Modelling SDEs in biology

Formulation, Numerical Simulation and Analysis.

Second day: Numerical methods for SDEs

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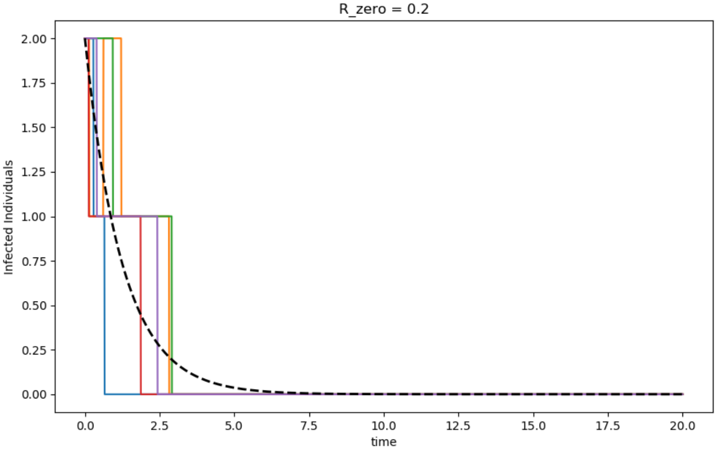
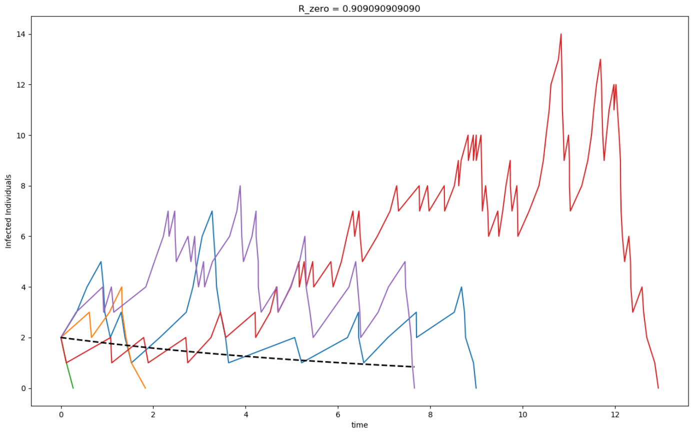
SIS-DTCM


R_zero = 0.2
Infected Individuals
2.00
1.75
1.50
1.25
1.00
0.75
0.50
0.25
0.00
0.0
2.5
5.0
7.5
10.0
12.5
15.0
17.5
20.0
time

Code

```
if 0 < infected[delta_t_i] < N-1:
    # Is there a birth?
    birth_probability = beta * infected[delta_t_i] * (N - infected[delta_t_i]) / N * delta_t
    death_probability = (b + gamma) * infected[delta_t_i] * delta_t
    complement_probability = 1.0 - (birth_probability + death_probability) * delta_t
    if np.random.rand() <= birth_probability:
        birth = True
    if np.random.rand() <= death_probability:
        death = True
    transition = 1 * birth - 1 * death
    infected[delta_t_i + 1] = infected[delta_t_i] + transition
# The evolution stops if we reach $0$ or $N$.
else:
    infected[delta_t_i + 1] = infected[delta_t_i]
```

SIS-CTCM

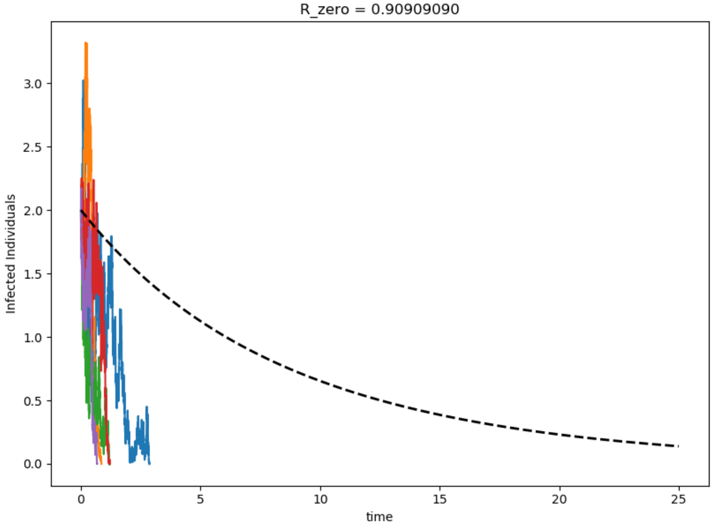


R_zero = 0.909090909090
Infected Individuals
time
0
2
4
6
8
10
12
14

Code

```
while (i[j] > 0 and t[j] < final_time):
    u_1 = np.random.rand()
    u_2 = np.random.rand()
    den = (beta / N) * i[j] * s[j] + (b + gamma) * i[j]
    infection_prob = (beta * s[j] / N) / (beta * s[j] / N +
        b + gamma)
    # exponential random time
    t.append(t[j] - np.log(u_1) / den)
    if u_1 <= infection_prob:
        i.append(i[j] + 1)
        s.append(s[j] - 1)
    else:
        i.append(i[j] - 1)
        s.append(s[j] + 1)
    j = j+1
```

SIS-SDE


R_zero = 0.90909090
Infected Individuals
3.0
2.5
2.0
1.5
1.0
0.5
0.0
0
5
10
15
20
25
time

Code

```
for k in np.arange(num_of_realizations):
    normal_sampler = np.zeros(array_size)
    normal_sampler[1:] = np.sqrt(delta_t) * np.random.randn(
        array_size -1)
    winner_inc = np.cumsum(normal_sampler)
    infected[0] = infected_0
    for i in np.arange(array_size - 1):
        delta_w_i = winner_inc[i+1] - winner_inc[i]
        euler_i = infected[i] + f(infected[i], 0) * delta_t \
        + g(infected[i]) * delta_w_i
        infected[i + 1] = euler_i
    plt.plot(time, infected)
# odeint parameters
```

Table of contents

Let, $I_t = i$. Denote by T_i the interevent time. $H_i(t)\mathbb{P}[T_i > t]$.

$$
\begin{aligned}
H_i(t+\Delta t) &= H_i(t) p_{ii}(\Delta t) \\
&= H_i(t)(1-(b(i)+d(i)))\Delta_t + o(\Delta t) \\
\frac{H_i(t+\Delta t) - H_i(t)}{\Delta t} &= -(b(i)+d(i))H_i(t) + o(\Delta t)
\end{aligned}
$$

Thus

$$
\frac{dH_i(t)}{dt} = -(b(i)+d(i))H_i(t)
$$

which have solution $H_i(t) = \exp(-(b(i)+d(i))t)$

Let $F_i(t) = \mathbb{P}[T_i \leq t] = 1 - exp(-(b(i) + d(i))t)$

Then using a uniform r.v. $u \sim [0, 1]$

$$\begin{aligned} \mathbb{P}\left[F_i^{-1}(u) \leq t\right] &= \mathbb{P}\left[F_i(F_i^{-1}(u)) \leq F_i(t)\right] \\ &= \mathbb{P}[u \leq F_i(t)] \\ &= F_i(t). \end{aligned}$$

Therefore

$$T_i = F_i^{-1}(u) = -\frac{-\log(u)}{b(i) + d(i)}$$

Notation

SDE

$$dx(t) = \underbrace{f(x(t),t)dt}_{\text{deriva}} + \underbrace{g(x(t),t)dB(t)}_{\text{difusión}},$$

$$f : \mathbb{R}^d \times [0,T] \to \mathbb{R}^d, \qquad g : \mathbb{R}^d \times [0,T] \to \mathbb{R}^{d\times m}$$

$$B(t) = (B_1(t), \ldots, B_m(t))^T, \quad (\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$$

Notation

SDE

$$dx(t) = \underbrace{f(x(t),t)dt}_{\text{deriva}} + \underbrace{g(x(t),t)dB(t)}_{\text{difusión}},$$
$$x_0 = x(0), \quad t \in [0,T].$$

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Notation

SDE

$$x(t) = x_0 + \int_0^t f(x(s), s)ds + \int_0^t g(x(s), s)dB(s)$$

$$f : \mathbb{R}^d \times [0, T] \to \mathbb{R}^d, \qquad g : \mathbb{R}^d \times [0, T] \to \mathbb{R}^{d \times m}$$

$$B(t) = (B_1(t), \dots, B_m(t))^T, \quad (\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t \geq 0}, \mathbb{P})$$

Construction idea

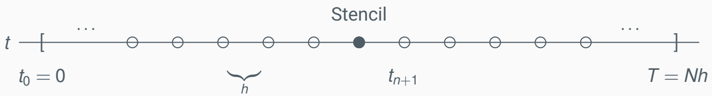
SDE

$$x(t) = x_0 + \int_0^t f(x(s), s)ds + \int_0^t g(x(s), s)dB(s)$$

Construction idea

SDE

$$x(t) = x_0 + \int_0^t f(x(s), s)\,ds + \int_0^t g(x(s), s)\,dB(s)$$



Construction idea

SDE

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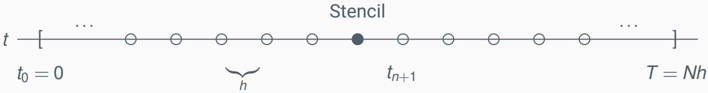

$$x(t_{n+1}) = x(t_n) + \int_{t_n}^{t_{n+1}} f(x(s), s)ds + \int_{t_n}^{t_{n+1}} g(x(s), s)dB(s)$$

Construction idea

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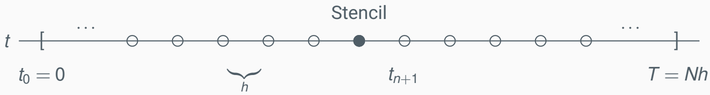

$$x(t_{n+1}) = x(t_n) + \underbrace{\int_{t_n}^{t_{n+1}} f(x(s), s)ds}_{\approx \text{det}} + \underbrace{\int_{t_n}^{t_{n+1}} g(x(s), s)dB(s)}_{\approx}$$

Construction idea

SDE

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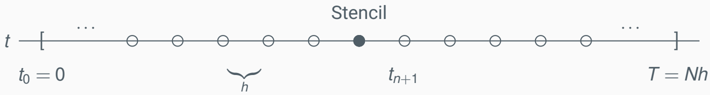

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SDE

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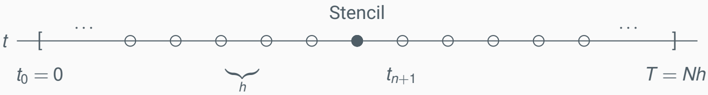

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Construction idea

SDE

$$x(t) = x_0 \quad \int^t \quad \int^t g(x(s),s)dB(s)$$

$$t \quad [\ \cdots \ \circ \ \circ \qquad \circ \ \circ \ \circ \ \circ \ \cdots \ ]$$

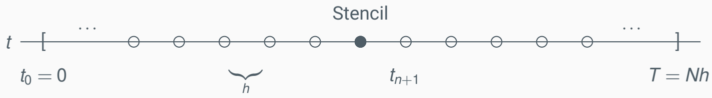
$t_0 = 0$ $\qquad\qquad$ $T = Nh$

$$x(t_{n+1}) = x(t_n) + \underbrace{\int_{t_n}^{t_{n+1}} f(x(s),s)ds}_{\approx f(x_n)h} + \underbrace{\int_{t_n}^{t_{n+1}} g(x(s),s)dB(s)}_{\substack{\approx g(x_n)\Delta B_n \\ \Delta B_n = B_{t_{n+1}} - B_{t_n}}}$$

Construction idea

SDE

$$\boxed{x(t) = x_0 + \int_0^t f(x(s),s)ds + \int_0^t g(x(s),s)dB(s)}$$



$$x(t_{n+1}) = x(t_n) + \underbrace{\int_{t_n}^{t_{n+1}} f(x(s),s)ds}_{\approx f(x_n)h} + \underbrace{\int_{t_n}^{t_{n+1}} g(x(s),s)dB(s)}_{\substack{\approx g(x_n)\Delta B_n \\ \Delta B_n = B_{t_{n+1}} - B_{t_n}}}$$

$$X_0 = x_0, \qquad X_n \approx x(t_n), \qquad n = 1\ldots, N-1$$
$$X_{n+1} = X_n + f(X_n)h + g(X_n)\Delta B_n,$$

Construction idea

SDE

$$x(t) = x_0 + \int_0^t f(x(s),s)\,ds + \int_0^t g(x(s),s)\,dB(s)$$

Stencil

$$x(t_{n+1}) = x(t_n) + \underbrace{\int_{t_n}^{t_{n+1}} f(x(s),s)\,ds}_{\approx f(x_n)h} + \underbrace{\int_{t_n}^{t_{n+1}} g(x(s),s)\,dB(s)}_{\substack{\approx g(x_n)\Delta B_n \\ \Delta B_n = B_{t_{n+1}} - B_{t_n}}}$$

$$X_0 = x_0, \qquad X_n \approx x(t_n), \qquad n = 1\dots, N-1$$

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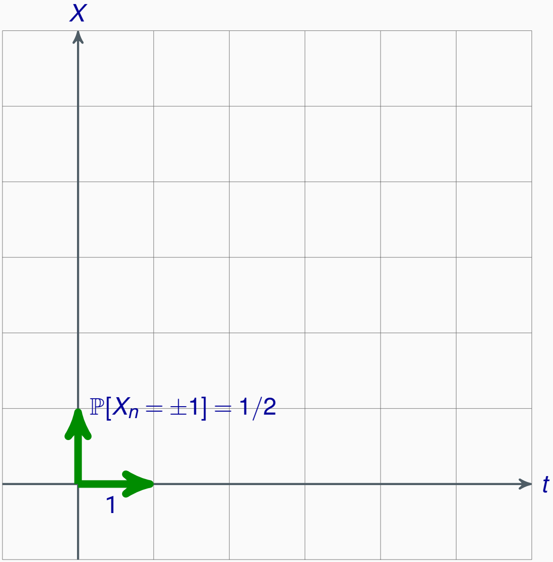
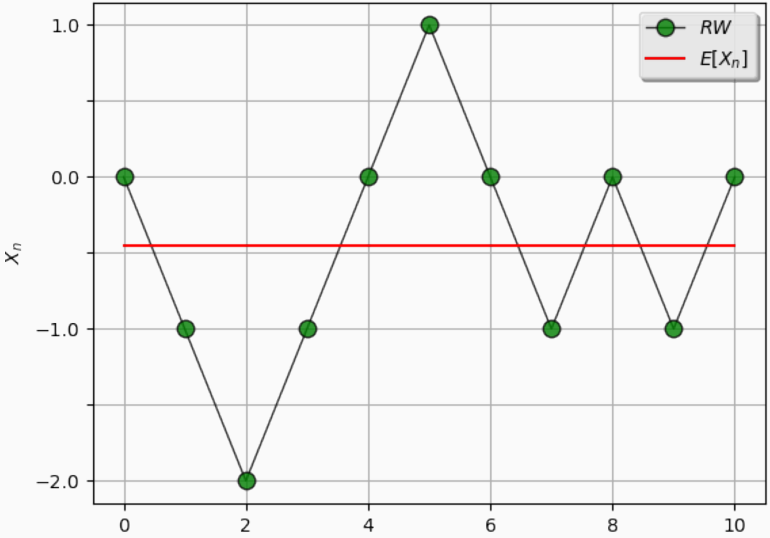
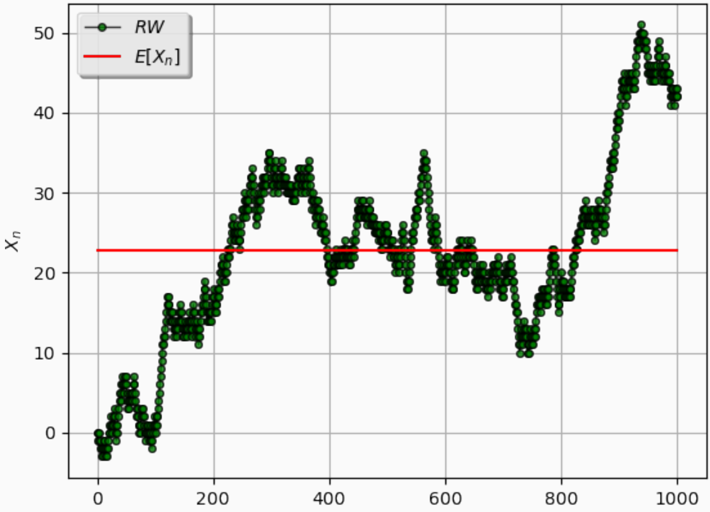
Random Walk



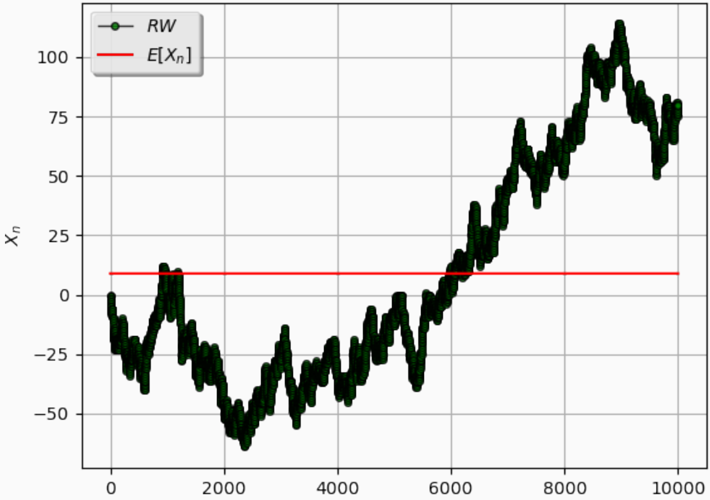


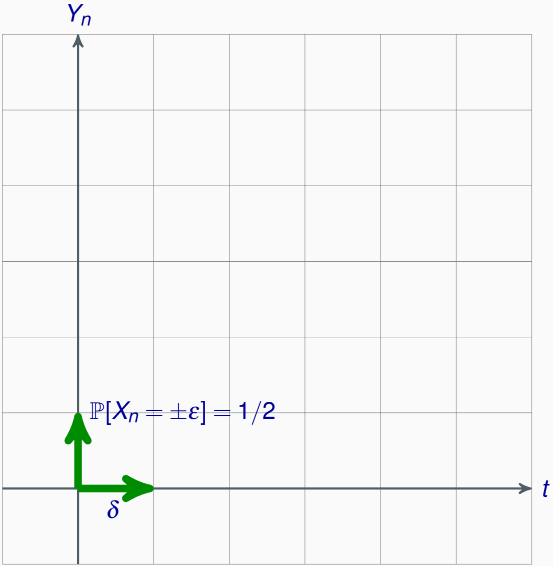
1.0
0.0
−1.0
−2.0
X_n
0
2
4
6
8
10
RW
$E[X_n]$



RW
$E[X_n]$
X_n
50
40
30
20
10
0
0
200
400
600
800
1000

RandomWalk





Y_n
$\mathbb{P}[X_n = \pm\varepsilon] = 1/2$
t
δ

$\{X_n\}_{n=1}^{\infty} \quad i.i.d$

$$\mathbb{P}\left[X_j = \pm\varepsilon\right] = \frac{1}{2}.$$

$$\{X_n\}_{n=1}^{\infty} \quad i.i.d$$

$$\mathbb{P}\left[X_j = \pm\varepsilon\right] = \frac{1}{2}.$$

$$\boxed{\begin{aligned} Y_{\delta,\varepsilon}(0) =& 0 \\ Y_{\delta,\varepsilon}(n\delta) =& X_1 + X_2 + \cdots + X_n. \end{aligned}}$$

$$\{X_n\}_{n=1}^{\infty} \quad i.i.d$$

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$$\begin{aligned} Y_{\delta,\varepsilon}(t) = &\frac{(n+1)\delta - t}{\delta} Y_{\delta,\varepsilon}(n\delta) \\ &+ \frac{t - n\delta}{\delta} Y_{\delta,\varepsilon}((n+1)\delta). \\ &n\delta < t < (n+1)\delta. \end{aligned}$$

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We want

$$\lim_{\substack{\delta \to 0 \\ \varepsilon \to 0}} Y_{\delta,\varepsilon}$$

$$\{X_n\}_{n=1}^{\infty} \quad i.i.d$$
$$\mathbb{P}\left[X_j = \pm\varepsilon\right] = \frac{1}{2}.$$

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We want

$$\lim_{\substack{\delta\to 0 \\ \varepsilon\to 0}} Y_{\delta,\varepsilon}$$

Let $\lambda \in \mathbb{R}$ fixed. Compute

▸ caracteristica $\displaystyle \lim_{\delta,\varepsilon\to 0} \mathbb{E}\left[e^{i\lambda Y_{\delta,\varepsilon}(t)}\right].$

$$\{X_n\}_{n=1}^{\infty} \quad i.i.d$$

$$\mathbb{P}\left[X_j = \pm\varepsilon\right] = \frac{1}{2}.$$

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$$n\delta < t < (n+1)\delta.$$

We want

$$\lim_{\substack{\delta \to 0 \\ \varepsilon \to 0}} Y_{\delta,\varepsilon}$$

$t = n\delta,$

$$\begin{aligned} \mathbb{E}\left[e^{i\lambda Y_{\delta,\varepsilon}(t)}\right] &= \prod_{j=1}^{n} \mathbb{E}\left[e^{i\lambda X_j}\right] \\ &= \left(\mathbb{E}\left[e^{i\lambda X_j}\right]\right)^n \\ &= \left(\frac{1}{2} e^{i\lambda\varepsilon} + \frac{1}{2} e^{-i\lambda\varepsilon}\right)^n \\ &= (\cos(\lambda\varepsilon))^n \\ &= (\cos(\lambda\varepsilon))^{\frac{t}{\delta}}. \end{aligned}$$

$$\{X_n\}_{n=1}^{\infty} \quad i.i.d$$

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We want

$$\lim_{\substack{\delta\to 0\\ \varepsilon\to 0}} Y_{\delta,\varepsilon}$$

$t = n\delta$, $\quad u = (cos(\lambda\varepsilon))^{\frac{1}{\delta}} \quad \ln(u) = \frac{1}{\delta}\ln(cos(\lambda\varepsilon))$
For x, ε small! $\ln(1+x) \approx x$

$$\cos(\lambda\varepsilon) \approx \underbrace{1 - \frac{1}{2}\lambda^2\varepsilon^2}_{1+x}.$$

Then

$$\{X_n\}_{n=1}^{\infty} \quad i.i.d$$

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We want

$$\lim_{\substack{\delta\to 0 \\ \varepsilon \to 0}} Y_{\delta,\varepsilon}$$

$t = n\delta, \quad u = (cos(\lambda\varepsilon))^{\frac{1}{\delta}}$

$$u \approx e^{-\frac{1}{2\delta}\lambda^2\varepsilon^2}$$

$$\mathbb{E}\left[e^{i\lambda Y_{\delta,\varepsilon}(t)}\right] \approx e^{-\frac{1}{2\delta}t\lambda^2\varepsilon^2}.$$

$\varepsilon^2 = \delta$

$$\lim_{\delta\to 0} \mathbb{E}\left[e^{i\lambda Y_{\delta,\sqrt{\delta}}(t)}\right] = e^{-\frac{1}{2}t\lambda^2}, \qquad \lambda \in \mathbb{R}.$$

$$\{X_n\}_{n=1}^{\infty} \quad i.i.d$$

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$$n\delta < t < (n+1)\delta.$$

We want

$$\lim_{\substack{\delta \to 0 \\ \varepsilon \to 0}} Y_{\delta,\varepsilon}$$

$t = n\delta, \quad u = (\cos(\lambda\varepsilon))^{\frac{1}{\delta}}$

$$u \approx e^{-\frac{1}{2\delta}\lambda^2\varepsilon^2}$$

$$\mathbb{E}\left[e^{i\lambda Y_{\delta,\varepsilon}(t)}\right] \approx e^{-\frac{1}{2\delta}t\lambda^2\varepsilon^2}.$$

$\varepsilon^2 = \delta$

$$\lim_{\delta \to 0} \mathbb{E}\left[e^{i\lambda Y_{\delta,\sqrt{\delta}}(t)}\right] = e^{-\frac{1}{2}t\lambda^2}, \qquad \lambda \in \mathbb{R}.$$

$$\boxed{\therefore B(t) \stackrel{\mathscr{D}}{=} \lim_{\delta \to 0} Y_{\delta,\sqrt{\delta}}(t)}$$

Construction

Theorem

Let $Y_{\delta,\varepsilon}(t)$ a random walk starting at 0, with jumps ε , $-\varepsilon$, of equal probability at $\delta, 2\delta, 3\delta, \ldots$. Assume $\varepsilon^2 = \delta$. Then for each $t \geq 0$, the limit

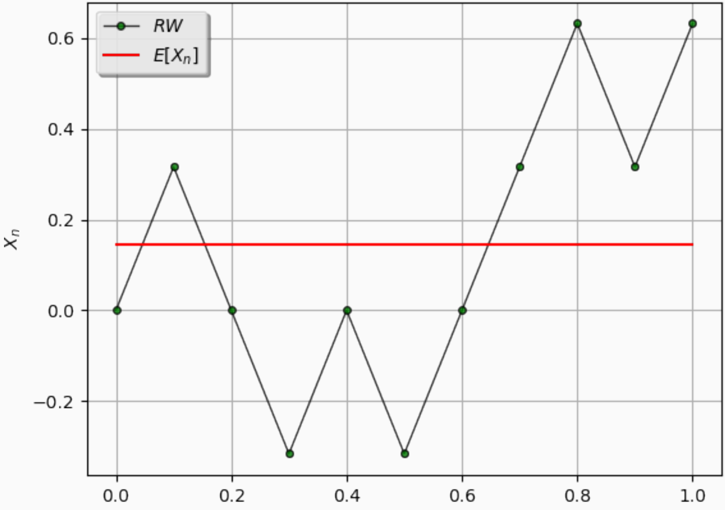
$$B(t) = \lim_{\delta \to 0} Y_{\delta,\sqrt{\delta}}(t),$$

exist in distribution,

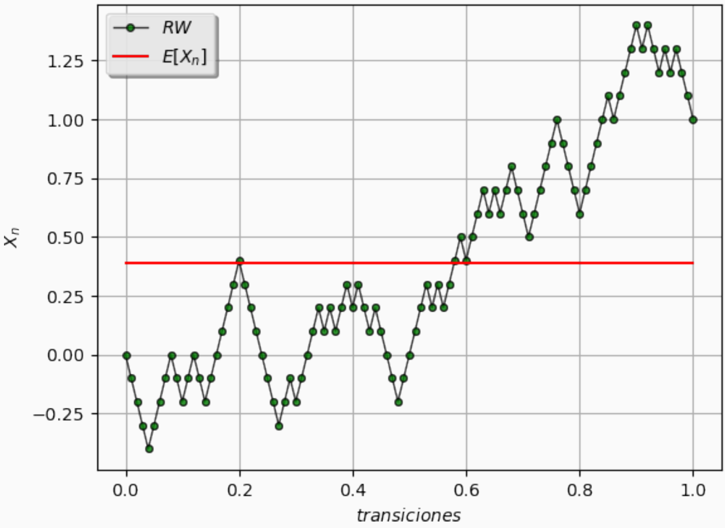
$$\mathbb{E}\left[e^{i\lambda B(t)}\right] = e^{-\frac{1}{2}t\lambda^2}, \qquad \lambda \in \mathbb{R}.$$

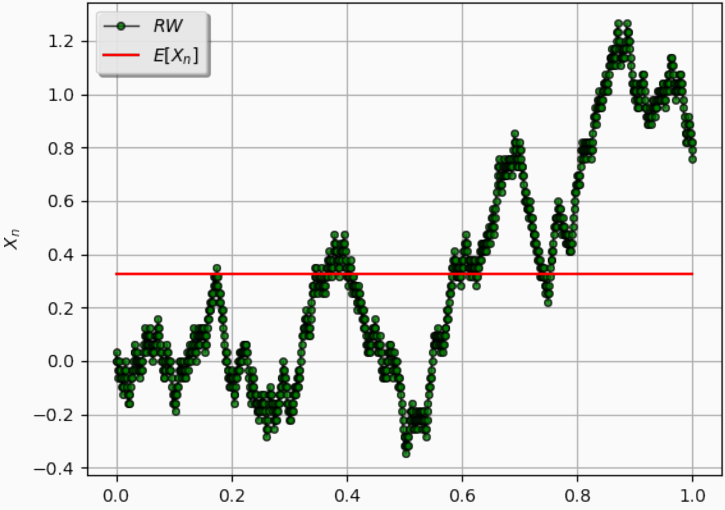
Code

```
N = 10
T = 1.0
delta = T/np.float(N)
eps = 1.0/np.sqrt(np.float(N))
t = np.linspace(0,T,N+1)
b = np.random.binomial(1,.5, N) # bernulli 0,1
omega = 2.0 * b - 1          # bernulli -1,1
Xn = eps * (omega.cumsum())  # bernulli -h,h
Xn = np.concatenate(([0], Xn))
```



RW
E[X_n]
0.6
0.4
0.2
0.0
−0.2
X_n
0.0
0.2
0.4
0.6
0.8
1.0


RW
$E[X_n]$
1.25
1.00
0.75
0.50
0.25
0.00
−0.25
X_n
0.0
0.2
0.4
0.6
0.8
1.0
transiciones

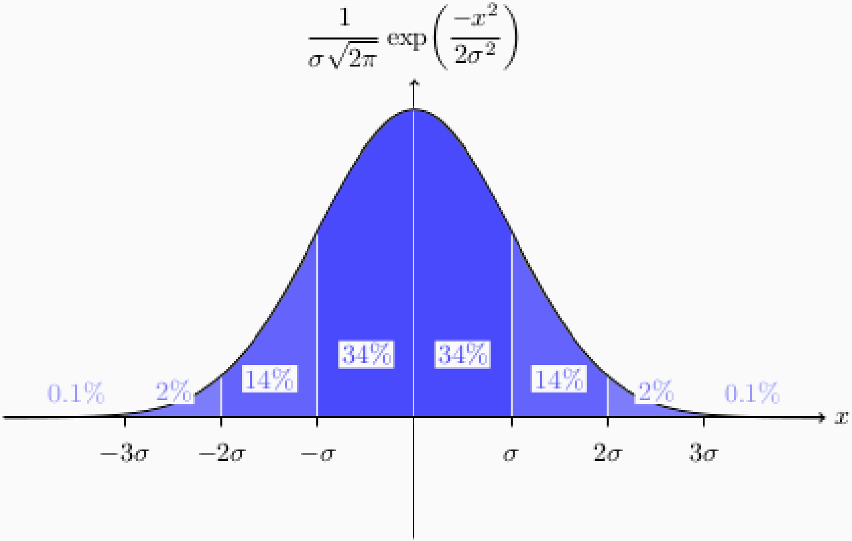

RW
E[X_n]
X_n
1.2
1.0
0.8
0.6
0.4
0.2
0.0
−0.2
−0.4
0.0
0.2
0.4
0.6
0.8
1.0

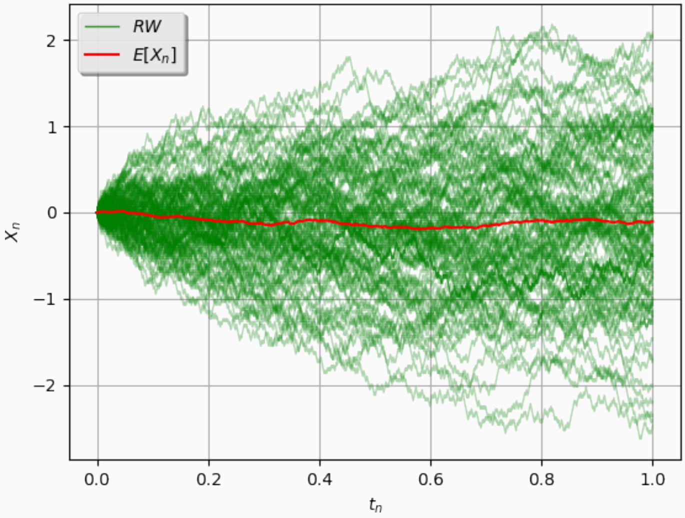
Construcción

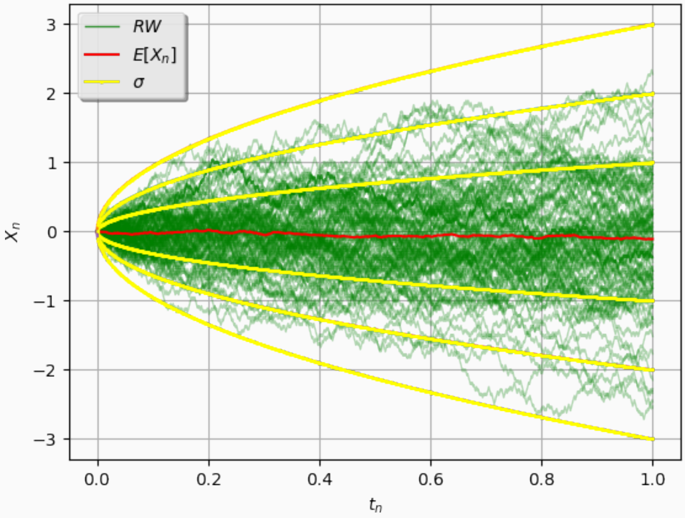
$$\varepsilon^2 = \delta$$

$$Y_{\delta,\varepsilon}(t) \xrightarrow[\delta,\varepsilon\to 0]{\mathscr{D}} B(t) \qquad \forall t \geq 0$$

$$\mathbb{E}\left[e^{i\lambda B(t)}\right] \xrightarrow{\delta,\varepsilon\to 0} e^{-\frac{1}{2}t\lambda^2}, \quad \lambda \in \mathbb{R}.$$


$\frac{1}{\sigma\sqrt{2\pi}} \exp\left(\frac{-x^2}{2\sigma^2}\right)$
0.1%
2%
14%
34%
34%
14%
2%
0.1%
x
-3σ
-2σ
$-\sigma$
σ
2σ
3σ


RW
$E[X_n]$
2
1
0
−1
−2
X_n
0.0
0.2
0.4
0.6
0.8
1.0
t_n



RW
$E[X_n]$
σ
3
2
1
0
−1
−2
−3
X_n
0.0
0.2
0.4
0.6
0.8
1.0
t_n

Strong approximation

Definition

Brownian motion B is the unique process that satisfies

(i) $B(0) = 0$ c.s.

(ii) Para $0 \leq s \leq t$, $B(t) - B(s) \sim \sqrt{t-s}N(0,1)$.

(iii) for each $t_0 \leq t_1 \leq \cdots \leq t_n \in [0, T]$, r.v. $B(t_i) - B(t_j)$ are independent

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$$0 = t_0 \leq t_1 \leq \cdots \leq t_{M-1} \leq t_M = t$$

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Let $\{t_n\}_{n=0}^{N}$, $t_n = nh$,

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$$B(t_n) \approx \sum_{j=0}^{n} \Delta B_j, \qquad \Delta B_0 := 0, \qquad \Delta B_j \sim \sqrt{h}N(0,1).$$

Weak vs Strong

given

$$dx(t) = f(x(t))dt + g(x(t))dB(t),$$
$$x(0) = x_0, \quad t \in [0, T]$$

Weak

$$X_{n+1} = X_n + f(X_n)h + g(X_n) \underbrace{\Delta B_n}_{\substack{\approx \sqrt{h}\varepsilon_n \\ \mathbb{P}[\varepsilon_n = \pm 1] = 1/2}}$$

strong

$$X_{n+1} = X_n + f(X_n)h + g(X_n) \underbrace{\Delta B_n}_{\substack{\approx \sqrt{h}\varepsilon_n \\ \varepsilon_n \sim N(0,1)}}$$

Some representative schemes

- **Implicite**:
  - θ-BEM
  - FBEM
- **Explicit**:
  - Tamed EM
  - Truncated
  - Sabanis

θ-Euler Maruyama

$$
\begin{aligned}
Y_{k+1} &= Y_k + h(1-\theta)f(Y_k) + \theta f(Y_{k+1}) + g(Y_k)\Delta W_k, \\
&\theta \in [0,1].
\end{aligned}
$$

Xuerong Mao and Lukasz Szpruch.
Strong convergence and stability of implicit numerical methods for stochastic differential equations with non-globally lipschitz continuous coefficients.
Journal of Computational and Applied Mathematics, 238:14–28, January 2013.

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Forward-Backward Euler Maruyama

$$
\begin{aligned}
Y_k &= Y_{k-1} + h(1-\theta)f(Y_{k-1}) + \theta f(Y_k) + g(Y_{k-1})\Delta W_{k-1}\\
\widehat{Y}_{k+1} &= \widehat{Y_k} + hf(Y_k) + g(Y_k)\Delta W_k, \qquad \theta \in [0,1].
\end{aligned}
$$

Xuerong Mao and Lukasz Szpruch.
Strong convergence and stability of implicit numerical methods for stochastic differential equations with non-globally lipschitz continuous coefficients.
Journal of Computational and Applied Mathematics, 238:14–28, January 2013.

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Tamed Euler Maruyama

$$Y_{k+1} = Y_k + \frac{hf(Y_k)}{1 + h\|f(Y_k)\|} + g(Y_k)\Delta W_k$$

Martin Hutzenthaler, Arnulf Jentzen, and Peter E. Kloeden.
Strong convergence of an explicit numerical method for sdes with nonglobally lipschitz continuous coefficients.
The Annals of Applied Probability, 22(4):1611–1641, August 2012.

Some representative schemes

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  - Sabanis

Truncated Euler Maruyama

$$
\begin{aligned}
Y_{k+1} &= Y_k + f_\Delta(Y_k)h + g_\Delta(Y_k)\Delta_k, \\
f_\Delta(x) &:= \left(|x| \wedge \mu^{-1}(h(\Delta))\frac{x}{|x|}\right), \\
g_\Delta(x) &:= \left(|x| \wedge \mu^{-1}(h(\Delta))\frac{x}{|x|}\right)
\end{aligned}
$$

Xuerong Mao.
The truncated euler-maruyama method for stochastic differential equations.
Journal of Computational and Applied Mathematics, 290:370 – 384, 2015.

Some representative schemes

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  - Sabanis

Euler Maruyama with varying coefficients

$$Y_{k+1} = Y_k + \frac{hf(Y_k) + g(Y_k)\Delta W_k}{1 + k^{-\alpha}(\|f(Y_k)\| + \|g(Y_k)\|)}, \quad \alpha \in (0, 1/2]$$

Sotirios Sabanis.
Euler approximations with varying coefficients : the case of superlinearly growing diffusion coefficients.
To appear in Annals of Applied Probability, 2015.

SDE solvers

- Julia: juliadiffeq
- Python:
    - StochPy
    - sdeint
    - Py3DE
- R:
    - sde
    - sde.sim.R
- MATLAB: SDE Models

Common hypothesis

$$\boxed{dy(t) = f(y(t))dt + g(y(t))dW_t, \quad Y_0 = y_0.} \qquad \text{(EDE)}$$

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Drifft

$$f : \mathbb{R}^d \to \mathbb{R}^d,$$
$$f = \left(f^{(1)}, \dots, f^{(d)}\right),$$

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Drifft

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$$f = \left(f^{(1)}, \dots, f^{(d)}\right),$$

Difusion

$$g : \mathbb{R}^d \to \mathbb{R}^{d \times m},$$
$$g = \left(g^{(i,j)}\right)_{\substack{i \in \{1,\dots,d\} \\ j \in \{1,\dots,m\}}}$$
$$W = \left(W^{(1)}, \dots, W^{(m)}\right)$$

Common hypothesis

$$\boxed{dy(t) = f(y(t))dt + g(y(t))dW_t, \quad Y_0 = y_0.} \tag{EDE}$$

Hypothesis

(EU-1) Local Lipschitz

(EU-2) Local Lipschitz

(EU-3) Monotony

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(EU-1) Local Lipschitz $\forall R > 0$,

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Hypothesis

(EU-1) Local Lipschitz $\forall R > 0, \quad \exists L_f = L_f(R) > 0$

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Hypothesis

(EU-1) Local Lipschitz $\forall R > 0, \quad \exists L_f = L_f(R) > 0$
$|f(u) - f(v)|^2 \leq L_f |u - v|^2$

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$\Rightarrow$ $\exists!\, y(t)$

EM results

$$dy(t) = f(y(t))dt + g(y(t))dW_t, \quad Y_0 = y_0. \qquad \text{(SDE)}$$

Hypothesis

(H1) $\forall R > 0, \exists\, C_R > 0$

$|f(x) - f(y)|^2 \vee |g(x) - g(y)|^2 \leq C_R |x - y|^2$

$\forall x, y \in \mathbb{R}^d |x| \vee |y| \leq R.$

(H2) Para algún $p > 2$, $\exists\, A > 0$ t.q.

$$\mathbb{E}\left[\sup_{0 \leq t \leq T} |\overline{Y}(t)|^p\right] \vee \mathbb{E}\left[\sup_{0 \leq t \leq T} |y(t)|^p\right] \leq A.$$

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Higham, D. J., Mao, X., and Stuart, A. M. (2002).
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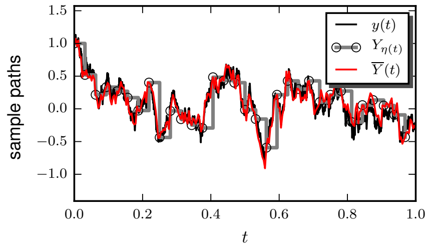
$$\begin{aligned} \overline{Y}(t) &:= Y_{\eta(t)} + (t - t_{\eta(t)}) f(Y_{\eta(t)}) \\ &\quad + g(Y_{\eta(t)})(W(t) - W_{\eta(t)}), \\ \eta(t) &:= k, \text{ for } t \in [t_k, t_{k+1}) \end{aligned}$$

EM results

$$dy(t) = f(y(t))dt + g(y(t))dW_t, \quad Y$$



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$\forall x, y \in \mathbb{R}^d |x| \vee |y| \leq R.$

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$$\overline{Y}(t) := Y_{\eta(t)} + (t - t_{\eta(t)}) f(Y_{\eta(t)}) + g(Y_{\eta(t)})(W(t) - W_{\eta(t)}),$$

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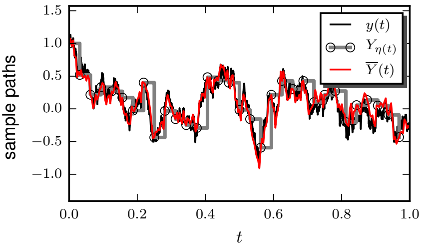
EM results

$$dy(t) = f(y(t))dt + g(y(t))dW_t, \quad Y$$



Hypothesis

(H1) $\forall R > 0, \exists\, C_R > 0$

$$|f(x) - f(y)|^2 \vee |g(x) - g(y)|^2 \leq C_R |x - y|^2$$

$\forall x, y \in \mathbb{R}^d |x| \vee |y| \leq R.$

(H2) Para algún $p > 2$, $\exists\, A > 0$ t.q.

$$\mathbb{E}\left[\sup_{0\leq t\leq T} |\overline{Y}(t)|^p\right] \vee \mathbb{E}\left[\sup_{0\leq t\leq T} |y(t)|^p\right] \leq A.$$

$$\begin{aligned}\overline{Y}(t) &:= Y_{\eta(t)} + (t - t_{\eta(t)}) f(Y_{\eta(t)}) \\ &\quad + g(Y_{\eta(t)})(W(t) - W_{\eta(t)}),\end{aligned}$$

$$\eta(t) := k, \text{ for } t \in [t_k, t_{k+1})$$

Theorem

EM converge

$$\lim_{h\to 0} \mathbb{E}\left[\sup_{0\leq t\leq T} |\overline{Y}(t) - y(t)|^2\right] = 0.$$

Definiciones y resultados previos

Kloeden, P. E. and Platen, E. (1991).
***Numerical Solution of Stochastic Differential Equations*.**
Applications of Matematics. Springer-Verlag.

EDE

$$dy(t) = f(y(t))dt + g(y(t))dW_t, \quad Y_0 = y_0 \tag{EDE}$$

Let Y^h a one step

max h.

$$\varepsilon(h) = \mathbb{E}\left(|y(T) - Y^h(T)|\right)$$

Definition

Y^h at times $(\tau)_h = \{\tau_n : n = 0, 1, \cdots\}$ is strong consistent, si $\exists\ C = C(h) \geq 0, \quad h_0$ s.t. $\forall Y_n^h, n = 1, 2, \ldots, N, \quad h \in (0, h_0)$

- $\lim\limits_{h \downarrow 0} C(h) = 0$
- $\mathbb{E}\left(\left|\mathbb{E}\left(\frac{Y_{n+1}^h - Y_n^h}{h} | \mathscr{F}_{\tau_n}\right) - f\left(Y_n^h\right)\right|^2\right) \leq C(h).$
- $\mathbb{E}\left(\frac{1}{h}\left|Y_{n+1}^h - Y_n^h - \mathbb{E}\left(\frac{Y_{n+1}^h - Y_n^h}{h} | \mathscr{F}_{\tau_n}\right) - g\left(Y_n^h\right)\Delta W_n\right|^2\right) \leq C(h).$

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EDE

$$dy(t) = f(y(t))dt + g(y(t))dW_t, \qquad y_0 = y(0)$$

Y^h a scheme with max h.

$$\varepsilon(h) = \mathbb{E}\left(|y(T) - Y^h(T)|\right)$$

Definition

Y^h is strong convergent to y if at final time T

$$\lim_{h\downarrow 0} \mathbb{E}\left(|y(T) - Y^h(T)|\right) = 0$$

EDE

$$dy(t) = f(y(t))dt + g(y(t))dW_t, \qquad y_0 = y(0)$$

Y^h a scheme with max h.

$$\varepsilon(h) = \mathbb{E}\left(|y(T) - Y^h(T)|\right)$$

Definition (convergence order)

Y^h is strong convergent with order γ, if $\exists$ C independent from h y h_0 s.t.

$$\varepsilon(h) = \mathbb{E}(|y(T) - Y(T)|) \leq Ch^{\gamma} \qquad \forall h \in (0, h_0).$$

SDE

$$dy(t) = f(y(t))dt + g(y(t))dW_t, \qquad y_0 = y(0)$$

Theorem

Consistency implies convergence

[Mao and Szpruch, 2013]

Theorem

(EU-1)-(EU-3) $\Rightarrow \exists! \, \{y(t)\}_{t\geq 0}, \forall y(0) = y_0 \in \mathbb{R}^d$.
Further $0 < T < \infty$,

- $\mathbb{E}\,[y(T)] < \left(|y_0|^2 + 2\alpha T\right) \exp(2\beta T)$,
- $\tau_n := \inf\{t \geq 0 : |y(t)| > n\}, \; n \in \mathbb{N}$,
- $\mathbb{E}\,[|y(t)|^p] \leq 2^{\frac{p-2}{2}} \left(1 + \mathbb{E}\,[|y_0|^p]\right) e^{Cpt}$.

Extension

Mao, X. and Szpruch, L. (2013).
Strong convergence and stability of implicit numerical methods for stochastic differential equations with non-globally lipschitz continuous coefficients.
Journal of Computational and Applied Mathematics, 238:14–28.

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