The Linear Steklov Method for SDEs with non-globally Lipschitz Coefficients: Strong convergence and simulation.☆

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Abstract

We present an explicit numerical method for solving stochastic differential equations with non-globally Lipschitz coefficients. A linear version of the Steklov average under a split-step formulation supports our new solver. The Linear Steklov method converges strongly with a standard one-half order. Also, we present numerical evidence that the explicit Linear Steklov reproduces almost surely stability solutions with high-accuracy for diverse application models even for stochastic differential systems with super-linear diffusion coefficients.

Keywords: stochastic differential equations; explicit methods; strong convergence; Steklov average.

1. Introduction

Applications of Monte Carlo simulations [6, 7] and Brownian Dynamics [1] require fast numerical methods with low computational cost, thus excludig the use of implicit schemes in most cases. The Euler-Maruyama method is the leader in such simulations due to its simple algebraic structure, a cheap computational cost and an acceptable convergence rate under global Lipschitz conditions. However, if the drift or diffusion coefficients of stochastic differential equations (SDEs) are non-globally Lipschitz functions, then the Euler-Maruyama approximation diverges in the mean square sense [10, 12]. In most cases, the coefficients of the stochastic models in finances, biology or physics have super-linear growth and locally Lipschitz coefficients. Therefore recent research has been focused on modifying the Euler-Maruyama method to obtain strong convergence under these conditions keeping its simple structure and a low computational cost. In the last years, several methods have been developed in this direction: the family of Tamed schemes [9, 11, 21, 25, 26], a special type of balanced method [24], the stopped scheme [15] and a truncated Euler method [17]. In these works, the strong convergence of the new method is proved using the theory developed by Higham, Stuart and Mao in [8] or by means of the new approach given by Hutzenthaler and Jentzen in [9]. Both techniques obtain the property of strong convergence by proving boundedness moments of the numerical and analytical solution of the underlying SDE. In spite of the recent work in this subject, it is still necessary to construct more accurate numerical methods for SDE under super-linear growth and non-globally Lipschitz coefficients.

In this paper, we develop an explicit method based on a linear version of the Steklov method proposed in [3] for the vector Itô stochastic differential equation:

$$dy(t) = f(y(t))dt + g(y(t))dW(t), \quad 0 \leq t \leq T, \quad y(0) = y_0, \tag{1}$$

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where $(f^{(1)},\ldots,f^{(d)}):\mathbb{R}^d\to\mathbb{R}^d$ is one sided Lipschitz and $g=(g^{(j,i)})_{j\in\{1,\ldots,d\},i\in\{1,\ldots,m\}}:\mathbb{R}^d\to\mathbb{R}^{d\times m}$ is global Lipschitz. Also we assume that each component function $f^{(j)}$ can be written of the form

$$f^{(j)}(y)=a_j(y)y^{(j)}+b_j(y^{(-j)}), \tag{2}$$

where a_j and b_j are two scalar functions in $\mathbb{R}^d$ and $y^{(-j)}=\left(y^{(1)},\ldots,y^{(j-1)},y^{(j+1)},\ldots y^{(d)}\right)$. The paper is organized as follows: In section 2 we give known results that are essential for our purposes. In section 3, we construct the new explicit method and prove the always existence of a succession of the Linear Steklov approximation as well as local Lipschitz conditions for its coefficients. In section 3 we prove the strong convergence of the LS method with one-half order using the Higham, Stuart and Mao (HSM) technique and in section 4 its convergence rate is obtained. In section 5 we analyze numerically the accuracy and efficiency of the proposed method applied to stochastic differential equations with super-linear growth and locally Lipschitz coefficients. Finally we give some conclusions.

2. General Settings

Throughout this paper, we work with a standard setup, that is, $y(t)\in\mathbb{R}^d$ for each t and $W(t)$ is a m-dimensional standard Brownian motion on a filtered and complete probability space $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\in[0,T]},\mathbb{P})$, with the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ generated by the Brownian process. Moreover, we denote the norm of a vector $y\in\mathbb{R}^d$ and the Frobenious norm of a matrix $G\in\mathbb{R}^{d\times m}$ by $|y|$ and $|G|$ respectively. The usual scalar product of two vectors $x,y\in\mathbb{R}^d$ is denoted by $\langle x,y\rangle$. The complement of a set A is denoted by A^c and the indicador function of the set Ω is denoted by $\mathbf{1}_{\{\Omega\}}$. In the following, we recall some classical results about the moment boundedness, existence and uniqueness of the solution of the stochastic differential system (1), see [8, 16, 20]. We also state some theorems about the strong convergence of the Euler-Maruyama method given by Higham et al. in [8] which will be useful to prove the strong convergence of the Linear Steklov method.

Let us assume the following:

Hypothesis 2.1. The coefficients of SDE (1) satisfy the conditions:

(H-1) The functions f,g are in the class $C^1(\mathbb{R}^d)$.

(H-2) **Local, global Lipschitz condition**. For each integer n, there is a positive constant $L_f=L_f(n)$ such that

$$|f(x)-f(y)|^2\leq L_f|x-y|^2\qquad\forall x,y\in\mathbb{R}^d,\qquad |x|\vee|y|\leq n,$$

and there is a positive constant L_g such that

$$|g(x)-g(y)|^2\leq L_g|x-y|^2,\qquad\forall x,y\in\mathbb{R}^d.$$

(H-3) **Monotone condition.** There exist two positive constants α and β such that

$$\langle x,f(x)\rangle+\frac{1}{2}|g(x)|^2\leq\alpha+\beta|x|^2,\qquad\forall x\in\mathbb{R}^d. \tag{3}$$

Under Hypothesis 2.1 we can assure existence and uniqueness of the solution of continuous system (1) in order to justify the development of a numerical approximation. Next we state the bounds on the moments of the solution of (1).

Theorem 2.1. *Let $p \geq 2$ and $x_0 \in L^p(\Omega, \mathbb{R}^d)$. Assume that there exits a constant $C > 0$ such that for all $(x,t) \in \mathbb{R}^d \times [t_0, T]$,*

$$\langle x, f(x,t)\rangle + \frac{p-1}{2}|g(x,t)|^2 \leq C(1+|x|^2).$$

Then

$$\mathbb{E}|y(t)|^p \leq 2^{\frac{p-2}{2}}\left(1+\mathbb{E}|y_0|^p\right)e^{Cpt} \qquad \text{for all } t \in [0,T].$$

and

$$\mathbb{E}\left[\sup_{0\leq t\leq T}|y(t)|^p\right] \leq C\left(1+\mathbb{E}|y_0|^p\right).$$

The next theorems give the necessary conditions to assure strong convergence and the convergence rate of the Euler-Maruyama (EM) method.

Theorem 2.2. *Assume Hypothesis 2.1 holds, then the EM scheme given by:*

$$Y_{k+1}^{EM} = Y_k^{EM} + hf(Y_k^{EM}) + g(Y_k^{EM})\Delta W_k, \tag{4}$$

where h is the step-size and its continuous-time extension

$$\overline{Y}^{EM}(t) := Y_0 + \int_0^t f(Y_{\eta(s)}^{EM})\,ds + \int_0^t g(Y_{\eta(s)}^{EM})\,dW(s), \tag{5}$$

where $\eta(t) := k$ for $t \in [t_k, t_{k+1})$, satisfies

$$\lim_{h\to 0}\mathbb{E}\left[\sup_{0\leq t\leq T}|\overline{Y}^{EM}(t) - y(t)|^2\right] = 0. \tag{6}$$

Under the following assumptions, we can get the rate of convergence of the EM scheme.

Hypothesis 2.2. There exist positive constants $L_f, D \in \mathbb{R}$ and $q \in \mathbb{Z}^+$ such that $\forall x, y \in \mathbb{R}^d$

$$\begin{aligned}
\langle x-y, f(x)-f(y)\rangle &\leq L_f|x-y|^2,\\
|f(x)-f(y)|^2 &\leq D_f(1+|x|^q+|y|^q)|x-y|^2,\\
|g(x)-g(y)|^2 &\leq D_g(1+|x|^q+|y|^q)|x-y|^2.
\end{aligned}$$

Hypothesis 2.3. The SDE (1) and the EM solution (4) satisfies

$$\mathbb{E}\left[\sup_{0\leq t\leq T}|y(t)|^p\right] < \infty, \quad \mathbb{E}\left[\sup_{0\leq t\leq T}|Y^{EM}(t)|^p\right] < \infty, \quad \mathbb{E}\left[\sup_{0\leq t\leq T}|\overline{Y}^{EM}(t)|^p\right] < \infty, \qquad \forall p \geq 1.$$

Theorem 2.3. *Under Hypotheses 2.2 and 2.3 the EM solution with continuous extension* (5) *satisfies*

$$\mathbb{E}\left[\sup_{0\leq t\leq T}|\overline{Y}^{EM}(t) - y(t)|^2\right] = \mathcal{O}(h). \tag{7}$$

3. Construction of the Linear Steklov method

In order to construct the LS method, we assume that for each component of the drift coefficient of (1) can be written as

$$f^{(j)}(y) = a_j(y)y^{(j)} + b_j(y^{(-j)}), \qquad y^{(-j)} = (y^{(1)}, \ldots, y^{(j-1)}, y^{(j+1)}, \ldots, y^{(d)}).$$

The Linear Steklov approximation consists in estimating the function f of the SDE (1) (for simplicity $d = j = 1$) by

$$f(y(t)) \approx \varphi_f(y(t_{\eta_+(t)})) = \left(\frac{1}{y(t_{\eta_+(t)}) - y(t_{\eta(t)})} \int\limits_{y(t_{\eta(t)})}^{y(t_{\eta_+(t)})} \frac{du}{a(y(t_{\eta(t)}))u + b} \right)^{-1}, \tag{8}$$

where $\eta_+(t) := k+1$ for $t \in [t_k, t_{k+1})$ and φ_f is the linearized Steklov average [3]. Now, using (8) we develop the new method as follows: First, let us take $0 = t_0 < t_1 < \cdots < t_N = T$ a partition of the interval $[0, T]$ with constant step-size $h = T/N$ and such that $t_k = kh$ for $k = 0, \ldots, N$. By discretization of the SDEs (1), we have for each component equation $j \in \{1, \ldots, d\}$ on the time iteration $k \in \{1, \ldots, N\}$, $a_{j,k} = a_j \left(Y_k^{(1)}, \ldots, Y_k^{(d)} \right)$ and $b_{j,k} = b_j \left(Y_k^{(-j)} \right)$. We define the LS method for the multivariate case under a split-step formulation as:

$$\begin{aligned} Y_k^{\star(j)} &= Y_k^{(j)} + h\, \varphi_{f^{(j)}}(Y_k^\star), \qquad (9)\\ Y_{k+1}^{(j)} &= Y_k^{\star(j)} + g^{(j)}(Y_k^\star)\, \Delta W_k, \qquad (10) \end{aligned}$$

with $Y_0 = y_0$ and $\varphi_f(Y_k^\star) = \left(\varphi_{f^{(1)}}(Y_k^\star), \ldots, \varphi_{f^{(d)}}(Y_k^\star) \right)$ where

$$\varphi_{f^{(j)}}(Y_k^\star) = \left(\frac{1}{Y_k^{\star(j)} - Y_k^{(j)}} \int_{Y_k^{(j)}}^{Y_k^{\star(j)}} \frac{du}{a_{j,k}u + b_{j,k}} \right)^{-1}. \tag{11}$$

The Linear Steklov algorithmic is an implicit scheme, but we can derive an explicit matrix version given by

$$Y_{k+1} = A^{(1)}(h, Y_k)\, Y_k + A^{(2)}(h, Y_k) b(Y_k) + g(Y_k)\, \Delta W_k, \tag{12}$$

where $A^{(1)} = A^{(1)}(h, y)$, $A^{(2)} = A^{(2)}(h, y)$ are defined by

$$A^{(1)} := \begin{pmatrix} e^{ha_1(y)} & & 0 \\ & \ddots & \\ 0 & & e^{ha_d(y)} \end{pmatrix},$$

$$A^{(2)} := \begin{pmatrix} \left(\frac{e^{ha_1(y)} - 1}{a_1(y)} \right) \mathbf{1}_{\{E_1^c\}} & & 0 \\ & \ddots & \\ 0 & & \left(\frac{e^{ha_d(y)} - 1}{a_d(y)} \right) \mathbf{1}_{\{E_d^c\}} \end{pmatrix} + h \begin{pmatrix} \mathbf{1}_{\{E_1\}} & & 0 \\ & \ddots & \\ 0 & & \mathbf{1}_{\{E_d\}} \end{pmatrix}. \tag{13}$$

and $b(u) = \left(b_1(y^{(-1)}), \ldots, b_d(y^{(-d)}) \right)^T$. It is worth mentioning that the explicit formulation (12)-(13) is well defined in the set $E_j := \{ y \in \mathbb{R}^d : a_j(y) = 0 \}$. In the following, we assume two important hypotheses which are crucial to prove existence of the explicit Linear Steklov method.

Hypothesis 3.1. For each component function $f^{(j)} : \mathbb{R}^d :\to \mathbb{R}$ with $j \in \{1, \ldots, d\}$:

(A-1) There are two locally Lipschitz functions of class $C^1(\mathbb{R}^d)$ denoted by $a_j : \mathbb{R}^d \to \mathbb{R}$ and $b_j : \mathbb{R}^{d-1} \to \mathbb{R}$ such that the j-component of the drift function can be rewritten as in (2).

(A-2) There is a positive constant L_a such that

$$a_j(y) \leq L_a, \qquad \forall y \in \mathbb{R}^d.$$

(A-3) Each function $b_j(\cdot)$ satisfies the linear growth condition

$$|b_j(y^{(-j)})|^2 \leq L_b(1+|y|^2), \qquad \forall y \in \mathbb{R}^d.$$

Hypothesis 3.2. Set E_j satisfies either:

(i) All points of $y \in E_j$ which are non isolated zeros of a_j and:

- The set $D_y := \{x \in B_r(y) : a_j(x) = 0\}$ is a smooth curve through y.
- The canonical vector e_j is not tangent to D_y.
- For each $y \in E_j$, there is an open ball with center on y and radio r, $B_r(y)$, such that

$$a_j \neq 0, \qquad \frac{\partial a_j(x)}{\partial x^{(j)}} \neq 0, \qquad \forall x \in B_r(y) \setminus D_y.$$

(ii) All points $y \in E_j$ which are isolated zeros of a_j and:

- For each $y \in E_j$, y is not a limit point of the set $E_\alpha := \{x \in \mathbb{R}^d : (a_j)_\alpha(x) = 0\}$.
- For each $y \in E_j$, there is a subset E_y such that the open segment $\overline{y\,x}$ is in E_y and the y directional derivative at x satisfies

$$(a_j)_\alpha(x) \neq 0, \qquad \forall x \in E_y.$$

By Hypothesis 3.1 there is a unique linear Steklov approximation and Hypothesis 3.2 assures the existence of the directional derivative at an isolated singularity, for references [4, 14]. Under the previous assumptions, we will show that the explicit Linear Steklov approximation (12)-(13) always exists, the function φ is bounded by the drift function f and also the coefficients φ and g satisfy a monotone condition. First, we will give a lemma.

Lemma 3.1. *Assume Hypotheses 2.1, 3.1 and 3.2 hold. The function $\Phi_j(y) = \Phi(h, a_j)(y)$ defined by*

$$\Phi_j(y) := \frac{e^{ha_j(y)} - 1}{ha_j(y)}, \tag{14}$$

is bounded on $\mathbb{R}^d$ for each $j \in \{1, \ldots, d\}$ and we denote its bound by L_Φ.

Proof. By Hypothesis 2.1, the operator Φ is continuous on E_j^c, thus

$$\lim_{h\to 0} \frac{e^{ha_j(y)} - 1}{ha_j} = 1, \tag{15}$$

for each fixed $\in E_j^c$. If $y^* \in E_j$ and fixing any h, by Hypothesis 3.2, we obtain one of the following cases:

$$\lim_{\substack{y\to y^*\\ y\in E_j^c}} \Phi(h, a_j)(y) = \lim_{\substack{y\to y^*\\ y\in E_j^c}} \frac{\frac{\partial a_j(y)}{\partial y^{(j)}} h e^{ha_j(y)}}{h\frac{\partial a_j(y)}{\partial y^{(j)}}} = 1, \tag{16}$$

or

$$\lim_{\substack{y\to y^*\\ y\in E_j^c}} \Phi(h, a_j)(y) = \lim_{\substack{y\to y^*\\ y\in E_j^c}} \frac{\left(e^{ha_j(y)} - 1\right)_\alpha}{\left(ha_j(y)\right)_\alpha} = 1, \qquad \alpha = 0, \pi, 2\pi, \ldots \tag{17}$$

From (15), (16) and (17) we can deduce that

$$\left|\frac{e^{ha_j(y)} - 1}{ha_j(y)}\right| \leq \left|\frac{e^{hL_a} - 1}{ha_j^*}\right|, \qquad \forall y \in \mathbb{R}^d. \tag{18}$$

where $a_j^* := \inf_{y\in E_j^c}\{|a_j(y)|\}$. If $a_j^* = 0$, we can use an argument similar to (16)–(17). $\square$

We can now state the following theorem.

Theorem 3.1. *Assume Hypotheses 2.1, 3.1 and 3.2 hold, and $A^{(1)}$, $A^{(2)}$, b defined by (13). Then given $u \in \mathbb{R}^d$, the equation*

$$v = u + h\varphi_f(v), \tag{19}$$

has a unique solution

$$v = A^{(1)}(h,u)u + A^{(2)}(h,u)b(u). \tag{20}$$

If we define the functions $F_h(\cdot)$, $\varphi_{f_h}(\cdot)$ and $g_h(\cdot)$ by

$$F_h(u) = v, \qquad \varphi_{f_h}(u) = \varphi_f(F_h(u)), \qquad g_h(u) = g(F_h(u)), \tag{21}$$

then $F_h(\cdot)$, $\varphi_{f_h}(\cdot)$, $g_h(\cdot)$ are local Lipschitz functions and for all $u \in \mathbb{R}^d$ and each h fixed, there is a positive constant L_Φ such that

$$|\varphi_{f_h}(u)| \leq L_\Phi |f(u)|. \tag{22}$$

Moreover, for each h fixed, there are positive constants α^* and β^* such that

$$\langle \varphi_{f_h}(u), u\rangle \vee |g_h(u)|^2 \leq \alpha^* + \beta^*|u|^2, \qquad \forall u \in \mathbb{R}^d. \tag{23}$$

Proof. Let us first prove that (20) is solution of equation (19). Note that

$$v^{(j)} = u^{(j)} + h\,\varphi_{f^{(j)}}(v), \tag{24}$$

for each $j \in \{1,\dots,d\}$ and using the linear Steklov function (8), we can derive that

$$v^{(j)} = e^{ha_j(u)}u^{(j)} + \left[h\Phi_j(u)\mathbf{1}_{\{E_j^c\}} + h\mathbf{1}_{\{E_j\}}\right] b_j(u^{(-j)}), \tag{25}$$

which is the j-component of the vector $A^{(1)}u + A^{(2)}b(u)$. Now we prove inequality (22). Given that $v = \varphi_f(F_h(u))$, we can also rewrite (24) as

$$\varphi_{f_h}^{(j)}(u) = \frac{F_h^{(j)}(u) - u^{(j)}}{\displaystyle\int_{u^{(j)}}^{F_h^{(j)}(u)} \frac{dz}{a_j(u)z + b_j(u^{(-j)})}}.$$

If $u \in E_j$ then $\varphi_{f_h}^{(j)}(u) = b_j(u^{(-j)}) = f^j(u)$, so $L_\Phi \geq 1$ fulfills (22). On the other hand, if $u \in E_j^c$ then

$$\varphi_{f_h}^{(j)}(u) = \frac{(F_h^{(j)}(u) - u^{(j)})a_j(u)}{\underbrace{\ln\left(a_j(u)F_h^{(j)}(u) + b_j(u^{(-j)})\right)}_{:=R_1} - \ln\left(a_j(u)u^{(j)} + b_j(u^{(-j)})\right)} = \Phi_j(u)f^j(u), \tag{26}$$

where

$$R_1 = \ln\left\{a_j(u)\left[e^{ha_j(u)}u^{(j)} + h\Phi_j(u)b_j(u^{(-j)})\right] + b_j\left(u^{(-j)}\right)\right\} = ha_j(u) + \ln\left(f^{(j)}(u)\right).$$

By lemma 3.1, inequality (22) is satisfied for all $u \in E_j \cup E_j^c$. As $g_h(x) = g(F_h(x))$, by Hypothesis 2.1, then

$$|g_h(u) - g_h(v)|^2 \leq L_g|F_h(u) - F_h(v)|^2 \leq 2L_g \underbrace{|A^{(1)}u - A^{(1)}v|^2}_{:=R_2} + 2L_g \underbrace{|A^{(2)}b(u) - A^{(2)}b(v)|^2}_{:=R_3}. \tag{27}$$

Let us consider each term of the right hand of inequality (27). First, note that $A^{(1)}$ is a continuous differentiable function on all $\mathbb{R}^d$, so using the mean value theorem, we have

$$R_2 \leq L_{A^{(1)}}|u-v|^2, \quad u,v \in \mathbb{R}^d, \quad |u|\vee|v|\leq n, \tag{28}$$

for a positive constant $L_{A^{(1)}} \geq \sup_{0\leq t\leq 1}|\partial A^{(1)}(h, u+t(v-u))|^2$. Meanwhile,

$$\begin{aligned} R_3 &= \sum_{j=1}^{d}\Big[\mathbf{1}_{\{E_j^c\}}(u)\Phi_j(u)b_j(u^{(-j)}) + h\mathbf{1}_{\{E_j\}}(u)b_j(u^{(-j)}) - \mathbf{1}_{\{E_j^c\}}(v)\Phi_j(v)b_j(u^{(-j)}) \\ &- h\mathbf{1}_{\{E_j\}}(v)b_j(v^{(-j)})\Big]^2 \leq 4\sum_{j=1}^{d}\Big[\left(\mathbf{1}_{\{E_j^c\}}(u)L_\Phi b_j(u^{(-j)})\right)^2 + \left(h\mathbf{1}_{\{E_j\}}(u)b_j(u^{(-j)})\right)^2 \\ &+ \left(\mathbf{1}_{\{E_j^c\}}(v)L_\Phi b_j(v^{(-j)})\right)^2 + \left(h\mathbf{1}_{\{E_j\}}(v)b_j(v^{(-j)}\right)^2\Big]. \end{aligned} \tag{29}$$

Since $b_j^2(\cdot)$ is a function of class $C^1(\mathbb{R}^d)$, there is a constant $L_b = L_b(n)$ such that

$$|b_j(u)|^2 \leq L_b, \quad \forall u \in \mathbb{R}^d, \quad |u|\vee|v|\leq n, \tag{30}$$

for each $j \in \{1, \cdots, d\}$. Using this bound in (29), we obtain

$$R_3 \leq 4\sum_{j=1}^{d}\left[2L_\Phi L_b + 2h^2 L_b\right] \leq L_0, \qquad \forall u,v \in \mathbb{R}^d, \quad |u|\vee|v|\leq n, \tag{31}$$

where $L_0 = 8dL_b(n)(L_\Phi + h^2)$. By inequalities (28) and (31), we get

$$|g_h(u) - g_h(v)|^2 \leq L_{g_h}(n)|u-v|^2, \qquad \forall u,v \in \mathbb{R}^d, \quad |u|\vee|v|\leq n, \tag{32}$$

where $L_{g_h}(n) \geq n^2 + 1 + L_0 + L_{A^{(1)}}$. Then $g_h(\cdot)$ is a locally Lipschitz function. Furthermore, note that under some modifications this argument can be used to prove that $F_h(\cdot)$ is also a locally Lipschitz function, which implies that φ_{f_h} is a locally Lipschitz function. Finally, we will demonstrate inequality (23). By Hypotheses 2.1 and 3.1, we have

$$\langle f(u), u\rangle = \sum_{j=1}^{d} a_j(u)\left(u^{(j)}\right)^2 + \sum_{j=1}^{d} b_j(u)u^{(j)} \leq \alpha + \beta|u|^2,$$

and

$$\langle b(u), u\rangle \leq \alpha + (\beta + L_a)|u|^2.$$

Using these inequalities and (22), we deduce that

$$\langle \varphi_{f_h}(u), u\rangle = \sum_{j=1}^{d}\Phi_j(u)f^{(j)}(u)u^{(j)} \leq L_\Phi L_a|u| + L_\Phi(\alpha + (L_a+\beta)|u|^2) \leq L_{\varphi_{f_h}}(1+|u|^2). \tag{33}$$

where $L_{\varphi_{f_h}} \geq 2L_\Phi \max\{L_a, \alpha, \beta\} + 1$. Meanwhile, since g is globally Lipschitz then

$$|g_h(u)|^2 \leq 2|g(F_h(u)) - g(F_h(0))|^2 + 2|g(F_h(0))|^2 \leq 4L_g|F_h(u)|^2 + 8L_g|F_h(0)|^2 + 4|g(0)|^2. \tag{34}$$

Now, we bound each term on the right-hand side of (34). By the monotone condition (3), $|g(0)|^2 \leq 2\alpha$. Moreover,

$$|F_h^{(j)}(0)| = h\Phi_j(0)\,|b_j(0)|\mathbf{1}_{\{E_j^c\}}(0) + h\,|b_j(0)|\mathbf{1}_{\{E_j\}}(0) \leq \frac{b_0^*}{a_0^*}e^{hL_a}(1+h), \quad \forall j \in \{1, \cdots, d\}.$$

where $a_0^* := \min_{\substack{j\in\{1,\cdots,d\}\\ a_j(0)\neq 0}} \{|a_j(0)|\}$ and $b_0^* := \max_{j\in\{1,\cdots,d\}} \{|b_j(0)|\}$. Then

$$|F_h(0)|^2 \leq d \left(\frac{b_0^*}{a_0^*}\right)^2 e^{2hL_a}(1+h)^2. \tag{35}$$

As Φ_j is bounded from (25), we get

$$F_h^{(j)}(u) \leq e^{ha_j(u)}|u^{(j)}| + hL_\Phi |b_j(u)| \mathbf{1}_{\{E_j^c\}}(u) + h|b_j(u)|\mathbf{1}_{\{E_j\}}(u).$$

And by Hypothesis 3.1,

$$|F_h^{(j)}(u)|^2 \leq 3e^{2hL_a}|u|^2 + (3h^2L_\Phi^2 L_b + 3h^2 L_b)(1+|u|^2) \leq L_F(1+|u|^2), \tag{36}$$

where $L_F \geq 3d\max\{e^{2hL_a}, h^2L_b(L_\Phi^2+1)\}$. Using (35) and (36) in inequality (34) yields

$$|g_h(u)|^2 \leq 4L_gL_F(1+|u|^2) + 8L_g d\left(\frac{b_0^*}{a_0^*}\right)^2 e^{2hL_a}(1+h)^2 + 8\alpha.$$

Therefore, if $L_{g_h} \geq 4L_gL_F + 8L_gd\left(\frac{b_0^*}{a_0^*}\right)^2 e^{2hL_a}(1+h)^2 + 8\alpha$ then

$$|g_h(u)|^2 \leq L_{g_h}(1+|u|^2). \tag{37}$$

Hence, from inequalities (33) and (37) and taking for each fixed $h > 0$, $\alpha^* := L_{\varphi_{f_h}} \vee L_{g_h}$ and $\beta^* := 2\alpha^*$, we obtain inequality (23). □

Remark 3.1. It is worth mentioning that if $b_j = 0$ in (A-1) then Hypotheses 3.1 and 3.2 are unnecessary to prove Theorem 3.1. Several applications as some stochastic Lotka-Volterra systems [18, 19], the Ginzburg-Landau SDE [13] or the damped Langevin Equations where the potential lacks of a constant term [11]. By other hand, if $bj \neq 0$ then SDE as the stochastic SIR [23], the noisy Duffing-Van der Pol Oscillator [22], the stochastic Lorenz equation [5], among others follow this structure.

Theorem **??** holds the key to demonstrate strong convergence of the explicit LS method.

4. Strong Convergence of the Linear Steklov Method

To prove strong convergence of the Linear Steklov method (12)-(13) for SDEs with locally Lipschitz coefficients, we rely on the Higham-Mao-Stuart (HMS) technique [8]. Recently, several works have used this procedure to establish strong convergence for some particular scheme among others [9–11, 20, 24]. The proof is rather technical and can be divided into the following steps:

Step 1: The explicit LS method (12)-(13) is equivalent to the Euler-Maruyama method (4) for the conveniently modified SDE

$$dy_h(t) = \varphi_{f_h}(y_h(t))dt + g_h(y_h(t))dW(t), \qquad y_h(0) = y_0, \qquad t \in [0,T], \tag{38}$$

which is a perturbation of SDE (1) given that $|f(x) - \varphi_{f_h}(x)| = \mathcal{O}(h)$.

Step 2: The solution of the modified SDE (38) has bounded moments and it is "close" to the solution of the SDEs (1) in the sense of the uniform mean square norm $\mathbb{E}\left[\sup_{0\leq t\leq T}|\cdot|^2\right]$.

Step 3: The explicit LS method has bounded moments.

Step 4: There is a continuous extension of the explicit LS method of the form:

$$\overline{Y}(t) := Y_0 + \int_0^t \varphi_{f_h}(Y_{\eta(s)})\, ds + \int_0^t g(Y_{\eta(s)})\, dW(s), \tag{39}$$

which has bounded moments.

Proving the above statements and using theorem 2.2, we can conclude the strong convergence of the explicit LS method. Step 1 is formalized in the following corollary.

Corollary 4.1. *Assume Hypotheses 2.1, 3.1 and 3.2 hold, then the explicit LS method* (12)-(13) *for SDE* (1) *is equivalent to the Euler-Maruyama scheme applied to the modified SDE* (38).

Proof. Using the functions $\varphi_{f_h}(\cdot)$ and $g_h(\cdot)$ defined in (21), we can rewrite the explicit LS method as

$$Y_{k+1} = Y_k + h\varphi_{f_h}(Y_k) + g_h(Y_k)\Delta W_k,$$

which is the EM approximation for the modified SDE (38). □

Now we proceed with step 2.

Lemma 4.1. *Assume Hypotheses 2.1, 3.1 and 3.2 hold, then there is a universal constant* $C = C(p,T) > 0$ *and a sufficiently small step size* h*, such that for all* $p > 2$

$$\mathbb{E}\left[\sup_{0\le t\le T} |y_h(t)|^p\right] \le C\left(1 + \mathbb{E}|y_0|^p\right). \tag{40}$$

Moreover

$$\lim_{h\to 0} \mathbb{E}\left[\sup_{0\le t\le T} |y(t) - y_h(t)|^2\right] = 0. \tag{41}$$

Proof. By theorem 2.1 and inequality (23), we have bound (40). On the other hand, to prove (41) we will use the properties of φ_{f_h} and the Higham's stopping time technique employed in [8]. Note that by relationship (26), we can rewrite the Steklov function as

$$\varphi_{f_h}(u) = f^{(j)}(u)\mathbf{1}_{\{E_j\}}(u) + \Phi_j(u) f^{(j)}(u)\mathbf{1}_{\{E_j^c\}}(u). \tag{42}$$

Let $|u|< n$ by lemma 3.1 we have $|\Phi(u)|\le L_\Phi$ and since $f \in C^1(\mathbb{R}^d)$, there is a positive constant R_n, which depends on n, such that

$$\begin{aligned}
|\varphi_{f_h}^{(j)}(u) - f^{(j)}(u)| &\le \mathbf{1}_{\{E_j^c\}}(u)|f^{(j)}(u)||\Phi_j(u) - 1| \\
&\le \mathbf{1}_{\{E_j^c\}}(u)\left(L_\Phi + 1\right)|f(u)| \\
&\le \mathbf{1}_{\{E_j^c\}}(u) R_n (L_\Phi + 1), \quad \forall u \in \mathbb{R}^d, \quad |u|\le n,
\end{aligned}$$

for each $j \in \{1,\ldots,d\}$. At the interface betweeen E_j and E_j^c, we know by the scalar L' Hôpital theorem that $\lim_{h\to 0} \Phi_j(u) = 1$ for each fixed $u \in E_j^c$. And if $u \in E_j$, by Hypothesis 3.2 and using L'Hôpital theorem for multivariable functions [4, 14], we have

$$\lim_{h\to 0} F_h^{(j)}(u) = \lim_{h\to 0} e^{h a_j(u)} u^{(j)} + \lim_{h\to 0}\left(\Phi_j(u)\mathbf{1}_{\{E_j^c\}}(u) + h\mathbf{1}_{\{E_j\}}(u)\right) b_j(u^{(j)}) = u^{(j)},$$

for each $j \in \{1,\ldots,d\}$, hence $\lim_{h\to 0} F_h(u) = u$. In addition, g_h and g are locally Lipschitz. So, given $n > 0$ there is a function $K_n(\cdot) : (0,\infty) \to (0,\infty)$ such that $K_n(h) \to 0$ when $h \to 0$ and

$$|\varphi_{f_h}(u) - f(u)|^2 \vee |g_h(u) - g(u)|^2\le K_n(h) \qquad \forall u \in \mathbb{R}^d, \quad |u|\le n. \tag{43}$$

As f, g are C^1 there is a constant $H_n > 0$ such that

$$|f(u) - f(v)|^2 \vee |g(u) - g(v)|^2\le H_n|u - v|^2 \qquad \forall u, v \in \mathbb{R}^d, \quad |u|\vee|v|\le n. \tag{44}$$

The rest of the proof run as in [8, Lem. 3.6]. □

In the following, we proceed with step 3 which establishes that LS method has bounded moments. This is the key step in the HMS technique.

Lemma 4.2. *Assume Hypotheses 2.1, 3.1 and 3.2 hold, then for each $p \geq 2$ there is a universal positive constant $C = C(p, T)$ such that for the explicit LS method* (12)-(13) *is satisfied*

$$\mathbb{E}\left[\sup_{kh\in[0,T]} |Y_k|^{2p}\right] \leq C.$$

Proof. Using a split formulation of the LS scheme (12)-(13) as follows:

$$\begin{aligned} Y_k^{\star(j)} &= A^{(1)}(h, Y_k)Y_k + A^{(2)}(h, Y_k)b(Y_k), \\ Y_{k+1}^{(j)} &= Y_k^{\star(j)} + g^{(j)}(Y_k^\star)\,\Delta W_k, \end{aligned}$$

from the first step of this split scheme, using (A-3) and the Cauchy-Schwartz inequality, we get

$$\begin{aligned} |Y_k^\star|^2 &\leq |A_k^{(1)}|^2|Y_k|^2 + 2\left\langle A_k^{(1)}Y_k, A_k^{(2)}Y_k b_k\right\rangle + |A_k^{(2)}|^2|b_k|^2 \\ &\leq |A_k^{(1)}|^2|Y_k|^2 + 2\sqrt{L_b}d|A_k^{(1)}||A_k^{(2)}||Y_k|(1+|Y_k|) + L_b|A_k^{(2)}|^2(1+|(Y_k)|^2). \end{aligned} \tag{45}$$

denoting by $A_k^{(i)} := A^{(i)}(h, Y_k)$ for $i = 1, 2$ and $b_k := b(Y_k)$. From (A-2), we can deduce that

$$|A_k^{(1)}|^2 = \left|\mathrm{diag}\left(e^{ha_1(Y_k)}, \ldots, e^{ha_d(Y_k)}\right)\right|^2 \leq M_{A^{(1)}}, \tag{46}$$

where $M_{A^{(1)}} = d\,e^{2TL_a}$ and also by (18), we can derive that

$$\begin{aligned} |A^{(2)}(h, Y_k)|^2 &= \left|h\,\mathrm{diag}\left(\mathbf{1}_{\{E_1\}}(Y_k) + \mathbf{1}_{\{E_1^c\}}(Y_k)\Phi_1(Y_k), \ldots, \mathbf{1}_{\{E_d\}}(Y_k) + \mathbf{1}_{\{E_d^c\}}(Y_k)\Phi_d(Y_k)\right)\right|^2 \\ &\leq \sum_{j=1}^{d}\left(\mathbf{1}_{\{E_j^c\}}|h\Phi_j(Y_k)|^2 + h^2\right) \leq 2e^{2L_aT}\sum_{j=1}^{d}\frac{1}{a_j^*} + dT^2 \leq M_{A^{(2)}}. \end{aligned} \tag{47}$$

Substituting (46) and (47) on inequality (45) yields

$$\begin{aligned} |Y_k^\star|^2 &\leq L_{A^{(1)}}|Y_k|^2 + 2d\sqrt{L_{A^{(1)}}L_{A^{(2)}}L_b}\,|Y_k|(1+|Y_k|) + L_{A^{(2)}}L_b(1+|Y_k|^2) \\ &\leq \widetilde{C}(3|Y_k|^2 + |Y_k| + 1) \leq C(1+|Y_k|^2), \end{aligned} \tag{48}$$

where $\widetilde{C} \geq L_{A^{(1)}} + 2d\sqrt{L_{A^{(1)}}L_{A^{(2)}}L_b} + L_{A^{(2)}}L_b$ and $C = 6\widetilde{C}$. Applying bound (48) in the second step of the split scheme, we get

$$|Y_{k+1}|^2 \leq C\left(|Y_k|^2 + 1\right) + 2\left\langle Y_k^\star, g(Y_k^\star)\Delta W_k\right\rangle + \left|g(Y_k^\star)\Delta W_k\right|^2.$$

Now, we choose two integers N, M such that $Nh \leq Mh \leq T$. So, adding backwards we obtain

$$|Y_N|^2 \leq S_N\left(\sum_{j=0}^{N-1}(1+|Y_j|^2) + 2\sum_{j=0}^{N-1}\left\langle Y_j^\star, g(Y_j^\star)\Delta W_j\right\rangle + \sum_{j=0}^{N-1}\left|g(Y_j^\star)\Delta W_j\right|^2\right),$$

where $S_N := \sum_{j=0}^{N-1} C^{N-j}$. Raising both sides to the power p, we can derive that

$$|Y_N|^{2p} \leq 6^p S_N^p\left(N^{p-1}\sum_{j=0}^{N-1}(1+|Y_j|^{2p}) + \left|\sum_{j=0}^{N-1}\left\langle Y_j^\star, g(Y_j^\star)\Delta W_j\right\rangle\right|^p + N^{p-1}\sum_{j=0}^{N-1}\left|g(Y_j^\star)\Delta W_j\right|^{2p}\right). \tag{49}$$

Now we will show that the second and third terms of inequality (49) are bounded. We denote by $C = C(p,T)$ a generic positive constant which does not depend on the step size h and whose value may change between occurrences. Next, applying the Bunkholder-Davis-Gundy inequality [16], we have

$$\begin{aligned}
\mathbb{E}\left[\sup_{0\le N\le M}\left|\sum_{j=0}^{N-1}\langle Y_j^\star, g(Y_j^\star)\Delta W_j\rangle\right|^p\right] &\le C\mathbb{E}\left[\sum_{j=0}^{N-1}|Y_j^\star|^2|g(Y_j^\star)|^2h\right]^{p/2} \\
&\le Ch^{p/2}M^{p/2-1}\mathbb{E}\sum_{j=0}^{M-1}|Y_j^\star|^p(\alpha+\beta|Y_j^\star|^2)^{p/2} \\
&\le 2^{p/2-1}CT^{p/2-1}h\mathbb{E}\sum_{j=0}^{M-1}(\alpha^{p/2}|Y_j^\star|^p+\beta^{p/2}|Y_j^\star|^{2p}) \\
&\le Ch\mathbb{E}\sum_{j=0}^{M-1}(1+2|Y_j^\star|^p+|Y_j^\star|^{2p}) \\
&\le C+Ch\sum_{j=0}^{M-1}\mathbb{E}|Y_j|^{2p}, \qquad (50)
\end{aligned}$$

Now, using the Cauchy-Schwartz inequality, the monotone condition (3) and bound (48), we obtain

$$\begin{aligned}
\mathbb{E}\left[\sup_{0\le N\le M}\sum_{j=0}^{N-1}\left|g(Y_j^\star)\Delta W_j\right|^{2p}\right] &\le \sum_{j=0}^{M-1}\mathbb{E}\left|g(Y_j^\star)\right|^{2p}\mathbb{E}\left|\Delta W_j\right|^{2p} \\
&\le Ch^p\sum_{j=0}^{M-1}\mathbb{E}\left[\alpha+\beta|Y_j^\star|^2\right]^p \\
&\le Ch^p\sum_{j=0}^{M-1}\mathbb{E}\left[\alpha^p+\beta^p|Y_j^\star|^{2p}\right] \\
&\le Ch^{p-1}+Ch^p\sum_{j=0}^{M-1}\mathbb{E}|Y_j|^{2p}. \qquad (51)
\end{aligned}$$

Thus, combining bounds (50) and (51) with inequality (49), we can assert that

$$\mathbb{E}\left[\sup_{0\le N\le M}|Y_N|^{2p}\right]\le C+C(1+h)\sum_{j=0}^{M-1}\mathbb{E}\left[\sup_{0\le N\le j}|Y_N|^{2p}\right]. \qquad (52)$$

Finally, using the discrete-type Gronwall inequality [16], we conclude that

$$\mathbb{E}\left[\sup_{0\le N\le M}|Y_N|^{2p}\right]\le Ce^{C(1+h)M}\le Ce^{C(1+T)}<C,$$

since the constant C does not depend on h, the proof is complete. □

Let $\{Y_k\}$ denote the explicit LS solution of SDE (1) defined by the recurrence equations (12)-(13). By corollary 4.1, we can give a continuous extension for the LS approximation from the time continuous EM extension (5). Moreover, we will also prove in the following theorem that the moments of the continuous LS extension remain bounded.

Corollary 4.2. *Assume Hypotheses 2.1, 3.1 and 3.2 hold and suppose* $0 < h < 1$ *and* $p \geq 2$. *Then there is a continuous extension* $\overline{Y}(t)$ *of the Linear Steklov recurrence* $\{Y_k\}$ *defined by* (12)-(13) *and a positive constant* $C = C(T,p)$ *such that*

$$\mathbb{E}\left[\sup_{0\leq t\leq T}|\overline{Y}(t)|^{2p}\right] \leq C.$$

Proof. We take $t = s + t_k$ in $[0,T]$ such that $\Delta W_k(s) := W(t_k+s) - W(t_k)$ for $0 \leq s < h$. Then we define

$$\overline{Y}(t_k+s) := Y_k + s\varphi_{f_h}(Y_k) + g_h(Y_k)\Delta W_k(s), \tag{53}$$

as a continuous extension of the LS scheme (12)-(13). We proceed to show that $\overline{Y}(t)$ has bounded moments. Using the split formulation, we have $Y_k^\star = Y_k + h\varphi_{f_h}(Y_k)$, that is

$$Y_k + s\varphi_{f_h}(Y_k) = \gamma Y_k^\star + (1-\gamma)Y_k,$$

where $\gamma = s/h$. Hence, we can rewrite the continuous extension (53) as

$$\overline{Y}(t) = \gamma Y_k^\star + (1-\gamma)Y_k + g_h(Y_k)\Delta W_k(s). \tag{54}$$

Using bound (48) in (54), we get

$$|\overline{Y}(t_k+s)|^2 \leq 3\left[\gamma C + (\gamma C + 1 - \gamma)|Y_k|^2 + |g_h(Y_k)\Delta W_k(s)|^2\right] \leq C + C\left(|Y_k|^2 + |g_h(Y_k)\Delta W_k(s)|^2\right).$$

Thus,

$$\sup_{0\leq t\leq T}|\overline{Y}(t)|^{2p} \leq \sup_{0\leq kh\leq T}\left[\sup_{0\leq s\leq h}|\overline{Y}(t_k+s)|^{2p}\right] \leq \sup_{0\leq kh\leq T}\left[\sup_{0\leq s\leq h} C\left(1+|Y_k|^{2p}+|g_h(Y_k)\Delta W_k(s)|^{2p}\right)\right], \tag{55}$$

for $t \in [0,T]$. Now, taking a non negative integer $0 \leq k \leq N$ such that $0 \leq Nh \leq T$, we obtain

$$\sup_{0\leq t\leq T}|\overline{Y}(t)|^{2p} \leq C\left(1 + \sup_{0\leq kh\leq T}|Y_k|^{2p} + \sup_{0\leq s\leq h}\sum_{j=0}^{N}|g_h(Y_j)\Delta W_j(s)|^{2p}\right). \tag{56}$$

So, using the Doob's Martingale inequality [16], lemma 4.2 and that g_h is a locally Lipschitz function, we can bound each term of the inequality (56) as follows

$$\mathbb{E}\left[\sup_{0\leq s\leq h}|g(Y_j)\Delta W_j(s)|^{2p}\right] \leq \left(\frac{2p}{2p-1}\right)^{2p}\mathbb{E}|g_h(Y_j)\Delta W_j(h)|^{2p} \leq Ch^p\left(1+\mathbb{E}|Y_j|^{2p}\right) \leq Ch, \tag{57}$$

for each $j \in \{0,\dots,N\}$. Since $Nh \leq T$ and combining bounds (56) and (57), we conclude the proof. □

We can now state the strong convergence theorem for the explicit Linear Steklov method.

Theorem 4.1. *Assume Hypotheses 2.1, 3.1 and 3.2 hold and consider the explicit LS method* (12)-(13) *for SDE* (1). *Then there exists a continuous-time extension* $\overline{Y}(t)$ *of the LS solution* $\{Y_k\}$ *for which* $\overline{Y}(t) = Y_k$ *and*

$$\lim_{h\to 0}\left[\sup_{0\leq t\leq T}|\overline{Y}(t) - y(t)|^2\right] = 0.$$

Proof. First, we bound the mean square error of the Steklov approximation as follows

$$\mathbb{E}\left[\sup_{0\leq t\leq T}|\overline{Y}(t)-y(t)|^2\right] \leq 2\mathbb{E}\left[\sup_{0\leq t\leq T}|\overline{Y}(t)-y_h(t)|^2\right] + 2\mathbb{E}\left[\sup_{0\leq t\leq T}|y_h(t)-y(t)|^2\right]. \tag{58}$$

By lemma 4.1, the solution of the modified SDE (38), y_h, has p-bounded moments ($p \geq 2$), and by corollary 4.2, the LS continuous extension for the SDE (1), $\overline{Y}(t)$, has bounded moments and it is equivalent to the EM extension for the modified SDE (38). Hence, we can apply theorem 2.2 to conclude that

$$\lim_{h\to 0} \mathbb{E}\left[\sup_{0\leq t\leq T} |\overline{Y}(t) - y_h(t)|^2\right] = 0. \tag{59}$$

Meanwhile by lemma 4.1,

$$\lim_{h\to 0} \mathbb{E}\left[\sup_{0\leq t\leq T} |y_h(t) - y(t)|^2\right] = 0. \tag{60}$$

Combining limits (59) and (60) with inequality (58), we have the desired conclusion. □

5. Convergence Rate

In this section we show that the explicit Linear Steklov method (12)-(13) converges with a standard order of one-half. For that, we use a similar procedure to that carried out in [8].

Lemma 5.1. *Under Hypotheses 2.2 and 2.3 and sufficiently small h, there exist constants $D' \in \mathbb{R}$ and $q' \in \mathbb{Z}$ such that for all $u, v \in \mathbb{R}^d$*

$$|\varphi_{f_h}(u) - \varphi_{f_h}(v)|^2 \leq D'\left(1 + |u|^{q'} + |v|^{q'}\right)|u - v|^2, \tag{61}$$

$$|f(u) - \varphi_{f_h}(u)|^2 \leq D'\left(1 + |u|^{q'}\right)h^2, \tag{62}$$

$$|g(u) - g_h(u)|^2 \leq D'\left(1 + |u|^{q'}\right)h^2. \tag{63}$$

Proof. From relationship (42) and inequality (22), we have

$$|\varphi_{f_h}(u) - \varphi_{f_h}(v)|^2 \leq (2 + L_\Phi)|f(u) - f(v)|^2 \leq (2 + L_\Phi)D(1 + |u|^q + |v|^q).$$

Moreover, if $u \in E_j$ then $\varphi_{f_h}(u) = f^{(j)}(u)$. On the other hand, if $u \in E_j^c$ then

$$|f(u) - \varphi_{f_h}(u)|^2 = \sum_{j=1}^{d} |1 - \Phi_j(u)|^2 |f^{(j)}(u)|^2,$$

By the L'Hôpital theorem, we get

$$\lim_{h\to 0} |1 - \Phi_j(u)| = \left|1 - \lim_{h\to 0} \frac{e^{ha_j(u)} - 1}{ha_j(u)}\right| \leq \left|1 - \lim_{h\to 0} e^{hL_a}\right| = 0.$$

Thus there is a sufficiently small $h > 0$ such that $|1 - \Phi_j(u)| < Ch$ for all $u \in E_j^c$ and

$$|f(u) - \varphi_{f_h}(u)|^2 \leq Ch^2 |f(u)|^2 \leq D'(1 + |u|^q)h^2,$$

as we require. Meanwhile, given that $g_h(u) = g(F_h(u))$ from theorem 3.1 it is followed

$$|g(u) - g_h(u)|^2 \leq L_g|u - u + h\varphi f_h(u)|^2 \leq 2(1 + L_\Phi)h^2|f(u)|^2 \leq 2(1 + L_\Phi)D(1 + |u|^q)h^2.$$

□

Lemma 5.2. *Assume Hypotheses 2.2 and 2.3 hold then the solution $y_h(t)$ of the modified SDE (38) satisfies*

$$\mathbb{E}\left[\sup_{0\leq t\leq T} |y_h(t) - y(y)|^2\right] = \mathcal{O}(h^2). \tag{64}$$

Proof. We define $e(t) := y(t) - y_h(t)$ where

$$\begin{aligned} y(t) &= y_0 + \int_0^t f(y(s))ds + \int_0^t g(y(s))dW(s), \\ y_h(t) &= y_0 + \int_0^t \varphi_{f_h}(y_h(s))ds + \int_0^t g_h(y_h(s))dW(s). \end{aligned}$$

Using Itô's formula over the function $V(t,x,y) = |x-y|^2$ for all $x, y \in \mathbb{R}^d$, we obtain

$$de(t) = (f(y(t)) - \varphi_{f_h}(y(t))dt) + (g(y(t)) - g_h(y_h(t)))\, dW(t),$$

Thus

$$\begin{aligned} |e(t)|^2 = 2 &\underbrace{\int_0^t \langle e(s), f(y(s)) - \varphi_{f_h}(y_h(s))\rangle\, ds}_{:=I_1} + \underbrace{\int_0^t |g(y(s)) - g_h(y_h(s))|^2 ds}_{:=I_2} \\ &+ 2\underbrace{\int_0^t \langle e(s), [g(y(s)) - g_h(y_h(s))]\, dW(s)\rangle}_{:=I_3}. \end{aligned} \tag{65}$$

Now we proceed to bound each integral of inequality (65). By Hypothesis 2.2 and the Young inequality, we get

$$\begin{aligned} I_1(t) &\le 2\int_0^t \langle y(s) - y_h(s), f(y(s)) - f(y_h(s))\rangle\, ds + \int_0^t \langle y(s) - y_h(s), f(y_h(s)) - \varphi_f(y_h(s))\rangle\, ds \\ &\le 3\int_0^t |y(s) - y_h(s)|^2 ds + D'h^2 \int_0^t 1 + |y_h(s)|^{q'} ds. \end{aligned}$$

Since $y_h(t)$ the has bonded moments, there exists a universal constant L which does not depends on h such that

$$\mathbb{E}\left[I_1(s)\right] \le L\int_0^t \mathbb{E}|e(s)|^2 ds + Lh^2. \tag{66}$$

Meanwhile, using Hypotheses 2.1 and 2.2 it is followed

$$I_2(t) \le 2L_g \int_0^t |y(s) - y_h(s)|^2 ds + 2D'h^2 \int_0^t 1 + |y_h(s)|^q ds,$$

thus

$$\mathbb{E}\left[I_2(s)\right] \le L\int_0^t \mathbb{E}|e(s)|^2 ds + Lh^2. \tag{67}$$

Note that $\mathbb{E}\left[I_3(t)\right] \le \mathbb{E}\left[\sup_{0\le s\le t}|I_3(s)|\right]$. From the Burkholder-Davis-Gaundy inequality, Hypotheses 2.1 and 2.2 and as $y_h(t)$ has bounded moments, we see that

$$\begin{aligned} \mathbb{E}\left[\sup_{0\le s\le t}|I_3(s)|\right] &\le 2^4\mathbb{E}\left[\sup_{0\le s\le t}|e(s)|^2 \int_0^t |g(y(s)) - g_h(y(s))|^2 ds\right]^{1/2} \\ &\le 2^4\mathbb{E}\left[\frac{1}{2\cdot 2^9}\left(\sup_{0\le s\le t}|e(s)|^2\right) + \frac{2^9}{2}\left(\int_0^t |g(y(s)) - g_h(y_h(s))|^2 ds\right)^2\right] \\ &\le 2L_g\mathbb{E}\left[\int_0^t |y(s) - y_h(s)|^2 ds\right] + D'Th^2 + D'Th^2\int_0^t \mathbb{E}|y_h(s)|^{q'} ds \\ &\le L\int_0^t \mathbb{E}|e(s)|^2 ds + Lh^2. \end{aligned} \tag{68}$$

Substituting inequalities (66), (67) and (68) on equation (65), we deduce that

$$\mathbb{E}\left[\sup_{0\le s\le t}|e(t)|^2\right]\le L\int_0^t \mathbb{E}|e(s)|^2ds+Lh^2\le L\int_0^t \mathbb{E}\left[\sup_{0\le r\le s}|e(s)|^2\right]ds+Lh^2.$$

By the Gronwall inequality, we conclude that

$$\mathbb{E}\left[\sup_{0\le t\le T}|e(t)|^2\right]\le L\exp(LT)h^2\le Ch^2.$$

□

We can now obtain the convergence rate of the explicit Linear Steklov method.

Theorem 5.1. *Under Hypotheses 2.1–2.2 and consider the explicit LS method* (12)-(13) *for the SDE* (1). *Then there exists a continuous-time extension* $\overline{Y}(t)$ *of the LS numerical approximation for which*

$$\mathbb{E}\left[\sup_{0\le t\le T}|\overline{Y}(t)-y(t)|^2\right]=\mathcal{O}(h). \tag{69}$$

Proof. Using bound (58) then by lemma 5.2 and since the LS continuous-time extension (39) is equivalent to the EM continuous-time extension (5), we can use theorem 2.3 and conclude (69). □

6. Numerical Simulation

Here we analyze the behavior of the explicit Linear Steklov for scalar and vector SDE. The tests are choosen to show numerically that the LS scheme reproduces almost surely stability solutions and has a high accuracy for stochastic differential systems with locally Lipschitz drift and diffusion coefficients. Moreover, we validate the efficiency of the new method by comparing with other actual methods like the Euler-Maruyama scheme, the Backward Euler scheme and some schemes of the Tamed Euler family schemes. All simulations are implemented in the Python lenguage using the Mersenne random number generator with a fixed seed of 100.

6.1. Scalar SDE

We consider the SDE with a super-linear growth reported by Tretyakov y Zhang in [24] given by

$$dy(t)=\left(1-y^5(t)+y^3(t)\right)dt+y^2(t)dW(t),\qquad y_0=0. \tag{70}$$

Note that the diffusion coefficient of (70) has a super-linear growth. In this work, the authors present numerical evidence that the Increment-Tamed (I-TEM) scheme defined by

$$X_{k+1}=X_k+\frac{f(X_k)h+g(X_k)\Delta W_k}{\max\left(1,h\left|hf(X_k)+g(X_k)\,\Delta W_k\right|\right)}, \tag{71}$$

produces spurious oscillations. The I-TEM scheme converges strongly under a linear growth diffusion however but does not converges under locally Lipschitz diffusion with a non-linear growth. Now we construct the explicit Linear Steklov method for (70) so defining by $a(x):=-x^4+x^2$, $b:=1$ and the set $E=\{-1,0,1\}$, we have

$$Y_{k+1}=e^{ha(Y_k)}Y_k+\frac{e^{ha(Y_k)}-1}{a(Y_k)}\mathbf{1}_{\{E^c\}}+h\mathbf{1}_{\{E\}}+Y_k^2\Delta W_k, \tag{72}$$

which is the LS approximation. Figure 1 shows the numerical solution of SDE (70) obtained with the I-TEM method (71), the LS method (72) and the basic Tamed (TEM) scheme proposed in [11]. Note that the I-TEM diverges but on the other hand the TEM and LS methods has a similar behaviour and reliably describe the profile of the solution.

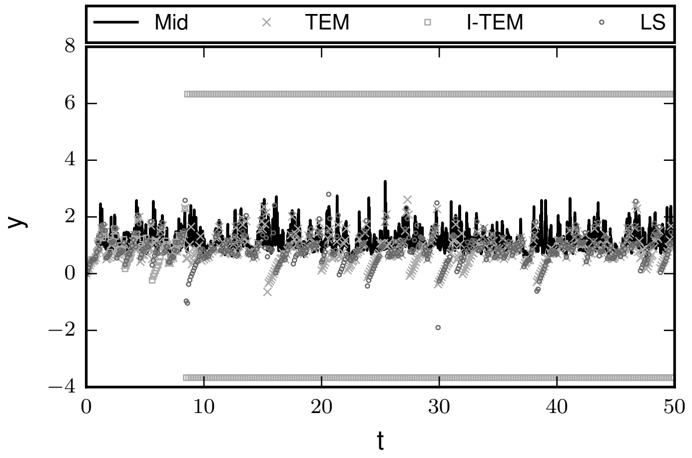

Figure 1: Numerical solution of SDE (70) with the I-TEM, LS and TEM methods with $h = 0.1$, the reference solution is obtained by a Midpoint rule approximation [24] with $h = 10^{-4}$.

6.2. Vector SDE

Example 1. Now we consider a benchmark of Brownian Dynamics [11] defined by the following d-dimensional SDE:

$$dy(t) = (y(t) - |y(t)|\cdot y(t))\,dt + dW(t), \qquad y(0) = 0. \tag{73}$$

This model describes the motion of Brownian d-particles of a unit mass each one which are immersed on the potential $U(x) = \frac{1}{4}|x|^4 - \frac{1}{2}|x|^2$. The Linear Steklov method is constructed taking $a_j(x) := 1 - |x|$ and $b_j = 0$ for $j \in 1 \dots d$ and we obtain

$$Y_{k+1} = \operatorname{diag}\left[e^{ha_1(Y_k)}, \dots, e^{ha_d(Y_k)})\right] Y_k + \Delta W_k. \tag{74}$$

Now we compare the convergence rate and the computational cost of the LS method and the basic tamed Euler scheme [11]. Table 1 shows the root means square errors given by

$$\sqrt{\mathbb{E}\left[|Y_N - y(T)|^2\right]} \approx \frac{1}{M}\left(\sum_{i=1}^{M} |y_i(T) - Y_{N,i}|^2\right)^{1/2}, \tag{75}$$

at a final time $T = 1$ over a sample of M =10 000 trajectories of the TEM, LS and Backward Euler (BEM) [20] approximations for the SDE (73) with $d = 10$. In this experiment we confirm that the LS method converges with standard order 1/2 and is almost equal accurate than the TEM. It is known that in Browninan Dynamics Simulations [1], the dimension of a SDE increases considerable the complexity and computational cost which excludes the use of implicit methods like the BEM scheme, see Figure 2. In this figure we observe that the runtime of BEM depends on dimension in a quadratic way, while the LS and TEM depends on linear form.

Example 2. Hutzenthaler et al. improve convergence of the Euler method by taming the drift increment term with the factor $\frac{1}{1+h|f(Y_k)|}$, as consequence, the norm of $\frac{hf(Y_k)}{1+h|f(Y_k)|}$, is bounded by 1, which controls the drift contribution of the TEM method at each step. This idea works very well over SDEs with drift contribution and initial condition that are comparable with this bound. However, we observed that on models where the drift contribution has other scales, the TEM over damps the drift contribution. To fix ideas, we consider the stochastic

	TEM		LS		BEM	
h	ms-error	ECO	ms-error	ECO	ms-error	ECO
2^{-2}	1.703 88	—	1.553 94	—	1.381 57	—
2^{-3}	1.169 77	0.54	1.107 75	0.48	1.053 09	0.39
2^{-7}	0.278 95	0.48	0.277 95	0.48	0.276 895	0.48
2^{-11}	0.070 10	0.50	0.070 09	0.50	0.070 07	0.50
2^{-15}	0.017 39	0.51	0.017 39	0.51	0.017 39	0.51

Table 1: Mean square errors and the experimental convergence order (ECO) for the SDE (73) with a TEM approximation with $h = 2^{-19}$ as the reference solution.

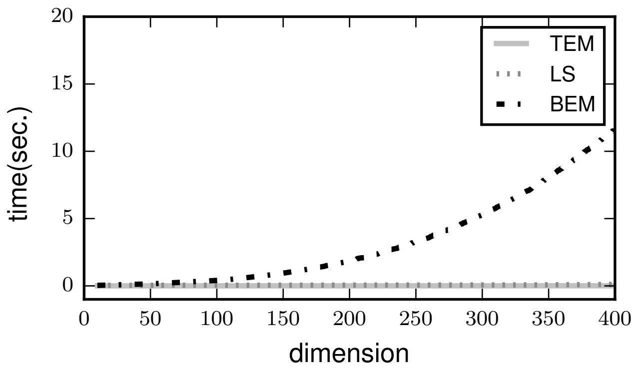

Figure 2: Runtime calculation of Y_N with $h = 2^{-17}$, using the BEM, LS and TEM methods for SDE (73).

model reported in [2],

$$\begin{aligned} dy_1(t) &= (\lambda - \delta y_1(t) - (1-\gamma)\beta y_1(t) y_3(t))\, dt - \sigma_1 y_1(t) dW_t^{(1)}, \\ dy_2(t) &= ((1-\gamma)\beta y_1(t) y_3(t) - \alpha y_2(t))\, dt - \sigma_1 y_2(t) dW_t^{(1)}, \\ dy_3(t) &= ((1-\eta) N_0 \alpha y_2(t) - \mu y_3(t) - (1-\gamma)\beta y_1(t) y_3(t))\, dt - \sigma_2 y_3(t) dW_t^{(2)}. \end{aligned} \tag{76}$$

Here we use the following LS method. Taking

$$\begin{aligned} E_1 &:= \left\{ (x,y,z)^T \in \mathbb{R}^3 : z = 0 \text{ or } z = \frac{-\delta}{\beta(1-\gamma)} \right\}, \quad E_2 := \emptyset, \\ E_3 &:= \left\{ (x,y,z)^T \in \mathbb{R}^3 : x = 0, \frac{-\mu}{\beta(1-\gamma)} \right\}, \end{aligned}$$

$$\begin{aligned} a_1(Y_k)) &:= -\left(\delta + (1-\gamma)\beta Y_k^{(3)}\right), & b_1(Y_k^{(-1)}) &:= \lambda, \\ a_2(Y_k) &:= -\alpha, & b_2(Y_k^{(-2)}) &:= (1-\gamma)\beta Y_k^{(1)} Y_k^{(3)}, \\ a_3(Y_k) &= -\left(\mu + (1-\gamma)\beta Y_k^{(1)}\right), & b_3(Y_k^{(-3)}) &:= (1-\eta)\, N_0 \alpha Y_k^{(2)}, \end{aligned}$$

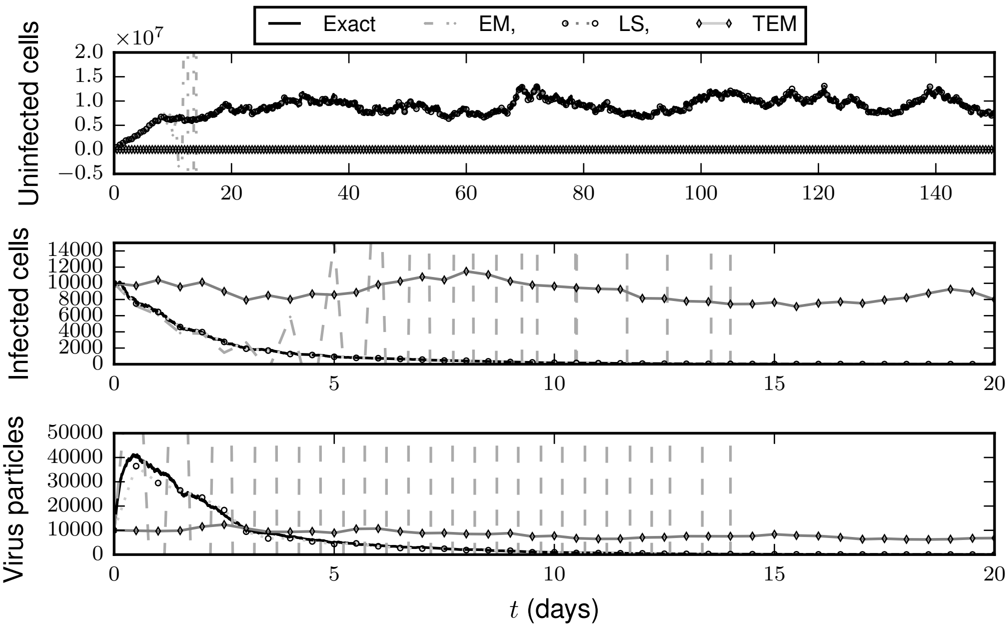

Figure 3: Likening between EM, LS, TEM approximations for SDE (76) with $\gamma = 0.5$, $\eta = 0.5$, $\lambda = 10^6$, $\delta = 0.1$, $\beta = 10^{-8}$, $\alpha = 0.5$, $N_0 = 100$, $\mu = 5$, $\sigma_1 = 0.1$, $\sigma_2 = 0.1$, $y_0 = (10\,000, 10\,000, 10\,000.)^T$, $h = 0.5$. Here the reference solution means a BEM simulation with the same parameters but with a step-size $h = 10^{-5}$.

the LS method for the stochastic model (76) is given by

$$Y_{k+1} = A^{(1)}(h, Y_k)\, Y_k + A^{(2)}(h, Y_k) b(Y_k) + g(Y_k)\,\Delta W_k, \qquad \Delta W_k = \left(W_k^{(1)}, W_k^{(2)}\right)^T,$$

$$A^{(1)}(h, Y_k) := \begin{pmatrix} e^{ha_1(Y_k)} & 0 & 0 \\ 0 & e^{ha_2(Y_k))} & 0 \\ 0 & 0 & e^{ha_3(Y_k))} \end{pmatrix},$$

$$A^{(2)} := \begin{pmatrix} h\Phi_1(Y_k)\mathbf{1}_{\{E_1^c\}} & 0 & 0 \\ 0 & \left(\dfrac{e^{-h\alpha} - 1}{\alpha}\right) & 0 \\ 0 & 0 & h\Phi_3(Y_k)\mathbf{1}_{\{E_3^c\}} \end{pmatrix} + h \begin{pmatrix} \mathbf{1}_{\{E_1\}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathbf{1}_{\{E_3\}} \end{pmatrix},$$

$$b(Y_k) := \begin{pmatrix} b_1(Y_k^{(-1)}) \\ b_2(Y_k^{(-2)}) \\ b_3(Y_k^{(-3)}) \end{pmatrix}, \qquad g(Y_k) := \begin{pmatrix} -\sigma_1 Y_k^{(1)} & 0 \\ -\sigma_1 Y_k^{(2)} & 0 \\ 0 & -\sigma_2 Y_k^{(3)} \end{pmatrix}.$$

Dalal et al. in [2] gives conditions over the parameter of SDE (76), which assure a.s. exponential stability — in the sense that the infected cells (y_2) and virus particles (y_3) will tend to their equilibrium value 0 exponentially with probability 1—and verify this asymptotic behavior by simulation with parameters reported in published literature. Figure 3 shows a simulation path with same parameters with the LS and TEM approximations. We observe how the TEM oscillates around of initial condition while the LS reproduce the underlying asymptotic behavior.

7. Conclusions

In this work we have extended the explicit Steklov scheme for vector SDE by developing a new version based on a linearized Steklov average. This method is constructed on the basis that the drift function can be rewritten in a linearized form which is unique. Moreover, strong order one-half convergence has been proved for our explicit linear method and we have presented several applications formulated with the LS scheme. Finally, high-performance of the Linear Steklov method have been analyzed in diverse problems for which other methods have failed.

Referece

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