

Manuscript Number:

Title: Initial conditions stability of a numerical approximation for  
Kolmogorov equations in infinite dimensions

Article Type: Research Paper

Section/Category: Numerical methods for differential equations (ruffo)

Keywords: Stability; spectral methods; Kolmogorov equation; stochastic  
parabolic equations

Corresponding Author: SAUL Diaz Infante Velasco,

Corresponding Author's Institution:

First Author: SAUL Diaz Infante Velasco

Order of Authors: SAUL Diaz Infante Velasco; Francisco Delgado Vences,  
Ph.D; ALAN D Matzumya-Zazueta, Master

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October 5, 2019

Saúl Díaz-Infante Velasco, PhD  
CONACYT-Universidad de Sonora,  
Department of Mathematics,  
Graduate division  
Blvd Luis Encinas y Rosales S/N,  
Hermosillo, Sonora C.P. 8300,  
Phone: +52(662) 2592219 ext. 2430  
Email: saul.diazinfante@unison.mx

Zhaosheng Feng  
Editor, Communications in Nonlinear  
Science and Numerical Simulation,  
University of Texas-Rio Grande Valley,  
School of Mathematical and Statistical Science,  
Edinburg, United States

Dear Professor Feng,

We are pleased to submit an original research article entitled "Initial conditions continuity of a numerical approximation for Kolmogorov equations," by Francisco Delgado-Vences, Alan Matsumiya-Zazueta, and Saúl Díaz-Infante to be considered for publication in Communications in Nonlinear Science and Numerical Simulation.

In this article, we derive stability theory for the approximation of Kolmogorov equations in infinite dimensions. Our results are for the stability respect to initial conditions of a weak spectral method.

To the best of our knowledge, this manuscript is the first work that addresses this kind of numerical stability for weak spectral schemes of parabolic SPDEs.

Our manuscript has not been published and is not under consideration for publication elsewhere. We have no conflicts of interest to disclose. We also confirm that the co-authors have agreed to the present submitted version.

Sincerely yours,



Saúl Díaz-Infante Velasco, PhD

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# 1 Initial conditions stability of a numerical approximation 2 for Kolmogorov equations in infinite dimensions

3 Francisco Delgado-Vences<sup>a</sup>, Alan Matsumiya-Zazueta<sup>c</sup>, Saúl Díaz-Infante<sup>b,\*</sup>

4 <sup>a</sup> CONACYT-UNAM, Instituto de Matemáticas, Sede Oaxaca, Antonio de León #2, altos, Col.  
5 Centro, Oaxaca de Juárez, CP. 68000 Phone: +52(951) 5160541

6 <sup>b</sup> CONACYT-Universidad de Sonora, Departamento de Matemáticas, Boulevard Luis Encinas y  
7 Rosales S/N, 83000, Hermosillo, Sonora, México.

8 <sup>c</sup> Universidad de Sonora Departamento de Matemáticas, Departamento de Matemáticas. Blvd  
9 Luis Encinas y Rosales S/N, Colonia Centro. Edificio 3K-1. C.P. 83000. Hermosillo, Sonora,  
10 México,

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## 11 Abstract

We establish conditions to ensure stability of initial conditions for Kolmogorov equations associated to some kind of stochastic partial differential equations in infinite dimensions. Our approach consists in solving the associated Kolmogorov equation of the underlying SPDE with a spectral method. The numerical solution of the Kolmogorov equation results to be a weak approximation, in probability sense, of a parabolic type of SPDEs. We illustrate our results with numerical experiments.

12 **Keywords:** numerical stability, weak approximation, Kolmogorov, spectral  
13 methods, SPDs .

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\*Corresponding author

Email addresses: delgado@im.unam.mx (Francisco Delgado-Vences ),  
alan.matsumiya@gmail.com (Alan Matsumiya-Zazueta ), saul.diazinfante@unison.mx  
(Saúl Díaz-Infante )

## 14 **1. Introduction**

15 Stochastic Partial Differential Equations (SPDEs) are important tools in mod-  
16 eling complex phenomena, they arise in many fields of knowledge like Physics,  
17 Biology, Economy, Finance, etc. Develop efficient numerical methods for simu-  
18 lating SPDEs is very important but also very difficult and challenging.

19 The Fokker-Planck-Kolmogorov (FPK) equation is a partial differential equa-  
20 tion that describes the time evolution of the probability density function of the  
21 velocity of a particle under the influence of drag forces and random forces, it is  
22 a kind of continuity equation for densities. Citing [1] “parabolic equations on  
23 Hilbert spaces appear in mathematical physics to model systems with infinitely  
24 many degrees of freedom. Typical examples are provided by spin configurations  
25 in statistical mechanics and by crystals in solid state theory. Infinite-dimensional  
26 parabolic equations provide an analytic description of infinite dimensional diffu-  
27 sion processes in such branches of applied mathematics as population biology,  
28 fluid dynamics, and mathematical finance.” This kind of equations have been  
29 deeply studied in the last years, see for instance [2, 3, 4] and the references therein.

30 Try to finding analytical solutions of FPK associated with SPDEs results im-  
31 practical. Thus, work with efficient and accurate numerical schemes is crucial. In  
32 this way, the spectral methods play an essential role to obtain better schemes—under  
33 certain conditions; this sort of methods are more accurate than finite differences of  
34 finite elements and need fewer grid points. Here the adjective “better” would be  
35 under accuracy, consistency, stability, and other targets properties. In this work,  
36 we explore the ability of the method reported in [5] to preserve the continuity re-  
37 spect to initial conditions. That is, if a given problem satisfies certain regularity  
38 conditions, then two of its solution remain closed if its initial function conditions

are close. So, we desire that a numerical method reproduce this behavior and if it is the case, we say that an underlying method is stable in this context.

Our main contribution is the characterization of mild conditions to assure the continuity respect to initial function conditions to a family of SPDEs and the stability of a regarding weak spectral approximation. To the best of our knowledge, this paper is the first in report numerical stability theory for Kolmogorov equations in infinite dimensions.

The stability theory for spectral methods is still under construction and is an active research area. We mention the seminal works of L.N. Trefethen and M.R. Trummer [6], D. Gottlieb et. al. [7] as reference for the deterministic case, and N. Li, J. Fiordilino, and X. Feng, [8] A. Lang, A. Petersson, and A. Thalhammer, [9] for the stochastic version.

This paper is organized as follows. In Section 2 we review the Fokker-Plank-Kolmogorov equation associated with SPDEs in a separable Hilbert space. Section 3 provides conditions to assure stability respect initial conditions and in Section 4 we illustrate our results with numerical experiments.

## 2. Kolmogorov equations for SPDEs in Hilbert spaces

Let  $\mathcal{H}$  be a separable infinite-dimensional Hilbert space with inner product  $(\cdot, \cdot)_{\mathcal{H}}$  and norm  $\|\cdot\|_{\mathcal{H}}$ . We define a Gaussian measure  $\mu$  with mean zero and nuclear covariance operator  $\Lambda$  with  $Tr(\Lambda) < +\infty$ .

We focus on the following Kolmogorov equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}Tr(QD^2u) + \langle Ax, Du \rangle_{\mathcal{H}} + \langle B(x), Du \rangle_{\mathcal{H}}, \quad x \in D(A). \quad (1)$$

Several authors have proved results on existence and uniqueness of the solution of the Kolmogorov equations, see for instance Da Prato [4] for a survey, Da

62 Prato-Debussche [10] for the Burgers equation, Barbu-Da Prato [11] for the 2D  
 63 Navier-Stokes stochastic flow in a channel.

#### 64 2.1. On the Ornstein-Uhlenbeck semigroup

65 Following [12], in  $\mathcal{H}$  we define a Gaussian measure  $\mu$  with mean zero and  
 66 nuclear covariance operator  $\Lambda$  with  $Tr(\Lambda) < +\infty$  and since  $\Lambda : \mathcal{H} \mapsto \mathcal{H}$  is a pos-  
 67 itive definite, self-adjoint operator then its square-root operator  $\Lambda^{1/2}$  is a positive  
 68 definite, self-adjoint Hilbert-Schmidt operator on  $\mathcal{H}$ .

69 Define the inner product  $(g, h)_0 := (\Lambda^{-1/2}g, \Lambda^{-1/2}h)_{\mathcal{H}}$ , for  $g, h \in \Lambda^{1/2}\mathcal{H}$ .  
 70 Let  $\mathcal{H}_0$  denote the Hilbert subspace of  $\mathcal{H}$ , which is the completion of  $\Lambda^{1/2}\mathcal{H}$  with  
 71 respect to the norm  $\|g\|_0 := (g, g)_0^{1/2}$ . Then  $\mathcal{H}_0$  is dense in  $\mathcal{H}$  and the inclusion map  
 72  $i : \mathcal{H}_0 \hookrightarrow \mathcal{H}$  is compact. The triple  $(i, \mathcal{H}_0, \mathcal{H})$  forms an abstract Wiener space.

73 Let  $\mathbb{H} = L^2(\mathcal{H}, \mu)$  denote the Hilbert space of Borel measurable functionals on  
 74 the probability space with inner product

$$[\Phi, \Psi]_{\mathbb{H}} := \int_{\mathcal{H}} \Phi(v)\Psi(v)\mu(dv), \quad \text{for } \Phi, \Psi \in \mathbb{H},$$

75 and norm  $\|\Phi\|_{\mathbb{H}} := [\Phi, \Phi]_{\mathbb{H}}^{1/2}$ . We choose a basis system  $\{\varphi_k\}$  for  $\mathcal{H}$ .

76 A functional  $\Phi : \mathcal{H} \mapsto \mathbb{R}$ , is said to be a smooth simple functional (or a  
 77 cylinder functional) if there exists a  $C^\infty$ -function  $\phi$  on  $\mathbb{R}^n$  and  $n$ -continuous linear  
 78 functional  $l_1, \dots, l_n$  on  $\mathcal{H}$  such that for  $h \in \mathcal{H}$

$$\Phi(h) = \phi(h_1, \dots, h_n) \quad \text{where} \quad h_i = l_i(h), \quad i = 1, \dots, n. \quad (2)$$

79 The set of all such functionals will be denoted by  $\mathcal{S}(\mathbb{H})$ . Denote by  $P_k(x)$  the  
 80 Hermite polynomial of degree  $k$  taking values in  $\mathbb{R}$ . Then,  $P_k(x)$  is given by the  
 81 following formula

$$P_k(x) = \frac{(-1)^k}{(k!)^{1/2}} e^{\frac{x^2}{2}} \frac{d^k}{dx^k} e^{-\frac{x^2}{2}}$$

82 with  $P_0 = 1$ . It is well-known that  $\{P_k(\cdot)\}_{k \in \mathbb{N}}$  is a complete orthonormal system for  
 83  $L^2(\mathbb{R}, \mu_1(dx))$  with  $\mu_1(dx) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$ . Define the set of infinite multi-index as

$$\mathcal{J} = \left\{ \alpha = (\alpha_i, i \geq 1) \mid \alpha_i \in \mathbb{N} \cup \{0\}, \quad |\alpha| := \sum_{i=1}^{\infty} \alpha_i < +\infty \right\}.$$

For  $\mathbf{n} \in \mathcal{J}$  define the *Hermite polynomial functionals* on  $\mathcal{H}$  by

$$H_{\mathbf{n}}(h) = \prod_{i=1}^{\infty} P_{n_i}(l_i(h)), \quad h \in \mathcal{H}_0, \quad \mathbf{n} \in \mathcal{J}, \quad (3)$$

84 and where  $l_i(h) = \langle h, \Lambda^{-1/2} \varphi_i \rangle_{\mathcal{H}}$ ,  $i = 1, 2, \dots$  where  $P_n(\xi)$  is the usual Hermite  
 85 polynomial for  $\xi \in \mathbb{R}$  and  $n \in \mathbb{N}$ .

86 **Remark.** Notice that  $l_i(h)$  is defined only for  $h \in \mathcal{H}_0$ . However, regarding  $h$  as  
 87 a  $\mu$ -random variable in  $\mathcal{H}$ , we have  $\mathbb{E}(l_i(h)) = \|\varphi_i\|^2 = 1$  and then  $l_k(h)$  can be  
 88 defined  $\mu$ -a.e.  $h \in \mathcal{H}$ , similar to defining a stochastic integral.

89 It is possible to identify the Hermite polynomial functionals defined in (3),  
 90 for  $h \in \mathcal{H}_0$ , as a deterministic version of the Wick polynomials defined on the  
 91 canonical Wiener space (for further details see [13] for instance).

92 We have the following result (See Theorems 9.1.5 and 9.1.7 in Da Prato-  
 93 Zabczyk [1] or Lemma 3.1 in chapter 9 from Chow [12]).

94 **Lemma 2.1.** For  $h \in \mathcal{H}$  let  $l_i(h) = \langle h, \Lambda^{-1/2} \varphi_i \rangle_{\mathcal{H}}$ ,  $i = 1, 2, \dots$ . The set  $\{H_{\mathbf{n}}\}$   
 95 of all Hermite polynomials on  $\mathcal{H}$  forms a complete orthonormal system for  $\mathbb{H}$ .  
 96 Hence the set of all functionals are dense in  $\mathbb{H}$ . Moreover, we have the direct  
 97 sum decomposition:  $\mathbb{H} = \bigoplus_{j=0}^{\infty} K_j$ , where  $K_j$  is the subspace of  $\mathbb{H}$  spanned by  
 98  $\{H_{\mathbf{n}} : |\mathbf{n}| = j\}$ .



Let  $\Phi$  be a smooth simple functional given by (2). Then the Fréchet derivatives,  $D\Phi = \Phi'$  and  $D_2\Phi = \Phi''$  in  $\mathcal{H}$  can be computed as follows:

$$(D\Phi(h), v) = \sum_{k=1}^n [\partial_k \phi(h_1, \dots, h_n)] l_k(v)$$

$$(D^2\Phi(h), v) = \sum_{j,k=1}^n [\partial_j \partial_k \phi(h_1, \dots, h_n)] l_j(v) l_k(v),$$

for any  $u, v \in \mathcal{H}$ , where  $\partial_k \phi = \frac{\partial}{\partial h_k} \phi$ . Similarly, for  $m > 2$ ,  $D^m\Phi(h)$  is a  $m$ -linear form on  $\mathcal{H}^m$  with inner product  $(\cdot, \cdot)_m$ . We have  $[D^m\Phi(h)](v_1, \dots, v_m) = (D^m\Phi(h), v_1 \otimes \dots \otimes v_m)_m$ , for  $h, v_1, \dots, v_m \in \mathcal{H}$ . Consider the following linear stochastic equation

$$du_t = Au_t dt + dW_t, \quad u_0 = h \in \mathcal{H}. \quad (4)$$

Where  $A : \mathcal{D}(A) \subset \mathcal{H} \rightarrow \mathcal{H}$  is the infinitesimal generator of a strongly continuous semigroup  $e^{tA}$  in  $\mathcal{H}$ .  $W_t$  is a  $Q$ -Wiener process in  $\mathcal{H}$ . Chow in [12, Lemma 9.4.1] has shown the following result.

**Lemma 2.2.** *Suppose that  $A$  and  $Q$  satisfy the following:*

1.  $A : \mathcal{D}(A) \subset \mathcal{H} \rightarrow \mathcal{H}$  is self-adjoint and there is  $\beta > 0$  such that

$$\langle Av, v \rangle_{\mathcal{H}} \leq -\beta \|v\|_{\mathcal{H}}^2 \quad \forall v \in \mathcal{H}.$$

2.  $A$  commutes with  $Q$  in  $\mathcal{D}(A) \subset \mathcal{H}$ .

Then (4) has a unique invariant measure  $\mu$  which is a Gaussian measure on  $\mathcal{H}$  with zero mean and covariance operator  $\Lambda = \frac{1}{2}Q(-A)^{-1} = \frac{1}{2}(-A)^{-1}Q$ .

Suppose that  $A$  and  $Q$  have the same eigenfunctions  $e_k$  with eigenvalues  $\lambda_k$  and  $\rho_k$  respectively.

115 It is well-know (See for instance Da Prato and Zabczyk [1]) that the solution  
 116 of (4) is a time-homogeneous Markov process with transition operator  $P_t$  defined  
 117 for  $\Phi \in \mathbb{H}$  given by

$$(P_t\Phi)(h) = \int_{\mathcal{H}} \Phi(v) \mu_t^h(dv) = \mathbb{E}[\Phi(u_t^h)]. \quad (5)$$

118 Let  $\Phi \in \mathcal{S}(\mathbb{H})$  be a smooth simple functional. By setting  $\varphi_k = e_k$  in (2), it takes the  
 119 form  $\Phi(h) = \phi(l_1(h), \dots, l_n(h))$ , where  $l_k(h) = (h, \Lambda^{-1/2}e_k)$ . Define a differential  
 120 operator  $A_0$  on  $\mathcal{S}(\mathbb{H})$  by

$$\mathcal{A}_0\Phi(v) = \frac{1}{2}Tr[RD^2\Phi(v)] + \langle Av, D\Phi(v) \rangle, \quad v \in H, \quad (6)$$

121 which is well defined, since  $D\Phi \in D(A)$  and  $\langle Av, D\Phi(v) \rangle = (v, AD\Phi(v))_{\mathcal{H}}$ .

122 The following results have been proved in [12].

123 **Lemma 2.3.** *Let  $P_t$  be the transition operator as defined by (4). Then the following*  
 124 *properties hold:*

- 125 1.  $P_t : \mathcal{S}(\mathbb{H}) \rightarrow \mathcal{S}(\mathbb{H})$  for  $t \geq 0$ .
- 126 2.  $\{P_t, t \geq 0\}$  is a strongly continuous semigroup on  $\mathcal{S}(\mathbb{H})$  so that, for any  $\Phi \in$   
 127  $\mathcal{S}(\mathbb{H})$ , we have  $P_0 = I$ ,  $P_{t+s}\Phi = P_tP_s\Phi$ , for all  $t, s \geq 0$ , and  $\lim_{t \downarrow 0} P_t\Phi = \Phi$ .
- 128 3.  $\mathcal{A}_0$  is the infinitesimal generator of  $P_t$  so that, for each  $\Phi \in \mathcal{S}(\mathbb{H})$ ,

$$\lim_{t \downarrow 0} \frac{1}{t}(P_t - I)\Phi = \mathcal{A}_0\Phi.$$

129 □

**Lemma 2.4.** *Let  $H_n(h)$  be a Hermite polynomial functional given by (3). Then the*  
*following hold:*

$$\mathcal{A}_0H_n(h) = -\lambda_n H_n(h), \quad (7)$$

$$P_t H_n(h) = \exp\{-\lambda_n t\} H_n(h), \quad (8)$$

130 for any  $\mathbf{n} \in \mathcal{J}$  and  $h \in H$ , where  $\lambda_{\mathbf{n}} = \sum_{i=1}^{\infty} n_i \lambda_i$ .

131 The following Theorem is a Green formula that we will need forward. Its  
132 proof can be seen, for instance, in [12, Thm. 3.3, Ch. 9].

**Theorem 2.5.** *Let  $\Phi \in \mathcal{S}(\mathbb{H})$  be a smooth simple functional and let  $\mu \sim N(0, \Lambda)$  be a Gaussian measure in  $\mathcal{H}$ . Then, for any  $g, h \in \mathcal{H}$  the following formula holds*

$$\int_{\mathcal{H}} (\Lambda h, D\Phi(v))_{\mathcal{H}} \mu(dv) = \int_{\mathcal{H}} (v, h)_{\mathcal{H}} \Phi(v) \mu(dv). \quad (9)$$

**Lemma 2.6.** *Assume the conditions for Lemma 2.4 hold. Then, for any  $\Phi, \Psi \in \mathcal{S}(\mathbb{H})$ , the following Green's formula holds:*

$$\int_{\mathcal{H}} (\mathcal{A}_0 \Phi) \Psi d\mu = \int_{\mathcal{H}} \Phi (\mathcal{A}_0 \Psi) d\mu = -\frac{1}{2} \int_{\mathcal{H}} (QD\Phi, D\Psi) d\mu. \quad (10)$$

By Lemma 2.1, for  $\Phi \in \mathbb{H}$ , it can be represented as

$$\Phi(v) = \sum_{n=0}^{\infty} \phi_{\mathbf{n}} H_{\mathbf{n}}(v), \quad (11)$$

where  $n = |\mathbf{n}|$  and  $\mathbf{n} \in \mathcal{J}$ . Notice that we can think in  $\mathbf{n}$  as a vector of  $r$  dimension, i.e.  $\mathbf{n} = (n_1, \dots, n_r)$ . Let  $\alpha_{\mathbf{n}} = \alpha_{n_1} \cdots \alpha_{n_r}$  be a sequence of positive numbers with  $\alpha_{\mathbf{n}} > 0$ , such that  $\alpha_{\mathbf{n}} \rightarrow \infty$  as  $n \rightarrow \infty$ . Define

$$\begin{aligned} |||\Phi|||_{k,\alpha} &= \left[ \sum_{\mathbf{n}} (1 + \alpha_{\mathbf{n}})^k |\phi_{\mathbf{n}}|^2 \right]^{1/2}, \\ |||\Phi|||_{0,\alpha} &= |||\Phi||| = \left[ \sum_{\mathbf{n}} |\phi_{\mathbf{n}}|^2 \right]^{1/2}, \end{aligned}$$

133 which is  $L^2(\mu)$ -norm of  $\Phi$ . For the given sequence  $\alpha = \{\alpha_n\}$ , let  $\mathbb{H}_{k,\alpha}$  denote  
134 the completion of  $\mathcal{S}(\mathbb{H})$  with respect to the norm  $||| \cdot |||_{k,\alpha}$ . Then  $\mathbb{H}_{k,\alpha}$  is called  
135 a Gauss–Sobolev space of order  $k$  with parameter  $\alpha$ . The dual space of  $\mathbb{H}_{k,\alpha}$  is

136  $\mathbb{H}_{-k,\alpha}$ . From now on, we will fix the sequence  $\alpha_n = \lambda_n$ , where  $\lambda_n$  is given in  
 137 Lemma 2.4. We shall simply denote  $\mathbb{H}_{k,\alpha}$  by  $\mathbb{H}_k$  and  $|||\Phi|||_{k,\alpha}$  by  $|||\Phi|||_k$ .

138 The following results ensure the existence of an extension for the operator  $\mathcal{A}_0$   
 139 to a domain containing  $\mathbb{H}_2$ . Their proofs can be found in [12] for instance.

140 **Theorem 2.7.** *Let the conditions on  $A$  and  $Q$  in Lemma 2.2 hold. Then*  
 141  *$P_t : \mathbb{H} \rightarrow \mathbb{H}$ , for  $t \geq 0$ , is a contraction semigroup with the infinitesimal gener-*  
 142 *ator  $\tilde{A}$ . The domain of  $\tilde{A}$  contains  $\mathbb{H}_2$  and we have  $\tilde{A} = \mathcal{A}_0$  in  $\mathcal{S}(\mathbb{H})$ .*

143 **Theorem 2.8.** *Let the conditions for Theorem 2.7 hold true. The differential oper-*  
 144 *ator  $\mathcal{A}_0$  defined by (6) in  $\mathcal{S}(\mathbb{H})$  can be extended to be a self-adjoint linear operator*  
 145  *$A$  in  $\mathbb{H}$  with domain  $\mathbb{H}_2$ .*

146 Since both  $\tilde{A}$  and  $A$  are extensions of  $\mathcal{A}_0$  to a domain containing  $\mathbb{H}_2$ , they must  
 147 coincide there.

148 Given the Gauss-Sobolev space  $\mathbb{H}_k$  with norm  $||| \cdot |||_k$  we denote its dual space  
 149 by  $\mathbb{H}_{-k}$  with norm  $||| \cdot |||_{-k}$ . Thus, we have the inclusions,  $\mathbb{H}_k \subset \mathbb{H} \subset \mathbb{H}_{-k}$ . We denote  
 150 the duality between  $\mathbb{H}_k$  and  $\mathbb{H}_{-k}$  by  $\langle\langle \Psi, \Phi \rangle\rangle_k$ ,  $\Phi \in \mathbb{H}_k$ ,  $\Psi \in \mathbb{H}_{-k}$ . We also  
 151 set  $\mathbb{H}_0 = \mathbb{H}$ , with  $||| \cdot |||_0 = ||| \cdot |||$  and  $\langle\langle \cdot, \cdot \rangle\rangle_1 = \langle\langle \cdot, \cdot \rangle\rangle$ ,  $\langle\langle \cdot, \cdot \rangle\rangle_0 = [\cdot, \cdot]$ .

## 152 2.2. A non linear Kolmogorov equation

153 Consider the following Kolmogorov equation,

$$\begin{aligned} \frac{\partial}{\partial t} \Psi(v, t) &= \mathcal{A} \Psi(v, t) + \langle B(v), D \Psi(v, t) \rangle_{\mathcal{H}}, \quad \text{a.e. } v \in \mathbb{H}_2, \\ \Psi(v, 0) &= \phi(v), \end{aligned}$$

154 where, as defined in Theorem 2.7,  $\mathcal{A} : \mathbb{H}_2 \rightarrow \mathbb{H}$  is given by

$$\mathcal{A} \Phi = \frac{1}{2} \text{Tr}[R D^2 \Phi(v)] + \langle A v, D \Phi(v) \rangle. \quad (12)$$

155 Hypothesis on  $B$  will be specified latter. For now, we will consider that it is a  
 156 locally Lipschitz function. The additional term  $\langle B(v), D\Psi(v, t) \rangle_{\mathcal{H}}$  is defined  $\mu$ -a.e.  
 157  $v \in \mathbb{H}_2$ . We will allow the initial datum  $\phi$  will be in  $\mathbb{H}$ .

158 We will study a mild solution of the equation (12). Let  $\lambda > 0$  be a parameter.  
 159 By changing  $\Psi$  to  $e^{\lambda t}\Psi$  in (12) we get the following equation:

$$\begin{aligned} \frac{\partial}{\partial t} \Psi(v, t) &= \mathcal{A}_\lambda \Psi(v, t) + \langle B(v), D\Psi(v, t) \rangle_{\mathcal{H}}, \quad \text{a.e. } v \in \mathbb{H}_2, \\ \Psi(v, 0) &= \phi(v), \end{aligned} \quad (13)$$

160 where  $\mathcal{A}_\lambda = \mathcal{A} - \lambda I$ , with  $I$  the identity operator in  $\mathbb{H}$ . Clearly, the problems  
 161 (12) and (13) are equivalent, as far for the existence and uniqueness questions are  
 162 concerned. We will work on the problem (13).

163 Denote by  $P_t$  the semigroup with infinitesimal generator  $\mathcal{A}_\lambda$ . The existence  
 164 of  $P_t$  is ensured by the Theorem 2.7. Then, we can rewrite the equation (13) in an  
 165 integral form by using the semigroup  $P_t$

$$\Psi(v, t) = e^{-\lambda t}(P_t \phi)(v) + \int_0^t e^{-\lambda(t-s)} [P_{t-s}(B, D\Psi_s)](v) ds, \quad (14)$$

166 where we denote  $\phi = \phi(\cdot)$  and  $\Psi_s = \Psi(\cdot, s)$ . Chow [12] had proved the following  
 167 lemma.

**Lemma 2.9.** *Let  $\Psi \in L^2((0, T); \mathbb{H})$  for some  $T > 0$ . Then, for any  $\lambda > 0$  there exists  $C_\lambda > 0$  such that*

$$\| \int_0^t e^{-\lambda(t-s)} P_{t-s} \Psi_s ds \|^2 \leq C_\lambda \int_0^t \| \Psi_s \|_{-1}^2 ds, \quad 0 < t \leq T. \quad (15)$$

168 We now prove the following theorem on existence and uniqueness of a mild  
 169 solution to (13).

**Theorem 2.10.** Suppose that  $B : \mathcal{H} \rightarrow \mathcal{H}_0$  satisfies  $(B, D\Phi) \in L^2((0, T); \mathbb{H})$  for any  $\Phi \in \mathbb{H}$  and

$$\sup_{v \in \mathcal{H}} \|\Lambda^{-1/2} B(v)\|_{\mathcal{H}} < +\infty.$$

170 Then,  $B$  satisfies

$$|||(B(v), D\Phi(v))|||_{-1}^2 \leq C |||\Phi(v)|||^2 \quad \text{for any } \Phi \in \mathbb{H}, \quad v \in \mathbb{H}_2, \quad (16)$$

171 for some  $C > 0$ . Moreover, for  $\Phi \in \mathbb{H}$ , the initial-value problem (13) has a unique  
172 mild solution  $\Psi \in C((0, T); \mathbb{H})$ .

173 For the part of the existence and uniqueness of the solution we will adapt the  
174 proof of the Theorem 5.2 in Chapter 9 from [12].

*Proof.* First we will prove (16). We have

$$|||(B(v), D\Phi(v))|||_{-1}^2 = \sum_{\mathbf{n}} (1 + \lambda_{\mathbf{n}})^{-1} |\phi_{\mathbf{n}}|^2,$$

with

$$\phi_{\mathbf{n}} = \left( (B(v), D\Phi(v))_{\mathcal{H}}, H_{\mathbf{n}}(v) \right)_{\mathbb{H}} = \int_{\mathcal{H}} (B(v), D\Phi(v))_{\mathcal{H}} H_{\mathbf{n}}(v) \mu(dv). \quad (17)$$

By the Theorem 2.5, in particular (9), we have

$$\int_{\mathcal{H}} (\Lambda h, D\Phi(v))_{\mathcal{H}} \mu(dv) = \int_{\mathcal{H}} (v, h)_{\mathcal{H}} \Phi(v) \mu(dv),$$

for all  $\Phi \in \mathcal{S}(\mathbb{H})$ ,  $g, h \in \mathcal{H}$  and  $\mu \sim N(0, \Lambda)$ . Then, in particular, in each direction  $H_{\mathbf{n}}$  this formula is still true, so we have

$$\int_{\mathcal{H}} (\Lambda h, D\Phi(v))_{\mathcal{H}} H_{\mathbf{n}}(v) \mu(dv) = \int_{\mathcal{H}} (v, h)_{\mathcal{H}} \Phi(v) H_{\mathbf{n}}(v) \mu(dv) .$$

Then, applying this last equality to (17) we get

$$\begin{aligned}
\phi_n &= \int_{\mathcal{H}} (\Lambda[\Lambda^{-1}B(v)], D\Phi(v))_{\mathcal{H}} H_{\mathbf{n}}(v) \mu(dv) \\
&= \int_{\mathcal{H}} (\Lambda^{-1}B(v), v)_{\mathcal{H}} \Phi(v) H_{\mathbf{n}}(v) \mu(dv) \\
&= \int_{\mathcal{H}} (\Lambda^{-1/2}B(v), \Lambda^{1/2}v)_{\mathcal{H}} \Phi(v) H_{\mathbf{n}}(v) \mu(dv) .
\end{aligned}$$

175 Thus,

$$\begin{aligned}
|\phi_n|^2 &= \left| \int_{\mathcal{H}} (\Lambda^{-1/2}B(v), \Lambda^{1/2}v)_{\mathcal{H}} \Phi(v) H_{\mathbf{n}}(v) \mu(dv) \right|^2 \\
&\leq \int_{\mathcal{H}} |(\Lambda^{-1/2}B(v), \Lambda^{1/2}v)_{\mathcal{H}}|^2 |H_{\mathbf{n}}(v)|^2 \mu(dv) \int_{\mathcal{H}} |\Phi(v)|^2 \mu(dv) .
\end{aligned} \tag{18}$$

We now focus on the first integral. Let  $I_1$  be the first integral of (18). Then,

$$\begin{aligned}
I_1 &\leq \int_{\mathcal{H}} \|\Lambda^{-1/2}B(v)\|_{\mathcal{H}}^2 \|\Lambda^{1/2}v\|_{\mathcal{H}}^2 |H_{\mathbf{n}}(v)|^2 \mu(dv) \\
&\leq \sup_{v \in \mathcal{H}} \|\Lambda^{-1/2}B(v)\|_{\mathcal{H}}^2 \int_{\mathcal{H}} \|\Lambda^{1/2}v\|_{\mathcal{H}}^2 |H_{\mathbf{n}}(v)|^2 \mu(dv) \\
&\leq C \int_{\mathcal{H}} \|v\|_{\mathcal{H}}^2 |H_{\mathbf{n}}(v)|^2 \mu(dv) \\
&\leq C .
\end{aligned}$$

The last inequality follows by using proposition 9.2.10 in page 198 from [14].

Then, by using this bound on (18) we have.

$$\begin{aligned}
|\phi_n|^2 &\leq C \int_{\mathcal{H}} |\Phi(v)|^2 \mu(dv) \\
&\leq C \|\Phi(v)\|^2 .
\end{aligned}$$

Thus,

$$\| (B(v), D\Phi(v)) \|_{-1}^2 \leq C \|\Phi(v)\|^2 \sum_{\mathbf{n}} (1 + \lambda_{\mathbf{n}})^{-1} \leq C \|\Phi(v)\|^2 ,$$

176 which proves (16).

We now prove the existence and uniqueness of a solution to the initial- value problem (13). Let  $\mathbb{X}_T$  denote the Banach space  $C([0, T]; \mathbb{H})$  with the sup-norm

$$\|\Psi\|_T := \sup_{0 \leq t \leq T} \|\Psi\|.$$

In  $\mathbb{X}_T$  define the linear operator  $\mathbb{Q}$  as

$$\mathbb{Q}\Psi = e^{-\lambda t} P_t \Phi + \int_0^t e^{-\lambda(t-s)} P_{t-s}(B, D\Psi_s) ds, \quad \text{for any } \Psi \in \mathbb{X}_T.$$

By Theorem 2.7  $P_t$  is a contraction semigroup, then using this fact and Lemma 2.9 we have

$$\begin{aligned} \|\mathbb{Q}\Psi\|^2 &\leq 2 \left[ \|e^{-\lambda t} P_t \Phi\|^2 + \left\| \int_0^t e^{-\lambda(t-s)} P_{t-s}(B, D\Psi_s) ds \right\|^2 \right] \\ &\leq 2 \left[ \|\Phi\|^2 + C_\lambda \int_0^t \|(B, D\Psi_s)\|_{-1}^2 ds \right] \\ &\leq 2\|\Phi\|^2 + C_1 \int_0^t \|\Psi_s\|^2 ds, \end{aligned}$$

for some  $C_1 > 0$ . Hence,  $\|\mathbb{Q}\Psi\|_T \leq C(1 + \|\Psi\|_T)$ , with  $C = C(\Phi, \lambda, T)$ . Then, the map  $\mathbb{Q} : \mathbb{X}_T \rightarrow \mathbb{X}_T$  is well defined. We now show that is a contraction for a small  $t$ . Let  $\Psi, \Psi' \in \mathbb{X}_T$ . Then

$$\begin{aligned} \|\mathbb{Q}\Psi - \mathbb{Q}\Psi'\|^2 &= \left\| \int_0^t e^{-\lambda(t-s)} P_{t-s}[(B, D\Psi_s) - (B, D\Psi'_s)] ds \right\|^2 \\ &\leq C_\lambda \int_0^t \|(B, D\Psi_s - D\Psi'_s)\|_{-1}^2 ds \\ &\leq C_2 \int_0^t \|\Psi_s - \Psi'_s\|^2 ds, \end{aligned}$$

177 for some  $C_2 > 0$ .

178 It follows that  $\|\mathbb{Q}\Psi - \mathbb{Q}\Psi'\|_T \leq \sqrt{C_2 T} \|\Psi - \Psi'\|_T$ . Then, for small  $T$ ,  $\mathbb{Q}$  is  
179 a contraction on  $\mathbb{X}_T$ . Hence the Cauchy problem (13) has a unique mild solution.

180

□



181 We now prove a theorem on the dependence on initial conditions for the mild  
182 solution of (13).

183 **Theorem 2.11.** *Suppose that  $B : \mathcal{H} \rightarrow \mathcal{H}_0$  satisfies  $(B, D\Phi) \in L^2((0, T); \mathbb{H})$  for  
184 any  $\Phi \in \mathbb{H}$  and*

$$\sup_{v \in \mathcal{H}} \|\Lambda^{-1/2} B(v)\|_{\mathcal{H}} < +\infty. \quad (19)$$

185 *Then, the unique mild solution  $\Psi \in C((0, T); \mathbb{H})$  for (13) depends continuously on  
186 the initial conditions.*

*Proof.* We know, with the assumption (19), that the existence of a unique mild solution for (13) is guaranteed by Theorem 2.10. We will denote by  $\Psi_t^\varphi$  its mild solution at time  $t$  with initial condition  $\varphi$ :

$$\Psi_t^\varphi = e^{-\lambda t} P_t \varphi + \int_0^t e^{-\lambda(t-s)} P_{t-s} (B, D\Psi_s^\varphi) ds.$$

Then,

$$\begin{aligned} \Psi_t^\varphi - \Phi_t^\psi &= e^{-\lambda t} P_t \varphi - e^{-\lambda t} P_t \psi + \int_0^t e^{-\lambda(t-s)} P_{t-s} (B, D\Psi_s^\varphi - D\Phi_s^\psi) ds \\ &= e^{-\lambda t} P_t (\varphi - \psi) + \int_0^t e^{-\lambda(t-s)} P_{t-s} (B, D\Psi_s^\varphi - D\Phi_s^\psi) ds. \end{aligned}$$

From this expression we get

$$\begin{aligned} \|\Psi_t^\varphi - \Phi_t^\psi\|^2 &\leq \|e^{-\lambda t} P_t (\varphi - \psi)\|^2 + \left\| \int_0^t e^{-\lambda(t-s)} P_{t-s} (B, D\Psi_s^\varphi - D\Phi_s^\psi) ds \right\|^2 \\ &\leq \|\varphi - \psi\|^2 + C_\lambda \int_0^t \|(B, D\Psi_s^\varphi - D\Phi_s^\psi)\|_{-1}^2 ds \\ &\leq \|\varphi - \psi\|^2 + C_2 \int_0^t \|\Psi_s^\varphi - \Phi_s^\psi\|^2 ds. \end{aligned}$$

Thus, by Gronwall's inequality we obtain

$$\|\Psi_t^\varphi - \Phi_t^\psi\|^2 \leq \exp(C_2 t) \|\varphi - \psi\|^2, \quad (20)$$

187 which implies,  $\|\Psi_t^\varphi - \Phi_t^\psi\| \leq \exp(Ct) \|\varphi - \psi\|$ . This completes the proof.  $\square$

### 188 3. Numerical stability respect to initial conditions

189 In this section, we prove the continuity with respect to the initial conditions for  
 190 a numerical approximation of the Kolmogorov equation associated with an SPDE.  
 191 Here we understand that a numerical scheme is stable respect to initial conditions  
 192 if this method reproduces the same behavior when the continuous problem satis-  
 193 fies continuity respect initial conditions.

194 Consider the stochastic differential equation in  $\mathcal{H}$

$$dX_t = AX_t dt + B(X_t)dt + \sqrt{Q}dW_t, \quad (21)$$

195 where the operator  $A : \mathcal{D}(A) \subset \mathcal{H} \rightarrow \mathcal{H}$  is the infinitesimal generator of a strongly  
 196 continuous semigroup  $e^{tA}$  in  $\mathcal{H}$ ,  $Q$  is a bounded operator from another Hilbert  
 197 space  $\mathcal{U}$  to  $\mathcal{H}$  and  $B : \mathcal{D}(B) \subset \mathcal{H} \rightarrow \mathcal{H}$  is a nonlinear mapping.

198 The equation (21) can be associated to a Kolmogorov equation in the next way,  
 199 we define

$$u(t, x) = \mathbb{E}[\varphi(X_t^x)], \quad (22)$$

200 where  $\varphi : \mathcal{H} \rightarrow \mathbb{R}$  and  $X_t^x$  is the solution to (21) with initial conditions  $X_0 = x$   
 201 where  $x \in \mathcal{H}$ . Then  $u$  satisfies the Kolmogorov equation (1).

We use Lemma 2.1 to write the solution  $\Psi_t^\varphi$  as in a Fourier-Hermite decompo-  
 sition:

$$\Psi_t^\varphi = \sum_{n \in \mathcal{J}} u_n(t) H_n(x), \quad x \in \mathcal{H}, \quad t \in [0, T]. \quad (23)$$

202 Note that the time-dependent coefficients  $u_n(t)$  depend on the functional and  
 203 on the initial condition but it is not a function of the initial condition. First we  
 204 prove an auxiliary result.

205 **Lemma 3.1.** Set  $\{P_k(\xi)\}_{k \in \mathbb{N}}$  the family of normalized Hermite polynomials in  $\mathbb{R}$ .

206 For every  $k \in \mathbb{N}$  and  $\xi, \eta \in \mathbb{R}$  such that  $\eta < \xi$  we have that

$$P_k(\xi) - P_k(\eta) = C(k)Pe_{k+1}(\gamma) \cdot (\xi - \eta), \quad (24)$$

207 where  $\gamma \in (\eta, \xi)$  and  $C(k) = \frac{(-1)^k}{(k+1)(k!)^{1/2}}$ . Moreover,  $Pe_k(x)$  is the unnormalized

208 Hermite polynomial of  $k$  degree.

*Proof.* We know that  $P_k(\xi) = \frac{(-1)^k}{(k!)^{1/2}} e^{\xi^2/2} \frac{d}{d\xi^k} e^{-\xi^2/2}$ . Set  $c(k) = (-1)^k (k!)^{-1/2}$ , then

$$\begin{aligned} P_k(\xi) - P_k(\eta) &= c(k) \left[ e^{\xi^2/2} \frac{d}{d\xi^k} e^{-\xi^2/2} - e^{\eta^2/2} \frac{d}{d\eta^k} e^{-\eta^2/2} \right] \\ &= c(k) \left[ e^{x^2/2} \frac{d}{dx^k} e^{-x^2/2} \Big|_{x=\eta}^\xi \right] \\ &= c(k) \int_\eta^\xi F_k(x) dx, \end{aligned}$$

where  $F_k$  is a continuous function such that  $F'_k(x) = e^{x^2/2} \frac{d}{dx^k} e^{-x^2/2}$ . In fact, denoting by  $Pe_k(x)$  the unnormalized Hermite polynomial of  $k$  degree, results

$$F'_k(x) = e^{x^2/2} \frac{d}{dx^k} e^{-x^2/2} = Pe_k(x),$$

and since the Hermite polynomials constitute an Appell sequence we have that

$$F'_k(x) = Pe_k(x) = \frac{1}{k+1} Pe'_{k+1}(x),$$

which implies that  $F_k(x) = \frac{1}{k+1} Pe_{k+1}(x)$ . Now, since  $F_k(x)$  is a continuous function, then there exists  $\gamma \in (\eta, \xi)$  such that

$$\int_\eta^\xi F_k(x) dx = F_k(\gamma) \cdot (\xi - \eta).$$

209 All these implies that  $P_k(\xi) - P_k(\eta) = c(k)F_k(\gamma) \cdot (\xi - \eta)$ . From this expression the

210 lemma follows immediately.  $\square$

211 We will use some technical results on the SPDE to prove the following re-  
 212 sult—the main result of this section.

213 **Theorem 3.2.** *Assume that the eigenvalues of  $\Lambda$ , satisfies that for every  $k \in \mathbb{N}$ ,  
 214  $\lambda_k < \lambda_{k+1} \rightarrow \infty$ . Assume that the functional  $\varphi$  is Lipschitz. Then, the numeric  
 215 approximation  $\Psi_t^\varphi$  (given by (23)) to the solution of the Kolmogorov equation  
 216  $\Psi \in C((0, T); \mathbb{H})$  depends continuously on the initial conditions.*

*Proof.* Let  $x, y \in H$  be two different initial values. We want to estimate  $\Psi_t^x - \Psi_t^y$ .  
 By definition,

$$\Psi_t^x = \sum_{\bar{n} \in \mathcal{J}} u_{\bar{n}}^x(t) H_{\bar{n}}(x) . \quad (25)$$

217 Thus,

$$\begin{aligned} \Psi_t^x - \Psi_t^y &= \sum_{\bar{n} \in \mathcal{J}} u_{\bar{n}}^x(t) H_{\bar{n}}(x) - \sum_{\bar{n} \in \mathcal{J}} u_{\bar{n}}^y(t) H_{\bar{n}}(y) \\ &= \sum_{\bar{n} \in \mathcal{J}} [u_{\bar{n}}^x(t) - u_{\bar{n}}^y(t)] H_{\bar{n}}(x) + \sum_{\bar{n} \in \mathcal{J}} u_{\bar{n}}^y(t) [H_{\bar{n}}(x) - H_{\bar{n}}(y)] . \end{aligned} \quad (26)$$

We focus on the first term in (26). From the definition of the initial condition  
 we obtain the following expression for the time-dependent coefficient

$$u_{\bar{n}}^x(t) = \int_{\mathcal{H}} H_{\bar{n}}(x) \mathbb{E}[\varphi(X_t^x)] \mu(dx) .$$

From this we get

$$\begin{aligned}
u_{\bar{n}}^x(t) - u_{\bar{n}}^y(t) &= \int_{\mathcal{H}} H_{\bar{n}}(x) \mathbb{E}[\varphi(X_t^x)] \mu(dx) - \int_{\mathcal{H}} H_{\bar{n}}(y) \mathbb{E}[\varphi(X_t^y)] \mu(dy) \\
&= \int_{\mathcal{H} \times \mathcal{H}} H_{\bar{n}}(x) \mathbb{E}[\varphi(X_t^x)] \mu(dx) \mu(dy) \\
&\quad - \int_{\mathcal{H} \times \mathcal{H}} H_{\bar{n}}(y) \mathbb{E}[\varphi(X_t^y)] \mu(dx) \mu(dy) \\
&= \int_{\mathcal{H} \times \mathcal{H}} H_{\bar{n}}(x) (\mathbb{E}[\varphi(X_t^x)] - \mathbb{E}[\varphi(X_t^y)]) \mu(dx) \mu(dy) \\
&\quad + \int_{\mathcal{H} \times \mathcal{H}} (H_{\bar{n}}(x) - H_{\bar{n}}(y)) \mathbb{E}[\varphi(X_t^y)] \mu(dx) \mu(dy) .
\end{aligned}$$

218 Then, by the Cauchy-Schwartz inequality, we obtain

$$\begin{aligned}
|u_{\bar{n}}^x(t) - u_{\bar{n}}^y(t)|^2 &\leq \left| \int_{\mathcal{H} \times \mathcal{H}} H_{\bar{n}}(x) (\mathbb{E}[\varphi(X_t^x)] - \mathbb{E}[\varphi(X_t^y)]) \mu(dx) \mu(dy) \right|^2 \\
&\quad + \left| \int_{\mathcal{H} \times \mathcal{H}} (H_{\bar{n}}(x) - H_{\bar{n}}(y)) \mathbb{E}[\varphi(X_t^y)] \mu(dx) \mu(dy) \right|^2 \\
&\leq \int_{\mathcal{H} \times \mathcal{H}} H_{\bar{n}}^2(x) \mu(dx) \mu(dy) \\
&\quad \times \int_{\mathcal{H} \times \mathcal{H}} |\mathbb{E}[\varphi(X_t^x)] - \mathbb{E}[\varphi(X_t^y)]|^2 \mu(dx) \mu(dy) \\
&\quad + \int_{\mathcal{H} \times \mathcal{H}} \mathbb{E}^2[\varphi(X_t^y)] \mu(dx) \mu(dy) \\
&\quad \times \int_{\mathcal{H} \times \mathcal{H}} |H_{\bar{n}}(x) - H_{\bar{n}}(y)|^2 \mu(dx) \mu(dy) \\
&= \int_{\mathcal{H} \times \mathcal{H}} |\mathbb{E}[\varphi(X_t^x)] - \mathbb{E}[\varphi(X_t^y)]|^2 \mu(dx) \mu(dy) \\
&\quad + \int_{\mathcal{H} \times \mathcal{H}} \mathbb{E}^2[\varphi(X_t^y)] \mu(dx) \mu(dy) \\
&\quad \times \int_{\mathcal{H} \times \mathcal{H}} |H_{\bar{n}}(x) - H_{\bar{n}}(y)|^2 \mu(dx) \mu(dy) .
\end{aligned} \tag{27}$$

219 We now estimate the norm of the expression (26) with the help of (27).

$$\begin{aligned}
& \|\Psi_t^x - \Psi_t^y\|_{(L^2(\mathcal{H}, \mu))^2}^2 = \int_{\mathcal{H} \times \mathcal{H}} |\Psi_t^x - \Psi_t^y|^2 \mu(dx) \mu(dy) \\
& \leq \int_{\mathcal{H} \times \mathcal{H}} \left| \sum_{\bar{n} \in \mathcal{J}} [u_{\bar{n}}^x(t) - u_{\bar{n}}^y(t)] H_{\bar{n}}(x) \right|^2 \mu(dx) \mu(dy) \\
& \quad + \int_{\mathcal{H} \times \mathcal{H}} \left| \sum_{\bar{n} \in \mathcal{J}} u_{\bar{n}}^y(t) [H_{\bar{n}}(x) - H_{\bar{n}}(y)] \right|^2 \mu(dx) \mu(dy) \\
& \leq \int_{\mathcal{H} \times \mathcal{H}} \sum_{\bar{n} \in \mathcal{J}} |u_{\bar{n}}^x(t) - u_{\bar{n}}^y(t)|^2 H_{\bar{n}}^2(x) \mu(dx) \mu(dy) \\
& \quad + \int_{\mathcal{H} \times \mathcal{H}} \sum_{\bar{n} \in \mathcal{J}} [u_{\bar{n}}^y(t)]^2 \sum_{\bar{n} \in \mathcal{J}} |H_{\bar{n}}(x) - H_{\bar{n}}(y)|^2 \mu(dx) \mu(dy) \\
& = \sum_{\bar{n} \in \mathcal{J}} |u_{\bar{n}}^x(t) - u_{\bar{n}}^y(t)|^2 \int_{\mathcal{H} \times \mathcal{H}} H_{\bar{n}}^2(x) \mu(dx) \mu(dy) \\
& \quad + \sum_{\bar{n} \in \mathcal{J}} [u_{\bar{n}}^y(t)]^2 \sum_{\bar{n} \in \mathcal{J}} \int_{\mathcal{H} \times \mathcal{H}} |H_{\bar{n}}(x) - H_{\bar{n}}(y)|^2 \mu(dx) \mu(dy) \quad (28) \\
& = \sum_{\bar{n} \in \mathcal{J}} [u_{\bar{n}}^y(t)]^2 \sum_{\bar{n} \in \mathcal{J}} \int_{\mathcal{H} \times \mathcal{H}} |H_{\bar{n}}(x) - H_{\bar{n}}(y)|^2 \mu(dx) \mu(dy) \\
& \quad + \sum_{\bar{n} \in \mathcal{J}} |u_{\bar{n}}^x(t) - u_{\bar{n}}^y(t)|^2 \\
& = \int_{\mathcal{H} \times \mathcal{H}} \left| \mathbb{E}[\varphi(X_t^x)] - \mathbb{E}[\varphi(X_t^y)] \right|^2 \mu(dx) \mu(dy) \\
& \quad + \int_{\mathcal{H} \times \mathcal{H}} \mathbb{E}^2[\varphi(X_t^y)] \mu(dx) \mu(dy) \\
& \quad \times \sum_{\bar{n} \in \mathcal{J}} \int_{\mathcal{H} \times \mathcal{H}} |H_{\bar{n}}(x) - H_{\bar{n}}(y)|^2 \mu(dx) \mu(dy) \\
& \quad + \sum_{\bar{n} \in \mathcal{J}} [u_{\bar{n}}^y(t)]^2 \sum_{\bar{n} \in \mathcal{J}} \int_{\mathcal{H} \times \mathcal{H}} |H_{\bar{n}}(x) - H_{\bar{n}}(y)|^2 \mu(dx) \mu(dy) .
\end{aligned}$$

220 Note that  $\mathbb{E}^2[\varphi(X_t^y)] = u^2(t, x) \in L^2(\mathcal{H}, \mu)$ , therefore the first integral in the  
221 second term is a continuous bounded function of  $t$ . Moreover,  $\sum_{\bar{n} \in \mathcal{J}} [u_{\bar{n}}^y(t)]^2$  is the  
222  $L^2(\mathcal{H}, \mu)$ -norm of the function  $u(t, x)$ , then the series converges and it is also a

223 continuous bounded function of  $t$ . Thus, from (28) we get

$$\begin{aligned} \|\Psi_t^x - \Psi_t^y\|_{(L^2(\mathcal{H}, \mu))^2}^2 &\leq \int_{\mathcal{H} \times \mathcal{H}} \left| \mathbb{E}[\varphi(X_t^x)] - \mathbb{E}[\varphi(X_t^y)] \right|^2 \mu(dx) \mu(dy) \\ &\quad + f(t) \sum_{\bar{n} \in \mathcal{J}} \int_{\mathcal{H} \times \mathcal{H}} |H_{\bar{n}}(x) - H_{\bar{n}}(y)|^2 \mu(dx) \mu(dy), \end{aligned} \quad (29)$$

224 where  $f(t) = \sum_{\bar{n} \in \mathcal{J}} [u_{\bar{n}}^y(t)]^2 + \int_{\mathcal{H}} \mathbb{E}^2[\varphi(X_t^y)] \mu(dy)$ .

225 From the proof of Theorem 2.11 (see (20)) we know that

$$\begin{aligned} |||\Psi_t^\varphi - \Phi_t^\psi|||^2 &= \int_{\mathcal{H} \times \mathcal{H}} \left| \mathbb{E}[\varphi(X_t^x)] - \mathbb{E}[\varphi(X_t^y)] \right|^2 \mu(dx) \mu(dy) \\ &\leq \exp(Ct) \int_{\mathcal{H} \times \mathcal{H}} \|x - y\|_{\mathcal{H}}^2 \mu(dx) \mu(dy) \\ &= \exp(Ct) |||x - y|||^2. \end{aligned} \quad (30)$$

226 Therefore the first term in the right side of (29) is bounded by (30).

We now focus on the second term in the last inequality. Notice that for every  $\bar{n} \in \mathcal{J}$  we have

$$H_{\bar{n}}(x) - H_{\bar{n}}(y) = \prod_{i=1}^{\infty} [P_{n_i}(\xi_i) - P_{n_i}(\eta_i)], \quad (31)$$

227 where  $\xi_i = \langle x, \Lambda^{-1/2} e_i \rangle_{\mathcal{H}}$  and  $\eta_i = \langle y, \Lambda^{-1/2} e_i \rangle_{\mathcal{H}}$  (see (3) and lines after that for the  
228 definition). Hence, applying Lemma 3.1 to equation (31) we have that

$$\begin{aligned} H_{\bar{n}}(x) - H_{\bar{n}}(y) &= \prod_{i=1}^{\infty} C(i) P e_{i+1}(\gamma_i) \cdot (\xi_i - \eta_i) \\ &= \prod_{i=1}^{\infty} C(i) P e_{i+1}(\gamma_i) \langle x - y, \Lambda^{-1/2} e_i \rangle_{\mathcal{H}}, \end{aligned} \quad (32)$$

229 here  $\gamma_i \in (\xi_i \wedge \eta_i, \xi_i \vee \eta_i)$  for every  $i \in \mathbb{N}$ . Then

$$\begin{aligned}
& \sum_{\bar{n} \in \mathcal{J}} \int_{\mathcal{H} \times \mathcal{H}} |H_{\bar{n}}(x) - H_{\bar{n}}(y)|^2 \mu(dx) \mu(dy) \\
&= \sum_{\bar{n} \in \mathcal{J}} \int_{\mathcal{H} \times \mathcal{H}} \left| \prod_{i=1}^{\infty} C(i) P e_{i+1}(\gamma_i) \langle x - y, \Lambda^{-1/2} e_i \rangle_{\mathcal{H}} \right|^2 \mu(dx) \mu(dy) \\
&= \sum_{\bar{n} \in \mathcal{J}} \int_{\mathcal{H} \times \mathcal{H}} \prod_{i=1}^{\infty} [C(i) P e_{i+1}(\gamma_i)]^2 \left| \langle x - y, \Lambda^{-1/2} e_i \rangle_{\mathcal{H}} \right|^2 \mu(dx) \mu(dy) \\
&\leq \sum_{\bar{n} \in \mathcal{J}} \int_{\mathcal{H} \times \mathcal{H}} \prod_{i=1}^{\infty} [C(i) P e_{i+1}(\gamma_i)]^2 \|x - y\|_{\mathcal{H}}^2 \|\Lambda^{-1/2} e_i\|_{\mathcal{H}}^2 \mu(dx) \mu(dy) \\
&= \sum_{\bar{n} \in \mathcal{J}} \int_{\mathcal{H} \times \mathcal{H}} \prod_{i=1}^{\infty} [C(i) P e_{i+1}(\gamma_i)]^2 \|x - y\|_{\mathcal{H}}^2 \lambda_i^{-1} \|e_i\|_{\mathcal{H}}^2 \mu(dx) \mu(dy) \\
&= \|x - y\|_{\mathcal{H}}^2 \sum_{\bar{n} \in \mathcal{J}} \prod_{i=1}^{\infty} [C(i)]^2 \lambda_i^{-1} \int_{\mathcal{H} \times \mathcal{H}} [P e_{i+1}(\gamma_i)]^2 \mu(dx) \mu(dy).
\end{aligned} \tag{33}$$

230 Recall that for every  $i \in \mathbb{N}$  we have that  $\gamma_i \in (\xi_i \wedge \eta_i, \xi_i \vee \eta_i)$ , set  $\hat{\gamma}_i \in (\xi_i \wedge \eta_i, \xi_i \vee \eta_i)$   
231 such that  $P e_i^2(\gamma_i) \leq P e_{i+1}^2(\hat{\gamma}_i)$  for every  $\gamma_i \in (\xi_i \wedge \eta_i, \xi_i \vee \eta_i)$ , notice that the existence  
232 of  $\hat{\gamma}_i$  is guaranteed since  $P e_{i+1}^2(\cdot)$  is a continuous function. Then, from (33) we get

233

$$\begin{aligned}
& \sum_{\bar{n} \in \mathcal{J}} \int_{\mathcal{H} \times \mathcal{H}} |H_{\bar{n}}(x) - H_{\bar{n}}(y)|^2 \mu(dx) \mu(dy) \\
&\leq \|x - y\|_{\mathcal{H}}^2 \sum_{\bar{n} \in \mathcal{J}} \prod_{i=1}^{\infty} [C(i)]^2 \lambda_i^{-1} [P e_{i+1}(\hat{\gamma}_i)]^2 \int_{\mathcal{H}} \int_{\mathcal{H}} \mu(dx) \mu(dy) \\
&= \|x - y\|_{\mathcal{H}}^2 \sum_{\bar{n} \in \mathcal{J}} \prod_{i=1}^{\infty} [C(i)]^2 \lambda_i^{-1} [P e_{i+1}(\hat{\gamma}_i)]^2.
\end{aligned} \tag{34}$$

Here, we recall that  $C(i) = \frac{(-1)^i}{(i+1)(i!)^{1/2}}$  then  $\frac{(-1)^i}{[(i+1)!]^{1/2}} P e_{i+1}(\hat{\gamma}_i)$  is the normalized Hermite polynomial of  $i + 1$  degree evaluated on  $\hat{\gamma}_i$  which is bounded by a constant  $C$



for every  $i \in \mathbb{N}$ . Moreover, since  $\lambda_k < \lambda_{k+1} \rightarrow \infty$  then this implies that

$$\sum_{\bar{n} \in \mathcal{J}} \prod_{i=1}^{\infty} [C(i)]^2 \lambda_i^{-1} [Pe_{i+1}(\hat{\gamma}_i)]^2 \leq C \sum_{\bar{n} \in \mathcal{J}} \prod_{i=1}^{\infty} \lambda_i^{-1} (i+1)^{-1} \leq C, \quad (35)$$

where  $C$  is a finite constant. Putting together (33) and (35) we get that

$$\sum_{\bar{n} \in \mathcal{J}} \int_{\mathcal{H} \times \mathcal{H}} |H_{\bar{n}}(x) - H_{\bar{n}}(y)|^2 \mu(dx) \mu(dy) \leq C \|x - y\|_{\mathcal{H}}. \quad (36)$$

Putting together inequalities (29), (30) and (36) we obtain

$$\|\Psi_t^x - \Psi_t^y\|_{(L^2(\mathcal{H}, \mu))^2}^2 \leq \exp(CT) \int_{\mathcal{H} \times \mathcal{H}} \|x - y\|_{\mathcal{H}}^2 \mu(dx) \mu(dy) + f(t) \|x - y\|_{\mathcal{H}}. \quad (37)$$

234 Now, if  $\|x - y\|_{\mathcal{H}} \leq \delta$ , then from (37) we get  $\|\Psi_t^x - \Psi_t^y\|_{(L^2(\mathcal{H}, \mu))^2} \leq G(t)\delta$ .  $\square$

235 **Remark.** If we consider in addition the supremum norm on  $t$ , then from (37) we

236 get

$$\begin{aligned} \sup_{0 \leq t \leq T} \|\Psi_t^x - \Psi_t^y\|_{(L^2(\mathcal{H}, \mu))^2}^2 &\leq C \|x - y\|_{\mathcal{H}}^2 \sup_{0 \leq t \leq T} f(t) \\ &\quad + \exp(CT) \int_{\mathcal{H} \times \mathcal{H}} \|x - y\|_{\mathcal{H}}^2 \mu(dx) \mu(dy). \end{aligned} \quad (38)$$

237 Notice that  $f(t)$  is differentiable and continuous, then  $\sup_{0 \leq t \leq T} f(t) \leq C$ , then from

238 (38) we obtain

$$\begin{aligned} \sup_{0 \leq t \leq T} \|\Psi_t^x - \Psi_t^y\|_{(L^2(\mathcal{H}, \mu))^2} &\leq C \|x - y\|_{\mathcal{H}} \\ &\quad + \exp(CT) \int_{\mathcal{H} \times \mathcal{H}} \|x - y\|_{\mathcal{H}}^2 \mu(dx) \mu(dy). \end{aligned} \quad (39)$$

239 From this inequality it is possible to show the continuous dependence on the

240 initial conditions for this norm.

## 241 4. Numerical experiments

242 In this section we run numerical experiments to illustrate that our scheme pre-  
 243 serves the underlying initial condition continuity. To this end, we solve a stochas-  
 244 tic version of the Fisher and Burgers PDEs with two near initial function condi-  
 245 tions  $x(\xi)$ ,  $\widehat{x}(\xi)$ . In [15] we provide a GitHub repository with a Python imple-  
 246 mentation to reproduce the following figures. We also provide in [16, 17], the 3D  
 247 on-line plotly versions of Figures 2 and 5.

### 248 *Stochastic Fisher-KPP equation in an interval*

249 Let  $\mathcal{H} = L^2(0, 1)$ . We consider the stochastic Fisher-KPP equation

$$\begin{aligned} dX(t, \xi) &= \left[ \nu \partial_\xi^2 X(t, \xi) + X(t, \xi)(1 - X(t, \xi)) \right] dt + dW(t, \xi), \\ X(t, 0) &= X(t, 1) = 0, \quad t > 0, \\ X(0, \xi) &\in \mathcal{H}, \quad \xi \in [0, 1], \end{aligned} \tag{40}$$

250 in the interval  $[0, 1]$  and with initial function conditions  $x(\xi)$  and  $\widehat{x}(\xi)$ . In order to  
 251 fix this initial function conditions close, we use for our experiments

$$x(\xi) := \text{sech}^2(5(\xi - 0.5)), \quad \widehat{x}(\xi) := \sum_{k=0}^N T_k(x(\xi)), \tag{41}$$

252 where  $T_k(\cdot)$  denotes the Chebyshev polynomial of the first kind. That is,  $\widehat{x}(\cdot)$  is the  
 253 Chebyshev truncated expansion of  $x(\cdot)$ .

254 Figure 1 displays the plots of this initial conditions. In Figure 2 we observe  
 255 how the mentioned approximations remains close—blue color scale denotes the  
 256 solution of equation 40 with initial function condition  $x(\xi)$ , while yellow color  
 257 corresponds to the approximation with initial condition  $\widehat{x}$ . Since we employ trans-  
 258 parency to obtain this 3D plot, the purple scale results from the closeness of the

259 solutions. Further, Figure 3 suggest the conclusion of Theorem 3.2, that is, the  
 260 solutions of equation (40) are continuous respect to initial conditions and satisfies  
 261 the estimation (30).

Figure 1: Numerical Solution of the Fisher-KPP eq. (40) with initial conditions  $x, \hat{x}$  at time  $t = 0$ .

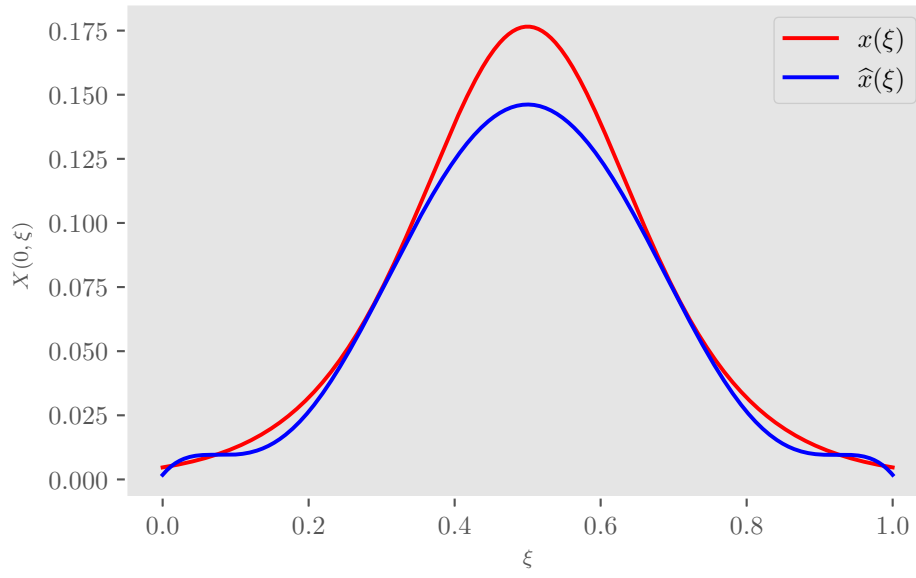
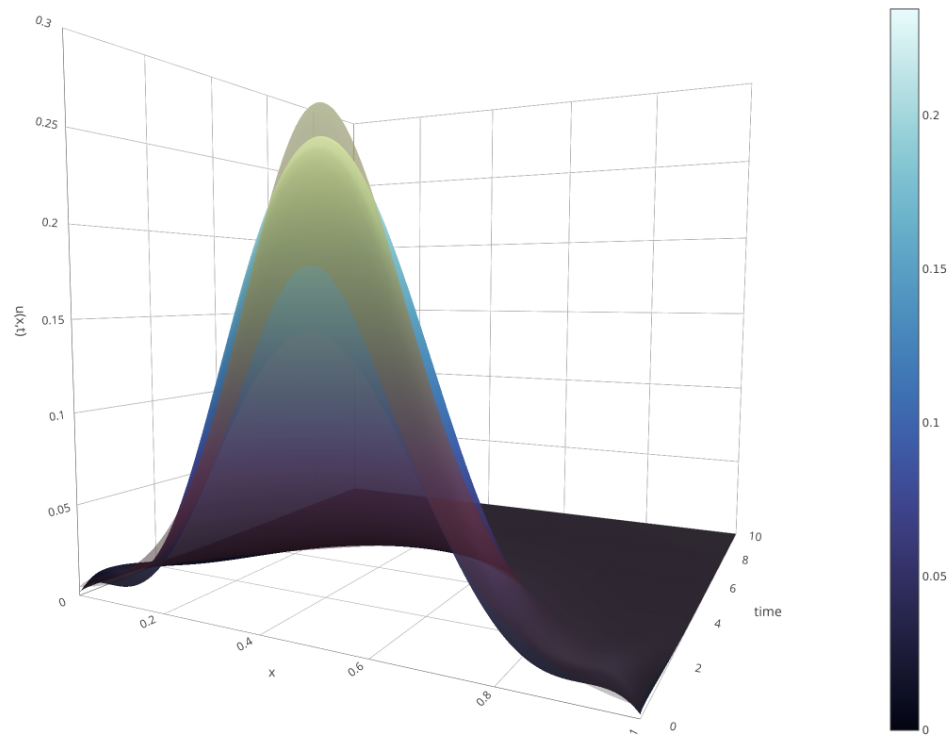


Figure 2: Likening between two solution with closed initial conditions  $x, \widehat{x}$  of the stochastic Fisher-KPP eq. (40). See [16] to obtain other camera perspectives.



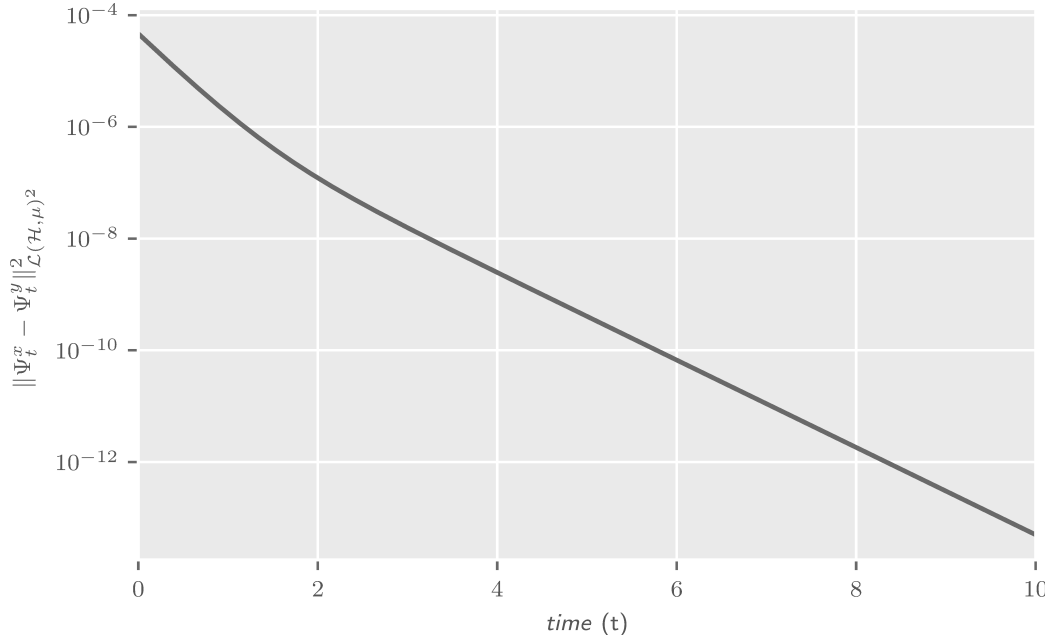


Figure 3:  $\mathcal{L}^2(\mathcal{H}, \mu)$  distance between two solutions of the stochastic Fisher PDE with initial conditions  $x = x(\xi)$ , and  $y = \widehat{x}(\xi)$ .

Figure 2 illustrates the distance between initial conditions. The yellow pallet with transparency and a blue scale highlight the zones where the two solutions are close. Thus, the zones where the color is purple denotes, where the two solutions of SPDEs are close. According to the  $\mathcal{L}(\mathcal{H}, \mu)$ -distance between the two underlying solutions, Figure 3 confirms the above argument.

267 *Stochastic Burgers equation*

268 Let  $\mathcal{H} = L^2(0, 1)$ , consider the stochastic Burgers equation in the interval  $[0, 1]$

269

$$\begin{aligned} dX(t, \xi) &= \left[ \nu \partial_\xi^2 X(t, \xi) + \frac{1}{2} \partial_\xi X^2(t, \xi) \right] dt + dW(t, \xi), \\ X(t, 0) &= X(t, 1) = 0, \quad t > 0, \\ X(0, \xi) &= x(\xi), \quad x \in \mathcal{H}. \end{aligned} \tag{42}$$

270 As in the above experiment, we use the initial conditions  $x(\xi)$  and its truncated  
271 Chebyshev expansion

$$x(\xi) := \sin(\pi\xi), \quad \widehat{x}(\xi) := \sum_{k=0}^N T_k x(\xi). \tag{43}$$

272 Figures 4 to 6 illustrate a similar argument presented in the above experiment.

Figure 4: Numerical Solution of the Burgers eq. (42) with initial conditions  $x(\xi), \widehat{x}(\xi)$ .

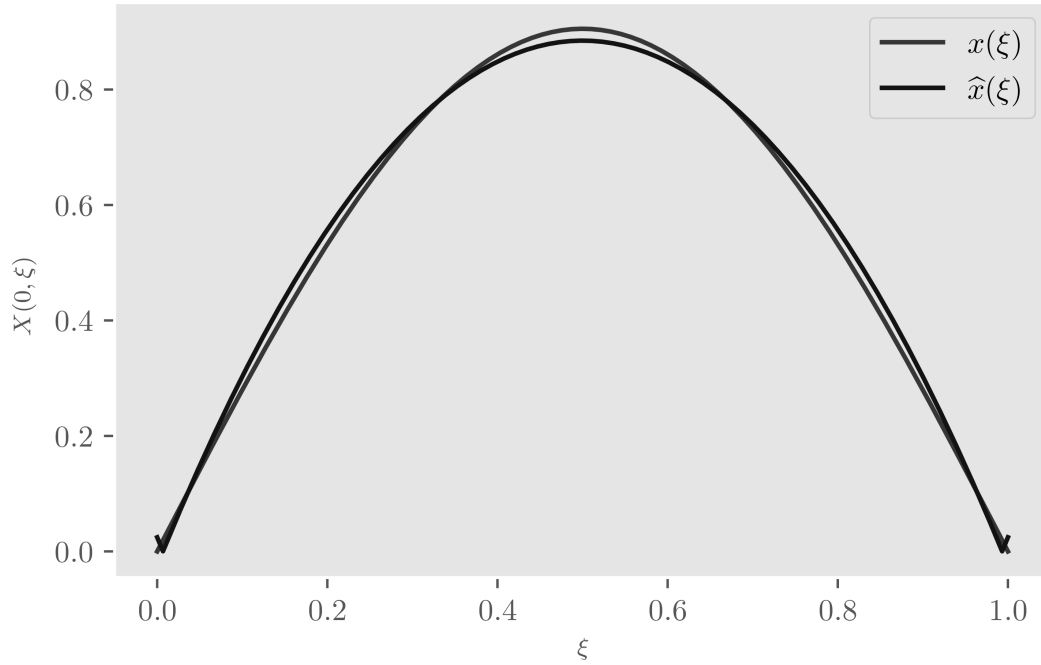


Figure 5: Likening between two solution with closed initial conditions  $x(\xi)$ , and  $\widehat{x}(\xi)$  of the stochastic Burgers eq. (42). See [16] to obtain other camera perspectives.

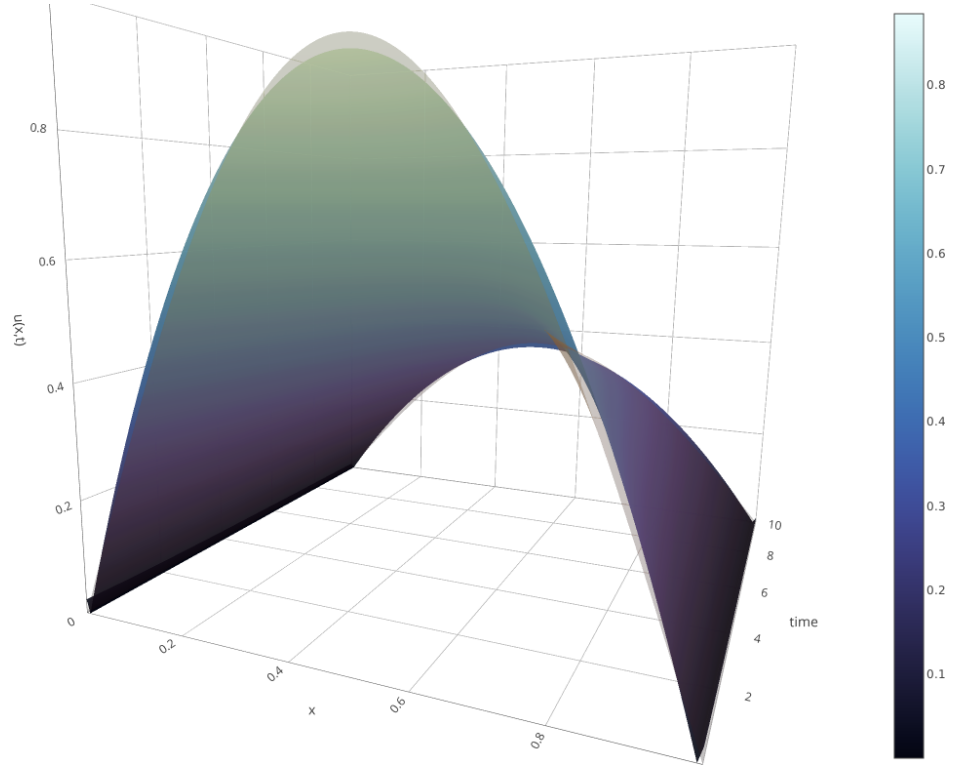
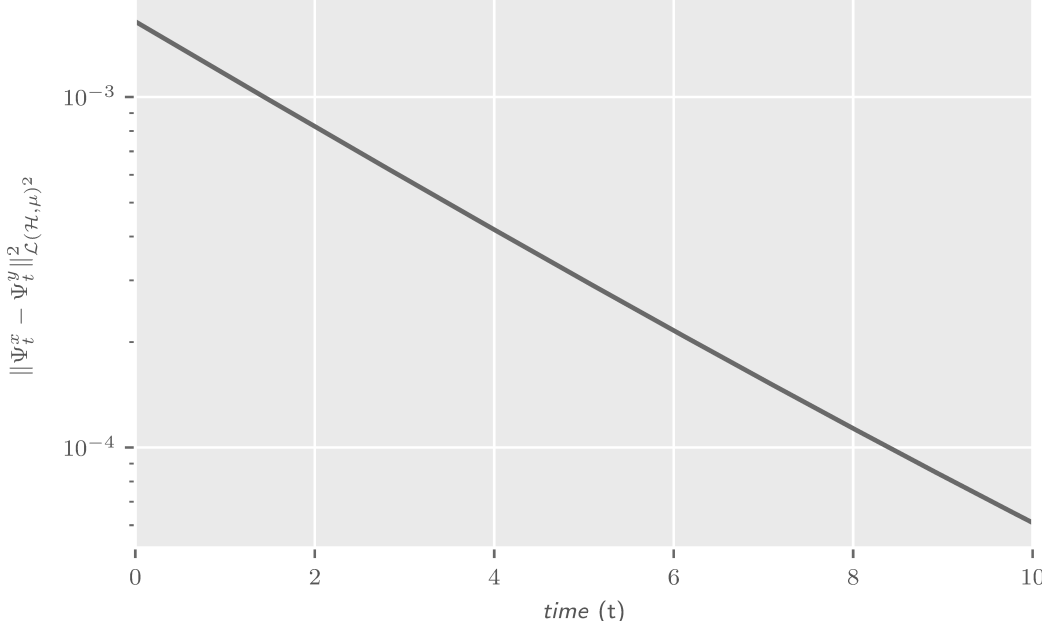


Figure 6: Distance between two solutions of the stochastic Burgers eq. (42) with initial conditions  $x = x(\xi)$ , and  $y = \widehat{x}(\xi)$ .



## 273 5. Conclusions

274 To the best of our knowledge, our results represent the first contribution on  
 275 the numeric stability respect to initial conditions of weak approximations of Kol-  
 276 mogorov equations in infinite dimensions. This kind of stability, combining with  
 277 the weak approximation approach, would save computation time. That is, since  
 278 our scheme asks specific conditions to obtain a weak numerical solution of an un-  
 279 derlying SPDE, we convert the stochastic problem into a deterministic ODE for  
 280 the first moment. This procedure overcome Montecarlo type simulations to ap-  
 281 proximate moments or distributions—simulate many realization of the numerical  
 282 stochastic process to approximate distributions or moments. Further, under our  
 283 setting, the regarding spectral approximation assure high precision and order of



284 convergence. Thus we guess that our method would improve the time and save  
285 resources of computation. We are preparing another article to confirm this con-  
286 jectures.

## 287 **6. Acknowledgments**

288 This research was supported by the Consejo Nacional de Ciencia y Tecnología  
289 (CONACYT).

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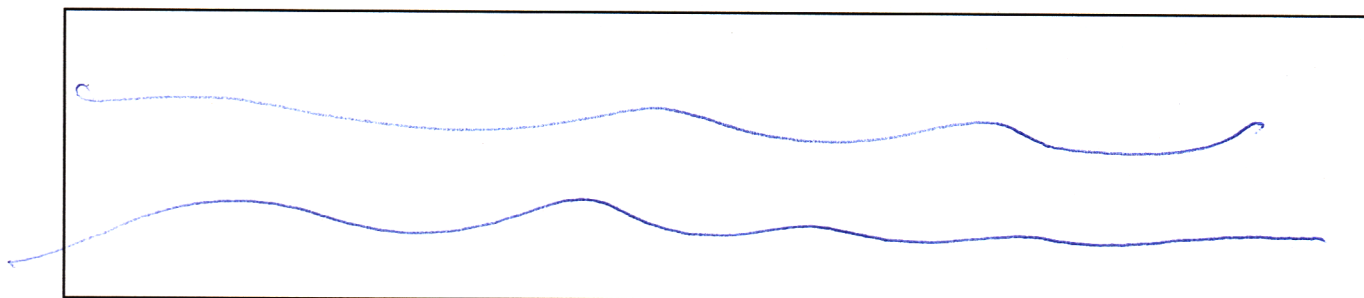
### Highlights:

- We present a spectral method to solve infinite-dimensional Kolmogorov equations associated with a particular family of SPDs
- We also establish theory to characterize the stability respect to initial conditions of the regarding the spectral method
- The numerical experiments illustrate our theory

**Declaration of interests**

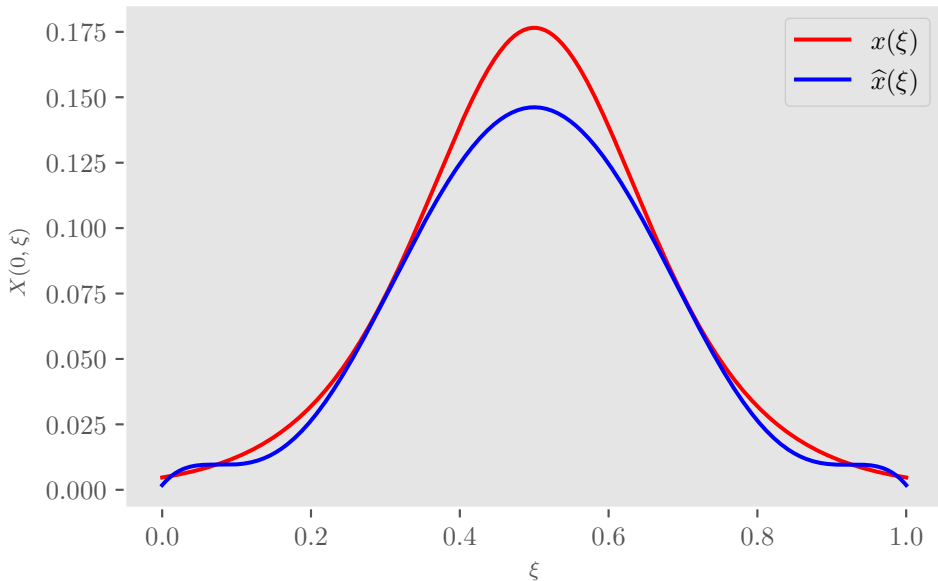
☒ The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

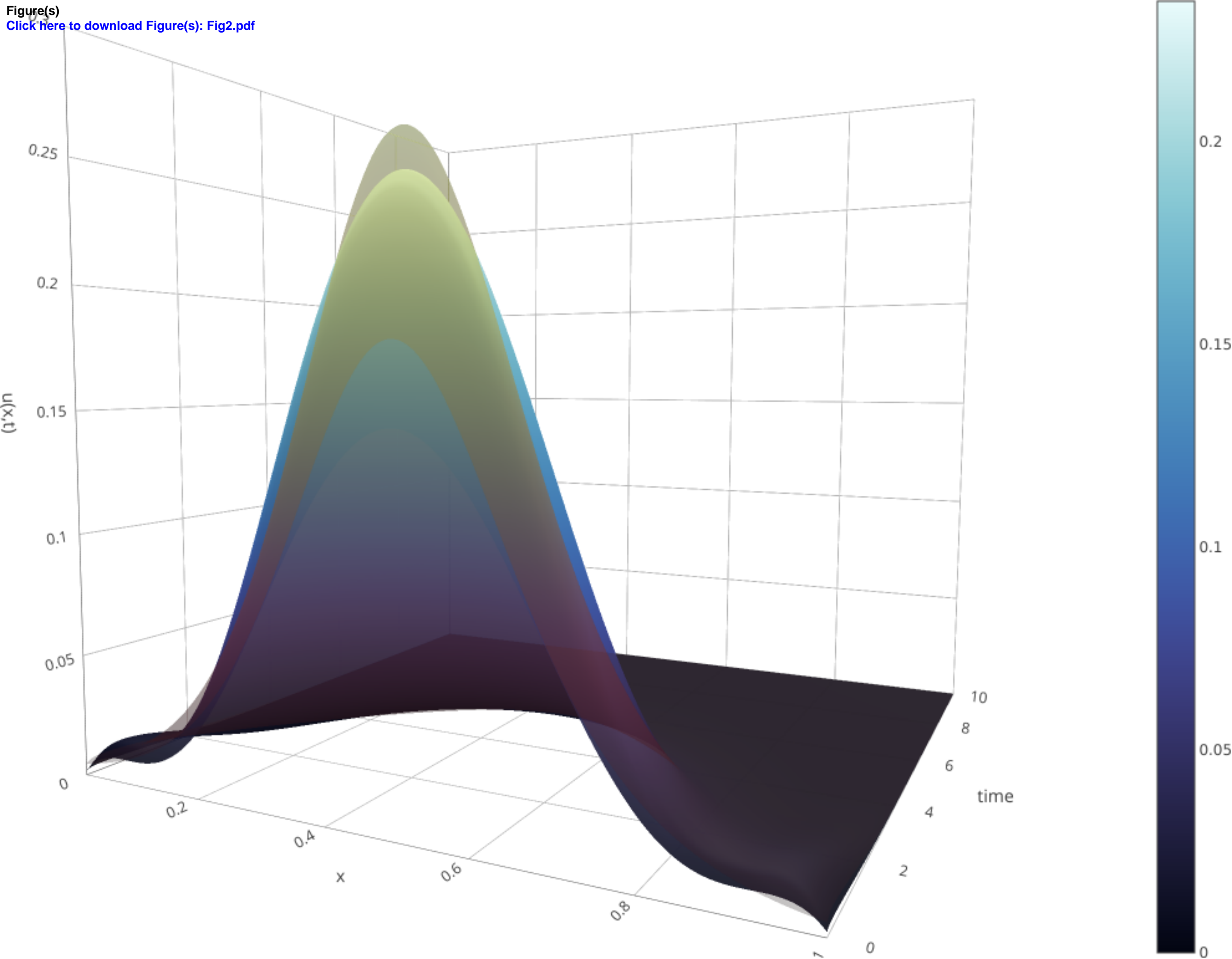
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A rectangular box with a black border. Inside the box, there are two horizontal wavy lines drawn in blue ink. The top line starts with a small loop on the left and ends with a small hook on the right. The bottom line is a continuous wavy line that spans the width of the box. The lines are approximately 10 pixels thick.

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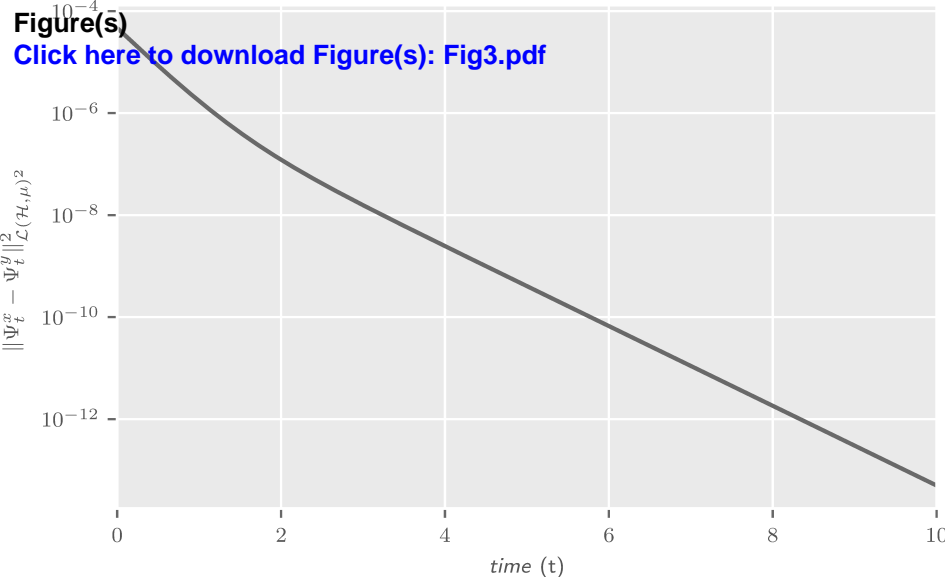




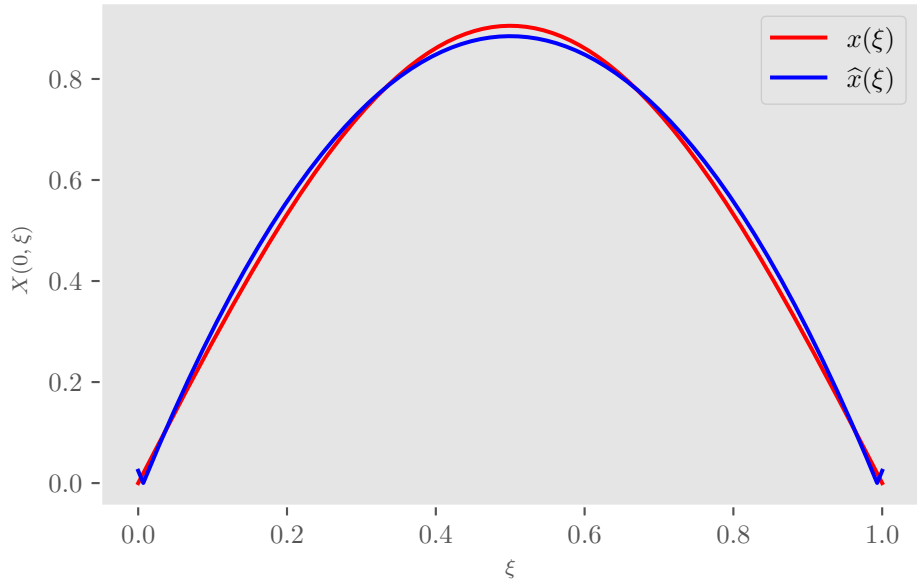


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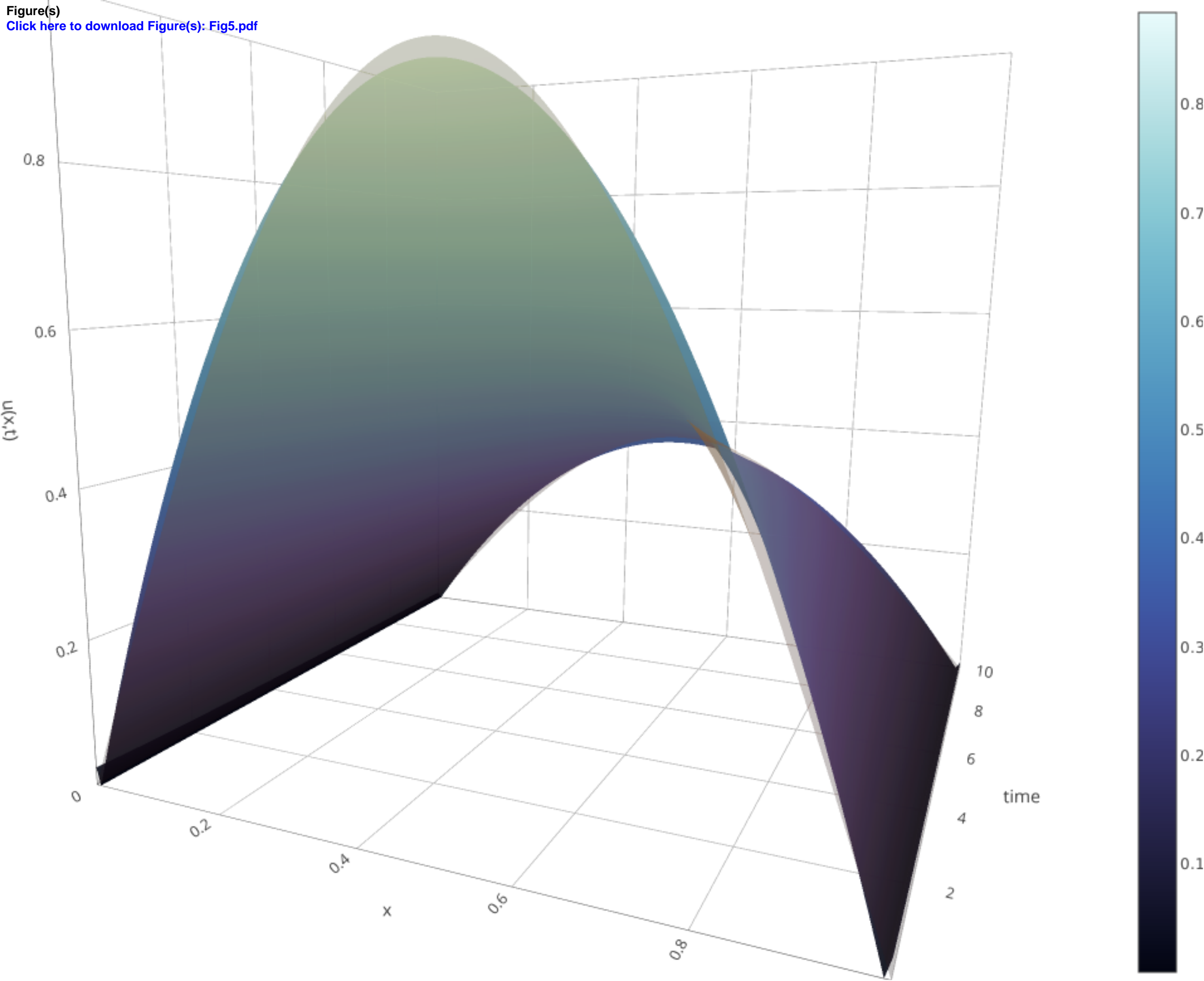


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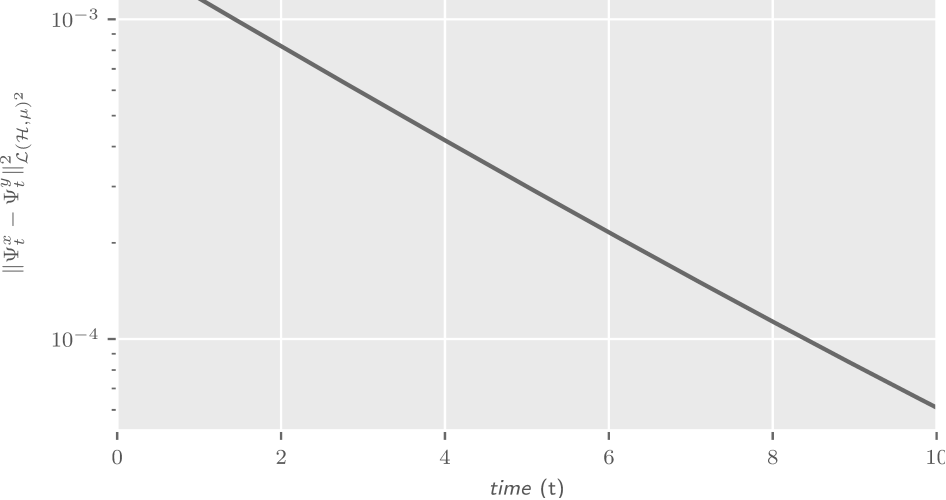
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