Strong Convergence and Almost Sure Stability of the Explicit Linear Steklov Method for SDEs under non-globally Lipschitz Coefficients.☆

S. Díaz-Infante[a], S. Jerez[a]

[a] *Split Step Linear Steklov Method Department of Applied Mathematics, CIMAT, Guanajuato, Gto., Mexico, 36240.*

Abstract

We present an explicit and easily implementable numerical method for solving stochastic differential equations (SDEs) with non-globally Lipschitz coefficients. A linear version of the Steklov average under a split-step formulation supports our new solver. The Linear Steklov (LS) method converges strongly with a standard one-half order. Also, we study the almost sure asymptotic stability and in order to emphasize the performance of the Linear Steklov discretization we use models from population dynamics and non linear stochastic oscillators.

Keywords: stochastic differential equations; explicit methods; strong convergence; almost surely asymptotic stability.

1. Introduction

Fundamental computational tools in applications, as Brownian Dynamics [6] or Monte Carlo simulations [11] requires simple and computational cheap numerical methods — in some cases this exclude the use of implicit schemes. In this context, the Euler Maruyama (EM) leads because has simple algebraic structure, cheap computational cost and acceptable convergence rate under global Lipschitz condition, but, if the drift or diffusion of a SDE grows faster than something linear, then the EM approximation diverges in mean square sense [14, 16, 19]. Moreover, Giles in [10] proposes an efficient variance reduction technique based on numerical strong convergence: the *multilevel* Monte Carlo method. Since applications in Finance, Biology or Physics consider stochastic differential equations (SDE) with super-linear grow and Locally Lipschitz coefficients, designing explicit strong convergent schemes (in L^p sense) attracts the actual stochastic numerics research.

Given its simple structure and low computational cost, results natural develop numerical schemes (on the above set up) by improving the EM. In this line arises the tamed schemes [15, 18, 31, 35, 36], a special type of balanced method [34], and the stopped scheme [23], which considers local Lipschitz drift and at most linear grow diffusion. Recently in [25] Mao, develops the truncated Euler method and Sabanis in [32] proposes a new tamed type scheme, both considers diffusion, and drift under locally Lipschitz and super-linear grow conditions. All of

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Email addresses: sauld@cimat.mx (S. Díaz-Infante), jerez@cimat.mx (S. Jerez)

this works prove convergence of their schemes following the theory developed by Higham Stuart and Mao in [13] or the new approach developed by Hutzenthaler and Jentzen in [15]. Both theories turn the problem of prove strong convergence into provide a bound for the moments of the numerical and continuous solution of a underlying SDE. The approach of Hutzenthaler and Jentzen add tools based on conveniently rare events to the well established stopping time technique of Higham et al.

Following the work of Higham, Mao, and Stuart, in this work we edevelop a new explicit method for non linear SDE with a specific structure, we name it the Linear Steklov method (LS). Our approach follow the same ideas of [8] in order to extend the explicit scalar Steklov method to a multidimensional setting. To motivate it, consider the vector Itô stochastic differential equation of the form

$$dy(t) = f(y(t))dt + g(y(t))dW(t), \quad 0 \leq t \leq T, \quad y(0) = y_0, \tag{1}$$

where $(f^{(1)}, \ldots, f^{(d)}) : \mathbb{R}^d \to \mathbb{R}^d$ is one sided Lipschitz and $g = (g^{(i,j)})_{i\in\{1,\ldots,d\}, j\in\{1,\ldots,m\}} : \mathbb{R}^d \to \mathbb{R}^{d\times m}$ is global Lipschitz. Also we assume that each component function $f^{(j)}$ has the structure

$$f^{(j)}(x) = a_j(x)x^{(j)} + b_j(x^{(-j)}), \qquad x \in \mathbb{R}^d, \qquad x^{(-j)} = \left(x^{(1)}, \ldots, x^{(j-1)}, x^{(j+1)}, \ldots x^{(d)}\right).$$

Note that Stochastic models as Lotka Volterra, Duffin - Van der Pol, Lorenz, SIR, SIS follow this form. We work on the standard setup, that is, $y(t) \in \mathbb{R}^d$ for each t and $W(t)$ is a m-dimensional standard Brownian motion on a filtered and complete probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$, with the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ generated by the Brownian process.

In section 2... Section 3 ...

Notation

2. General Settings

In this section we remind some classical results about the moment boundedness, existence and uniqueness of the solution of the stochastic differential system (1), see [13, 24, 26]. Moreover, we state some theorems about the strong convergence of the Euler-Maruyama given by Higham et al. in [13] which will be useful to prove the strong convergence of the Linear Steklov method. Finally, we give several definitions and theorems of the multivariable calculus that we consider to justify the existence of the Linear Steklov approximation, for references [9, 22]. Let us assume the following:

Hypothesis 2.1. The coefficients of SDE (1) satisfy the conditions:

(H-1) The functions f, g are in the class $C^1(\mathbb{R}^d)$.

(H-2) **Local, global Lipschitz condition**. For each integer n, there is a positive constant $L_f = L_f(n)$ such that

$$|f(x) - f(y)|^2 \leq L_f|x - y|^2 \qquad \forall x, y \in \mathbb{R}^d, \qquad |x|\vee|y| \leq n,$$

and there is a positive constant L_g such that

$$|g(x) - g(y)|^2 \leq L_g|x - y|^2, \qquad \forall x, y \in \mathbb{R}^d.$$

(H-3) **Monotone condition.** There exist two positive constants α and β such that

$$\langle x, f(x)\rangle + \frac{1}{2}|g(x)|^2 \leq \alpha + \beta|x|^2, \qquad \forall x \in \mathbb{R}^d. \tag{2}$$

Under Hypothesis 2.1 we can assure existence and uniqueness of the solution of continuous system (1) as well as bounds on its moments in order to justify the development of a numerical approximation. Next we enumerate the classical results that we mentioned above.

Theorem 2.1. *Assume Hypothesis 2.1 then for all $y(0) = y_0 \in \mathbb{R}^d$ there exists a unique global solution $\{y(t)\}_{t\geq 0}$ to SDE (1). Moreover, the solution has the following properties for any $T > 0$,*

$$\mathbb{E}\,|y(T)|^2 < \left(|y_0|^2 + 2\alpha T\right) e^{2\beta T},$$

and

$$\mathbb{P}\left[\tau_n \leq T\right] \leq \frac{\left(|y_0|^2 + 2\alpha T\right) e^{2\beta T}}{n},$$

where n is any positive integer and $\tau_n := \inf\{t \geq 0 : |y(t)| > n\}$.

Theorem 2.2. *Let $p \geq 2$ and $x_0 \in L^p(\Omega, \mathbb{R}^d)$. Assume that there exits a constant $C > 0$ such that for all $(x, t) \in \mathbb{R}^d \times [t_0, T]$,*

$$\langle x, f(x,t)\rangle + \frac{p-1}{2}|g(x,t)|^2 \leq C(1 + |x|^2).$$

Then

$$\mathbb{E}|y(t)|^p \leq 2^{\frac{p-2}{2}} \left(1 + \mathbb{E}|y_0|^p\right) e^{Cpt} \quad \text{for all } t \in [0, T].$$

Lemma 2.1. *Assuming Hypothesis 2.1, for each $p \geq 2$, there is a $C = C(p, T)$ such that*

$$\mathbb{E}\left[\sup_{0\leq t\leq T} |y(t)|^p\right] \leq C\left(1 + \mathbb{E}|y_0|^p\right).$$

Now we state some results of the multivariable calculus like the L'Hôpital Rule and the existence of the directional derivative at an isolated singularity that will be used throughout the paper.

Definition 2.1. Let $u, \mathbf{q} \in \mathbb{R}^2$ and α the positive angle formed by the x-axis and the segment $\overline{u\mathbf{q}}$. We denote by

$$f_\alpha(u) = \cos(\alpha)\frac{\partial f}{\partial u^{(1)}}(u) + \sin(\alpha)\frac{\partial f}{\partial u^{(2)}}(u) = \frac{\langle q - u, \nabla f(u)\rangle}{|u - q|},$$

the $\mathbf{q}$ *directional derivative at u.*

Definition 2.2. A set $S \subset \mathbb{R}^2$ is *star-like* with respect to a point $\mathbf{q}$, if for each point $s \in S$ the open segment $\overline{s\mathbf{q}}$ is in S.

Theorem 2.3. *Let $\mathcal{N}$ be a neighborhood in $\mathbb{R}^2$ of a point $\mathbf{q}$ for which the two differentiable functions $f : \mathcal{N} \to \mathbb{R}$ and $g : \mathcal{N} \to \mathbb{R}$. Set*

$$C = \{x \in \mathcal{N} : f(x) = g(x) = 0\},$$

assuming that C is a smooth curve through $\mathbf{q}$ and there exist a vector $\mathbf{v}$ not tangent to C at $\mathbf{q}$ such that the directional derivative $D_{\mathbf{v}}g$ of g in the direction of $\mathbf{v}$ is never zero within $\mathcal{N}$ and $\mathbf{q}$ is a limit point of $\mathcal{N} \setminus C$. Then

$$\lim_{(x,y)\to\mathbf{q}} \frac{f(x,y)}{g(x,y)} = \lim_{\substack{(x,y)\to\mathbf{q} \\ (x,y)\in\mathcal{N}\setminus C}} \frac{D_{\mathbf{v}}f}{D_{\mathbf{v}}g},$$

if the latter limit exists.

Theorem 2.4. *Let $\mathbf{q} \in \mathbb{R}^2$ and let f,g be functions whose domains include a set $S \subset \mathbb{R}^2$ which is star-like with respect to the point $\mathbf{q}$. Suppose that on S the functions are differentiable and that the directional derivative of g with respect to $\mathbf{q}$ is never zero. With the understanding that all limits are taken from within on S at $\mathbf{q}$ and if*

$$f(\mathbf{q}) = g(\mathbf{q}) = 0 \qquad \text{and} \qquad \lim_{x\to\mathbf{q}} \frac{f_\alpha(x)}{g_\alpha(x)} = L,$$

then

$$\lim_{x\to\mathbf{q}} \frac{f(x)}{g(x)} = L.$$

Finally, the next theorems gives the necessary conditions to assure strong convergence and the convergence rate of the Euler-Maruyama (EM) method.

Theorem 2.5. *Assume Hypothesis 2.1 holds, then the EM scheme given by:*

$$Y_{k+1}^{EM} = Y_k^{EM} + hf(Y_k^{EM}) + g(Y_k^{EM})\Delta W_k, \tag{3}$$

where h is the step-size and its continuous extension

$$\overline{Y}^{EM}(t) := Y_0 + \int_0^t f(Y_{\eta(s)}^{EM})\, ds + \int_0^t g(Y_{\eta(s)}^{EM})\, dW(s), \tag{4}$$

where $\eta(t) := k$ for $t \in [t_k, t_{k+1})$, satisfies

$$\lim_{h\to 0} \mathbb{E}\left[\sup_{0\le t\le T} |\overline{Y}^{EM}(t) - y(t)|^2\right] = 0. \tag{5}$$

Under the following assumptions, we can get the rate of convergence of the EM scheme.

Hypothesis 2.2. There exist constants $L_f, D \in \mathbb{R}$ and $q \in \mathbb{Z}^+$ such that $\forall u, v \in \mathbb{R}^d$

$$\begin{aligned} \langle u - v, f(u) - f(v)\rangle &\le L_f|u - v|^2, \\ |f(u) - f(v)|^2 &\le D(1 + |u|^q + |v|^q)|u - v|^2. \end{aligned}$$

Hypothesis 2.3. The SDE (1) and the EM solution (3) satisfies

$$\mathbb{E}\left[\sup_{0\le t\le T}|y(t)|^p\right],\quad \mathbb{E}\left[\sup_{0\le t\le T}|Y^{EM}(t)|^p\right]<\infty,\quad \mathbb{E}\left[\sup_{0\le t\le T}|\overline{Y}^{EM}(t)|^p\right]<\infty,\qquad \forall p\ge 1.$$

Theorem 2.6. *Under Hypotheses 2.2 and 2.3 the EM solution with continuous extension* (4) *satisfies*

$$\mathbb{E}\left[\sup_{0\le t\le T}|\overline{Y}^{EM}(t)-y(t)|^2\right]=\mathcal{O}(h^2). \tag{6}$$

3. Construction of the Linear Steklov method

In order to construct the LS method, we assume that for each component of the drift coefficient of (1) there exist two locally Lipschitz functions $a_j:\mathbb{R}^d\to\mathbb{R}$, and $b_j:\mathbb{R}^{d-1}\to\mathbb{R}$ such that

$$f^{(j)}(y)=a_j(y)y^{(j)}+b_j(y^{(-j)}),\qquad y^{(-j)}=(y^{(1)},\dots,y^{(j-1)},y^{(j+1)},\dots,y^{(d)}). \tag{7}$$

The Linear Steklov approximation consists in estimating the function f of the SDEs (1) (for simplicity $d=j=1$) by

$$f(y(t))\approx\varphi_f(y(t_{\eta_+(t)}))=\left(\frac{1}{y(t_{\eta_+(t)})-y(t_{\eta(t)})}\int\limits_{y(t_{\eta(t)})}^{y(t_{\eta_+(t)})}\frac{du}{a(y(t_{\eta(t)}))u+b}\right)^{-1},\qquad t\in[0,T], \tag{8}$$

where φ_f is the linearized Steklov average [8]. Now, using (8) we develop the new method as follows: First, let us take $0=t_0<t_1<\cdots<t_N=T$ a partition of the interval $[0,T]$ with constant step-size $h=T/N$ and such that $t_k=kh$ for $k=0,\dots,N$. By discretization of the SDEs (1), we have for each component equation $j\in\{1,\dots,d\}$ on the time iteration $k\in\{1,\dots,N\}$, $a_{j,k}=a_j\left(Y_k^{(1)},\dots,Y_k^{(d)}\right)$ and $b_{j,k}=b_j\left(Y_k^{(-j)}\right)$. We define the LS method for the multivariate case under a split-step formulation as:

$$\begin{aligned} Y_k^{\star(j)} &= Y_k+h\,\varphi_{f^{(j)}}(Y_k^\star), \qquad (9)\\ Y_{k+1}^{(j)} &= Y_k^{\star(j)}+g^{(j)}(Y_k^\star)\,\Delta W_k, \qquad (10)\end{aligned}$$

with $Y_0=y_0$ and $\varphi_f(Y_k^\star)=\left(\varphi_{f^{(1)}}(Y_k^\star),\dots,\varphi_{f^{(d)}}(Y_k^\star)\right)$ where

$$\varphi_{f^{(j)}}(Y_k^\star)=\left(\frac{1}{Y_k^{\star(j)}-Y_k^{(j)}}\int_{Y_k^{(j)}}^{Y_k^{\star(j)}}\frac{du}{a_{j,k}u+b_{j,k}}\right)^{-1}. \tag{11}$$

The Linear Steklov algorithmic is an implicit scheme, but we can derive an explicit matrix version given by

$$Y_{k+1}=A^{(1)}(h,Y_k)\,Y_k+A^{(2)}(h,Y_k)b(Y_k)+g(Y_k)\,\Delta W_k, \tag{12}$$

where $A^{(1)} = A^{(1)}(h,u)$, $A^{(2)} = A^{(2)}(h,u)$ and $b(u) = \left(b_1(u^{(-1)}), \ldots, b_d(u^{(-d)})\right)^T$ are defined by

$$A^{(1)} := \begin{pmatrix} e^{ha_1(u)} & & 0 \\ & \ddots & \\ 0 & & e^{ha_d(u)} \end{pmatrix},$$

$$A^{(2)} := \begin{pmatrix} \left(\dfrac{e^{ha_1(u)}-1}{a_1(u)}\right)\mathbf{1}_{\{E_1^c\}} & & 0 \\ & \ddots & \\ 0 & & \left(\dfrac{e^{ha_d(u)}-1}{a_d(u)}\right)\mathbf{1}_{\{E_d^c\}} \end{pmatrix} + h \begin{pmatrix} \mathbf{1}_{\{E_1\}} & & 0 \\ & \ddots & \\ 0 & & \mathbf{1}_{\{E_d\}} \end{pmatrix}. \tag{13}$$

It is worth mentioning that the explicit formulation (12)-(13) is well defined in the set $E_j := \{x \in \mathbb{R}^d : a_j(x) = 0\}$. In the following, we assume two important hypotheses which are crucial to prove existence of the explicit Linear Steklov method.

Hypothesis 3.1. For each component function $f^{(j)} : \mathbb{R}^d :\to \mathbb{R}$, $j \in \{1, \ldots, d\}$:

(A-1) There are two locally Lipschitz functions $a_j : \mathbb{R}^d \to \mathbb{R}$, and $b_j : \mathbb{R}^{d-1} \to \mathbb{R}$ such that

$$f^{(j)}(y) = a_j(y)y^{(j)} + b_j(y^{(-j)}), \qquad y^{(-j)} = (y^{(1)}, \ldots, y^{(j-1)}, y^{(j+1)}, \ldots, y^{(d)}).$$

(A-2) There is a positive constant L_a such that

$$a_j(x) \leq L_a, \qquad \forall x \in \mathbb{R}^d, \quad j = 1, \ldots, d.$$

(A-3) Each function $b_j(\cdot)$ satisfies the linear growth condition

$$|b_j(u^{(-j)})|^2 \leq L_b(1 + |u|^2), \qquad \forall x \in \mathbb{R}^d, \quad j = 1, \ldots, d.$$

(A-4) For each $x \in E_j$ there is an open ball of positive radius and center x, $B_r(x)$, such that

$$\frac{\partial a_j(u)}{\partial u^{(j)}} \neq 0, \qquad \forall u \in B_r(x) \setminus E_j.$$

Hypothesis 3.2. The set $E_j := \{x \in \mathbb{R}^d : a_j(x) = 0\}$ satisfies either:

(i) All points of $x \in E_j$ which are non isolated zeros of a_j and:

- The set $D := \{u \in B_r(x) : a_j(u) = e^{ha_j(u)} - 1 = 0\}$ is a smooth curve through x.
- The canonical vector e_j is not tangent to D.
- For each $x \in E_j$, there is an open ball with center on x and radio r, $B_r(x)$, such that

$$a_j \neq 0, \qquad \frac{\partial a_j(u)}{\partial u^{(j)}} \neq 0, \qquad \forall u \in B_r(x) \setminus D.$$

(ii) All points $x \in E_j$ which are isolated zeros of a_j and:

- For each $x \in E_j$, x is not a limit point of the set $E_\alpha := \{u \in \mathbb{R}^d : (a_j)_\alpha(u) = 0\}$.
- For each $x \in E_j$ there is a star-like set, $x \in E_x$, such that the x directional derivative at u satisfies

$$(a_j)_\alpha(u) \neq 0, \qquad \forall u \in E_x.$$

Next we prove our main theorem, which under the previous assumptions, shows that the explicit LS method (12)-(13) always exists, the function φ is bounded by the drift function f and also the coefficients φ and g satisfy a monotone condition. First, we will give a lemma.

Lemma 3.1. *Assume Hypotheses 2.1, 3.1 and 3.2 hold. The function $\Phi_j(u) = \Phi(h, a_j)(u)$ defined by*

$$\Phi_j(u) := \frac{e^{ha_j(u)} - 1}{ha_j(u)}, \tag{14}$$

is bounded on $\mathbb{R}^d$ for each $j \in \{1, \dots, d\}$ and we denote its bound by L_Φ.

Proof. By Hypothesis 2.1, the operator Φ is continuous on E_j^c, thus

$$\lim_{h \to 0} \frac{e^{ha_j(u)} - 1}{ha_j} = 1, \tag{15}$$

for each fixed $u \in E_j^c$. If $u^* \in E_j$ and fixing any h, by Hypothesis 3.2, we obtain one of the following cases:

$$\lim_{\substack{u \to u^* \\ u \in E_j^c}} \Phi(h, a_j)(u) = \lim_{\substack{u \to u^* \\ u \in E_j^c}} \frac{\frac{\partial a_j(u)}{\partial u^{(j)}} h e^{ha_j(u)}}{h \frac{\partial a_j(u)}{\partial u^{(j)}}} = 1, \tag{16}$$

or

$$\lim_{\substack{u \to u^* \\ u \in E_j^c}} \Phi(h, a_j)(u) = \lim_{\substack{u \to u^* \\ u \in E_j^c}} \frac{\left(e^{ha_j(u)} - 1\right)_\alpha}{\left(ha_j(u)\right)_\alpha} = 1, \qquad \alpha = 0, \pi, 2\pi, \dots \tag{17}$$

From (15), (16) and (17) we can deduce that

$$\left| \frac{e^{ha_j(u)} - 1}{ha_j(u)} \right| \leq \left| \frac{e^{hL_a} - 1}{ha_j^*} \right|, \qquad \forall u \in \mathbb{R}^d. \tag{18}$$

where $a_j^* := \inf_{u \in E_j^c}\{|a_j(u)|\}$. Even though $a_j^* = 0$, we can use an argument similar to (16)–(17). $\square$

We can now state the main theorem.

Theorem 3.1. *Assume Hypotheses 2.1, 3.1 and 3.2 hold, and $A^{(1)}$, $A^{(2)}$, b defined by (13). Then given $u \in \mathbb{R}^d$, the equation*

$$v = u + h\varphi_f(v), \tag{19}$$

has a unique solution

$$v = A^{(1)}(h, u)u + A^{(2)}(h, u)b(u). \tag{20}$$

If we define the functions $F_h(\cdot)$, $\varphi_{f_h}(\cdot)$ and $g_h(\cdot)$ by

$$F_h(u) = v, \qquad \varphi_{f_h}(u) = \varphi_f(F_h(u)), \qquad g_h(u) = g(F_h(u)), \tag{21}$$

then $F_h(\cdot)$, $\varphi_{f_h}(\cdot)$, $g_h(\cdot)$ are local Lipschitz functions and for all $u \in \mathbb{R}^d$ and each h fixed, there is a positive constant C_h such that

$$|\varphi_{f_h}(u)| \leq L_\Phi |f(u)|. \tag{22}$$

Moreover, for each h fixed, there are positive constants α^* and β^* such that

$$\langle \varphi_{f_h}(u), u\rangle \vee |g_h(u)|^2 \leq \alpha^* + \beta^* |u|^2, \qquad \forall u \in \mathbb{R}^d. \tag{23}$$

Proof. Let us first prove that (20) is solution of equation (19). Note that

$$v^{(j)} = u^{(j)} + h\, \varphi_{f^{(j)}}(v), \tag{24}$$

for each $j \in \{1, \ldots, d\}$ and using the linear Steklov function (11), we can derive that

$$v^{(j)} = e^{h a_j(u)} u^{(j)} + \left[h\Phi_j(u) \mathbf{1}_{\{E_j^c\}} + h \mathbf{1}_{\{E_j\}} \right] b_j(u^{(-j)}), \tag{25}$$

which is the j-component of the vector $A^{(1)}u + A^{(2)}b(u)$. Now we prove inequality (22). Given that $v = \varphi_f(F_h(u))$, we can also rewrite (24) as

$$\varphi_{f_h}^{(j)}(u) = \frac{F_h^{(j)}(u) - u^{(j)}}{\displaystyle\int_{u^{(j)}}^{F_h^{(j)}(u)} \frac{dz}{a_j(u) z + b_j(u^{(-j)})}}.$$

If $u \in E_j$ then $\varphi_{f_h}^{(j)}(u) = b_j(u^{(-j)}) = f(u)$, so $L_\Phi \geq 1$ fulfills (22). On the other hand, if $u \in E_j^c$ then

$$\varphi_{f_h}^{(j)}(u) = \frac{(F_h^{(j)}(u) - u^{(j)}) a_j(u)}{\underbrace{\ln\left(a_j(u) F_h^{(j)}(u) + b_j(u^{(-j)})\right)}_{:=R_1} - \ln\left(a_j(u) u^{(j)} + b_j(u^{(-j)})\right)} = \Phi_j(u) f^j(u), \tag{26}$$

where

$$R_1 = \ln\left\{ a_j(u) \left[e^{h a_j(u)} u^{(j)} + h\Phi_j(u) b_j(u^{(-j)}) \right] + b_j\left(u^{(-j)}\right) \right\} = h a_j(u) + \ln\left(f^{(j)}(u)\right).$$

By lemma 3.1, inequality (22) is satisfied for all $u \in E_j \cup E_j^c$. As $g_h(x) = g\left(F_h(x)\right)$, by Hypothesis 2.1, then

$$|g_h(u) - g_h(v)|^2 \leq L_g |F_h(u) - F_h(v)|^2 \leq 2L_g \underbrace{|A^{(1)}u - A^{(1)}v|^2}_{:=R_2} + 2L_g \underbrace{|A^{(2)}b(u) - A^{(2)}b(v)|^2}_{:=R_3}. \tag{27}$$

Let us consider each term of the right hand of inequality (27). First, note that $A^{(1)}$ is a continuous differentiable function on all $\mathbb{R}^d$, so using the mean value theorem, we have

$$R_2 \leq L_{A^{(1)}} |u - v|^2, \quad u, v \in \mathbb{R}^d, \quad |u| \vee |v| \leq n, \tag{28}$$

for a positive constant $L_{A^{(1)}} = \sup_{0\le t\le 1}|\partial A^{(1)}(h, u + t(v-u))|^2$. Meanwhile,

$$
\begin{aligned}
R_3 &= \sum_{j=1}^{d}\Big[\mathbf{1}_{\{E_j^c\}}(u)\Phi_j(u)b_j(u^{(-j)}) + h\mathbf{1}_{\{E_j\}}(u)b_j(u^{(-j)}) - \mathbf{1}_{\{E_j^c\}}(v)\Phi_j(v)b_j(u^{(-j)}) \\
&- h\mathbf{1}_{\{E_j\}}(v)b_j(v^{(-j)})\Big]^2 \le 4\sum_{j=1}^{d}\Big[\Big(\mathbf{1}_{\{E_j^c\}}(u)L_\Phi b_j(u^{(-j)})\Big)^2 + \Big(h\mathbf{1}_{\{E_j\}}(u)b_j(u^{(-j)})\Big)^2 \\
&+ \Big(\mathbf{1}_{\{E_j^c\}}(v)L_\Phi b_j(v^{(-j)})\Big)^2 + \Big(h\mathbf{1}_{\{E_j\}}(v)b_j(v^{(-j)}\Big)^2\Big]. \qquad (29)
\end{aligned}
$$

Since $b_j^2(\cdot)$ is a function of class $C^1(\mathbb{R}^d)$, there is a constant $M_b = M_b(n)$ such that

$$
|b_j(u)|^2 \le L_b, \quad \forall u \in \mathbb{R}^d, \quad |u|\vee|v|\le n, \qquad (30)
$$

for each $j \in \{1, \cdots, d\}$. Using this bound in (29), we obtain

$$
R_3 \le 4\sum_{j=1}^{d}\left[2L_\Phi L_b + 2h^2 L_b\right] \le L_0, \qquad \forall u, v \in \mathbb{R}^d, \quad |u|\vee|v|\le n, \qquad (31)
$$

where $L_0 = 8dL_b(n)(L_\Phi + h^2)$. By inequalities (28) and (31), we get

$$
|g_h(u) - g_h(v)|^2\le L_{g_h}(n)|u-v|^2, \qquad \forall u, v \in \mathbb{R}^d, \quad |u|\vee|v|\le n, \qquad (32)
$$

where $L_{g_h}(n) \ge n^2 + 1 + L_0 + L_{A^{(1)}}$. Then $g_h(\cdot)$ is a locally Lipschitz function. Furthermore, note that under some modifications this argument can be used to prove that $F_h(\cdot)$ is also a locally Lipschitz function, which implies that φ_{f_h} is a locally Lipschitz function. Finally, we will demonstrate inequality (23). By Hypotheses 2.1 and 3.1, we have

$$
\langle f(u), u\rangle = \sum_{j=1}^{d} a_j(u)\left(u^{(j)}\right)^2 + \sum_{j=1}^{d} b_j(u)u^{(j)} \le \alpha + \beta|u|^2,
$$

and

$$
\langle b(u), u\rangle \le \alpha + (\beta + L_a)|u|^2.
$$

Using these inequalities and (22), we deduce that

$$
\langle \varphi_{f_h}(u), u\rangle = \sum_{j=1}^{d}\Phi_j(u)f^{(j)}(u)u^{(j)} \le L_\Phi L_a|u|+L_\Phi(\alpha + (L_a+\beta)|u|^2) \le L_{\varphi_{f_h}}(1+|u|^2). \qquad (33)
$$

where $L_{\varphi_{f_h}} \ge 2L_\Phi \max\{L_a, \alpha, \beta\} + 1$. Meanwhile, since g is globally Lipschitz then

$$
|g_h(u)|^2\le 2|g(F_h(u)) - g(F_h(0))|^2+2|g(F_h(0))|^2\le 4L_g|F_h(u)|^2+8L_g|F_h(0)|^2+4|g(0)|^2. \qquad (34)
$$

Now, we bound each term on the right-hand side of (34). By the monotone condition (2), $|g(0)|^2\le 2\alpha$. Moreover,

$$
|F_h^{(j)}(0)|= h\Phi_j(0)\,|b_j(0)|\mathbf{1}_{\{E_j^c\}}(0) + h\,|b_j(0)|\mathbf{1}_{\{E_j\}}(0) \le \frac{b_0^*}{a_0^*}e^{hL_a}(1+h), \quad \forall j \in \{1, \cdots, d\}.
$$

where $a_0^* := \min_{\substack{j\in\{1,\cdots,d\}\\ a_j(0)\neq 0}} \{|a_j(0)|\}$ and $b_0^* := \max_{j\in\{1,\cdots,d\}} \{|b_j(0)|\}$. Then

$$|F_h(0)|^2 \leq d\left(\frac{b_0^*}{a_0^*}\right)^2 e^{2hL_a}(1+h)^2. \tag{35}$$

As Φ_j is bounded, we get

$$\begin{aligned} F_h^{(j)}(u) &= e^{ha_j(u)}u^{(j)} + h\Phi_j(u)b_j(u)\mathbf{1}_{\{E_j^c\}}(u) + hb_j(u)\mathbf{1}_{\{E_j\}}(u) \\ &\leq e^{ha_j(u)}|u^{(j)}| + hL_\Phi|b_j(u)|\mathbf{1}_{\{E_j^c\}}(u) + h|b_j(u)|\mathbf{1}_{\{E_j\}}(u). \end{aligned}$$

And by Hypothesis 3.1,

$$\begin{aligned} |F_h^{(j)}(u)|^2 &\leq 3e^{2hL_a}|u|^2 + (3h^2L_\Phi^2L_b + 3h^2L_b)(1+|u|^2) \\ &\leq 3h^2L_b(1+L_\Phi^2) + 3\left(e^{2hL_a} + h^2L_b(L_\Phi^2+1)\right)|u|^2 \\ &\leq L_F(1+|u|^2), \end{aligned} \tag{36}$$

where $L_F \geq 3d\max\{e^{2hL_a}, h^2L_b(L_\Phi^2+1)\}$. Using (35) and (36) in inequality (34) yields

$$|g_h(u)|^2 \leq 4L_gL_F(1+|u|^2) + 8L_gd\left(\frac{b_0^*}{a_0^*}\right)^2 e^{2hL_a}(1+h)^2 + 8\alpha.$$

Therefore, taking $L_{g_h} \geq 4L_gL_F + 8L_gd\left(\frac{b_0^*}{a_0^*}\right)^2 e^{2hL_a}(1+h)^2 + 8\alpha$ then

$$|g_h(u)|^2 \leq L_{g_h}(1+|u|^2). \tag{37}$$

Hence, from inequalities (33) and (37) and taking for each fixed $h > 0$, $\alpha^* := L_{\varphi_{f_h}} \vee L_{g_h}$ and $\beta^* := 2\alpha^*$, we obtain inequality (23). $\square$

Remark 3.1. It is worth mentioning that if $b_j = 0$ in (A-1) then Hypotheses 3.1 and 3.2 are superfluous to prove Theorem 3.1. Several applications as stochastic Lotka-Volterra systems [27, 28], the Ginzburg-Landau SDE [20, Sec. 4.4], the damped Langevin Equations where the potential lacks of a constant term (for example $U(x) = \frac{1}{4}|x|^4 - \frac{1}{2}|x|^2$) [17].

This theorem holds the key to demonstrate strong convergence of the explicit LS method.

4. Strong Convergence of the Linear Steklov Method

To prove strong convergence of the Linear Steklov method (12)-(13) for SDEs with locally Lipschitz coefficients, we rely on the Higham-Mao-Stuart (HMS) technique [13]. Recently, several works have used this procedure to establish strong convergence for some particular scheme among others [2, 12, 14, 15, 17, 21, 26, 33]. The proof is rather technical and can be divided into the following steps:

Step 1: The explicit LS method (12)-(13) is equivalent to the Euler-Maruyama method (3) for the conveniently modified SDE

$$dy_h(t) = \varphi_{f_h}(y_h(t))dt + g_h(y_h(t))dW(t), \qquad y_h(0) = y_0, \qquad t\in[0,T], \tag{38}$$

which is a perturbation of SDE (1) given that $|f(x) - \varphi_{f_h}(x)| = \mathcal{O}(h)$.

Step 2: The solution of the modified SDE (38) has bounded moments and it is "close" to the solution of the SDEs (1) in the sense of the uniform mean square norm $\mathbb{E}\left[\sup_{0\le t\le T}|\cdot|^2\right]$.

Step 3: The explicit LS method has bounded moments.

Step 4: There is a continuous extension of the explicit LS method of the form:

$$\overline{Y}(t) := Y_0 + \int_0^t \varphi_{f_h}(Y_{\eta(s)})\, ds + \int_0^t g(Y_{\eta(s)})\, dW(s), \qquad \eta(t) := k \ \text{ for } t\in[t_k, t_{k+1}),$$

which has bounded moments.

Proving the above statements and using theorem 2.5, it is straightforward to conclude the strong convergence of the explicit LS method. Step 1 is formalized in the following corollary.

Corollary 4.1. *Assume Hypotheses 2.1, 3.1 and 3.2 hold, then the explicit LS method* (12)-(13) *for SDE* (1) *is equivalent to the Euler-Maruyama scheme applied to the modified SDE* (38)*.*

Proof. Using the functions $\varphi_{f_h}(\cdot)$ and $g_h(\cdot)$ defined in (21), we can rewrite the explicit LS method as

$$Y_{k+1} = Y_k + h\varphi_{f_h}(Y_k) + g_h(Y_k)\Delta W_k,$$

which is the EM approximation for the modified SDE (38). □

Now we proceed with step 2.

Lemma 4.1. *Assume Hypotheses 2.1, 3.1 and 3.2 hold, then there is a constant $C = C(p,T) > 0$ and a sufficiently small step size h, such that for all $p > 2$*

$$\mathbb{E}\left[\sup_{0\le t\le T}|y_h(t)|^p\right] \le C\left(1 + \mathbb{E}|y_0|^p\right). \tag{39}$$

Moreover

$$\lim_{h\to 0}\mathbb{E}\left[\sup_{0\le t\le T}|y(t) - y_h(t)|^2\right] = 0. \tag{40}$$

Proof. By theorem 2.2 and inequality (23), we have bound (39). On the other hand, to prove (40) we will use the properties of φ_{f_h} and the Higham's stopping time technique employed in [13, Thm 2.2]. Note that by relationship (26), we can rewrite the Steklov function as

$$\varphi_{f_h}(u) = f^{(j)}(u)\mathbf{1}_{\{E_j\}}(u) + \Phi_j(u) f^{(j)}(u)\mathbf{1}_{\{E_j^c\}}(u). \tag{41}$$

If $u\in E_j^c$, given that $f\in C^1(\mathbb{R}^d)$ and $\Phi_j(\cdot)$ is a bounded function, there is a positive constant K_n, which depends on n, such that

$$\begin{aligned}
|\varphi_{f_h}^{(j)}(u) - f^{(j)}(u)| &\le \mathbf{1}_{\{E_j^c\}}(u)|f^{(j)}(u)||\Phi_j(u) - 1| \\
&\le \mathbf{1}_{\{E_j^c\}}(u)\left(L_\Phi + 1\right)|f(u)| \\
&\le \mathbf{1}_{\{E_j^c\}}(u) K_n (L_\Phi + 1), \quad \forall u\in\mathbb{R}^d, \quad |u|\le n,
\end{aligned}$$

for each $j \in \{1, \ldots, d\}$. At the interface betweeen E_j and E_j^c, we know by the scalar L' Hôpital theorem that $\lim_{h\to 0} \Phi_j(u) = 1$ for $u \in E_j^c$. And if $u \in E_j$, by Hypothesis 3.2 and using theorems 2.3 and 2.4, we have

$$\lim_{h\to 0} F_h^{(j)}(u) = \lim_{h\to 0} e^{ha_j(u)} u^{(j)} + \lim_{h\to 0} \left(\Phi_j(u) \mathbf{1}_{\{E_j^c\}}(u) + h \mathbf{1}_{\{E_j\}}(u) \right) b_j(u^{(j)}) = u^{(j)},$$

for each $j \in \{1, \ldots, d\}$, hence $\lim\limits_{h\to 0} F_h(u) = u$. Consequently, given $n > 0$ there is a function $K_n(\cdot) : (0, \infty) \to (0, \infty)$ such that $K_n(h) \to 0$ when $h \to 0$ and

$$|\varphi_{f_h}(u) - f(u)|^2 \vee |g_h(u) - g(u)|^2 \leq K_n(h) \qquad \forall u \in \mathbb{R}^d, \quad |u| \leq n. \tag{42}$$

Meanwhile, since f, g are C^1 there is a constant $H_n > 0$ such that

$$|f(u) - f(v)|^2 \vee |g(u) - g(v)|^2 \leq H_n |u - v|^2 \qquad \forall u, v \in \mathbb{R}^d, \quad |u| \vee |v| \leq n. \tag{43}$$

Now, using the stopping time technique given in [13, Lem. 3.6], we get the desired conclusion. □

In the following, we proceed with step 3 which establishes that LS method has bounded moments. This is the key step in the HSM technique.

Lemma 4.2. *Assume Hypotheses 2.1, 3.1 and 3.2 hold, then for each $p \geq 2$ there is a universal positive constant $C = C(p, T)$ such that for the explicit LS method* (12)-(13) *is satisfied*

$$\mathbb{E}\left[\sup_{kh\in[0,T]} |Y_k|^{2p} \right] \leq C.$$

Proof. Using a split formulation of the LS scheme (12)-(13) as follows:

$$\begin{aligned} Y_k^{\star(j)} &= A^{(1)}(h, Y_k) Y_k + A^{(2)}(h, Y_k) b(Y_k), \\ Y_{k+1}^{(j)} &= Y_k^{\star(j)} + g^{(j)}(Y_k^\star)\, \Delta W_k, \end{aligned}$$

from the first step of this split scheme, using (A-3) and the Cauchy-Schwartz inequality, we get

$$\begin{aligned} |Y_k^\star|^2 &\leq |A^{(1)}(h, Y_k)|^2 |Y_k|^2 + 2 \left\langle A^{(1)}(h, Y_k) Y_k, A^{(2)}(h, Y_k) Y_k b(Y_k) \right\rangle + |A^{(2)}(h, Y_k)|^2 |b(Y_k)|^2 \\ &\leq |A^{(1)}(h, Y_k)|^2 |Y_k|^2 + 2\sqrt{L_b} d |A^{(1)}(h, Y_k)| |A^{(2)}(h, Y_k)| |Y_k| (1 + |Y_k|) \\ &\quad + L_b |A^{(2)}(h, Y_k)|^2 (1 + |(Y_k)|^2). \end{aligned} \tag{44}$$

From (A-2), we can deduce that

$$|A^{(1)}(h, Y_k)|^2 = \left| \operatorname{diag}\left(e^{ha_1(Y_k)}, \ldots, e^{ha_d(Y_k)} \right) \right|^2 \leq M_{A^{(1)}}, \tag{45}$$

where $M_{A^{(1)}} = d\, e^{2TL_a}$ and also by (18), we can derive that

$$\begin{aligned} |A^{(2)}(h, Y_k)|^2 &= \left| h \operatorname{diag}\left(\mathbf{1}_{\{E_1\}}(Y_k) + \mathbf{1}_{\{E_1^c\}}(Y_k) \Phi_1(Y_k), \ldots, \mathbf{1}_{\{E_d\}}(Y_k) + \mathbf{1}_{\{E_d^c\}}(Y_k) \Phi_d(Y_k) \right) \right|^2 \\ &\leq \sum_{j=1}^d \left(\mathbf{1}_{\{E_j^c\}} |h \Phi_j(Y_k)|^2 + h^2 \right) \leq 2e^{2L_aT} \sum_{j=1}^d \frac{1}{a_j^*} + dT^2 \leq M_{A^{(2)}}. \end{aligned} \tag{46}$$

Substituting (45) and (46) on inequality (44) yields

$$\begin{aligned} |Y_k^\star|^2 &\le M_{A^{(1)}}|Y_k|^2 + 2d\sqrt{M_{A^{(1)}}M_{A^{(2)}}L_b}\,|Y_k|(1+|Y_k|) + M_{A^{(2)}}L_b(1+|Y_k|^2) \\ &\le \widetilde{C}(3|Y_k|^2+|Y_k|+1) \le C(1+|Y_k|^2), \end{aligned} \tag{47}$$

where $\widetilde{C} \ge M_{A^{(1)}} + 2d\sqrt{M_{A^{(1)}}M_{A^{(2)}}L_b} + M_{A^{(2)}}L_b$ and $C = 6\widetilde{C}$. Applying the bound (47) in the second step of the split scheme, we get

$$|Y_{k+1}|^2 \le C\left(|Y_k|^2+1\right) + 2\left\langle Y_k^\star, g(Y_k^\star)\Delta W_k\right\rangle + |g(Y_k^\star)\Delta W_k|^2 .$$

Now, we choose two integers N, M such that $Nh \le Mh \le T$. So, adding backwards we obtain

$$|Y_N|^2 \le S_N\left(\sum_{j=0}^{N-1}(1+|Y_j|^2) + 2\sum_{j=0}^{N-1}\left\langle Y_j^\star, g(Y_j^\star)\Delta W_j\right\rangle + \sum_{j=0}^{N-1}\left|g(Y_j^\star)\Delta W_j\right|^2\right),$$

where $S_N := \sum_{j=0}^{N-1} C^{N-j}$. Raising both sides to the power p and using the standard inequality, we obtain

$$|Y_N|^{2p} \le 6^p S_N^p\left(N^{p-1}\sum_{j=0}^{N-1}(1+|Y_j|^{2p}) + \left|\sum_{j=0}^{N-1}\left\langle Y_j^\star, g(Y_j^\star)\Delta W_j\right\rangle\right|^p + N^{p-1}\sum_{j=0}^{N-1}\left|g(Y_j^\star)\Delta W_j\right|^{2p}\right). \tag{48}$$

Now we will show that the second and third terms of inequality (48) are bounded. We denote by $C = C(p,T)$ a generic positive constant which does not depend on the step size h and whose value may change between occurrences. Next, applying the Bunkholder-Davis-Gundy inequality, we see that

$$\begin{aligned} \mathbb{E}\left[\sup_{0\le N\le M}\left|\sum_{j=0}^{N-1}\left\langle Y_j^\star, g(Y_j^\star)\Delta W_j\right\rangle\right|^p\right] &\le C\mathbb{E}\left[\sum_{j=0}^{N-1}|Y_j^\star|^2|g(Y_j^\star)|^2h\right]^{p/2} \\ &\le Ch^{p/2}M^{p/2-1}\mathbb{E}\sum_{j=0}^{M-1}|Y_j^\star|^p(\alpha+\beta|Y_j^\star|^2)^{p/2} \\ &\le 2^{p/2-1}CT^{p/2-1}h\mathbb{E}\sum_{j=0}^{M-1}(\alpha^{p/2}|Y_j^\star|^p+\beta^{p/2}|Y_j^\star|^{2p}) \\ &\le Ch\mathbb{E}\sum_{j=0}^{M-1}(1+2|Y_j^\star|^p+|Y_j^\star|^{2p}) \\ &\le Ch\sum_{j=0}^{M-1}\left[1+\mathbb{E}|Y_j^\star|^{2p}\right] \\ &\le C + Ch\sum_{j=0}^{M-1}\mathbb{E}|Y_j|^{2p}, \end{aligned} \tag{49}$$

Now, using the Cauchy-Schwartz inequality, the monotone condition (2) and the bound (47), we obtain

$$\begin{aligned}
\mathbb{E}\left[\sup_{0\le N\le M}\sum_{j=0}^{N-1}\left|g(Y_j^\star)\Delta W_j\right|^{2p}\right] &= \mathbb{E}\sum_{j=0}^{M-1}\left|g(Y_j^\star)\Delta W_j\right|^{2p}\\
&\le \sum_{j=0}^{M-1}\mathbb{E}\left|g(Y_j^\star)\right|^{2p}\mathbb{E}\left|\Delta W_j\right|^{2p}\\
&\le Ch^p\sum_{j=0}^{M-1}\mathbb{E}\left[\alpha+\beta|Y_j^\star|^2\right]^p\\
&\le Ch^p\sum_{j=0}^{M-1}\mathbb{E}\left[\alpha^p+\beta^p|Y_j^\star|^{2p}\right]\\
&\le Ch^{p-1}+Ch^p\sum_{j=0}^{M-1}\mathbb{E}|Y_j|^{2p}. \qquad (50)
\end{aligned}$$

Thus, combining bounds (49) and (50) with the inequality (48), we can assert that

$$\mathbb{E}\left[\sup_{0\le N\le M}|Y_N|^{2p}\right]\le C+C(1+h)\sum_{j=0}^{M-1}\mathbb{E}\left[\sup_{0\le N\le j}|Y_N|^{2p}\right]. \qquad (51)$$

Finally, using the discrete-type Gronwall inequality, we conclude that

$$\mathbb{E}\left[\sup_{0\le N\le M}|Y_N|^{2p}\right]\le Ce^{C(1+h)M}\le Ce^{C(1+T)}<C,$$

since the constant C does not depend on h, the proof is complete. □

Let $\{Y_k\}$ denote the explicit LS solution of SDE (1) defined by the recurrence equations (12)-(13). By corollary 4.1, we can give a continuous extension for the LS approximation from the time continuous EM extension (4). Moreover, we will also prove in the following theorem that the moments of the continuous LS extension remains bounded.

Corollary 4.2. *Assume Hypotheses 2.1, 3.1 and 3.2 hold and suppose* $0<h<1$ *and* $p\ge 2$. *Then there is a continuous extension* $\overline{Y}(t)$ *of the Linear Steklov recurrence* $\{Y_k\}$ *defined by* (12)-(13) *and a positive constant* $C=C(T,p)$ *such that*

$$\mathbb{E}\left[\sup_{0\le t\le T}|\overline{Y}(t)|^{2p}\right]\le C.$$

PROOF. We take $t=s+t_k$ in $[0,T]$ such that $\Delta W_k(s):=W(t_k+s)-W(t_k)$ for $0\le s<h$. Then we define

$$\overline{Y}(t_k+s):=Y_k+s\varphi_{f_h}(Y_k)+g_h(Y_k)\Delta W_k(s), \qquad (52)$$

as a continuous extension of the LS scheme (12)-(13). We proceed to show that $\overline{Y}(t)$ has bounded moments. Using the split formulation, we have $Y_k^\star = Y_k + h\varphi_{f_h}(Y_k)$, that is

$$Y_k + s\varphi_{f_h}(Y_k) = \gamma Y_k^\star + (1-\gamma)Y_k,$$

where $\gamma = s/h$. Hence, we can rewrite the continuous extension (52) as

$$\overline{Y}(t) = \gamma Y_k^\star + (1-\gamma)Y_k + g_h(Y_k)\Delta W_k(s). \tag{53}$$

Using bound (47) in (53), we get

$$|\overline{Y}(t_k+s)|^2 \leq 3\left[\gamma C + (\gamma C + 1 - \gamma)|Y_k|^2 + |g_h(Y_k)\Delta W_k(s)|^2\right] \leq C + C\left(|Y_k|^2 + |g_h(Y_k)\Delta W_k(s)|^2\right).$$

Thus,

$$\sup_{0\leq t\leq T}|\overline{Y}(t)|^{2p} \leq \sup_{0\leq kh\leq T}\left[\sup_{0\leq s\leq h}|\overline{Y}(t_k+s)|^{2p}\right] \leq \sup_{0\leq kh\leq T}\left[\sup_{0\leq s\leq h} C\left(1+|Y_k|^{2p}+|g_h(Y_k)\Delta W_k(s)|^{2p}\right)\right], \tag{54}$$

for $t\in[0,T]$. Now, taking a non negative integer $0\leq k\leq N$ such that $0\leq Nh\leq T$, we obtain

$$\sup_{0\leq t\leq T}|\overline{Y}(t)|^{2p} \leq C\left(1 + \sup_{0\leq kh\leq T}|Y_k|^{2p} + \sup_{0\leq s\leq h}\sum_{j=0}^{N}|g_h(Y_j)\Delta W_j(s)|^{2p}\right). \tag{55}$$

So, using the Doob's Martingale inequality, lemma 4.2 and that g_h is a locally Lipschitz function, we can bound each term of the inequality (55) as follows

$$\mathbb{E}\left[\sup_{0\leq s\leq h}|g(Y_j)\Delta W_j(s)|^{2p}\right] \leq \left(\frac{2p}{2p-1}\right)^{2p}\mathbb{E}|g_h(Y_j)\Delta W_j(h)|^{2p} \leq Ch^p\left(1+\mathbb{E}|Y_j|^{2p}\right) \leq Ch, \tag{56}$$

for each $j\in\{0,\dots,N\}$. Since $Nh\leq T$ and combining bounds (55) and (56), we conclude the proof. □

We can now state the strong convergence theorem for the explicit Linear Steklov method.

Theorem 4.1. *Assume Hypotheses 2.1, 3.1 and 3.2 hold, consider the explicit LS method* (12)-(13) *for SDE* (1)*. Then there exist a continuous-time extension $\overline{Y}(t)$ of the LS solution $\{Y_k\}$ for which $\overline{Y}(t) = Y_k$ and*

$$\lim_{h\to 0}\left[\sup_{0\leq t\leq T}|\overline{Y}(t)-y(t)|^2\right] = 0.$$

Proof. First, we bound the mean square error of the Steklov approximation as follows

$$\mathbb{E}\left[\sup_{0\leq t\leq T}|\overline{Y}(t)-y(t)|^2\right] \leq 2\mathbb{E}\left[\sup_{0\leq t\leq T}|\overline{Y}(t)-y_h(t)|^2\right] + 2\mathbb{E}\left[\sup_{0\leq t\leq T}|y_h(t)-y(t)|^2\right]. \tag{57}$$

By lemma 4.1, the solution of the modified SDE (38), y_h, has p-bounded moments ($p\geq 2$), and by corollary 4.2, the LS continuous extension for the SDE (1), $\overline{Y}(t)$, has bounded moments and it is equivalent to the EM extension for the modified SDE (38). Hence, we can apply theorem 2.5 to conclude that

$$\lim_{h\to 0}\mathbb{E}\left[\sup_{0\leq t\leq T}|\overline{Y}(t)-y_h(t)|^2\right] = 0, \tag{58}$$

meanwhile by lemma 4.1,

$$\lim_{h\to 0}\mathbb{E}\left[\sup_{0\le t\le T}|y_h(t)-y(t)|^2\right]=0. \tag{59}$$

Combining limits (58) and (59) with inequality (57), we have the desired conclusion. $\square$

5. Convergence Rate

In this section, we show that the explicit Linear Steklov method (12)-(13) converges with a standard order of one-half. For that, we use an analogous procedure as in [13].

Lemma 5.1. *Under Hypotheses 2.2 and 2.3 and sufficiently small h, there exist constants $D' \in \mathbb{R}$ and $q' \in \mathbb{Z}$ such that for all $u, v \in \mathbb{R}^d$*

$$|\varphi_{f_h}(u)-\varphi_{f_h}(v)|^2 \le D'\left(1+|u|^{q'}+|v|^{q'}\right)|u-v|^2, \tag{60}$$

$$|f(u)-\varphi_{f_h}(u)|^2 \le D'\left(1+|u|^{q'}\right)h^2, \tag{61}$$

$$|g(u)-g_h(u)|^2 \le D'\left(1+|u|^{q'}\right)h^2. \tag{62}$$

Proof. From relationship (41) and inequality (22), we have

$$|\varphi_{f_h}(u)-\varphi_{f_h}(v)|^2 \le (2+L_\Phi)|f(u)-f(v)|^2 \le (2+L_\Phi)D(1+|u|^q+|v|^q).$$

Moreover, if $u\in E_j$ then $\varphi_{f_h}(u)=f^{(j)}(u)$. On the other hand, if $u\in E_j^c$ then

$$|f(u)-\varphi_{f_h}(u)|^2=\sum_{j=1}^{d}|1-\Phi_j(u)|^2|f^{(j)}(u)|^2,$$

By the L'Hôpital theorem, we get

$$\lim_{h\to 0}|1-\Phi_j(u)|=\left|1-\lim_{h\to 0}\frac{e^{ha_j(u)}-1}{ha_j(u)}\right|\le\left|1-\lim_{h\to 0}e^{hL_a}\right|=0.$$

Thus there is a sufficiently small $h>0$ such that $|1-\Phi_j(u)|< Ch$ for all $u\in E_j^c$ and

$$|f(u)-\varphi_{f_h}(u)|^2 \le Ch^2|f(u)|^2 \le D'(1+|u|^q)h^2,$$

as we require. Meanwhile, given that $g_h(u)=g(F_h(u))$ from Theorem 3.1 it is followed

$$|g(u)-g_h(u)|^2 \le L_g|u-u+h\varphi f_h(u)|^2 \le 2(1+L_\Phi)h^2|f(u)|^2 \le 2(1+L_\Phi)D(1+|u|^q)h^2.$$

$\square$

Lemma 5.2. *Assume Hypotheses 2.2 and 2.3 hold then the solution $y_h(t)$ of the modified SDE (38) satisfies*

$$\mathbb{E}\left[\sup_{0\le t\le T}|y_h(t)-y(y)|^2\right]=\mathcal{O}(h^2). \tag{63}$$

PROOF. We define $e(t) := y(t) - y_h(t)$ where

$$\begin{aligned} y(t) &= y_0 + \int_0^t f(y(s))ds + \int_0^t g(y(s))dW(s), \\ y_h(t) &= y_0 + \int_0^t \varphi_{f_h}(y_h(s))ds + \int_0^t g_h(y_h(s))dW(s). \end{aligned}$$

Using Itô's formula over the function $V(t,x,y) = |x-y|^2$ for all $x, y \in \mathbb{R}^d$, we obtain

$$de(t) = (f(y(t)) - \varphi_{f_h}(y(t))dt) + (g(y(t)) - g_h(y_h(t)))\, dW(t),$$

Thus

$$\begin{aligned} |e(t)|^2 = 2 &\underbrace{\int_0^t \langle e(s), f(y(s)) - \varphi_{f_h}(y_h(s))\rangle\, ds}_{:=I_1} + \underbrace{\int_0^t |g(y(s)) - g_h(y_h(s))|^2 ds}_{:=I_2} \\ &+ 2 \underbrace{\int_0^t \langle e(s), [g(y(s)) - g_h(y_h(s))]\, dW(s)\rangle}_{:=I_3}. \end{aligned} \tag{64}$$

Now we proceed to bound each integral of inequality (64). By Hypothesis 2.2, the Young and standard inequalities, we get

$$\begin{aligned} I_1(t) &\leq 2 \int_0^t \langle y(s) - y_h(s), f(y(s)) - f(y_h(s))\rangle\, ds + \int_0^t \langle y(s) - y_h(s), f(y_h(s)) - \varphi_f(y_h(s))\rangle\, ds \\ &\leq 3 \int_0^t |y(s) - y_h(s)|^2 ds + D'h^2 \int_0^t 1 + |y_h(s)|^{q'} ds. \end{aligned}$$

Since $y_h(t)$ the has bonded moments, there exists a universal constant L which does not depends on h such that

$$\mathbb{E}\left[I_1(s)\right] \leq L \int_0^t \mathbb{E}|e(s)|^2 ds + Lh^2. \tag{65}$$

Meanwhile, using Hypotheses 2.1 and 2.2 it is followed

$$I_2(t) \leq 2L_g \int_0^t |y(s) - y_h(s)|^2 ds + 2D'h^2 \int_0^t 1 + |y_h(s)|^q ds.$$

and thus

$$\mathbb{E}\left[I_2(s)\right] \leq L \int_0^t \mathbb{E}|e(s)|^2 ds + Lh^2. \tag{66}$$

Note that $\mathbb{E}\left[I_3(t)\right] \leq \mathbb{E}\left[\sup_{0\leq s\leq t} |I_3(s)|\right]$. From the Burkholder-Davis-Gaundy inequality, Hypotheses 2.1 and 2.2 and as $y_h(t)$ has bounded moments, we see that

$$
\begin{aligned}
\mathbb{E}\left[\sup_{0\le s\le t}|I_3(s)|\right] &\le 2^4\mathbb{E}\left[\sup_{0\le s\le t}|e(s)|^2\int_0^t |g(y(s))-g_h(y(s))|^2 ds\right]^{1/2} \\
&\le 2^4\mathbb{E}\left[\frac{1}{2\cdot 2^9}\left(\sup_{0\le s\le t}|e(s)|^2\right)+\frac{2^9}{2}\left(\int_0^t |g(y(s))-g_h(y_h(s))|^2 ds\right)^2\right] \\
&\le \frac{1}{2}\mathbb{E}\left[\sup_{0\le s\le t}|e(s)|^2\right]+2^8\mathbb{E}\left[\int_0^t |g(y(s))-g_h(y_h(s))|^2 ds\right] \\
&\le 2L_g\mathbb{E}\left[\int_0^t |y(s)-y_h(s)|^2 ds\right]+D'Th^2+D'Th^2\int_0^t \mathbb{E}|y_h(s)|^{q'} ds \\
&\le L\int_0^t \mathbb{E}|e(s)|^2 ds+Lh^2. \qquad (67)
\end{aligned}
$$

Substituting inequalities (65), (66) and (67) on equation (64), we deduce that

$$
\mathbb{E}\left[\sup_{0\le s\le t}|e(t)|^2\right]\le L\int_0^t \mathbb{E}|e(s)|^2 ds+Lh^2\le L\int_0^t \mathbb{E}\left[\sup_{0\le r\le s}|e(s)|^2\right]ds+Lh^2.
$$

By the Gronwall inequality, we conclude that

$$
\mathbb{E}\left[\sup_{0\le t\le T}|e(t)|^2\right]\le L\exp(LT)h^2\le Ch^2.
$$

□

We can now obtain the convergence rate of the explicit Linear Steklov method.

Theorem 5.1. *Consider the LS method applied to SDE (1) and under Hypotheses 2.1–2.2. Then there exist a continuous extension $\overline{Y}(t)$ of the LS numerical approximation for which*

$$
\mathbb{E}\left[\sup_{0\le t\le T}|\overline{Y}(t)-y(t)|^2\right]=\mathcal{O}(h).
$$

Proof. We can bound

$$
\mathbb{E}\left[\sup_{0\le t\le T}|\overline{Y}(t)-y(t)|^2\right]\le 2\mathbb{E}\left[\sup_{0\le t\le T}|\overline{Y}(t)-y_h(t)|^2\right]+2\mathbb{E}\left[\sup_{0\le t\le T}|y_h(t)-y(t)|^2\right].
$$

By Lemma 5.2 we have

$$
\mathbb{E}\left[\sup_{0\le t\le T}|y_h(t)-y(t)|^2\right]=\mathcal{O}(h^2),
$$

then it remains to prove that

$$
\mathbb{E}\left[\sup_{0\le t\le T}|\overline{Y}(t)-y_h(t)|^2\right],
$$

but $\overline{Y}(t)$ can be regarded as the EM applied to the modified SDE (**??**), and by Lemmas (4.1),(4.2) and Corollary 4.2 y_h ,Y_k and $\overline{Y}(t)$ has bounded moments, then adapting the proof of **??** (see [13, Thm. 4.4]) we can deduce that

$$\mathbb{E}\left[\sup_{0\le t\le T}|\overline{Y}(t)-y_h(t)|^2\right]=\mathcal{O}(h).$$

as we require. □

6. Numerical Simulation

We performed all simulations using python 2.7 on a laptop MSI-ge70 with Intel CPU 4700HQ (Core i7, 2.40 GHz) under Linux Debian. All experiments runs using the Mersenne-Twister random number generator with seed at 100 and the package scipy.optimize to solve the algebraic equations of the implicit methods. We estimate mean square error by

$$\sqrt{\mathbb{E}\left[|Y_N-y(T)|^2\right]}\approx\left(\frac{1}{M}\sum_{i=1}^{M}|y_i(T)-Y_{N,i}|^2\right)^{1/2}. \tag{68}$$

6.1. Scalar SDE

Example 1. Now we examine the LS method using a SDE that with super-linear grow diffusion. We consider the SDE reported by Tretyakov and Zhang in [33, Eq. (5.6)]

$$dy(t)=\left(1-y^5(t)+y^3(t)\right)dt+y^2(t)dW(t),\qquad y_0=0. \tag{69}$$

Tretyakov and Zhang shows via simulation of (69) that the increment-tamed scheme [15, Eq(1.5)]

$$X_{k+1}=X_k+\frac{f(X_k)h+g(X_k)\Delta W_k}{\max\left(1,h\left|hf(X_k)+g(X_k)\Delta W_k\right|\right)} \tag{70}$$

produces spurious oscillations. Hutzenthaler and Jentzen prove the convergence of this scheme under linear growth condition over diffusion. So, this suggest us that only certain kind of explicit schemes with convergence under globally Lipschitz and linear growth diffusion conditions can extended their convergence to a locally Lipschitz diffusion and other kind of growth bound. Using $a(x):=-x^4+x^2$, $b:=1$ and $E=\{-1,0,1\}$, we construct the LS method

$$Y_{k+1}=\exp(ha(Y_k))Y_k+\frac{\exp(ha(Y_k))-1}{a(Y_k)}\mathbf{1}_{\{E^c\}}+h\mathbf{1}_{\{E\}}+Y_k^2\Delta W_k. \tag{71}$$

Figure 1 shows the numerical solution of SDE (69) with the Increment-Tamed (I-TEM) (70), LS method (71), and the Tamed-Sabanis (TEM-S) scheme with $\alpha=1/2$. We consider the implicit Midpoint scheme [33, Eq.(5.3)] with $h=10^{-4}$ as reference.

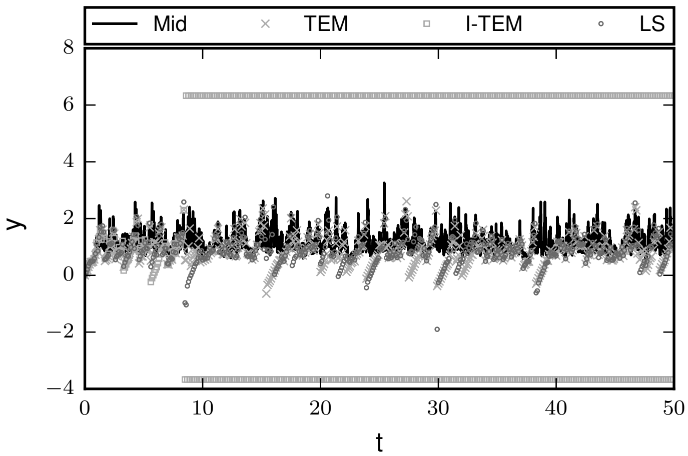

Figure 1: Numerical solution of SDE (69) using the I-TEM (70), LS method (71) and TEM with $h = 0.1$, the reference solutions is a Midpoint rule approximation with $h = 10^{-4}$

6.2. Stochastic Differential Systems

Example 1. Now we compare the order of convergence and the run time of the LS method withthe TEM scheme as in [17]. That is, we consider a Langevin equation under the d-dimensional potential $U(x) = \frac{1}{4}|x|^4 - \frac{1}{2}|x|^2$, and d-dimensional Brownian additive noise. The corresponding SDE reads

$$dy(t) = (y(t) - |y(t)| \cdot y(t))\, dt + dW(t), \qquad y(0) = 0. \tag{72}$$

This model describes the motion of a Brownian particle of unit mass immersed on the potential $U(x)$. Taking $a_j(x) := 1 - |x|$ and $b_j = 0$, $j \in 1 \dots d$ on we obtain the LS method

$$Y_{k+1} = \operatorname{diag}\left[e^{ha_1(Y_k)}, \dots, e^{ha_d(Y_k)})\right] Y_k + \Delta W_k \tag{73}$$

Table 1 shows the root means square errors at final time $T = 1$ over a sample of M =10 000 trajectories of the TEM, LS and BEM solutions of SDE (72) with dimension $d = 10$. We consider the TEM solution with step $h = 2^{-19}$ as reference solution. The TEM is faster and almost equal accurate than the LS method.

In some application as in Browninan Dynamics Simulations [6], the dimension of a SDE increases considerable the complexity and computational cost — this prohibits the use of implicit methods. Figure 2 shows this (for SDE(72)): the runtime of BEM depends on dimension in a quadratic way , while the LS and TEM depends on linear form.

Example 2. Hutzenthaler et al. improve convergence of the Euler method by taming the drift increment term with the factor $\frac{1}{1+h|f(Y_k)|}$, as consequence, the norm of $\frac{hf(Y_k)}{1+h|f(Y_k)|}$, is bounded by 1, which controls the drift contribution of the TEM method at each step. This idea works very well over SDEs with drift contribution and initial condition that are comparable with this bound. However, we observed that on models where the drift contribution has other

	TEM		LS		BEM	
h	ms-error	ECO	ms-error	ECO	ms-error	ECO
2^{-2}	1.703 88	—	1.553 94	—	1.381 57	—
2^{-3}	1.169 77	0.54	1.107 75	0.48	1.053 09	0.39
2^{-7}	0.278 95	0.48	0.277 95	0.48	0.276 895	0.48
2^{-11}	0.070 10	0.50	0.070 09	0.50	0.070 07	0.50
2^{-15}	0.017 39	0.51	0.017 39	0.51	0.017 39	0.51
2^{-18}	0.006 17	0.50	0.006 17	0.50	0.006 17	0.50

Table 1: Mean square errors and the experimental convergence order (ECO) for the SDE (72) with a TEM with $h = 2^{-19}$ as reference solution.

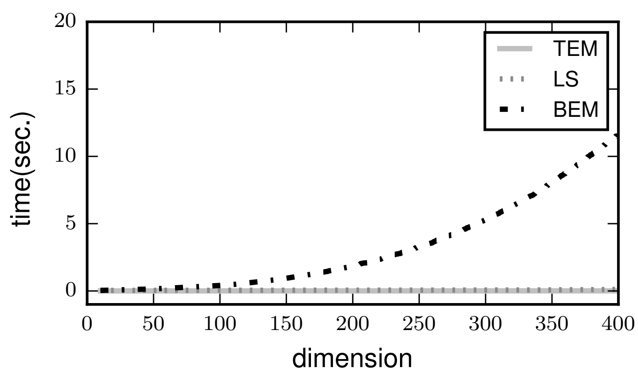

Figure 2: Runtime calculation of Y_N with $h = 2^{-17}$, using the BEM, LS and TEM methods for SDE (72).

scales, the TEM over damps the drift contribution. To fix ideas, we consider the stochastic model reported in [7],

$$\begin{aligned} dy_1(t) &= (\lambda - \delta y_1(t) - (1-\gamma)\beta y_1(t) y_3(t))\, dt - \sigma_1 y_1(t) dW_t^{(1)}, \\ dy_2(t) &= ((1-\gamma)\beta y_1(t) y_3(t) - \alpha y_2(t))\, dt - \sigma_1 y_2(t) dW_t^{(1)}, \\ dy_3(t) &= ((1-\eta) N_0 \alpha y_2(t) - \mu y_3(t) - (1-\gamma)\beta y_1(t) y_3(t))\, dt - \sigma_2 y_3(t) dW_t^{(2)}. \end{aligned} \tag{74}$$

Here we use the following LS method. Taking

$$E_1 := \left\{ (x,y,z)^T \in \mathbb{R}^3 : z = 0 \text{ or } \frac{-\delta}{\beta(1-\gamma)} \right\}, \quad E_2 := \emptyset, \quad E_3 := \left\{ (x,y,z)^T \in \mathbb{R}^3 : x = 0 \right\},$$

$$\begin{aligned} a_1(Y_k)) &:= -\left(\delta + (1-\gamma)\beta Y_k^{(3)}\right), & b_1(Y_k^{(-1)}) &:= \lambda, \\ a_2(Y_k) &:= -\alpha, & b_2(Y_k^{(-2)}) &:= (1-\gamma)\beta Y_k^{(1)} Y_k^{(3)}, \\ a_3(Y_k) &= -\left(\mu + (1-\gamma)\beta Y_k^{(1)}\right), & b_3(Y_k^{(-3)}) &:= (1-\eta)\, N_0 \alpha Y_k^{(2)}, \end{aligned}$$

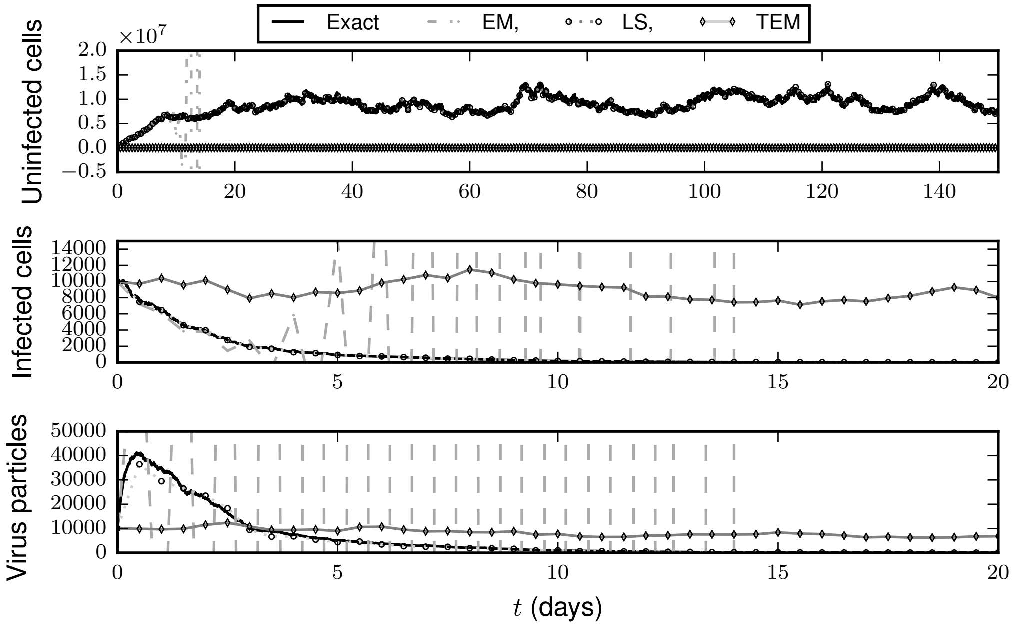

Figure 3: Likening between EM, LS, TEM approximations for SDE (74) with $\gamma = 0.5$, $\eta = 0.5$, $\lambda = 10^6$, $\delta = 0.1$, $\beta = 10^{-8}$, $\alpha = 0.5$, $N_0 = 100$, $\mu = 5$, $\sigma_1 = 0.1$, $\sigma_2 = 0.1$, $y_0 = (10\,000, 10\,000, 10\,000.)^T$, $h = 0.5$. Here the reference solution means a BEM simulation with the same parameters but with a step-size $h = 10^{-5}$.

the LS method for the stochastic model (74) reads,

$$
\begin{aligned}
Y_{k+1} &= A^{(1)}(h, Y_k)\, Y_k + A^{(2)}(h, Y_k) b(Y_k) + g(Y_k)\, \Delta W_k, \qquad \Delta W_k = \left(W_k^{(1)}, W_k^{(2)}\right)^T, \\
A^{(1)}(h, Y_k) &:= \begin{pmatrix} e^{ha_1(Y_k)} & 0 & 0 \\ 0 & e^{ha_2(Y_k))} & 0 \\ 0 & 0 & e^{ha_3(Y_k))} \end{pmatrix}, \\
A^{(2)} &:= \begin{pmatrix} \left(\dfrac{e^{ha_1(Y_k))}-1}{a_1(Y_k))}\right)\mathbf{1}_{\{E_1^c\}} & 0 & 0 \\ 0 & \left(\dfrac{e^{-h\alpha}-1}{\alpha}\right) & 0 \\ 0 & 0 & \left(\dfrac{e^{ha_3(u)}-1}{a_3(Y_k))}\right)\mathbf{1}_{\{E_3^c\}} \end{pmatrix} \\
&\quad + h \begin{pmatrix} \mathbf{1}_{\{E_1\}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathbf{1}_{\{E_3\}} \end{pmatrix}, \\
b(Y_k) &:= \begin{pmatrix} b_1(Y_k^{(-1)}) \\ b_2(Y_k^{(-2)}) \\ b_3(Y_k^{(-3)}) \end{pmatrix}, \qquad g(Y_k) := \begin{pmatrix} -\sigma_1 Y_k^{(1)} & 0 \\ -\sigma_1 Y_k^{(2)} & 0 \\ 0 & -\sigma_2 Y_k^{(3)} \end{pmatrix}.
\end{aligned} \tag{75}
$$

Dalal et al. in [7] simulates SDE (74) with parameters reported in published literature [3, 4, 29, 30]. Figure 3 shows a simulation path with same parameters. We observe how the TEM oscillates around of initial condition while the LS follows the reference solution.

7. Conclusions

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