$$
p_{ji}(\Delta t) := \begin{cases} \frac{\beta i(N-i)}{N}\Delta t, & j = i+1 \\ (b+\gamma) i \Delta t, & j = i-1 \\ 1 - \left[\frac{\beta i(N-i)}{N} + (b+\gamma) i\right]\Delta t, & j = i \\ 0 & \text{otherwise} \end{cases}
$$

$$
\begin{pmatrix}
1 & d(1)\Delta t & 0 & \cdots & 0 & 0 \\
0 & 1-(b+d)(1)\Delta t & d(2)\Delta t & \cdots & 0 & 0 \\
0 & b(1)\Delta t & 1-(b+d)(2)\Delta t & \cdots & 0 & 0 \\
0 & 0 & b(2)\Delta t & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & d(N-1)\Delta t & 0 \\
0 & 0 & 0 & \cdots & 1-(b+d)(N-1)\Delta t & d(N)\Delta t \\
0 & 0 & 0 & \cdots & d(N-1)\Delta t & 1-d(N)\Delta t
\end{pmatrix}
$$

$$\mathbb{E}(I_{t+\Delta t}) = \sum_{i=0}^{N} i p_i(t+\Delta t)$$

$$= \sum_{i=0}^{N} i p_{i-1}(t) b(i-1)\Delta t + \sum_{i=0}^{N-1} i p_{i+1}(t) d(i+1)\Delta t$$

$$+ \sum_{i=0}^{N} i p_i(t)\Delta t - \sum_{i=0}^{N} i p_i(t) b(i)\Delta t - \sum_{i=0}^{N} i p_i(t) d(i)\Delta t$$

$$\mathbb{E}(I_{t+\Delta t}) = \mathbb{E}(I_t) + \sum_{i=1}^{N} p_{i-1}(t)\frac{\beta(i-1)(N-[i-1])}{N}\Delta t - \sum_{i=0}^{N-1} p_{i+1}(t)(b+\gamma)(i+1)\Delta t$$

$$= \mathbb{E}(I_t) + [\beta-(b+\gamma)]\Delta t\,\mathbb{E}(I_t) - \frac{\beta}{N}\Delta t\,\underbrace{\mathbb{E}(I_t^2)}_{\geq \mathbb{E}^2(I_t)}$$

$$\frac{\mathbb{E}(I_{t+\Delta t}) - \mathbb{E}(I_t)}{\Delta t} \leq [\beta-(b+\gamma)]\mathbb{E}(I_t) - \frac{\beta}{N}\mathbb{E}^2(I_t)$$

$$\frac{d\mathbb{E}(I_t)}{dt} \leq [\beta-(b+\gamma)]\mathbb{E}(I_t) - \frac{\beta}{N}\mathbb{E}^2(I_t)$$

$$= \frac{\beta}{N}[N-\mathbb{E}(I_t)]\mathbb{E}(I_t) - (b+\gamma)\mathbb{E}(I_t)$$

Formulating a SIS-CTMC

Let

$$\{I_t\}_{t\geq 0},$$

$$p_i(t) = \mathbb{P}[I_t = i].$$

Thus, the Markov porperty becomes in

$$\mathbb{P}\left[I_{t_{n+1}} \middle| I_{t_0}, \cdots, I_{t_n}\right] = \mathbb{P}\left[I_{t_{n+1}} \middle| I_{t_n}\right]$$

for all $t_0 < t_1 < \cdots < t_n$

$$p_{ji}(\Delta t) := \begin{cases} \frac{\beta i(N-i)}{N}\Delta t + o(\Delta t), & j = i+1 \\ (b+\gamma)i\Delta t + o(\Delta t), & j = i-1 \\ 1 - \left[\frac{\beta i(N-i)}{N} + (b+\gamma)i\right]\Delta t + o(\Delta t), & j = i \\ o(\Delta t) & \text{otherwise} \end{cases}$$

$$\lim_{t\to\infty} \frac{o(\Delta t)}{\Delta t} = 0$$

Using the notation for birth and death processes, we have

$$p_{ij}(\Delta t) := \begin{cases} b(i)\Delta t + o(\Delta t), & j = i+1 \\ d(i)\Delta t + o(\Delta t), & j = i-1 \\ 1 - [b(i) + d(i)]\Delta t + o(\Delta t), & j = i \\ 0 & \text{otherwise} \end{cases}$$

Assuming $\mathbb{P}[I_0 = i_0] = 1$, then

$$\begin{aligned} p_i(t+\Delta t) = & p_{i-1}(t)b(i-1)\Delta t \\ & + p_{i+1}(t)d(i+1)\Delta t \\ & + p_i(t)[1 - (b(i) + d(i))]\Delta t + o(\Delta t) \\ & i = 1, 2, \ldots, N \end{aligned}$$

Thus

$$\begin{aligned} \frac{p_i(t-\Delta t) - p_i(t)}{\Delta t} = & p_{i-1}(t)b(i-1) + p_{i+1}(t)d(i-1) \\ & - p_i[b(i) + d(i)] \\ & i = 1, 2, \cdots, N. \end{aligned}$$

Hence, letting $\Delta t \to 0$, we obtain

$$\begin{aligned} \frac{dp_i(t)}{dt} = & p_{i-1}(t)b(i-1) + p_{i+1}(t)d(i+1) \\ & - p_i[b(i) + d(i)] \\ & i = 1, 2, \cdots, N. \end{aligned}$$

$$\boxed{\begin{array}{l} FKE: \dfrac{dp}{dt} = Qp \\ p(t) = (p_0(t), \ldots, p_N(t))^\top \end{array}}$$

$$Q = \begin{pmatrix} 0 & d(1) & 0 & \cdots & 0 \\ 0 & -[b(1)+d(1)] & d(2) & \cdots & 0 \\ 0 & b(1) & -[b(2)+d(2)] & \cdots & 0 \\ 0 & 0 & b(2) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & d(N) \\ 0 & 0 & 0 & \cdots & -d(N) \end{pmatrix}$$

Results that

$$\lim_{t\to\infty} p(t) = (1, 0, \ldots, 0)^\top$$

and

$$Q = \lim_{\Delta t \to 0} \frac{P(\Delta t) - I}{\Delta t}$$

Expected value of the SIS-CTMC

Considering the m.g.f

$$
\begin{aligned}
M(\theta, t) &:= \mathbb{E}[\exp(\theta I_t)] \\
&= \sum_{i=0}^{N} p_i(t) \exp(i\theta)
\end{aligned}
$$

Results that

$$
\mathbb{E}[I_t^k] = \frac{\partial^k M}{\partial \theta^k}\Big|_{\theta=0}, \quad k = 1, 2, \ldots,
$$

Now we deduce a differential equation for the momments of our sis-CTMC.

$$\frac{\partial M}{t} = \sum_{i=0}^{N} \frac{dp_i \exp(i\theta)}{dt}$$

from FKE

$$= \exp(\theta) \sum_{i=0}^{N} p_{i-1} \exp[(i-1)\theta] b(i-1)$$

$$+ \exp(-\theta) \sum_{i=0}^{N} p_{i+1} \exp[(i+1)\theta] d(i+1)$$

$$- \sum_{i=0}^{N} p_i \exp(i\theta)(b(i) + d(i))$$