Stochastic Asymptotic Analysis of a Multi-Host Model with Vector Transmission.

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Abstract

We present a stochastic extension of a disease vector transmission model with multi-hosts. Our deterministic dynamics considers vectors and his interacting with two kind of hosts. Applying a general stochastic perturbation to a vector biting rate, we extend to a stochastic differential equation system (SDEs), which model the influence of environmental noise on the vector transmission.

Keywords: Multi-host, persistence, extinction, stochastic perturbation, vector transmission.

1. Introduction

Motivation. Now days, vector diseases represents the principal cause of death in many sub development countries. A combination of low resources, climate variations and lack of efficient health services, amplifies its prevalence. So, design tools to planing strategies under real conditions is crucial. However this phenomena depends on many and very intricate variables. By this reason, not exist a general model which describe or predict in a acceptable way, the essence of this complex system. Instead, a lot of literature focus on describe a specific characteristic under restrictive or unrealistic conditions. We believe that to incorporate uncertainty is an clever option — resume the effect of many variables as "noise" —to produce a more realistic model.

Find other diseases, structure and use Chagas like example, leismaniasis peste (sin tratamiento

Previous and related work. Essentially, in literature exist two main alternatives for incorporate environmental noise. The first alternative, considers as step one, to describe the disease transitions with a discrete Markov chain. Next, letting discrete time to zero, one gets a stochastic differential equation (SDE). We refer to [Allen's work] and reference there in. The other approach, perturbs a target parameter ϕ of a given ordinary differential equation (ODE) with a Wienner process. To be precise, for a differential time $(t + dt)$ one stochastically perturbs ϕdt with a Wienner process of intensity σ. So, one substitute ϕdt by $\phi dt + \sigma dW_t$ to get a SDE model. We mention as representative works in this line to [Gray Mao, Greeendhald, Liu Cai].

Environmental noise could dramatically changes the nature of a deterministic equilibrium. For example, Mao et al. (2002) found that even the presence of a tiny noise can suppress a potential population explosion. Hence, it is natural to investigate effects of environment random fluctuations in a population dynamics.

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[1]Since 1880.

Results. We follow ideas of [Schurz] in order to extend a SIS deterministic epidemic model applying a general stochastic perturbation. In this sense our results follows the same style of [Tosun, Cai, Gray.]. However all mentioned literature do not consider multi host structure o vectorial disease transmission. In this line we direct our research. Concretely, our main contributions are:

- The study a disease transmitted by vector with two different hosts, which is a new approach on epidemic SDE models.
- We consider a family of functions dependent on the states variables as noise intensities. In this way, we obtain sufficient criteria to guarantee extinction and persistence of the disease.
- According to our numeric simulations, we observe a mean behavior which is very similar to our deterministic base dynamics. For example, our numeric mean estimations suggest that deterministic free disease and endemics equilibriums remains close in large time from its stochastic extension.

Outline.

2. Stochastic Multi-Host Model

2.1. Deterministic SI Multi-Host Model

We consider a vector Multi-Host SI structure, that is, human, animal and vector populations respectively splits susceptible S and infected I classes in S_h, I_h, S_a, I_a, and S_v, I_v subclasses. For modeling propose, we assume that whole populations of humans H and animals A are constants and the total vector population $T_v = S_v + I_v$ follows a logistic dynamics [e.g. 1]. To be precise, let K carrying capacity, Λ_v vector birth, μ_v vector death rates and $r = \Lambda_v - \mu_v$ vector grow rate and $\mu_v + (r/K)T_v$, the vector disease mortality (independent from the illness). Then, the total vector population follows the dynamics described by

$$\dot{T}_v = \Lambda_v T_v - [\mu_v + (r/K)T_v] T_v. \tag{1}$$

Hosts dynamics. We consider that when an infected vector bites a susceptible host, this moves to the infected class with certain probability. Set $f(S, I_v)$ for the infection force and $N = S + I$ for total host population, hence we describe the host dynamics by

$$\begin{aligned} \dot{S} &= \mu N - f(S, I_v) - \mu S \\ \dot{I} &= f(S, I_v) - \mu I. \end{aligned} \tag{2}$$

Note that N remains constant, so we replace S by $N - I$, and rewrite the infected equation in (2) as

$$\dot{I} = f(N - I, I_v) - \mu I. \tag{3}$$

Model Formulation. Therefore, combining ideas to the infection forces from [1], the vector mortality formulation of [2] and the dynamics described by (3), we propose our vector multi-host deterministic base model (see Table 1 for parameters details)

$$\begin{aligned} \dot{I}_h &= z_h \theta_h I_v (H - I_h) - \mu_h I_h \\ \dot{I}_a &= z_a \theta_a I_v (A - I_a) - \mu_a I_a \\ \dot{S}_v &= \Lambda_v T_v - (z_h \theta_{v_h} I_h + z_a \theta_{v_a} I_a) S_v - (\mu_v + r_K T_v) T_v \frac{S_v}{T_v} \\ \dot{I}_v &= (z_h \theta_{v_h} I_h + z_a \theta_{v_a} I_a) S_v - (\mu_v + r_K T_v) T_v \frac{I_v}{T_v} \end{aligned} \tag{4}$$

$$\theta_h = \frac{\pi_h}{H}, \qquad \theta_a = \frac{\pi_a}{A}, \qquad \theta_{v_h} = \frac{\pi_{v_h}}{H}, \qquad \theta_{v_a} = \frac{\pi_{v_a}}{A}, \qquad r_K = \frac{r}{K}.$$

Our model describes the vector transmission dynamics of diseases with human—animal multi-host vector structure and where the host population lacks of recover classes — for example, due to permanent infection, and absent or inefficient of treatment. Along with others, we mention Chagas, Plague, Leishmaniasis [3–5] as diseases with this characteristics. In the following section we extend this deterministic base to a more realistic description.

Table 1: Model parameters

Parameter	Units	Description
H, A, K	human, animal, vector	Maximum allowable number of humans, animals and vectors.
μ_h, μ_a, μ_v	year^{-1}	Mortality rate of humans, animals and vectors.
z_h, z_a	$\text{bite}\,\text{year}^{-1}\,\text{vector}^{-1}$	Human and Animal bitting rates per vector.
π_h , π_a	$\text{human}\,\text{bite}^{-1}, \text{animal}\,\text{bite}^{-1}$	Proportion of contacts between an infected vector and a susceptible human (animal) that result in infection.
π_{v_h} , π_{v_a}	$\text{vector}\,\text{bite}^{-1}$	Fraction of contacts between susceptible vectors and infected humans (animals) that produces new infections.
Λ_v	year^{-1}	Vector birth rate
r	year^{-1}	Vector net growth rate

2.2. Stochastic Extension

Randomness of the environmental factors influence on the biting rates of vectors. Motivated for this, we perturb bitting parameters z_h and z_a with functional stochastic noise intensities,

$$\begin{aligned} z_h dt &\to z_h dt + F_1\left(I_h(t), I_a(t), S_v(t), I_v(t)\right) dW_1(t),\\ z_a dt &\to z_a dt + F_2\left(I_h(t), I_a(t), S_v(t), I_v(t)\right) dW_2(t), \end{aligned}$$

where W_i are independent Wiener processes defined on a filtered complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, and $F_i\left(I_h, I_a, S_v, I_v\right)$ represents intensity of the noise on bitting parameters. Thus, our stochastic extension of (4) reads

$$\begin{aligned} dI_h &= \left[\alpha_h I_v\left(H - I_h\right) - \mu_h I_h\right] dt + F_1\left(I_h, I_a, S_v, I_v\right)\theta_h I_v\left(H - I_h\right) dW_1(t)\\ dI_a &= \left[\alpha_a I_v\left(A - I_a\right) - \mu_a I_a\right] dt + F_2\left(I_h, I_a, S_v, I_v\right)\theta_a I_v\left(A - I_a\right) dW_2(t)\\ dS_v &= \left[\Lambda_v T_v - \left(\alpha_{v_h} I_h + \alpha_{v_a} I_a\right) S_v - \left(\mu_v + r_K T_v\right) S_v\right] dt\\ &\quad - F_1\left(I_h, I_a, S_v, I_v\right)\theta_{v_h} I_h S_v dW_1(t) - F_2\left(I_h, I_a, S_v, I_v\right)\theta_{v_a} I_a S_v dW_2(t)\\ dI_v &= \left[\left(\alpha_{v_h} I_h + \alpha_{v_a} I_a\right) S_v - \left(\mu_v + r_K T_v\right) I_v\right] dt + F_1\left(I_h, I_a, S_v, I_v\right)\theta_{v_h} I_h S_v dW_1(t)\\ &\quad + F_2\left(I_h, I_a, S_v, I_v\right)\theta_{v_a} I_a S_v dW_2(t)\ . \end{aligned} \tag{5}$$

where

$$\alpha_h = z_h\theta_h, \qquad \alpha_a = z_a\theta_a, \qquad \alpha_{v_h} = z_{v_h}\theta_{v_h}, \qquad \alpha_{v_a} = z_{v_a}\theta_{v_a},$$

In order to assure existence, uniqueness and positivity of solutions for the above stochastic differential equation system (SDEs) we made the following Assumption.

Assumption 1. *Taking*

$$\mathbf{D} := \left\{(I_h, I_a, S_v, I_v) \in \mathbb{R}^4 :\ 0 \leq t_0,\ 0 \leq I_h < H,\ 0 \leq I_a < A,\ 0 < S_v \leq K,\ 0 \leq I_v < K,\ S_v + I_v \leq K\right\}$$

as our working set, we ask the following.

(A-1) *The coefficients of SDEs* (5) *are Lipschitz on* $\mathbf{D}$.

(A-2) *The initial condition* $X_0 := (I_h(0), I_a(0), I_v(0), S_v(0))$ *lies on* $\mathbf{D}$.

(A-3) *Each* $F_i\left(I_h, I_a, S_v, I_v\right)$ *have the form* $I_v G_i\left(I_h, I_a, S_v, I_v\right)$, *where* G_i *are functions locally Lipschitz-continuous on* $\mathbb{R}^4$.

2.3. Existence and Regularity of Solutions

For begin with the analysis of SDE (5), we guarantee the existence and uniqueness of the solutions, and since we study populations, these solution have to be positive. The following Theorem establish this.

Theorem 2.1. *Under Assumption 1 there is an unique solution* $(I_h(t), I_a(t), S_v(t), I_v(t))$ *to SDE* (5) *for* $t \geq 0$ *and the solution will remain in* $\mathbf{D}$ *with probability 1, namely* $(I_h(t), I_a(t), S_v(t), I_v(t)) \in \mathbf{D}$ *for all* $t \geq 0$ *almost surely.*

Proof. Let us first outline of the proof. We first going to guarantee the local existence and uniqueness of the solution, after that, we prove the globality of this and the invariance of $\mathbf{D}$ employing the Corollary 3.1 of Khasminskii [6, p. 76].

Let $X(t) = (I_h(t), I_a(t), S_v(t), I_v(t))$, $F_1 = F_1(I_h(t), I_a(t), S_v(t), I_v(t))$, $F_2 = F_2(I_h(t), I_a(t), S_v(t), I_v(t))$. For each $n \in \mathbb{N}$ define,

$$\begin{aligned}\mathbf{D}_n := \{(I_h, I_a, S_v, I_v) \in \mathbb{R}^4 : \quad & e^{-n} < I_h < H - e^{-n}, \quad e^{-n} < I_a < A - e^{-n}, \\ & e^{-n} < S_v < K - e^{-n}, e^{-n} < I_v < K - e^{-n}, S_v + I_v \leq K\}.\end{aligned}$$

Let us denote by $\tau(\mathbf{D}_n)$ the random time of first exit of stochastic process $(I_h(t), I_a(t), S_v(t), I_v(t))$ from the set $\mathbf{D}_n$. Since $b_0(X(t))$ and $b_1(X(t))$ are locally Lipschitz-continuous and satisfy linear growth condition on $\mathbf{D}_n$, there is an unique local solution on $t \in [0, \tau(\mathbf{D}_n))$ for any initial value $(I_h(0), I_a(0), S_v(0), I_v(0)) \in \mathbf{D}_n$.

To prove the global existence of the solution, we follow ideas of [7]. Let,

$$\begin{aligned}V(I_h, I_a, S_v, I_v) = {} & I_h + (H - I_h) - \ln(H - I_h) + I_a + (A - I_a) - \ln(A - I_a) + (K - S_v) \\ & - \ln(K - S_v) + S_v - \ln S_v + I_v + (K - I_v) - \ln(K - I_v),\end{aligned}$$

defined on

$$\widetilde{\mathbf{D}} := \{(I_h, I_a, S_v, I_v) \in \mathbb{R}^4;\ 0 \leq t :\ 0 < I_h < H,\ 0 < I_a < A,\ 0 < S_v < K, 0 < I_v < K,\ S_v + I_v \leq K\}.$$

Since $y - \ln y \geq 1$ for all $y \in (0, \infty)$, it follows that $V(I_h, I_a, S_v, I_v) \geq 5$ for $(I_h, I_a, S_v, I_v) \in \widetilde{\mathbf{D}}$ Also, we define

$$\begin{aligned}c = {} & \frac{1}{5}(\alpha_h + \alpha_a) K + \frac{2}{5}(\alpha_{v_h} H + \alpha_{v_a} A) + \frac{1}{5}(2\Lambda_v + r) + \frac{1}{10} \sup_{(I_h, I_a, S_v, I_v) \in \widetilde{\mathbf{D}}} \left\{ \left(\theta_h^2 K^4 + 3\theta_{v_h}^2 H^2 K^2\right) G_1^2 \right\} \\ & + \frac{1}{10} \sup_{(I_h, I_a, S_v, I_v) \in \widetilde{\mathbf{D}}} \left\{ \left(\theta_a^2 K^4 + 3\theta_{v_a}^2 A^2 K^2\right) G_2^2 \right\} .\end{aligned}$$

Applying the infinitesimal generator (eq. (A.2)) to our Liapunov function $V\left(I_h, I_a, S_v, I_v\right)$, yields

$$
\begin{aligned}
\mathcal{L}V\left(I_h, I_a, S_v, I_v\right) = & \left[\alpha_h I_v\left(H-I_h\right)-\mu_h I_h\right]\frac{\partial V}{\partial I_h}+\left[\alpha_a I_v\left(A-I_a\right)-\mu_a I_a\right]\frac{\partial V}{\partial I_a} \\
& +\left[\Lambda_v T_v-\left(\alpha_{v_h} I_h+\alpha_{v_a} I_a\right)S_v-\left(\mu_v+r_k T_v\right)S_v\right]\frac{\partial V}{\partial S_v} \\
& +\left[\left(\alpha_{v_h} I_h+\alpha_{v_a} I_a\right)S_v-\left(\mu_v+r_k T_v\right)I_v\right]\frac{\partial V}{\partial I_v} \\
& +\frac{1}{2}F_1^2\theta_h^2 I_v^2\left(H-I_h\right)^2\frac{\partial^2 V}{\partial I_h^2}+\frac{1}{2}F_2^2\theta_a^2 I_v^2\left(A-I_a\right)^2\frac{\partial^2 V}{\partial I_a^2} \\
& +\frac{1}{2}\left[F_1^2\theta_{v_h}^2 I_h^2 S_v^2+F_2^2\theta_{v_a}^2 I_a^2 S_v^2\right]\left(\frac{\partial^2 V}{\partial S_v^2}+\frac{\partial^2 V}{\partial I_v^2}\right) \\
& =\left(\frac{\alpha_h I_v\left(H-I_h\right)-\mu_h I_h}{H-I_h}\right)+\left(\frac{\alpha_a I_v\left(A-I_a\right)-\mu_a I_a}{A-I_a}\right) \\
& +\left(\frac{\Lambda_v T_v-\left(\alpha_{v_h} I_h+\alpha_{v_a} I_a\right)S_v-\left(\mu_v+r_k T_v\right)S_v}{-S_v}\right) \\
& +\left(\frac{\Lambda_v T_v-\left(\alpha_{v_h} I_h+\alpha_{v_a} I_a\right)S_v-\left(\mu_v+r_k T_v\right)S_v}{K-S_v}\right) \\
& +\left(\frac{\left(\alpha_{v_h} I_h+\alpha_{v_a} I_a\right)S_v-\left(\mu_v+r_k T_v\right)I_v}{K-I_v}\right) \\
& +\frac{1}{2}\left[\left(\frac{F_1^2\theta_h^2 I_v^2\left(H-I_h\right)^2}{(H-I_h)^2}\right)+\left(\frac{F_2^2\theta_a^2 I_v^2\left(A-I_a\right)^2}{(A-I_a)^2}\right)\right] \\
& +\frac{1}{2}\left[F_1^2\theta_{v_h}^2 I_h^2 S_v^2+F_2^2\theta_{v_a}^2 I_a^2 S_v^2\right]\left(\frac{1}{S_v^2}+\frac{1}{(K-S_v)^2}+\frac{1}{(K-I_v)^2}\right).
\end{aligned}
$$

Substituting $F_i = I_v G_i$ on the above relation, we obtain

$$
\begin{aligned}
\mathcal{L}V\left(I_h, I_a, S_v, I_v\right) = & \left(\frac{\alpha_h I_v\left(H-I_h\right)-\mu_h I_h}{H-I_h}\right)+\left(\frac{\alpha_a I_v\left(A-I_a\right)-\mu_a I_a}{A-I_a}\right) \\
& +\left(\frac{\Lambda_v T_v-s\left(\alpha_{v_h} I_h+\alpha_{v_a} I_a\right)S_v-\left(\mu_v+r_k T_v\right)S_v}{-S_v}\right) \\
& +\left(\frac{\Lambda_v T_v-\left(\alpha_{v_h} I_h+\alpha_{v_a} I_a\right)S_v-\left(\mu_v+r_k T_v\right)S_v}{K-S_v}\right) \\
& +\left(\frac{\left(\alpha_{v_h} I_h+\alpha_{v_a} I_a\right)S_v-\left(\mu_v+r_k T_v\right)I_v}{K-I_v}\right) \\
& +\frac{1}{2}\left[\left(\frac{I_v^2 G_1^2\theta_h^2 I_v^2\left(H-I_h s\right)^2}{(H-I_h)^2}\right)+\left(\frac{I_v^2 G_2^2\theta_a^2 I_v^2\left(A-I_a\right)^2}{(A-I_a)^2}\right)\right] \\
& +\frac{1}{2}\left[I_v^2 G_1^2\theta_{v_h}^2 I_h^2 S_v^2+I_v^2 G_2^2\theta_{v_a}^2 I_a^2 S_v^2\right]\left(\frac{1}{S_v^2}+\frac{1}{(K-S_v)^2}+\frac{1}{(K-I_v)^2}\right).
\end{aligned}
$$

Discarding negative terms, we bound the above relation by

$$\begin{aligned}
\mathcal{L}V\left(I_h, I_a, S_v, I_v\right) &\leq \left(\alpha_h + \alpha_a\right) I_v + 2\left(\alpha_{v_h} I_h + \alpha_{v_a} I_a\right) + \left(\mu_v + r_k T_v\right) \\
&+ \left(r + \Lambda_v\right) + \frac{1}{2}\left(G_1^2 \theta_h^2 + G_2^2 \theta_a^2\right) I_v^4 \\
&+ \frac{1}{2}\left(2 I_v^2 G_1^2 \theta_{v_h}^2 I_h^2 + 2 I_v^2 G_2^2 \theta_{v_a}^2 I_a^2 + G_1^2 \theta_{v_h}^2 I_h^2 S_v^2 + G_2^2 \theta_{v_a}^2 I_a^2 S_v^2\right) \\
&\leq \left(\alpha_h + \alpha_a\right) K + 2\left(\alpha_{v_h} H + \alpha_{v_a} A\right) + \left(2\Lambda_v + r\right) \\
&+ \sup_{(I_h, I_a, S_v, I_v) \in \widetilde{\mathbf{D}}} \frac{1}{2}\left\{\left(\theta_h^2 K^4 + 3\theta_{v_h}^2 H^2 K^2\right) G_1^2\right\} \\
&+ \sup_{(I_h, I_a, S_v, I_v) \in \widetilde{\mathbf{D}}} \frac{1}{2} s\left\{\left(\theta_a^2 K^4 + 3\theta_{v_a}^2 A^2 K^2\right) G_2^2\right\} \\
&= 5c\ .
\end{aligned}$$

As we know, $V\left(I_h, I_a, S_v, I_v\right) \geq 5$ for all $\left(I_h, I_a, S_v, I_v\right) \in \widetilde{\mathbf{D}}$, and since $5c \geq V\left(I_h, I_a, S_v, I_v\right)$, we deduce that $cV\left(I_h, I_a, S_v, I_v\right) \geq \mathcal{L}V\left(I_h, I_a, S_v, I_v\right)$.

Now we construct a crescent collection of subsets of $\mathbf{D}$ in order to satisfies the conditions of Theorem A.2. For each $n \in \mathbb{N}$ define

$$\mathbf{D}_n := \left[e^{-n}, H - e^{-n}\right] \times \left[e^{-n}, A - e^{-n}\right] \times \left[e^{-n}, K - e^{-n}\right] \times \left[e^{-n}, K - e^{-n}\right].$$

Let

$$\widetilde{\mathbf{D}} \backslash \mathbf{D}_n = (0, e^{-n}) \bigcup (H - e^{-n}, H) \times (0, e^{-n}) \bigcup (A - e^{-n}, A) \times (0, e^{-n}) \bigcup (K - e^{-n}, K) \times (0, e^{-n}) \bigcup (K - e^{-n}, K).$$

Since the real valued function $f(x) := x - \ln(x)$ satisfies

$$f(x) \geq 1, \qquad \forall x > 0,$$

we deduce that

$$V\left(I_h, I_a, S_v, I_v\right) \geq (K - S_v) - \ln(K - S_v) + S_v - \ln S_v + 3, \qquad \text{for all } \left(I_h, I_a, S_v, I_v\right) \in \widetilde{\mathbf{D}} \backslash \mathbf{D}_n.$$

Moreover, the function $f(x)$ is increasing when $x \in (K - e^{-n}, K)$, then

$$\begin{aligned}
V\left(I_h, I_a, S_v, I_v\right) &\geq (K - K + e^{-n}) - \ln(K - K + e^{-n}) + 4, \\
&\geq e^{-n} + n + 4, \\
&> n + 4.
\end{aligned}$$

Likewise, note that $f(x)$ decreases when $x \in (0, e^{-n})$, so

$$\begin{aligned}
V\left(I_h, I_a, S_v, I_v\right) &\geq e^{-n} - \ln(e^{-n}) + 4, \\
&\geq e^{-n} + n + 4, \\
&> n + 4.
\end{aligned}$$

Thus,

$$V\left(I_h, I_a, S_v, I_v\right) > n + 4, \qquad \text{for all } \left(I_h, I_a, S_v, I_v\right) \in \widetilde{\mathbf{D}} \backslash \mathbf{D}_n\ .$$

Hence, combining the above inequalities we arrive at,

$$\inf_{(I_h, I_a, S_v, I_v) \in \widetilde{\mathbf{D}} \backslash \mathbf{D}_n} V\left(I_h, I_a, S_v, I_v\right) > n + 4, \quad \text{for each } n \in \mathbb{N}\ . \tag{6}$$

Next we prove that $(I_h(t), I_a(t), S_v(t), I_v(t))$ remains in $\widetilde{D}$. Let $W(I_h, I_a, S_v, I_v, t) = e^{-c(t-t_0)} V(I_h, I_a, S_v, I_v)$ defined on $\widetilde{\mathbf{D}} \times [t_0, \infty)$ for $t_0 \geq 0$ fixed. Denote by $\tau(\mathbf{D}_n)$ the first exit time of solution (I_h, I_a, S_v, I_v) from the set $\mathbf{D}_n$ and $\tau_n(t) := \min\{t, \tau(\mathbf{D}_n)\}$. Since $\mathcal{L}V(I_h, I_a, S_v, I_v) \leq cV(I_h, I_a, S_v, I_v)$, we have that, $\mathcal{L}W(I_h, I_a, S_v, I_v, t) \leq 0$ for $(I_h, I_a, S_v, I_v, t) \in \widetilde{\mathbf{D}} \times [t_0, \infty)$. In order to get a upper bound for

$$\mathbb{E}W(I_h(\tau_n), I_a(\tau_n), S_v(\tau_n), I_v(\tau_n), \tau_n),$$

we apply the Dynkin's formula, then

$$\begin{aligned}
\mathbb{E}W(I_h(\tau_n), I_a(\tau_n), S_v(\tau_n), I_v(\tau_n), \tau_n) &= \mathbb{E}W(I_h(t_0), I_a(t_0), S_v(t_0), I_v(t_0), t_0) \\
&\quad + \mathbb{E}\int_{\tau_n}^{t_0} \mathcal{L}W(I_h(s), I_a(s), S_v(s), I_v(s), s)\, ds \\
&\leq \mathbb{E}W(I_h(t_0), I_a(t_0), S_v(t_0), I_v(t_0), t_0) \\
&= \mathbb{E}V(I_h(t_0), I_a(t_0), S_v(t_0), I_v(t_0)).
\end{aligned}$$

Since $\mathbf{D}_n \subset \mathbf{D}_{n+1}$, we have that

$$\begin{aligned}
\mathbb{P}\left(\tau(\widetilde{\mathbf{D}}) < t\right) &\leq \mathbb{P}\left(\tau(\widetilde{\mathbf{D}}_n) < t\right) \\
&= \mathbb{E}(\mathbb{1}_{\tau_n < t}) \\
&\leq \mathbb{E}\left(e^{c(t-\tau_n)} \frac{V(I_h(\tau_n), I_a(\tau_n), S_v(\tau_n), I_v(\tau_n))}{\inf\limits_{(I_h, I_a, S_v, I_v) \in \widetilde{\mathbf{D}} \setminus \mathbf{D}_n} V(I_h, I_a, S_v, I_v)} \mathbb{1}_{\tau_n < t}\right) \\
&\leq \frac{e^{c(t-t_0)} \mathbb{E}V(I_h(t_0), I_a(t_0), S_v(t_0), I_v(t_0))}{\inf\limits_{(I_h, I_a, S_v, I_v) \in \widetilde{\mathbf{D}} \setminus \mathbf{D}_n} V(I_h, I_a, S_v, I_v)} \\
&\leq \frac{e^{c(t-t_0)} \mathbb{E}V(I_h(t_0), I_a(t_0), S_v(t_0), I_v(t_0))}{n+4},
\end{aligned}$$

then, letting $n \to \infty$, we see that $\mathbb{P}(\tau_n < t) \to 0$, for all $(I_h(t_0), I_a(t_0), S_v(t_0), I_v(t_0)) \in \mathbf{D}_n$ and fixed $t \in [t_0, \infty)$. Furthermore, note that

$$\mathbb{P}(\tau(\widetilde{\mathbf{D}}) < t) = \lim_{n \to \infty} \mathbb{P}(\tau(\mathbf{D}_n) < t) = 0,$$

so, we conclude that,

$$\mathbb{P}(\tau(\widetilde{\mathbf{D}}) = \infty) = 1.$$

With this, we have proved that for all $t \geq 0$ and initial condition on $\widetilde{\mathbf{D}}$, the solution (I_h, I_a, S_v, I_v) remains on $\widetilde{\mathbf{D}}$ with probability one — in other words, the solution process $\{(I_h(t), I_a(t), S_v(t), I_v(t)), t \geq t_0 \geq 0\}$ is a.s. regular (or invariant) on $\widetilde{\mathbf{D}}$ in the sense of Schurz [8, Definition 2.1]. Moreover, for each set $\mathbf{E}_n := \{(t, x) : t \geq t_0, x \in \widetilde{\mathbf{D}}_n\}$, the coefficients $b_0(X(t))$, $b_1(X(t))$ satisfies all conditions of Theorem A.1, then exist a unique continuous solution $\left\{\left(I_h(t), I_a(t), S_v(t), I_{v(t)}\right)_n, t \geq t_0 \geq 0\right\}$, which coincides a.s. with (I_h, I_a, S_v, I_v) up to time τ_n, that is

$$\mathbb{P}\left\{\sup_{t_0 \leq t \leq \tau_n} \|(I_h(t), I_a(t), S_v(t), I_v(t)) - (I_h(t), I_a(t), S_v(t), I_v(t))_n\| > 0\right\} = 0.$$

Furthermore, $\widetilde{\mathbf{D}}_n$ satisfies conditions of Theorem A.2, then we concluded that $\left(I_h(t), I_a(t), S_v(t), I_{v(t)}\right)$ is the unique global continuous solution of SDE's (5) and this solution is invariant respect to $\widetilde{\mathbf{D}}$.

Note that sets with constraints as, $I_h = 0$, $I_a = 0$, $I_v = 0$ and $S_v = K$ are not in $\widetilde{\mathbf{D}}$. So, we discuss these in the following cases

(CASE-I) To fix ideas, we first examine the $I_h = 0$ constraint. In this case SDE (5) becomes

$$\begin{aligned} dI_a &= \left[\alpha_a I_v \left(A - I_a\right) - \mu_a I_a\right] dt + F_2 \theta_a I_v \left(A - I_a\right) dW_2(t) \\ dS_v &= \left[\Lambda_v T_v - \alpha_{v_a} I_a S_v - \left(\mu_v + r_k T_v\right) S_v\right] dt - F_2 \theta_{v_a} I_a S_v dW_2(t) \\ dI_v &= \left[\alpha_{v_a} I_a S_v - \left(\mu_v + r_k T_v\right) I_v\right] dt + F_2 \theta_{v_a} I_a S_v dW_2(t) \ . \end{aligned} \tag{7}$$

Let $\mathbf{D}_1 = \{(I_a, S_v, I_v) \in \mathbb{R}^3;\ 0 \leq t:\ 0 < I_a < A,\ 0 < S_v < K,\ 0 < I_v < K,\ S_v + I_v \leq K\}$ the set over which is defined the SDE (7), and

$$V_1\left(I_a, S_v, I_v\right) = I_a + (A - I_a) - \ln(A - I_a) + (K - S_v) - \ln(K - S_v) + S_v - \ln S_v + I_v + (K - I_v) - \ln(K - I_v).$$

Following the above argument, we can prove the invariance of $\mathbf{D}_1$, global existence, continuous and uniqueness of the solution. Furthermore, this reasoning applies to the cases with $I_a = 0$ and $I_v = 0$ constraints.

(CASE-II) Now, if $S_v = K$, then SDE (5) becomes

$$\begin{aligned} dI_h &= -\mu_h I_h dt \\ dI_a &= -\mu_a I_a dt, \end{aligned} \tag{8}$$

since $S_v = K$ implies $I_v = 0$. Also, note that Equation (8) is lineal, and consequently has a unique continuous global solution in $\mathbf{D}_5 = \{(I_h, I_a) \in \mathbb{R}^2;\ 0 \leq t:\ 0 < I_h < H,\ 0 < I_a < A\}$.

The same proof works for sub-domains with combinations of mentioned constraints. □

3. Global Stochastic Asymptotic Stability of the Disease-Free Fixed Point

In an epidemiology context, one of the goals is to study sceneries where a disease extinguishes or persists. So in the following sections we stay results in this direction. Let the threshold parameter

$$\mathcal{R} := \frac{\alpha_h \alpha_{v_h}}{\mu_h \mu_v} HK + \frac{\alpha_a \alpha_{v_a}}{\mu_a \mu_v} AK.$$

In the next theorem, we give suitable conditions related with $\mathcal{R}$ value to assure disease extinction. That is, under definition of stochastic asymptotic stability in the large (definition B.3) and conditions of the SDE (5) parameters we establish this kind of stability for the disease-free equilibrium point.

Theorem 3.1. *If $\mathcal{R} < 1$, then the disease-free equilibrium point, $x^* = (0, 0, K, 0)$, of SDE (5) is globally stochastically asymptotically stable in the large.*

Proof. The main idea of the proof is to propose a Lyapunov function (V) and verify the hypotheses of Theorem B.1. Let

$$\begin{gathered} V(I_h, I_a, S_v, I_v) := \frac{1}{2}\left(S_v + I_v - K\right)^2 + p_1 I_h + p_2 I_a + I_v, \\ p_1 = \frac{\alpha_{v_h}}{\mu_h} K, \quad p_2 = \frac{\alpha_{v_a}}{\mu_a} K. \end{gathered}$$

Note that function $V(I_h, I_a, S_v, I_v)$ is non-negative definite on $\mathbf{D}$. In order to satisfies Theorem B.1, we show that function $V(I_h, I_a, S_v, I_v)$ is radially unbounded. Define

$$q = \min\left\{\frac{1}{2}, p_1, p_2\right\},$$

and arbitrarily fix $M > 0$. Taking $N > \max\left\{\frac{2M + K^2}{q}, \sqrt{\frac{8M + 4K^2}{q}}\right\} > 0$ such that $N < \|(I_h, I_a, S_v, I_v)\|$ and non negative I_h, I_a, S_v, I_v, we have that

$$\frac{1}{2}\left(S_v + I_v - K\right)^2 + p_1 I_h + p_2 I_a + I_v > M.$$

Hence, letting $M \to \infty$, we deduce that $V(I_h, I_a, S_v, I_v)$ is radially unbounded in the sense of Definition B.4. Now, applying the infinitesimal generator $\mathcal{L}$ (defined on Appendix A) to function $V(I_h, I_a, S_v, I_v)$, we obtain

$$\begin{aligned}
\mathcal{L}V(I_h, I_a, S_v, I_v) = {} & p_1 \left[\alpha_h I_v (H - I_h) - \mu_h I_h\right] + p_2 \left[\alpha_a I_v (A - I_a) - \mu_a I_a\right] \\
& + (S_v + I_v - K) \left[\Lambda_v T_v - (\alpha_{v_h} I_h + \alpha_{v_a} I_a) S_v - (\mu_v + r_k T_v) S_v\right] \\
& + (S_v + I_v - K) \left[(\alpha_{v_h} I_h + \alpha_{v_a} I_a) S_v - (\mu_v + r_k T_v) I_v\right] \\
& + \left[(\alpha_{v_h} I_h + \alpha_{v_a} I_a) S_v - (\mu_v + r_k T_v) I_v\right] \\
& + \frac{1}{2} (1 - 1 - 1 + 1) S_v^2 \left(F_1^2 \theta_{v_h}^2 I_h^2 + F_2^2 \theta_{v_a}^2 I_a^2\right) \\
& \le (S_v + I_v - K) r T_v \left(1 - \frac{T_v}{K}\right) + (p_1 \alpha_h H + p_2 \alpha_a A - \mu_v) I_v \\
& - r_k T_v I_v + (K \alpha_{v_h} - p_1 \mu_h) I_h + (K \alpha_{v_a} - p_2 \mu_a) I_a \\
& - (p_1 \alpha_h + \alpha_{v_h}) I_v I_h - (p_2 \alpha_a + \alpha_{v_a}) I_v I_a \ .
\end{aligned}$$

Using p_1 and p_2 values, we observe that

$$\begin{aligned}
\alpha_{v_h} K - p_1 \mu_h &= 0 \\
\alpha_{v_a} K - p_2 \mu_a &= 0 \ .
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
\mathcal{L}V(I_h, I_a, S_v, I_v) \le {} & (S_v + I_v - K) r T_v \left(1 - \frac{T_v}{K}\right) + \mu_v (\mathcal{R} - 1) I_v - r_k T_v I_v \\
& - (p_1 \alpha_h + \alpha_{v_h}) I_v I_h - (p_2 \alpha_a + \alpha_{v_a}) I_v I_a \ .
\end{aligned}$$

Finally, since $\mathcal{R} < 1$, we concluded that $\mathcal{L}V(I_h, I_a, S_v, I_v)$ is negative-definite on $\mathbf{D} - \{x^*\}$. In this way, we have guaranteed the hypotheses of Theorem B.1. □

Remark. *In mathematical epidemiology context, the basic reproduction number ($\mathcal{R}_0$) yields when an epidemic occurs. Applying the method of the next generation matrix [9], we compute $\mathcal{R}_0$ for the deterministic system* (4)

Introductio
for R_0

$$\mathcal{R}_0 = \sqrt{\left(\frac{z_h \theta_h H}{\mu_v + r}\right)\left(\frac{z_h \theta_{v_h} K}{\mu_h}\right) + \left(\frac{z_a \theta_{v_a} K}{\mu_a}\right)\left(\frac{z_a \theta_a A}{\mu_v + r}\right)} \ . \tag{9}$$

We interpret this value as the average number of infected individuals indirectly generated by one infected vector during its whole infectious period. Qualitative analysis of the deterministic system (4)*, shows that if $\mathcal{R}_0$ is lower than one, then exist two equilibriums — the trivial and disease-free points — while if $\mathcal{R}_0$ is greater than one, then there is an endemic fixed point. Note that threshold parameter $\mathcal{R}_0$ always is lower than $\mathcal{R}$, therefore, if $\mathcal{R}$ is less than one, our stochastic system* (5) *preserves this threshold behavior.*

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Can we say something about its stability?

4. Stochastic Persistence

In the above section, we give conditions to assure disease extinction. Now, we study other important characteristic in epidemiology — the disease persistence. We know that if $\mathcal{R}_0 > 1$, then system (4) has an unique endemic equilibrium. Four stochastic extension (5), we give conditions to guarantee infected population persistence in order to establish disease presence. To this end, first we enunciate the following.

Definition 1 (Liu and Wang [10, pg. 936])**.** *Let us denote by $p(t)$ the population size at time t. If $\liminf_{t \to \infty} \frac{1}{t} \int_0^t p(s) ds > 0$ a.s., then $p(t)$ is said to be stochastic strongly persistent in the mean.*

Lemma 4.1 (Ji and Jiang [11, Lemma 5.1]). *Let $p \in \mathcal{C}([0,+\infty)\times\Omega,(0,+\infty))$. If there exist positive constants λ_0, λ, such that*

$$\ln(p(t)) \geq \lambda t - \lambda_0 \int_0^t p(s)ds + F(t), \ \ a.s.$$

for all $t \geq 0$, where $F \in \mathcal{C}([0,+\infty)\times\Omega,\mathbb{R})$ and $\lim_{t\to\infty} \frac{F(t)}{t} = 0$ a.s., then

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t p(s)ds \geq \frac{\lambda}{\lambda_0} \ \ a.s.$$

To simplify notation, we write

$$c_1 = \min\left\{\alpha_h H + \alpha_a A, \frac{\alpha_{v_h} K}{2}, \frac{\alpha_{v_a} K}{2}\right\}, \qquad c_2 = \max\{\mu_h, \mu_a, \Lambda_v\}, \qquad c_3 = \max\{c_{31}, c_{32}, c_{33}, c_{34}, c_{35}\},$$

where,

$$c_{31} = \sup_{(I_h,I_a,S_v,I_v)\in\mathbf{D}} \left\{F_1^2\theta_h^2 H^2 + F_2^2\theta_a^2 A^2\right\}, \qquad c_{32} = \sup_{(I_h,I_a,S_v,I_v)\in\mathbf{D}} \left\{F_1^2\theta_{v_h}^2 K^2\right\},$$

$$c_{33} = \sup_{(I_h,I_a,S_v,I_v)\in\mathbf{D}} \left\{F_2^2\theta_{v_a}^2 K^{s2}\right\}, \qquad c_{34} = \sup_{(I_h,I_a,S_v,I_v)\in\mathbf{D}} \left\{F_1^2\theta_h H\theta_{v_h} K\right\},$$

$$c_{35} = \sup_{(I_h,I_a,S_v,I_v)\in\mathbf{D}} \left\{F_2^2\theta_a A\theta_{v_a} K\right\}.$$

Theorem 4.2. *Let $(I_h(t_0),, I_a(t_0), S_v(t_0), I_v(t_0)) \in \mathbf{D}$. If $2c_1 - 2c_2 - c_3 > 0$, then the infected population of SDE (5) $I_h(t) + I_a(t) + I_v(t)$ is stochastic strong persistent in the mean.*

Proof. The main idea of the proof is to apply Lemma 4.1. To this end, we use our constants c_1, c_2, c_3 in order to argue the required hypothesis. By Itô's formula

$$\begin{aligned} d(\ln(I_h + I_a + I_v)) = &\left[\left(\frac{\alpha_h I_v(H - I_h) - \mu_h}{I_h + I_a + I_v}\right) + \left(\frac{\alpha_a I_v(A - I_a) - \mu_a I_a}{I_h + I_a + I_v}\right)\right. \\ &+ \left.\left(\frac{(\alpha_{v_h} I_h + \alpha_{v_a} I_a)S_v}{I_h + I_a + I_v}\right) - \left(\frac{(\mu_v + r_k T_v)I_v}{I_h + I_a + I_v}\right)\right] dt \\ &- \frac{1}{2}\left[\left(\frac{F_1(\theta_h I_v(H - I_h) + \theta_{v_h} I_h S_v)}{I_h + I_a + I_v}\right)^2 + \left(\frac{F_2(\theta_a I_v(A - I_a) + \theta_{v_a} I_a S_v)}{I_h + I_a + I_v}\right)^2\right] dt \\ &+ \left(\frac{F_1(\theta_h I_v(H - I_h) + \theta_{v_h} I_h S_v)}{I_h + I_a + I_v}\right) dW_1 + \left(\frac{F_2(\theta_a I_v(A - I_a) + \theta_{v_a} I_a S_v)}{I_h + I_a + I_v}\right) dW_2 \ . \end{aligned}$$

Next, using $-(\mu_v + r_k T_v) \geq -\Lambda_v$, we get

$$\begin{aligned} d(\ln(I_h + I_a + I_v)) \geq &\left[\left(\frac{\alpha_h I_v(H - I_h) - \mu_h}{I_h + I_a + I_v}\right) + \left(\frac{\alpha_a I_v(A - I_a) - \mu_a I_a}{I_h + I_a + I_v}\right)\right. \\ &- \left.\left(\frac{\Lambda_v I_v}{I_h + I_a + I_v}\right) + \left(\frac{(\alpha_{v_h} I_h + \alpha_{v_a} I_a)(T_v - I_v)}{I_h + I_a + I_v}\right)\right] dt \\ &- \frac{1}{2}\left[\left(\frac{F_1(\theta_h I_v(H - I_h) + \theta_{v_h} I_h S_v)}{I_h + I_a + I_v}\right)^2 + \left(\frac{F_2(\theta_a I_v(A - I_a) + \theta_{v_a} I_a S_v)}{I_h + I_a + I_v}\right)^2\right] dt \\ &+ \left(\frac{F_1(\theta_h I_v(H - I_h) + \theta_{v_h} I_h S_v)}{I_h + I_a + I_v}\right) dW_1 + \left(\frac{F_2(\theta_a I_v(A - I_a) + \theta_{v_a} I_a S_v)}{I_h + I_a + I_v}\right) dW_2 \ . \end{aligned}$$

Applying definition of c_2 and c_3, we have

$$\begin{aligned} d(\ln(I_h+I_a+I_v)) \geq & \left(\frac{(\alpha_h H+\alpha_a A)\, I_v+(\alpha_{v_h} I_h+\alpha_{v_a} I_a) T_v}{I_h+I_a+I_v}\right) dt-\left(\frac{(\alpha_h+\alpha_{v_h})\, I_v I_h}{I_h+I_a+I_v}\right) dt \\ & -\left(\frac{(\alpha_a+\alpha_{v_a})\, I_v I_a}{I_h+I_a+I_v}\right) dt-c_2 dt-\frac{c_3}{2}\left(\frac{I_v^2+I_h^2+I_a^2+2 I_v I_h+2 I_v I_a}{(I_h+I_a+I_v)^2}\right) dt \\ & +\left(\frac{F_1(\theta_h I_v(H-I_h)+\theta_{v_h} I_h S_v)}{I_h+I_a+I_v}\right) dW_1+\left(\frac{F_2(\theta_a I_v(A-I_a)+\theta_{v_a} I_a S_v)}{I_h+I_a+I_v}\right) dW_2 \ . \end{aligned}$$

We now integrate on both sides to deduce that,

$$\begin{aligned} \ln(I_h+I_a+I_v) \geq & \ln(I_h(0)+I_a(0)+I_v(0))-\left(c_2+\frac{c_3}{2}\right) t+\int_0^t\left(\frac{(\alpha_h H+\alpha_a A)\, I_v}{I_h+I_a+I_v}\right) ds \\ & +\int_0^t\left(\frac{(\alpha_{v_h} I_h+\alpha_{v_a} I_a) T_v}{I_h+I_a+I_v}\right) ds-\left(\frac{\alpha_h+\alpha_{v_h}+\alpha_a+\alpha_{v_a}}{4}\right) \int_0^t (I_h+I_a+I_v) ds \\ & +\int_0^t\left(\frac{F_1(\theta_h I_v(H-I_h)+\theta_{v_h} I_h S_v)}{I_h+I_a+I_v}\right) dW_1(s)+\int_0^t\left(\frac{F_2(\theta_a I_v(A-I_a)+\theta_{v_a} I_a S_v)}{I_h+I_a+I_v}\right) dW_2(s) \ . \end{aligned}$$

Since function $T_v(\cdot)$ is increasing for all initial condition $T_v(0) \in (0, K]$, there exists $t_K \geq 0$, such that

$$T_v(s) \geq \frac{K}{2}, \quad \text{for all } s \geq t_K \ .$$

Consequently, if $t \geq t_K$, then

$$\begin{aligned} \ln(I_h+I_a+I_v) \geq & \ln(I_h(0)+I_a(0)+I_v(0))-\left(c_2+\frac{c_3}{2}\right) t+\int_{t_K}^t\left(\frac{(\alpha_h H+\alpha_a A)\, I_v}{I_h+I_a+I_v}\right) ds \\ & +\int_{t_K}^t\left(\frac{(\alpha_{v_h} I_h+\alpha_{v_a} I_a) \frac{K}{2}}{I_h+I_a+I_v}\right) ds-\left(\frac{\alpha_h+\alpha_{v_h}+\alpha_a+\alpha_{v_a}}{4}\right) \int_0^t (I_h+I_a+I_v) ds \\ & +\int_0^t\left(\frac{F_1(\theta_h I_v(H-I_h)+\theta_{v_h} I_h S_v)}{I_h+I_a+I_v}\right) dW_1(s)+\int_0^t\left(\frac{F_2(\theta_a I_v(A-I_a)+\theta_{v_a} I_a S_v)}{I_h+I_a+I_v}\right) dW_2(s) \ . \end{aligned} \tag{10}$$

For abbreviation, we write

$$\begin{aligned} M_0^{t_K} &:= \ln(I_h(0)+I_a(0)+I_v(0))-c_1 t_K, \\ M_1(t) &:= \int_0^t\left(\frac{F_1(\theta_h I_v(H-I_h)+\theta_{v_h} I_h S_v)}{I_h+I_a+I_v}\right) dW_1(s)+\int_0^t\left(\frac{F_2(\theta_a I_v(A-I_a)+\theta_{v_a} I_a S_v)}{I_h+I_a+I_v}\right) dW_2(s) \ . \end{aligned}$$

Since t_K is fixed and finite, we see that

$$\lim_{t\to\infty} \frac{M_0^{t_K}}{t} = 0 \ .$$

Furthermore, the strong law of large numbers for martingales [see e.g., 12, Thm. 3.4] implies

$$\lim_{t\to\infty} \frac{M_1(t)}{t} = 0 \quad \text{a.s.}$$

Note that by hypothesis $4c_1 - 4c_2 - 2c_3 > 0$. Therefore, applying Lemma 4.1, with

$$p(t) = I_h(t)+I_a(t)+I_v(t), \quad F(t) = M_0^{t_K}+M_1(t), \quad \lambda_0 = \alpha_h+\alpha_a+\alpha_{v_h}+\alpha_{v_a}, \quad \lambda = 4c_1-4c_2-2c_3,$$

115 we show stochastic persistence in the mean for our infected populations. □

5. Numerical Results

In this section, we illustrate our results, by numerical experiments. Taking as example Chagas disease, we confirm its extinction and persistence under real literature parameters. Also we expose how noise perturbation affect our deterministic base dynamics under specific functional responses.

Chagas disease resides in multiple hosts —more than 100 species of mammals [1], including humans and more than one vector species [13]. A human can get the disease via bite of infected bug, blood transfusion, organ transplantation, ingestion of contaminated food, vertical transmission and accidental laboratory exposure [14]. So, for our experiments, we consider humans and one animal specie as hosts. For vectors, we use entomological information of *Triatoma infestans* and since the main transmission route of Chagas disease is via bite of infected bug [15], we only consider this. Also, we put environmental noise in this kind of transmission, to be precise, we perturb vector bitting parameters z_h, z_a as in Section 2.2 with noise intensities

$$F_1(I_h, I_a, S_v, I_v) = \frac{\sigma_{z_h} H}{K} I_v, \qquad\qquad F_2(I_h, I_a, S_v, I_v) = \frac{\sigma_{z_a} A}{K} I_v \ . \tag{11}$$

Hence, our SDE model (5) becomes

$$\begin{aligned}
dI_h &= \left[\alpha_h I_v \left(H - I_h\right) - \mu_h I_h\right] dt + \sigma_{z_h} \left(\frac{\theta_h H}{K}\right) I_v^2 \left(H - I_h\right) dW_1(t) \\
dI_a &= \left[\alpha_a I_v \left(A - I_a\right) - \mu_a I_a\right] dt + \sigma_{z_a} \left(\frac{\theta_a A}{K}\right) I_v^2 \left(A - I_a\right) dW_2(t) \\
dS_v &= \left[\Lambda_v \left(S_v + I_v\right) - \left(\alpha_{v_h} I_h + \alpha_{v_a} I_a\right) S_v - \left(\mu_v + r_K \left(S_v + I_v\right)\right) S_v\right] dt \\
&\quad - \sigma_{z_h} \left(\frac{\theta_{v_h} H}{K}\right) I_h I_v S_v dW_1(t) - \sigma_{z_a} \left(\frac{\theta_{v_a} A}{K}\right) I_a I_v S_v dW_2(t) \\
dI_v &= \left[\left(\alpha_{v_h} I_h + \alpha_{v_a} I_a\right) S_v - \left(\mu_v + r_K \left(S_v + I_v\right)\right) I_v\right] dt \\
&\quad + \sigma_{z_h} \left(\frac{\theta_{v_h} H}{K}\right) I_h I_v S_v dW_1(t) + \sigma_{z_a} \left(\frac{\theta_{v_a} A}{K}\right) I_a I_v S_v dW_2(t) \ .
\end{aligned} \tag{12}$$

As example, we consider a rural community — we think in a farmer community with less than 10000 habitants as human host population. Likewise, we assume enough initial infected density to sustain our different scenarios. So, it is reasonable to consider $(I_h(0), I_a(0), S_v(0), I_v(0)) = (10, 50, 1170, 30)$ as our initial populations. In this way, we illustrate extinction and persistence behavior of SDE (12), in the context of Theorems 3.1 and 4.2. Also, we contrast these stochastic scenes with our deterministic base ($\sigma_{z_h} = 0$, $\sigma_{z_a} = 0$) dynamics.

Since the coefficients of our model (12) not satisfies globally, the Lipschitz and linear growth conditions, we implement the Linear-Steklov (LS) method reported in [16] — see Appendix C for details.

5.1. Experiment 1: Extinction — Disease-Free Equilibrium Numeric Analysis

Under context of Theorem 3.1, we use some parameter values from literature (see Table 3) and tune the rest parameters in order to satisfies our conditions of global stochastic asymptotic stability. Thus, we consider a whole human density of $H = 1500$ habitants which sustain $A = 2500$ domestic animals. The maximum allowable number of vectors is assumed 20 times host population H. Likewise, we assume a human and animals lifetime of 70 and 6 years, respectively, which yields the mortalities rates $\mu_h = 0.014\,285\,7$ year^{-1} and $\mu_a = 0.166\,666\,6$ year^{-1}. The vector mortality is $\mu_v = 281.1$ year^{-1}.

In Figure 1 we contras the effect of noise between infected population of humans and vectors. Remember that we perturb bitting parameters, so is feasible to obtain a most notable effect over this population. However, as Theorem 3.1 states, we recover the extinction of disease. We confirm these observation with the mean and variance estimation. As we see in Figure 2, the expected value of the stochastic solution practically follows our deterministic dynamics, for this reason we get low numerical variance at long time — see Figure 3. We generate 10 000 trajectories to estimate the mean and variance of the infected populations. In addition, we register the first time when each one of this realizations achieves a level less than one individual for all infected populations. Figure 4 shows how the 10 000 concentrates between 220 and 230 years — this suggest that Chagas extinguishes (with high probability) after 250 years.

5.2. Experiment 2: Persistence — Illness Subservience

Now we illustrate that Chagas survive under hypothesis of Theorem 4.2. For guarantee these conditions, we use the paramters values of Table 4. Here, we chose the step size and initial conditions from Table 2. In Figure 5, we show a sample path solution of SDE (12), as we see, our stochastic extension similarly follows the deterministic dynamics of (4). To confirm this claim, we generate 10 000 trajectories to estimate statistic of the solution process to SDE (12). Figure 6 shows the mean value. The histograms of Figure 7 suggest that almost all individuals became infected. Finally in Figure 8 we show how conforming decrease the variance solutions respect to the noise intensities $\eta_i := (\sigma^i_{z_h}, \sigma^i_{z_a})$.

Tabla4 π_{v_h} unico de referencia ref[17]

μ_v poner re 17

6. Conclusions and final comments

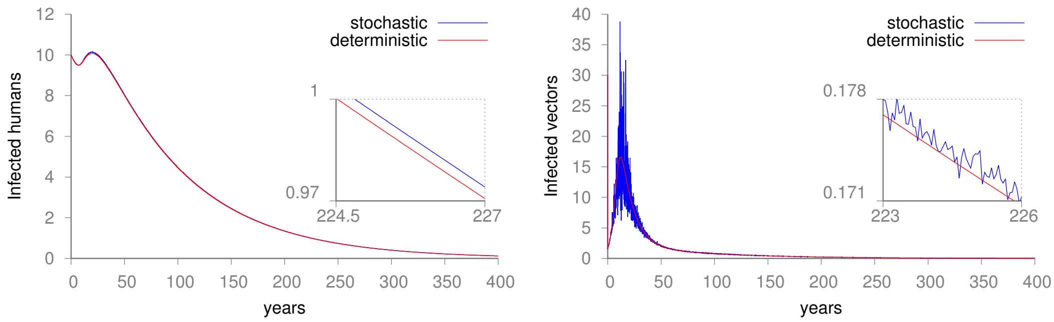
Future works. As we know, in the disease modeling, there are many manner to consider the infection forces. Hence, we believe that generalize this characteristic will help to complete the present study, which we leave as future work.

Set up parameters	
Initial Conditions	Solver Parameters
$I_h(0) = 10$	$h = 10^{-4}$
$I_a(0) = 50$	$h_{res} = 10^{-6}$
$S_v(0) = 1170$	
$I_v(0) = 30$	

Table 2: Set up parameters.

Table 3: Parameters for Experiment 1 (Survival numerics)

Parameters for numerical experiment 1: disease free equilibrium			Reference
$H = 1500$ humans	$A = 2500$ animals	$K = 80\,000$ vectors	—
$\mu_h = 0.0142857$ year^{-1}	$\mu_a = 0.1666666$ year^{-1}	$\mu_v = 281.1$ year^{-1}	—
$z_h = 14.6$ bite year^{-1} vector^{-1}	$z_a = 40.15$ bite year^{-1} vector^{-1}	$\pi_a = 0.0009$ animal bite^{-1}	[4]
$\pi_h = 0.0009$ human bite^{-1}	$\pi_{v_h} = 0.03$ vector bite^{-1}	$\pi_{v_a} = 0.49$ vector bite^{-1}	[17]
		$\Lambda_v = 281.561$ year^{-1}	[18]
		$T = 400$ years	—
		$\sigma_{z_a} = 2.2$ bite vector^{-1} human^{-1} year^{-1}	vector human and vector animal biting intensity noise
		$\sigma_{z_h} = 2.3$ bite vector^{-1} animal^{-1} year^{-1}	



(a) Extinction of the stochastic infected human population.

(b) Stochstic infected vector population.

Figure 1: A realization of the stochastic process solution under hypothesis of Theorem 3.1 for $t \in [0, 400]$ years, see Tables 2 and 3 for initial conditions and parameters values.

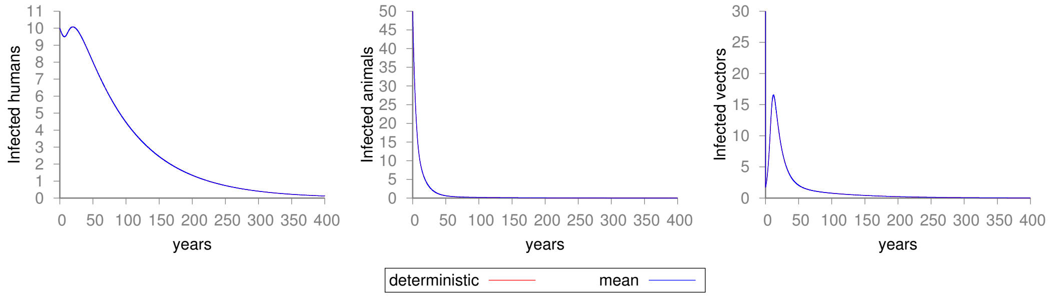


Figure 2: Numerical mean of 10 000 solution trajectories under conditions of Theorem 3.1.

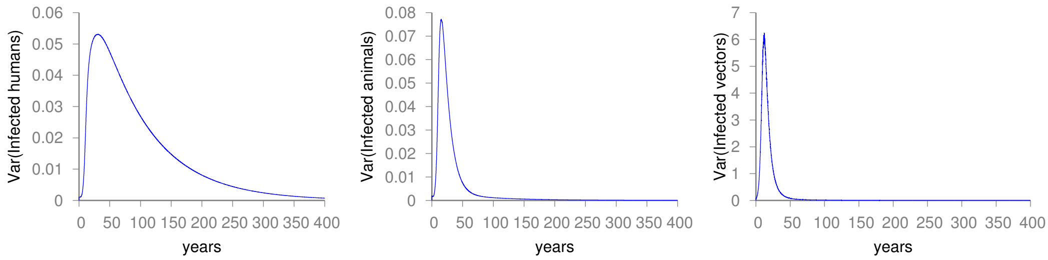


Figure 3: Variance of 10 000 solution trajectories under conditions of Theorem 3.1.

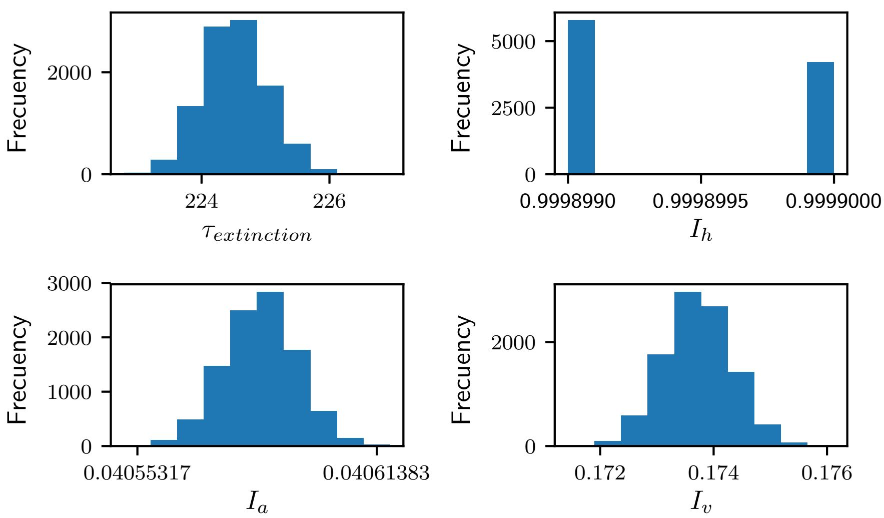
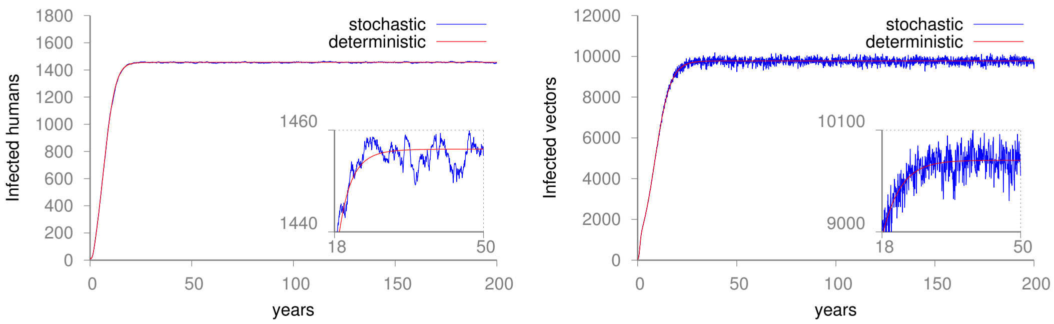


Figure 4: Histograms of 10 000 sample paths under sufficient conditions for disease extinction see Theorem 3.1 and Table 3. As we see at upper left, the first time when all population are less than one $\tau_{extinction}$ distributes its 10 000 occurrences around 220 and 230 years.

Table 4: Parameters for Experiment2 (Persistence numerics)

Parameters for numerical experiment 2: Persistence of the disease			Reference
$H = 1500$ humans	$A = 3000$ animals	$K = 12\,000$ vectors	—
$\mu_h = 0.014\,285\,7$ year^{-1}	$\mu_a = 0.1666666$ year^{-1}	$\pi_a = 0.02$ animal bite^{-1}	—
$\pi_h = 0.002$ human bite^{-1}	$\pi_{v_h} = 0.03$ vector bite^{-1}	$\Lambda_v = 2.7$ year^{-1}	—
$T = 200$ years	$z_a = 182.5$ bite year^{-1} vector^{-1}	$z_h = 36.5$ bite year^{-1} vector^{-1}	—
		$\pi_{v_a} = 0.06$ vector bite^{-1}	[17]
		$\mu_v = 2.4455$ year^{-1}	[18]
Figures 5 and 6	Figure 8		
$\sigma_{z_h} = 0.003\,95$ $\sigma_{z_a} = 0.001\,95$	$\sigma^1_{z_h} = 0.003\,95$, $\sigma^2_{z_h} = 0.002$ $\sigma^1_{z_a} = 0.001\,95$, $\sigma^2_{z_a} = 0.001$	bite vector^{-1} human^{-1} year^{-1} bite vector^{-1} animal^{-1} year^{-1}	vector human and vector animal biting amplitude noise



(a) Stochstic sample path of infected humans under conditions of Theorem 4.2. We made a zoom to enphasize the stochastic perturbation.

(b) Sotchastic bheavior of a infected animal path.

Figure 5: A sample path of the stochastic solution process of SDE (12) under conditions of Theorem 4.2. Parameters values in Table 4.

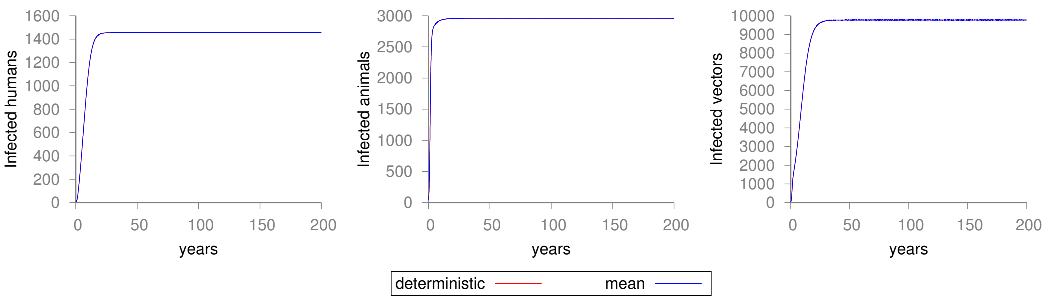


Figure 6: Likening between deterministic solution and the mean of 10 000 realizations of the stochastic solution process. As we see, our mean value follows the deterministic profile.

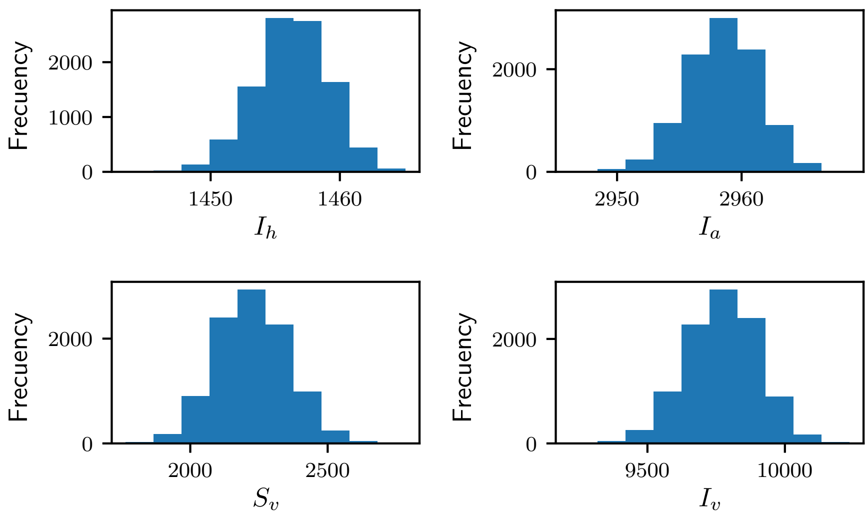


Figure 7: Histograms of 10 000 sample solution paths of SDE (12) under hypothesis of Theorem 4.2 and at fixed time t = 200 years.

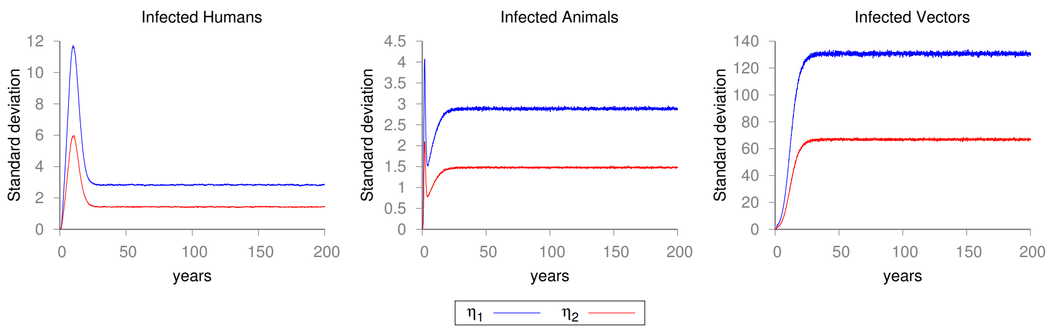


Figure 8: The noise effect amplitude over variance. of 10 000 sample paths. Here we decrease the initial noise intensities $\eta_1 := (\sigma^1_{z_h}, \sigma^1_{z_a}) = (0.003\,95, 0.001\,95)$ until $\eta_2 := (\sigma^2_{z_h}, \sigma^2_{z_a}) = (0.002, 0.001)$. As we see, this figure suggest that noise intensity can modulates the solution variance of SDE (12).

Appendix A. Existence and Regularity of Solutions

Consider the following SDE,

$$\begin{aligned} dX(t) &= f(t,X)dt + g(t,X)dW(t), \\ X(t_0) &= x_0, \quad t \in [t_0,T]. \end{aligned} \tag{A.1}$$

with a non random initial condition $X(t_0) = x_0$, finite T, and real coefficients $f : [t_0,T] \times \mathbb{R}^d \to \mathbb{R}^d$, $g : [t_0,T] \times \mathbb{R}^d \to \mathbb{R}^{d\times m}$, driven by m independent standard one-dimensional Wiener processes W_j, defined on a complete and filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, \mathbb{P})$. We say that Equation (A.1) has unique solution up to equivalence if for any two solutions $X_1(t)$ and $X_2(t)$ $\mathbb{P}\{X_1(t) = X_2(t)$ for all $t \in [t_0,T]\} = 1$.

Let $\mathcal{L}$ denotes the infinitesimal generator associated to the stochastic SDE (A.1), and defined by

$$\mathcal{L} = \frac{\partial}{\partial t} + \sum_{i=1}^{d} f_i(t,X(t))\frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{m}\left(g(t,X(t))g^T(t,X(t))\right)_{ij}\frac{\partial^2}{\partial x_i \partial x_j}. \tag{A.2}$$

Under the above, we use a part of the result for existence and unique of solutions enunciated by Khasminskii in [6].

Theorem A.1 (Khasminskii [6, Theorem 3.4])**.** *Let the vectors*

$$f(t,x), \quad g_1(t,x),\dots,g_m(t,x), \quad t\in[t_0,T], \qquad x\in\mathbb{R}^d,$$

be continuous functions of (t,x), such that for some constant B the following conditions hold in the entire domain of definition:

$$\begin{aligned} \|f(t,x)-f(t,y)\| + \sum_{r=1}^{m}\|g_r(t,x)-g_r(t,y)\| &\le B\|x-y\|, \\ \|f(t,x)\| + \sum_{r=1}^{m}\|g_r(t,x)\| &\le B(1+\|x\|) \ . \end{aligned} \tag{A.3}$$

Then, for every random variable $X(t_0)$ independent of the process $W_r(t) - W_r(t_0)$ there exists a solution $X(t)$ of (A.1) *which is almost surely continuous stochastic process and is unique up to equivalence.*

Also, we applied the following result in order to assure regularity of solutions to our SDE model.

Theorem A.2 (Khasminskii [6, Corollary 3.1])**.** *Let D and D_n, for each positive integer n, be open sets in $\mathbb{R}^d$, such that*

$$\mathsf{D}_n \subseteq \mathsf{D}_{n+1}, \quad \overline{\mathsf{D}}_n \subseteq \mathsf{D}, \quad \textit{and} \quad \mathsf{D} = \bigcup_n \mathsf{D}_n,$$

Suppose that in every cylinder $[t_0,\infty)\times\mathsf{D}_n$, the functions $f(t,x)$ and $g_r(t,x)$ satisfy conditions (A.3) *and there exists a function $V(t,x)$, twice continuously differentiable in x and continuously differentiable in t in the domain $[t_0,\infty)\times\mathsf{D}$, such that for some positive constant c it holds that $\mathcal{L}V \le cV$ and,*

$$\inf_{t\ge t_0, x\in\mathsf{D}\setminus\mathsf{D}_n} V(t,x) \to \infty, \qquad \textit{as} \quad n\to\infty,$$

then, the conclusion of Theorem A.1 holds provided that also $\mathbb{P}(x(t_0)\in\mathsf{D}) = 1$. Moreover, the solution satisfies that

$$\mathbb{P}(x(t)\in\mathsf{D}) = 1, \ \textit{for all } t\ge t_0 \ .$$

Appendix B. Stability

Definition B.2. *We say that x^* is an equilibrium position of the SDE*(A.1) *if $f(t,x^*) = 0$ and $g(t,x^*) = 0$ for all $t \geq t_0$.*

Definition B.3 (Khasminskii [6, Section 5])**.** *Let x^* as above definition and X^{t_0,x_0} the solution of* (A.1) *whit initial condition $X(t_0) = x_0$.*

(i) *The equilibrium solution x^* of equation* (A.1) *is said to be stable in probability for $t \geq 0$ if for any $t_0 \geq 0$ and $\epsilon > 0$*

$$\lim_{x_0 \to x^*} \mathbb{P}\left\{\sup_{t>t_0} \|X^{t_0,x_0} - x^*\| > \epsilon\right\} = 0.$$

(ii) *The equilibrium solution x^* is said to be asymptotically stable in probability if it is stable in probability and*

$$\lim_{x_0 \to x^*} \mathbb{P}\left\{\lim_{t\to\infty} X^{t_0,x_0}(t) = x^*\right\} = 1.$$

(iii) *The equilibrium solution x^* is said to be (asymptotically) stable in the large if it is stable in probability and also for all t_0, x_0*

$$\mathbb{P}\left\{\lim_{t\to\infty} X^{t_0,x_0}(t) = x^*\right\} = 1.$$

Definition B.4 (Mao [12, p. 108])**.** *A function $V(x,t)$ defined on $\mathbb{R}^d \times [t_0, \infty)$ is said to be radially unbounded if*

$$\lim_{\|x\|\to\infty} \inf_{t \geq t_0} V(x,t) = \infty.$$

Theorem B.1 (Mao [12, Theorem 2.4])**.** *Suppose that the assumptions of Theorem A.1 holds for $t \geq t_0$, and let $S_h = \{X \in \mathbb{R}^d : \|X - x^*\| < h\}$. If there exists a positive-definite decrescent radially unbounded function $V(x,t) \in C^{2,1}(S_h \times [t_0,\infty);\mathbb{R}_+)$, such that $\mathcal{L}V(x,t)$ is negative-definite, then the equilibrium solution x^* of equation* (A.1) *is stochastically asymptotically stable in the large.*

Lemma B.2 (Strong Law of Large Numbers for Martingales [12, Theorem 3.4])**.** *Let $= \{M_t\}_{t\geq 0}$ be a real–valued continuous local martingale vanishing at $t = 0$. If M has finite quadratic variation, then*

$$\lim_{t\to\infty} \frac{M_t}{t} = 0.$$

Appendix C. Linear-Steklov Method

According to SDE (5), let

$$y(t) := (I_h(t), I_a(t), S_v(t), I_v(t))^T, \qquad F_1 = F_1\left(I_h(t), I_a(t), S_v(t), I_v(t)\right),$$

$$F_2 = F_2\left(I_h(t), I_a(t), S_v(t), I_v(t)\right)$$

$$f(y) := \begin{bmatrix} \alpha_h I_v (H - I_h) - \mu_h I_h \\ \alpha_a I_v (A - I_a) - \mu_a I_a \\ \Lambda_v T_v - (\alpha_{v_h} I_h + \alpha_{v_a} I_a) S_v - (\mu_v + r_K T_v) S_v \\ (\alpha_{v_h} I_h + \alpha_{v_a} I_a) S_v - (\mu_v + r_K T_v) I_v \end{bmatrix}, \quad g(y) := \begin{bmatrix} F_1 \theta_h I_v (H - I_h) & 0 \\ 0 & F_2 \theta_a I_v (A - I_a) \\ -F_1 \theta_{v_h} I_h S_v & -F_2 \theta_{v_a} I_a S_v \\ F_1 \theta_{v_h} I_h S_v & F_2 \theta_{v_a} I_a S_v \end{bmatrix}.$$

Following the notation of [16] and after some manipulations, we rewrite each drift component function $f^{(j)} : \mathbb{R}^4 :\to \mathbb{R}$, $j \in \{1, \dots, 4\}$ of SDE (12) as

$$f^{(j)}(y) = a_j(y) y^{(j)} + b_j(y^{(-j)}), \qquad y^{(-j)} = (y^{(1)}, \dots, y^{(j-1)}, y^{(j+1)}, \dots, y^{(d)})^T,$$

where

$$\begin{aligned}
a_1(y) &:= -\left(\alpha_h y^{(4)} + \mu_h\right), & b_1(y) &:= \alpha_h y^{(4)} H,\\
a_2(y) &:= -\left(\alpha_a y^{(4)} + \mu_a\right), & b_2(y) &:= \alpha_a y^{(4)} A,\\
a_3(y) &:= -\left(\alpha_{v_h} y^{(1)} + \alpha_{v_a} y^{(2)} + r_K\left(y^{(3)} + y^{(4)}\right) + \mu_v - \Lambda_v\right), & b_3(y) &:= \Lambda_v y^{(4)},\\
a_4(y) &:= -\left(\mu_v + r_k\left(y^{(3)} + y^{(4)}\right)\right), & b_4(y) &:= \left(\alpha_{v_h} y^{(1)} + \alpha_{v_a} y^{(2)}\right) y^{(3)} \ .
\end{aligned}$$

Let h the step size. Consider the matrices $A^{(1)} = A^{(1)}(h, y)$, $A^{(2)} = A^{(2)}(h, y)$ defined by

$$A^{(1)} := \begin{pmatrix} e^{ha_1(y)} & & 0 \\ & \ddots & \\ 0 & & e^{ha_4(y)} \end{pmatrix},$$

$$A^{(2)} := \begin{pmatrix} \left(\dfrac{e^{ha_1(y)} - 1}{a_1(y)}\right) \mathbb{1}_{\{E_1^c\}} & & 0 \\ & \ddots & \\ 0 & & \left(\dfrac{e^{ha_4(y)} - 1}{a_4(y)}\right) \mathbb{1}_{\{E_4^c\}} \end{pmatrix} + h \begin{pmatrix} \mathbb{1}_{\{E_1\}} & & 0 \\ & \ddots & \\ 0 & & \mathbb{1}_{\{E_4\}} \end{pmatrix},$$

$$E_j := \{y \in \mathbb{R}^4 : a_j(y) = 0\}, \qquad b(y) := \left(b_1(y^{(-1)}), \ldots, b_4(y^{(-4)})\right)^T.$$

Thus, given $Y_0 = y(0)$, and $h = T/N$, the LS method for SDE (12) reads

$$Y_{k+1} = A^{(1)}(h, Y_k) Y_k + A^{(2)}(h, Y_k) b(Y_k) + g(Y_k) \Delta W_k, \qquad k = 0, \ldots, N-1. \tag{C.1}$$

Since Theorem 2.1 ensures positivity of the SDE (5) solution, in order to preserves this property, we modify the scheme (C.1) as

$$\begin{aligned}
Y_k^{\star} &= A^{(1)}(h, Y_k) Y_k + A^{(2)}(h, Y_k) b(Y_k) + g(Y_k) \Delta W_k \\
Y_{k+1} &= \left(\mathbb{1}_{\{Y_k^{\star} \geq 0\}} - \mathbb{1}_{\{Y_k^{\star} < 0\}}\right) Y_k^{\star} \ .
\end{aligned} \tag{C.2}$$

review if our diffusions satisfies dissipative conditions

References

1. Crawford, B., Kribs-Zaleta, C.. A metapopulation model for sylvatic T. cruzi transmission with vector migration. *Math Biosci Eng* 2014;11(3):471–509. doi:10.3934/mbe.2014.11.471.
2. Mena-Lorca, J., Velasco-Hernández, J.X., Marquet, P.A.. Coexistence in metacommunities: A tree-species model. *Math Biosci* 2006;202(1):42–56. doi:10.1016/j.mbs.2006.04.005.
3. Alvar, J., Vlez, I.D., Bern, C., Herrero, M., Desjeux, P., Cano, J., Jannin, J., Boer, M.d., the WHO Leishmaniasis Control Team, . Leishmaniasis worldwide and global estimates of its incidence. *PLOS ONE* 2012;7(5):1–12. URL: https://doi.org/10.1371/journal.pone.0035671. doi:10.1371/journal.pone.0035671.
4. Cruz-Pacheco, G., Esteva, L., Vargas, C.. Control measures for chagas disease. *Mathematical Biosciences* 2012;237(12):49 – 60. URL: http://www.sciencedirect.com/science/article/pii/S0025556412000363. doi:https://doi.org/10.1016/j.mbs.2012.03.005.
5. Singh, N., Mishra, J., Singh, R., Singh, S.. Animal reservoirs of visceral leishmaniasis in india. *The Journal of parasitology* 2013;99(1):64–67.
6. Khasminskii, R.. Stochastic Stability of Differential Equations; vol. 66 of *Stochastic Modelling and Applied Probability*. Berlin, Heidelberg: Springer Berlin Heidelberg; 2012. ISBN 978-3-642-23279-4. URL: http://link.springer.com/10.1007/978-3-642-23280-0. doi:10.1007/978-3-642-23280-0.
7. Schurz, H., Tosun, K.. Stochastic Asymptotic Stability of SIR Model with Variable Diffusion Rates. *J Dyn Differ Equations* 2015;27(1):69–82. doi:10.1007/s10884-014-9415-9.
8. Schurz, H.. Modeling, analysis and discretization of stochastic logistic equations. *International Journal of Numerical Analysis and Modeling* 2007;4(2):178–197.
9. Van Den Driessche, P., Watmough, J.. Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmission. *Math Biosci* 2002;180:29–48. doi:10.1016/S0025-5564(02)00108-6.
10. Liu, M., Wang, K.. Persistence and extinction of a stochastic single-specie model under regime switching in a polluted environment. *J Theor Biol* 2010;264(3):934–944. doi:10.1016/j.jtbi.2010.03.008.

11. Ji, C., Jiang, D.. Threshold behaviour of a stochastic SIR model. *Appl Math Model* 2014;38(21):5067–5079. doi:10.1016/j.apm.2014.03.037.
12. Mao, X.. Stochastic differential equations and applications. 2007. ISBN 9781904275343.
13. Steverding, D.. The history of Chagas disease. *Parasit Vectors* 2014;7:317. doi:10.1186/1756-3305-7-317.
14. Pereira Nunes, M.C., Dones, W., Morillo, C.A., Encina, J.J., Ribeiro, A.L.. Chagas disease: An overview of clinical and epidemiological aspects. *J Am Coll Cardiol* 2013;62(9):767–776. doi:10.1016/j.jacc.2013.05.046.
15. Nouvellet, P., Cucunubá, Z.M., Gourbiere, S.. Ecology, Evolution and Control of Chagas Disease: A Century of Neglected Modelling and a Promising Future. *Adv Parasitol* 2015;87:135–191. doi:10.1016/bs.apar.2014.12.004.
16. Daz-Infante, S., Jerez, S.. Convergence and asymptotic stability of the explicit steklov method for stochastic differential equations. *Journal of Computational and Applied Mathematics* 2016;291:36 – 47. URL: http://www.sciencedirect.com/science/article/pii/S037704271500028X. doi:https://doi.org/10.1016/j.cam.2015.01.016; mathematical Modeling and Computational Methods.
17. Cohen, J.E., Gürtler, R.E.. Modeling household transmission of American trypanosomiasis. *Science* 2001;293(5530):694–698. doi:10.1126/science.1060638.
18. Rabinovich, J.E.. Vital Statistics of Triatominae (Hemiptera : Reduviidae) Under Laboratory Conditions. *J Med Entomol* 1972;9(4):351–370. doi:10.1093/jmedent/9.4.351.