Abstract

We characterize the stability respect to initial conditions of a weak numerical scheme to approximate the solution of a family of SPDEs. Our approach consists in solving the associated Kolmogorov equation of the underlying SPDE whit a spectral method. We illustrate our results with stochastic versions of the Burgers and Fisher equations.

Keywords: Stability, spectral method, Kolmogorov equation, parabolic stochastic partial differential equations.

1 Introduction

Stochastic Partial Differential Equations (SPDEs) are important tools in modeling complex phenomena, they arise in many fields of knowledge like Physics, Biology, Economy, Finance, etc. Develop efficient numerical methods for simulating SPDEs is very important but also very difficult and challenging.

The Fokker-Planck-Kolmogorov (FPK) equation is a partial differential equation that describes the time evolution of the probability density function of the velocity of a particle under the influence of drag forces and random forces, it is a kind of continuity equation for densities. Citing [8] "parabolic equations on Hilbert spaces appear in mathematical physics to model systems with infinitely many degrees of freedom. Typical examples are provided by spin configurations in statistical mechanics and by crystals in solid state theory. Infinite-dimensional parabolic equations provide an analytic description of infinite dimensional diffusion processes in such branches of applied mathematics as population biology, fluid dynamics, and mathematical finance." This kind of equations have been deeply studied in the last years, see for instance [2, 4, 5] and the references therein.

Analytical solutions of SPDEs are rarely available, then using numerical methods to approximate these solutions are essential. Numerical methods for SPDEs have been developed during the last decades, most of them are strong approximations, in the probabilistic sense. The list of references is extensive, here we mention just a few on spectral approaches. For other methods, we refer to the book [17] and the references therein.

The literature body of stochastic spectral methods identifies two important families, according to the Karhunen-Loeve expansion and the Wiener-Chaos expansion. However, since the former approach converges slower than the latter for non-linear SPDEs, schemes based in the Wiener-Chaos expansion are more convenient, see [20] for further details.

The numerical analysis of SPDEs based on weak approximations, in the probability sense, is a virgin research field. There are just a few works in this direction. For example, Schwab and Süli proposed in [18] a variational space-time method to approximate the solution of an infinite-dimensional Kolmogorov-type equation. However, their article lacks numerical experiments.

Our contribution is closed related to [9], where the authors report a numerical method for Kolmogorov equations associated with SPDEs, that is, a scheme based on weak approximations.

Work with efficient and accurate numerical schemes is crucial. In this way, the spectral methods play an essential role to obtain better schemes—under certain conditions this sort of methods are more accurate than finite differences of finite elements and need fewer grid points. Here the adjective "better" would be under accuracy, consistency, stability, and other targets properties. In this work, we explore the ability of the method reported in [9] to preserve the continuity respect to initial conditions. That is, if a given problem satisfies certain regularity conditions, then two of its solution remain closed if its initial function conditions are close. So, we desire that a numerical method reproduce this behavior and if it is the case, we say that an underlying method is stable in this context.

Our main contribution is the characterization of mild conditions to assure the continuity respect to initial function conditions to a family of SPDEs and the stability of a regarding weak spectral approximation. To the best of our knowledge, this paper is the first in report numerical stability theory for Kolmogorov equations in infinite dimensions.

The stability theory for spectral methods is still under construction and is an active research area. We mention the seminal works of L.N. Trefethen and M.R. Trummer [19], D. Gottlieb et. al. [12] as reference for the deterministic case, and N. Li, J. Fiordilino, and X. Feng, [15] A. Lang, A. Petersson, and A. Thalhammer, [14] for the stochastic version.

This paper is organized as follows. In Section 2 we review the Fokker-Plank-Kolmogorov equation associated with SPDEs in a separable Hilbert space. Section 3 provides conditions to assure stability respect initial conditions and in Section 4 we illustrate our results with numerical experiments.

2 Kolmogorov equations for SPDEs in Hilbert spaces

Let $\mathcal{H}$ be a separable infinite-dimensional Hilbert space with inner product $(,)_{\mathcal{H}}$ and norm $\|\cdot\|_{\mathcal{H}}$. We define a Gaussian measure μ with mean zero and nuclear covariance operator Λ with $Tr(\Lambda) < +\infty$.

We focus on the following Kolmogorov equation

$$\frac{\partial u}{\partial t} = \frac{1}{2} Tr(QD^2 u) + \langle Ax, Du \rangle_{\mathcal{H}} + \langle B(x), Du \rangle_{\mathcal{H}}, \qquad x \in D(A). \tag{1}$$

Several authors have proved results on existence and uniqueness of the solution of the Kolmogorov equations, see for instance Da Prato [5] for a survey, Da Prato-Debussche [6] for the Burgers equation, Barbu-Da Prato [1] for the 2D Navier-Stokes stochastic flow in a channel.

2.1 On the Ornstein-Uhlenbeck semigroup

Following [3], in $\mathcal{H}$ we define a Gaussian measure μ with mean zero and nuclear covariance operator Λ with $Tr(\Lambda) < +\infty$ and since $\Lambda : \mathcal{H} \mapsto \mathcal{H}$ is a positive definite, self-adjoint operator then its square-root operator $\Lambda^{1/2}$ is a positive definite, self-adjoint Hilbert-Schmidt operator on $\mathcal{H}$.

Define the inner product $(g,h)_0 := \left(\Lambda^{-1/2} g, \Lambda^{-1/2} h\right)_{\mathcal{H}}$, for $g, h \in \Lambda^{1/2}\mathcal{H}$. Let $\mathcal{H}_0$ denote the Hilbert subspace of $\mathcal{H}$, which is the completion of $\Lambda^{1/2}\mathcal{H}$ with respect to the norm $\|g\|_0 := (g,g)_0^{1/2}$. Then $\mathcal{H}_0$ is dense in $\mathcal{H}$ and the inclusion map $i : \mathcal{H}_0 \hookrightarrow \mathcal{H}$ is compact. The triple $(i, \mathcal{H}_0, \mathcal{H})$ forms an abstract Wiener space.

Let $\mathbb{H} = L^2(\mathcal{H}, \mu)$ denote the Hilbert space of Borel measurable functionals on the probability space with inner product

$$\left[\Phi, \Psi\right]_{\mathbb{H}} := \int_{\mathcal{H}} \Phi(v)\Psi(v)\mu(dv), \qquad \text{for } \Phi, \Psi \in \mathbb{H},$$

and norm $\|\Phi\|_{\mathbb{H}} := \left[\Phi, \Phi\right]_{\mathbb{H}}^{1/2}$. We choose a basis system $\{\varphi_k\}$ for $\mathcal{H}$.

A functional $\Phi : \mathcal{H} \mapsto \mathbb{R}$, is said to be a smooth simple functional (or a cylinder functional) if there exists a C^∞-function ϕ on $\mathbb{R}^n$ and n-continuous linear functional $l_1, \ldots, l_n$ on $\mathcal{H}$ such that for $h \in \mathcal{H}$

$$\Phi(h) = \phi(h_1, \ldots, h_n) \quad \text{where} \qquad h_i = l_i(h), \quad i = 1, \ldots, n. \tag{2}$$

The set of all such functionals will be denoted by $\mathcal{S}(\mathbb{H})$. Denote by $P_k(x)$ the Hermite polynomial of degree k taking values in $\mathbb{R}$. Then, $P_k(x)$ is given by the following formula

$$P_k(x) = \frac{(-1)^k}{(k!)^{1/2}} e^{\frac{x^2}{2}} \frac{d^k}{dx^k} e^{-\frac{x^2}{2}}$$

with $P_0 = 1$. It is well-known that $\{P_k(\cdot)\}_{k\in\mathbb{N}}$ is a complete orthonormal system for $L^2(\mathbb{R}, \mu_1(dx))$ with $\mu_1(dx) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$. Define the set of infinite multi-index as

$$\mathcal{J} = \Big\{ \boldsymbol{\alpha} = (\alpha_i, i \geq 1) \quad \Big| \quad \alpha_i \in \mathbb{N} \cup \{0\}, \quad |\boldsymbol{\alpha}| := \sum_{i=1}^{\infty} \alpha_i < +\infty \Big\}.$$

For $\boldsymbol{n} \in \mathcal{J}$ define the *Hermite polynomial functionals* on $\mathcal{H}$ by

$$H_{\boldsymbol{n}}(h) = \prod_{i=1}^{\infty} P_{n_i}(l_i(h)), \quad h \in \mathcal{H}_0, \quad \boldsymbol{n} \in \mathcal{J}, \tag{3}$$

and where $l_i(h) = \langle h, \Lambda^{-1/2}\varphi_i \rangle_{\mathcal{H}}, \quad i = 1, 2, \dots$ where $P_n(\xi)$ is the usual Hermite polynomial for $\xi \in \mathbb{R}$ and $n \in \mathbb{N}$.

Remark 1. Notice that $l_i(h)$ is defined only for $h \in \mathcal{H}_0$. However, regarding h as a μ-random variable in $\mathcal{H}$, we have $\mathbb{E}\Big(l_i(h)\Big) = \|\varphi_i\|^2 = 1$ and then $l_k(h)$ can be defined μ-a.e. $h \in \mathcal{H}$, similar to defining a stochastic integral.

It is possible to identify the Hermite polynomial functionals defined in (3), for $h \in \mathcal{H}_0$, as a deterministic version of the Wick polynomials defined on the canonical Wiener space (for further details see [13] for instance).

We have the following result (See Theorems 9.1.5 and 9.1.7 in Da Prato-Zabczyk [8] or Lemma 3.1 in Chapter 9 from Chow [3]).

Lemma 2.1. *For $h \in \mathcal{H}$ let $l_i(h) = \langle h, \Lambda^{-1/2}\varphi_i \rangle_{\mathcal{H}}$, $i = 1, 2, \dots$. The set $\{H_{\boldsymbol{n}}\}$ of all Hermite polynomials on $\mathcal{H}$ forms a complete orthonormal system for $\mathbb{H}$. Hence the set of all functionals are dense in $\mathbb{H}$. Moreover, we have the direct sum decomposition: $\mathbb{H} = \bigoplus_{j=0}^{\infty} K_j$, where K_j is the subspace of $\mathbb{H}$ spanned by $\{H_{\boldsymbol{n}} : |\boldsymbol{n}| = j\}$.*

Let Φ be a smooth simple functional given by (2). Then the Fréchet derivatives, $D\Phi = \Phi'$ and $D_2\Phi = \Phi''$ in $\mathcal{H}$ can be computed as follows:

$$(D\Phi(h), v) = \sum_{k=1}^{n} \Big[\partial_k \phi(h_1, \dots, h_n)\Big] l_k(v)$$
$$(D^2\Phi(h), v) = \sum_{j,k=1}^{n} \Big[\partial_j \partial_k \phi(h_1, \dots, h_n)\Big] l_j(v) l_k(v),$$

for any $u, v \in \mathcal{H}$, where $\partial_k \phi = \frac{\partial}{\partial h_k}\phi$. Similarly, for $m > 2$, $D^m\Phi(h)$ is a $m-$linear form on $\mathcal{H}^m$ with inner product $(\cdot,\cdot)_m$. We have $[D^m\Phi(h)](v_1, \cdots, v_m) = (D^m\Phi(h), v_1 \otimes \cdots \otimes v_m)_m$, for $h, v_1, \dots, v_m \in \mathcal{H}$. Consider the following linear stochastic equation

$$du_t = Au_t dt + dW_t, \quad u_0 = h \in \mathcal{H}. \tag{4}$$

Where $A : \mathcal{D}(A) \subset \mathcal{H} \to \mathcal{H}$ is the infinitesimal generator of a strongly continuous semigroup e^{tA} in $\mathcal{H}$. W_t is a Q-Wiener process in $\mathcal{H}$. Chow in [3, Lemma 9.4.1] has shown the following result.

Lemma 2.2. *Suppose that A and Q satisfy the following:*

1. *$A : \mathcal{D}(A) \subset \mathcal{H} \to \mathcal{H}$ is self-adjoint and there is $\beta > 0$ such that*

$$\langle Av, v\rangle_{\mathcal{H}} \leq -\beta \|v\|_{\mathcal{H}} \quad \forall v \in \mathcal{H}.$$

2. *A commutes with Q in $\mathcal{D}(A) \subset \mathcal{H}$.*

Then (4) has a unique invariant measure μ which is a Gaussian measure on $\mathcal{H}$ with zero mean and covariance operator $\Lambda = \frac{1}{2}Q(-A)^{-1} = \frac{1}{2}(-A)^{-1}Q$.

Suppose that A and Q have the same eigenfunctions e_k with eigenvalues λ_k and ρ_k respectively.

It is well-know (See for instance Da Prato and Zabczyk [8]) that the solution of (4) is a time-homogeneous Markov process with transition operator P_t defined for $\Phi \in \mathbb{H}$ given by

$$(P_t\Phi)(h) = \int_{\mathcal{H}} \Phi(v)\mu_t^h(dv) = \mathbb{E}\Big[\Phi(u_t^h)\Big]. \tag{5}$$

Let $\Phi \in \mathcal{S}(\mathbb{H})$ be a smooth simple functional. By setting $\varphi_k = e_k$ in (2), it takes the form $\Phi(h) = \phi(l_1(h), \cdots, l_n(h))$, where $l_k(h) = (h, \Lambda^{-1/2}e_k)$. Define a differential operator A_0 on $\mathcal{S}(\mathbb{H})$ by

$$\mathcal{A}_0\Phi(v) = \tfrac{1}{2}Tr[RD^2\Phi(v)] + \langle Av, D\Phi(v)\rangle, \qquad v \in H, \tag{6}$$

which is well defined, since $D\Phi \in D(A)$ and $\langle Av, D\Phi(v)\rangle = (v, AD\Phi(v))_{\mathcal{H}}$.

The following results have been proved in [3].

Lemma 2.3. *Let P_t be the transition operator as defined by (4). Then the following properties hold:*

1. *$P_t : \mathcal{S}(\mathbb{H}) \to \mathcal{S}(\mathbb{H})$ for $t \geq 0$.*
2. *$\{P_t, t \geq 0\}$ is a strongly continuous semigroup on $\mathcal{S}(\mathbb{H})$ so that, for any $\Phi \in \mathcal{S}(\mathbb{H})$, we have $P_0 = I$, $P_{t+s}\Phi = P_t P_s \Phi$, for all $t, s \geq 0$, and $\lim_{t\downarrow 0} P_t\Phi = \Phi$.*
3. *$\mathcal{A}_0$ is the infinitesimal generator of P_t so that, for each $\Phi \in \mathcal{S}(\mathbb{H})$,*

$$\lim_{t\downarrow 0} \tfrac{1}{t}\Big(P_t - I\Big)\Phi = \mathcal{A}_0\Phi.$$

□

Lemma 2.4. *Let $H_n(h)$ be a Hermite polynomial functional given by (3). Then the following hold:*

$$\mathcal{A}_0 H_{\mathbf{n}}(h) = -\lambda_{\mathbf{n}} H_{\mathbf{n}}(h), \tag{7}$$

$$P_t H_{\mathbf{n}}(h) = \exp\{-\lambda_{\mathbf{n}} t\} H_{\mathbf{n}}(h), \tag{8}$$

for any $\mathbf{n} \in \mathcal{J}$ and $h \in H$, where $\lambda_{\mathbf{n}} = \sum_{i=1}^{\infty} n_i \lambda_i$.

The following Theorem is a Green formula that we will need forward. Its proof can be seen, for instance, in [3, Thm. 3.3, Ch. 9].

Theorem 2.5. *Let $\Phi \in \mathcal{S}(\mathbb{H})$ be a smooth simple functional and let $\mu \sim N(0, \Lambda)$ be a Gaussian measure in $\mathcal{H}$. Then, for any $g, h \in \mathcal{H}$ the following formula holds*

$$\int_{\mathcal{H}} (\Lambda h, D\Phi(v))_{\mathcal{H}} \mu(dv) = \int_{\mathcal{H}} (v, h)_{\mathcal{H}} \Phi(v) \mu(dv) \ . \tag{9}$$

Lemma 2.6. *Assume the conditions for Theorem 2.4 hold. Then, for any $\Phi, \Psi \in \mathcal{S}(\mathbb{H})$, the following Green's formula holds:*

$$\int_{\mathcal{H}} (\mathcal{A}_0\Phi)\Psi d\mu = \int_{\mathcal{H}} \Phi(\mathcal{A}_0\Psi) d\mu = -\frac{1}{2}\int_{\mathcal{H}} (QD\Phi, D\Psi) d\mu \ . \tag{10}$$

By Theorem 2.1, for $\Phi \in \mathbb{H}$, it can be represented as

$$\Phi(v) = \sum_{n=0}^{\infty} \phi_{\mathbf{n}} H_{\mathbf{n}}(v), \tag{11}$$

where $n = |\mathbf{n}|$ and $\mathbf{n} \in \mathcal{J}$. Notice that we can think in $\mathbf{n}$ as a vector of r dimension, i.e. $\mathbf{n} = (n_1, \ldots, n_r)$. Let $\alpha_{\mathbf{n}} = \alpha_{n_1} \cdots \alpha_{n_r}$ be a sequence of positive numbers with $\alpha_{\mathbf{n}} > 0$, such that $\alpha_{\mathbf{n}} \to \infty$ as $n \to \infty$. Define

$$|||\Phi|||_{k,\alpha} = \left[\sum_{\mathbf{n}} \Big(1 + \alpha_{\mathbf{n}}\Big)^k |\phi_n|^2 \right]^{1/2},$$
$$|||\Phi|||_{0,\alpha} = |||\Phi||| = \left[\sum_{\mathbf{n}} |\phi_n|^2 \right]^{1/2},$$

which is $L^2(\mu)$-norm of Φ. For the given sequence $\alpha = \{\alpha_n\}$, let $\mathbb{H}_{k,\alpha}$ denote the completion of $\mathcal{S}(\mathbb{H})$ with respect to the norm $||| \cdot |||_{k,\alpha}$. Then $\mathbb{H}_{k,\alpha}$ is called a Gauss–Sobolev space of order k with parameter α. The dual space of $\mathbb{H}_{k,\alpha}$ is $\mathbb{H}_{-k,\alpha}$. From now on, we will fix the sequence $\alpha_{\mathbf{n}} = \lambda_{\mathbf{n}}$, where $\lambda_{\mathbf{n}}$ is given in Theorem 2.4. We shall simply denote $\mathbb{H}_{k,\alpha}$ by $\mathbb{H}_k$ and $|||\Phi|||_{k,\alpha}$ by $|||\Phi|||_k$.

The following results ensure the existence of an extension for the operator $\mathcal{A}_0$ to a domain containing $\mathbb{H}_2$. Their proofs can be found in [3] for instance.

Theorem 2.7. *Let the conditions on A and Q in Theorem 2.2 hold. Then $P_t : \mathbb{H} \to \mathbb{H}$, for $t \geq 0$, is a contraction semigroup with the infinitesimal generator $\tilde{A}$. The domain of $\tilde{A}$ contains $\mathbb{H}_2$ and we have $\tilde{A} = \mathcal{A}_0$ in $\mathcal{S}(\mathbb{H})$.*

Theorem 2.8. *Let the conditions of Theorem 2.7 hold, then the differential operator $\mathcal{A}_0$ defined by (6) in $\mathcal{S}(\mathbb{H})$ can be extended to be a self-adjoint linear operator A in $\mathbb{H}$ with domain $\mathbb{H}_2$.*

Since both $\tilde{A}$ and A are extensions of $\mathcal{A}_0$ to a domain containing $\mathbb{H}_2$, they must coincide there.

Given the Gauss-Sobolev space $\mathbb{H}_k$ with norm $||| \cdot |||_k$ we denote its dual space by $\mathbb{H}_{-k}$ with norm $|||\cdot|||_{-k}$. Thus, we have the inclusions, $\mathbb{H}_k \subset \mathbb{H} \subset \mathbb{H}_k$. We denote the duality between $\mathbb{H}_k$ and $\mathbb{H}_{-k}$ by $\langle\langle \Psi, \Phi \rangle\rangle_k, \quad \Phi \in \mathbb{H}_k, \quad \Psi \in \mathbb{H}_{-k}$. We also set $\mathbb{H}_0 = \mathbb{H}$, with $||| \cdot |||_0 = ||| \cdot |||$ and $\langle\langle \cdot, \cdot \rangle\rangle_1 = \langle\langle \cdot, \cdot \rangle\rangle$, $\langle\langle \cdot, \cdot \rangle\rangle_0 = [\cdot, \cdot]$.

2.2 A non linear Kolmogorov equation

Consider the following Kolmogorov equation,

$$\frac{\partial}{\partial t}\Psi(v,t) = \mathcal{A}\Psi(v,t) + \langle B(v), D\Psi(v,t)\rangle_{\mathcal{H}}, \quad \text{a.e. } v \in \mathbb{H}_2,$$
$$\Psi(v,0) = \phi(v)\ ,$$

where, as defined in Theorem 2.7, $\mathcal{A} : \mathbb{H}_2 \to \mathbb{H}$ is given by

$$\mathcal{A}\Phi = \tfrac{1}{2}Tr[RD^2\Phi(v)] + \langle Av, D\Phi(v)\rangle\ . \tag{12}$$

Hypothesis on B will be specified latter. For now, we will consider that it is a locally Lipschitz function. The additional term $\langle B(v), D\Psi(v,t)\rangle_{\mathcal{H}}$ is defined μ-a.e. $v \in \mathbb{H}_2$. We will allow the initial datum ϕ will be in $\mathbb{H}$.

We will study a mild solution of the equation (12). Let $\lambda > 0$ be a parameter. By changing Ψ to $e^{\lambda t}\Psi$ in (12) we get the following equation:

$$\begin{aligned}\frac{\partial}{\partial t}\Psi(v,t) &= \mathcal{A}_\lambda\Psi(v,t) + \langle B(v), D\Psi(v,t)\rangle_{\mathcal{H}}, \qquad \text{a.e. } v \in \mathbb{H}_2,\\ \Psi(v,0) &= \phi(v)\ ,\end{aligned} \tag{13}$$

where $\mathcal{A}_\lambda = \mathcal{A} - \lambda I$, with I the identity operator in $\mathbb{H}$. Clearly, the problems (12) and (13) are equivalent, as far for the existence and uniqueness questions are concerned. We will work on the problem (13).

Denote by P_t the semigroup with infinitesimal generator $\mathcal{A}_\lambda$. The existence of P_t is ensured by the Theorem 2.7.Then, we can rewrite the equation (13) in an integral form by using the semigroup P_t

$$\Psi(v,t) = e^{-\lambda t}(P_t\phi)(v) + \int_0^t e^{-\lambda(t-s)}[P_{t-s}(B, D\Psi_s)](v)ds, \tag{14}$$

where we denote $\phi = \phi(\cdot)$ and $\Psi_s = \Psi(\cdot, s)$. Chow [3] had proved the following lemma.

Lemma 2.9. *Let $\Psi \in L^2((0,T);\mathbb{H})$ for some $T > 0$. Then, for any $\lambda > 0$ there exists $C_\lambda > 0$ such that*

$$|||\int_0^t e^{-\lambda(t-s)}P_{t-s}\Psi_s ds|||^2 \le C_\lambda \int_0^T |||\Psi_s|||_{-1}^2 ds, \qquad 0 < t \le T\ . \tag{15}$$

We now prove the following theorem on existence and uniqueness of a mild solution to (13).

Theorem 2.10. *Suppose that $B : \mathcal{H} \to \mathcal{H}_0$ satisfies $(B, D\Phi) \in L^2((0,T); \mathbb{H})$ for any $\Phi \in \mathbb{H}$ and*

$$\sup_{v \in \mathcal{H}} ||\Lambda^{-1/2} B(v)||_{\mathcal{H}} < +\infty.$$

Then, B satisfies

$$|||\Big(B(v), D\Phi(v)\Big)|||_{-1}^2 \leq C|||\Phi(v)|||^2 \quad \textit{for any } \Phi \in \mathbb{H}, \quad v \in \mathbb{H}_2, \tag{16}$$

for some $C > 0$. Moreover, for $\Phi \in \mathbb{H}$, the initial-value problem (13) *has a unique mild solution $\Psi \in C((0,T); \mathbb{H})$.*

For the part of the existence and uniqueness of the solution we will adapt the proof of the Theorem 5.2 in Chapter 9 from [3].

Proof. First we will prove (16). We have

$$|||\Big(B(v), D\Phi(v)\Big)|||_{-1}^2 = \sum_{\mathbf{n}} (1 + \lambda_{\mathbf{n}})^{-1} |\phi_n|^2,$$

with

$$\phi_n = \Big((B(v), D\Phi(V))_{\mathcal{H}}, H_{\mathbf{n}}(v)\Big)_{\mathbb{H}} = \int_{\mathcal{H}} (B(v), D\Phi(v))_{\mathcal{H}} H_{\mathbf{n}}(v) \mu(dv). \tag{17}$$

By the Theorem 2.5, in particular (9), we have

$$\int_{\mathcal{H}} (\Lambda h, D\Phi(v))_{\mathcal{H}} \mu(dv) = \int_{\mathcal{H}} (v, h)_{\mathcal{H}} \Phi(v) \mu(dv),$$

for all $\Phi \in \mathcal{S}(\mathbb{H})$, $g, h \in \mathcal{H}$ and $\mu \sim N(0, \Lambda)$. Then, in particular, in each direction $H_{\mathbf{n}}$ this formula is still true, so we have

$$\int_{\mathcal{H}} (\Lambda h, D\Phi(v))_{\mathcal{H}} H_{\mathbf{n}}(v) \mu(dv) = \int_{\mathcal{H}} (v, h)_{\mathcal{H}} \Phi(v) H_{\mathbf{n}}(v) \mu(dv) \ .$$

Then, applying this last equality to (17) we get

$$\begin{aligned}
\phi_n &= \int_{\mathcal{H}} \Big(\Lambda[\Lambda^{-1} B(v)], D\Phi(v)\Big)_{\mathcal{H}} H_{\mathbf{n}}(v) \mu(dv) \\
&= \int_{\mathcal{H}} \Big(\Lambda^{-1} B(v), v\Big)_{\mathcal{H}} \Phi(v) H_{\mathbf{n}}(v) \mu(dv) \\
&= \int_{\mathcal{H}} \Big(\Lambda^{-1/2} B(v), \Lambda^{1/2} v\Big)_{\mathcal{H}} \Phi(v) H_{\mathbf{n}}(v) \mu(dv) \ .
\end{aligned}$$

Thus,

$$
\begin{aligned}
|\phi_n|^2 &= \left| \int_{\mathcal{H}} \left(\Lambda^{-1/2}B(v), \Lambda^{1/2}v\right)_{\mathcal{H}} \Phi(v) H_{\mathbf{n}}(v) \mu(dv) \right|^2 \\
&\le \int_{\mathcal{H}} \left| \left(\Lambda^{-1/2}B(v), \Lambda^{1/2}v\right)_{\mathcal{H}} \right|^2 \left| H_{\mathbf{n}}(v) \right|^2 \mu(dv) \int_{\mathcal{H}} \left| \Phi(v) \right|^2 \mu(dv) \ .
\end{aligned} \tag{18}
$$

We now focus on the first integral. Let I_1 be the first integral of (18). Then,

$$
\begin{aligned}
I_1 &\le \int_{\mathcal{H}} \left|\left|\Lambda^{-1/2}B(v)\right|\right|_{\mathcal{H}}^2 \left|\left|\Lambda^{1/2}v\right|\right|_{\mathcal{H}}^2 \left|H_{\mathbf{n}}(v)\right|^2 \mu(dv) \\
&\le \sup_{v\in\mathcal{H}} \left|\left|\Lambda^{-1/2}B(v)\right|\right|_{\mathcal{H}}^2 \int_{\mathcal{H}} \left|\left|\Lambda^{1/2}v\right|\right|_{\mathcal{H}}^2 \left|H_{\mathbf{n}}(v)\right|^2 \mu(dv) \\
&\le C \int_{\mathcal{H}} \left|\left|v\right|\right|_{\mathcal{H}}^2 \left|H_{\mathbf{n}}(v)\right|^2 \mu(dv) \\
&\le C \ .
\end{aligned}
$$

The last inequality follows by using proposition 9.2.10 in page 198 from [7]. Then, by using this bound on (18) we have.

$$
\begin{aligned}
|\phi_n|^2 &\le C \int_{\mathcal{H}} \left|\Phi(v)\right|^2 \mu(dv) \\
&\le C|||\Phi(v)|||^2.
\end{aligned}
$$

Thus,

$$
|||\Big(B(v), D\Phi(v)\Big)|||_{-1}^2 \le C|||\Phi(v)|||^2 \sum_{\mathbf{n}} (1+\lambda_{\mathbf{n}})^{-1} \le C|||\Phi(v)|||^2,
$$

which proves (16).

We now prove the existence and uniqueness of a solution to the initial-value problem (13). Let $\mathbb{X}_T$ denote the Banach space $\mathcal{C}([0,T];\mathbb{H})$ with the sup-norm

$$
|||\Psi|||_T := \sup_{0\le t\le T} |||\Psi||| \ .
$$

In $\mathbb{X}_T$ define the linear operator $\mathbb{Q}$ as

$$
\mathbb{Q}\Psi = e^{-\lambda t} P_t \Phi + \int_0^t e^{-\lambda(t-s)} P_{t-s}(B, D\Psi_s) ds, \quad \text{for any } \Psi \in \mathbb{X}_T.
$$

By Theorem 2.7 P_t is a contraction semigroup, then using this fact and Lemma 2.9 we have

$$\begin{aligned}|||\mathbb{Q}\Psi|||^2 &\le 2\Big[|||e^{-\lambda t}P_t\Phi|||^2 + |||\int_0^t e^{-\lambda(t-s)}P_{t-s}(B, D\Psi_s)ds|||^2\Big] \\ &\le 2\Big[|||\Phi|||^2 + C_\lambda \int_0^t |||(B, D\Psi_s)|||_{-1}^2 ds\Big] \\ &\le 2|||\Phi|||^2 + C_1\int_0^t |||\Psi_s|||^2 ds,\end{aligned}$$

for some $C_1 > 0$. Hence, $|||\mathbb{Q}\Psi|||_T \le C(1 + |||\Psi|||_T)$, with $C = C(\Phi, \lambda, T)$. Then, the map $\mathbb{Q} : \mathbb{X}_T \to \mathbb{X}_T$ is well defined. We now show that is a contraction for a small t. Let $\Psi, \Psi' \in \mathbb{X}_T$. Then

$$\begin{aligned}|||\mathbb{Q}\Psi - \mathbb{Q}\Psi'|||^2 &= |||\int_0^t e^{-\lambda(t-s)}P_{t-s}\Big[(B, D\Psi_s) - (B, D\Psi'_s)\Big]ds|||^2 \\ &\le C_\lambda \int_0^t |||(B, D\Psi_s - D\Psi')|||_{-1}^2 ds \\ &\le C_2\int_0^t |||\Psi_s - \Psi'|||^2 ds,\end{aligned}$$

for some $C_2 > 0$.

It follows that $|||\mathbb{Q}\Psi - \mathbb{Q}\Psi'|||_T \le \sqrt{C_2 T}|||\Psi_s - \Psi'|||_T$. Then, for small T, $\mathbb{Q}$ is a contraction on $\mathbb{X}_T$. Hence the Cauchy problem (13) has a unique mild solution. □

We now prove a theorem on the dependence on initial conditions for the mild solution of (13).

Theorem 2.11. *Suppose that $B : \mathcal{H} \to \mathcal{H}_0$ satisfies $(B, D\Phi) \in L^2((0,T); \mathbb{H})$ for any $\Phi \in \mathbb{H}$ and*

$$\sup_{v\in\mathcal{H}} ||\Lambda^{-1/2}B(v)||_{\mathcal{H}} < +\infty. \tag{19}$$

Then, the unique mild solution $\Psi \in C((0,T); \mathbb{H})$ for (13) depends continuously on the initial conditions.

Proof. We know, with the assumption (19), that the existence of a unique mild solution for (13) is guaranteed by Theorem 2.10. We will denote by Ψ_t^φ its mild solution at time t with initial condition φ:

$$\Psi_t^\varphi = e^{-\lambda t}P_t\varphi + \int_0^t e^{-\lambda(t-s)}P_{t-s}(B, D\Psi_s^\varphi)ds\ .$$

Then,

$$\begin{aligned}\Psi_t^\varphi - \Phi_t^\psi &= e^{-\lambda t} P_t \varphi - e^{-\lambda t} P_t \psi + \int_0^t e^{-\lambda(t-s)} P_{t-s}(B, D\Psi_s^\varphi - D\Phi_s^\psi) ds \\ &= e^{-\lambda t} P_t(\varphi - \psi) + \int_0^t e^{-\lambda(t-s)} P_{t-s}(B, D\Psi_s^\varphi - D\Phi_s^\psi) ds.\end{aligned}$$

From this expression we get

$$\begin{aligned}|||\Psi_t^\varphi - \Phi_t^\psi|||^2 &\leq |||e^{-\lambda t} P_t(\varphi - \psi)|||^2 + |||\int_0^t e^{-\lambda(t-s)} P_{t-s}(B, D\Psi_s^\varphi - D\Phi_s^\psi)|||^2 ds \\ &\leq |||\varphi - \psi|||^2 + C_\lambda \int_0^t |||(B, D\Psi_s^\varphi - D\Phi_s^\psi)|||_{-1}^2 ds \\ &\leq |||\varphi - \psi|||^2 + C_2 \int_0^t |||\Psi_s^\varphi - \Phi_s^\psi|||^2 ds.\end{aligned}$$

Thus, by Gronwall's inequality we obtain

$$|||\Psi_t^\varphi - \Phi_t^\psi|||^2 \leq \exp(C_2 t)|||\varphi - \psi|||^2, \tag{20}$$

which implies, $|||\Psi_t^\varphi - \Phi_t^\psi||| \leq \exp(Ct)|||\varphi - \psi|||$. This completes the proof. □

3 Numerical stability respect to initial conditions

In this section, we prove the continuity with respect to the initial conditions for a numerical approximation of the Kolmogorov equation associated with an SPDE. Here we understand that a numerical scheme is stable respect to initial conditions if this method reproduces the same behavior when the continuous problem satisfies continuity respect initial conditions.

Consider the stochastic differential equation in $\mathcal{H}$

$$dX_t = AX_t dt + B(X_t)dt + \sqrt{Q} dW_t, \tag{21}$$

where the operator $A : \mathcal{D}(A) \subset \mathcal{H} \to \mathcal{H}$ is the infinitesimal generator of a strongly continuous semigroup e^{tA} in $\mathcal{H}$, Q is a bounded operator from another Hilbert space $\mathcal{U}$ to $\mathcal{H}$ and $B : \mathcal{D}(B) \subset \mathcal{H} \to \mathcal{H}$ is a nonlinear mapping.

The equation (21) can be associated to a Kolmogorov equation in the next way, we define

$$u(t,x) = \mathbb{E}\Big[\varphi(X_t^x)\Big], \tag{22}$$

where $\varphi : \mathcal{H} \to \mathbb{R}$ and X_t^x is the solution to (21) with initial conditions $X_0 = x$ where $x \in \mathcal{H}$. Then u satisfies the Kolmogorov equation (1).

We use Theorem 2.1 to write the solution Ψ_t^φ as in a Fourier-Hermite decomposition:

$$\Psi_t^\varphi = \sum_{\boldsymbol{n}\in\mathcal{J}} u_{\boldsymbol{n}}(t) H_{\boldsymbol{n}}(x), \qquad x \in \mathcal{H}, \quad t \in [0,T] \ . \tag{23}$$

Note that the time-dependent coefficients $u_{\boldsymbol{n}}(t)$ depend on the functional and on the initial condition but it is not a function of the initial condition. First we prove an auxiliary result.

Lemma 3.1. *Set $\{P_k(\xi)\}_{k\in\mathbb{N}}$ the family of normalized Hermite polynomials in $\mathbb{R}$. For every $k \in \mathbb{N}$ and $\xi, \eta \in \mathbb{R}$ such that $\eta < \xi$ we have that*

$$P_k(\xi) - P_k(\eta) = C(k) Pe_{k+1}(\gamma) \cdot (\xi - \eta), \tag{24}$$

where $\gamma \in (\eta, \xi)$ and $C(k) = \frac{(-1)^k}{(k+1)(k!)^{1/2}}$. Moreover, $Pe_k(x)$ is the unnormalized Hermite polynomial of k degree.

Proof. We know that $P_k(\xi) = \frac{(-1)^k}{(k!)^{1/2}} e^{\xi^2/2} \frac{d}{d\xi^k} e^{-\xi^2/2}$. Set $c(k) = (-1)^k (k!)^{-1/2}$, then

$$\begin{aligned} P_k(\xi) - P_k(\eta) &= c(k)\Big[e^{\xi^2/2}\frac{d}{d\xi^k}e^{-\xi^2/2} - e^{\eta^2/2}\frac{d}{d\eta^k}e^{-\eta^2/2}\Big] \\ &= c(k)\left[e^{x^2/2}\frac{d}{dx^k}e^{-x^2/2}\Big|_{x=\eta}^{\xi}\right] \\ &= c(k)\int_\eta^\xi F_k(x)dx, \end{aligned}$$

where F_k is a continuous function such that $F_k'(x) = e^{x^2/2}\frac{d}{dx^k}e^{-x^2/2}$. In fact, denoting by $Pe_k(x)$ the unnormalized Hermite polynomial of k degree, results

$$F_k'(x) = e^{x^2/2}\frac{d}{dx^k}e^{-x^2/2} = Pe_k(x),$$

and since the Hermite polynomials constitute an Appell sequence we have that

$$F_k'(x) = Pe_k(x) = \frac{1}{k+1} Pe_{k+1}'(x),$$

which implies that $F_k(x) = \frac{1}{k+1} Pe_{k+1}(x)$. Now, since $F_k(x)$ is a continuous function, then there exists $\gamma \in (\eta, \xi)$ such that

$$\int_\eta^\xi F_k(x)dx = F_k(\gamma) \cdot (\xi - \eta).$$

All these implies that $P_k(\xi) - P_k(\eta) = c(k)F_k(\gamma) \cdot (\xi - \eta)$. From this expression the lemma follows immediately. □

We will use some technical results on the SPDE to prove the following result—the main result of this section.

Theorem 3.2. *Assume that the eigenvalues of* Λ*, satisfies that for every* $k \in \mathbb{N}$*,* $\lambda_k < \lambda_{k+1} \to \infty$*. Assume that the functional* φ *is Lipschitz. Then, the numeric approximation* Ψ_t^φ *(given by* (23)*) to the solution of the Kolmogorov equation* $\Psi \in C((0,T); \mathbb{H})$ *depends continuously on the initial conditions.*

Proof. Let $x, y \in H$ be two different initial values. We want to estimate $\Psi_t^x - \Psi_t^y$. By definition,

$$\Psi_t^x = \sum_{\bar{n} \in \mathcal{J}} u_{\bar{n}}^x(t) H_{\bar{n}}(x) \; . \tag{25}$$

Thus,

$$\begin{aligned} \Psi_t^x - \Psi_t^y &= \sum_{\bar{n} \in \mathcal{J}} u_{\bar{n}}^x(t) H_{\bar{n}}(x) - \sum_{\bar{n} \in \mathcal{J}} u_{\bar{n}}^y(t) H_{\bar{n}}(y) \\ &= \sum_{\bar{n} \in \mathcal{J}} \Big[u_{\bar{n}}^x(t) - u_{\bar{n}}^y(t) \Big] H_{\bar{n}}(x) + \sum_{\bar{n} \in \mathcal{J}} u_{\bar{n}}^y(t) \Big[H_{\bar{n}}(x) - H_{\bar{n}}(y) \Big]. \end{aligned} \tag{26}$$

We focus on the first term in (26). From the definition of the initial condition we obtain the following expression for the time-dependent coefficient

$$u_{\bar{n}}^x(t) = \int_{\mathcal{H}} H_{\bar{n}}(x) \mathbb{E}\Big[\varphi(X_t^x)\Big] \mu(dx) \; .$$

From this we get

$$\begin{aligned}
u_{\bar n}^x(t)-u_{\bar n}^y(t) &= \int_{\mathcal H} H_{\bar n}(x)\mathbb{E}\Big[\varphi(X_t^x)\Big]\mu(dx) - \int_{\mathcal H} H_{\bar n}(y)\mathbb{E}\Big[\varphi(X_t^y)\Big]\mu(dy) \\
&= \int_{\mathcal H\times\mathcal H} H_{\bar n}(x)\mathbb{E}\Big[\varphi(X_t^x)\Big]\mu(dx)\mu(dy) \\
&\quad - \int_{\mathcal H\times\mathcal H} H_{\bar n}(y)\mathbb{E}\Big[\varphi(X_t^y)\Big]\mu(dx)\mu(dy) \\
&= \int_{\mathcal H\times\mathcal H} H_{\bar n}(x)\Big(\mathbb{E}\Big[\varphi(X_t^x)\Big]-\mathbb{E}\Big[\varphi(X_t^y)\Big]\Big)\mu(dx)\mu(dy) \\
&\quad + \int_{\mathcal H\times\mathcal H} \Big(H_{\bar n}(x)-H_{\bar n}(y)\Big)\mathbb{E}\Big[\varphi(X_t^y)\Big]\mu(dx)\mu(dy)\ .
\end{aligned}$$

Then, by the Cauchy-Schwartz inequality, we obtain

$$\begin{aligned}
|u_{\bar n}^x(t)-u_{\bar n}^y(t)|^2 &\le \left|\int_{\mathcal H\times\mathcal H} H_{\bar n}(x)\left(\mathbb{E}\left[\varphi(X_t^x)\right]-\mathbb{E}\left[\varphi(X_t^y)\right]\right)\mu(dx)\mu(dy)\right|^2 \\
&\quad + \left|\int_{\mathcal H\times\mathcal H} \left(H_{\bar n}(x)-H_{\bar n}(y)\right)\mathbb{E}\left[\varphi(X_t^y)\right]\mu(dx)\mu(dy)\right|^2 \\
&\le \int_{\mathcal H\times\mathcal H} H_{\bar n}^2(x))\mu(dx)\mu(dy) \\
&\quad \times \int_{\mathcal H\times\mathcal H} \left|\mathbb{E}\left[\varphi(X_t^x)\right]-\mathbb{E}\left[\varphi(X_t^y)\right]\right|^2\mu(dx)\mu(dy) \\
&\quad + \int_{\mathcal H\times\mathcal H} \mathbb{E}^2\left[\varphi(X_t^y)\right]\mu(dx)\mu(dy) \qquad (27) \\
&\quad \times \int_{\mathcal H\times\mathcal H} \left|H_{\bar n}(x)-H_{\bar n}(y)\right|^2\mu(dx)\mu(dy) \\
&= \int_{\mathcal H\times\mathcal H} \left|\mathbb{E}\left[\varphi(X_t^x)\right]-\mathbb{E}\left[\varphi(X_t^y)\right]\right|^2\mu(dx)\mu(dy) \\
&\quad + \int_{\mathcal H\times\mathcal H} \mathbb{E}^2\left[\varphi(X_t^y)\right]\mu(dx)\mu(dy) \\
&\quad \times \int_{\mathcal H\times\mathcal H} \left|H_{\bar n}(x)-H_{\bar n}(y)\right|^2\mu(dx)\mu(dy)\ .
\end{aligned}$$

We now estimate the norm of the expression (26) with the help of (27).

$$
\begin{aligned}
\|\Psi_t^x - \Psi_t^y\|^2_{\left(L^2(\mathcal{H},\mu)\right)^2} &= \int_{\mathcal{H}\times\mathcal{H}} |\Psi_t^x - \Psi_t^y|^2 \mu(dx)\mu(dy) \\
&\le \int_{\mathcal{H}\times\mathcal{H}} \Big| \sum_{\bar{n}\in\mathcal{J}} \Big[u_{\bar{n}}^x(t) - u_{\bar{n}}^y(t)\Big] H_{\bar{n}}(x)\Big|^2 \mu(dx)\mu(dy) \\
&\quad + \int_{\mathcal{H}\times\mathcal{H}} \Big| \sum_{\bar{n}\in\mathcal{J}} u_{\bar{n}}^y(t)\Big[H_{\bar{n}}(x) - H_{\bar{n}}(y)\Big]\Big|^2 \mu(dx)\mu(dy) \\
&\le \int_{\mathcal{H}\times\mathcal{H}} \sum_{\bar{n}\in\mathcal{J}} \Big|u_{\bar{n}}^x(t) - u_{\bar{n}}^y(t)\Big|^2 H_{\bar{n}}^2(x)\mu(dx)\mu(dy) \\
&\quad + \int_{\mathcal{H}\times\mathcal{H}} \sum_{\bar{n}\in\mathcal{J}} \Big[u_{\bar{n}}^y(t)\Big]^2 \sum_{\bar{n}\in\mathcal{J}} \Big|H_{\bar{n}}(x) - H_{\bar{n}}(y)\Big|^2 \mu(dx)\mu(dy) \\
&= \sum_{\bar{n}\in\mathcal{J}} \Big|u_{\bar{n}}^x(t) - u_{\bar{n}}^y(t)\Big|^2 \int_{\mathcal{H}\times\mathcal{H}} H_{\bar{n}}^2(x)\mu(dx)\mu(dy) \\
&\quad + \sum_{\bar{n}\in\mathcal{J}} \Big[u_{\bar{n}}^y(t)\Big]^2 \sum_{\bar{n}\in\mathcal{J}} \int_{\mathcal{H}\times\mathcal{H}} \Big|H_{\bar{n}}(x) - H_{\bar{n}}(y)\Big|^2 \mu(dx)\mu(dy) \qquad (28) \\
&= \sum_{\bar{n}\in\mathcal{J}} \Big[u_{\bar{n}}^y(t)\Big]^2 \sum_{\bar{n}\in\mathcal{J}} \int_{\mathcal{H}\times\mathcal{H}} \Big|H_{\bar{n}}(x) - H_{\bar{n}}(y)\Big|^2 \mu(dx)\mu(dy) \\
&\quad + \sum_{\bar{n}\in\mathcal{J}} \Big|u_{\bar{n}}^x(t) - u_{\bar{n}}^y(t)\Big|^2 \\
&= \int_{\mathcal{H}\times\mathcal{H}} \Big|\mathbb{E}\Big[\varphi(X_t^x)\Big] - \mathbb{E}\Big[\varphi(X_t^y)\Big]\Big|^2 \mu(dx)\mu(dy) \\
&\quad + \int_{\mathcal{H}\times\mathcal{H}} \mathbb{E}^2\Big[\varphi(X_t^y)\Big]\mu(dx)\mu(dy) \\
&\quad \times \sum_{\bar{n}\in\mathcal{J}} \int_{\mathcal{H}\times\mathcal{H}} \Big|H_{\bar{n}}(x) - H_{\bar{n}}(y)\Big|^2 \mu(dx)\mu(dy) \\
&\quad + \sum_{\bar{n}\in\mathcal{J}} \Big[u_{\bar{n}}^y(t)\Big]^2 \sum_{\bar{n}\in\mathcal{J}} \int_{\mathcal{H}\times\mathcal{H}} \Big|H_{\bar{n}}(x) - H_{\bar{n}}(y)\Big|^2 \mu(dx)\mu(dy)\ .
\end{aligned}
$$

Note that $\mathbb{E}^2\Big[\varphi(X_t^y)\Big] = u^2(t,x) \in L^2(\mathcal{H},\mu)$, therefore the first integral in the second term is a continuous bounded function of t. Moreover, $\sum_{\bar{n}\in\mathcal{J}} \Big[u_{\bar{n}}^y(t)\Big]^2$ is the $L^2(\mathcal{H},\mu)$-norm of the function $u(t,x)$, then the series converges and it is also a continuous bounded function of t. Thus, from (28) we get

$$\|\Psi_t^x - \Psi_t^y\|^2_{\left(L^2(\mathcal{H},\mu)\right)^2} \leq \int_{\mathcal{H}\times\mathcal{H}} \Big|\mathbb{E}\Big[\varphi(X_t^x)\Big] - \mathbb{E}\Big[\varphi(X_t^y)\Big]\Big|^2 \mu(dx)\mu(dy) \\ + f(t) \sum_{\bar{n}\in\mathcal{J}} \int_{\mathcal{H}\times\mathcal{H}} \Big|H_{\bar{n}}(x) - H_{\bar{n}}(y)\Big|^2 \mu(dx)\mu(dy) \ , \tag{29}$$

where $f(t) = \sum_{\bar{n}\in\mathcal{J}} \Big[u^y_{\bar{n}}(t)\Big]^2 + \int_{\mathcal{H}} \mathbb{E}^2\Big[\varphi(X_t^y)\Big]\mu(dy)$.

From the proof of Theorem 2.11 (see (20)) we know that

$$\begin{aligned} |||\Psi_t^\varphi - \Phi_t^\psi|||^2 &= \int_{\mathcal{H}\times\mathcal{H}} \Big|\mathbb{E}\Big[\varphi(X_t^x)\Big] - \mathbb{E}\Big[\varphi(X_t^y)\Big]\Big|^2 \mu(dx)\mu(dy) \\ &\leq \exp(Ct) \int_{\mathcal{H}\times\mathcal{H}} \|x-y\|^2_{\mathcal{H}} \mu(dx)\mu(dy) \\ &= \exp(Ct)|||x-y|||^2. \end{aligned} \tag{30}$$

Therefore the first term in the right side of (29) is bounded by (30).

We now focus on the second term in the last inequality. Notice that for every $\bar{n} \in \mathcal{J}$ we have

$$H_{\bar{n}}(x) - H_{\bar{n}}(y) = \prod_{i=1}^{\infty} \Big[P_{n_i}(\xi_i) - P_{n_i}(\eta_i)\Big], \tag{31}$$

where $\xi_i = \langle x, \Lambda^{-1/2} e_i\rangle_{\mathcal{H}}$ and $\eta_i = \langle y, \Lambda^{-1/2} e_i\rangle_{\mathcal{H}}$ (see (3) and lines after that for the definition). Hence, applying Theorem 3.1 to equation (31) we have that

$$\begin{aligned} H_{\bar{n}}(x) - H_{\bar{n}}(y) &= \prod_{i=1}^{\infty} C(i) Pe_{i+1}(\gamma_i) \cdot (\xi_i - \eta_i) \\ &= \prod_{i=1}^{\infty} C(i) Pe_{i+1}(\gamma_i) \langle x-y, \Lambda^{-1/2} e_i\rangle_{\mathcal{H}}, \end{aligned} \tag{32}$$

here $\gamma_i \in \Big(\xi_i \wedge \eta_i, \xi_i \vee \eta_i\Big)$ for every $i \in \mathbb{N}$. Then

$$
\begin{aligned}
&\sum_{\bar{n}\in\mathcal{J}} \int_{\mathcal{H}\times\mathcal{H}} \Big| H_{\bar{n}}(x) - H_{\bar{n}}(y) \Big|^2 \mu(dx)\mu(dy) \\
&\qquad = \sum_{\bar{n}\in\mathcal{J}} \int_{\mathcal{H}\times\mathcal{H}} \Big| \prod_{i=1}^{\infty} C(i) Pe_{i+1}(\gamma_i) \langle x-y, \Lambda^{-1/2} e_i \rangle_{\mathcal{H}} \Big|^2 \mu(dx)\mu(dy) \\
&\qquad = \sum_{\bar{n}\in\mathcal{J}} \int_{\mathcal{H}\times\mathcal{H}} \prod_{i=1}^{\infty} \Big[C(i) Pe_{i+1}(\gamma_i) \Big]^2 \Big| \langle x-y, \Lambda^{-1/2} e_i \rangle_{\mathcal{H}} \Big|^2 \mu(dx)\mu(dy) \\
&\qquad \leq \sum_{\bar{n}\in\mathcal{J}} \int_{\mathcal{H}\times\mathcal{H}} \prod_{i=1}^{\infty} \Big[C(i) Pe_{i+1}(\gamma_i) \Big]^2 \|x-y\|_{\mathcal{H}}^2 \|\Lambda^{-1/2} e_i\|_{\mathcal{H}}^2 \mu(dx)\mu(dy) \\
&\qquad = \sum_{\bar{n}\in\mathcal{J}} \int_{\mathcal{H}\times\mathcal{H}} \prod_{i=1}^{\infty} \Big[C(i) Pe_{i+1}(\gamma_i) \Big]^2 \|x-y\|_{\mathcal{H}}^2 \lambda_i^{-1} \|e_i\|_{\mathcal{H}}^2 \mu(dx)\mu(dy) \\
&\qquad = \|x-y\|_{\mathcal{H}}^2 \sum_{\bar{n}\in\mathcal{J}} \prod_{i=1}^{\infty} \Big[C(i) \Big]^2 \lambda_i^{-1} \int_{\mathcal{H}\times\mathcal{H}} \Big[Pe_{i+1}(\gamma_i) \Big]^2 \mu(dx)\mu(dy).
\end{aligned}
\tag{33}
$$

Recall that for every $i \in \mathbb{N}$ we have that $\gamma_i \in \Big(\xi_i \wedge \eta_i, \xi_i \vee \eta_i\Big)$, set $\hat{\gamma}_i \in \Big(\xi_i \wedge \eta_i, \xi_i \vee \eta_i\Big)$ such that $Pe_i^2(\gamma_i) \leq Pe_{i+1}^2(\hat{\gamma}_i)$ for every $\gamma_i \in \Big(\xi_i \wedge \eta_i, \xi_i \vee \eta_i\Big)$, notice that the existence of $\hat{\gamma}_i$ is guaranteed since $Pe_{i+1}^2(\cdot)$ is a continuous function. Then, from (33) we get

$$
\begin{aligned}
&\sum_{\bar{n}\in\mathcal{J}} \int_{\mathcal{H}\times\mathcal{H}} \Big| H_{\bar{n}}(x) - H_{\bar{n}}(y) \Big|^2 \mu(dx)\mu(dy) \\
&\qquad\qquad \leq \|x-y\|_{\mathcal{H}}^2 \sum_{\bar{n}\in\mathcal{J}} \prod_{i=1}^{\infty} \Big[C(i) \Big]^2 \lambda_i^{-1} \Big[Pe_{i+1}(\hat{\gamma}_i) \Big]^2 \int_{\mathcal{H}} \int_{\mathcal{H}} \mu(dx)\mu(dy) \\
&\qquad\qquad = \|x-y\|_{\mathcal{H}}^2 \sum_{\bar{n}\in\mathcal{J}} \prod_{i=1}^{\infty} \Big[C(i) \Big]^2 \lambda_i^{-1} \Big[Pe_{i+1}(\hat{\gamma}_i) \Big]^2.
\end{aligned}
\tag{34}
$$

Here, we recall that $C(i) = \frac{(-1)^i}{(i+1)(i!)^{1/2}}$ then $\frac{(-1)^i}{\Big[(i+1)!\Big]^{1/2}} Pe_{i+1}(\hat{\gamma}_i)$ is the normalized Hermite polynomial of $i+1$ degree evaluated on $\hat{\gamma}_i$ which is bounded by a constant C for every $i \in \mathbb{N}$. Moreover, since $\lambda_k < \lambda_{k+1} \to \infty$ then this implies that

$$
\sum_{\bar{n}\in\mathcal{J}} \prod_{i=1}^{\infty} \Big[C(i) \Big]^2 \lambda_i^{-1} \Big[Pe_{i+1}(\hat{\gamma}_i) \Big]^2 \leq C \sum_{\bar{n}\in\mathcal{J}} \prod_{i=1}^{\infty} \lambda_i^{-1} (i+1)^{-1} \leq C,
\tag{35}
$$

where C is a finite constant. Putting together (33) and (35) we get that

$$\sum_{\bar{n}\in\mathcal{J}} \int_{\mathcal{H}\times\mathcal{H}} \Big|H_{\bar{n}}(x) - H_{\bar{n}}(y)\Big|^2 \mu(dx)\mu(dy) \leq C\|x-y\|_{\mathcal{H}} \ . \tag{36}$$

Putting together inequalities (29), (30) and (36) we obtain

$$\|\Psi_t^x - \Psi_t^y\|^2_{\left(L^2(\mathcal{H},\mu)\right)^2} \leq \exp(Ct)\int_{\mathcal{H}\times\mathcal{H}} \|x-y\|^2_{\mathcal{H}}\mu(dx)\mu(dy) + f(t)\|x-y\|_{\mathcal{H}}. \tag{37}$$

Now, if $\|x-y\|_{\mathcal{H}} \leq \delta$, then from (37) we get $\|\Psi_t^x - \Psi_t^y\|_{\left(L^2(\mathcal{H},\mu)\right)^2} \leq G(t)\delta$. □

Remark 2. If we consider in addition the supremum norm on t, then from (37) we get

$$\begin{aligned}\sup_{0\leq t\leq T} \|\Psi_t^x - \Psi_t^y\|^2_{\left(L^2(\mathcal{H},\mu)\right)^2} \leq & C\|x-y\|^2_{\mathcal{H}} \sup_{0\leq t\leq T} f(t) \\ & + \exp(CT)\int_{\mathcal{H}\times\mathcal{H}} \|x-y\|^2_{\mathcal{H}}\mu(dx)\mu(dy) \ .\end{aligned} \tag{38}$$

Notice that $f(t)$ is differentiable and continuous, then $\sup_{0\leq t\leq T} f(t) \leq C$, then from (38) we obtain

$$\begin{aligned}\sup_{0\leq t\leq T} \|\Psi_t^x - \Psi_t^y\|_{\left(L^2(\mathcal{H},\mu)\right)^2} \leq & \ C\|x-y\|_{\mathcal{H}} \\ & + \exp(CT)\int_{\mathcal{H}\times\mathcal{H}} \|x-y\|^2_{\mathcal{H}}\mu(dx)\mu(dy) \ .\end{aligned} \tag{39}$$

From this inequality it is possible to show the continuously dependence on the initial conditions for this norm.

4 Numerical experiments

In this section we run numerical experiments to illustrate that our scheme preserves the underlying initial condition continuity. To this end, we solve a stochastic version of the Fisher and Burgers PDEs with two near initial function conditions $x(\xi)$, $\widehat{x}(\xi)$. In [16] we provide a GitHub repository with a Python implementation to reproduce the following figures. We also provide in [10, 11], the 3D on-line plotly versions of Figures 2 and 5.

Stochastic Fisher-KPP equation in an interval

Let $\mathcal{H} = L^2(0,1)$. We consider the stochastic Fisher-KPP equation

$$\begin{aligned} dX(t,\xi) &= \left[\nu \partial_\xi^2 X(t,\xi) + X(t,\xi)(1 - X(t,\xi))\right] dt + dW(t,\xi), \\ X(t,0) &= X(t,1) = 0, \quad t > 0, \\ X(0,\xi) &\in \mathcal{H},\ \xi \in [0,1], \end{aligned} \tag{40}$$

in the interval $[0,1]$ and with initial function conditions $x(\xi))$ and $\widehat{x}(\xi)$. In order to fix this initial function conditions close, we use for our experiments

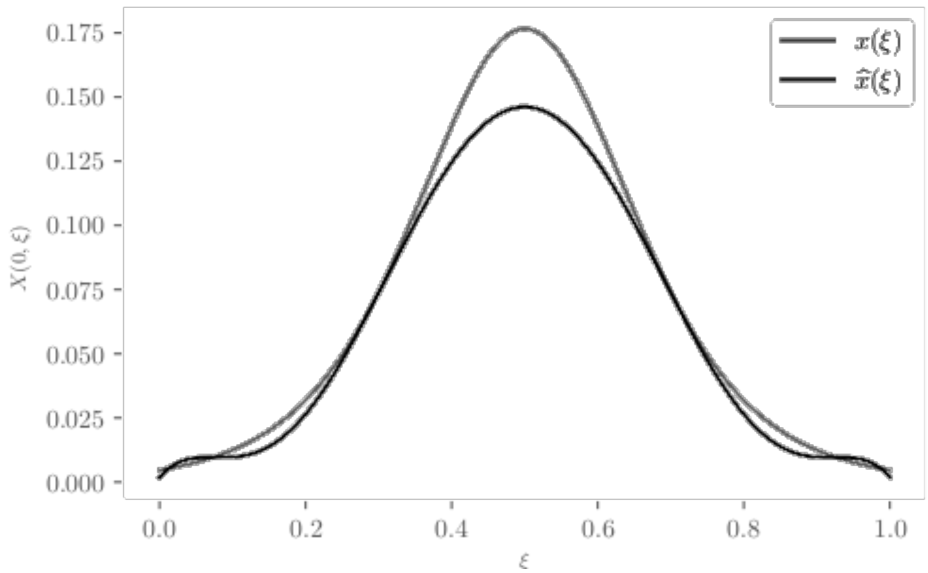
$$x(\xi) := \operatorname{sech}^2(5(\xi - 0.5)), \qquad \widehat{x}(\xi) := \sum_{k=0}^{N} T_k(x(\xi)), \tag{41}$$

where $T_k(\cdot)$ denotes the Chebyshev polynomial of the first kind. That is, $\widehat{x}(\cdot)$ is the Chebyshev truncated expansion of $x(\cdot)$.

Figure 1 displays the plots of this initial conditions. In Figure 2 we observe how the mentioned approximations remains close—blue color scale denotes the solution of equation 40 with initial function condition $x(\xi)$, while yellow color corresponds to the approximation with initial condition $\widehat{x}$. Since we employ transparency to obtain this 3D plot, the purple scale results from the closeness of the solutions. Further, Figure 3 suggest the conclusion of Theorem 3.2, that is, the solutions of equation (40) are continuous respect to initial conditions and satisfies the estimation (30).

Figure 2 illustrates the distance between initial conditions. The yellow pallet with transparency and a blue scale highlight the zones where the two solutions are close. Thus, the zones where the color is purple denotes, where the two solutions of SPDEs are close. According to the $\mathcal{L}(\mathcal{H}, \mu)$-distance between the two underlying solutions, Figure 3 confirms the above argument.

Figure 1: Numerical Solution of the Fisher-KPP eq. (40) with initial conditions x, $\widehat{x}$ at time $t = 0$.



Stochastic Burgers equation

Let $\mathcal{H} = L^2(0,1)$, consider the stochastic Burgers equation in the interval $[0,1]$

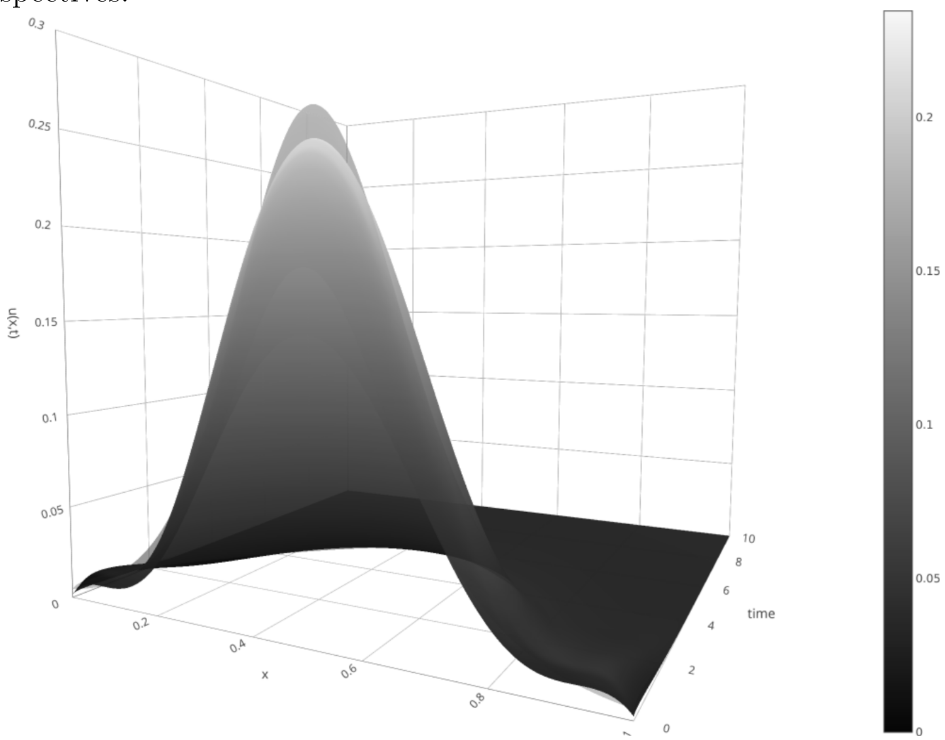
$$\begin{aligned} dX(t,\xi) &= \left[\nu\partial_\xi^2 X(t,\xi) + \frac{1}{2}\partial_\xi X^2(t,\xi)\right] dt + dW(t,\xi), \\ X(t,0) &= X(t,1) = 0, \quad t > 0, \\ X(0,\xi) &= x(\xi), \quad x \in \mathcal{H}\ . \end{aligned} \tag{42}$$

As in the above experiment, we use the initial conditions $x(\xi)$ and its truncated Chebyshev expansion

$$x(\xi) := \sin(\pi\xi), \qquad \widehat{x}(\xi) := \sum_{k=0}^{N} T_k x(\xi). \tag{43}$$

Figures 4 to 6 illustrate a similar argument presented in the above experiment.

Figure 2: Likening between two solution with closed initial conditions x, $\widehat{x}$ of the stochastic Fisher-KPP eq. (40). See [11] to obtain other camera perspectives.



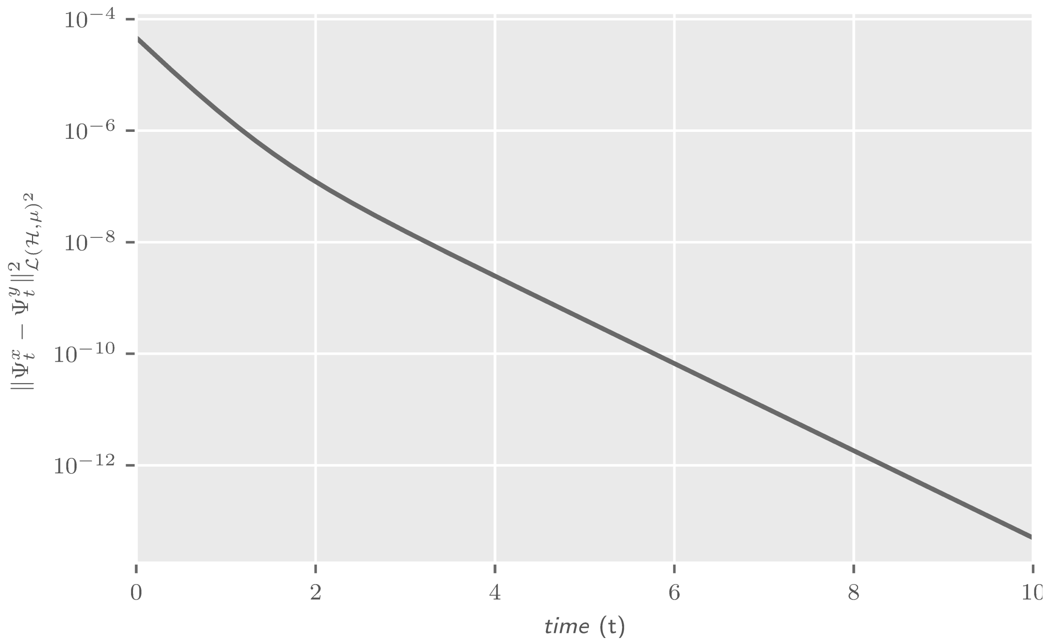


Figure 3: $\mathcal{L}^2(\mathcal{H}, \mu)$ distance between two solutions of the stochastic Fisher PDE with initial conditions $x = x(\xi)$, and $y = \widehat{x}(\xi)$.

Figure 4: Numerical Solution of the Burgers eq. (42) with initial conditions $x(\xi)$, $\widehat{x}(\xi)$.

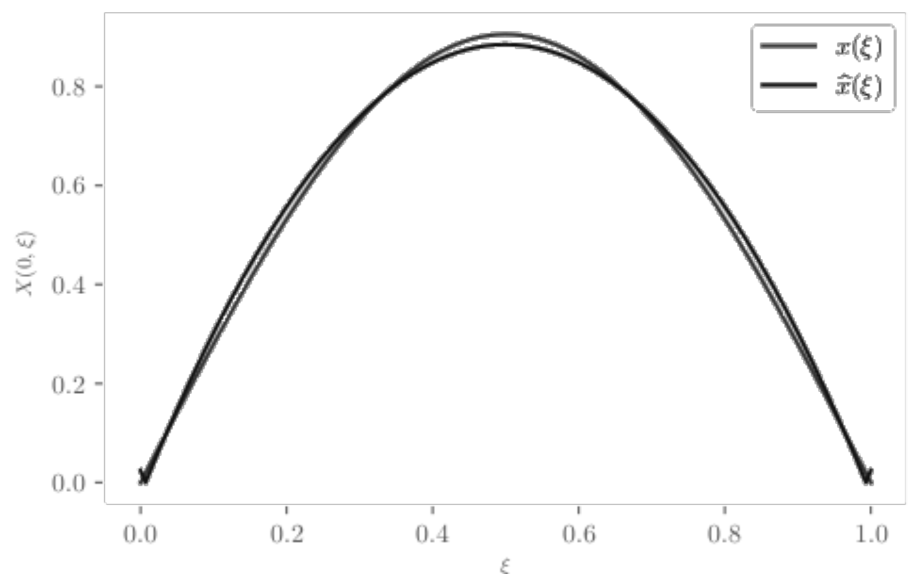


Figure 5: Likening between two solution with closed initial conditions $x(\xi)$, and $\widehat{x}(\xi)$ of the stochastic Burgers eq. (42). See [11] to obtain other camera perspectives.

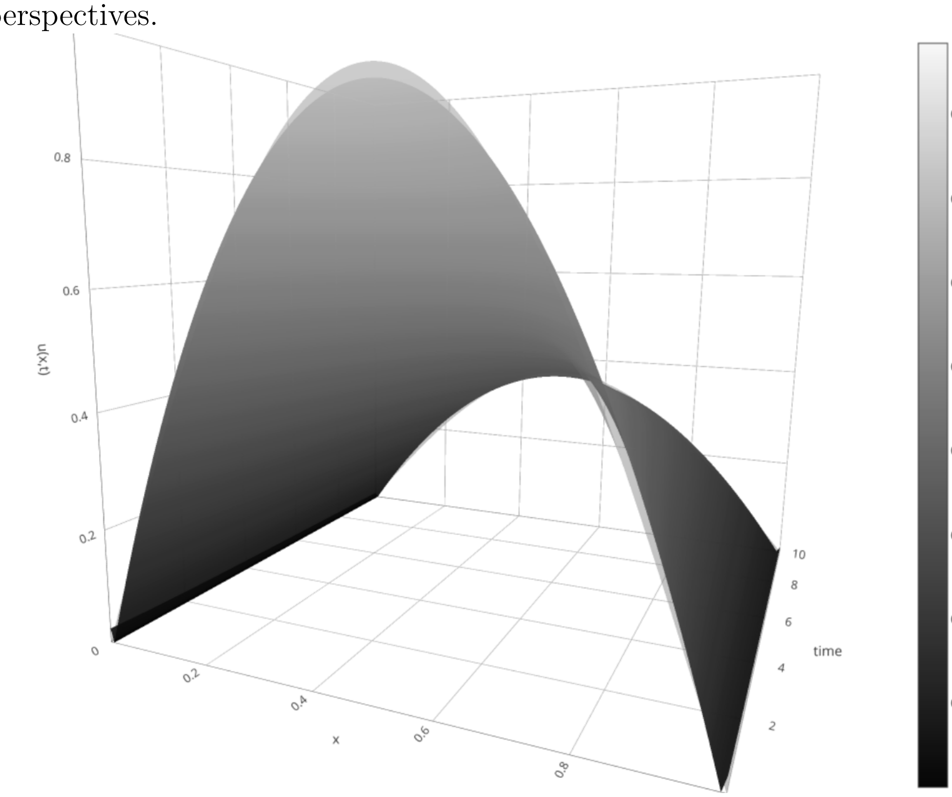
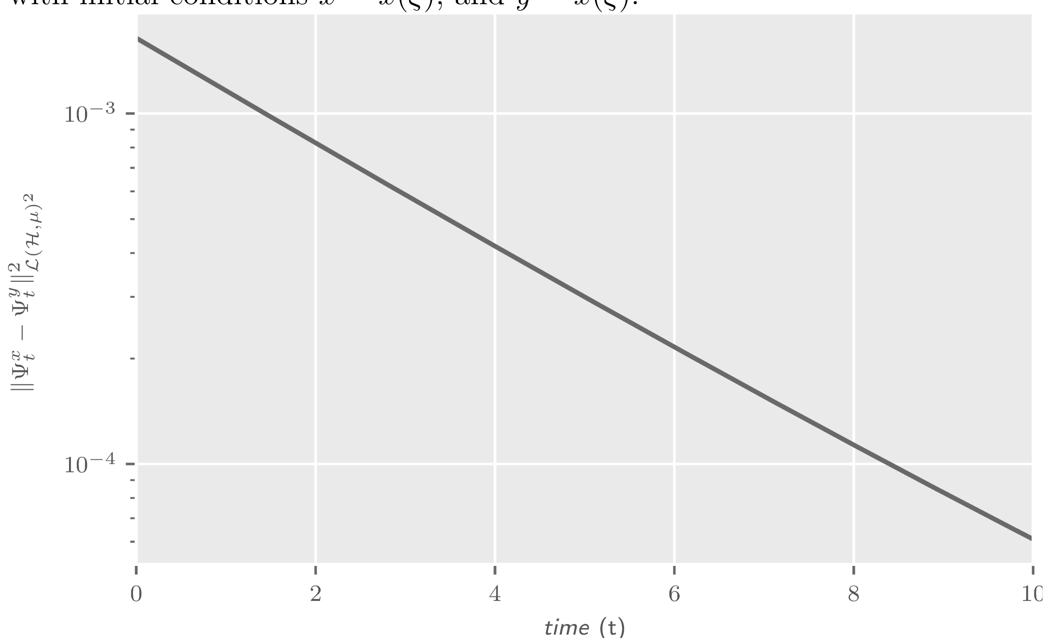


Figure 6: Distance between two solutions of the stochastic Burgers eq. (42) with initial conditions $x = x(\xi)$, and $y = \widehat{x}(\xi)$.



5 Conclusions

To the best of our knowledge, our results represent the first contribution to the numeric stability respect to initial conditions of weak approximations of Kolmogorov equations in infinite dimensions. This kind of stability, combining with the weak approximation approach, would save computation time. That is, since our scheme asks specific conditions to obtain a weak numerical solution of an underlying SPDE, we convert the stochastic problem into a deterministic ODE for the first moment. This procedure overcome Montecarlo type simulations to approximate moments or distributions—simulate many realization of the numerical stochastic process to approximate distributions or moments. Further, under our setting, the regarding spectral approximation assures high precision and order of convergence. Thus we guess that our method would improve the time and save resources of computation. We are preparing another article to confirm this conjectures.

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