Modelling with SDEs in Biology

Formulation, Numerical Simulation and Analysis

First day: Formaulation of Models with SDEs

Saúl Díaz Infante Velasco

CONACYT-UNIVERSIDAD de SONORA, Cimat, Guanajuato Gto

UQ

$$ODE + noise = Better\ Model$$

UQ

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Population growth

$$\frac{dN}{dt} = a(t)N(t) \qquad N_0 = N(0) = cte.$$

UQ

$$ODE + noise = Better\ Model$$

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$$\frac{dN}{dt} = a(t)N(t) \qquad N_0 = N(0) = cte.$$

$$\boxed{a(t) = r(t) + \text{"}noise\text{"}}$$

UQ

$ODE + noise = Better\ Model$

Electric Circuits

$$L \cdot Q''(t) + R \cdot Q'(t) + \frac{1}{C} \cdot Q(t) = F(t)$$

$$Q(0) = Q_0$$

$$Q'(0) = I_0$$

UQ

ODE + noise = Better Model

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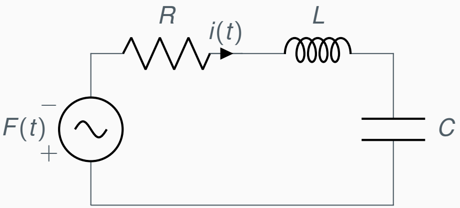

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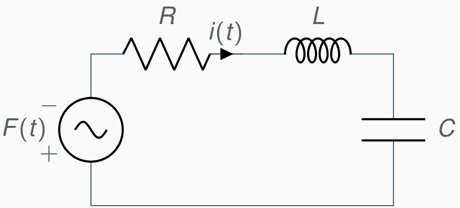

$$\boxed{F(t) = G(t) + \text{"noise"}}$$

To fix ideas

Example

$$dN(t) = aN(t)dt$$

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Perturb in $[t, t + dt)$

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Get a SDE

$$dN(t) = aN(t)dt + \sigma N(t)dB(t)$$

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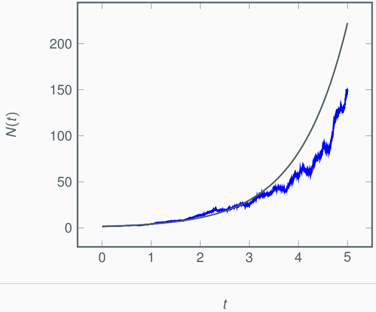

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Some Important applications

- Finance
- Physics
- Chemistry
- Biology
- Epidemiology

Henston

$$dS_t = \mu S_t dt + \sqrt{V_t} S_t \left(\sqrt{1-\rho^2} dW_t^{(1)} + \rho dW^{(2)} \right)$$

$$dV_t = \kappa(\lambda - V_t) dt + \theta \sqrt{V_t} dW_t^{(2)}$$

Hutzenthaler, M. and Jentzen, A. (2015).
Numerical approximations of stochastic differential equations with non-globally lipschitz continuous coefficients.
Memoirs of the American Mathematical Society, 236(1112).

Some Important applications

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Langevin

$$dX_t = -(\nabla U)(X_t)dt + \sqrt{2\varepsilon}dW_t$$

Hutzenthaler, M. and Jentzen, A. (2015).
Numerical approximations of stochastic differential equations with non-globally lipschitz continuous coefficients.
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Some Important applications

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Brusselator

$$dX_t = \left[\delta - (\alpha+1)X_t + Y_t X_t^2\right] dt + g_1(X_t) dW_t^{(1)}$$
$$dY_t = \left[\alpha X_t + Y_t X_t^2\right] dt + g_2(X_t) dW_t^{(2)}$$

Hutzenthaler, M. and Jentzen, A. (2015).
Numerical approximations of stochastic differential equations with non-globally lipschitz continuous coefficients.
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- Finance
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- Chemistry
- Biology
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Lotka Volterra

$$dX_t = (\lambda X_t - kX_t Y_t)dt + \sigma X_t dW_t$$
$$dY_t = (kX_t Y_t - mY_t)dt$$

Hutzenthaler, M. and Jentzen, A. (2015).
Numerical approximations of stochastic differential equations with non-globally lipschitz continuous coefficients.
Memoirs of the American Mathematical Society, 236(1112).

Some Important applications

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SIR

$$
\begin{aligned}
dS_t &= (-\alpha S_t I_t - \delta S_t + \delta)dt - \beta S_t I_t dW_t \\
dI_t &= (\alpha S_t I_t - (\gamma + \delta) I_t)dt + \beta S_t I_t dW_t \\
dR_t &= (\gamma I_t - \delta R_t)dt
\end{aligned}
$$

Main objective

To present two of the common approaches in stochastic modeling with SDEs.

Allen's approach

$$DTMC \to CTMC \to SDE$$

Stochastic perturbation

$$dN(t)/dt = aN(t)$$

$$adt \rightsquigarrow adt + \sigma dB(t)$$

$$dN(t) = aN(t)dt + \sigma N(t)dB(t)$$

Table of contents

Why noise?

Environmental effects

- Extinction
- Outbreaks

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Environmental effects

- Extinction
- Outbreaks

Environmental Brownian noise suppresses explosions.

Mao, X., Marion, G., and Renshaw, E. (2002).
Environmental brownian noise suppresses explosions in population dynamics.
Stochastic Processes and their Applications, 97(1):95–110.

Why noise?

Environmental effects

- Extinction
- Outbreaks

Noise color induces extinction

Ripa, J. and Lundberg, P. (1996).
Noise Colour and the Risk of Population Extinctions.
Proceedings of the Royal Society B: Biological Sciences, 263(1377):1751–1753.

Why noise?

Environmental effects

- Extinction
- Outbreaks

$\mathscr{R}_0$: Endemic g.a.e. $\rightarrow$ periodic oscillations

Allen, L. and van den Driessche, P. (2013).
Relations between deterministic and stochastic thresholds for disease extinction in continuous- and discrete-time infectious disease models.
Mathematical Biosciences, 243(1):99–108.

In Biology

- **DTMC, CTMC**
- Stochastic perturbation of parameters
- Mean reverting processes

DTMC $+$ CTMC $+$ ME $\rightarrow$ *SDE*

Allen, L. J. (2017).
A primer on stochastic epidemic models: Formulation, numerical simulation, and analysis.
Infectious Disease Modelling, 2(2):128–142.

In Biology

- DTMC, CTMC
- **Stochastic perturbation of parameters**
- Mean reverting processes

$\varphi dt \rightsquigarrow \varphi dt + \sigma dB_t$

Gray, A., Greenhalgh, D., Hu, L., Mao, X., and Pan, J. (2011).
A Stochastic Differential Equation SIS Epidemic Model.
SIAM Journal on Applied Mathematics, 71(3):876–902.

In Biology

- DTMC, CTMC
- **Stochastic perturbation of parameters**
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$\varphi dt \rightsquigarrow \varphi dt + F(x)dB_t$

Schurz, H. and Tosun, K. (2015). **Stochastic Asymptotic Stability of SIR Model with Variable Diffusion Rates.** *Journal of Dynamics and Differential Equations*, 27(1):69–82.

Alternativas

In Biology

- DTMC, CTMC
- Stochastic perturbation of parameters
- Mean reverting processes

$$d\varphi_t = (\varphi_e - \varphi_t)dt + \sigma_\varphi dBt$$

Allen, E. (2016).
Environmental variability and mean-reverting processes.
Discrete and Continuous Dynamical Systems - Series B, 21(7):2073–2089.

To fix ideas, we recall the deterministic philosophy to formulate ODEs

- We study a process in an small interval of time Δt
- Describe the resulting information in Δt
- Letting $\Delta t \to 0$ gives an ODE

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Formulation of SDE: $DTMC \to CTMC + ME \to SDE$

To fix ideas, we recall the deterministic philosophy to formulate ODEs

- We study a process in an small interval of time Δt
- Describe the resulting information in Δt
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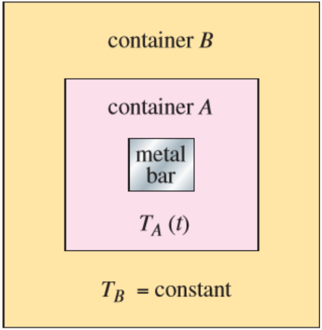
Newton's Cooling Law

$$T_A(t+\Delta t) - T_A(t) = \alpha(T_B - T_A(t))\Delta t$$

$$\frac{T_A(t+\Delta t) - T_A(t)}{\Delta t} = \alpha(T_B - T_A(t))$$

Letting $\Delta t \to 0$

$$\frac{dT_A(t)}{dt} = \alpha(T_B - T_A(t))$$



A stochastic analogy.

i) Formulate a discrete stochastic model for the dynamical system understudy, which describe changes in a small time interva Δt
ii) Compute the expected value and covariance for the difference in a short time Δt
iii) Letting $\Delta t \to 0$, he above information leads to the CTMC
iv) Thus, we infer the SDE from the similarities in the forward-backward-Kolmogorov equation between the discrete and Continuous Markov Chain

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Example: Formulation of a stochastic SIS model

Consider the deterministic SI model:

$$
\begin{aligned}
\frac{dS}{dt} &= -\frac{\beta}{N} SI + (b+\gamma) I \\
\frac{dI}{dt} &= \frac{\beta}{N} SI - (b+\gamma) I \\
N &= S(t) + I(t)
\end{aligned}
$$

Where N is constant and $S(t) = N - I(t)$.

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Where N is constant and $S(t) = N - I(t)$.

$$\mathscr{R}_0 = \frac{\beta}{b+\gamma}$$
$$\mathscr{R}_0 \leq 1$$
$$\Rightarrow \lim_{t\to\infty} (S(t), N(t)) = (N, 0)$$
$$\mathscr{R}_0 > 1$$
$$\Rightarrow \lim_{t\to\infty} (S(t), N(t)) = (S^*, I^*)$$

Example: Formulation of a stochastic SIS model

Consider the process $\{\mathscr{I}_t\}_{t=0}^{\infty}$ with time discrete and space of states $\{0, 1, \cdots, N\}$.

(H-1)

$$
\begin{aligned}
p_i(t) &:= \mathbb{P}[\mathscr{I}(t) = i] \\
i &= 0, 1, 2 \ldots, N \\
t &= 0, \Delta t, 2\Delta t, \cdots,
\end{aligned}
$$

(H-2) Markov property

$$\mathbb{P}\left[I_{t+\Delta t} \middle| I_0, I_{\Delta t}, \cdots, I_t\right] = \mathbb{P}\left[I_{t+\Delta t} \middle| I_t\right]$$

(H-3) The unique transitions with positive probability

$$p_{ji}(t+\Delta t) = \mathbb{P}\left[I_{t+\Delta t} = j \middle| I_t = i\right]$$

are

$$i \to i+1,\ i \to i-1,\ i \to i$$

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$$
i \to i+1, \; i \to i-1, \; i \to i
$$

$$
p_{ji}(\Delta t) := \begin{cases}
\frac{\beta i(N-i)}{N}\Delta t, & j = i+1 \\
(b+\gamma) i \Delta t, & j = i-1 \\
1 - \left[\frac{\beta i(N-i)}{N} + (b+\gamma) i\right]\Delta t, & j = i, \\
0 & \text{otherwise}
\end{cases}
$$

where Δt is sufficiently small s.t.

$$
\max_{i \in \{1, \ldots, N\}} \{[b(i) + d(i)]\Delta t\} \leq 1
$$

Example: Formulation of a stochastic SIS model

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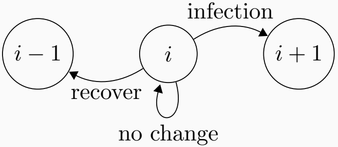
where Δt is sufficiently small s.t.

$$\max_{i\in\{1,\ldots,N\}} \{[b(i)+d(i)]\Delta t\} \leq 1$$



Example: Formulation of a stochastic SIS model

Example: Formulation of a stochastic SIS model

Letting

$$
\begin{aligned}
b(i) &:= \frac{\beta i(N-i)}{N}\Delta t \\
d(i) &:= (b+\gamma) i \Delta t
\end{aligned}
$$

Example: Formulation of a stochastic SIS model

Letting

$$b(i) := \frac{\beta i(N-i)}{N}\Delta t$$
$$d(i) := (b+\gamma) i \Delta t$$

FKE

$$\begin{aligned} p_i(t+\Delta t) = p_{i-1}(t) b(i-1)\Delta t \\ + p_{i+1}(t) d(i+1)\Delta t \\ + p_i(t)(1-[b(i)+d(i)]\Delta t) \end{aligned}$$

Example: Formulation of a stochastic SIS model

Letting

$$b(i) := \frac{\beta i(N-i)}{N}\Delta t$$
$$d(i) := (b+\gamma) i \Delta t$$

FKE

$$\begin{aligned} p_i(t+\Delta t) = {} & p_{i-1}(t) b(i-1)\Delta t \\ & + p_{i+1}(t) d(i+1)\Delta t \\ & + p_i(t)(1-[b(i)+d(i)]\Delta t) \end{aligned}$$

Thus $P(\Delta t) =$

$$\begin{pmatrix} 1 & d(1)\Delta t & 0 & \cdots & 0 & 0 \\ 0 & 1-(b+d)(1)\Delta t & d(2)\Delta t & \cdots & 0 & 0 \\ 0 & b(1)\Delta t & 1-(b+d)(2)\Delta t & \cdots & 0 & 0 \\ 0 & 0 & b(2)\Delta t & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & d(N-1)\Delta t & 0 \\ 0 & 0 & 0 & \cdots & 1-(b+d)(N-1)\Delta t & d(N)\Delta t \\ 0 & 0 & 0 & \cdots & d(N-1)\Delta t & 1-d(N)\Delta t \end{pmatrix}$$

$$p(t+\Delta t) = P(\Delta t)p(t) = P^{n+1}(\Delta t)p(0), \qquad t = n\Delta t$$

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Expected Value of the SIS-DTMC

Expected Value of the SIS-DTMC

$$\mathbb{E}(I_{t+\Delta t}) = \sum_{i=0}^{N} i p_i(t+\Delta t)$$

Expected Value of the SIS-DTMC

$$
\begin{aligned}
\mathbb{E}(I_{t+\Delta t}) &= \sum_{i=0}^{N} i p_i(t+\Delta t) \\
&= \sum_{i=0}^{N} i p_{i-1}(t) b(i-1)\Delta t + \sum_{i=0}^{N-1} i p_{i+1}(t) d(i+1)\Delta t \\
&+ \sum_{i=0}^{N} i p_i(t)\Delta t - \sum_{i=0}^{N} i p_i(t) b(i)\Delta t - \sum_{i=0}^{N} i p_i(t) d(i)\Delta t
\end{aligned}
$$

Expected Value of the SIS-DTMC

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&= \sum_{i=0}^{N} i p_{i-1}(t) b(i-1)\Delta t + \sum_{i=0}^{N-1} i p_{i+1}(t) d(i+1)\Delta t \\
&+ \sum_{i=0}^{N} i p_i(t)\Delta t - \sum_{i=0}^{N} i p_i(t) b(i)\Delta t - \sum_{i=0}^{N} i p_i(t) d(i)\Delta t
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}(I_{t+\Delta t}) &= \mathbb{E}(I_t) + \sum_{i=1}^{N} p_{i-1}(t) \frac{\beta(i-1)(N-[i-1])}{N}\Delta t \\
&- \sum_{i=0}^{N-1} p_{i+1}(t)(b+\gamma)(i+1)\Delta t \\
&= \mathbb{E}(I_t) + [\beta-(b+\gamma)]\Delta t \mathbb{E}(I_t) - \frac{\beta}{N}\Delta t \underbrace{\mathbb{E}(I_t^2)}_{\geq \mathbb{E}^2(I_t)}
\end{aligned}
$$

Expected Value of the SIS-DTMC

$$\frac{\mathbb{E}(I_{t+\Delta t}) - \mathbb{E}(I_t)}{\Delta t} \leq [\beta - (b+\gamma)]\mathbb{E}(I_t) - \frac{\beta}{N}\mathbb{E}^2(I_t)$$

Expected Value of the SIS-DTMC

$$\frac{\mathbb{E}(I_{t+\Delta t}) - \mathbb{E}(I_t)}{\Delta t} \leq [\beta - (b+\gamma)]\mathbb{E}(I_t) - \frac{\beta}{N}\mathbb{E}^2(I_t)$$

$$\begin{aligned}
\frac{d\mathbb{E}(I_t)}{dt} &\leq [\beta - (b+\gamma)]\mathbb{E}(I_t) - \frac{\beta}{N}\mathbb{E}^2(I_t) \\
&= \frac{\beta}{N}[N - \mathbb{E}(I_t)]\mathbb{E}(I_t) - (b+\gamma)\mathbb{E}(I_t)
\end{aligned}$$

Expected Value of the SIS-DTMC

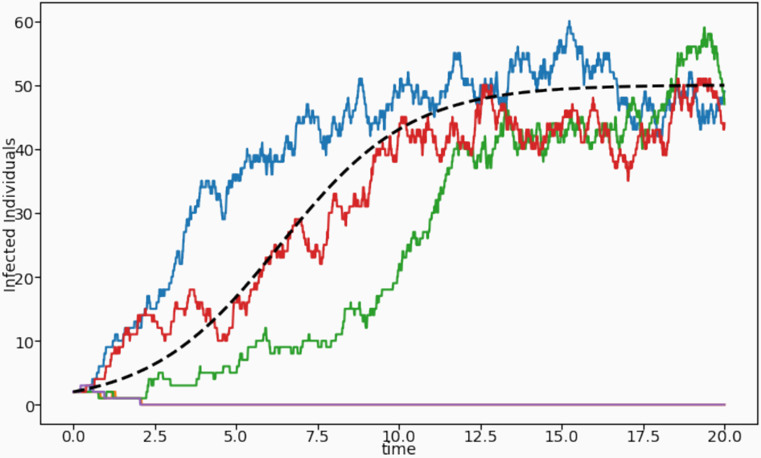
$$\frac{\mathbb{E}(I_{t+\Delta t})-\mathbb{E}(I_t)}{\Delta t} \leq [\beta-(b+\gamma)]\mathbb{E}(I_t)-\frac{\beta}{N}\mathbb{E}^2(I_t)$$

$$\begin{aligned}\frac{d\mathbb{E}(I_t)}{dt} &\leq [\beta-(b+\gamma)]\mathbb{E}(I_t)-\frac{\beta}{N}\mathbb{E}^2(I_t)\\ &= \frac{\beta}{N}[N-\mathbb{E}(I_t)]\mathbb{E}(I_t)-(b+\gamma)\mathbb{E}(I_t)\end{aligned}$$

$$\boxed{\frac{d\mathbb{E}(I_t)}{dt} \leq \frac{\beta}{N}\mathbb{E}(S_t)\mathbb{E}(I_t)-(b+\gamma)\mathbb{E}(I_t)}$$

Example: Formulation of a stochastic SIS model

$N = 100,\ \Delta t = 0.01,\ \beta = 1,$
$b = 0.25\ \gamma = 0.25$



$\mathcal{R}_0 = 2,$
$\bar{I} = 50$

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Formulating a SIS-CTMC

Let $\{I_t\}_{t\geq 0}, p_i(t) = \mathbb{P}[I_t = i]$.

Formulating a SIS-CTMC

Let $\{I_t\}_{t\geq 0}, p_i(t) = \mathbb{P}[I_t = i]$. Thus, the Markov property becomes in

$$\mathbb{P}\left[I_{t_{n+1}} \middle| I_{t_0}, \cdots, I_{t_n}\right] = \mathbb{P}\left[I_{t_{n+1}} \middle| I_{t_n}\right]$$

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$$p_{ji}(\Delta t) := \begin{cases} \frac{\beta i(N-i)}{N}\Delta t + o(\Delta t), & j = i+1 \\ (b+\gamma)i\Delta t + o(\Delta t), & j = i-1 \\ 1 - \left[\frac{\beta i(N-i)}{N} + (b+\gamma)i\right]\Delta t + o(\Delta t), & j = i \\ o(\Delta t) & \text{otherwise} \end{cases}$$

$$\lim_{t\to\infty} \frac{o(\Delta t)}{\Delta t} = 0$$

Using the notation for birth and death processes, we have

$$p_{ij}(\Delta t) := \begin{cases} b(i)\Delta t + o(\Delta t), & j = i+1 \\ d(i)\Delta t + o(\Delta t), & j = i-1 \\ 1 - [b(i) + d(i)]\,\Delta t + o(\Delta t), & j = i \\ 0 & \text{otherwise.} \end{cases}$$

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$\mathbb{P}[I_0 = i_0] = 1,$

$$\begin{aligned} p_i(t+\Delta t) =& p_{i-1}(t)b(i-1)\Delta t \\ & + p_{i+1}(t)d(i+1)\Delta t \\ & + p_i(t)[1-(b(i)+d(i))]\Delta t + o(\Delta t) \\ i =& 1,2,\ldots,N \end{aligned}$$

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Thus

$$\begin{aligned} \frac{p_i(t-\Delta t) - p_i(t)}{\Delta t} = & p_{i-1}(t) b(i-1) + p_{i+1}(t) d(i+1) \\ & - p_i[b(i) + d(i)] + o(\Delta t) \\ & i = 1, 2, \cdots, N. \end{aligned}$$

Hence, letting $\Delta t \to 0$, we obtain

$$\begin{aligned}\frac{dp_i(t)}{dt} =& p_{i-1}(t)b(i-1) + p_{i+1}(t)d(i+1) \\ & - p_i[b(i) + d(i)] \\ & i = 1, 2, \cdots, N.\end{aligned}$$

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$$Q = \begin{pmatrix} 0 & d(1) & 0 & \ldots & 0 \\ 0 & -[b(1)+d(1)] & d(2) & \ldots & 0 \\ 0 & b(1) & -[b(2)+d(2)] & \ldots & 0 \\ 0 & 0 & b(2) & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & d(N) \\ 0 & 0 & 0 & \ldots & -d(N) \end{pmatrix}$$

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Results that

$$\lim_{t \to \infty} p(t) = (1, 0, \ldots, 0)^\top$$

and

$$Q = \lim_{\Delta t \to 0} \frac{P(\Delta t) - I}{\Delta t}$$

Expected value of the SIS-CTMC

Consider the m.g.f

$$
\begin{aligned}
M(\theta, t) &:= \mathbb{E}[\exp(\theta I_t)] \\
&= \sum_{i=0}^{N} p_i(t) \exp(i\theta)
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Now we deduce a differential equation for the moments of our sis-CTMC.

$$\frac{\partial M}{\partial t} = \sum_{i=0}^{N} \frac{dp_i}{dt} \exp(i\theta)$$

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from r.h.s of FKE

$$\begin{aligned}
&= \exp(\theta) \sum_{i=0}^{N} p_{i-1} \exp[(i-1)\theta] b(i-1) \\
&\quad + \exp(-\theta) \sum_{i=0}^{N} p_{i+1} \exp[(i+1)\theta] d(i+1) \\
&\quad - \sum_{i=0}^{N} p_i \exp(i\theta)(b(i) + d(i))
\end{aligned}$$

Substituting definition of b, d we obtain

$$
\begin{aligned}
\frac{\partial M}{\partial t} =& \beta(exp(\theta) - 1) \sum_{i=1}^{N} ip_i \exp(i\theta) \\
& + (b + \gamma)(\exp(-\theta) - 1) \sum_{i=1}^{N} ip_i \exp(i\theta) \\
& - \frac{\beta}{N}(\exp(\theta) - 1) \sum_{i=1}^{N} i^2 p_i \exp(i\theta)
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&- \frac{\beta}{N}(\exp(\theta)-1)\sum_{i=1}^{N} i^2 p_i \exp(i\theta) \\
=& [\beta(exp(\theta)-1)(b+\gamma)(\exp(-\theta)-1)]\frac{\partial M}{\partial \theta} \\
&- \frac{\beta}{N}(\exp(\theta)-1)\frac{\partial^2 M}{\partial \theta^2}
\end{aligned}
$$

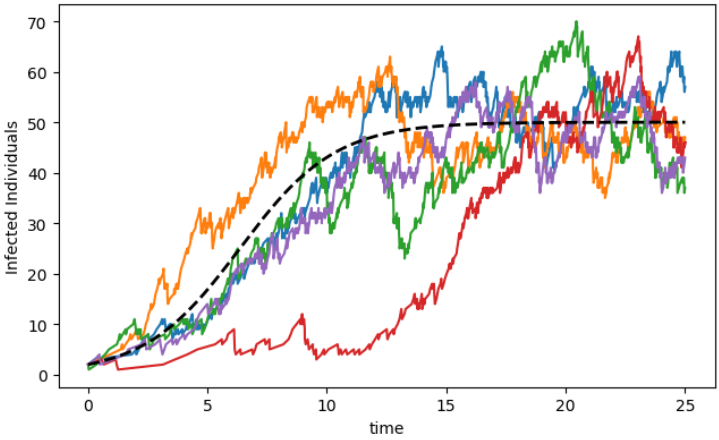
Following [Bailey, 1964] we can deduce from the above equation

$$\frac{d\mathbb{E}(I_t)}{dt} = [\beta - (b + \gamma)]\mathbb{E}(I_t) - \frac{\beta}{N}\mathbb{E}(I_t^2).$$

Then we conclude as in the SIS-DTMC.

Bailey, N. T. J. (1964).
The elements of stochastic processes with applications to the natural sciences.
John Wiley & Sons, Inc., New York-London-Sydney.

Using the Guillespie algorithm



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Denote the transition p.d.f as

$$p(y, t+\Delta t; x, t)$$
$$y = I_{t+\Delta t}, \quad x = I_t$$

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\frac{dp_i}{dt} =& p_{i-1} b(i-1) + p_{i+1} d(i+1) - p_i [b(i) + d(i)] \\
=& -\frac{p_{i+1}[d(i+1) - d(i+1)] - p_{i-1}[d(i-1) - d(i-1)]}{2\Delta i} \\
&+ \frac{1}{2} \frac{p_{i+1}[d(i+1) + d(i+1)] - 2p_i[b(i) + d(i)] + p_{i-1}[d(i-1) + d(i-1)]}{(\Delta_i)^2}
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Set $\Delta i = 1$, then

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Let $i = x$, $\Delta i = \Delta x$ and $pi(t) = p(x,t)$. Thus, letting $\Delta x \to 0$, we obtain the FKE

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$$\begin{aligned}\frac{\partial p(x,t)}{\partial t} =& \frac{\partial}{\partial x}\{[b(x) - d(x)]p(x,t)\} + \frac{1}{2}\frac{\partial^2}{\partial t^2}\{b(x) + d(x)p(x,t)\} \\ =& \frac{\partial}{\partial x}\left\{\left[\frac{\beta}{N} x(N-x) - (b+\gamma)x\right] p(x,t)\right\} \\ &+ \frac{1}{2}\frac{\partial^2}{\partial x^2}\left\{\left[\frac{\beta}{N} x(N-x) + (\beta+\gamma)x\right] p(x,t)\right\}\end{aligned}$$

Using the SIS-CTMC probablity transition kernel

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p_{ji}(\Delta t) := \begin{cases} \frac{\beta i(N-i)}{N}\Delta t + o(\Delta t), & j = i+1 \\ (b+\gamma) i \Delta t + o(\Delta t), & j = i-1 \\ 1 - \left[\frac{\beta i(N-i)}{N} + (b+\gamma) i\right]\Delta t + o(\Delta t), & j = i \\ o(\Delta t) & \text{otherwise.} \end{cases}
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If increment Δt follows a exponential distribution and is suficiently small. Results that increment

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Fix time t s.t $I_t = i$

$$\begin{aligned} \mathbb{E}\Delta I =& b(I_t)\Delta t - d(I_t)\Delta t + o(\Delta t) \\ =& \underbrace{[b(I)_t - d(I_t)]}_{:=\mu(I_t)} + o(\Delta t) \end{aligned}$$

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Since $\Delta I_t \sim \mathcal{N}(\mu(I_t)\Delta t, \sigma^2(I_t)\Delta t)$, we see that

$$\begin{aligned}I_{t+\Delta t} &= I_t + \Delta I_t \\ &\approx I_t + \mu(I_t)\Delta t + \sigma(I_t)\sqrt{\Delta t}\eta \\ \eta &\sim \mathcal{N}(0,1)\end{aligned}$$

The Euler-Maruyama's recurrence equation.

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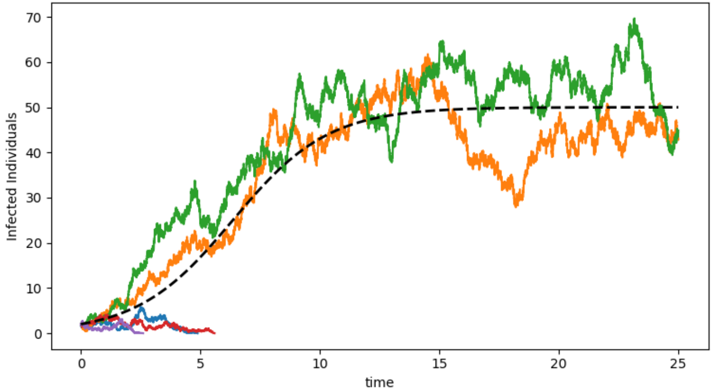
Sustituting, the notation for birth and death processes

$$dI_t = \frac{\beta}{N} I_t(N - I_t) - (b+\gamma) I_t + \sqrt{\frac{\beta}{N} I_t(N - I_t) + (b+\gamma) I_t} dW_t.$$


Infected Individuals
70
60
50
40
30
20
10
0
0
5
10
15
20
25
time