$$p_{ji}(\Delta t) := \begin{cases} \frac{\beta i(N-i)}{N}\Delta t, & j = i+1 \\ (b+\gamma) i \Delta t, & j = i-1 \\ 1 - \left[\frac{\beta i(N-i)}{N} + (b+\gamma) i\right]\Delta t, & j = i \\ 0 & \text{otherwise} \end{cases}$$

$$
\begin{pmatrix}
1 & d(1)\Delta t & 0 & \cdots & 0 & 0 \\
0 & 1-(b+d)(1)\Delta t & d(2)\Delta t & \cdots & 0 & 0 \\
0 & b(1)\Delta t & 1-(b+d)(2)\Delta t & \cdots & 0 & 0 \\
0 & 0 & b(2)\Delta t & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & d(N-1)\Delta t & 0 \\
0 & 0 & 0 & \cdots & 1-(b+d)(N-1)\Delta t & d(N)\Delta t \\
0 & 0 & 0 & \cdots & d(N-1)\Delta t & 1-d(N)\Delta t
\end{pmatrix}
$$

$$
\begin{aligned}
\mathbb{E}(I_{t+\Delta t}) &= \sum_{i=0}^{N} i p_i(t+\Delta t) \\
&= \sum_{i=0}^{N} i p_{i-1}(t) b(i-1)\Delta t + \sum_{i=0}^{N} i p_{i+1}(t) d(i+1)\Delta t \\
&\quad + \sum_{i=0}^{N} i p_i(t)\Delta t - \sum_{i=0}^{N} i p_i(t) b(i)\Delta t - \sum_{i=0}^{N} i p_i(t) d(i)\Delta t
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}(I_{t+\Delta t}) &= \mathbb{E}(I_t) + \sum_{i=1}^{N} p_{i-1}(t) \frac{\beta(i-1)(N-[i-1])}{N}\Delta t - \sum_{i=0}^{N-1} p_{i+1}(t)(b+\gamma)(i+1)\Delta t \\
&= \mathbb{E}(I_t) + [\beta-(b+\gamma)]\Delta t \mathbb{E}(I_t) - \frac{\beta}{N}\Delta t \underbrace{\mathbb{E}(I_t^2)}_{\geq \mathbb{E}^2(I_t)}
\end{aligned}
$$

$$
\frac{\mathbb{E}(I_{t+\Delta t}) - \mathbb{E}(I_t)}{\Delta t} \leq [\beta-(b+\gamma)]\mathbb{E}(I_t) - \frac{\beta}{N}\mathbb{E}^2(I_t)
$$

$$
\begin{aligned}
\frac{d\mathbb{E}(I_t)}{dt} &\leq [\beta-(b+\gamma)]\mathbb{E}(I_t) - \frac{\beta}{N}\mathbb{E}^2(I_t) \\
&= \frac{\beta}{N}[N-\mathbb{E}(I_t)]\mathbb{E}(I_t) - (b+\gamma)\mathbb{E}(I_t)
\end{aligned}
$$

Formulating a SIS-CTMC

Let

$$\{I_t\}_{t\geq 0},$$
$$p_i(t) = \mathbb{P}[I_t = i].$$

Thus, the Markov porperty becomes in

$$\mathbb{P}\left[I_{t_{n-1}} \middle| I_{t_0}, \cdots, I_{t_n}\right] = \mathbb{P}\left[I_{t_{n+1}} \middle| I_{t_n}\right]$$