Strong Convergence and Almost Sure Stability of the Explicit Linear Steklov Method for SDEs under non-globally Lipschitz Coefficients.☆

S. Díaz-Infante[a], S. Jerez[a]

[a] *Split Step Linear Steklov Method Department of Applied Mathematics, CIMAT, Guanajuato, Gto., Mexico, 36240.*

Abstract

We present an explicit and easily implementable numerical method for solving stochastic differential equations (SDEs) with non-globally Lipschitz coefficients. A linear version of the Steklov average under a split-step formulation supports our new solver. The Linear Steklov (LS) method converges strongly with a standard one-half order. Also, we study the almost sure asymptotic stability and in order to emphasize the performance of the Linear Steklov discretization we use models from population dynamics and non linear stochastic oscillators.

Keywords: stochastic differential equations; explicit methods; strong convergence; almost surely asymptotic stability.

1. Introduction

In this chapter we study numerical approximations of vector Itô stochastic differential equations (SDEs) with the form

$$dy(t) = f(y(t))dt + g(y(t))dW(t), \quad 0 \leq t \leq T, \quad y(0) = y_0. \tag{1}$$

Here $(f^{(1)}, \ldots, f^{(d)}) : \mathbb{R}^d \to \mathbb{R}^d$ and $g = (g^{(i,j)})_{i\in\{1,\ldots,d\}, j\in\{1,\ldots,m\}} : \mathbb{R}^d \to \mathbb{R}^{d\times m}$. We will work with the standard setup, that is, $y(t) \in \mathbb{R}^d$ for each t and $W(t)$ is a m-dimensional standard Brownian motion on a filtered and complete probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$, with the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ generated by the Brownian process. Also, we require the following assumptions over the coefficients f, g.

Hypothesis 1.1. The coefficients of SDE (1) satisfy the following conditions:

(H-1) The functions f, g are in the class $C^1(\mathbb{R}^d)$.

(H-2) **Local, global Lipschitz condition**. For each integer n, there is a positive constant $L_f = L_f(n)$ such that

$$|f(x) - f(y)|^2 \leq L_f|x - y|^2 \qquad \forall x, y \in \mathbb{R}^d, \qquad |x| \vee |y| \leq n,$$

and there is a positive constant L_g such that

$$|g(x) - g(y)|^2 \leq L_g|x - y|^2, \qquad \forall x, y \in \mathbb{R}^d.$$

☆This work has been partially supported by CONACYT project *****

Email addresses: sauld@cimat.mx (S. Díaz-Infante), jerez@cimat.mx (S. Jerez)

(H-3) **Monotone condition.** There exist two positive constants α and β such that

$$\langle x, f(x)\rangle + \frac{1}{2}|g(x)|^2 \leq \alpha + \beta|x|^2, \qquad \forall x \in \mathbb{R}^d. \tag{2}$$

Hypothesis 1.2. Also, we require a special structure over the drift coefficient.

(A-1) For each component function $f^{(j)} : \mathbb{R}^d :\to \mathbb{R}$, $j \in \{1, \ldots, d\}$ there are two locally Lipschitz functions $a_j : \mathbb{R}^d \to \mathbb{R}$, and $b_j : \mathbb{R}^{d-1} \to \mathbb{R}$ such that

$$f^{(j)}(y) = a_j(y)y^{(j)} + b_j(y^{(-j)}), \qquad y^{(-j)} = (y^{(1)}, \ldots, y^{(j-1)}, y^{(j+1)}, \ldots, y^{(d)}). \tag{3}$$

(A-2) There is a positive constant L_a such that

$$a_j(x) \leq L_a, \qquad \forall x \in \mathbb{R}^d, \quad j = 1, \ldots, d. \tag{4}$$

(A-3) Each function $b_j(\cdot)$ satisfy the linear growth condition

$$|b_j(u^{(-j)})|^2 \leq L_b(1 + |u|^2), \qquad \forall x \in \mathbb{R}^d, \quad j = 1, \ldots, d. \tag{5}$$

(A-4) Let $E_j := \{x \in \mathbb{R}^d : a_j(x) = 0\}$. Then, for each $x \in E_j$ there is an open ball of positive radius and center x, $B_r(x)$, such that

$$\frac{\partial a_j(u)}{\partial u^{(j)}} \neq 0, \qquad \forall u \in B_r(x) \setminus E_j.$$

In the next section, we will present the results for the existence and uniqueness of the solution for the continuous problem (1).

2. Existence and uniqueness of the solution

To assure existence and uniqueness of the solution of the SDE (1), we recall a classical result reported by Mao and Szpruch in [15]. Also, we remind two results that establish bonds for the moments of the solution see [7, 14].

Theorem 2.1 ([15, Thm. 2.2])**.** *Let Hypothesis 1.1 holds. Then for all $y(0) = y_0 \in \mathbb{R}^d$ given, there exist a unique global solution $\{y(t)\}_{t\geq 0}$ to SDE(1). Moreover, the solution has the following properties for any $T > 0$,*

$$\mathbb{E}\,|y(T)|^2 < \left(|y_0|^2 + 2\alpha T\right)\exp(2\beta T),$$

and

$$\mathbb{P}\left[\tau_n \leq T\right] \leq \frac{\left(|y_0|^2 + 2\alpha T\right)\exp(2\beta T)}{n},$$

where n is any positive integer and $\tau_n := \inf\{t \geq 0 : |y(t)| > n\}$.

Theorem 2.2 ([14, Thm. 2.4.1])**.** *Let $p \geq 2$ and $x_0 \in L^p(\Omega, \mathbb{R}^d)$. Assume that there exits a constant $C > 0$ such that for all $(x,t) \in \mathbb{R}^d \times [t_0, T]$,*

$$\langle x, f(x,t)\rangle + \frac{p-1}{2}|g(x,t)|^2 \leq C(1 + |x|^2).$$

Then

$$\mathbb{E}|y(t)|^p \leq 2^{\frac{p-2}{2}}\left(1 + \mathbb{E}|y_0|^p\right)\exp(Cpt) \qquad \textit{for all } t \in [0,T].$$

Lemma 2.1 ([7, Lem 3.2])**.** *Under Hypothesis 1.1, for each $p \geq 2$, there is a $C = C(p,T)$ such that*

$$\mathbb{E}\left[\sup_{0\leq t\leq T} |y(t)|^p\right] \leq C\left(1 + \mathbb{E}|y_0|^p\right).$$

3. Construction of the Linear Steklov method

For simplicity, we begin the construction of the Linear Steklov method (LS) considering the scalar case of SDE (1), that is, when $d = m = 1$, also, to shorten notation we use a, b instead a_j, b_j. Tough this ideas, we will generalize to higher dimensions. Let $0 = t_0 < t_1 < \cdots < t_N = T$ a partition of the interval $[0, T]$ with constant step-size $h = T/N$ and such that $t_k = kh$ for $k = 0, \ldots, N$. The main idea of the LS approximation consists in estimating the drift coefficient of (1) by

$$f(y(t)) \approx \varphi_f(y(t_{\eta_+(t)})) = \left(\frac{1}{y(t_{\eta_+(t)}) - y(t_{\eta(t)})} \int_{y(t_{\eta(t)})}^{y(t_{\eta_+(t)})} \frac{du}{a(y(t_{\eta(t)}))u + b} \right)^{-1}, \qquad t \in [0, T], \tag{6}$$

where

$$\begin{aligned} \eta(t) &:= k \text{ for } t \in [t_k, t_{k+1}), \quad k \geq 0, \\ \eta_+(t) &:= k + 1 \text{ for } t \in [t_k, t_{k+1}), \quad k \geq 0. \end{aligned}$$

So we define the LS method for the scalar version of the SDE (1) using a split-step formulation as follows

$$Y_k^\star = Y_k + h\varphi_f(Y_k^\star), \tag{7}$$
$$Y_{k+1} = Y_k^\star + g(Y_k^\star)\Delta W_k, \tag{8}$$

with $Y_0 = y_0$ and $\varphi_f\left(Y_k^\star\right)$ defined by

$$\varphi_f\left(Y_k^\star\right) = \left(\frac{1}{Y_k^\star - Y_k} \int_{Y_k}^{Y_k^\star} \frac{du}{a(Y_k)u + b} \right)^{-1} \tag{9}$$

This scheme combines a split-step technique with a linear version of an exact deterministic method see [4, 16]. In detail, first we compute the discrete value $Y_k^\star$ using the Linear Steklov approximation (7), and next, Y_{k+1} is obtained by adding the stochastic increment $g(Y_k^\star)\Delta W_k$.

To higher dimensions, we adapt the same split step scheme (7)–(8) as follows. For each component equation $j \in \{1, \ldots, d\}$, on the iteration $k \in \{1, \ldots, N\}$ take

$$a_{j,k} = a_j\left(Y_k^{(1)}, \ldots, Y_k^{(d)}\right), \qquad b_{j,k} = b_j\left(Y_k^{(-j)}\right). \tag{10}$$

So, define $\varphi_f(Y_k^\star) = \left(\varphi_{f^{(1)}}(Y_k^\star), \ldots, \varphi_{f^{(d)}}(Y_k^\star)\right)$ by

$$\varphi_{f^{(j)}}\left(Y_k^\star\right) = \left(\frac{1}{Y_k^{\star(j)} - Y_k^{(j)}} \int_{Y_k^{(j)}}^{Y_k^{\star(j)}} \frac{du}{a_{j,k}u + b_{j,k}} \right)^{-1}. \tag{11}$$

It is worth mentioning that even this formulation is semi implicit, we always can derive a explicit version. The next result deals with this issue. To simplify notation, we define $A^{(1)} = A^{(1)}(h, u)$,

$A^{(2)} = A^{(2)}(h,u)$ and $b = b(u)$ by

$$A^{(1)} := \begin{pmatrix} e^{ha_1(u)} & & 0 \\ & \ddots & \\ 0 & & e^{ha_d(u)} \end{pmatrix},$$

$$A^{(2)} := \begin{pmatrix} \left(\frac{e^{ha_1(u)}-1}{a_1(u)}\right)\mathbf{1}_{\{E_1^c\}} & & 0 \\ & \ddots & \\ 0 & & \left(\frac{e^{ha_d(u)}-1}{a_d(u)}\right)\mathbf{1}_{\{E_d^c\}} \end{pmatrix} + h\begin{pmatrix} \mathbf{1}_{\{E_1\}} & & 0 \\ & \ddots & \\ 0 & & \mathbf{1}_{\{E_d\}} \end{pmatrix}, \tag{12}$$

$$E_j := \{x \in \mathbb{R}^d : a_j(x) = 0\}, \qquad b(u) := \left(b_1(u^{(-1)}), \ldots, b_d(u^{(-d)})\right)^T.$$

Also we will need the following results from [12, Thm 2.1], [5, Thm. 1]. The first theorem will help us with the singularities of set E_j in the case where all elements of this set are limit points.

Theorem 3.1 (Multivariate L'hôpital's Rule)**.** *Let $\mathcal{N}$ be a neighborhood in $\mathbb{R}^2$ containing a point $\mathbf{q}$ at which two differentiable functions $f : \mathcal{N} \to \mathbb{R}$ and $g : \mathcal{N} \to \mathbb{R}$ are zero. Set*

$$C = \{x \in \mathcal{N} : f(x) = g(x) = 0\},$$

and suppose that C is a smooth curve through $\mathbf{q}$.

Suppose there exist a vector $\mathbf{v}$ not tangent to C at $\mathbf{q}$ such that the directional derivative $D_{\mathbf{v}}g$ of g in the direction of $\mathbf{v}$ is never zero within $\mathcal{N}$. Also we assume that $\mathbf{q}$ is a limit point of $\mathcal{N} \setminus C$. Then

$$\lim_{(x,y)\to\mathbf{q}} \frac{f(x,y)}{g(x,y)} = \lim_{\substack{(x,y)\to\mathbf{q} \\ (x,y)\in\mathcal{N}\setminus C}} \frac{D_{\mathbf{v}}f}{D_{\mathbf{v}}g}$$

if the latter limit exist.

For the second auxiliary we will need the following concepts.

Definition 3.1 (Directional derivative referred at a point)**.** Let $u, \mathbf{q} \in \mathbb{R}^2$ and α the positive angle respect to the x-axis and the segment $\overline{u\mathbf{q}}$. We denote by

$$f_\alpha(u) = \cos(\alpha)\frac{\partial f}{\partial u^{(1)}}(u) + \sin(\alpha)\frac{\partial f}{\partial u^{(2)}}(u) = \frac{\langle q-u, \nabla f(u)\rangle}{|u-q|}$$

the *directional derivative respect to the point $\mathbf{q}$ on u.*

Definition 3.2 (Star-like set)**.** A set $S \subset \mathbb{R}^2$ is *star-like* with respect a point $\mathbf{q}$, if for each point $s \in S$ the open segment $\overline{s\mathbf{q}}$ is in S.

Whit this in mine, second theorem give us a way to analyze isolated singularities.

Theorem 3.2. *Let $\mathbf{q} \in \mathbb{R}^2$ and let f,g be functions whose domains include a set $S \subset \mathbb{R}^2$ which is star-like with respect to the point $\mathbf{q}$. Suppose that on S the functions are differentiable and that the directional derivative of g with respect to $\mathbf{q}$ is never zero. With the understanding that all limits are taken from within on S at $\mathbf{q}$ and if*

(i) $f(\mathbf{q}) = g(\mathbf{q}) = 0$,

(ii) $\displaystyle\lim_{x\to\mathbf{q}} \frac{f_\alpha(x)}{g_\alpha(x)} = L$,

then

$$\lim_{x\to\mathbf{q}}\frac{f(x)}{g(x)}=L.$$

With this on mind, we additionally require the following.

Hypothesis 3.1. The set $E_j := \{x \in \mathbb{R}^d : a_j(x) = 0\}$ satisfies either:

(i) All point $q \in E_j$ is a non isolated zero of a_j and:

- the set
$$D := \{u \in B_r(q) : e^{ha_j(u)} - 1 = a_j(u) = 0\},$$
is a smooth curve through q.
- The canonical vector e_j is not tangent to D.
- For each $q \in E_j$, there is an open ball with center on q and radio r $B_r(q)$, such that and
$$a_j \neq 0, \qquad \frac{\partial a_j(u)}{\partial u^{(j)}} \neq 0, \qquad \forall u \in B_r(q)\setminus D.$$

(ii) All point $q \in E_j$ is a isolated zero of a_j and:

- For each $q \in E_j$, q is not a limit point of the set $E_\alpha := \{x \in \mathbb{R}^d : (a_j)_\alpha(x) = 0\}$.
- For each $q \in E_j$ there is a star-like set respect to q E_q, such that the directional derivative respect to q satisfies
$$(a_j)_\alpha(x) \neq 0, \qquad \forall x \in E_q.$$

Lemma 3.1. *Let Hypothesis 1.1 , Hypothesis 1.2, Hypothesis 3.1 holds, and $A^{(1)}$, $A^{(2)}$, b defined by* (12)*. Then given $u \in \mathbb{R}^d$, the equation*

$$v = u + h\varphi_f(v), \tag{13}$$

has a unique solution

$$v = A^{(1)}(h,u)u + A^{(2)}(h,u)b(u). \tag{14}$$

If we define the functions $F_h(\cdot)$, $\varphi_{f_h}(\cdot)$ and $g_h(\cdot)$ by

$$F_h(u) = v, \qquad \varphi_{f_h}(u) = \varphi_f(F_h(u)), \qquad g_h(u) = g(F_h(u)), \tag{15}$$

then $F_h(\cdot)$, $\varphi_{f_h}(\cdot)$, $g_h(\cdot)$ are local Lipschitz functions and for all $u \in \mathbb{R}^d$ and each h fixed, there is a positive constant C_h such that

$$|\varphi_{f_h}(u)| \leq C_h|f(u)|. \tag{16}$$

Moreover, for each h fixed, there are positive constants α^* and β^* such that

$$\langle \varphi_{f_h}(u), u\rangle \vee |g_h(u)|^2 \leq \alpha^* + \beta^*|u|^2, \qquad \forall u \in \mathbb{R}^d. \tag{17}$$

PROOF. Let us first, prove that (14) is solution of Equation (13). To this end note that for each $j \in \{1,\ldots,d\}$

$$v^{(j)} = u^{(j)} + h\varphi_{f^{(j)}}(v)$$

So, using in Equation (14) that

$$\varphi_{f^{(j)}}(v) = \left(\frac{1}{v^{(j)} - u^{(j)}}\int_{u^{(j)}}^{v^{(j)}} \frac{dz}{a_j(u)z + b_j(u^{(-j)})}\right)^{-1}. \tag{18}$$

After some algebraic manipulations we arrive at

$$v^{(j)} = \exp(ha_j(u))u^{(j)} + \left[\left(\frac{\exp(ha_j(u)) - 1}{a_j(u)}\right)\mathbf{1}_{\{E_j^c\}} + h\mathbf{1}_{\{E_j\}}\right] b_j(u^{(-j)}), \tag{19}$$

for each $j \in \{1, \ldots, d\}$, which is the j-component of the vector $A^{(1)}(h,u)u + A^{(2)}(h,u)b(u)$. Next, we will prove the inequality (16). Since

$$\varphi_{f_h}^{(j)}(u) = \frac{F_h^{(j)}(u) - u^{(j)}}{\displaystyle\int_{u^{(j)}}^{F_h^{(j)}(u)} \frac{dz}{a_j(u)z + b_j(u^{(-j)})}}.$$

We first see that if $u \in E_j$, then $\varphi_{f_h}^{(j)}(u) = b_j(u^{(-j)}) = f(u)$. Then taking $C \geq 1$ we have the conclusion. On the other hand, if $u \in E_j^c$, then

$$\varphi_{f_h}^{(j)}(u) = \frac{(F_h^{(j)}(u) - u^{(j)})a_j(u)}{\underbrace{\ln\left(a_j(u)F_h^{(j)}(u) + b_j(u^{(-j)})\right)}_{:=Ter1} - \ln\left(a_j(u)u^{(j)} + b_j(u^{(-j)})\right)} \tag{20}$$

Now consider in Equation (20) the term labeled $Ter1$, and observe that

$$\begin{aligned} Ter1 &= \ln\left\{a_j(u)\left[e^{ha_j(u)}u^{(j)} + \left(\frac{e^{ha_j(u)} - 1}{a_j(u)}\right) b_j(u^{(-j)})\right] + b_j\left(u^{(-j)}\right)\right\} \\ &= ha_j(u) + \ln\left(f^{(j)}(u)\right). \end{aligned} \tag{21}$$

Combining the relations (20) and (21), after algebraic manipulations we arrive at

$$\varphi_{f_h}^{(j)}(u) = \left(\frac{e^{ha_j(u)} - 1}{ha_j(u)}\right) f^{(j)}(u), \quad \forall u \in E_j^c. \tag{22}$$

Hence, it remains to prove that

$$\Phi(h, a_j)(u) := \frac{e^{ha_j(u)} - 1}{ha_j(u)}, \tag{23}$$

is bounded on $\mathbb{R}^d$ for each $j \in \{1, \ldots, d\}$. First wee see that under the Hypothesis 1.1 the operator Φ is continuous on E_j^c. Furthermore, for each fixed $u \in E_j^c$

$$\lim_{h \to 0} \frac{e^{ha_j(u)} - 1}{ha_j} = 1.$$

On the other hand, for each fixed h, by Hypothesis 3.1 we obtain one of the following cases:

CASE I:

$$\lim_{\substack{u \to u^* \\ u \in E_j^c}} \Phi(h, a_j)(u) = \lim_{\substack{u \to u^* \\ u \in E_j^c}} \frac{\frac{\partial a_j(u)}{\partial u^{(j)}} he^{ha_j(u)}}{h\frac{\partial a_j(u)}{\partial u^{(j)}}} = 1. \tag{24}$$

CASE II:

$$\lim_{\substack{u \to u^* \\ u \in E_j^c}} \Phi(h, a_j)(u) = \lim_{\substack{u \to u^* \\ u \in E_j^c}} \frac{\left(e^{ha_j(u)} - 1\right)_\alpha}{\left(ha_j(u)\right)_\alpha} = 1, \qquad \alpha = 0, \pi, 2\pi, \ldots \tag{25}$$

So, under this situation, we can say for each j that the function

$$f^{(j)}(u)\mathbf{1}_{\{E_j\}} + \left(\frac{e^{ha_j(u)}-1}{ha_j(u)}\right) f^{(j)}(u)\mathbf{1}_{\{E_j^c\}}$$

is continuous on $\mathbb{R}^d$ and bounded on E_j. Now, let

$$a_j^* := \inf_{u\in E_j^c}\{|a_j(u)|\}.$$

So, a_j^* satisfy one of the two following cases:

CASE I: $0 < a_j^* \leq L_a$.

CASE II: $a_j^* = 0$.

In the first case we see that

$$\frac{e^{ha_j(u)}-1}{ha_j(u)} \leq \frac{e^{hL_a}-1}{ha_j^*(u)} < \infty, \qquad \forall h \in (0,\infty).$$

For CASE II, we can apply an argument similar as in (24)–(25). Then there is $C_h > 0$ such that

$$\left|\frac{e^{ha_j(u)}-1}{ha_j(u)}\right| < C_h, \qquad \forall u \in \mathbb{R}^d. \tag{26}$$

Combining this fact with Equation (22), we obtain

$$|\varphi_{f_h^{(j)}}(u)| \leq \left|\frac{e^{ha_j(u)}-1}{ha_j(u)}\right| |f^{(j)}(u)| < C_h|f^{(j)}(u)|, \qquad \forall u \in \mathbb{R}^d,$$

which prove inequality (16).

No, we prove the g_h is a locally Lipschitz function. By (H-1) in Hypothesis 1.1, g is a globally Lipschitz function, so

$$g_h(x) = g\left(F_h(x)\right),$$

is the composition of a continuous bounded function and a globally Lipschitz function, furthermore, note that

$$\begin{aligned}|g_h(u)-g_h(v)|^2 &\leq L_g|F_h(u)-F_h(v)|^2\\ &\leq 2L_g \underbrace{|A^{(1)}(h,u)u - A^{(1)}(h,v)v|^2}_{:=Ter_1} + 2L_g \underbrace{|A^{(2)}(h,u)b(u) - A^{(2)}(h,v)b(v)|^2}_{:=Ter_2}.\end{aligned} \tag{27}$$

Now, we work with each term of the right hand of inequality (27). First note that $A^{(1)}$ is a continuous differentiable function on all $\mathbb{R}^d$ so by the mean value Theorem we have

$$|A^{(1)}(h,u)u - A^{(1)}(h,v)v|^2 \leq \sup_{0\leq t\leq 1} |\partial A^{(1)}(h, u+t(v-u))|^2|u-v|^2,$$

then, there is a positive constant $L_{A^{(1)}} = L_{A^{(1)}}(h,n)$ such that

$$|A^{(1)}(h,u)u - A^{(1)}(h,v)v|^2 \leq L_{A^{(1)}}|u-v|^2, \quad u,v\in\mathbb{R}^d, \quad |u|\vee|v| \leq n. \tag{28}$$

In the other hand,

$$
\begin{aligned}
Ter_2 = \sum_{j=1}^{d} \Big[& \mathbf{1}_{\{E_j^c\}}(u)\Phi(h,a_j)(u)b_j(u^{(-j)}) + h\mathbf{1}_{\{E_j\}}(u)b_j(u^{(-j)}) - \mathbf{1}_{\{E_j^c\}}(v)\Phi(h,a_j)(v)b_j(u^{(-j)}) \\
& - h\mathbf{1}_{\{E_j\}}(v)b_j(v^{(-j)})\Big]^2 \\
\leq 4\sum_{j=1}^{d} \Big[& \Big(\mathbf{1}_{\{E_j^c\}}(u)\Phi(h,a_j)(u)b_j(u^{(-j)})\Big)^2 + \Big(h\mathbf{1}_{\{E_j\}}(u)b_j(u^{(-j)})\Big)^2 \\
& + \Big(\mathbf{1}_{\{E_j^c\}}(v)\Phi(h,a_j)(v)b_j(v^{(-j)})\Big)^2 + \Big(h\mathbf{1}_{\{E_j\}}(v)b_j(v^{(-j)}\Big)^2\Big] \\
\leq 4\sum_{j=1}^{d} \Big[& \Big(\mathbf{1}_{\{E_j^c\}}(u)C_h b_j(u^{(-j)})\Big)^2 + \Big(h\mathbf{1}_{\{E_j\}}(u)b_j(u^{(-j)})\Big)^2 + \Big(\mathbf{1}_{\{E_j^c\}}(v)C_h b_j(v^{(-j)})\Big)^2 \\
& + \Big(h\mathbf{1}_{\{E_j\}}(v)b_j(v^{(-j)}\Big)^2\Big].
\end{aligned}
\tag{29}
$$

Since each $b_j^2(\cdot)$ function is of class $C^1(\mathbb{R}^d)$, there is a constant $M_b = M_b(n)$ such that

$$|b_j(u)|^2 \leq M_b, \quad \forall u \in \mathbb{R}^d, \quad |u| \vee |v| \leq n, \quad j \in \{1, \cdots, d\}\ . \tag{30}$$

Putting this bound in inequality (29), we deduce that

$$
\begin{aligned}
Ter_2 &\leq 4\sum_{j=1}^{d}\left[2C_h M_b + 2h^2 M_b\right] \\
&\leq \underbrace{8dM_b(n)(C_h + h^2)}_{:=L_0}, \quad \forall u, v \in \mathbb{R}^d, \quad |u| \vee |v| \leq n.
\end{aligned}
\tag{31}
$$

Then, combining inequalities (28) and (31) we arrive at

$$|g_h(u) - g_h(v)|^2 \leq L_{A^{(1)}}|u - v|^2 + L_0.$$

So, taking $L_{g_h} = L_{g_h}(h,n) \geq n^2 + 1 + L_0 + L_{A^{(1)}}$, we see that

$$|g_h(u) - g_h(v)|^2 \leq L_{g_h}(n)|u - v|^2, \quad \forall u, v \in \mathbb{R}^d, \quad |u| \vee |v| \leq n.$$

Then $g_h(\cdot)$ is a locally Lipschitz function. Furthermore, note that under some modifications, this argument also prove that $F_h(\cdot)$ is a Locally Lipschitz function, which also implies that φ_{f_h} is a locally Lipschitz function. Finally, we prove the inequality (17). We first observe that by Hypothesis 1.1 and Hypothesis 1.2

$$
\begin{aligned}
\langle f(u), u\rangle &= \sum_{j=1}^{d} a_j(u)\left(u^{(j)}\right)^2 + \sum_{j=1}^{d} b_j(u)u^{(j)} \\
&\leq \alpha + \beta|u|^2,
\end{aligned}
$$

then

$$\langle b(u), u\rangle \leq \alpha + (\beta + L_a)|u|^2. \tag{32}$$

With this on mind, and using the inequality (16), we deduce that

$$
\begin{aligned}
\langle \varphi_{f_h}(u), u\rangle &= \sum_{j=1}^{d} \Phi(h, aj)(u) f^{(j)}(u)u^{(j)} \\
&\leq \sum_{j=1}^{d} C_h L_a |u^{(j)}| + C_h \langle b(u), u\rangle \\
&\leq C_h L_a |u| + C_h(\alpha + (L_a + \beta)|u|^2).
\end{aligned}
$$

So, taking $L_{\varphi_{f_h}} = L_{\varphi_{f_h}}(h) \geq 2C_h \cdot \max\{L_a, \alpha, \beta\} + 1$, we obtain

$$\langle \varphi_{f_h}(u), u\rangle \leq L_{\varphi_{f_h}}(1+|u|^2), \quad \forall u \in \mathbb{R}^d. \tag{33}$$

On the other hand, since g is globally Lipschitz it follows that

$$\begin{aligned} |g_h(u)|^2 &\leq 2|g(F_h(u)) - g(F_h(0)|^2 + 2|g(F_h(0)|^2 \\ &\leq 2L_g|F_h(u) - F_h(0)|^2 + 4|g(F_h(0)) - g(0)|^2 + 4|g(0)|^2 \\ &\leq 4L_g|F_h(u)|^2 + 8L_g|F_h(0)|^2 + 4|g(0)|^2. \end{aligned} \tag{34}$$

Now we bound each term in the right hand side of inequality (34). Let us first observe that by the monotone condition in Hypothesis 1.1

$$|g(0)|^2 \leq 2\alpha. \tag{35}$$

On the other hand

$$F_h^{(j)}(0) = \frac{e^{ha_j(0)} - 1}{a_j(0)} b_j(0)\mathbf{1}_{\{E_j^c\}}(0) + hb_j(0)\mathbf{1}_{\{E_j\}}(0).$$

So, taking

$$a_0^* := \min_{\substack{j\in\{1,\cdots,d\} \\ a_j(0)\neq 0}} \{|a_j(0)|\}, \qquad b_0^* := \max_{j\in 1,\cdots,d} \{|b_j(0)|\}$$

we can deduce that

$$|F_h^{(j)}(0)| \leq \frac{b_0^*}{a_0^*} e^{hL_a}(1+h), \quad \forall j \in \{1,\cdots,d\}.$$

Then

$$|F_h(0)|^2 \leq d\left(\frac{b_0^*}{a_0^*}\right)^2 e^{2hL_a}(1+h)^2. \tag{36}$$

In this line, since the operator $\Phi(h, a_j)$ is bounded, it follows that

$$\begin{aligned} F_h^{(j)}(u) &= e^{ha_j(u)}u^{(j)} + h\Phi(h,a_j)(u)b_j(u)\mathbf{1}_{\{E_j^c\}}(u) + hb_j(u)\mathbf{1}_{\{E_j\}}(u) \\ &\leq e^{ha_j(u)}|u^{(j)}| + hC_h|b_j(u)|\mathbf{1}_{\{E_j^c\}}(u) + h|b_j(u)|\mathbf{1}_{\{E_j\}}(u). \end{aligned}$$

Then we deduce by Hypothesis 1.2 that

$$\begin{aligned} |F_h^{(j)}(u)|^2 &\leq 3e^{2hL_a}|u|^2 + (3h^2C_h^2L_b + 3h^2L_b)(1+|u|^2) \\ &\leq 3h^2L_b(1+C_h^2) + 3\left(e^{2hL_a} + h^2L_b(C_h^2+1)\right)|u|^2. \end{aligned}$$

So, taking $L_F \geq 3d\max\{\exp(2hL_a), h^2L_b(C_h^2+1)\}$, we obtain that

$$|F_h^{(j)}(u)|^2 \leq L_F(1+|u|^2). \tag{37}$$

Then, combining bounds (35),(36) and (37) we arrive at

$$|g_h(u)|^2 \leq 4L_gL_F(1+|u|^2) + 8L_gd\left(\frac{b_0^*}{a_0^*}\right)^2 e^{2hL_a}(1+h)^2 + 8\alpha.$$

Therefore, if $L_{g_h} \geq 4L_gL_F + 8L_gd\left(\frac{b_0^*}{a_0^*}\right)^2 e^{2hL_a}(1+h)^2 + 8\alpha$ wee see that

$$|g_h(u)|^2 \leq L_{g_h}(1+|u|^2) \tag{38}$$

Hence, from the inequalities (33) and (38) and taking for each fixed $h > 0$

$$\alpha^* := L_{\varphi_{f_h}} \vee L_{g_h}, \qquad \beta^* := 2\alpha^*$$

we have the desired conclusion. $\square$

Remark 3.1. Note that by Lemma 3.1, we have that $|f(x) - \varphi_{f_h}(x)| = \mathcal{O}(h)$. Hence it is convenient to consider the following modified SDE

$$dy_h(t) = \varphi_{f_h}(y_h(t))dt + g_h(y_h(t))dW(t), \qquad y_h(0) = y_0, \qquad t \in [0, T],$$

as a perturbation of SDE (1). Moreover, the functions $\varphi_{f_h}(\cdot)$ and $g_h(\cdot)$ in (15) are respectively defined as the functions φ_f and g, but evaluated in the solution of $c = d + h\varphi(c)$, then we can rewrite the LS method (7)–(8) as

$$\begin{aligned} Y_k^\star &= Y_k + h\varphi_{f_h}(Y_k), \\ Y_{k+1} &= Y_k^\star + g_h(Y_k)\Delta W_k. \end{aligned}$$

We formalize these ideas in the following sections.

4. Strong Convergecne using the Higham-Mao-Stuart technique

For the sake of clarity, we discuss in this section a technique to prove strong convergence of stochastic numerical methods, reported by Higham, Mao, and Stuart in [7]. In this seminal paper, the authors assume non-globally Lipschitz conditions. This kind of analysis is useful whenever moment bounds can be established for both, the Euler-Maruyama scheme and other method that can be shown to be "close" to it. A vast amount of literature has been cited this technique, some of these works are [3], [6], [9], [10], [8] [11], [15], [17], just to mention a few.

To review this technique, we begin recalling the definition of the Euler-Maruyama (EM) approximation for SDE (1),

$$X_{k+1} = X_k + hf(X_k) + g(X_k)\Delta W_k, \tag{39}$$

and two versions of a conveniently continuous extension used by Higham et al. in [7]

$$\begin{aligned} \overline{X}(t) &:= X_{\eta(t)} + (t - t_{\eta(t)})f(X_{\eta(t)}) + g(X_{\eta(t)})(W(t) - W_{\eta(t)}) \\ \eta(t) &:= k, \text{ for } t \in [t_k, t_{k+1}), \end{aligned} \tag{40}$$

and

$$\overline{X}(t) := X_0 + \int_0^t f(X_{\eta(s)})ds + \int_0^t g(X_{\eta(s)})dW(s).$$

So, with this notation we have $\overline{X}(t_k) = X_k$

Using the continuous extension (40) and the uniform mean square norm, the authors work with a stronger version of the ms-error

$$\mathbb{E}\left[\sup_{0\le t\le t} |y(t) - \overline{X}(t)|^2\right].$$

So, in order to prove the strong convergence of the EM method, the following assumptions are required.

Assumption 4.1. For each $R > 0$ there is a constant C_R, depending only on R, such that

$$|f(x) - f(y)|^2 \vee |g(x) - g(y)|^2 \le C_R|x - y|^2, \quad \forall x, y \in \mathbb{R}^d \text{ with } |x| \vee |y| \le R. \tag{41}$$

And for some $p > 2$, there is a constant A such that

$$\mathbb{E}\left[\sup_{0\le t\le T} |\overline{X}(t)|^p\right] \vee \mathbb{E}\left[\sup_{0\le t\le T} |y(t)|^p\right] \le A. \tag{42}$$

In [7], the authors prove that the Assumption 4.1 is sufficient to ensure strong convergence for the EM scheme, namely

Theorem 4.1 ([7, Thm 2.2])**.** *Under Assumption 4.1, the EM scheme* (39) *with continuous extension* (40) *satisfies*

$$\lim_{h\to 0} \mathbb{E}\left[\sup_{0\le t\le T} |\overline{X}(t) - y(t)|^2\right] = 0. \tag{43}$$

Applying this result, the authors prove the strong convergence of an implicit split-step variant of the EM, the SSEM method. Their technique consist in proving each assertion of the following steps.

Step 1: The SSEM for SDE (1) is equivalent to the EM for the following conveniently SDE

$$dx_h(t) = f_h(x_h(t))dt + g_h(x_h(t))dW(t). \tag{44}$$

Step 2: The solution of the modified SDE (44) has bounded moments and it is "close" to y the sense of the uniform mean square norm $\mathbb{E}\left[\sup_{0\le t\le T} |\cdot|^2\right]$.

Step 3: Show that the SSEM method for the SDE (1) has bounded moments.

Step 4: There is a continuous extension of the SSEM, $\overline{Z}(t)$, with bounded moments.

Step 5: Use the above steps and Theorem 4.1 to conclude that

$$\lim_{h\to 0} \left\{ \mathbb{E}\left[\sup_{0\le t\le T} |x_h(t) - y(t)|^2\right] + \mathbb{E}\left[\sup_{0\le t\le T} |\overline{Z}(t) - y_h(t)|^2\right] \right\} = 0. \tag{45}$$

In the next section, using Theorem 4.1 and this technique, we will prove the strong convergence of the LS method (7)–(8).

5. Strong Convergence of the Linear Steklov Method

Here, we state and prove the main result of this chapter, the strong convergence of the LS method (7)–(8) for the solution of SDE (1). The main idea of the proof consist in applying the technique discussed in the previous section. We begin establishing the underlying convergence theorem.

Theorem 5.1. *Let Hypothesis 1.1 and Hypothesis 1.2 hold, consider the LS method (7)–(8) for the SDE* (1)*. Then there is a continuous-time extension $\overline{Y}(t)$ of the LS solution $\{Y_k\}$ for which $\overline{Y}(t_k) = Y_k$ and*

$$\lim_{h\to 0} \mathbb{E}\left[\sup_{0\le t\le T} |\overline{Y}(t) - y(t)|^2\right] = 0.$$

To proof this result, we initiate with the first step of the HMS technique, that is, we will show that the LS method for SDE (1) is equivalent to the EM scheme applied to the conveniently modified SDE

$$dy_h(t) = \varphi_{f_h}(y_h(t))dt + g_h(y_h(t))dW(t), \qquad y_h(0) = y_0, \qquad t \in [0, T]. \tag{46}$$

We formalize this as a Corollary of Lemma 3.1.

Corollary 5.1. *Let Hypothesis 1.1 and Hypothesis 1.2 holds, then the LS method for SDE* (1) *is equivalent to the EM scheme applied to the modified SDE* (46)*.*

PROOF. Using the functions $\varphi_{f_h}(\cdot)$ and $g_h(\cdot)$ defined in (15) of Lemma 3.1, we can rewrite the LS method (7)–(8) as

$$Y_{k+1} = Y_k + h\varphi_{f_h}(Y_k) + g_h(Y_k)\Delta W_k,$$

which is the EM approximation for the modified SDE (46). $\square$

Now we proceed with the Step 2, that is, we will prove that the solution of the modified SDE (46) has bounded moments and is close in uniform mean square norm to the solution of the SDE (1).

Lemma 5.1. *Let Hypothesis 1.1 and Hypothesis 1.2 holds. Then there is a constant $C = C(p,T) > 0$ and a sufficiently small step size h, such that for all $p > 2$*

$$\mathbb{E}\left[\sup_{0\le t\le T} |y_h(t)|^p\right] \le C\left(1 + \mathbb{E}|y_0|^p\right). \tag{47}$$

Moreover

$$\lim_{h\to 0} \mathbb{E}\left[\sup_{0\le t\le T} |y(t) - y_h(t)|^2\right] = 0. \tag{48}$$

PROOF. The bound (47) follows from inequality (17) in Lemma 3.1 and Theorem 2.2. On the other hand, to prove (48) we will use the properties of φ_{f_h} and the Higham's stopping time technique employed in [7, Thm 2.2]. Note that by Lemma 3.1

$$\varphi_{f_h}(x) = \Phi(h, a_j)(u) f^{(j)}(u)\mathbf{1}_{\{E_j^c\}}(u) + f^{(j)}(u)\mathbf{1}_{\{E_j\}}(u).$$

Furthermore, by Hypothesis 3.1 and since $f \in C^1(\mathbb{R}^d)$, $\Phi(h,a_j)(\cdot)$ is bounded, hence, there is a positive constant $K_n$which depends only n such that for each $j \in \{1,\dots,d\}$

$$\begin{aligned}
|\varphi_{f_h}^{(j)}(u) - f^{(j)}(u)| &\le \mathbf{1}_{\{E_j^c\}}(u)|f^{(j)}(u)||\Phi(h,a_j)(u) - 1| \\
&\le \mathbf{1}_{\{E_j^c\}}(u)\left(C_h + 1\right)|f(u)| \\
&\le \mathbf{1}_{\{E_j^c\}}(u)K_n(C_h + 1), \quad \forall u \in \mathbb{R}^d, \quad |u|\le n, \quad \forall j \in \{1,\dots,d\}.
\end{aligned}$$

Moreover, we know by the proof of Lemma 3.1 that

$$\lim_{\substack{h\to 0\\ u\in E_j^c}} \Phi(h,a_j)(u) = 1.$$

Also, we note that for each $j \in \{1,\dots,d\}$

$$\lim_{h\to 0} F_h^{(j)}(u) = \lim_{h\to 0} e^{ha_j(u)}u^{(j)} + \lim_{h\to 0}\left(\frac{e^{ha_j(u)} - 1}{a_j(u)}\mathbf{1}_{\{E_j^c\}}(u) + h\mathbf{1}_{\{E_j\}}(u)\right) b_j(u^{(j)}) = u^{(j)},$$

hence $\lim_{h\to 0} F_h(u) = u$. Consequently, given $n > 0$ there is a function $K_n(\cdot) : (0,\infty) \to (0,\infty)$, such that $K_n(h) \to 0$ when $h \to 0$ and

$$|\varphi_{f_h}(u) - f(u)|^2 \vee |g_h(u) - g(u)|^2 \le K_n(h) \qquad \forall u \in \mathbb{R}^d, \quad |u|\le n. \tag{49}$$

Now, using that both f, g are C^1, there is a constant $H_n > 0$ such that

$$|f(u) - f(v)|^2 \vee |g(u) - g(v)|^2 \le H_n|u - v|^2 \qquad \forall u, v \in \mathbb{R}^d, |u|\vee|v|\le n. \tag{50}$$

On the other hand, by Lemma 2.1 and inequality (47) we obtain

$$\mathbb{E}\left[\sup_{0\le t\le T} |y(t)|^p\right] \vee \mathbb{E}\left[\sup_{0\le t\le T} |y_h(t)|^p\right] \le K := C\left(1 + \mathbb{E}|y_0|^p\right).$$

Now, we define the stopping times

$$\tau_n := \inf\{t \geq 0 : |y(t)| \geq n\}, \qquad \rho_n := \inf\{t \geq 0 : |y_h(t)| \geq n\}, \qquad \theta_n := \tau_n \wedge \rho_n, \tag{51}$$

and the difference function

$$e_h(t) := y(t) - y_h(t).$$

From the Young's inequality (A.2), we deduce that for any $\delta > 0$

$$\begin{aligned} \mathbb{E}\left[\sup_{0\leq t\leq T} |e_h(t)|^2\right] &= \mathbb{E}\left[\sup_{0\leq t\leq T} |e_h(t)|^2 \mathbf{1}_{\{\tau_n > T, \rho_n > T\}}\right] + \mathbb{E}\left[\sup_{0\leq t\leq T} |e_h(t)|^2 \mathbf{1}_{\{\tau_n \leq T \text{ or } \rho_n \leq T\}}\right] \\ &\leq \mathbb{E}\left[\sup_{0\leq t\leq T} |e_h(t\wedge\theta_n)|^2 \mathbf{1}_{\{\theta_n \geq T\}}\right] + \frac{2\delta}{p}\mathbb{E}\left[\sup_{0\leq t\leq T} |e_h(t)|^p\right] \\ &+ \frac{1-2/p}{\delta^{2/(p-2)}}\mathbb{P}\left[\tau_n \leq T \text{ or } \rho_n \leq T\right]. \end{aligned} \tag{52}$$

We proceed to bound each term on the right-hand side of inequality (52). By Lemma 2.1, $y(t)$ has bounded moments, hence there is a positive constant A such that

$$\mathbb{P}\left[\tau_n \leq T\right] = \mathbb{E}\left[\mathbf{1}_{\{\tau_n < T\}} \frac{|y(\tau_n)|^p}{n^p}\right] \leq \frac{1}{n^p}\mathbb{E}\left[\sup_{0\leq t\leq T} |y(t)|^p\right] \leq \frac{A}{n^p}, \qquad \text{for } p \geq 2. \tag{53}$$

The same conclusion can be drawn for ρ_n, then

$$\mathbb{P}\left[\tau_n \leq T \text{ or } \rho_n \leq T\right] \leq \mathbb{P}\left[\tau_n \leq T\right] + \mathbb{P}\left[\rho_n \leq T\right] \leq \frac{2A}{n^p}. \tag{54}$$

Now, using the inequality (A.4) and Lemma 2.1 we have

$$\mathbb{E}\left[\sup_{0\leq t\leq T} |e_h(t)|^p\right] \leq 2^{p-1}\mathbb{E}\left[\sup_{0\leq t\leq T} \left(|y(t)|^p + |y_h(t)|^p\right)\right] \leq 2^p A. \tag{55}$$

So, combining the bound (54) with (55) in inequality (52) we obtain

$$\mathbb{E}\left[\sup_{0\leq t\leq T} |e_h(t)|^2\right] \leq \mathbb{E}\left[\sup_{0\leq t\leq T} |e_h(t\wedge\theta_n)|^2 \mathbf{1}_{\{\theta_n \geq T\}}\right] + \frac{2^{p+1}\delta A}{p} + \frac{2(p-2)A}{p\delta^{2/(p-2)}n^p}. \tag{56}$$

Next, we show that the first term of (56) is bounded. Adding conveniently terms yields

$$\begin{aligned} e_h(t\wedge\theta_n) &= \int_0^{t\wedge\theta_n} \left[f(y(s)) - f(y_h(s)) + f(y_h(s)) - \varphi_{f_h}(y_h(s))\right] ds \\ &+ \int_0^{t\wedge\theta_n} \left[g(y(s)) - g(y_h(s)) + g(y_h(s)) - g_h(y_h(s))\right] dW(s). \end{aligned}$$

Using the bounds (49) and (50), the Cauchy-Schwarz, and Doob martingale inequalities, we get

$$\mathbb{E}\left[\sup_{0\leq t\leq \tau} |e_h(t\wedge\theta_n)|^2\right] \leq 4H_n(T+4)\int_0^{\tau} \mathbb{E}\left[\sup_{0\leq t\leq \tau} |e_h(t\wedge\theta_n)|^2\right] ds + 4T(T+4)K_n(h).$$

The Gronwall inequality now yields

$$\mathbb{E}\left[\sup_{0\leq t\leq T} |e_h(t\wedge\theta_R)|^2\right] \leq 4T(T+4)K_n(h)\exp(4H_n(T+4)T).$$

Hence, given $\epsilon > 0$ for any $\delta > 0$ such that $2^{p+1}\delta A/p < \epsilon/3$, we can take $n > 0$ verifying $(p-2)2A/(p\delta^{2/(p-2)}n^p) < \epsilon/3$. Moreover, we can take h sufficiently small such that $4T(T+4)K_n(h)\exp(4H_n(T+4)T) < \epsilon/3$. It follows immediately that

$$\mathbb{E}\left[\sup_{0\leq t\leq T}|e_h(t)|^2\right] < \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon,$$

which is the desired conclusion. □

Next, we proceed with Step 3, in which we establish that LS method has bounded moments.

Lemma 5.2. *Let Hypothesis 1.1, Hypothesis 1.2 and Hypothesis 3.1 holds. Then for each $p \geq 2$ there is a universal positive constant $C = C(p,T)$ such that for the LS method*

$$\mathbb{E}\left[\sup_{kh\in[0,T]}|Y_k|^{2p}\right] \leq C.$$

Proof. Using the split formulation of the LS scheme (7)–(8) we get

$$|Y_k^{\star}|^2 \leq |A^{(1)}(h,Y_k)|^2|Y_k|^2 + 2\left\langle A^{(1)}(h,Y_k)Y_k, A^{(2)}(h,Y_k)Y_k b(Y_k)\right\rangle + |A^{(2)}(h,Y_k)|^2|b(Y_k)|^2. \quad (57)$$

Then, applying the Cauchy-Schwartz inequality and Hypothesis 1.2 we arrive at

$$|Y_k^{\star}|^2 \leq |A^{(1)}(h,Y_k)|^2|Y_k|^2 + 2\sqrt{L_b}d|A^{(1)}(h,Y_k)||A^{(2)}(h,Y_k)||Y_k|(1+|Y_k|) + L_b|A^{(2)}(h,Y_k)|^2(1+|(Y_k)|^2).$$

Using (1.2) we see that

$$|A^{(1)}(h,Y_k)|^2 = \left|\operatorname{diag}\left(e^{ha_1(Y_k)},\ldots,e^{ha_d(Y_k)}\right)\right|^2 \leq \underbrace{d\exp(2TL_a)}_{:=L_{A^{(1)}}}. \quad (58)$$

In similar way, we deduce from the bound (26) that

$$\begin{aligned}|A^{(2)}(h,Y_k)|^2 &= \left|h\operatorname{diag}\left(\mathbf{1}_{\{E_1\}}(Y_k) + \mathbf{1}_{\{E_1^c\}}(Y_k)\Phi(h,a_1)(Y_k),\ldots,\mathbf{1}_{\{E_d\}}(Y_k) + \mathbf{1}_{\{E_d^c\}}(Y_k)\Phi(h,a_d)(Y_k)\right)\right|^2 \\ &\leq \sum_{j=1}^{d}\left(\mathbf{1}_{\{E_j^c\}}|h\Phi(h,a_1)(Y_k)|^2 + h^2\right) \leq \underbrace{2\exp(2L_aT)\sum_{j=1}^{d}\frac{1}{a_j^*} + dT^2}_{:=L_{A^{(2)}}}. \quad (59)\end{aligned}$$

Combining (5) of Hypothesis 1.2 with bounds (58) and (59) yields

$$|Y_k^{\star}|^2 \leq L_{A^{(1)}}|Y_k|^2 + 2d\sqrt{L_{A^{(1)}}L_{A^{(2)}}L_b}|Y_k|(1+|Y_k|) + L_{A^{(2)}}L_b(1+|Y_k|^2).$$

So, taking $\widetilde{C} \geq L_{A^{(1)}} + 2d\sqrt{L_{A^{(1)}}L_{A^{(2)}}L_b} + L_{A^{(2)}}L_b$ we can assert that

$$|Y_k^{\star}|^2 \leq \widetilde{C}(3|Y_k|^2 + |Y_k| + 1) \leq 6\widetilde{C}\left(|Y_k|^2 + 1\right) \leq C(1+|Y_k|^2). \quad (60)$$

Then, applying bound (60) in eq. (8) we arrive at

$$|Y_{k+1}|^2 \leq C\left(|Y_k|^2 + 1\right) + 2\left\langle Y_k^{\star}, g(Y_k^{\star})\Delta W_k\right\rangle + |g(Y_k^{\star})\Delta W_k|^2$$

Now, we choose two integers N, M such that $Nh \leq Mh \leq T$. So, adding backwards we get

$$
\begin{aligned}
|Y_N|^2 &\leq S_N \left(\sum_{j=0}^{N-1} (1 + |Y_j|^2) + 2 \sum_{j=0}^{N-1} \left\langle Y_j^\star, g(Y_j^\star)\Delta W_j \right\rangle + \sum_{j=0}^{N-1} \left| g(Y_j^\star)\Delta W_j \right|^2 \right) \\
S_N &:= \sum_{j=0}^{N-1} C^{N-j}
\end{aligned}
$$

Raising both sides to the power p and using the standard inequality (A.4) we obtain

$$
|Y_N|^{2p} \leq 6^p S_N^p \left(N^{p-1} \sum_{j=0}^{N-1} (1 + |Y_j|^{2p}) + \left| \sum_{j=0}^{N-1} \left\langle Y_j^\star, g(Y_j^\star)\Delta W_j \right\rangle \right|^p + N^{p-1} \sum_{j=0}^{N-1} \left| g(Y_j^\star)\Delta W_j \right|^{2p} \right) \tag{61}
$$

Now we will show that the second and third terms of the inequality (61) are bounded. We denote by $C = C(p,T)$ an generic positive constant that not depends on the step size h and whose value may changes between occurrences. Next, using the Bunkholder-Davis-Gundy inequality, (A.6) we see that

$$
\begin{aligned}
\mathbb{E}\left[\sup_{0 \leq N \leq M} \left| \sum_{j=0}^{N-1} \left\langle Y_j^\star, g(Y_j^\star)\Delta W_j \right\rangle \right|^p \right] &\leq C\mathbb{E}\left[\sum_{j=0}^{N-1} |Y_j^\star|^2 |g(Y_j^\star)|^2 h \right]^{p/2} \\
&\leq C h^{p/2} M^{p/2-1} \mathbb{E} \sum_{j=0}^{M-1} |Y_j^\star|^p (\alpha + \beta |Y_j^\star|^2)^{p/2} \\
&\leq 2^{p/2-1} C T^{p/2-1} h \mathbb{E} \sum_{j=0}^{M-1} (\alpha^{p/2} |Y_j^\star|^p + \beta^{p/2} |Y_j^\star|^{2p}) \\
&\leq Ch\mathbb{E} \sum_{j=0}^{M-1} (1 + 2|Y_j^\star|^p + |Y_j^\star|^{2p}) \\
&\leq Ch \sum_{j=0}^{M-1} \left[1 + \mathbb{E}|Y_j^\star|^{2p} \right] \\
&\leq C + Ch \sum_{j=0}^{M-1} \mathbb{E}|Y_j|^{2p}.
\end{aligned} \tag{62}
$$

Now, note that

$$
\mathbb{E}\left[\sup_{0 \leq N \leq M} \sum_{j=0}^{N-1} |Y_j|^{2p} \right] = \sum_{j=0}^{M-1} \mathbb{E}|Y_j|^{2p}. \tag{63}
$$

Hence, using Cauchy-Schwartz inequality, the monotone condition (3), bound (60) and the standard inequality (A.4), we obtain

$$\begin{aligned}
\mathbb{E}\left[\sup_{0\le N\le M}\sum_{j=0}^{N-1}\left|g(Y_j^\star)\Delta W_j\right|^{2p}\right] &= \mathbb{E}\sum_{j=0}^{M-1}\left|g(Y_j^\star)\Delta W_j\right|^{2p}\\
&\le \sum_{j=0}^{M-1}\mathbb{E}\left|g(Y_j^\star)\right|^{2p}\mathbb{E}\left|\Delta W_j\right|^{2p}\\
&\le Ch^p\sum_{j=0}^{M-1}\mathbb{E}\left[\alpha+\beta|Y_j^\star|^2\right]^p\\
&\le Ch^p\sum_{j=0}^{M-1}\mathbb{E}\left[\alpha^p+\beta^p|Y_j^\star|^{2p}\right]\\
&\le Ch^{p-1}+Ch^p\sum_{j=0}^{M-1}\mathbb{E}|Y_j|^{2p}. \qquad (64)
\end{aligned}$$

Thus, combining the bounds (62) and (64) with the inequality (61), we can assert that

$$\begin{aligned}
\mathbb{E}\left[\sup_{0\le N\le M}|Y_N|^{2p}\right] &\le C(M,T)+C(M,T)(1+h)\sum_{j=0}^{M-1}\mathbb{E}|Y_j|^{2p}\\
&\le C+C(1+h)\sum_{j=0}^{M-1}\mathbb{E}\left[\sup_{0\le N\le j}|Y_N|^{2p}\right]. \qquad (65)
\end{aligned}$$

Finally, using the discrete-type Gronwall inequality (A.9), we conclude that

$$\mathbb{E}\left[\sup_{0\le N\le M}|Y_N|^{2p}\right]\le C\exp(C(1+h)M)\le C\exp(C(1+T))<C,$$

since the constant C does not depend on h, the proof is complete. □

As the LS scheme has bounded moments, we now proceed whit Step 4, so we will obtain a convenient continuous extension of the LS method with bounded moments. Let $\{Y_k\}$ denote the LS solution of SDE (1). By Corollary 5.1, it is feasible to constructs a continuous extension for the LS approximation, from the time continuous extension of the EM (40). Moreover, we would expect that the moments of this new extension remains bounded.

Corollary 5.2. *Let Hypothesis 1.1, Hypothesis 1.2 and Hypothesis 3.1 holds and suppose* $0<h<1$ *and* $p\ge 2$. *Then there is a continuous extension* $\overline{Y}(t)$ *of* $\{Y_k\}$ *and a positive constant* $C=C(T,p)$ *such that*

$$\mathbb{E}\left[\sup_{0\le t\le T}|\overline{Y}(t)|^{2p}\right]\le C.$$

Proof. We take $t=s+t_k$ in $[0,T]$, $\Delta W_k(s):=W(t_k+s)-W(t_k)$ and $0\le s<h$. Then we define

$$\overline{Y}(t_k+s):=Y_k+s\varphi_{f_h}(Y_k)+g_h(Y_k)\Delta W_k(s), \qquad (66)$$

as a continuous extension of the LS scheme. We proceed to show that $\overline{Y}(t)$ has bounded moments. By Lemma 3.1, we have $Y_k^\star=Y_k+h\varphi_{f_h}(Y_k)$. Then for $\gamma=s/h$, it follows that

$$\begin{aligned}
Y_k+s\varphi_{f_h}(Y_k) &= \gamma(Y_k+h\varphi_{f_h}(Y_k))+(1-\gamma)Y_k\\
&=\gamma Y_k^\star+(1-\gamma)Y_k.
\end{aligned}$$

Hence, we can rewrite the continuous extension (66) as

$$\overline{Y}(t) = \gamma Y_k^\star + (1-\gamma)Y_k + g_h(Y_k)\Delta W_k(s).$$

Combining this relation with the inequalities (60) and (A.4), we arrive at

$$\begin{aligned}|\overline{Y}(t_k+s)|^2 &\le 3\left[\gamma C + (\gamma C + 1 - \gamma)\,|Y_k|^2 + |g_h(Y_k)\Delta W_k(s)|^2\right]\\ &\le C + C\left(|Y_k|^2 + |g_h(Y_k)\Delta W_k(s)|^2\right).\end{aligned}$$

Thus,

$$\begin{aligned}\sup_{0\le t\le T}|\overline{Y}(t)|^{2p} &\le \sup_{0\le kh\le T}\left[\sup_{0\le s\le h}|\overline{Y}(t_k+s)|^{2p}\right]\\ &\le \sup_{0\le kh\le T}\left[\sup_{0\le s\le h} C\left(1+|Y_k|^{2p}+|g_h(Y_k)\Delta W_k(s)|^{2p}\right)\right],\end{aligned} \tag{67}$$

for $t\in[0,T]$. Now taking a non negative integer $0\le k\le N$ such that $0\le Nh\le T$. From the bond (67), we get

$$\sup_{0\le t\le T}|\overline{Y}(t)|^{2p} \le C\left(1+\sup_{0\le kh\le T}|Y_k|^{2p}+\sup_{0\le s\le h}\sum_{j=0}^{N}|g_h(Y_j)\Delta W_j(s)|^{2p}\right). \tag{68}$$

So, using the Doob's Martingale inequality (A.5), Lemma 5.2 and that g_h is a locally Lipschitz function, we can bound each term of the inequality (68), as follows

$$\begin{aligned}\mathbb{E}\left[\sup_{0\le s\le h}|g(Y_j)\Delta W_j(s)|^{2p}\right] &\le \left(\frac{2p}{2p-1}\right)^{2p}\mathbb{E}|g_h(Y_j)\Delta W_j(h)|^{2p}\\ &\le C\mathbb{E}|g_h(Y_j)|^{2p}\mathbb{E}|\Delta W_j(h)|^{2p}\\ &\le Ch^p\left(1+\mathbb{E}|Y_j|^{2p}\right)\\ &\le Ch,\end{aligned} \tag{69}$$

for each $j\in\{0,\dots,N\}$. Since $Nh\le T$, combining the bounds (68) and (69) we get the desired conclusion. □

Once we have carried out all the previous steps, we can prove the Theorem 5.1 by Step 5.

Proof (of Theorem 5.1). First, note that by inequality (A.4), we have

$$\mathbb{E}\left[\sup_{0\le t\le T}|\overline{Y}(t)-y(t)|^2\right] \le 2\mathbb{E}\left[\sup_{0\le t\le T}|\overline{Y}(t)-y_h(t)|^2\right] + 2\mathbb{E}\left[\sup_{0\le t\le T}|y_h(t)-y(t)|^2\right]. \tag{70}$$

Using Lemma 5.1, which was established in the Step 2, yields

$$\lim_{h\to 0}\mathbb{E}\left[\sup_{0\le t\le T}|y_h(t)-y(t)|^2\right] = 0. \tag{71}$$

It remains to prove that the first term of the right hand side in inequality (70) decreases to zero when h tends to zero. Recalling that:

i) By Lemma 5.1, the solution of the modified SDE (46), y_h, has p-bounded moments ($p\ge 2$).

ii) By Corollary 5.2, the LS continuous extension for the SDE (1), $\overline{Y}(t)$, has bounded moments and it is equivalent to the EM extension for the modified SDE (46).

Hence, we can apply Theorem 4.1 to conclude that

$$\lim_{h\to 0} \mathbb{E}\left[\sup_{0\le t\le T} |\overline{Y}(t) - y_h(t)|^2\right] = 0. \tag{72}$$

Finally, combining the limits (71) and (72) with inequality (70) gives

$$\begin{aligned}\lim_{h\to 0} \mathbb{E}\left[\sup_{0\le t\le T} |\overline{Y}(t) - y(t)|^2\right] \le & 2 \lim_{h\to 0} \mathbb{E}\left[\sup_{0\le t\le T} |\overline{Y}(t) - y_h(t)|^2\right] \\ & + 2 \lim_{h\to 0} \mathbb{E}\left[\sup_{0\le t\le T} |y_h(t) - y(t)|^2\right] = 0,\end{aligned}$$

which proves the theorem. □

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Appendix A Useful Inequalities

Hölder.

$$\mathbb{E}[X^T Y] \leq \left(\mathbb{E}|X|^p\right)^{\frac{1}{p}} \left(\mathbb{E}|X|^q\right)^{\frac{1}{q}} . \tag{A.1}$$

Young.

$$|a||b| \leq \frac{\delta}{p}|a|^r + \frac{\delta}{q\delta^{q/p}}|b|^q. \tag{A.2}$$

Minkowski.

$$\left(\mathbb{E}|X+Y|^p\right)^{\frac{1}{p}} \leq \left(\mathbb{E}|X|^p\right)^{\frac{1}{p}} + \left(\mathbb{E}|Y|^p\right)^{\frac{1}{p}}. \tag{A.3}$$

A standard inequality. Fix $1 < p < \infty$ and consider a sequence of real numbers $\{a_i\}_{i=1}^N$ with $N \in \mathbb{N}$. Then one can formulate this usefully inequality

$$\left(\sum_{j=1}^N a_j\right)^p \leq N^{p-1}\sum_{j=1}^N a_j^p. \tag{A.4}$$

Doob's Martingale Inequality. Let $\{M_t\}_{t\geq 0}$ be a $\mathbb{R}^d$-valued martingale. Let $[a,b]$ be a bounded interval in $\mathbb{R}_+$. If $p > 1$ and $M_t \in L^p(\Omega;\mathbb{R}^d)$ then

$$\mathbb{E}\left(\sup_{a\leq t\leq b}|M_t|^p\right) \leq \left(\frac{p}{p-1}\right)^p \mathbb{E}|M_b|^p. \tag{A.5}$$

Burkholder–Davis–Gundy inequality. Let $g \in \mathcal{L}(\mathbb{R}_+;\mathbb{R}^{d\times m})$. Define for $t \geq 0$

$$x(t) = \int_0^t g(s)dW(s) \quad \text{and} \qquad A(t) = \int_0^t |g(s)|^2 ds. \tag{A.6}$$

Then for all $p > 0$, there exist universal positive constants c_p, C_p such that

$$c_p\mathbb{E}|A(t)|^{\frac{p}{2}} \leq \mathbb{E}\left[\sup_{0\leq s\leq t}|x(s)|^p\right] \leq C_p\mathbb{E}|A(t)|^{\frac{p}{2}}, \tag{A.7}$$

for all $t \leq 0$. In particular, one may take

$$\begin{aligned}
&c_p = (p/2)^p, && C_p = (32/p)^{\frac{p}{2}} && \text{if } 0 < p < 2;\\
&c_p = 1, && C_p = (32/p)^{\frac{p}{2}} && \text{if } p = 2;\\
&c_p = (2p)^{-\frac{p}{2}}, && C_p = \frac{p+1}{2(p-1)^{\frac{p}{2}}} && \text{if } p > 2.
\end{aligned}$$

Gronwall inequality. Let $T > 0$ and $c \geq 0$. Let $u(\Delta)$ be a Borel measurable bounded nonnegative function on $[0,T]$, and let v be a nonnegative integrable function on $[0,T]$ If

$$u(t) \leq c + \int_0^t v(s)u(s)ds \qquad \forall t \in [0,T],$$

then

$$u(t) \leq c\exp\left(\int_0^t v(s)ds\right) \qquad \forall t \in [0,T]. \tag{A.8}$$

Discrete Gronwall Inequality. Let M be a positive integer. Let u_k and v_k be non-negative numbers for $k = 0,1,\ldots,M$. If

$$u_k \leq u_0 + \sum_{j=0}^{k-1} u_j v_j$$

then

$$u_k \leq u_0 \exp\left(\sum_{j=0}^{k-1} v_j\right). \tag{A.9}$$