NoMMA
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Instituto de

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Extinción, Persistencia y Comportamiento Umbral en Modelos Compartimentales Estocásticmente Perturbados

UNAM, Juriquilla, Queretaro
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Saúl Díaz Infante Velasco

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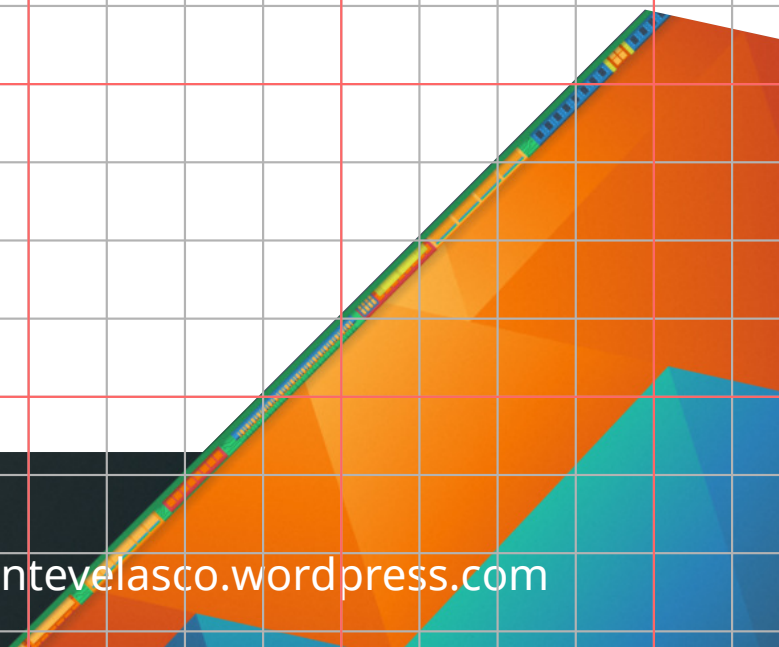

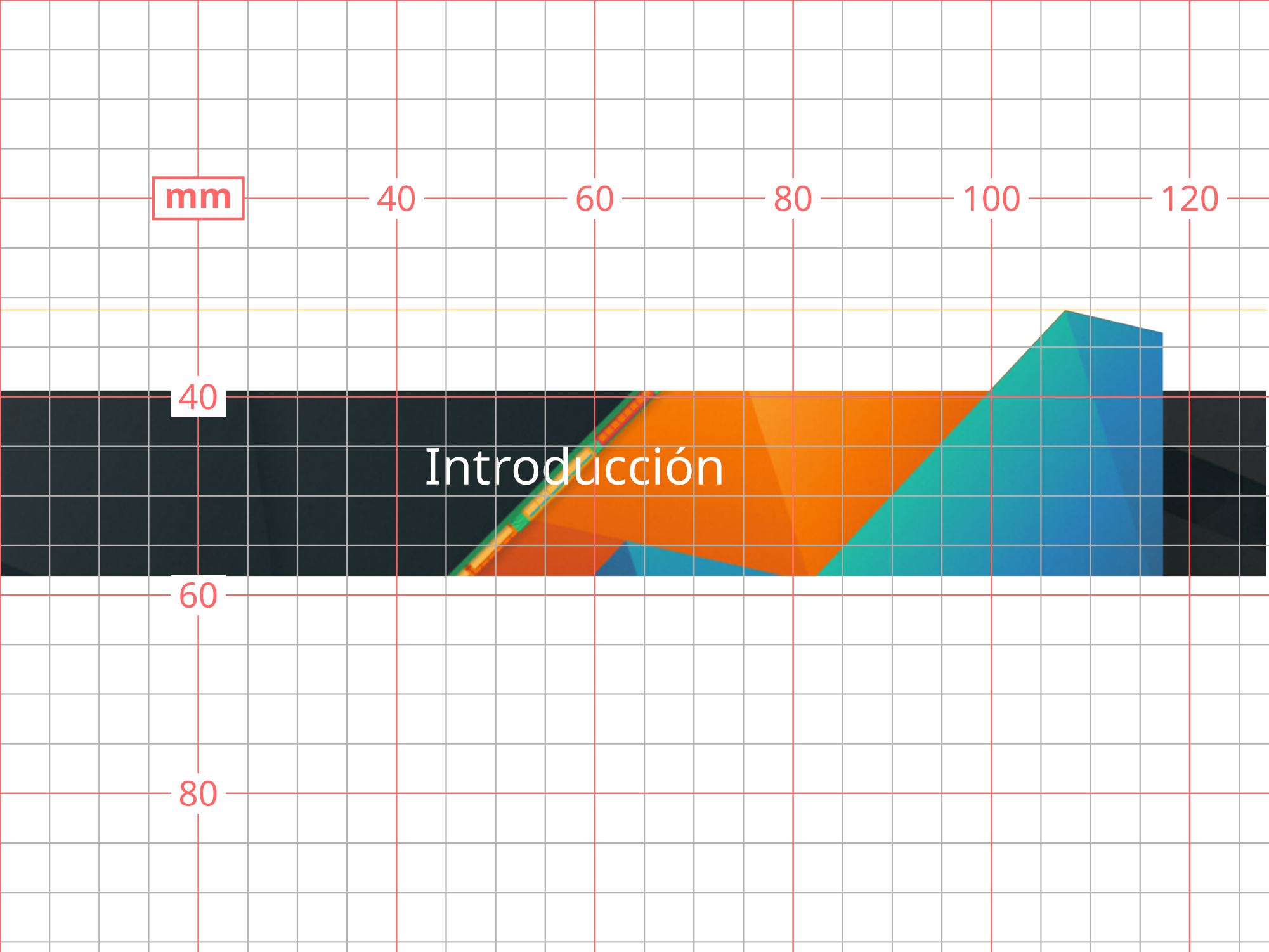
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Introducción

Para fijar ideas

$$\dot{S}(t) = \Lambda - \mu S(t) - \beta S(t) I(t) - \delta S(t)$$

$$\dot{I}(t) = \beta S(t) I(t) - (\mu + \gamma + \varepsilon) I(t)$$

$$\dot{R}(t) = \gamma I(t) - \mu R(t) + \delta S(t)$$

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Umbral determinista

$$\mathscr{R}_0 = \frac{\beta \Lambda}{(\mu + \gamma + \varepsilon)(\mu + \delta)}$$

$\mathscr{R}_0 < 1 \Rightarrow FDE$: (g.a.s)

$\mathscr{R}_0 > 1 \Rightarrow EE$: (g.a.s)

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$$\beta dt \rightsquigarrow \beta dt + \sigma dB_t$$

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Umbral estocástico

$\mathscr{R}_0^S = ?$

$\mathscr{R}_0^S < 1 \Rightarrow$ extinción

$\mathscr{R}_0^S > 1 \Rightarrow$ persistencia

Para fijar ideas

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Ver:

Y. Zhang, Y. Li, Q. Zhang, and A. Li.
Behavior of a stochastic SIR epidemic model with saturated incidence and vaccination rules.
Physica A: Statistical Mechanics and its Applications, 501:178–187, 2018.

¿Cuando considerar Modelos Estocásticos?

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Sean importantes

- Poblaciones pequeñas
- Variabilidad demográfica
- Variabilidad ambiental

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Según:

Allen, L. J. (2017). A primer on stochastic epidemic models: Formulation, numerical simulation, and analysis. *Infectious Disease Modelling*, 2(2):128–142.

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Ejemplo

Transmisión, recuperación, nacimientos, muertes.

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Sean importantes

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Ejemplo

Transmisión, recuperación, nacimientos, muertes.

Condiciones territoriales, acuáticas: enfermedades vectoriales, zoonóticas transmitidas por alimentos.

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Alternativas

Modelos

- (D/C)–TMCs
- Perturbación de parámetros
  - Procesos reversibles en media
  - β_t^H $H \in (0.5, 1)$
- Random Diff. Eq.

Herramientas

- Gillespie
- Kloeden-Methods
- Hermite-PC

MC + ME $\rightarrow$ *SDE*

L. J. Allen.
A primer on stochastic epidemic models: Formulation, numerical simulation, and analysis.
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$\varphi dt \rightsquigarrow \varphi dt + \sigma dB_t$

Gray, A., Greenhalgh, D., Hu, L., Mao, X., and Pan, J. (2011). A Stochastic Differential Equation SIS Epidemic Model. *SIAM Journal on Applied Mathematics*, 71(3):876–902.

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$\varphi dt \rightsquigarrow \varphi dt + F(x) dB_t$

Schurz, H. and Tosun, K. (2015). Stochastic Asymptotic Stability of SIR Model with Variable Diffusion Rates. *Journal of Dynamics and Differential Equations*, 27(1):69–82.

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$$d\varphi_t = (\varphi_e - \varphi_t)dt + \sigma_\varphi dBt$$

Allen, E. (2016). Environmental variability and mean-reverting processes. *Discrete and Continuous Dynamical Systems - Series B*, 21(7):2073–2089.

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Ma, Y., Zhang, Q., and Ye, M. (2017).
Mean-square dissipativity of numerical methods for a class of resource-competition models with fractional brownian motion.
Systems Science & Control Engineering, 5(1):268–277.

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parametros son v.a.

Chen-Charpentier, B.-M., Cortés, J.-C., Licea, J.-A., Romero, J.-V., Roselló, M.-D., Santonja, F.-J., and Villanueva, R.-J. (2015). Constructing adaptive generalized polynomial chaos method to measure the uncertainty in continuous models: A computational approach. *Mathematics and Computers in Simulation*, 109:113 – 129.

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Objetivo

Ilustrar las ideas de $\varphi dt \rightsquigarrow \varphi dt + \sigma dB_t$

- Modelación
- Análisis y Simulación
- Ideas al aire

Esquema de Charla

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1. Introducción

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2. Perturbación con MB

3. Propiedades del proceso soulución

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4. Umbral: $R_0^S := R_0^D - f(\sigma, \cdot)$

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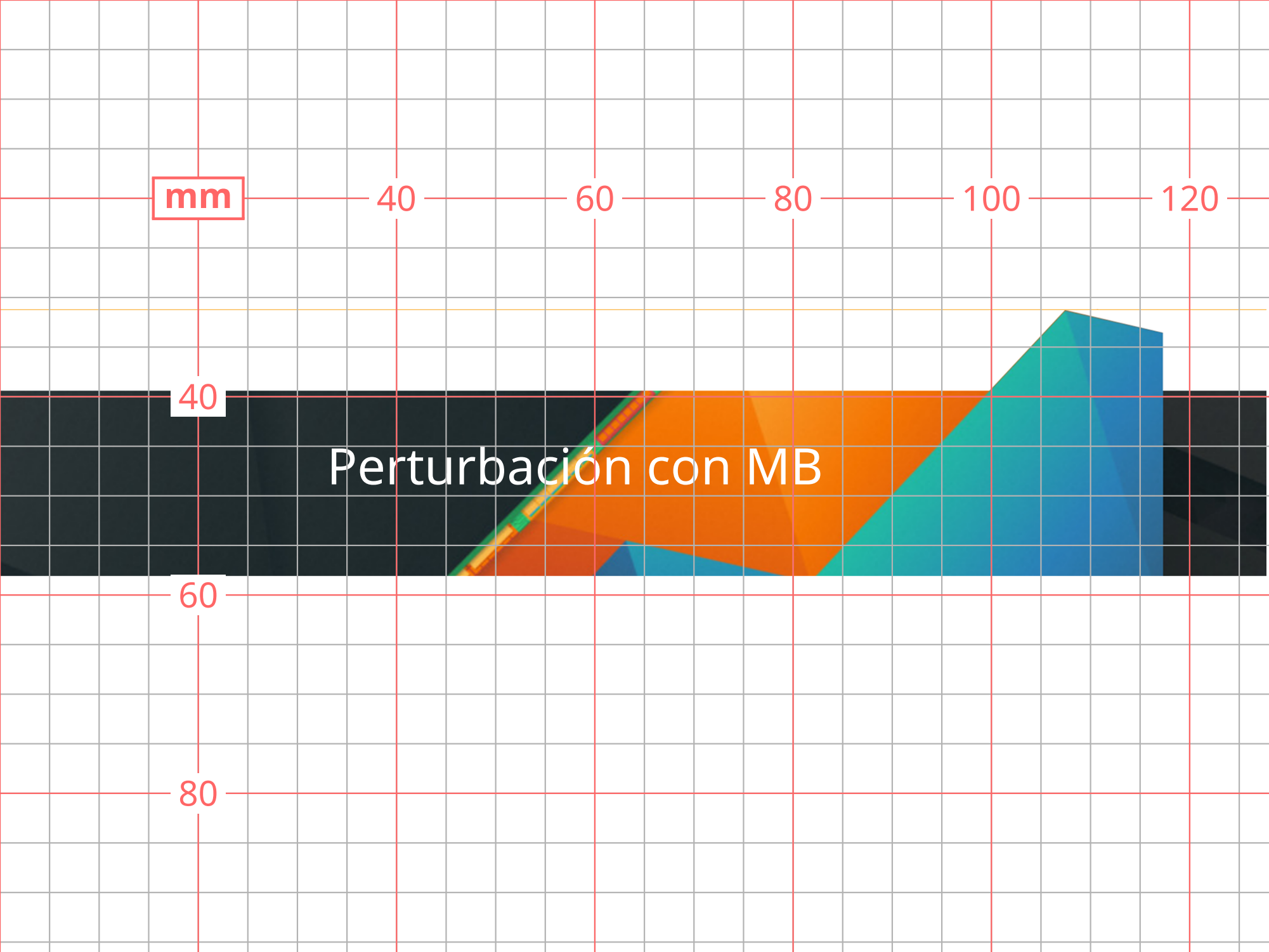
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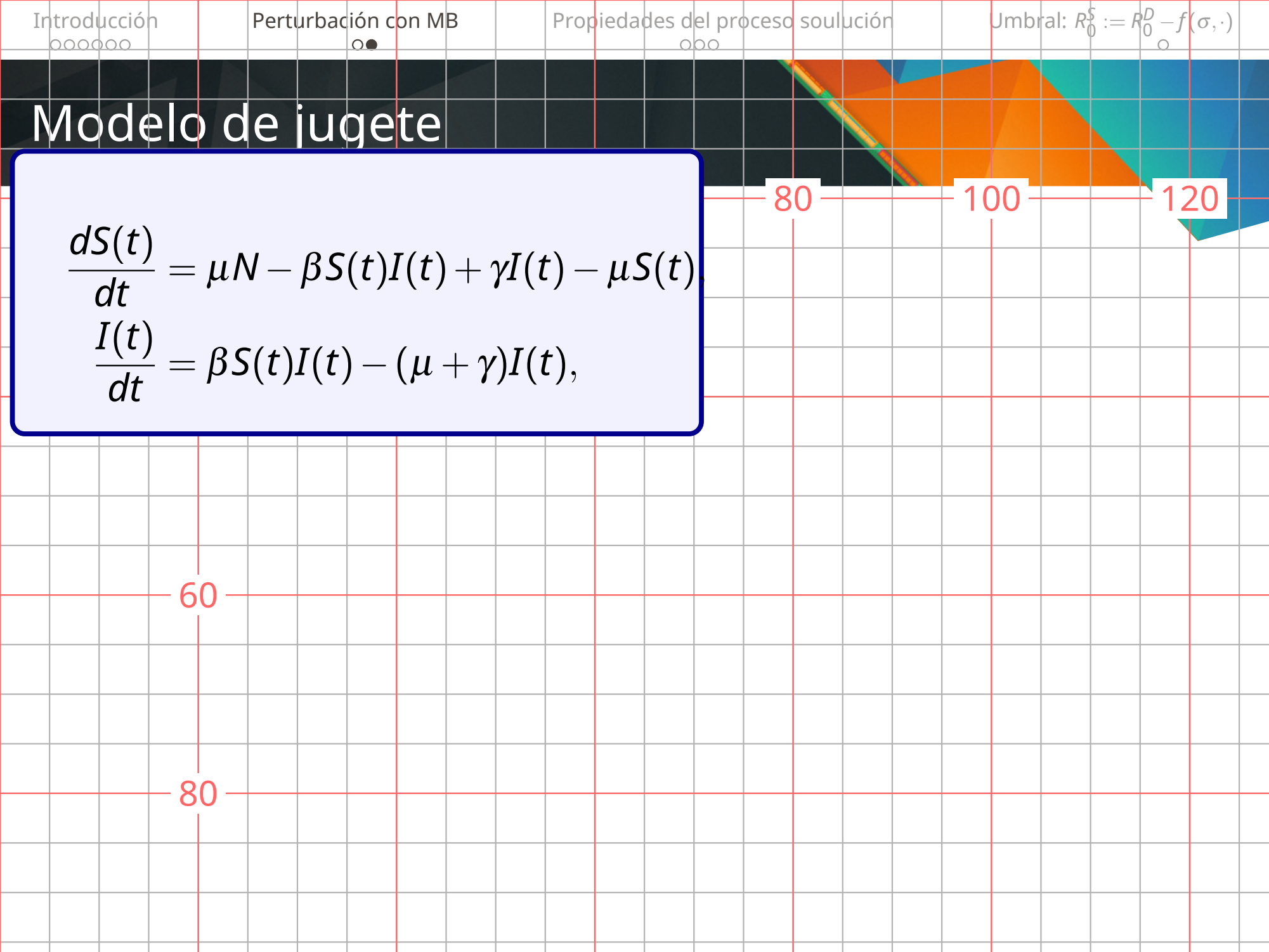
Perturbación con MB

Modelo de jugete

$$\frac{dS(t)}{dt} = \mu N - \beta S(t)I(t) + \gamma I(t) - \mu S(t),$$
$$\frac{I(t)}{dt} = \beta S(t)I(t) - (\mu + \gamma)I(t),$$



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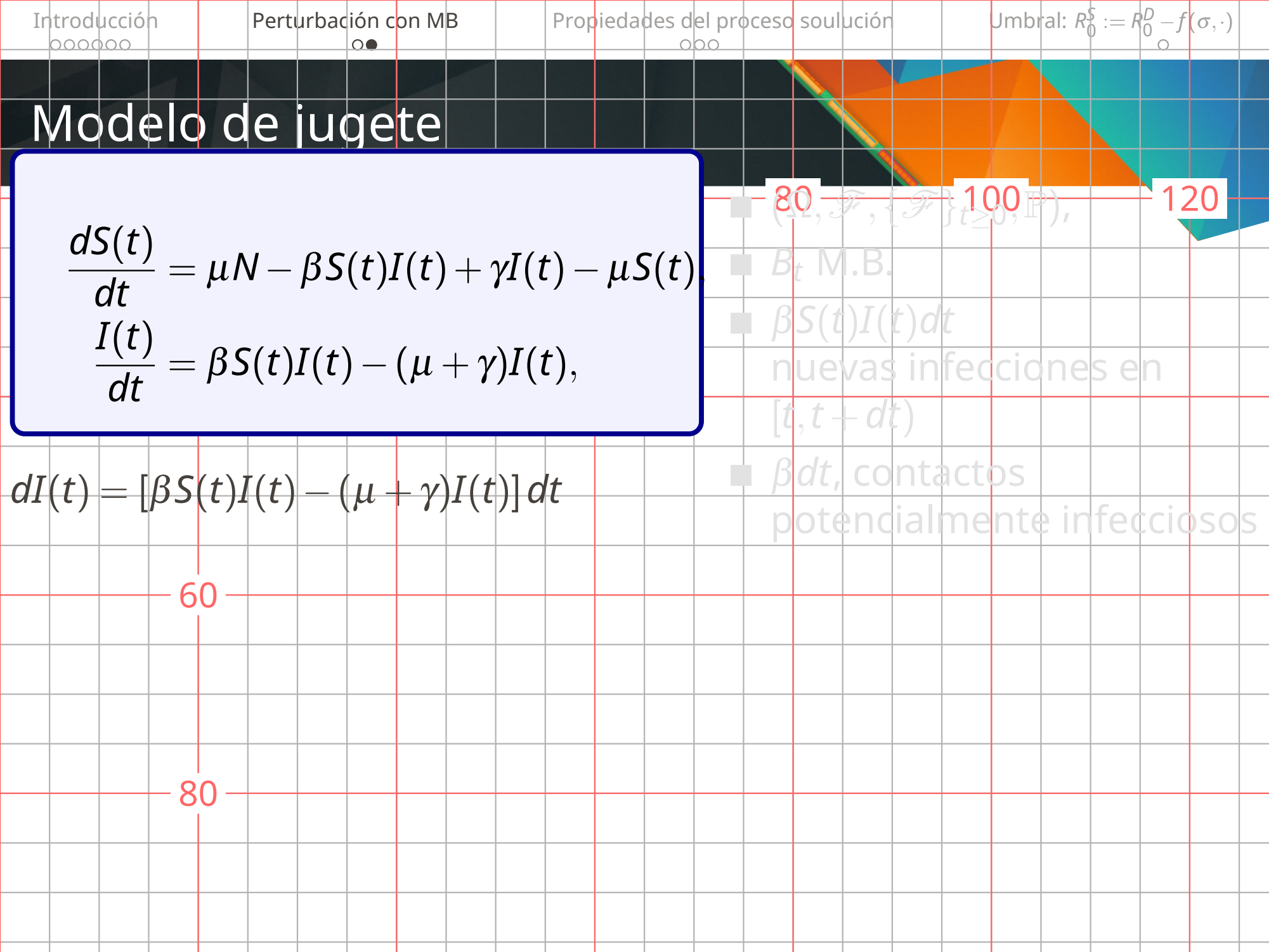
- $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{t\geq 0}, \mathbb{P})$,
- B_t M.B.
- $\beta S(t)I(t)dt$ nuevas infecciones en $[t, t+dt)$
- βdt, contactos potencialmente infecciosos



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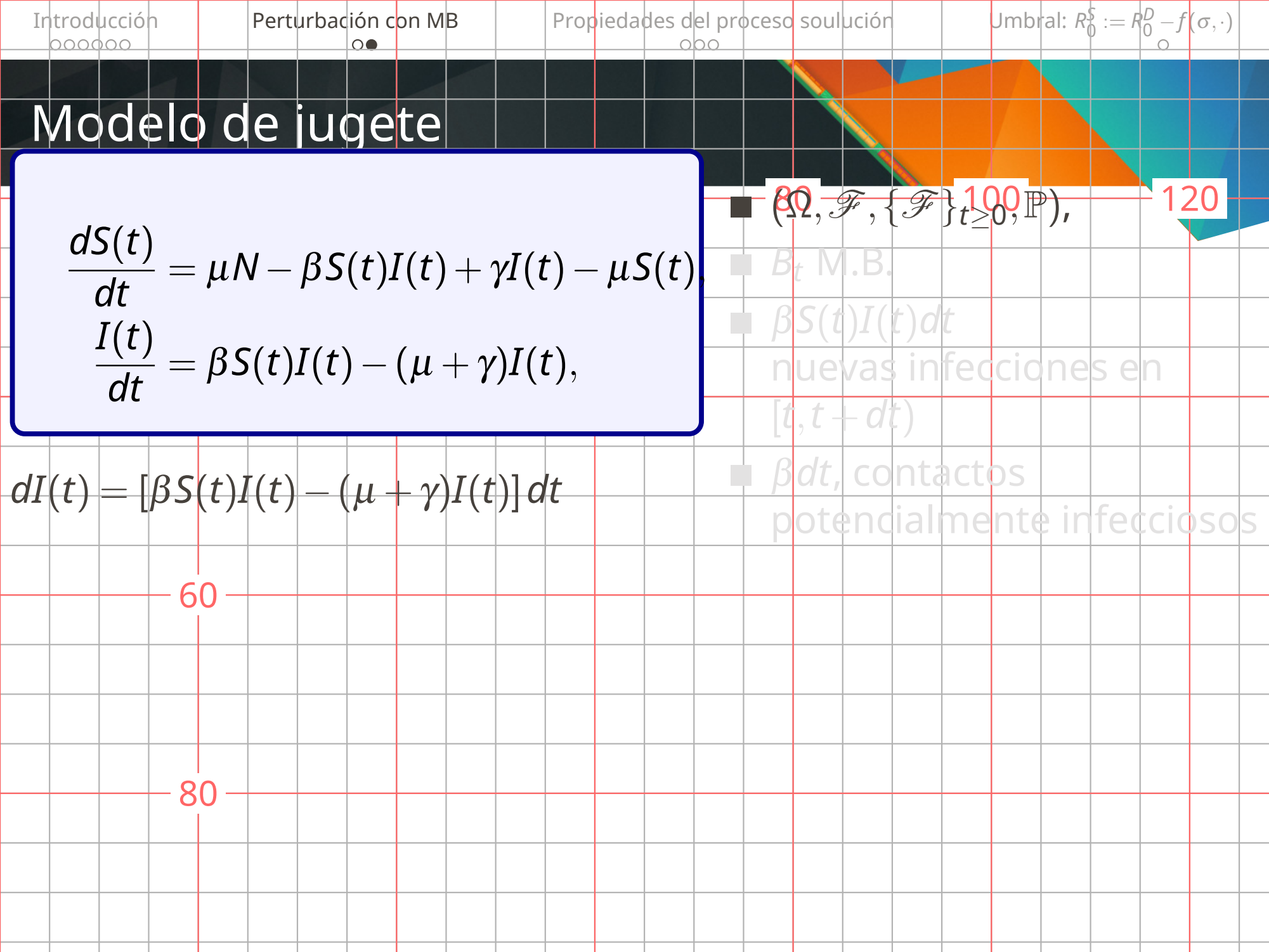

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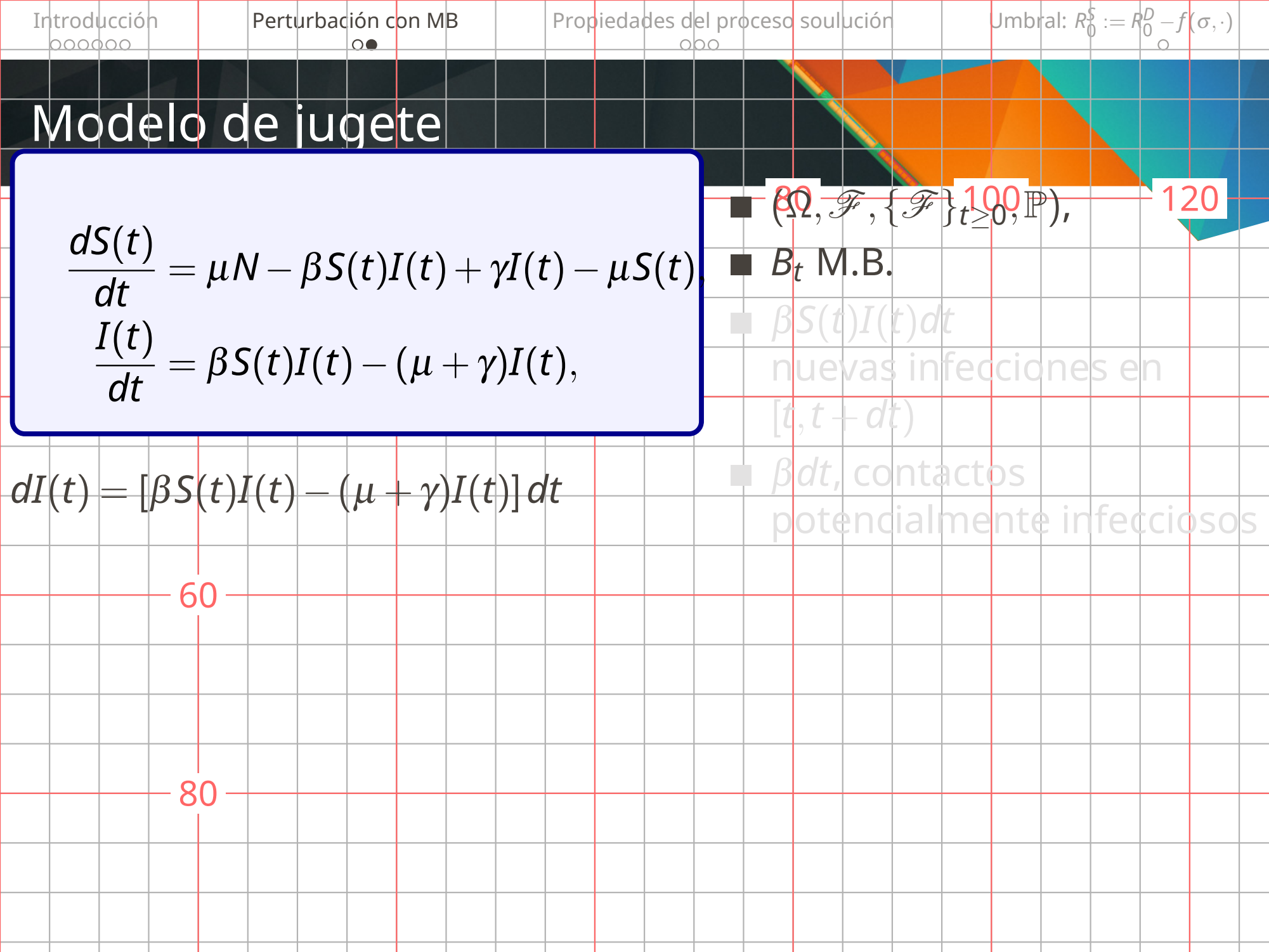

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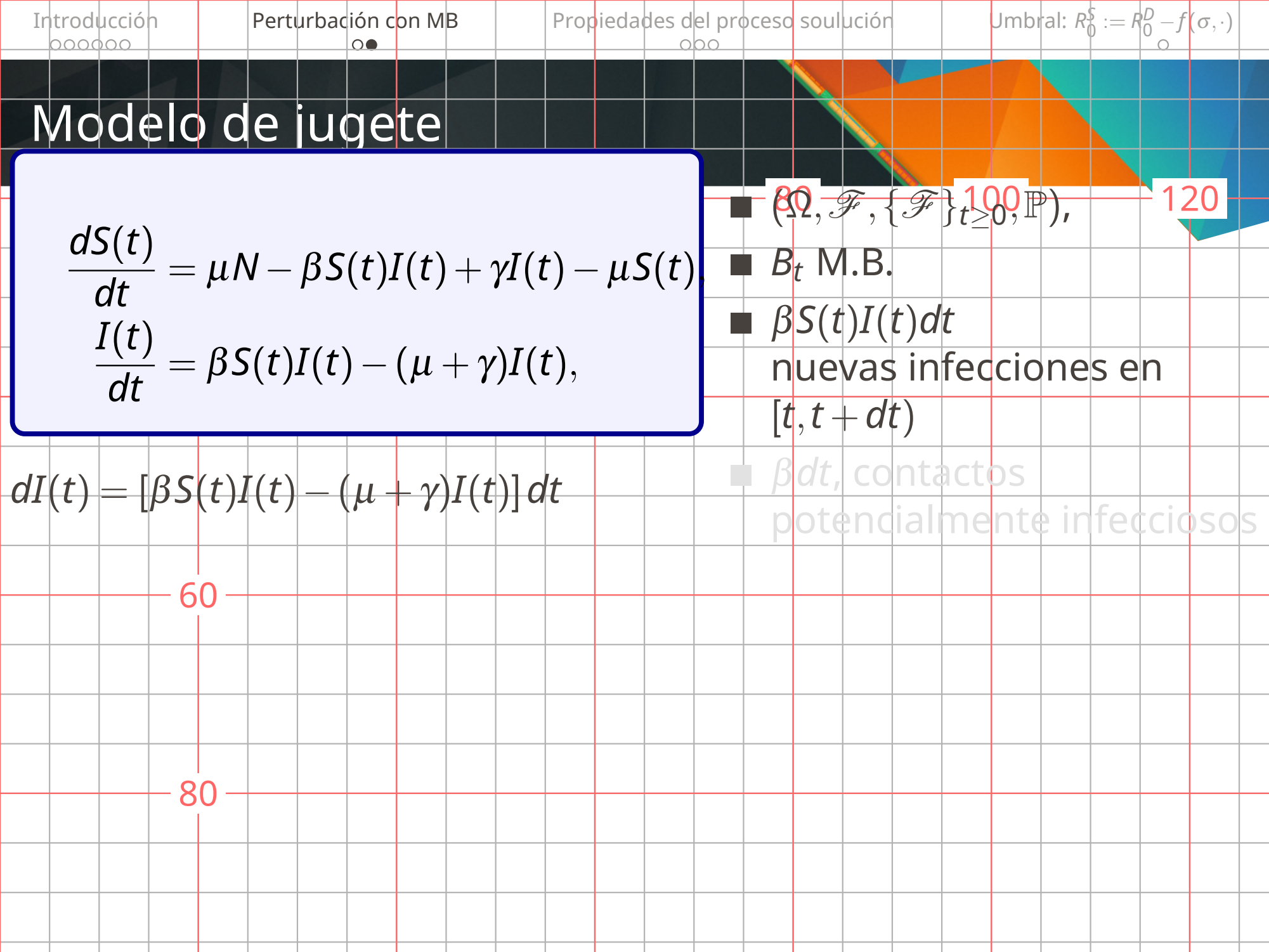

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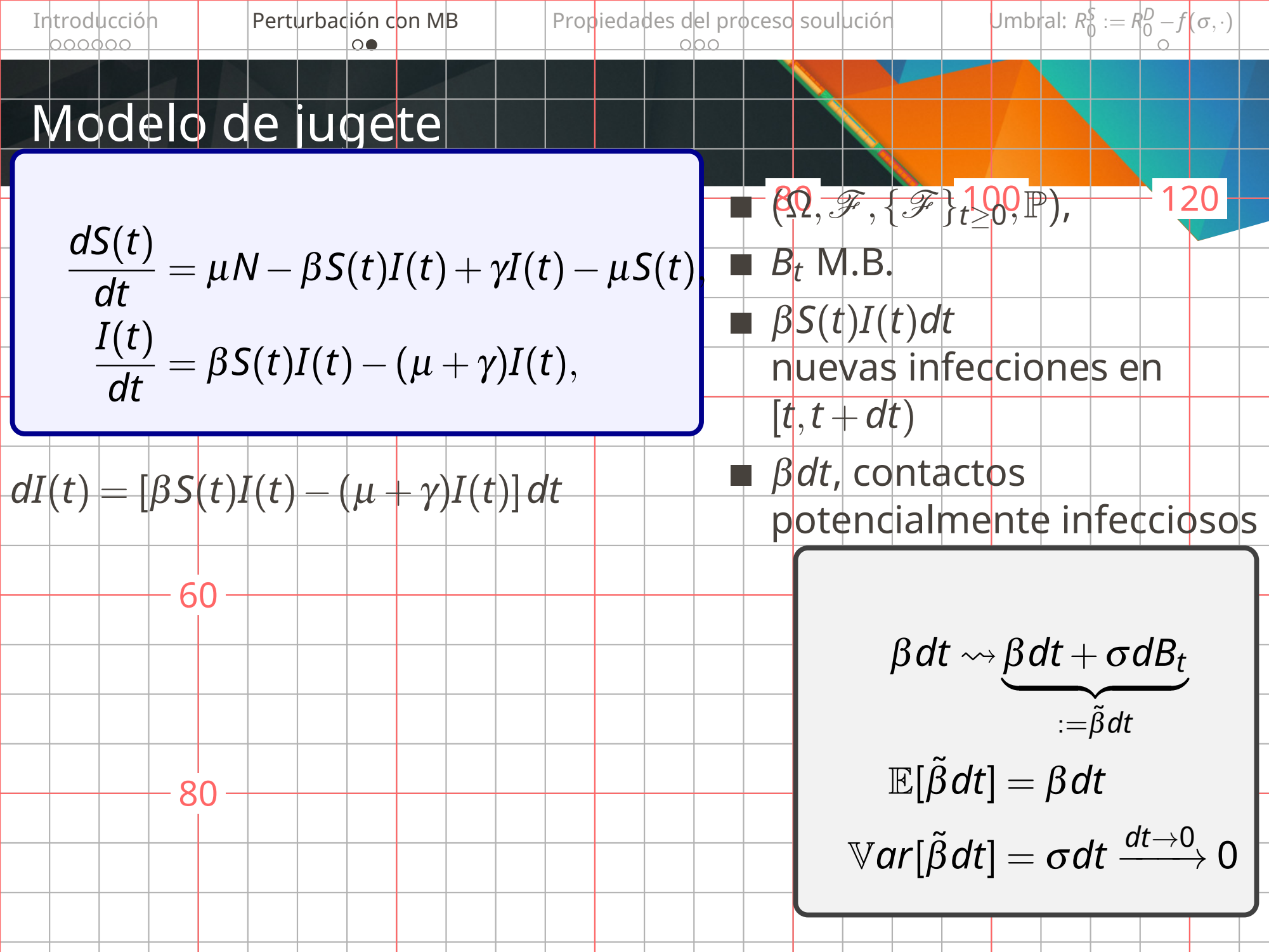

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$$\beta dt \rightsquigarrow \underbrace{\beta dt + \sigma dB_t}_{:=\tilde{\beta} dt}$$
$$\mathbb{E}[\tilde{\beta} dt] = \beta dt$$
$$\mathbb{V}ar[\tilde{\beta} dt] = \sigma dt \xrightarrow{dt\to 0} 0$$

Modelo de jugete

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$$N = S(t) + I(t) = \text{cte.}$$

$$\frac{dI(t)}{dt} = I(t)\left([\beta(N - I(t)) - \mu - \gamma]\right)dt - \sigma(N - I(t))dB_t$$


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Propiedades del proceso soulución
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Existencia de solución única positiva

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Teorema

- $I(0) \in (0, N)$

existe c.s. **solución global única positiva** e **invariante** a EDE($\star$)

$$\Pr\{I(t) \in (0, N)\ \forall t \geq 0\} = 1.$$

$$dI(t) = I(t)\{[\beta(N - I(t)) - \mu - \gamma]dt + \sigma(N - I(t))dB_t\} \qquad (\star)$$

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Extinción por ruido ambiental

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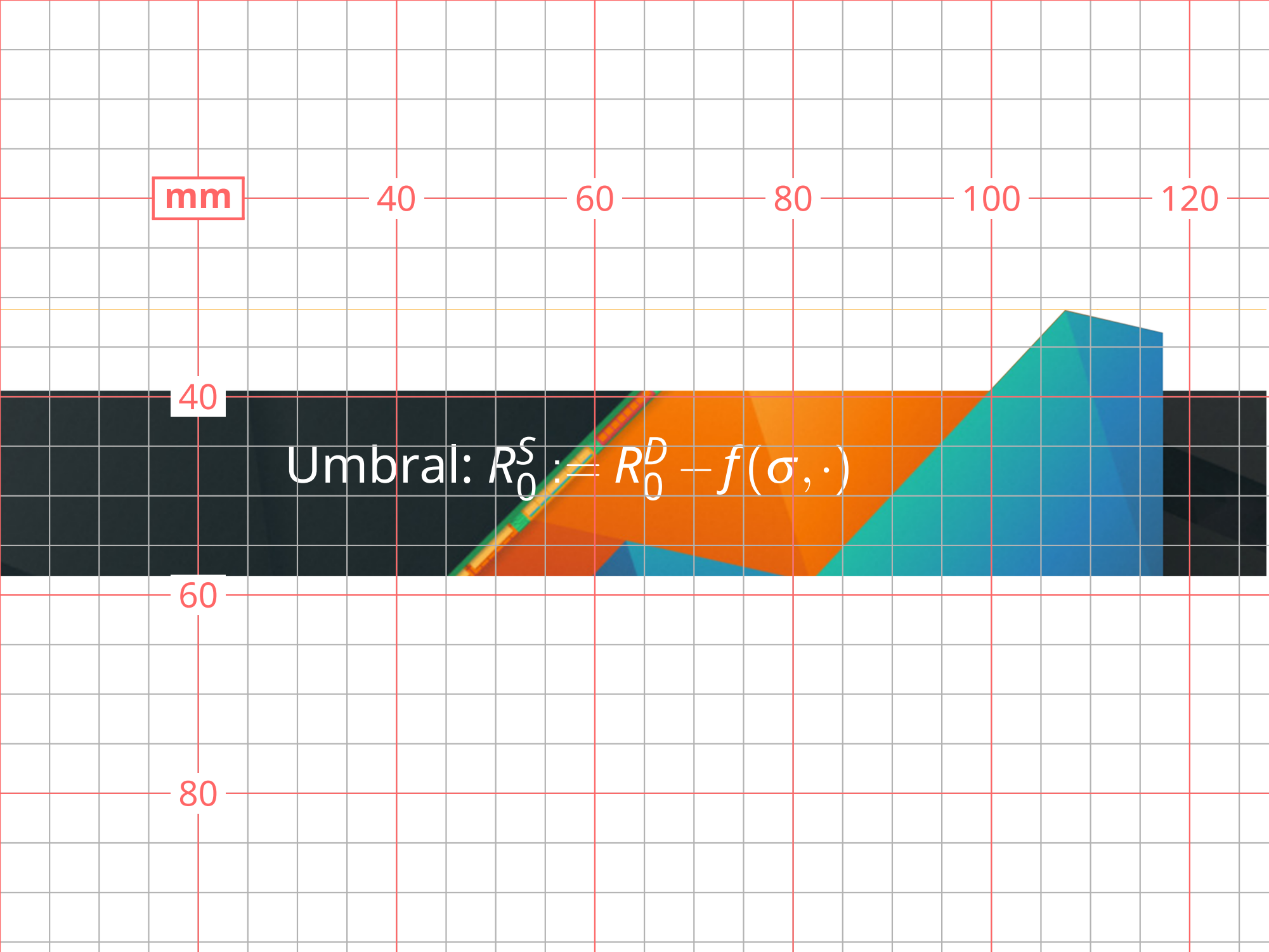
Teorema

- $\sigma^2 > \max\left\{\frac{\beta}{N}, \frac{\beta^2}{2(\mu+\gamma)}\right\}$

para toda $I(0) \in (0,N)$ la solución de EDE($\star$) cumple

$$\limsup_{t\to\infty} \frac{1}{t}\log I(t) \leq \underbrace{-\mu-\gamma+\frac{\beta^2}{2\sigma^2}}_{<0}$$

$$dI(t) = I(t)\{[\beta(N-I(t))-\mu-\gamma]dt + \sigma(N-I(t))dB_t\} \qquad (\star)$$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Umbral: $R_0^S := R_0^D - f(\sigma, \cdot)$