Initial conditions stability of a numerical approximation for Kolmogorov equations in infinite dimension

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Abstract

Delgado and Flandoli report in [Infin. Dimens. Anal. Quantum Probab. Relat. Top. **19** (2016), no. 3, 1650020, 37 pp.], a weak-numerical approximation for a parabolic family of infinite-dimensional SPDE. According to this method, we characterize numerical stability for initial conditions. We deduce sufficient conditions related with the Fourier-Hermite spectral decomposition of the Ornstein-Uhlenbeck operator associated with the underlying SPDE. Under these mild conditions, our results assure initial stability for both—the underlying solution and the numerical approximation. We believe this contribution is an invitation to study other numerical stability definitions in the infinite-dimensional setting.

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1 Introduction

Solutions of SPDEs are rarely available, then the use of numerical methods to approximate these solutions are essential. The literature body of stochastic spectral methods identifies two essential families, according to the Karhunen-Loeve expansion and the Wiener-Chaos expansion. However, since the former approach converges slower for non-linear SPDEs, schemes based in the Wiener-Chaos expansion are more convenient, see [20] for further details.

The numerical analysis of SPDEs based on weak approximations, in the probability sense, is a virgin research field. There are just a few works in this direction. Schwab and Süli propose in [18] a variational space-time method to approximate the solution of an infinite-dimensional Kolmogorov-type equation. However, their article lacks numerical experiments.

Our contribution is closed related to [9], where the authors report a numerical method for Kolmogorov equations associated with SPDEs, that is, a scheme based on weak approximations.

Work with efficient and accurate numerical schemes is crucial. In this way, spectral methods play an essential role in obtaining better schemes—under certain conditions, this sort of method is more accurate than finite differences of finite elements and need fewer grid points. Here the adjective "better" would be under accuracy, consistency, stability, and other targets properties.

We aim to explore the ability of the method reported in [9] to preserve the continuity respect to initial conditions. If a given problem satisfies certain regularity conditions, then two of its solution remains closed if its initial function conditions are close. So, we desire a numerical method to reproduce this behavior, and if it is the case, we say that an underlying method is IC-stable.

Our main contribution is the characterization of mild conditions to assure the continuity respect to initial function conditions to a family of SPDEs and the stability of a regarding weak spectral approximation. To the best of our knowledge, this paper is the first to report numerical stability theory for Kolmogorov equations in infinite dimensions.

The stability theory for spectral methods is still under construction and is an activeresearch area. We mention the seminal works of L.N. Trefethen and

M.R. Trummer [19], D. Gottlieb et al. [12] as reference for the deterministic case, and N. Li, J. Fiordilino, and X. Feng, [15] A. Lang, A. Petersson, and A. Thalhammer, [14] for the stochastic version.

After this introduction, in Section 2 we briefly review the Fokker-Plank-Kolmogorov equation associated with SPDEs in a separable Hilbert space. Section 3 shows some preliminary results to prove in Section 4, our main result—conditions to assure stability respect initial conditions. Section 5 confirms and illustrates our results with the numerical simulation of Fisher and Burgers SPDEs. The last section encloses the conclusions and final remarks.

2 Fokker-Planck-Kolmogorov equations for SPDEs in Hilbert spaces

The Fokker-Planck-Kolmogorov (FPK) equation is a partial differential equation describing the time evolution of the probability density function of the particle's velocity under the influence of drag forces irregular forces, and it is a kind of continuity equation for densities. Citing [8] 'parabolic equations on Hilbert spaces appear in mathematical physics to model systems with infinitely many degrees of freedom. Typical examples are provided by spin configurations in statistical mechanics and by crystals in solid-state theory. Infinite-dimensional parabolic equations provide an analytic description of infinite-dimensional diffusion processes in such branches of applied mathematics as population biology, fluid dynamics, and mathematical finance." This kind of equation has been intensely studied in recent years, for instance, [2, 4, 5] and the references therein.

Let $\mathcal{H}$ be a separable infinite-dimensional Hilbert space with inner product $(,)_{\mathcal{H}}$ and norm $\|\cdot\|_{\mathcal{H}}$. Consider the stochastic differential equation in $\mathcal{H}$

$$dX_t = AX_t dt + B(X_t)dt + \sqrt{Q}dW_t, \tag{1}$$

where the operator $A : \mathcal{D}(A) \subset \mathcal{H} \to \mathcal{H}$ is the infinitesimal generator of a strongly continuous semigroup e^{tA} in $\mathcal{H}$, Q is a bounded operator from another Hilbert space $\mathcal{U}$ to $\mathcal{H}$ and $B : \mathcal{D}(B) \subset \mathcal{H} \to \mathcal{H}$ is a nonlinear mapping.

We define

$$u(t,x) = \mathbb{E}\Big[\varphi(X_t^x)\Big], \tag{2}$$

where $\varphi : \mathcal{H} \to \mathbb{R}$ and X_t^x is the solution to (1) with initial conditions $X_0 = x$ with $x \in \mathcal{H}$. Then u satisfies the Kolmogorov equation:

$$\frac{\partial u}{\partial t} = \frac{1}{2} Tr(QD^2 u) + \langle Ax, Du \rangle_{\mathcal{H}} + \langle B(x), Du \rangle_{\mathcal{H}}, \qquad x \in D(A). \tag{3}$$

Several authors have proved results on existence and uniqueness of the solution of the Kolmogorov equations, see for instance Da Prato [5] for a survey, Da Prato-Debussche [6] for the Burgers equation, Barbu-Da Prato [1] for the 2D Navier-Stokes stochastic flow in a channel.

2.1 On the numerical scheme for the Kolmogorov equation

Following [3], in $\mathcal{H}$ we define a Gaussian measure μ with mean zero and nuclear covariance operator Λ with $Tr(\Lambda) < +\infty$ and since $\Lambda : \mathcal{H} \mapsto \mathcal{H}$ is a positive definite, self-adjoint operator then its square-root operator $\Lambda^{1/2}$ is a positive definite, self-adjoint Hilbert-Schmidt operator on $\mathcal{H}$.

Define the inner product $(g,h)_0 := \left(\Lambda^{-1/2} g, \Lambda^{-1/2} h\right)_{\mathcal{H}}$, for $g, h \in \Lambda^{1/2}\mathcal{H}$. Let $\mathcal{H}_0$ denote the Hilbert subspace of $\mathcal{H}$, which is the completion of $\Lambda^{1/2}\mathcal{H}$ with respect to the norm $\|g\|_0 := (g,g)_0^{1/2}$. Then $\mathcal{H}_0$ is dense in $\mathcal{H}$ and the inclusion map $i : \mathcal{H}_0 \hookrightarrow \mathcal{H}$ is compact. The triple $(i, \mathcal{H}_0, \mathcal{H})$ forms an abstract Wiener space.

Let $\mathbb{H} = L^2(\mathcal{H}, \mu)$ denote the Hilbert space of Borel measurable functionals on the probability space with inner product

$$\left[\Phi, \Psi\right]_{\mathbb{H}} := \int_{\mathcal{H}} \Phi(v)\Psi(v)\mu(dv), \qquad \text{for } \Phi, \Psi \in \mathbb{H},$$

and norm $\|\Phi\|_{\mathbb{H}} := \left[\Phi, \Phi\right]_{\mathbb{H}}^{1/2}$. We choose a basis system $\{\varphi_k\}$ for $\mathcal{H}$.

A functional $\Phi : \mathcal{H} \mapsto \mathbb{R}$, is said to be a smooth simple functional (or a cylinder functional) if there exists a C^∞-function ϕ on $\mathbb{R}^n$ and n-continuous linear functional $l_1, \ldots, l_n$ on $\mathcal{H}$ such that for $h \in \mathcal{H}$

$$\Phi(h) = \phi(h_1, \ldots, h_n) \quad \text{where} \qquad h_i = l_i(h), \quad i = 1, \ldots, n. \tag{4}$$

The set of all such functionals will be denoted by $\mathcal{S}(\mathbb{H})$. Denote by $P_k(x)$ the Hermite polynomial of degree k taking values in $\mathbb{R}$. Then, $P_k(x)$ is given by the following formula

$$P_k(x) = \frac{(-1)^k}{(k!)^{1/2}} e^{\frac{x^2}{2}} \frac{d^k}{dx^k} e^{-\frac{x^2}{2}}$$

with $P_0 = 1$. It is well-known that $\{P_k(\cdot)\}_{k\in\mathbb{N}}$ is a complete orthonormal system for $L^2(\mathbb{R}, \mu_1(dx))$ with $\mu_1(dx) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$. Define the set of infinite multi-index as

$$\mathcal{J} = \Big\{ \boldsymbol{\alpha} = (\alpha_i, i \geq 1) \quad \Big| \quad \alpha_i \in \mathbb{N} \cup \{0\}, \quad |\boldsymbol{\alpha}| := \sum_{i=1}^{\infty} \alpha_i < +\infty \Big\}.$$

For $\boldsymbol{n} \in \mathcal{J}$ define the *Hermite polynomial functionals* on $\mathcal{H}$ by

$$H_{\boldsymbol{n}}(h) = \prod_{i=1}^{\infty} P_{n_i}(l_i(h)), \quad h \in \mathcal{H}_0, \quad \boldsymbol{n} \in \mathcal{J}, \tag{5}$$

and where $l_i(h) = \langle h, \Lambda^{-1/2}\varphi_i\rangle_{\mathcal{H}}, \quad i = 1, 2, \ldots$ where $P_n(\xi)$ is the usual Hermite polynomial for $\xi \in \mathbb{R}$ and $n \in \mathbb{N}$.

Remark 1. Notice that $l_i(h)$ is defined only for $h \in \mathcal{H}_0$. However, regarding h as a μ-random variable in $\mathcal{H}$, we have $\mathbb{E}\Big(l_i(h)\Big) = \|\varphi_i\|^2 = 1$ and then $l_k(h)$ can be defined μ-a.e. $h \in \mathcal{H}$, similar to defining a stochastic integral.

It is possible to identify the Hermite polynomial functionals defined in (5), for $h \in \mathcal{H}_0$, as a deterministic version of the Wick polynomials defined on the canonical Wiener space (for further details see [13] for instance).

We need the following result (See Theorems 9.1.5 and 9.1.7 in Da Prato-Zabczyk [8] or Lemma 3.1 in Chapter 9 from Chow [3]).

Lemma 2.1. *For $h \in \mathcal{H}$ let $l_i(h) = \langle h, \Lambda^{-1/2}\varphi_i\rangle_{\mathcal{H}} \quad, i = 1, 2, \ldots$. The set $\{H_{\boldsymbol{n}}\}$ of all Hermite polynomials on $\mathcal{H}$ forms a complete orthonormal system for $\mathbb{H}$. Hence the set of all functionals are dense in $\mathbb{H}$. Moreover, we have the direct sum decomposition: $\mathbb{H} = \bigoplus_{j=0}^{\infty} K_j$, where K_j is the subspace of $\mathbb{H}$ spanned by $\{H_{\boldsymbol{n}} : |\boldsymbol{n}| = j\}$.*

Remark 2. Using this lemma, we write the solution of the Kolmogorov equation (3) as

$$u(t, x) = \sum_{\boldsymbol{n}\in\mathcal{J}} u_{\boldsymbol{n}}(t) H_{\boldsymbol{n}}(x), \qquad x \in \mathcal{H}, \quad t \in [0, T], \tag{6}$$

where $u_{\boldsymbol{n}} : [0, T] \mapsto \mathbb{R}$ and $H_{\boldsymbol{n}}(x)$ are the Hermite functionals. This spectral decomposition is used in [9] to construct a numerical scheme for approximate the solution of the Kolmogorov equation. The aim of this paper is to prove the stability with respect to the initial condition of this scheme.

3 Some preliminary results

Let Φ be a smooth simple functional given by (4). Then the Fréchet derivatives, $D\Phi = \Phi'$ and $D_2\Phi = \Phi''$ in $\mathcal{H}$ can be computed as follows:

$$(D\Phi(h), v) = \sum_{k=1}^{n} \Big[\partial_k \phi(h_1, \dots, h_n)\Big] l_k(v)$$
$$(D^2\Phi(h), v) = \sum_{j,k=1}^{n} \Big[\partial_j \partial_k \phi(h_1, \dots, h_n)\Big] l_j(v) l_k(v),$$

for any $u, v \in \mathcal{H}$, where $\partial_k \phi = \frac{\partial}{\partial h_k}\phi$. Similarly, for $m > 2$, $D^m\Phi(h)$ is a $m-$linear form on $\mathcal{H}^m$ with inner product $(\cdot,\cdot)_m$. We have $[D^m\Phi(h)](v_1, \cdots, v_m) = (D^m\Phi(h), v_1 \otimes \cdots \otimes v_m)_m$, for $h, v_1, \dots, v_m \in \mathcal{H}$. Consider the following linear stochastic equation

$$du_t = Au_t dt + dW_t, \quad u_0 = h \in \mathcal{H}. \tag{7}$$

Where $A : \mathcal{D}(A) \subset \mathcal{H} \to \mathcal{H}$ is the infinitesimal generator of a strongly continuous semigroup e^{tA} in $\mathcal{H}$. W_t is a Q-Wiener process in $\mathcal{H}$. Chow in [3, Lemma 9.4.1] has shown the following result.

Lemma 3.1. *Suppose that A and Q satisfy the following:*

1. *$A : \mathcal{D}(A) \subset \mathcal{H} \to \mathcal{H}$ is self-adjoint and there is $\beta > 0$ such that*

$$\langle Av, v\rangle_{\mathcal{H}} \leq -\beta \|v\|_{\mathcal{H}} \quad \forall v \in \mathcal{H}.$$

2. *A commutes with Q in $\mathcal{D}(A) \subset \mathcal{H}$.*

Then (7) *has a unique invariant measure μ which is a Gaussian measure on $\mathcal{H}$ with zero mean and covariance operator $\Lambda = \frac{1}{2}Q(-A)^{-1} = \frac{1}{2}(-A)^{-1}Q$.*

Suppose that A and Q have the same eigenfunctions e_k with eigenvalues λ_k and ρ_k respectively. It is well-know (See for instance Da Prato and Zabczyk [8]) that the solution of (7) is a time-homogeneous Markov process with transition operator P_t defined for $\Phi \in \mathbb{H}$ given by

$$(P_t\Phi)(h) = \int_{\mathcal{H}} \Phi(v)\mu_t^h(dv) = \mathbb{E}\Big[\Phi(u_t^h)\Big]. \tag{8}$$

Let $\Phi \in \mathcal{S}(\mathbb{H})$ be a smooth simple functional. By setting $\varphi_k = e_k$ in (4), it takes the form $\Phi(h) = \phi(l_1(h), \cdots, l_n(h))$, where $l_k(h) = (h, \Lambda^{-1/2}e_k)$. Define a differential operator A_0 on $\mathcal{S}(\mathbb{H})$ by

$$\mathcal{A}_0\Phi(v) = \tfrac{1}{2}Tr[RD^2\Phi(v)] + \langle Av, D\Phi(v)\rangle, \qquad v \in H, \tag{9}$$

which is well defined, since $D\Phi \in D(A)$ and $\langle Av, D\Phi(v)\rangle = (v, AD\Phi(v))_{\mathcal{H}}$.

The following results have been proved in [3].

Lemma 3.2. *Let P_t be the transition operator as defined by (7). Then the following properties hold:*

1. *$P_t : \mathcal{S}(\mathbb{H}) \to \mathcal{S}(\mathbb{H})$ for $t \geq 0$.*
2. *$\{P_t, t \geq 0\}$ is a strongly continuous semigroup on $\mathcal{S}(\mathbb{H})$ so that, for any $\Phi \in \mathcal{S}(\mathbb{H})$, we have $P_0 = I$, $P_{t+s}\Phi = P_tP_s\Phi$, for all $t, s \geq 0$, and $\lim_{t\downarrow 0} P_t\Phi = \Phi$.*
3. *$\mathcal{A}_0$ is the infinitesimal generator of P_t so that, for each $\Phi \in \mathcal{S}(\mathbb{H})$,*

$$\lim_{t\downarrow 0} \tfrac{1}{t}\Big(P_t - I\Big)\Phi = \mathcal{A}_0\Phi.$$

□

Lemma 3.3. *Let $H_n(h)$ be a Hermite polynomial functional given by (5). Then the following hold:*

$$\mathcal{A}_0 H_{\mathbf{n}}(h) = -\lambda_{\mathbf{n}} H_{\mathbf{n}}(h), \tag{10}$$

$$P_t H_{\mathbf{n}}(h) = \exp\{-\lambda_{\mathbf{n}} t\} H_{\mathbf{n}}(h), \tag{11}$$

for any $\mathbf{n} \in \mathcal{J}$ and $h \in H$, where $\lambda_{\mathbf{n}} = \sum_{i=1}^{\infty} n_i\lambda_i$.

The following Theorem is a Green formula that we will need forward. Its proof can be seen, for instance, in [3, Thm. 3.3, Ch. 9].

Theorem 3.4. *Let $\Phi \in \mathcal{S}(\mathbb{H})$ be a smooth simple functional and let $\mu \sim N(0, \Lambda)$ be a Gaussian measure in $\mathcal{H}$. Then, for any $g, h \in \mathcal{H}$ the following formula holds*

$$\int_{\mathcal{H}} (\Lambda h, D\Phi(v))_{\mathcal{H}} \mu(dv) = \int_{\mathcal{H}} (v, h)_{\mathcal{H}} \Phi(v)\mu(dv)\ . \tag{12}$$

Lemma 3.5. *Assume the conditions for Lemma 3.3 hold. Then, for any* $\Phi, \Psi \in \mathcal{S}(\mathbb{H})$*, the following Green's formula holds:*

$$\int_{\mathcal{H}} (\mathcal{A}_0 \Phi)\Psi d\mu = \int_{\mathcal{H}} \Phi(\mathcal{A}_0 \Psi) d\mu = -\frac{1}{2} \int_{\mathcal{H}} (QD\Phi, D\Psi) d\mu \ . \tag{13}$$

By Lemma 2.1, for $\Phi \in \mathbb{H}$, it can be represented as

$$\Phi(v) = \sum_{n=0}^{\infty} \phi_{\mathbf{n}} H_{\mathbf{n}}(v), \tag{14}$$

where $n = |\mathbf{n}|$ and $\mathbf{n} \in \mathcal{J}$. Notice that we can think in $\mathbf{n}$ as a vector of r dimension, i.e. $\mathbf{n} = (n_1, \ldots, n_r)$. Let $\alpha_{\mathbf{n}} = \alpha_{n_1} \cdots \alpha_{n_r}$ be a sequence of positive numbers with $\alpha_{\mathbf{n}} > 0$, such that $\alpha_{\mathbf{n}} \to \infty$ as $n \to \infty$. Define

$$|||\Phi|||_{k,\alpha} = \left[\sum_{\mathbf{n}} \left(1 + \alpha_{\mathbf{n}}\right)^k |\phi_n|^2 \right]^{1/2},$$
$$|||\Phi|||_{0,\alpha} = |||\Phi||| = \left[\sum_{\mathbf{n}} |\phi_n|^2 \right]^{1/2},$$

which is $L^2(\mu)$-norm of Φ. For the given sequence $\alpha = \{\alpha_n\}$, let $\mathbb{H}_{k,\alpha}$ denote the completion of $\mathcal{S}(\mathbb{H})$ with respect to the norm $||| \cdot |||_{k,\alpha}$. Then $\mathbb{H}_{k,\alpha}$ is called a Gauss–Sobolev space of order k with parameter α. The dual space of $\mathbb{H}_{k,\alpha}$ is $\mathbb{H}_{-k,\alpha}$. From now on, we will fix the sequence $\alpha_{\mathbf{n}} = \lambda_{\mathbf{n}}$, where $\lambda_{\mathbf{n}}$ is given in Lemma 3.3. We shall simply denote $\mathbb{H}_{k,\alpha}$ by $\mathbb{H}_k$ and $|||\Phi|||_{k,\alpha}$ by $|||\Phi|||_k$.

The following results ensure the existence of an extension for the operator $\mathcal{A}_0$ to a domain containing $\mathbb{H}_2$. Their proofs can be found in [3] for instance.

Theorem 3.6. *Let the conditions on A and Q in Lemma 3.1 hold. Then* $P_t : \mathbb{H} \to \mathbb{H}$*, for* $t \geq 0$*, is a contraction semigroup with the infinitesimal generator* $\tilde{A}$*. The domain of* $\tilde{A}$ *contains* $\mathbb{H}_2$ *and we have* $\tilde{A} = \mathcal{A}_0$ *in* $\mathcal{S}(\mathbb{H})$*.*

Theorem 3.7. *Let the conditions of Theorem 3.6 hold, then the differential operator* $\mathcal{A}_0$ *defined by* (9) *in* $\mathcal{S}(\mathbb{H})$ *can be extended to be a self-adjoint linear operator* A *in* $\mathbb{H}$ *with domain* $\mathbb{H}_2$*.*

Since both $\tilde{A}$ and A are extensions of $\mathcal{A}_0$ to a domain containing $\mathbb{H}_2$, they must coincide there.

Given the Gauss-Sobolev space $\mathbb{H}_k$ with norm $|||\cdot|||_k$ we denote its dual space by $\mathbb{H}_{-k}$ with norm $|||\cdot|||_{-k}$. Thus, we have the inclusions, $\mathbb{H}_k \subset \mathbb{H} \subset \mathbb{H}_k$. We denote the duality between $\mathbb{H}_k$ and $\mathbb{H}_{-k}$ by $\langle\langle \Psi, \Phi \rangle\rangle_k, \quad \Phi \in \mathbb{H}_k, \quad \Psi \in \mathbb{H}_{-k}$. We also set $\mathbb{H}_0 = \mathbb{H}$, with $|||\cdot|||_0 = |||\cdot|||$ and $\langle\langle \cdot, \cdot \rangle\rangle_1 = \langle\langle \cdot, \cdot \rangle\rangle$, $\langle\langle \cdot, \cdot \rangle\rangle_0 = [\cdot, \cdot]$. Consider the following Kolmogorov equation,

$$\begin{aligned} \frac{\partial}{\partial t}\Psi(v,t) &= \mathcal{A}\Psi(v,t) + \langle B(v), D\Psi(v,t)\rangle_{\mathcal{H}}, \quad \text{a.e. } v \in \mathbb{H}_2, \\ \Psi(v,0) &= \phi(v) \ , \end{aligned}$$

where, as defined in Theorem 3.6, $\mathcal{A} : \mathbb{H}_2 \to \mathbb{H}$ is given by

$$\mathcal{A}\Phi = \tfrac{1}{2} Tr[RD^2\Phi(v)] + \langle Av, D\Phi(v)\rangle \ . \tag{15}$$

Hypothesis on B will be specified latter. For now, we will consider that it is a locally Lipschitz function. The additional term $\langle B(v), D\Psi(v,t)\rangle_{\mathcal{H}}$ is defined μ-a.e. $v \in \mathbb{H}_2$. We will allow the initial datum ϕ will be in $\mathbb{H}$.

We will study a mild solution of the equation (15). Let $\lambda > 0$ be a parameter. By changing Ψ to $e^{\lambda t}\Psi$ in (15) we get the following equation:

$$\begin{aligned} \frac{\partial}{\partial t}\Psi(v,t) &= \mathcal{A}_\lambda\Psi(v,t) + \langle B(v), D\Psi(v,t)\rangle_{\mathcal{H}}, \qquad \text{a.e. } v \in \mathbb{H}_2, \\ \Psi(v,0) &= \phi(v) \ , \end{aligned} \tag{16}$$

where $\mathcal{A}_\lambda = \mathcal{A} - \lambda I$, with I the identity operator in $\mathbb{H}$. Clearly, the problems (15) and (16) are equivalent, as far for the existence and uniqueness questions are concerned. We will work on the problem (16).

Denote by P_t the semigroup with infinitesimal generator $\mathcal{A}_\lambda$. The existence of P_t is ensured by the Theorem 3.6.Then, we can rewrite the equation (16) in an integral form by using the semigroup P_t

$$\Psi(v,t) = e^{-\lambda t}(P_t\phi)(v) + \int_0^t e^{-\lambda(t-s)}[P_{t-s}(B, D\Psi_s)](v)ds, \tag{17}$$

where we denote $\phi = \phi(\cdot)$ and $\Psi_s = \Psi(\cdot, s)$. Chow [3] had proved the following lemma.

Lemma 3.8. *Let $\Psi \in L^2((0,T);\mathbb{H})$ for some $T > 0$. Then, for any $\lambda > 0$ there exists $C_\lambda > 0$ such that*

$$|||\int_0^t e^{-\lambda(t-s)} P_{t-s}\Psi_s ds|||^2 \le C_\lambda \int_0^T |||\Psi_s|||_{-1}^2 ds, \qquad 0 < t \le T \ . \tag{18}$$

We now prove the following theorem on existence and uniqueness of a mild solution to (16).

Theorem 3.9. *Suppose that* $B : \mathcal{H} \to \mathcal{H}_0$ *satisfies* $(B, D\Phi) \in L^2((0,T);\mathbb{H})$ *for any* $\Phi \in \mathbb{H}$ *and*

$$\sup_{v\in\mathcal{H}} ||\Lambda^{-1/2}B(v)||_{\mathcal{H}} < +\infty.$$

Then, B satisfies

$$|||\Big(B(v), D\Phi(v)\Big)|||_{-1}^2 \leq C|||\Phi(v)|||^2 \quad \text{for any } \Phi \in \mathbb{H}, \quad v \in \mathbb{H}_2, \tag{19}$$

for some $C > 0$*. Moreover, for* $\Phi \in \mathbb{H}$*, the initial-value problem* (16) *has a unique mild solution* $\Psi \in C((0,T);\mathbb{H})$*.*

For the part of the existence and uniqueness of the solution we will adapt the proof of the Theorem 5.2 in Chapter 9 from [3].

Proof. First we will prove (19). We have

$$|||\Big(B(v), D\Phi(v)\Big)|||_{-1}^2 = \sum_{\mathbf{n}} (1+\lambda_{\mathbf{n}})^{-1}|\phi_n|^2,$$

with

$$\phi_n = \Big((B(v), D\Phi(V))_{\mathcal{H}}, H_{\mathbf{n}}(v)\Big)_{\mathbb{H}} = \int_{\mathcal{H}} (B(v), D\Phi(v))_{\mathcal{H}} H_{\mathbf{n}}(v)\mu(dv). \tag{20}$$

By the Theorem 3.4, in particular (12), we have

$$\int_{\mathcal{H}} (\Lambda h, D\Phi(v))_{\mathcal{H}}\mu(dv) = \int_{\mathcal{H}} (v,h)_{\mathcal{H}}\Phi(v)\mu(dv),$$

for all $\Phi \in \mathcal{S}(\mathbb{H})$, $g, h \in \mathcal{H}$ and $\mu \sim N(0,\Lambda)$. Then, in particular, in each direction $H_{\mathbf{n}}$ this formula is still true, so we have

$$\int_{\mathcal{H}} (\Lambda h, D\Phi(v))_{\mathcal{H}} H_{\mathbf{n}}(v)\mu(dv) = \int_{\mathcal{H}} (v,h)_{\mathcal{H}}\Phi(v)H_{\mathbf{n}}(v)\mu(dv) \ .$$

Then, applying this last equality to (20) we get

$$\begin{aligned}
\phi_n &= \int_{\mathcal{H}} \Big(\Lambda[\Lambda^{-1}B(v)], D\Phi(v)\Big)_{\mathcal{H}} H_{\mathbf{n}}(v)\mu(dv) \\
&= \int_{\mathcal{H}} \Big(\Lambda^{-1}B(v), v\Big)_{\mathcal{H}} \Phi(v)H_{\mathbf{n}}(v)\mu(dv) \\
&= \int_{\mathcal{H}} \Big(\Lambda^{-1/2}B(v), \Lambda^{1/2}v\Big)_{\mathcal{H}} \Phi(v)H_{\mathbf{n}}(v)\mu(dv) \ .
\end{aligned}$$

Thus,

$$
\begin{aligned}
|\phi_n|^2 &= \left| \int_{\mathcal{H}} \Big(\Lambda^{-1/2}B(v), \Lambda^{1/2}v\Big)_{\mathcal{H}} \Phi(v) H_{\mathbf{n}}(v)\mu(dv) \right|^2 \\
&\leq \int_{\mathcal{H}} \Big|\Big(\Lambda^{-1/2}B(v), \Lambda^{1/2}v\Big)_{\mathcal{H}}\Big|^2 \Big|H_{\mathbf{n}}(v)\Big|^2 \mu(dv) \int_{\mathcal{H}} \Big|\Phi(v)\Big|^2 \mu(dv) \ .
\end{aligned}
\tag{21}
$$

We now focus on the first integral. Let I_1 be the first integral of (21). Then,

$$
\begin{aligned}
I_1 &\leq \int_{\mathcal{H}} \Big|\Big|\Lambda^{-1/2}B(v)\Big|\Big|_{\mathcal{H}}^2 \Big|\Big|\Lambda^{1/2}v\Big|\Big|_{\mathcal{H}}^2 \Big|H_{\mathbf{n}}(v)\Big|^2 \mu(dv) \\
&\leq \sup_{v\in\mathcal{H}} \Big|\Big|\Lambda^{-1/2}B(v)\Big|\Big|_{\mathcal{H}}^2 \int_{\mathcal{H}} \Big|\Big|\Lambda^{1/2}v\Big|\Big|_{\mathcal{H}}^2 \Big|H_{\mathbf{n}}(v)\Big|^2 \mu(dv) \\
&\leq C \int_{\mathcal{H}} \Big|\Big|v\Big|\Big|_{\mathcal{H}}^2 \Big|H_{\mathbf{n}}(v)\Big|^2 \mu(dv) \\
&\leq C \ .
\end{aligned}
$$

The last inequality follows by using proposition 9.2.10 in page 198 from [7]. Then, by using this bound on (21) we have.

$$
\begin{aligned}
|\phi_n|^2 &\leq C \int_{\mathcal{H}} \Big|\Phi(v)\Big|^2 \mu(dv) \\
&\leq C|||\Phi(v)|||^2.
\end{aligned}
$$

Thus,

$$
|||\Big(B(v), D\Phi(v)\Big)|||_{-1}^2 \leq C|||\Phi(v)|||^2 \sum_{\mathbf{n}} (1+\lambda_{\mathbf{n}})^{-1} \leq C|||\Phi(v)|||^2,
$$

which proves (19).

We now prove the existence and uniqueness of a solution to the initial-value problem (16). Let $\mathbb{X}_T$ denote the Banach space $\mathcal{C}([0,T];\mathbb{H})$ with the sup-norm

$$
|||\Psi|||_T := \sup_{0\leq t\leq T} |||\Psi||| \ .
$$

In $\mathbb{X}_T$ define the linear operator $\mathbb{Q}$ as

$$
\mathbb{Q}\Psi = e^{-\lambda t} P_t \Phi + \int_0^t e^{-\lambda(t-s)} P_{t-s}(B, D\Psi_s) ds, \quad \text{for any } \Psi \in \mathbb{X}_T.
$$

By Theorem 3.6 P_t is a contraction semigroup, then using this fact and Lemma 3.8 we have

$$\begin{aligned}|||\mathbb{Q}\Psi|||^2 &\leq 2\left[|||e^{-\lambda t}P_t\Phi|||^2 + |||\int_0^t e^{-\lambda(t-s)}P_{t-s}(B, D\Psi_s)ds|||^2\right]\\ &\leq 2\Big[|||\Phi|||^2 + C_\lambda\int_0^t |||(B, D\Psi_s)|||_{-1}^2 ds\Big]\\ &\leq 2|||\Phi|||^2 + C_1\int_0^t |||\Psi_s|||^2 ds,\end{aligned}$$

for some $C_1 > 0$. Hence, $|||\mathbb{Q}\Psi|||_T \leq C(1 + |||\Psi|||_T)$, with $C = C(\Phi, \lambda, T)$. Then, the map $\mathbb{Q} : \mathbb{X}_T \to \mathbb{X}_T$ is well defined. We now show that is a contraction for a small t. Let $\Psi, \Psi' \in \mathbb{X}_T$. Then

$$\begin{aligned}|||\mathbb{Q}\Psi - \mathbb{Q}\Psi'|||^2 &= |||\int_0^t e^{-\lambda(t-s)}P_{t-s}\Big[(B, D\Psi_s) - (B, D\Psi'_s)\Big]ds|||^2\\ &\leq C_\lambda\int_0^t |||(B, D\Psi_s - D\Psi')|||_{-1}^2 ds\\ &\leq C_2\int_0^t |||\Psi_s - \Psi'|||^2 ds,\end{aligned}$$

for some $C_2 > 0$.

It follows that $|||\mathbb{Q}\Psi - \mathbb{Q}\Psi'|||_T \leq \sqrt{C_2 T}|||\Psi_s - \Psi'|||_T$. Then, for small T, $\mathbb{Q}$ is a contraction on $\mathbb{X}_T$. Hence the Cauchy problem (16) has a unique mild solution. □

4 Numerical stability respect to initial conditions

In this section, we prove the continuity with respect to the initial conditions for a numerical approximation of the Kolmogorov equation associated with an SPDE. Here we understand that a numerical scheme is stable respect to initial conditions if this method reproduces the same behavior when the continuous problem satisfies continuity respect initial conditions.

We first prove a theorem on the dependence on initial conditions for the mild solution of (16).

Theorem 4.1. *Suppose that $B : \mathcal{H} \to \mathcal{H}_0$ satisfies $(B, D\Phi) \in L^2((0,T); \mathbb{H})$ for any $\Phi \in \mathbb{H}$ and*

$$\sup_{v \in \mathcal{H}} ||\Lambda^{-1/2} B(v)||_{\mathcal{H}} < +\infty. \tag{22}$$

Then, the unique mild solution $\Psi \in C((0,T); \mathbb{H})$ for (16) depends continuously on the initial conditions.

Proof. We know, with the assumption (22), that the existence of a unique mild solution for (16) is guaranteed by Theorem 3.9. We will denote by Ψ_t^φ its mild solution at time t with initial condition φ:

$$\Psi_t^\varphi = e^{-\lambda t} P_t \varphi + \int_0^t e^{-\lambda(t-s)} P_{t-s}(B, D\Psi_s^\varphi) ds \ .$$

Then,

$$\begin{aligned}\Psi_t^\varphi - \Phi_t^\psi &= e^{-\lambda t} P_t \varphi - e^{-\lambda t} P_t \psi + \int_0^t e^{-\lambda(t-s)} P_{t-s}(B, D\Psi_s^\varphi - D\Phi_s^\psi) ds \\ &= e^{-\lambda t} P_t(\varphi - \psi) + \int_0^t e^{-\lambda(t-s)} P_{t-s}(B, D\Psi_s^\varphi - D\Phi_s^\psi) ds.\end{aligned}$$

From this expression we get

$$\begin{aligned}|||\Psi_t^\varphi - \Phi_t^\psi|||^2 &\le |||e^{-\lambda t} P_t(\varphi - \psi)|||^2 + ||| \int_0^t e^{-\lambda(t-s)} P_{t-s}(B, D\Psi_s^\varphi - D\Phi_s^\psi)|||^2 ds \\ &\le |||\varphi - \psi|||^2 + C_\lambda \int_0^t |||(B, D\Psi_s^\varphi - D\Phi_s^\psi)|||_{-1}^2 ds \\ &\le |||\varphi - \psi|||^2 + C_2 \int_0^t |||\Psi_s^\varphi - \Phi_s^\psi|||^2 ds.\end{aligned}$$

Thus, by Gronwall's inequality we obtain

$$|||\Psi_t^\varphi - \Phi_t^\psi|||^2 \le \exp(C_2 t) |||\varphi - \psi|||^2, \tag{23}$$

which implies, $|||\Psi_t^\varphi - \Phi_t^\psi||| \le \exp(Ct)|||\varphi - \psi|||$. This completes the proof. □

We now will study the stability of the numerical scheme. As before, we use Lemma 2.1 to write the solution Ψ_t^φ as in a Fourier-Hermite decomposition:

$$\Psi_t^\varphi = \sum_{\boldsymbol{n} \in \mathcal{J}} u_{\boldsymbol{n}}(t) H_{\boldsymbol{n}}(x), \qquad x \in \mathcal{H}, \quad t \in [0,T] \ . \tag{24}$$

Now, we prove an auxiliary result.

Lemma 4.2. *Set $\{P_k(\xi)\}_{k\in\mathbb{N}}$ the family of normalized Hermite polynomials in $\mathbb{R}$. For every $k \in \mathbb{N}$ and $\xi, \eta \in \mathbb{R}$ such that $\eta < \xi$ we have that*

$$P_k(\xi) - P_k(\eta) = C(k)Pe_{k+1}(\gamma) \cdot (\xi - \eta), \tag{25}$$

where $\gamma \in (\eta, \xi)$ and $C(k) = \frac{(-1)^k}{(k+1)(k!)^{1/2}}$. Moreover, $Pe_k(x)$ is the unnormalized Hermite polynomial of k degree.

Proof. We know that $P_k(\xi) = \frac{(-1)^k}{(k!)^{1/2}} e^{\xi^2/2} \frac{d}{d\xi^k} e^{-\xi^2/2}$. Set $c(k) = (-1)^k (k!)^{-1/2}$, then

$$\begin{aligned} P_k(\xi) - P_k(\eta) &= c(k)\Big[e^{\xi^2/2}\frac{d}{d\xi^k}e^{-\xi^2/2} - e^{\eta^2/2}\frac{d}{d\eta^k}e^{-\eta^2/2}\Big] \\ &= c(k)\left[e^{x^2/2}\frac{d}{dx^k}e^{-x^2/2}\Big|_{x=\eta}^{\xi}\right] \\ &= c(k)\int_\eta^\xi F_k(x)dx, \end{aligned}$$

where F_k is a continuous function such that $F_k'(x) = e^{x^2/2}\frac{d}{dx^k}e^{-x^2/2}$. In fact, denoting by $Pe_k(x)$ the unnormalized Hermite polynomial of k degree, results

$$F_k'(x) = e^{x^2/2}\frac{d}{dx^k}e^{-x^2/2} = Pe_k(x),$$

and since the Hermite polynomials constitute an Appell sequence we have that

$$F_k'(x) = Pe_k(x) = \frac{1}{k+1}Pe_{k+1}'(x),$$

which implies that $F_k(x) = \frac{1}{k+1}Pe_{k+1}(x)$. Now, since $F_k(x)$ is a continuous function, then there exists $\gamma \in (\eta, \xi)$ such that

$$\int_\eta^\xi F_k(x)dx = F_k(\gamma)\cdot(\xi - \eta).$$

All these implies that $P_k(\xi) - P_k(\eta) = c(k)F_k(\gamma)\cdot(\xi - \eta)$. From this expression the lemma follows immediately. □

We will use some technical results on the SPDE to prove the following result—the main result of this work.

Theorem 4.3. *Assume that the eigenvalues of* Λ*, satisfies that for every* $k \in \mathbb{N}$*,* $\lambda_k < \lambda_{k+1} \to \infty$*. Assume that the functional* φ *is Lipschitz. Then, the numeric approximation* Ψ_t^{φ} *(given by (24)) to the solution of the Kolmogorov equation* $\Psi \in C((0,T);\mathbb{H})$ *depends continuously on the initial conditions.*

Proof. Let $x, y \in H$ be two different initial values. We want to estimate $\Psi_t^x - \Psi_t^y$. By definition,

$$\Psi_t^x = \sum_{\bar{n}\in\mathcal{J}} u_{\bar{n}}^x(t) H_{\bar{n}}(x) \ . \tag{26}$$

Thus,

$$\begin{aligned} \Psi_t^x - \Psi_t^y &= \sum_{\bar{n}\in\mathcal{J}} u_{\bar{n}}^x(t) H_{\bar{n}}(x) - \sum_{\bar{n}\in\mathcal{J}} u_{\bar{n}}^y(t) H_{\bar{n}}(y) \\ &= \sum_{\bar{n}\in\mathcal{J}} \Big[u_{\bar{n}}^x(t) - u_{\bar{n}}^y(t)\Big] H_{\bar{n}}(x) + \sum_{\bar{n}\in\mathcal{J}} u_{\bar{n}}^y(t) \Big[H_{\bar{n}}(x) - H_{\bar{n}}(y)\Big]. \end{aligned} \tag{27}$$

We focus on the first term in (27). From the definition of the initial condition we obtain the following expression for the time-dependent coefficient

$$u_{\bar{n}}^x(t) = \int_{\mathcal{H}} H_{\bar{n}}(x) \mathbb{E}\Big[\varphi(X_t^x)\Big] \mu(dx) \ .$$

From this we get

$$\begin{aligned} u_{\bar{n}}^x(t) - u_{\bar{n}}^y(t) &= \int_{\mathcal{H}} H_{\bar{n}}(x) \mathbb{E}\Big[\varphi(X_t^x)\Big] \mu(dx) - \int_{\mathcal{H}} H_{\bar{n}}(y) \mathbb{E}\Big[\varphi(X_t^y)\Big] \mu(dy) \\ &= \int_{\mathcal{H}\times\mathcal{H}} H_{\bar{n}}(x) \mathbb{E}\Big[\varphi(X_t^x)\Big] \mu(dx)\mu(dy) \\ &\quad - \int_{\mathcal{H}\times\mathcal{H}} H_{\bar{n}}(y) \mathbb{E}\Big[\varphi(X_t^y)\Big] \mu(dx)\mu(dy) \\ &= \int_{\mathcal{H}\times\mathcal{H}} H_{\bar{n}}(x) \Big(\mathbb{E}\Big[\varphi(X_t^x)\Big] - \mathbb{E}\Big[\varphi(X_t^y)\Big]\Big) \mu(dx)\mu(dy) \\ &\quad + \int_{\mathcal{H}\times\mathcal{H}} \Big(H_{\bar{n}}(x) - H_{\bar{n}}(y)\Big) \mathbb{E}\Big[\varphi(X_t^y)\Big] \mu(dx)\mu(dy) \ . \end{aligned}$$

Then, by the Cauchy-Schwartz inequality, we obtain

$$
\begin{aligned}
|u_{\bar n}^x(t) - u_{\bar n}^y(t)|^2 \leq & \left| \int_{\mathcal{H}\times\mathcal{H}} H_{\bar n}(x) \left(\mathbb{E}\left[\varphi(X_t^x)\right] - \mathbb{E}\left[\varphi(X_t^y)\right] \right) \mu(dx)\mu(dy) \right|^2 \\
& + \left| \int_{\mathcal{H}\times\mathcal{H}} \left(H_{\bar n}(x) - H_{\bar n}(y) \right) \mathbb{E}\left[\varphi(X_t^y)\right] \mu(dx)\mu(dy) \right|^2 \\
\leq & \int_{\mathcal{H}\times\mathcal{H}} H_{\bar n}^2(x))\mu(dx)\mu(dy) \\
& \times \int_{\mathcal{H}\times\mathcal{H}} \left| \mathbb{E}\left[\varphi(X_t^x)\right] - \mathbb{E}\left[\varphi(X_t^y)\right] \right|^2 \mu(dx)\mu(dy) \\
& + \int_{\mathcal{H}\times\mathcal{H}} \mathbb{E}^2\left[\varphi(X_t^y)\right] \mu(dx)\mu(dy) \qquad (28) \\
& \times \int_{\mathcal{H}\times\mathcal{H}} \left| H_{\bar n}(x) - H_{\bar n}(y) \right|^2 \mu(dx)\mu(dy) \\
= & \int_{\mathcal{H}\times\mathcal{H}} \left| \mathbb{E}\left[\varphi(X_t^x)\right] - \mathbb{E}\left[\varphi(X_t^y)\right] \right|^2 \mu(dx)\mu(dy) \\
& + \int_{\mathcal{H}\times\mathcal{H}} \mathbb{E}^2\left[\varphi(X_t^y)\right] \mu(dx)\mu(dy) \\
& \times \int_{\mathcal{H}\times\mathcal{H}} \left| H_{\bar n}(x) - H_{\bar n}(y) \right|^2 \mu(dx)\mu(dy) \ .
\end{aligned}
$$

We now estimate the norm of the expression (27) with the help of (28).

$$
\begin{aligned}
\|\Psi_t^x - \Psi_t^y\|^2_{\left(L^2(\mathcal{H},\mu)\right)^2} &= \int_{\mathcal{H}\times\mathcal{H}} |\Psi_t^x - \Psi_t^y|^2 \mu(dx)\mu(dy) \\
&\leq \int_{\mathcal{H}\times\mathcal{H}} \Big| \sum_{\bar{n}\in\mathcal{J}} \Big[u_{\bar{n}}^x(t) - u_{\bar{n}}^y(t)\Big] H_{\bar{n}}(x)\Big|^2 \mu(dx)\mu(dy) \\
&\quad + \int_{\mathcal{H}\times\mathcal{H}} \Big| \sum_{\bar{n}\in\mathcal{J}} u_{\bar{n}}^y(t)\Big[H_{\bar{n}}(x) - H_{\bar{n}}(y)\Big]\Big|^2 \mu(dx)\mu(dy) \\
&\leq \int_{\mathcal{H}\times\mathcal{H}} \sum_{\bar{n}\in\mathcal{J}} \Big|u_{\bar{n}}^x(t) - u_{\bar{n}}^y(t)\Big|^2 H_{\bar{n}}^2(x)\mu(dx)\mu(dy) \\
&\quad + \int_{\mathcal{H}\times\mathcal{H}} \sum_{\bar{n}\in\mathcal{J}} \Big[u_{\bar{n}}^y(t)\Big]^2 \sum_{\bar{n}\in\mathcal{J}} \Big|H_{\bar{n}}(x) - H_{\bar{n}}(y)\Big|^2 \mu(dx)\mu(dy) \\
&= \sum_{\bar{n}\in\mathcal{J}} \Big|u_{\bar{n}}^x(t) - u_{\bar{n}}^y(t)\Big|^2 \int_{\mathcal{H}\times\mathcal{H}} H_{\bar{n}}^2(x)\mu(dx)\mu(dy) \\
&\quad + \sum_{\bar{n}\in\mathcal{J}} \Big[u_{\bar{n}}^y(t)\Big]^2 \sum_{\bar{n}\in\mathcal{J}} \int_{\mathcal{H}\times\mathcal{H}} \Big|H_{\bar{n}}(x) - H_{\bar{n}}(y)\Big|^2 \mu(dx)\mu(dy) \qquad (29) \\
&= \sum_{\bar{n}\in\mathcal{J}} \Big[u_{\bar{n}}^y(t)\Big]^2 \sum_{\bar{n}\in\mathcal{J}} \int_{\mathcal{H}\times\mathcal{H}} \Big|H_{\bar{n}}(x) - H_{\bar{n}}(y)\Big|^2 \mu(dx)\mu(dy) \\
&\quad + \sum_{\bar{n}\in\mathcal{J}} \Big|u_{\bar{n}}^x(t) - u_{\bar{n}}^y(t)\Big|^2 \\
&= \int_{\mathcal{H}\times\mathcal{H}} \Big|\mathbb{E}\Big[\varphi(X_t^x)\Big] - \mathbb{E}\Big[\varphi(X_t^y)\Big]\Big|^2 \mu(dx)\mu(dy) \\
&\quad + \int_{\mathcal{H}\times\mathcal{H}} \mathbb{E}^2\Big[\varphi(X_t^y)\Big]\mu(dx)\mu(dy) \\
&\quad \times \sum_{\bar{n}\in\mathcal{J}} \int_{\mathcal{H}\times\mathcal{H}} \Big|H_{\bar{n}}(x) - H_{\bar{n}}(y)\Big|^2 \mu(dx)\mu(dy) \\
&\quad + \sum_{\bar{n}\in\mathcal{J}} \Big[u_{\bar{n}}^y(t)\Big]^2 \sum_{\bar{n}\in\mathcal{J}} \int_{\mathcal{H}\times\mathcal{H}} \Big|H_{\bar{n}}(x) - H_{\bar{n}}(y)\Big|^2 \mu(dx)\mu(dy) \ .
\end{aligned}
$$

Note that $\mathbb{E}^2\Big[\varphi(X_t^y)\Big] = u^2(t,x) \in L^2(\mathcal{H},\mu)$, therefore the first integral in the second term is a continuous bounded function of t. Moreover, $\sum_{\bar{n}\in\mathcal{J}} \Big[u_{\bar{n}}^y(t)\Big]^2$ is the $L^2(\mathcal{H},\mu)$-norm of the function $u(t,x)$, then the series converges and it is also a continuous bounded function of t. Thus, from (29) we get

$$\begin{aligned}\|\Psi_t^x - \Psi_t^y\|^2_{\left(L^2(\mathcal{H},\mu)\right)^2} \leq & \int_{\mathcal{H}\times\mathcal{H}} \Big|\mathbb{E}\Big[\varphi(X_t^x)\Big] - \mathbb{E}\Big[\varphi(X_t^y)\Big]\Big|^2 \mu(dx)\mu(dy) \\ & + f(t) \sum_{\bar{n}\in\mathcal{J}} \int_{\mathcal{H}\times\mathcal{H}} \Big|H_{\bar{n}}(x) - H_{\bar{n}}(y)\Big|^2 \mu(dx)\mu(dy) \ ,\end{aligned} \tag{30}$$

where $f(t) = \sum_{\bar{n}\in\mathcal{J}} \Big[u_{\bar{n}}^y(t)\Big]^2 + \int_{\mathcal{H}} \mathbb{E}^2\Big[\varphi(X_t^y)\Big]\mu(dy)$.

From the proof of Theorem 4.1 (see (23)) we know that

$$\begin{aligned}|||\Psi_t^\varphi - \Phi_t^\psi|||^2 &= \int_{\mathcal{H}\times\mathcal{H}} \Big|\mathbb{E}\Big[\varphi(X_t^x)\Big] - \mathbb{E}\Big[\varphi(X_t^y)\Big]\Big|^2 \mu(dx)\mu(dy) \\ &\leq \exp(Ct) \int_{\mathcal{H}\times\mathcal{H}} \|x-y\|^2_{\mathcal{H}} \mu(dx)\mu(dy) \\ &= \exp(Ct)|||x-y|||^2.\end{aligned} \tag{31}$$

Therefore the first term in the right side of (30) is bounded by (31).

We now focus on the second term in the last inequality. Notice that for every $\bar{n} \in \mathcal{J}$ we have

$$H_{\bar{n}}(x) - H_{\bar{n}}(y) = \prod_{i=1}^{\infty} \Big[P_{n_i}(\xi_i) - P_{n_i}(\eta_i)\Big], \tag{32}$$

where $\xi_i = \langle x, \Lambda^{-1/2} e_i\rangle_{\mathcal{H}}$ and $\eta_i = \langle y, \Lambda^{-1/2} e_i\rangle_{\mathcal{H}}$ (see (5) and lines after that for the definition). Hence, applying Lemma 4.2 to equation (32) we have that

$$\begin{aligned}H_{\bar{n}}(x) - H_{\bar{n}}(y) &= \prod_{i=1}^{\infty} C(i) Pe_{i+1}(\gamma_i) \cdot (\xi_i - \eta_i) \\ &= \prod_{i=1}^{\infty} C(i) Pe_{i+1}(\gamma_i) \langle x-y, \Lambda^{-1/2} e_i\rangle_{\mathcal{H}},\end{aligned} \tag{33}$$

here $\gamma_i \in \Big(\xi_i \wedge \eta_i, \xi_i \vee \eta_i\Big)$ for every $i \in \mathbb{N}$. Then

$$\begin{aligned}
&\sum_{\bar{n}\in\mathcal{J}} \int_{\mathcal{H}\times\mathcal{H}} \Big| H_{\bar{n}}(x) - H_{\bar{n}}(y) \Big|^2 \mu(dx)\mu(dy) \\
&\qquad = \sum_{\bar{n}\in\mathcal{J}} \int_{\mathcal{H}\times\mathcal{H}} \Big| \prod_{i=1}^{\infty} C(i) Pe_{i+1}(\gamma_i) \langle x-y, \Lambda^{-1/2} e_i \rangle_{\mathcal{H}} \Big|^2 \mu(dx)\mu(dy) \\
&\qquad = \sum_{\bar{n}\in\mathcal{J}} \int_{\mathcal{H}\times\mathcal{H}} \prod_{i=1}^{\infty} \Big[C(i) Pe_{i+1}(\gamma_i) \Big]^2 \Big| \langle x-y, \Lambda^{-1/2} e_i \rangle_{\mathcal{H}} \Big|^2 \mu(dx)\mu(dy) \\
&\qquad \leq \sum_{\bar{n}\in\mathcal{J}} \int_{\mathcal{H}\times\mathcal{H}} \prod_{i=1}^{\infty} \Big[C(i) Pe_{i+1}(\gamma_i) \Big]^2 \|x-y\|_{\mathcal{H}}^2 \|\Lambda^{-1/2} e_i\|_{\mathcal{H}}^2 \mu(dx)\mu(dy) \\
&\qquad = \sum_{\bar{n}\in\mathcal{J}} \int_{\mathcal{H}\times\mathcal{H}} \prod_{i=1}^{\infty} \Big[C(i) Pe_{i+1}(\gamma_i) \Big]^2 \|x-y\|_{\mathcal{H}}^2 \lambda_i^{-1} \|e_i\|_{\mathcal{H}}^2 \mu(dx)\mu(dy) \\
&\qquad = \|x-y\|_{\mathcal{H}}^2 \sum_{\bar{n}\in\mathcal{J}} \prod_{i=1}^{\infty} \Big[C(i) \Big]^2 \lambda_i^{-1} \int_{\mathcal{H}\times\mathcal{H}} \Big[Pe_{i+1}(\gamma_i) \Big]^2 \mu(dx)\mu(dy).
\end{aligned} \tag{34}$$

Recall that for every $i \in \mathbb{N}$ we have that $\gamma_i \in \Big(\xi_i \wedge \eta_i, \xi_i \vee \eta_i\Big)$, set $\hat{\gamma}_i \in \Big(\xi_i \wedge \eta_i, \xi_i \vee \eta_i\Big)$ such that $Pe_i^2(\gamma_i) \leq Pe_{i+1}^2(\hat{\gamma}_i)$ for every $\gamma_i \in \Big(\xi_i \wedge \eta_i, \xi_i \vee \eta_i\Big)$, notice that the existence of $\hat{\gamma}_i$ is guaranteed since $Pe_{i+1}^2(\cdot)$ is a continuous function. Then, from (34) we get

$$\begin{aligned}
&\sum_{\bar{n}\in\mathcal{J}} \int_{\mathcal{H}\times\mathcal{H}} \Big| H_{\bar{n}}(x) - H_{\bar{n}}(y) \Big|^2 \mu(dx)\mu(dy) \\
&\qquad \leq \|x-y\|_{\mathcal{H}}^2 \sum_{\bar{n}\in\mathcal{J}} \prod_{i=1}^{\infty} \Big[C(i) \Big]^2 \lambda_i^{-1} \Big[Pe_{i+1}(\hat{\gamma}_i) \Big]^2 \int_{\mathcal{H}} \int_{\mathcal{H}} \mu(dx)\mu(dy) \\
&\qquad = \|x-y\|_{\mathcal{H}}^2 \sum_{\bar{n}\in\mathcal{J}} \prod_{i=1}^{\infty} \Big[C(i) \Big]^2 \lambda_i^{-1} \Big[Pe_{i+1}(\hat{\gamma}_i) \Big]^2.
\end{aligned} \tag{35}$$

Here, we recall that $C(i) = \frac{(-1)^i}{(i+1)(i!)^{1/2}}$ then $\frac{(-1)^i}{[(i+1)!]^{1/2}} Pe_{i+1}(\hat{\gamma}_i)$ is the normalized Hermite polynomial of $i+1$ degree evaluated on $\hat{\gamma}_i$ which is bounded by a constant C for every $i \in \mathbb{N}$. Moreover, since $\lambda_k < \lambda_{k+1} \to \infty$ then this implies that

$$\sum_{\bar{n}\in\mathcal{J}} \prod_{i=1}^{\infty} \Big[C(i) \Big]^2 \lambda_i^{-1} \Big[Pe_{i+1}(\hat{\gamma}_i) \Big]^2 \leq C \sum_{\bar{n}\in\mathcal{J}} \prod_{i=1}^{\infty} \lambda_i^{-1} (i+1)^{-1} \leq C, \tag{36}$$

where C is a finite constant. Putting together (34) and (36) we get that

$$\sum_{\bar{n}\in\mathcal{J}}\int_{\mathcal{H}\times\mathcal{H}}\left|H_{\bar{n}}(x)-H_{\bar{n}}(y)\right|^2\mu(dx)\mu(dy)\leq C\|x-y\|_{\mathcal{H}}\ . \tag{37}$$

Putting together inequalities (30), (31) and (37) we obtain

$$\|\Psi_t^x-\Psi_t^y\|^2_{\left(L^2(\mathcal{H},\mu)\right)^2}\leq\exp(Ct)\int_{\mathcal{H}\times\mathcal{H}}\|x-y\|^2_{\mathcal{H}}\mu(dx)\mu(dy)+f(t)\|x-y\|_{\mathcal{H}}. \tag{38}$$

Now, if $\|x-y\|_{\mathcal{H}}\leq\delta$, then from (38) we get $\|\Psi_t^x-\Psi_t^y\|_{\left(L^2(\mathcal{H},\mu)\right)^2}\leq G(t)\delta$. $\square$

Remark 3. If we consider in addition the supremum norm on t, then from (38) we get

$$\begin{aligned}\sup_{0\leq t\leq T}\|\Psi_t^x-\Psi_t^y\|^2_{\left(L^2(\mathcal{H},\mu)\right)^2}\leq & C\|x-y\|^2_{\mathcal{H}}\sup_{0\leq t\leq T}f(t)\\ &+\exp(CT)\int_{\mathcal{H}\times\mathcal{H}}\|x-y\|^2_{\mathcal{H}}\mu(dx)\mu(dy)\ .\end{aligned} \tag{39}$$

Notice that $f(t)$ is differentiable and continuous, then $\sup_{0\leq t\leq T}f(t)\leq C$, then from (39) we obtain

$$\begin{aligned}\sup_{0\leq t\leq T}\|\Psi_t^x-\Psi_t^y\|_{\left(L^2(\mathcal{H},\mu)\right)^2}\leq & C\|x-y\|_{\mathcal{H}}\\ &+\exp(CT)\int_{\mathcal{H}\times\mathcal{H}}\|x-y\|^2_{\mathcal{H}}\mu(dx)\mu(dy)\ .\end{aligned} \tag{40}$$

From this inequality it is possible to show the continuously dependence on the initial conditions for this norm.

5 Numerical experiments

In this section we run numerical experiments to illustrate that our scheme preserves the underlying initial condition continuity. To this end, we solve a stochastic version of the Fisher and Burgers PDEs with two near initial function conditions $x(\xi)$, $\widehat{x}(\xi)$.

The numerical scheme for Kolmogorov equations associated with these SPDEs has been studied by Delgado-Vences and Flandoli in [9]. They rewrite these models as the abstract SPDE (1) and define its associated Kolmogorov equation (3). Hence we assume that, for the following Fisher and Burgers SPDEs, the abstract form (1) and its associated Kolmogorov equation (3) are well-defined.

In [16] we provide a GitHub repository with a Python implementation to reproduce the following figures. We also provide in [10, 11], the 3D color on-line plotly versions of the regarding solutions.

Stochastic Fisher-KPP equation in an interval

Let $\mathcal{H} = L^2(0,1)$. We consider the stochastic Fisher-KPP equation

$$\begin{aligned} dX(t,\xi) &= \left[\nu\partial_\xi^2 X(t,\xi) + X(t,\xi)(1-X(t,\xi))\right] dt + dW(t,\xi), \\ X(t,0) &= X(t,1) = 0, \quad t>0, \\ X(0,\xi) &\in \mathcal{H},\ \xi\in[0,1], \end{aligned} \tag{41}$$

in the interval $[0,1]$ and with initial function conditions $x(\xi))$ and $\widehat{x}(\xi)$. In order to fix this initial function conditions close, we use for our experiments

$$x(\xi) := \operatorname{sech}^2(5(\xi-0.5)), \qquad \widehat{x}(\xi) := \sum_{k=0}^{N} T_k(x(\xi)), \tag{42}$$

where $T_k(\cdot)$ denotes the Chebyshev polynomial of the first kind. That is, $\widehat{x}(\cdot)$ is the Chebyshev truncated expansion of $x(\cdot)$.

Figure 1 suggest the conclusion of Theorem 4.3, that is, the solutions of equation (41) are continuous respect to initial conditions and satisfies the estimation (31).

Stochastic Burgers equation

Let $\mathcal{H} = L^2(0,1)$, consider the stochastic Burgers equation in the interval $[0,1]$

$$\begin{aligned} dX(t,\xi) &= \left[\nu\partial_\xi^2 X(t,\xi) + \frac{1}{2}\partial_\xi X^2(t,\xi)\right] dt + dW(t,\xi), \\ X(t,0) &= X(t,1) = 0, \quad t>0, \\ X(0,\xi) &= x(\xi), \quad x\in\mathcal{H}\ . \end{aligned} \tag{43}$$

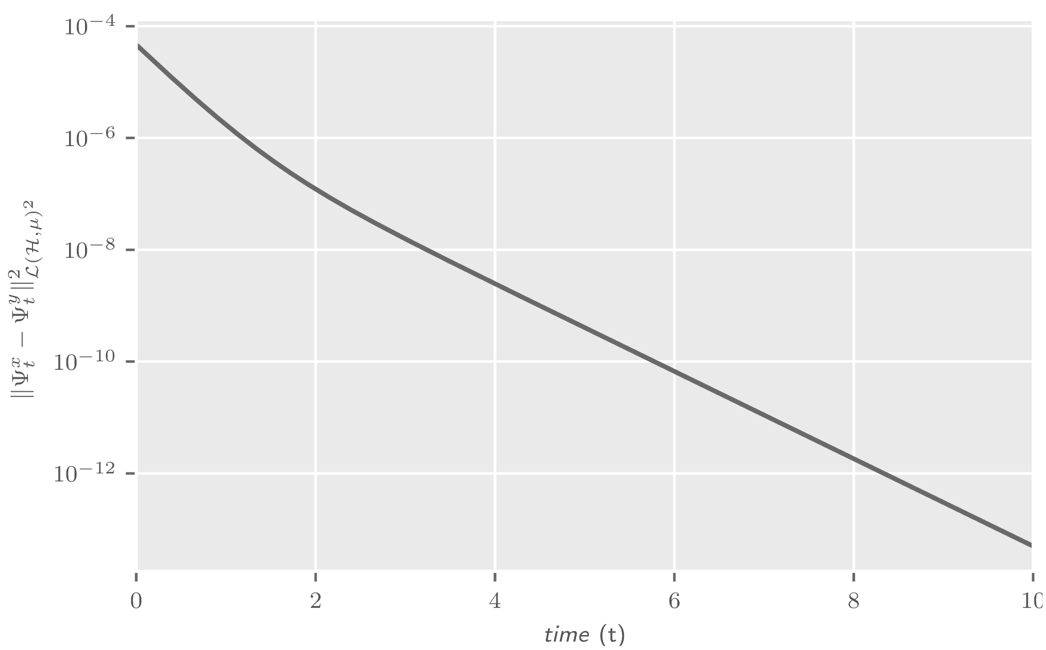


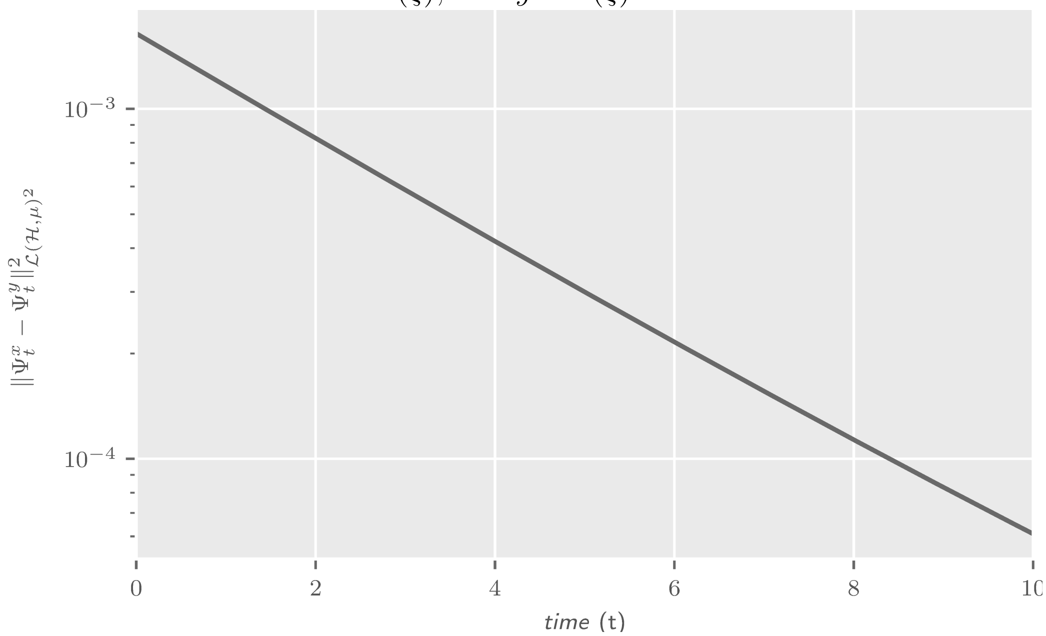
Figure 1: $\mathcal{L}^2(\mathcal{H},\mu)$ distance between two solutions of the stochastic Fisher PDE with initial conditions $x = x(\xi)$, and $y = \widehat{x}(\xi)$.

We use the initial conditions $x(\xi)$ and its truncated Chebyshev expansion

$$x(\xi) := \sin(\pi\xi), \qquad \widehat{x}(\xi) := \sum_{k=0}^{N} T_k x(\xi). \tag{44}$$

According to the $\mathcal{L}(\mathcal{H},\mu)$-distance between the two underlying solutions, Figure 2 confirms the conclusion of Theorem 4.3.

Figure 2: Distance between two solutions of the stochastic Burgers eq. (43) with initial conditions $x = x(\xi)$, and $y = \widehat{x}(\xi)$.



6 Conclusions

To the best of our knowledge, our results represent the first contribution to the numeric stability respect to initial conditions of weak approximations of Kolmogorov equations in infinite dimensions. This kind of stability, combining with the weak approximation approach, would save computation time. That is, since our scheme asks specific conditions to obtain a weak numerical solution of an underlying SPDE, we convert the stochastic problem into a deterministic ODE for the first moment. This procedure overcome Montecarlo type simulations to approximate moments or distributions—simulate many realization of the numerical stochastic process to approximate distributions or moments. Further, under our setting, the regarding spectral approximation assures high precision and order of convergence. Thus we guess that our method would improve the time and save resources of computation. We are preparing another article to confirm this conjectures.

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