NoMMA
Nodo Multidisciplinario de Matemáticas Aplicadas
Instituto d

Extinción, Persistencia y Comportamiento Umbral en Modelos Compartimentales

Estocásticmente Perturbados

UNAM, Juriquilla, Queretaro
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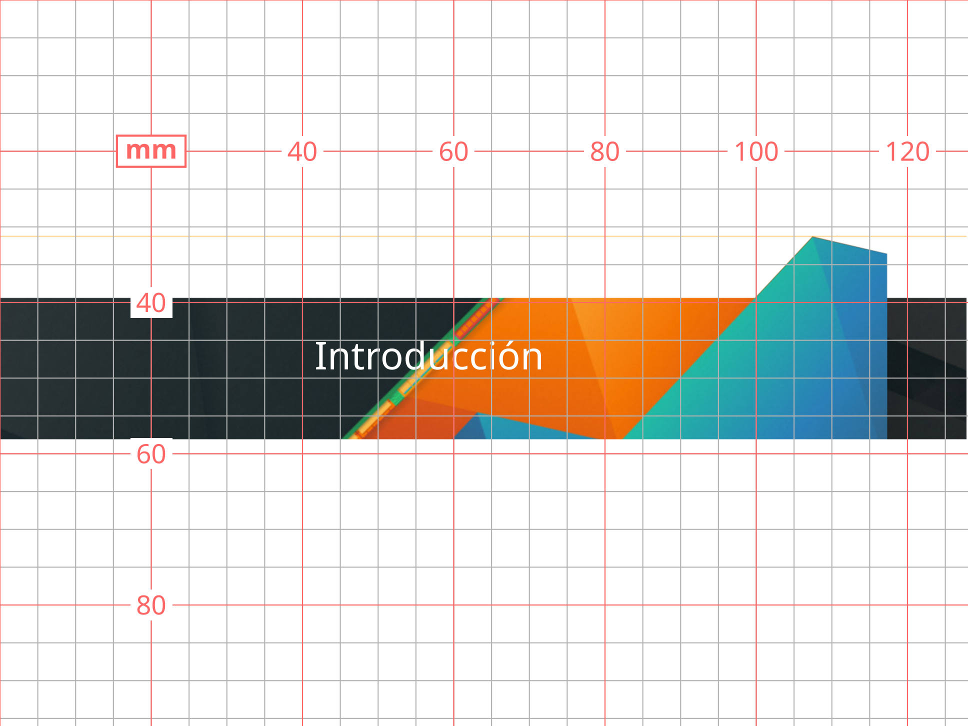
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Introducción
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Para fijar ideas

$$\dot{S}(t) = \Lambda - \mu S(t) - \beta S(t)I(t) - \delta S(t)$$

$$\dot{I}(t) = \beta S(t)I(t) - (\mu + \gamma + \varepsilon)I(t)$$

$$\dot{R}(t) = \gamma I(t) - \mu R(t) + \delta S(t)$$

Para fijar ideas

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\end{aligned}
$$

Umbral determinista

$$\mathscr{R}_0 = \frac{\beta \Lambda}{(\mu + \gamma + \varepsilon)(\mu + \delta)}$$

$\mathscr{R}_0 < 1 \Rightarrow$ *FDE* : (g.a.s)

$\mathscr{R}_0 > 1 \Rightarrow$ *EE* : (g.a.s)

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$$\dot{S}(t) = \Lambda - \mu S(t) - \beta S(t)I(t) - \delta S(t)$$
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$$\beta dt \rightsquigarrow \beta dt + \sigma dB_t$$

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Para fijar ideas

mm 40 60 80 100 120

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$$\mathscr{R}_0 = \frac{\beta\Lambda}{(\mu + \gamma + \varepsilon)(\mu + \delta)}$$
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$\mathscr{R}_0 > 1 \Rightarrow$ *EE* : (g.a.s)

Umbral estocástico

$\mathscr{R}_0^S = \ ?$

$\mathscr{R}_0^S < 1 \Rightarrow$ extinción

$\mathscr{R}_0^S > 1 \Rightarrow$ persistencia

Para fijar ideas

mm 40 60 80 100 120

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Ver:

Y. Zhang, Y. Li, Q. Zhang, and A. Li.
Behavior of a stochastic SIR epidemic model with saturated incidence and vaccination rules.
Physica A: Statistical Mechanics and its Applications, 501:178–187, 2018.

¿Cuando considerar Modelos Estocásticos?

mm 40 60 80 100 120

Sean importantes

- Poblaciones pequeñas
- Variabilidad demográfica
- Variabilidad ambiental

Según:

Allen, L. J. (2017). A primer on stochastic epidemic models: Formulation, numerical simulation, and analysis. *Infectious Disease Modelling*, 2(2):128–142.

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Ejemplo

Transmisión, recuperación, nacimientos, muertes.

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Ejemplo

Transmisión, recuperación, nacimientos, muertes.

Condiciones territoriales, acuáticas: enfermedades vectoriales, zoonóticas transmitidas por alimentos.

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Alternativas

Modelos

- (D/C)–TMCs
- Perturbación de parámetros
  - Procesos reversibles en media
  - β_t^H $H \in (0.5, 1)$
- Random Diff. Eq.

Herramientas

- Gillespie
- Kloeden-Methods
- Hermite-PC

MC + ME → *SDE*

L. J. Allen.
A primer on stochastic epidemic models: Formulation, numerical simulation, and analysis.
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$\varphi dt \rightsquigarrow \varphi dt + \sigma dB_t$

Gray, A., Greenhalgh, D., Hu, L., Mao, X., and Pan, J. (2011). A Stochastic Differential Equation SIS Epidemic Model. *SIAM Journal on Applied Mathematics*, 71(3):876–902.

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$\varphi dt \rightsquigarrow \varphi dt + F(x) dB_t$

Schurz, H. and Tosun, K. (2015). Stochastic Asymptotic Stability of SIR Model with Variable Diffusion Rates. *Journal of Dynamics and Differential Equations*, 27(1):69–82.

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$d\varphi_t = (\varphi_e - \varphi_t)dt + \sigma_\varphi dBt$

Allen, E. (2016). Environmental variability and mean-reverting processes. *Discrete and Continuous Dynamical Systems - Series B*, 21(7):2073–2089.

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Ma, Y., Zhang, Q., and Ye, M. (2017).
Mean-square dissipativity of numerical methods for a class of resource-competition models with fractional brownian motion.
Systems Science & Control Engineering, 5(1):268–277.

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parametros son v.a.

Chen-Charpentier, B.-M., Cortés, J.-C., Licea, J.-A., Romero, J.-V., Roselló, M.-D., Santonja, F.-J., and Villanueva, R.-J. (2015).
Constructing adaptive generalized polynomial chaos method to measure the uncertainty in continuous models: A computational approach.
Mathematics and Computers in Simulation, 109:113 – 129.

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Objetivo

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Ilustrar las ideas de $\varphi dt \rightsquigarrow \varphi dt + \sigma dB_t$

- Modelación
- Análisis y Simulación
- Ideas al aire

Esquema de Charla

1. Introducción
2. Perturbación con MB
3. Propiedades del proceso soulución
4. Umbral: $R_0^S := R_0^D - f(\sigma, \cdot)$
5. Ideas

Perturbación con MB

Modelo de jugete

$$
\begin{aligned}
\frac{dS(t)}{dt} &= \mu N - \beta S(t)I(t) + \gamma I(t) - \mu S(t),\\
\frac{I(t)}{dt} &= \beta S(t)I(t) - (\mu + \gamma)I(t),\\
R_0^D &= \frac{\beta N}{\mu + \gamma}
\end{aligned}
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- $(\Omega, \mathscr{F}, \{\mathscr{F}\}_{t\geq 0}, \mathbb{P})$,
- B_t M.B.
- $\beta S(t)I(t)dt$ nuevas infecciones en $[t, t+dt)$
- βdt, contactos potencialmente infecciosos

$$dI(t) = [\beta S(t)I(t) - (\mu+\gamma)I(t)]\,dt$$

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$$dI(t) = [\beta S(t)I(t) - (\mu + \gamma)I(t)]\,dt$$

$$\beta dt \rightsquigarrow \underbrace{\beta dt + \sigma dB_t}_{:=\tilde{\beta} dt}$$
$$\mathbb{E}[\tilde{\beta} dt] = \beta dt$$
$$\mathbb{V}ar[\tilde{\beta} dt] = \sigma^2 dt \xrightarrow{dt \to 0} 0$$

Modelo de jugete

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\begin{aligned}
\frac{dS(t)}{dt} &= \mu N - \beta S(t)I(t) + \gamma I(t) - \mu S(t), \\
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$$
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\end{aligned}
$$

$$N = S(t) + I(t) = \text{cte.}$$

$$\frac{dI(t)}{dt} = I(t)\left(\left[\beta(N - I(t)) - \mu - \gamma\right]\right)dt - \sigma(N - I(t))dB_t$$

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Propiedades del proceso soulución

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Existencia de solución única positiva

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Teorema

- $\forall I(0) \in (0, N)$

existe c.s. **solución global única positiva** e **invariante** a EDE$(\star)$

$$\Pr\{I(t) \in (0, N)\ \forall t \geq 0\} = 1.$$

$$\begin{aligned} dI(t) = I(t)\{&[\beta(N - I(t)) - \mu - \gamma]dt \\ &+ \sigma(N - I(t))dB_t\} \end{aligned} \qquad (\star)$$

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Extinción por ruido ambiental

$$dI(t) = I(t)\{[\beta(N - I(t)) - \mu - \gamma]dt + \sigma(N - I(t))dB_t\} \qquad (\star)$$

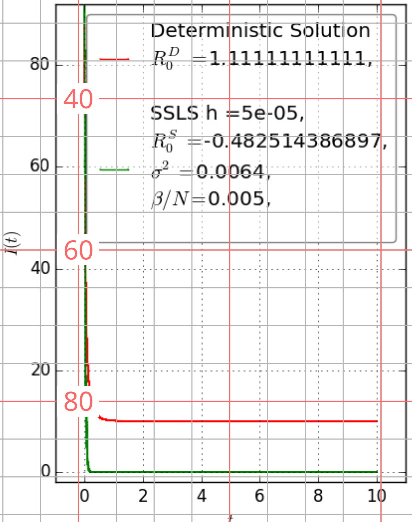
Teorema

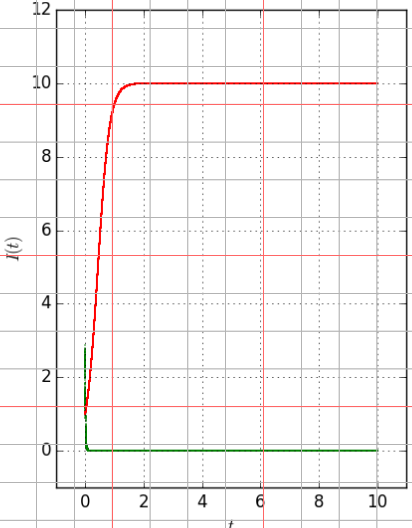
- $\sigma^2 > \max\left\{\dfrac{\beta}{N}, \dfrac{\beta^2}{2(\mu+\gamma)}\right\}$

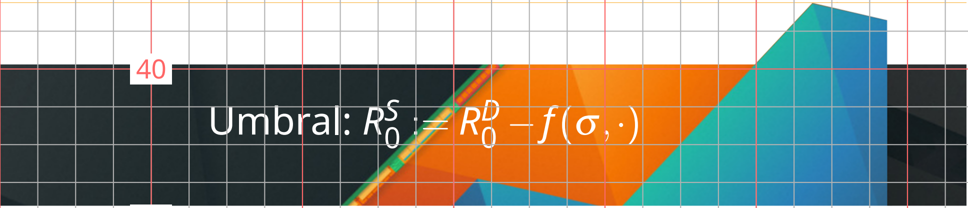
para toda $I(0) \in (0, N)$ la solución de EDE($\star$) cumple

$$\limsup_{t\to\infty} \frac{1}{t}\log I(t) \leq \underbrace{-\mu - \gamma + \frac{\beta^2}{2\sigma^2}}_{<0}$$

Simulación: Extinción por ruido ambiental






Umbral: $R_0^S := R_0^D - f(\sigma,\cdot)$

Extinción

Teorema (Extinción)

- $R_0^S := R_0^D - \dfrac{\sigma^2}{2(\mu+\gamma)} < 1$
- $\sigma^2 \leq \dfrac{\beta}{N}$

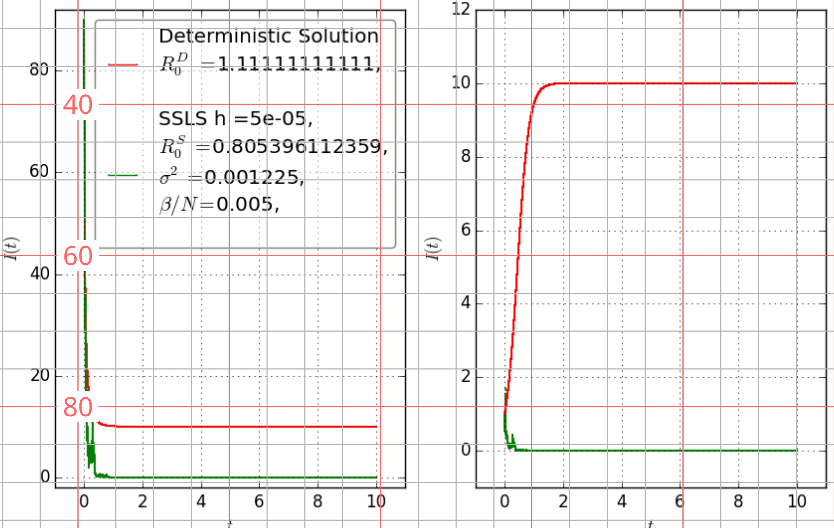
para toda $I(0) \in (0, N)$

$$\limsup_{t\to\infty} \frac{1}{t} \log I(t) \leq \kappa, \quad c.s.$$

$$\kappa := \beta N - \mu - \gamma - \frac{\sigma^2 N^2}{2\sigma^2} < 0$$

$$dI(t) = I(t)\{[\beta(N - I(t)) - \mu - \gamma]dt + \sigma(N - I(t))dB_t\} \qquad (\star)$$

Simulación: Extinción por condición umbral



Persistencia

Teorema (Persistencia)

- $R_0^S := R_0^D - \dfrac{\sigma^2}{2(\mu+\gamma)} > 1$

para toda $I(0) \in (0, N)$

$$\limsup_{t\to\infty} \frac{1}{t} \log I(t) \geq \varepsilon, \quad c.s.$$

$$\liminf_{t\to\infty} \frac{1}{t} \log I(t) \leq \varepsilon, \quad c.s.$$

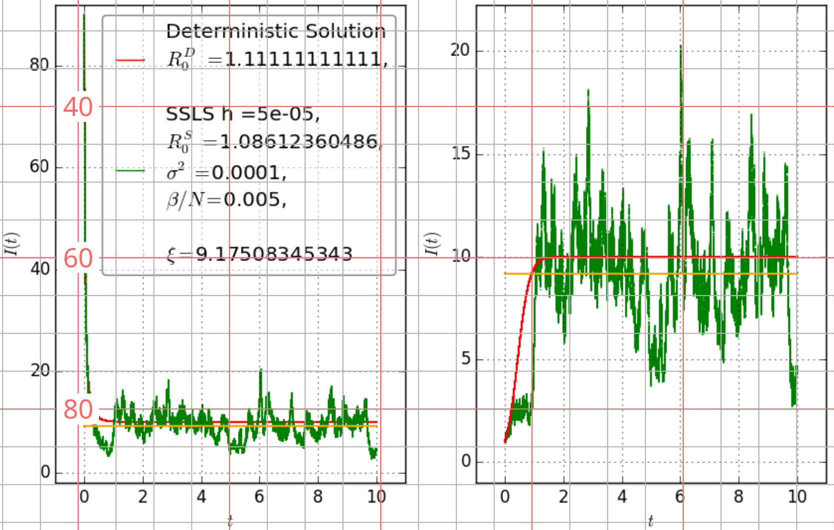
$$dI(t) = I(t)\{[\beta(N - I(t)) - \mu - \gamma]dt + \sigma(N - I(t))dB_t\} \qquad (\star)$$

$R_0^D : \beta N/(\mu+\gamma)$

$$R_0^D < 1 \ \Rightarrow \lim_{t\to\infty} I(t) = 0$$

$$R_0^D > 1 \ \Rightarrow \lim_{t\to\infty} I(t) = N\left(1 - \frac{1}{R_0^D}\right)$$

Simulación: Persistencia



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