandoli. A spectral-based numerical method for Kolmogorov equations in Hilbert spaces. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.*, 19(3):1650020, 37, 2016. ISSN 0219-0257. doi: 10.1142/S021902571650020X. URL https://doi.org/10.1142/S021902571650020X.

[10] Saul Diaz-Infante. Likening of two solutions of the burgers equation with two initial function conditions. https://plot.ly/ sauldiazinfante/30/ [2019/05/11], . URL https://plot.ly/~sauldiazinfante/30/.

[11] Saul Diaz-Infante. Likening of two solutions of the fisher equation with two initial function conditions. https://plot.ly/ sauldiazinfante/28/ [2019/05/11], . URL https://plot.ly/~sauldiazinfante/28/.

[12] David Gottlieb, Liviu Lustman, and Eitan Tadmor. Stability analysis of spectral methods for hyperbolic initial-boundary value systems. *SIAM J. Numer. Anal.*, 24 (2):241–256, 1987. ISSN 0036-1429. doi: 10.1137/0724020. URL https://doi.org/10.1137/0724020.

[13] Peter Imkeller. *Malliavin's calculus and applications in stochastic control and finance*, volume 1 of *IMPAN Lecture Notes*. Polish Academy of Sciences, Institute of Mathematics, Warsaw, 2008. ISBN 978-83-86806-02-7.

[14] Annika Lang, Andreas Petersson, and Andreas Thalhammer. Mean-square stability analysis of approximations of stochastic differential equations in infinite dimensions. *BIT Numerical Mathematics*, 57(4):963–990, Dec 2017. ISSN 1572-9125. doi: 10.1007/s10543-017-0684-7. URL https://doi.org/10.1007/s10543-017-0684-7.

[15] Ning Li, Joseph Fiordilino, and Xinlong Feng. Ensemble Time-Stepping Algorithm for the Convection-Diffusion Equation with Random Diffusivity. *Journal of Scientific Computing*, 79(2):1271–1293, May 2019. ISSN 1573-7691. doi: 10.1007/s10915-018-0890-8. URL https://doi.org/10.1007/s10915-018-0890-8.

[16] Alan Daniel Matzumiya. Github of the python implementation of the weak spectral method. `https://github.com/alanmatzumiya/Paper`, May 2019. URL `https://github.com/alanmatzumiya/Paper`.

[17] Grigori Noah Milstein and Michael V Tretyakov. *Stochastic numerics for mathematical physics*. Springer Science & Business Media, 2013.

[18] Christoph Schwab and Endre Süli. Adaptive galerkin approximation algorithms for kolmogorov equations in infinite dimensions. *Stochastic Partial Differential Equations: Analysis and Computations*, 1(1):204–239, 2013.

[19] Lloyd N. Trefethen and Manfred R. Trummer. An instability phenomenon in spectral methods. *SIAM J. Numer. Anal.*, 24(5):1008–1023, 1987. ISSN 0036-1429. doi: 10.1137/0724066. URL `https://doi.org/10.1137/0724066`.

[20] Zhongqiang Zhang and George Karniadakis. *Numerical methods for stochastic partial differential equations with white noise*. Springer, 2017.