Strong Convergence and Almost Sure Stability of the Explicit Linear Steklov Method for SDEs under non-globally Lipschitz Coefficients.☆

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Abstract

We present an explicit and easily implementable numerical method for solving stochastic differential equations (SDEs) with non-globally Lipschitz coefficients. A linear version of the Steklov average under a split-step formulation supports our new solver. The Linear Steklov (LS) method converges strongly with a standard one-half order. Also, we study the almost sure asymptotic stability and to emphasize his performance we use models from population dynamics and non linear stochastic oscillators.

Keywords: stochastic differential equations; explicit methods; strong convergence; almost surely asymptotic stability.

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1. Introduction

Applications of Monte Carlo type simulations [11, 12] as Brownian Dynamics [6] require fast numerical methods with low computational cost — excluding the use of implicit schemes in the majority of cases. The Euler-Maruyama (EM) method is the most popular in such simulations due to its simple algebraic structure, cheap computational cost and acceptable convergence rate under global Lipschitz conditions. However, if the drift or diffusion coefficients of stochastic differential equations (SDEs) are super-linear, then the EM approximation diverges in the mean square sense [17, 19]. In most applications, the coefficients of the stochastic models in finances, biology or physics have locally Lipschitz coefficients with super-linear growth. Therefore, recent research has been focused on modifying the EM method to obtain strong convergence under these conditions keeping its simple structure and its low computational cost. In the last years, several methods have been developed in this direction: the family of Tamed schemes [16, 18, 32, 36, 37], a special type of balanced method [35], the stopped scheme [23] and a truncated Euler method [25]. In these works, the strong convergence of the proposed method is proved using the theory developed by Higham, Stuart and Mao in [14] or by means of the new approach given by Hutzenthaler and Jentzen in [16]. Both techniques obtain the property of strong convergence by proving boundedness moments of the numerical and analytical solution of the underlying SDE. In spite of the recent work in this subject, it is still necessary to get more accurate numerical methods for SDE under super-linear growth and non-globally Lipschitz coefficients.

In this chapter, we develop an explicit method based on a linear version of the Steklov method proposed in [8] for the vector Itô stochastic differential equation:

$$dy(t) = f(y(t))dt + g(y(t))dW(t), \quad 0 \leq t \leq T, \quad y(0) = y_0, \tag{1}$$

where $(f^{(1)}, \ldots, f^{(d)}) : \mathbb{R}^d \to \mathbb{R}^d$ is one sided Lipschitz and $g = (g^{(j,i)})_{j\in\{1,\ldots,d\}, i\in\{1,\ldots,m\}} : \mathbb{R}^d \to \mathbb{R}^{d\times m}$ is global Lipschitz. Also we assume that each component function $f^{(j)}$ can be written of the form

$$f^{(j)}(x) = a_j(x)x^{(j)} + b_j(x^{(-j)}), \tag{2}$$

where a_j and b_j are two scalar functions in $\mathbb{R}^d$ and $x^{(-j)} = \left(x^{(1)}, \ldots, x^{(j-1)}, x^{(j+1)}, \ldots x^{(d)}\right)$. The paper is organized as follows: In section 2, we give known results that are essential for our purposes. In section 3, we construct the new explicit method and prove the always existence of a succession of the Linear Steklov approximation as well as local Lipschitz conditions for its coefficients. In section 4, we prove the strong convergence of the LS method with one-half order using the Higham, Mao and Mao (HMS) technique and in section 5, its convergence rate is obtained. In section 6, we analyze numerically the accuracy and efficiency of the proposed method applied to stochastic differential equations with super-linear growth and locally Lipschitz coefficients. Finally we give some conclusions.

2. General Settings

Throughout this paper, we work with a standard setup, that is, $y(t) \in \mathbb{R}^d$ for each t and $W(t)$ is a m-dimensional standard Brownian motion on a filtered and complete probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$, with the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ generated by the Brownian process. Moreover, we denote the norm of a vector $y \in \mathbb{R}^d$ and the Frobenious norm of a matrix $G \in \mathbb{R}^{d\times m}$ by $|y|$ and $|G|$ respectively. The usual scalar product of two vectors $x, y \in \mathbb{R}^d$ is denoted by

$\langle x, y\rangle$. The complement of a set E is denoted by E^c and the indicator function of the set E is denoted by $\mathbf{1}_{\{E\}}$. In the following, we recall some classical results about the moment boundedness, existence and uniqueness of the solution of the stochastic differential system (1), see [14, 24, 26]. We also state some theorems about the strong convergence of the Euler-Maruyama method given by Higham et al. in [14] which will be useful to prove the strong convergence of the Linear Steklov method.

Let us start by assuming the following:

Hypothesis 2.1. The coefficients of SDE (1) satisfy the conditions:

(H-1) The functions f, g are in the class $C^1(\mathbb{R}^d)$.

(H-2) **Local, global Lipschitz condition**. For each integer n, there is a positive constant $L_f = L_f(n)$ such that

$$|f(u) - f(v)|^2 \leq L_f |u - v|^2 \qquad \forall u, v \in \mathbb{R}^d, \qquad |u| \vee |v| \leq n,$$

and there is a positive constant L_g such that

$$|g(u) - g(v)|^2 \leq L_g |u - v|^2, \qquad \forall u, v \in \mathbb{R}^d.$$

(H-3) **Monotone condition.** There exist two positive constants α and β such that

$$\langle u, f(u)\rangle + \frac{1}{2}|g(u)|^2 \leq \alpha + \beta |u|^2, \qquad \forall u \in \mathbb{R}^d. \tag{3}$$

Under Hypothesis 2.1 we can assure existence and uniqueness of the solution of continuous system (1). Next we state the bounds on the moments of the solution of (1).

Theorem 2.1. *Assume Hypothesis 2.1 then for all $y(0) = y_0 \in \mathbb{R}^d$ there exists a unique global solution $\{y(t)\}_{t\geq 0}$ to SDE (1). Moreover, the solution has the following properties for any $T > 0$,*

$$\mathbb{E}\,|y(T)|^2 < \left(|y_0|^2 + 2\alpha T\right) e^{2\beta T},$$

and

$$\mathbb{P}\left[\tau_n \leq T\right] \leq \frac{\left(|y_0|^2 + 2\alpha T\right) e^{2\beta T}}{n},$$

where n is any positive integer and $\tau_n := \inf\{t \geq 0 : |y(t)| > n\}$.

Theorem 2.2. *Let $p \geq 2$ and $x_0 \in L^p(\Omega, \mathbb{R}^d)$. Assume that there exits a constant $C > 0$ such that for all $(x, t) \in \mathbb{R}^d \times [t_0, T]$,*

$$\langle x, f(x, t)\rangle + \frac{p-1}{2}|g(x, t)|^2 \leq C(1 + |x|^2).$$

Then

$$\begin{aligned} \mathbb{E}\left[\sup_{0\leq t\leq T} |y(t)|^p\right] &\leq C\left(1 + \mathbb{E}|y_0|^p\right), \\ \mathbb{E}|y(t)|^p &\leq 2^{\frac{p-2}{2}}\left(1 + \mathbb{E}|y_0|^p\right) e^{Cpt}, \qquad \forall t \in [0, T]. \end{aligned}$$

Hypothesis 2.2. The SDE (1) and the EM solution (37) satisfies

$$\mathbb{E}\left[\sup_{0\le t\le T}|y(t)|^p\right]<\infty,\quad \mathbb{E}\left[\sup_{0\le t\le T}|Y^{EM}(t)|^p\right]<\infty,\quad \mathbb{E}\left[\sup_{0\le t\le T}|\overline{Y}^{EM}(t)|^p\right]<\infty,\qquad \forall p\ge 1.$$

Theorem 2.3 ([26, Thm. 2.2]). *Let Hypothesis 2.1 holds. Then for all $y(0)=y_0\in\mathbb{R}^d$ given, there exist a unique global solution $\{y(t)\}_{t\ge 0}$ to SDE(1). Moreover, the solution has the following properties for any $T>0$,*

$$\mathbb{E}\,|y(T)|^2<\left(|y_0|^2+2\alpha T\right)\exp(2\beta T),$$

and

$$\mathbb{P}\left[\tau_n\le T\right]\le\frac{\left(|y_0|^2+2\alpha T\right)\exp(2\beta T)}{n},$$

where n is any positive integer and $\tau_n:=\inf\{t\ge 0:|y(t)|>n\}$.

Theorem 2.4 ([24, Thm. 2.4.1]). *Let $p\ge 2$ and $x_0\in L^p(\Omega,\mathbb{R}^d)$. Assume that there exits a constant $C>0$ such that for all $(x,t)\in\mathbb{R}^d\times[t_0,T]$,*

$$\langle x,f(x,t)\rangle+\frac{p-1}{2}|g(x,t)|^2\le C(1+|x|^2).$$

Then

$$\mathbb{E}|y(t)|^p\le 2^{\frac{p-2}{2}}\left(1+\mathbb{E}|y_0|^p\right)\exp(Cpt)\qquad \text{for all } t\in[0,T].$$

Lemma 2.1 ([14, Lem 3.2]). *Under Hypothesis 2.1, for each $p\ge 2$, there is a $C=C(p,T)$ such that*

$$\mathbb{E}\left[\sup_{0\le t\le T}|y(t)|^p\right]\le C\left(1+\mathbb{E}|y_0|^p\right).$$

3. Construction of the Linear Steklov method

For simplicity, we begin the construction of the Linear Steklov method (LS) considering the scalar case of SDE (1), that is, when $d=m=1$, also, to shorten notation we use a,b instead a_j,b_j. Tough this ideas, we will generalize to higher dimensions. Let $0=t_0<t_1<\cdots<t_N=T$ a partition of the interval $[0,T]$ with constant step-size $h=T/N$ and such that $t_k=kh$ for $k=0,\ldots,N$. The main idea of the LS approximation consists in estimating the drift coefficient of (1) by

$$f(y(t))\approx\varphi_f(y(t_{\eta_+(t)}))=\left(\frac{1}{y(t_{\eta_+(t)})-y(t_{\eta(t)})}\int_{y(t_{\eta(t)})}^{y(t_{\eta_+(t)})}\frac{du}{a(y(t_{\eta(t)}))u+b}\right)^{-1},\qquad t\in[0,T], \tag{4}$$

where

$$\eta(t) := k \text{ for } t \in [t_k, t_{k+1}), \quad k \geq 0,$$
$$\eta_+(t) := k+1 \text{ for } t \in [t_k, t_{k+1}), \quad k \geq 0.$$

So we define the LS method for the scalar version of the SDE (1) using a split-step formulation as follows

$$Y_k^\star = Y_k + h\varphi_f(Y_k^\star), \tag{5}$$
$$Y_{k+1} = Y_k^\star + g(Y_k^\star)\Delta W_k, \tag{6}$$

with $Y_0 = y_0$ and $\varphi_f\left(Y_k^\star\right)$ defined by

$$\varphi_f\left(Y_k^\star\right) = \left(\frac{1}{Y_k^\star - Y_k}\int_{Y_k}^{Y_k^\star}\frac{du}{a(Y_k)u+b}\right)^{-1} \tag{7}$$

This scheme combines a split-step technique with a linear version of an exact deterministic method see [8, 29]. In detail, first we compute the discrete value $Y_k^\star$ using the Linear Steklov approximation (5), and next, Y_{k+1} is obtained by adding the stochastic increment $g(Y_k^\star)\Delta W_k$.

To higher dimensions, we adapt the same split step scheme (5)–(6) as follows. For each component equation $j \in \{1, \ldots, d\}$, on the iteration $k \in \{1, \ldots, N\}$ take

$$a_{j,k} = a_j\left(Y_k^{(1)}, \ldots, Y_k^{(d)}\right), \qquad b_{j,k} = b_j\left(Y_k^{(-j)}\right). \tag{8}$$

So, define $\varphi_f(Y_k^\star) = \left(\varphi_{f^{(1)}}(Y_k^\star), \ldots, \varphi_{f^{(d)}}(Y_k^\star)\right)$ by

$$\varphi_{f^{(j)}}\left(Y_k^\star\right) = \left(\frac{1}{Y_k^{\star(j)} - Y_k^{(j)}}\int_{Y_k^{(j)}}^{Y_k^{\star(j)}}\frac{du}{a_{j,k}u+b_{j,k}}\right)^{-1}. \tag{9}$$

It is worth mentioning that even this formulation is semi implicit, we always can derive a explicit version. The next result deals with this issue. To simplify notation, we define $A^{(1)} = A^{(1)}(h,u)$, $A^{(2)} = A^{(2)}(h,u)$ and $b = b(u)$ by

$$A^{(1)} := \begin{pmatrix} e^{ha_1(u)} & & 0 \\ & \ddots & \\ 0 & & e^{ha_d(u)} \end{pmatrix},$$
$$A^{(2)} := \begin{pmatrix} \left(\dfrac{e^{ha_1(u)}-1}{a_1(u)}\right)\mathbf{1}_{\{E_1^c\}} & & 0 \\ & \ddots & \\ 0 & & \left(\dfrac{e^{ha_d(u)}-1}{a_d(u)}\right)\mathbf{1}_{\{E_d^c\}} \end{pmatrix} + h\begin{pmatrix} \mathbf{1}_{\{E_1\}} & & 0 \\ & \ddots & \\ 0 & & \mathbf{1}_{\{E_d\}} \end{pmatrix}, \tag{10}$$
$$E_j := \{x \in \mathbb{R}^d : a_j(x) = 0\}, \qquad b(u) := \left(b_1(u^{(-1)}), \ldots, b_d(u^{(-d)})\right)^T.$$

Also we will need the following results from [22, Thm 2.1], [9, Thm. 1]. The first theorem will help us with the singularities of set E_j in the case where all elements of this set are limit points.

Theorem 3.1 (Multivariate L'hôpital's Rule)**.** *Let $\mathcal{N}$ be a neighborhood in $\mathbb{R}^2$ containing a point $\mathbf{q}$ at which two differentiable functions $f : \mathcal{N} \to \mathbb{R}$ and $g : \mathcal{N} \to \mathbb{R}$ are zero. Set*

$$C = \{x \in \mathcal{N} : f(x) = g(x) = 0\},$$

and suppose that C is a smooth curve through $\mathbf{q}$.

Suppose there exist a vector $\mathbf{v}$ not tangent to C at $\mathbf{q}$ such that the directional derivative $D_{\mathbf{v}}g$ of g in the direction of $\mathbf{v}$ is never zero within $\mathcal{N}$. Also we assume that $\mathbf{q}$ is a limit point of $\mathcal{N} \setminus C$. Then

$$\lim_{(x,y)\to\mathbf{q}} \frac{f(x,y)}{g(x,y)} = \lim_{\substack{(x,y)\to\mathbf{q} \\ (x,y)\in\mathcal{N}\setminus C}} \frac{D_{\mathbf{v}}f}{D_{\mathbf{v}}g}$$

if the latter limit exist.

For the second auxiliary we will need the following concepts.

Definition 3.1 (Directional derivative referred at a point)**.** Let $u, \mathbf{q} \in \mathbb{R}^2$ and α the positive angle respect to the x-axis and the segment $\overline{u\mathbf{q}}$. We denote by

$$f_\alpha(u) = \cos(\alpha)\frac{\partial f}{\partial u^{(1)}}(u) + \sin(\alpha)\frac{\partial f}{\partial u^{(2)}}(u) = \frac{\langle q-u, \nabla f(u)\rangle}{|u-q|}$$

the *directional derivative respect to the point* $\mathbf{q}$ *on* u.

Definition 3.2 (Star-like set)**.** A set $S \subset \mathbb{R}^2$ is *star-like* with respect a point $\mathbf{q}$, if for each point $s \in S$ the open segment $\overline{s\mathbf{q}}$ is in S.

Whit this in mine, second theorem give us a way to analyze isolated singularities.

Theorem 3.2. *Let $\mathbf{q} \in \mathbb{R}^2$ and let f,g be functions whose domains include a set $S \subset \mathbb{R}^2$ which is star-like with respect to the point $\mathbf{q}$. Suppose that on S the functions are differentiable and that the directional derivative of g with respect to $\mathbf{q}$ is never zero. With the understanding that all limits are taken from within on S at $\mathbf{q}$ and if*

(i) $f(\mathbf{q}) = g(\mathbf{q}) = 0$,

(ii) $\displaystyle\lim_{x\to\mathbf{q}} \frac{f_\alpha(x)}{g_\alpha(x)} = L$,

then

$$\lim_{x\to\mathbf{q}} \frac{f(x)}{g(x)} = L.$$

With this on mind, we additionally require the following.

Hypothesis 3.1. For each component function $f^{(j)} : \mathbb{R}^d :\to \mathbb{R}$ with $j \in \{1, \ldots, d\}$:

(A-1) There are two locally Lipschitz functions of class $C^1(\mathbb{R}^d)$ denoted by $a_j : \mathbb{R}^d \to \mathbb{R}$ and $b_j : \mathbb{R}^{d-1} \to \mathbb{R}$ such that the j-component of the drift function can be rewritten as in (2).

(A-2) There is a positive constant L_a such that

$$a_j(x) \le L_a, \qquad \forall x \in \mathbb{R}^d.$$

(A-3) Each function $b_j(\cdot)$ satisfies the linear growth condition

$$|b_j(x^{(-j)})|^2 \leq L_b(1+|x|^2), \qquad \forall x \in \mathbb{R}^d.$$

Hypothesis 3.2. The set $E_j := \{x \in \mathbb{R}^d : a_j(x) = 0\}$ satisfies either:

(i) All point $q \in E_j$ is a non isolated zero of a_j and:

- the set
$$D := \{u \in B_r(q) : e^{ha_j(u)} - 1 = a_j(u) = 0\},$$
is a smooth curve through q.
- The canonical vector e_j is not tangent to D.
- For each $q \in E_j$, there is an open ball with center on q and radio r $B_r(q)$, such that and
$$a_j \neq 0, \qquad \frac{\partial a_j(u)}{\partial u^{(j)}} \neq 0, \qquad \forall u \in B_r(q) \setminus D.$$

(ii) All point $q \in E_j$ is a isolated zero of a_j and:

- For each $q \in E_j$, q is not a limit point of the set $E_\alpha := \{x \in \mathbb{R}^d : (a_j)_\alpha(x) = 0\}$.
- For each $q \in E_j$ there is a star-like set respect to q E_q, such that the directional derivative respect to q satisfies
$$(a_j)_\alpha(x) \neq 0, \qquad \forall x \in E_q.$$

Lemma 3.1. *Let Hypotheses 2.1, 3.1 and 3.2 holds, and $A^{(1)}$, $A^{(2)}$, b defined by (10). Then given $u \in \mathbb{R}^d$, the equation*

$$v = u + h\varphi_f(v), \tag{11}$$

has a unique solution

$$v = A^{(1)}(h,u)u + A^{(2)}(h,u)b(u). \tag{12}$$

If we define the functions $F_h(\cdot)$, $\varphi_{f_h}(\cdot)$ and $g_h(\cdot)$ by

$$F_h(u) = v, \qquad \varphi_{f_h}(u) = \varphi_f(F_h(u)), \qquad g_h(u) = g(F_h(u)), \tag{13}$$

then $F_h(\cdot)$, $\varphi_{f_h}(\cdot)$, $g_h(\cdot)$ are local Lipschitz functions and for all $u \in \mathbb{R}^d$ and each h fixed, there is a positive constant L_Φ such that

$$|\varphi_{f_h}(u)| \leq L_\Phi |f(u)|. \tag{14}$$

Moreover, for each h fixed, there are positive constants α^* and β^* such that

$$\langle \varphi_{f_h}(u), u\rangle \vee |g_h(u)|^2 \leq \alpha^* + \beta^*|u|^2, \qquad \forall u \in \mathbb{R}^d. \tag{15}$$

Proof. Let us first, prove that (12) is solution of Equation (11). To this end note that for each $j \in \{1, \ldots, d\}$

$$v^{(j)} = u^{(j)} + h\varphi_{f^{(j)}}(v)$$

So, using in Equation (12) that

$$\varphi_{f^{(j)}}(v) = \left(\frac{1}{v^{(j)} - u^{(j)}} \int_{u^{(j)}}^{v^{(j)}} \frac{dz}{a_j(u)z + b_j(u^{(-j)})} \right)^{-1}. \tag{16}$$

After some algebraic manipulations we arrive at

$$v^{(j)} = \exp(ha_j(u))u^{(j)} + \left[\left(\frac{\exp(ha_j(u)) - 1}{a_j(u)} \right) \mathbf{1}_{\{E_j^c\}} + h\mathbf{1}_{\{E_j\}} \right] b_j(u^{(-j)}), \tag{17}$$

for each $j \in \{1, \ldots, d\}$, which is the j-component of the vector $A^{(1)}(h,u)u + A^{(2)}(h,u)b(u)$. Next, we will prove the inequality (14). Since

$$\varphi_{f_h}^{(j)}(u) = \frac{F_h^{(j)}(u) - u^{(j)}}{\displaystyle\int_{u^{(j)}}^{F_h^{(j)}(u)} \frac{dz}{a_j(u)z + b_j(u^{(-j)})}} \qquad .$$

We first see that if $u \in E_j$, then $\varphi_{f_h}^{(j)}(u) = b_j(u^{(-j)}) = f(u)$. Then taking $C \geq 1$ we have the conclusion. On the other hand, if $u \in E_j^c$, then

$$\varphi_{f_h}^{(j)}(u) = \frac{(F_h^{(j)}(u) - u^{(j)})a_j(u)}{\underbrace{\ln\left(a_j(u)F_h^{(j)}(u) + b_j(u^{(-j)})\right)}_{:=Ter1} - \ln\left(a_j(u)u^{(j)} + b_j(u^{(-j)})\right)} \tag{18}$$

Now consider in Equation (18) the term labeled $Ter1$, and observe that

$$\begin{aligned} Ter1 &= \ln\left\{ a_j(u) \left[e^{ha_j(u)}u^{(j)} + \left(\frac{e^{ha_j(u)} - 1}{a_j(u)} \right) b_j(u^{(-j)}) \right] + b_j\left(u^{(-j)}\right) \right\} \\ &= ha_j(u) + \ln\left(f^{(j)}(u)\right). \end{aligned} \tag{19}$$

Combining the relations (18) and (19), after algebraic manipulations we arrive at

$$\varphi_{f_h}^{(j)}(u) = \left(\frac{e^{ha_j(u)} - 1}{ha_j(u)} \right) f^{(j)}(u), \quad \forall u \in E_j^c. \tag{20}$$

Hence, it remains to prove that

$$\Phi(h, a_j)(u) := \frac{e^{ha_j(u)} - 1}{ha_j(u)}, \tag{21}$$

is bounded on $\mathbb{R}^d$ for each $j \in \{1, \ldots, d\}$. First wee see that under the Hypothesis 2.1 the operator Φ is continuous on E_j^c. Furthermore, for each fixed $u \in E_j^c$

$$\lim_{h \to 0} \frac{e^{ha_j(u)} - 1}{ha_j} = 1.$$

On the other hand, for each fixed h, by Hypothesis 3.2 we obtain one of the following cases:

CASE I:

$$\lim_{\substack{u\to u^*\\ u\in E_j^c}} \Phi(h,a_j)(u) = \lim_{\substack{u\to u^*\\ u\in E_j^c}} \frac{\frac{\partial a_j(u)}{\partial u^{(j)}} h e^{ha_j(u)}}{h\frac{\partial a_j(u)}{\partial u^{(j)}}} = 1. \tag{22}$$

CASE II:

$$\lim_{\substack{u\to u^*\\ u\in E_j^c}} \Phi(h,a_j)(u) = \lim_{\substack{u\to u^*\\ u\in E_j^c}} \frac{\left(e^{ha_j(u)}-1\right)_\alpha}{\left(ha_j(u)\right)_\alpha} = 1, \qquad \alpha = 0,\pi,2\pi,\dots \tag{23}$$

So, under this situation, we can say for each j that the function

$$f^{(j)}(u)\mathbf{1}_{\{E_j\}} + \left(\frac{e^{ha_j(u)}-1}{ha_j(u)}\right) f^{(j)}(u)\mathbf{1}_{\{E_j^c\}}$$

is continuous on $\mathbb{R}^d$ and bounded on E_j. Now, let

$$a_j^* := \inf_{u\in E_j^c}\{|a_j(u)|\}.$$

So, a_j^* satisfy one of the two following cases:

CASE a: $0 < a_j^* \leq L_a$.

CASE b: $a_j^* = 0$.

In the first case we see that

$$\frac{e^{ha_j(u)}-1}{ha_j(u)} \leq \frac{e^{hL_a}-1}{ha_j^*(u)} < \infty, \qquad \forall h\in(0,\infty).$$

For CASE II, we can apply an argument similar as in (22)–(23). Then there is $L_\Phi > 0$ such that

$$\left|\frac{e^{ha_j(u)}-1}{ha_j(u)}\right| < L_\Phi, \qquad \forall u\in\mathbb{R}^d. \tag{24}$$

Combining this fact with Equation (20), we obtain

$$|\varphi_{f_h^{(j)}}(u)| \leq \left|\frac{e^{ha_j(u)}-1}{ha_j(u)}\right| |f^{(j)}(u)| < L_\Phi |f^{(j)}(u)|, \qquad \forall u\in\mathbb{R}^d,$$

which prove inequality (14).

No, we prove the g_h is a locally Lipschitz function. By (H-1) in Hypothesis 2.1, g is a globally Lipschitz function, so

$$g_h(x) = g\left(F_h(x)\right),$$

is the composition of a continuous bounded function and a globally Lipschitz function, furthermore, note that

$$\begin{aligned}|g_h(u)-g_h(v)|^2 &\leq L_g|F_h(u)-F_h(v)|^2\\ &\leq 2L_g \underbrace{|A^{(1)}(h,u)u - A^{(1)}(h,v)v|^2}_{:=Ter_1} + 2L_g \underbrace{|A^{(2)}(h,u)b(u) - A^{(2)}(h,v)b(v)|^2}_{:=Ter_2}.\end{aligned} \tag{25}$$

Now, we work with each term of the right hand of inequality (25). First note that $A^{(1)}$ is a continuous differentiable function on all $\mathbb{R}^d$ so by the mean value Theorem we have

$$|A^{(1)}(h,u)u - A^{(1)}(h,v)v|^2 \leq \sup_{0\leq t\leq 1} |\partial A^{(1)}(h, u+t(v-u))|^2 |u-v|^2,$$

then, there is a positive constant $L_{A^{(1)}} = L_{A^{(1)}}(h,n)$ such that

$$|A^{(1)}(h,u)u - A^{(1)}(h,v)v|^2 \leq L_{A^{(1)}}|u-v|^2, \quad u,v\in\mathbb{R}^d, \quad |u|\vee|v|\leq n. \tag{26}$$

In the other hand,

$$\begin{aligned} Ter_2 &= \sum_{j=1}^{d} \Big[\mathbf{1}_{\{E_j^c\}}(u)\Phi(h,a_j)(u)b_j(u^{(-j)}) + h\mathbf{1}_{\{E_j\}}(u)b_j(u^{(-j)}) - \mathbf{1}_{\{E_j^c\}}(v)\Phi(h,a_j)(v)b_j(u^{(-j)}) \\ &\qquad\qquad - h\mathbf{1}_{\{E_j\}}(v)b_j(v^{(-j)}) \Big]^2 \\ &\leq 4\sum_{j=1}^{d} \Big[\Big(\mathbf{1}_{\{E_j^c\}}(u)\Phi(h,a_j)(u)b_j(u^{(-j)})\Big)^2 + \Big(h\mathbf{1}_{\{E_j\}}(u)b_j(u^{(-j)})\Big)^2 \\ &\qquad\qquad + \Big(\mathbf{1}_{\{E_j^c\}}(v)\Phi(h,a_j)(v)b_j(v^{(-j)})\Big)^2 + \Big(h\mathbf{1}_{\{E_j\}}(v)b_j(v^{(-j)}\Big)^2 \Big] \\ &\leq 4\sum_{j=1}^{d} \Big[\Big(\mathbf{1}_{\{E_j^c\}}(u)L_\Phi b_j(u^{(-j)})\Big)^2 + \Big(h\mathbf{1}_{\{E_j\}}(u)b_j(u^{(-j)})\Big)^2 + \Big(\mathbf{1}_{\{E_j^c\}}(v)L_\Phi b_j(v^{(-j)})\Big)^2 \\ &\qquad\qquad + \Big(h\mathbf{1}_{\{E_j\}}(v)b_j(v^{(-j)}\Big)^2 \Big]. \end{aligned} \tag{27}$$

Since each $b_j^2(\cdot)$ function is of class $C^1(\mathbb{R}^d)$, there is a constant $M_b = M_b(n)$ such that

$$|b_j(u)|^2 \leq M_b, \quad \forall u\in\mathbb{R}^d, \quad |u|\vee|v|\leq n, \quad j\in\{1,\cdots,d\}\ . \tag{28}$$

Putting this bound in inequality (27), we deduce that

$$\begin{aligned} Ter_2 &\leq 4\sum_{j=1}^{d}\left[2L_\Phi M_b + 2h^2 M_b\right] \\ &\leq \underbrace{8dM_b(n)(L_\Phi + h^2)}_{:=L_0}, \quad \forall u,v\in\mathbb{R}^d, \quad |u|\vee|v|\leq n. \end{aligned} \tag{29}$$

Then, combining inequalities (26) and (29) we arrive at

$$|g_h(u) - g_h(v)|^2 \leq L_{A^{(1)}}|u-v|^2 + L_0.$$

So, taking $L_{g_h} = L_{g_h}(h,n) \geq n^2 + 1 + L_0 + L_{A^{(1)}}$, we see that

$$|g_h(u) - g_h(v)|^2 \leq L_{g_h}(n)|u-v|^2, \quad \forall u,v\in\mathbb{R}^d, \quad |u|\vee|v|\leq n.$$

Then $g_h(\cdot)$ is a locally Lipschitz function. Furthermore, note that under some modifications, this argument also prove that $F_h(\cdot)$ is a Locally Lipschitz function, which also implies that φ_{f_h}

is a locally Lipschitz function. Finally, we prove the inequality (15). We first observe that by Hypotheses 2.1 and 3.1

$$\begin{aligned}\langle f(u), u\rangle &= \sum_{j=1}^{d} a_j(u)\left(u^{(j)}\right)^2 + \sum_{j=1}^{d} b_j(u)u^{(j)} \\ &\leq \alpha + \beta|u|^2,\end{aligned}$$

then

$$\langle b(u), u\rangle \leq \alpha + (\beta + L_a)|u|^2. \tag{30}$$

With this on mind, and using the inequality (14), we deduce that

$$\begin{aligned}\langle \varphi_{f_h}(u), u\rangle &= \sum_{j=1}^{d} \Phi(h, aj)(u) f^{(j)}(u)u^{(j)} \\ &\leq \sum_{j=1}^{d} L_\Phi L_a|u^{(j)}|+L_\Phi \langle b(u), u\rangle \\ &\leq L_\Phi L_a|u|+L_\Phi(\alpha + (L_a + \beta)|u|^2).\end{aligned}$$

So, taking $L_{\varphi_{f_h}} = L_{\varphi_{f_h}}(h) \geq 2L_\Phi \cdot \max\{L_a, \alpha, \beta\} + 1$, we obtain

$$\langle \varphi_{f_h}(u), u\rangle \leq L_{\varphi_{f_h}}(1 + |u|^2), \quad \forall u \in \mathbb{R}^d. \tag{31}$$

On the other hand, since g is globally Lipschitz it follows that

$$\begin{aligned}|g_h(u)|^2 &\leq 2|g(F_h(u)) - g(F_h(0)|^2+2|g(F_h(0)|^2 \\ &\leq 2L_g|F_h(u) - F_h(0)|^2+4|g(F_h(0)) - g(0)|^2+4|g(0)|^2 \\ &\leq 4L_g|F_h(u)|^2+8L_g|F_h(0)|^2+4|g(0)|^2.\end{aligned} \tag{32}$$

Now we bound each term in the right hand side of inequality (32). Let us first observe that by the monotone condition in Hypothesis 2.1

$$|g(0)|^2\leq 2\alpha. \tag{33}$$

On the other hand

$$F_h^{(j)}(0) = \frac{e^{ha_j(0)} - 1}{a_j(0)} b_j(0)\mathbf{1}_{\{E_j^c\}}(0) + hb_j(0)\mathbf{1}_{\{E_j\}}(0).$$

So, taking

$$a_0^* := \min_{\substack{j\in\{1,\cdots,d\} \\ a_j(0)\neq 0}} \{|a_j(0)|\}, \qquad b_0^* := \max_{j\in 1,\cdots,d} \{|b_j(0)|\}$$

we can deduce that

$$|F_h^{(j)}(0)|\leq \frac{b_0^*}{a_0^*} e^{hL_a}(1 + h), \quad \forall j \in \{1, \cdots, d\}.$$

Then

$$|F_h(0)|^2 \leq d\left(\frac{b_0^*}{a_0^*}\right)^2 e^{2hL_a}(1+h)^2. \tag{34}$$

In this line, since the operator $\Phi(h, a_j)$ is bounded, it follows that

$$\begin{aligned}F_h^{(j)}(u) &= e^{ha_j(u)}u^{(j)} + h\Phi(h,a_j)(u)b_j(u)\mathbf{1}_{\{E_j^c\}}(u) + hb_j(u)\mathbf{1}_{\{E_j\}}(u)\\ &\leq e^{ha_j(u)}|u^{(j)}| + hL_\Phi|b_j(u)|\mathbf{1}_{\{E_j^c\}}(u) + h|b_j(u)|\mathbf{1}_{\{E_j\}}(u).\end{aligned}$$

Then we deduce by Hypothesis 3.1 that

$$\begin{aligned}|F_h^{(j)}(u)|^2 &\leq 3e^{2hL_a}|u|^2 + (3h^2L_\Phi^2L_b + 3h^2L_b)(1+|u|^2)\\ &\leq 3h^2L_b(1+L_\Phi^2) + 3\left(e^{2hL_a} + h^2L_b(L_\Phi^2+1)\right)|u|^2.\end{aligned}$$

So, taking $L_F \geq 3d\max\{\exp(2hL_a), h^2L_b(L_\Phi^2+1)\}$, we obtain that

$$|F_h^{(j)}(u)|^2 \leq L_F(1+|u|^2). \tag{35}$$

Then, combining bounds (33),(34) and (35) we arrive at

$$|g_h(u)|^2 \leq 4L_gL_F(1+|u|^2) + 8L_gd\left(\frac{b_0^*}{a_0^*}\right)^2 e^{2hL_a}(1+h)^2 + 8\alpha.$$

Therefore, if $L_{g_h} \geq 4L_gL_F + 8L_gd\left(\frac{b_0^*}{a_0^*}\right)^2 e^{2hL_a}(1+h)^2 + 8\alpha$ wee see that

$$|g_h(u)|^2 \leq L_{g_h}(1+|u|^2) \tag{36}$$

Hence, from the inequalities (31) and (36) and taking for each fixed $h > 0$

$$\alpha^* := L_{\varphi_{f_h}} \vee L_{g_h}, \qquad \beta^* := 2\alpha^*$$

we have the desired conclusion. □ □

Remark 3.1. Note that if $b_j = 0$ in (A-1) then Hypotheses 3.1 and 3.2 are unnecessary to prove Lemma 3.1. Several applications as some stochastic Lotka-Volterra systems [27, 28], the Ginzburg-Landau SDE [20] or the damped Langevin Equations where the potential lacks of a constant term [18] have this form. By other hand, if $bj \neq 0$ then SDE as the stochastic SIR [34], the noisy Duffing-Van der Pol Oscillator [33], the stochastic Lorenz equation [10], among others follow this structure.

Remark 3.2. Note that by Lemma 3.1, we have that $\lim_{h\to 0}|f(x) - \varphi_{f_h}(x)| = 0$. Hence it is convenient to consider the following modified SDE

$$dy_h(t) = \varphi_{f_h}(y_h(t))dt + g_h(y_h(t))dW(t), \qquad y_h(0) = y_0, \qquad t \in [0,T],$$

as a perturbation of SDE (1). Moreover, the functions $\varphi_{f_h}(\cdot)$ and $g_h(\cdot)$ in (13) are respectively defined as the functions φ_f and g, but evaluated in the solution of $c = d + h\varphi(c)$, then we can rewrite the LS method (5)–(6) as

$$\begin{aligned}Y_k^\star &= Y_k + h\varphi_{f_h}(Y_k),\\ Y_{k+1} &= Y_k^\star + g_h(Y_k)\Delta W_k.\end{aligned}$$

We formalize these ideas in the following sections.

4. Strong Convergecne using the Higham-Mao-Stuart's technique

For the sake of clarity, we discuss in this section a technique to prove strong convergence of stochastic numerical methods, reported by Higham, Mao, and Stuart in [14]. In this seminal paper, the authors assume non-globally Lipschitz conditions. This kind of analysis is useful whenever moment bounds can be established for both, the Euler-Maruyama scheme and other method that can be shown to be "close" to it. A vast amount of literature has been cited this technique, some of these works are [3, 13, 15, 16, 18, 21, 26, 35], among others.

To review this technique, we begin recalling the definition of the Euler-Maruyama (EM) approximation and its continuous extension for SDE (1), which for this section, we denote by

$$X_{k+1} = X_k + hf(X_k) + g(X_k)\Delta W_k, \tag{37}$$

and two versions of a conveniently continuous extension used by Higham et al. in [14]

$$\begin{aligned}\overline{X}(t) &:= X_{\eta(t)} + (t - t_{\eta(t)})f(X_{\eta(t)}) + g(X_{\eta(t)})(W(t) - W_{\eta(t)}) \\ \eta(t) &:= k, \text{ for } t \in [t_k, t_{k+1}),\end{aligned} \tag{38}$$

and

$$\overline{X}(t) := X_0 + \int_0^t f(X_{\eta(s)})ds + \int_0^t g(X_{\eta(s)})dW(s).$$

So, with this notation we have $\overline{X}(t_k) = X_k$, see Figure 1.

Using the continuous extension (38) and the uniform mean square norm, the authors work with a stronger version of the ms-error

$$\mathbb{E}\left[\sup_{0\leq t\leq t} |y(t) - \overline{X}(t)|^2\right].$$

So, in order to prove the strong convergence of the EM method, the following assumptions are required.

Assumption 4.1. For each $R > 0$ there is a constant C_R, depending only on R, such that

$$|f(x) - f(y)|^2 \vee |g(x) - g(y)|^2 \leq C_R|x - y|^2, \quad \forall x, y \in \mathbb{R}^d \text{ with } |x| \vee |y| \leq R. \tag{39}$$

And for some $p > 2$, there is a constant A such that

$$\mathbb{E}\left[\sup_{0\leq t\leq T} |\overline{X}(t)|^p\right] \vee \mathbb{E}\left[\sup_{0\leq t\leq T} |y(t)|^p\right] \leq A. \tag{40}$$

In [14], the authors prove that the Assumption 4.1 is sufficient to ensure strong convergence for the EM scheme, namely

Theorem 4.1 ([14, Thm 2.2])**.** *Under Assumption 4.1, the EM scheme* (37) *with continuous extension* (38) *satisfies*

$$\lim_{h\to 0} \mathbb{E}\left[\sup_{0\leq t\leq T} |\overline{X}(t) - y(t)|^2\right] = 0. \tag{41}$$

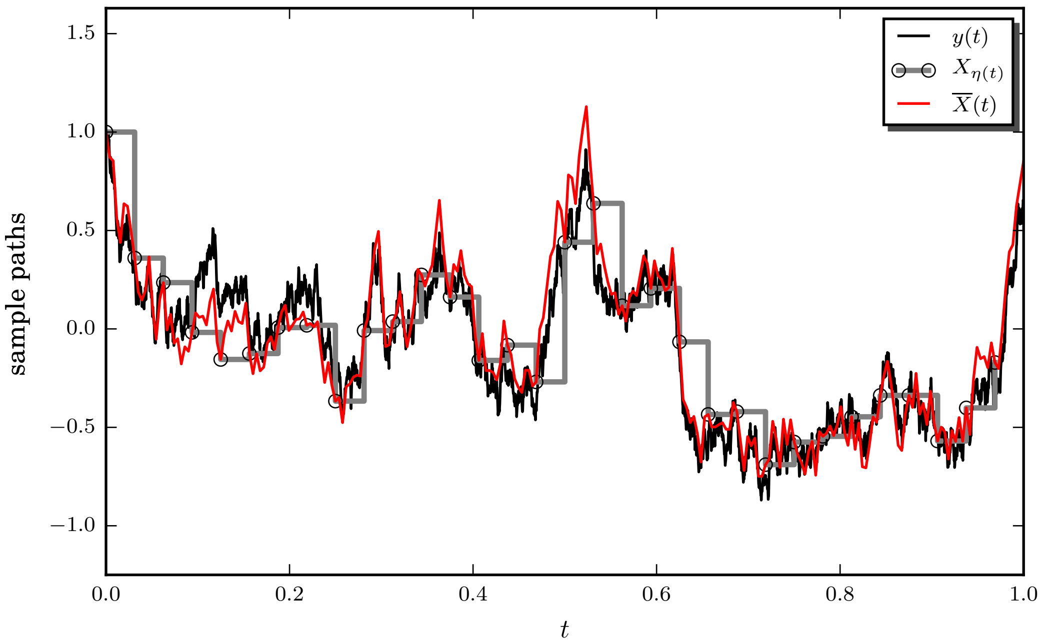

Figure 1: The red line represents the continuous extension of the EM scheme. The continuous gray line is the $X_{\eta(t)}$ process defined in (37).

Applying this result, the authors prove the strong convergence of an implicit split-step variant of the EM, the SSEM method. Their technique consist in proving each assertion of the following steps.

Step 1: The SSEM for SDE (1) is equivalent to the EM for the following conveniently SDE

$$dx_h(t) = f_h(x_h(t))dt + g_h(x_h(t))dW(t). \tag{42}$$

Step 2: The solution of the modified SDE (42) has bounded moments and it is "close" to y the sense of the uniform mean square norm $\mathbb{E}\left[\sup_{0\le t\le T}|\cdot|^2\right]$.

Step 3: Show that the SSEM method for the SDE (1) has bounded moments.

Step 4: There is a continuous extension of the SSEM, $\overline{Z}(t)$, with bounded moments.

Step 5: Use the above steps and Theorem 4.1 to conclude that

$$\lim_{h\to 0}\left\{\mathbb{E}\left[\sup_{0\le t\le T}|x_h(t)-y(t)|^2\right]+\mathbb{E}\left[\sup_{0\le t\le T}|\overline{Z}(t)-y_h(t)|^2\right]\right\}=0. \tag{43}$$

In the next section, using Theorem 4.1 and this technique, we will prove the strong convergence of the LS method (5)–(6).

5. Strong Convergence of the Linear Steklov Method

Here, we state and prove the main result of this chapter, the strong convergence of the LS method (5)–(6) for the solution of SDE (1). The main idea of the proof consist in applying the technique discussed in the previous section. We begin establishing the underlying convergence theorem.

Theorem 5.1. *Let Hypotheses 2.1 and 3.1 hold, consider the LS method (5)–(6) for the SDE* (1)*. Then there is a continuous-time extension $\overline{Y}(t)$ of the LS solution $\{Y_k\}$ for which $\overline{Y}(t_k) = Y_k$ and*

$$\lim_{h\to 0} \mathbb{E}\left[\sup_{0\le t\le T} |\overline{Y}(t) - y(t)|^2\right] = 0.$$

To proof this result, we initiate with the first step of the HMS technique, that is, we will show that the LS method for SDE (1) is equivalent to the EM scheme applied to the conveniently modified SDE

$$dy_h(t) = \varphi_{f_h}(y_h(t))dt + g_h(y_h(t))dW(t), \qquad y_h(0) = y_0, \qquad t \in [0, T]. \tag{44}$$

We formalize this as a Corollary of Lemma 3.1.

Corollary 5.1. *Let Hypotheses 2.1 and 3.1 holds, then the LS method for SDE* (1) *is equivalent to the EM scheme applied to the modified SDE* (44)*.*

Proof. Using the functions $\varphi_{f_h}(\cdot)$ and $g_h(\cdot)$ defined in (13) of Lemma 3.1, we can rewrite the LS method (5)–(6) as

$$Y_{k+1} = Y_k + h\varphi_{f_h}(Y_k) + g_h(Y_k)\Delta W_k,$$

which is the EM approximation for the modified SDE (44). □

Now we proceed with the Step 2, that is, we will prove that the solution of the modified SDE (44) has bounded moments and is close in uniform mean square norm to the solution of the SDE (1). In what follows we denote by C universal constant, that is, a positive constant independent on h which value could change in occurrences.

Lemma 5.1. *Let Hypotheses 2.1 and 3.1 holds. Then there is a constant $C = C(p, T) > 0$ and a sufficiently small step size h, such that for all $p > 2$*

$$\mathbb{E}\left[\sup_{0\le t\le T} |y_h(t)|^p\right] \le C\left(1 + \mathbb{E}|y_0|^p\right). \tag{45}$$

Moreover

$$\lim_{h\to 0} \mathbb{E}\left[\sup_{0\le t\le T} |y(t) - y_h(t)|^2\right] = 0. \tag{46}$$

Proof. The bound (45) follows from inequality (15) in Lemma 3.1 and Theorem 2.4. On the other hand, to prove (46) we will use the properties of φ_{f_h} and the Higham's stopping time technique employed in [14, Thm 2.2]. Note that by Lemma 3.1

$$\varphi_{f_h}(x) = \Phi(h, a_j)(u) f^{(j)}(u)\mathbf{1}_{\{E_j^c\}}(u) + f^{(j)}(u)\mathbf{1}_{\{E_j\}}(u).$$

Furthermore, by Hypothesis 3.2 and since $f \in C^1(\mathbb{R}^d)$, $\Phi(h,a_j)(\cdot)$ is bounded, hence, there is a positive constant $K_n$which depends only n such that for each $j \in \{1,\dots,d\}$

$$\begin{aligned}
|\varphi_{f_h}^{(j)}(u) - f^{(j)}(u)| &\leq \mathbf{1}_{\{E_j^c\}}(u)|f^{(j)}(u)||\Phi(h,a_j)(u) - 1| \\
&\leq \mathbf{1}_{\{E_j^c\}}(u)\left(L_\Phi + 1\right)|f(u)| \\
&\leq \mathbf{1}_{\{E_j^c\}}(u) K_n (L_\Phi + 1), \quad \forall u \in \mathbb{R}^d, \quad |u|\leq n, \quad \forall j \in \{1,\dots,d\}.
\end{aligned}$$

Moreover, we know by the proof of Lemma 3.1 that

$$\lim_{\substack{h\to 0\\ u\in E_j^c}} \Phi(h,a_j)(u) = 1.$$

Also, we note that for each $j \in \{1,\dots,d\}$

$$\lim_{h\to 0} F_h^{(j)}(u) = \lim_{h\to 0} e^{ha_j(u)}u^{(j)} + \lim_{h\to 0}\left(\frac{e^{ha_j(u)}-1}{a_j(u)}\mathbf{1}_{\{E_j^c\}}(u) + h\mathbf{1}_{\{E_j\}}(u)\right) b_j(u^{(j)}) = u^{(j)},$$

hence $\lim\limits_{h\to 0} F_h(u) = u$. Consequently, given $n > 0$ there is a function $K_n(\cdot) : (0,\infty) \to (0,\infty)$, such that $K_n(h) \to 0$ when $h \to 0$ and

$$|\varphi_{f_h}(u) - f(u)|^2 \vee |g_h(u) - g(u)|^2 \leq K_n(h) \qquad \forall u \in \mathbb{R}^d, \quad |u|\leq n. \tag{47}$$

Now, using that both f, g are C^1, there is a constant $H_n > 0$ such that

$$|f(u) - f(v)|^2 \vee |g(u) - g(v)|^2 \leq H_n|u-v|^2 \qquad \forall u,v \in \mathbb{R}^d, |u|\vee|v|\leq n. \tag{48}$$

On the other hand, by Lemma 2.1 and inequality (45) we obtain

$$\mathbb{E}\left[\sup_{0\leq t\leq T}|y(t)|^p\right] \vee \mathbb{E}\left[\sup_{0\leq t\leq T}|y_h(t)|^p\right] \leq K := C\left(1 + \mathbb{E}|y_0|^p\right).$$

Now, we define the stopping times

$$\tau_n := \inf\{t \geq 0 : |y(t)|\geq n\}, \qquad \rho_n := \inf\{t \geq 0 : |y_h(t)|\geq n\}, \qquad \theta_n := \tau_n \wedge \rho_n, \tag{49}$$

and the difference function

$$e_h(t) := y(t) - y_h(t).$$

From the Young's inequality (A.2), we deduce that for any $\delta > 0$

$$\begin{aligned}
\mathbb{E}\left[\sup_{0\leq t\leq T}|e_h(t)|^2\right] &= \mathbb{E}\left[\sup_{0\leq t\leq T}|e_h(t)|^2\mathbf{1}_{\{\tau_n>T,\rho_n>T\}}\right] + \mathbb{E}\left[\sup_{0\leq t\leq T}|e_h(t)|^2\mathbf{1}_{\{\tau_n\leq T \text{ or } \rho_n\leq T\}}\right] \\
&\leq \mathbb{E}\left[\sup_{0\leq t\leq T}|e_h(t\wedge\theta_n)|^2\mathbf{1}_{\{\theta_n\geq T\}}\right] + \frac{2\delta}{p}\mathbb{E}\left[\sup_{0\leq t\leq T}|e_h(t)|^p\right] \\
&+ \frac{1-2/p}{\delta^{2/(p-2)}}\mathbb{P}\left[\tau_n \leq T \text{ or } \rho_n \leq T\right].
\end{aligned} \tag{50}$$

We proceed to bound each term on the right-hand side of inequality (50). By Lemma 2.1, $y(t)$ has bounded moments, hence there is a positive constant A such that

$$\mathbb{P}\left[\tau_n \leq T\right] = \mathbb{E}\left[\mathbf{1}_{\{\tau_n < T\}} \frac{|y(\tau_n)|^p}{n^p}\right] \leq \frac{1}{n^p} \mathbb{E}\left[\sup_{0\leq t\leq T} |y(t)|^p\right] \leq \frac{A}{n^p}, \qquad \text{for } p \geq 2. \tag{51}$$

The same conclusion can be drawn for ρ_n, then

$$\mathbb{P}\left[\tau_n \leq T \text{ or } \rho_n \leq T\right] \leq \mathbb{P}\left[\tau_n \leq T\right] + \mathbb{P}\left[\rho_n \leq T\right] \leq \frac{2A}{n^p}. \tag{52}$$

Now, using the inequality (A.4) and Lemma 2.1 we have

$$\mathbb{E}\left[\sup_{0\leq t\leq T} |e_h(t)|^p\right] \leq 2^{p-1} \mathbb{E}\left[\sup_{0\leq t\leq T} (|y(t)|^p + |y_h(t)|^p)\right] \leq 2^p A. \tag{53}$$

So, combining the bound (52) with (53) in inequality (50) we obtain

$$\mathbb{E}\left[\sup_{0\leq t\leq T} |e_h(t)|^2\right] \leq \mathbb{E}\left[\sup_{0\leq t\leq T} |e_h(t\wedge\theta_n)|^2 \mathbf{1}_{\{\theta_n \geq T\}}\right] + \frac{2^{p+1}\delta A}{p} + \frac{2(p-2)A}{p\delta^{2/(p-2)} n^p}. \tag{54}$$

Next, we show that the first term of (54) is bounded. Adding conveniently terms yields

$$\begin{aligned} e_h(t\wedge\theta_n) &= \int_0^{t\wedge\theta_n} \left[f(y(s)) - f(y_h(s)) + f(y_h(s)) - \varphi_{f_h}(y_h(s))\right] ds \\ &\quad + \int_0^{t\wedge\theta_n} \left[g(y(s)) - g(y_h(s)) + g(y_h(s)) - g_h(y_h(s))\right] dW(s). \end{aligned}$$

Using the bounds (47) and (48), the Cauchy-Schwarz, and Doob martingale inequalities, we get

$$\mathbb{E}\left[\sup_{0\leq t\leq \tau} |e_h(t\wedge\theta_n)|^2\right] \leq 4H_n(T+4)\int_0^\tau \mathbb{E}\left[\sup_{0\leq t\leq \tau} |e_h(t\wedge\theta_n)|^2\right] ds + 4T(T+4)K_n(h).$$

The Gronwall inequality now yields

$$\mathbb{E}\left[\sup_{0\leq t\leq T} |e_h(t\wedge\theta_R)|^2\right] \leq 4T(T+4)K_n(h)\exp(4H_n(T+4)T).$$

Hence, given $\epsilon > 0$ for any $\delta > 0$ such that $2^{p+1}\delta A/p < \epsilon/3$, we can take $n > 0$ verifying $(p-2)2A/(p\delta^{2/(p-2)}n^p) < \epsilon/3$. Moreover, we can take h sufficiently small such that $4T(T+4)K_n(h)\exp(4H_n(T+4)T) < \epsilon/3$. It follows immediately that

$$\mathbb{E}\left[\sup_{0\leq t\leq T} |e_h(t)|^2\right] < \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon,$$

which is the desired conclusion. □

Next, we proceed with Step 3, in which we establish that LS method has bounded moments.

Lemma 5.2. *Let Hypotheses 2.1, 3.1 and 3.2 holds. Then for each $p \geq 2$ there is a universal positive constant $C = C(p,T)$ such that for the LS method*

$$\mathbb{E}\left[\sup_{kh\in[0,T]} |Y_k|^{2p}\right] \leq C.$$

Proof. Using the split formulation of the LS scheme (5)–(6) we get

$$|Y_k^\star|^2 \leq |A^{(1)}(h,Y_k)|^2|Y_k|^2 + 2\left\langle A^{(1)}(h,Y_k)Y_k, A^{(2)}(h,Y_k)Y_k b(Y_k)\right\rangle + |A^{(2)}(h,Y_k)|^2|b(Y_k)|^2. \quad (55)$$

Then, applying the Cauchy-Schwartz inequality and Hypothesis 3.1 we arrive at

$$|Y_k^\star|^2 \leq |A^{(1)}(h,Y_k)|^2|Y_k|^2 + 2\sqrt{L_b}d|A^{(1)}(h,Y_k)||A^{(2)}(h,Y_k)||Y_k|(1+|Y_k|) + L_b|A^{(2)}(h,Y_k)|^2(1+|(Y_k)|^2).$$

Using (3.1) we see that

$$|A^{(1)}(h,Y_k)|^2 = \left|\operatorname{diag}\left(e^{ha_1(Y_k)},\ldots,e^{ha_d(Y_k)}\right)\right|^2 \leq \underbrace{d\exp(2TL_a)}_{:=L_{A^{(1)}}}. \quad (56)$$

In similar way, we deduce from the bound (24) that

$$\begin{aligned}|A^{(2)}(h,Y_k)|^2 &= \left|h\operatorname{diag}\left(\mathbf{1}_{\{E_1\}}(Y_k) + \mathbf{1}_{\{E_1^c\}}(Y_k)\Phi(h,a_1)(Y_k),\ldots,\mathbf{1}_{\{E_d\}}(Y_k) + \mathbf{1}_{\{E_d^c\}}(Y_k)\Phi(h,a_d)(Y_k)\right)\right|^2 \\ &\leq \sum_{j=1}^d \left(\mathbf{1}_{\{E_j^c\}}|h\Phi(h,a_1)(Y_k)|^2 + h^2\right) \leq \underbrace{2\exp(2L_aT)\sum_{j=1}^d \frac{1}{a_j^*} + dT^2}_{:=L_{A^{(2)}}}. \end{aligned} \quad (57)$$

Combining (3) of Hypothesis 3.1 with bounds (56) and (57) yields

$$|Y_k^\star|^2 \leq L_{A^{(1)}}|Y_k|^2 + 2d\sqrt{L_{A^{(1)}}L_{A^{(2)}}L_b}|Y_k|(1+|Y_k|) + L_{A^{(2)}}L_b(1+|Y_k|^2).$$

So, taking $C \geq L_{A^{(1)}} + 2d\sqrt{L_{A^{(1)}}L_{A^{(2)}}L_b} + L_{A^{(2)}}L_b$ we can assert that

$$|Y_k^\star|^2 \leq C(3|Y_k|^2 + |Y_k| + 1) \leq 6C\left(|Y_k|^2 + 1\right) \leq C(1+|Y_k|^2). \quad (58)$$

Then, applying bound (58) in eq. (6) we arrive at

$$|Y_{k+1}|^2 \leq C\left(|Y_k|^2+1\right) + 2\left\langle Y_k^\star, g(Y_k^\star)\Delta W_k\right\rangle + |g(Y_k^\star)\Delta W_k|^2$$

Now, we choose two integers N, M such that $Nh \leq Mh \leq T$. So, adding backwards we get

$$|Y_N|^2 \leq S_N\left(\sum_{j=0}^{N-1}(1+|Y_j|^2) + 2\sum_{j=0}^{N-1}\left\langle Y_j^\star, g(Y_j^\star)\Delta W_j\right\rangle + \sum_{j=0}^{N-1}\left|g(Y_j^\star)\Delta W_j\right|^2\right)$$

$$S_N := \sum_{j=0}^{N-1} C^{N-j}$$

Raising both sides to the power p and using the standard inequality (A.4) we obtain

$$|Y_N|^{2p} \le 6^p S_N^p \left(N^{p-1} \sum_{j=0}^{N-1} (1+|Y_j|^{2p}) + \left| \sum_{j=0}^{N-1} \langle Y_j^\star, g(Y_j^\star)\Delta W_j \rangle \right|^p + N^{p-1} \sum_{j=0}^{N-1} \left| g(Y_j^\star)\Delta W_j \right|^{2p} \right) \tag{59}$$

Now we will show that the second and third terms of the inequality (59) are bounded. Next, using the Bunkholder-Davis-Gundy inequality [24, Thm 7.3 pg. 40], (A.6) we see that

$$\begin{aligned}
\mathbb{E}\left[\sup_{0\le N\le M} \left| \sum_{j=0}^{N-1} \langle Y_j^\star, g(Y_j^\star)\Delta W_j \rangle \right|^p \right] &\le C\mathbb{E}\left[\sum_{j=0}^{N-1} |Y_j^\star|^2 |g(Y_j^\star)|^2 h \right]^{p/2} \\
&\le C h^{p/2} M^{p/2-1} \mathbb{E} \sum_{j=0}^{M-1} |Y_j^\star|^p (\alpha + \beta |Y_j^\star|^2)^{p/2} \\
&\le 2^{p/2-1} C T^{p/2-1} h \mathbb{E} \sum_{j=0}^{M-1} (\alpha^{p/2} |Y_j^\star|^p + \beta^{p/2} |Y_j^\star|^{2p}) \\
&\le C h \mathbb{E} \sum_{j=0}^{M-1} (1 + 2|Y_j^\star|^p + |Y_j^\star|^{2p}) \\
&\le C h \sum_{j=0}^{M-1} \left[1 + \mathbb{E}|Y_j^\star|^{2p} \right] \\
&\le C + C h \sum_{j=0}^{M-1} \mathbb{E}|Y_j|^{2p}.
\end{aligned} \tag{60}$$

Now, note that

$$\mathbb{E}\left[\sup_{0\le N\le M} \sum_{j=0}^{N-1} |Y_j|^{2p} \right] = \sum_{j=0}^{M-1} \mathbb{E}|Y_j|^{2p}. \tag{61}$$

Hence, using Cauchy-Schwartz inequality, the monotone condition (3), bound (58) and the standard inequality (A.4), we obtain

$$\begin{aligned}\mathbb{E}\left[\sup_{0\le N\le M}\sum_{j=0}^{N-1}\left|g(Y_j^\star)\Delta W_j\right|^{2p}\right] &= \mathbb{E}\sum_{j=0}^{M-1}\left|g(Y_j^\star)\Delta W_j\right|^{2p}\\ &\le \sum_{j=0}^{M-1}\mathbb{E}\left|g(Y_j^\star)\right|^{2p}\mathbb{E}\left|\Delta W_j\right|^{2p}\\ &\le Ch^p\sum_{j=0}^{M-1}\mathbb{E}\left[\alpha+\beta|Y_j^\star|^2\right]^p\\ &\le Ch^p\sum_{j=0}^{M-1}\mathbb{E}\left[\alpha^p+\beta^p|Y_j^\star|^{2p}\right]\\ &\le Ch^{p-1}+Ch^p\sum_{j=0}^{M-1}\mathbb{E}|Y_j|^{2p}. \end{aligned}\tag{62}$$

Thus, combining the bounds (60) and (62) with the inequality (59), we can assert that

$$\begin{aligned}\mathbb{E}\left[\sup_{0\le N\le M}|Y_N|^{2p}\right] &\le C(M,T)+C(M,T)(1+h)\sum_{j=0}^{M-1}\mathbb{E}|Y_j|^{2p}\\ &\le C+C(1+h)\sum_{j=0}^{M-1}\mathbb{E}\left[\sup_{0\le N\le j}|Y_N|^{2p}\right]. \end{aligned}\tag{63}$$

Finally, using the discrete-type Gronwall inequality (A.9), we conclude that

$$\mathbb{E}\left[\sup_{0\le N\le M}|Y_N|^{2p}\right]\le C\exp(C(1+h)M)\le C\exp(C(1+T))<C,$$

since the constant C does not depend on h, the proof is complete. □

As the LS scheme has bounded moments, we now proceed whit Step 4, that is, we will obtain a convenient continuous extension of the LS method with bounded moments. Let $\{Y_k\}$ denote the LS solution of SDE (1). By Corollary 5.1, we conveniently made a continuous extension for the LS approximation, from the time continuous extension of the EM (38). Moreover, we prove that the moments of this extension remains bounded.

Corollary 5.2. *Let Hypotheses 2.1, 3.1 and 3.2 holds and suppose $0<h<1$ and $p\ge 2$. Then there is a continuous extension $\overline{Y}(t)$ of $\{Y_k\}$ and a positive constant $C=C(T,p)$ such that*

$$\mathbb{E}\left[\sup_{0\le t\le T}|\overline{Y}(t)|^{2p}\right]\le C.$$

Proof. We take $t=s+t_k$ in $[0,T]$, $\Delta W_k(s):=W(t_k+s)-W(t_k)$ and $0\le s<h$. Then we define

$$\overline{Y}(t_k+s):=Y_k+s\varphi_{f_h}(Y_k)+g_h(Y_k)\Delta W_k(s),\tag{64}$$

as a continuous extension of the LS scheme. We proceed to show that $\overline{Y}(t)$ has bounded moments. By Lemma 3.1, we have $Y_k^\star = Y_k + h\varphi_{f_h}(Y_k)$. Then for $\gamma = s/h$, it follows that

$$\begin{aligned} Y_k + s\varphi_{f_h}(Y_k) &= \gamma(Y_k + h\varphi_{f_h}(Y_k)) + (1-\gamma)Y_k \\ &= \gamma Y_k^\star + (1-\gamma)Y_k. \end{aligned}$$

Hence, we can rewrite the continuous extension (64) as

$$\overline{Y}(t) = \gamma Y_k^\star + (1-\gamma)Y_k + g_h(Y_k)\Delta W_k(s).$$

Combining this relation with the inequalities (58) and (A.4), we arrive at

$$\begin{aligned} |\overline{Y}(t_k+s)|^2 &\leq 3\left[\gamma C + (\gamma C + 1 - \gamma)\,|Y_k|^2 + |g_h(Y_k)\Delta W_k(s)|^2\right] \\ &\leq C + C\left(|Y_k|^2 + |g_h(Y_k)\Delta W_k(s)|^2\right). \end{aligned}$$

Thus,

$$\begin{aligned} \sup_{0\leq t\leq T} |\overline{Y}(t)|^{2p} &\leq \sup_{0\leq kh\leq T}\left[\sup_{0\leq s\leq h} |\overline{Y}(t_k+s)|^{2p}\right] \\ &\leq \sup_{0\leq kh\leq T}\left[\sup_{0\leq s\leq h} C\left(1 + |Y_k|^{2p} + |g_h(Y_k)\Delta W_k(s)|^{2p}\right)\right], \end{aligned} \tag{65}$$

for $t\in[0,T]$. Now taking a non negative integer $0\leq k\leq N$ such that $0\leq Nh\leq T$. From the bond (65), we get

$$\sup_{0\leq t\leq T} |\overline{Y}(t)|^{2p} \leq C\left(1 + \sup_{0\leq kh\leq T} |Y_k|^{2p} + \sup_{0\leq s\leq h}\sum_{j=0}^{N} |g_h(Y_j)\Delta W_j(s)|^{2p}\right). \tag{66}$$

So, using the Doob's Martingale inequality (A.5), Lemma 5.2 and that g_h is a locally Lipschitz function, we can bound each term of the inequality (66), as follows

$$\begin{aligned} \mathbb{E}\left[\sup_{0\leq s\leq h} |g(Y_j)\Delta W_j(s)|^{2p}\right] &\leq \left(\frac{2p}{2p-1}\right)^{2p} \mathbb{E}|g_h(Y_j)\Delta W_j(h)|^{2p} \\ &\leq C\mathbb{E}|g_h(Y_j)|^{2p}\mathbb{E}|\Delta W_j(h)|^{2p} \\ &\leq Ch^p\left(1 + \mathbb{E}|Y_j|^{2p}\right) \\ &\leq Ch, \end{aligned} \tag{67}$$

for each $j\in\{0,\ldots,N\}$. Since $Nh\leq T$, combining the bounds (66) and (67) we get the desired conclusion. □

Once we have carried out all the previous steps, we can prove the Theorem 5.1 by Step 5.

Proof (of Theorem 5.1). First, note that by inequality (A.4), we have

$$\mathbb{E}\left[\sup_{0\leq t\leq T} |\overline{Y}(t) - y(t)|^2\right] \leq 2\mathbb{E}\left[\sup_{0\leq t\leq T} |\overline{Y}(t) - y_h(t)|^2\right] + 2\mathbb{E}\left[\sup_{0\leq t\leq T} |y_h(t) - y(t)|^2\right]. \tag{68}$$

Using Lemma 5.1, which was established in the Step 2, yields

$$\lim_{h\to 0} \mathbb{E}\left[\sup_{0\le t\le T} |y_h(t) - y(t)|^2\right] = 0. \tag{69}$$

It remains to prove that the first term of the right hand side in inequality (68) decreases to zero when h tends to zero. Recalling that:

i) By Lemma 5.1, the solution of the modified SDE (44), y_h, has p-bounded moments ($p \ge 2$).

ii) By Corollary 5.2, the LS continuous extension for the SDE (1), $\overline{Y}(t)$, has bounded moments and it is equivalent to the EM extension for the modified SDE (44).

Hence, we can apply Theorem 4.1 to conclude that

$$\lim_{h\to 0} \mathbb{E}\left[\sup_{0\le t\le T} |\overline{Y}(t) - y_h(t)|^2\right] = 0. \tag{70}$$

Finally, combining the limits (69) and (70) with inequality (68) gives

$$\begin{aligned}\lim_{h\to 0} \mathbb{E}\left[\sup_{0\le t\le T} |\overline{Y}(t) - y(t)|^2\right] &\le 2\lim_{h\to 0} \mathbb{E}\left[\sup_{0\le t\le T} |\overline{Y}(t) - y_h(t)|^2\right] \\ &\quad + 2\lim_{h\to 0} \mathbb{E}\left[\sup_{0\le t\le T} |y_h(t) - y(t)|^2\right] = 0,\end{aligned}$$

which proves the theorem. □

6. Convergence Rate

In this section we show that the explicit Linear Steklov method eqs. (5) and (6) converges with a standard order of one-half. For that, we use a similar procedure as in [14]. In addition to Hypotheses 2.1, 3.1 and 3.2 we also require the following.

Hypothesis 6.1. There exist constants $L_f, D \in \mathbb{R}$ and $q \in \mathbb{Z}^+$ such that $\forall u, v \in \mathbb{R}^d$

$$\langle u - v, f(u) - f(v)\rangle \le L_f |u - v|^2, \tag{71}$$

$$|f(u) - f(v)|^2 \le D(1 + |u|^q + |v|^q)|u - v|^2. \tag{72}$$

Hypothesis 6.2. The SDE (1) and the EM solutions satisfy

$$\mathbb{E}\left[\sup_{0\le t\le T} |y(t)|^p\right], \quad \mathbb{E}\left[\sup_{0\le t\le T} |X(t)|^p\right], \quad \mathbb{E}\left[\sup_{0\le t\le T} |\overline{X}(t)|^p\right] < \infty, \qquad \forall p \ge 1. \tag{73}$$

Theorem 6.1. *[Higham et al. [14, Thm 4.4]] Under Hypotheses 3.1–6.1 the EM solution with continuous extension $\overline{X}$ satisfies*

$$\mathbb{E}\left[\sup_{0\le t\le T} |\overline{X}(t) - y(t)|^2\right] = \mathcal{O}(h^2). \tag{74}$$

Lemma 6.1. *Under Hypotheses 6.1 and 6.2 and sufficiently small h, there exist constants $D' \in \mathbb{R}$ and $q' \in \mathbb{Z}$ such that for all $u, v \in \mathbb{R}^d$*

$$|\varphi_{f_h}(u) - \varphi_{f_h}(v)|^2 \leq D' \left(1 + |u|^{q'} + |v|^{q'}\right) |u - v|^2, \tag{75}$$

$$|f(u) - \varphi_{f_h}(u)|^2 \leq D' \left(1 + |u|^{q'}\right) h^2, \tag{76}$$

$$|g(u) - g_h(u)|^2 \leq D' \left(1 + |u|^{q'}\right) h^2. \tag{77}$$

PROOF. From inequality (14), we have

$$|\varphi_{f_h}(u) - \varphi_{f_h}(v)|^2 \leq (2 + L_\Phi)|f(u) - f(v)|^2 \leq (2 + L_\Phi) D(1 + |u|^q + |v|^q).$$

Moreover, if $u \in E_j$ then $\varphi_{f_h}(u) = f^{(j)}(u)$. On the other hand, if $u \in E_j^c$ then

$$|f(u) - \varphi_{f_h}(u)|^2 = \sum_{j=1}^{d} |1 - \Phi(h, a_j)(u)|^2 |f^{(j)}(u)|^2,$$

By the L'Hôpital theorem, we get

$$\lim_{h \to 0} |1 - \Phi(h, a_j)(u)| = \left|1 - \lim_{h \to 0} \frac{e^{h a_j(u)} - 1}{h a_j(u)}\right| \leq \left|1 - \lim_{h \to 0} e^{h L_a}\right| = 0.$$

Thus, there is a sufficiently small $h > 0$ such that $|1 - \Phi_j(u)| < Ch$ for all $u \in E_j^c$ and

$$|f(u) - \varphi_{f_h}(u)|^2 \leq C h^2 |f(u)|^2 \leq D'(1 + |u|^q) h^2,$$

as we require. Given that $g_h(u) = g(F_h(u))$ from theorem 3.1 we get

$$|g(u) - g_h(u)|^2 \leq L_g |u - u + h\varphi f_h(u)|^2 \leq 2(1 + L_\Phi) h^2 |f(u)|^2 \leq 2(1 + L_\Phi) D(1 + |u|^q) h^2.$$

□

Lemma 6.2. *Assume Hypotheses 6.1 and 6.2 hold then the solution $y_h(t)$ of the modified SDE* (42) *satisfies*

$$\mathbb{E}\left[\sup_{0 \leq t \leq T} |y_h(t) - y(t)|^2\right] = \mathcal{O}(h^2). \tag{78}$$

PROOF. We define $e(t) := y(t) - y_h(t)$ where

$$\begin{aligned} y(t) &= y_0 + \int_0^t f(y(s))ds + \int_0^t g(y(s))dW(s), \\ y_h(t) &= y_0 + \int_0^t \varphi_{f_h}(y_h(s))ds + \int_0^t g_h(y_h(s))dW(s). \end{aligned}$$

Using Itô's formula over the function $V(t, x, y) = |x - y|^2$ for all $x, y \in \mathbb{R}^d$, we obtain

$$de(t) = (f(y(t)) - \varphi_{f_h}(y(t))dt) + (g(y(t)) - g_h(y_h(t)))\, dW(t),$$

Thus,

$$
\begin{aligned}
|e(t)|^2 = {} & 2\underbrace{\int_0^t \langle e(s), f(y(s)) - \varphi_{f_h}(y_h(s))\rangle \, ds}_{:=I_1} + \underbrace{\int_0^t |g(y(s)) - g_h(y_h(s))|^2 ds}_{:=I_2} \\
& + 2\underbrace{\int_0^t \langle e(s), [g(y(s)) - g_h(y_h(s))] \, dW(s)\rangle}_{:=I_3}. \qquad (79)
\end{aligned}
$$

Now we proceed to bound each integral of inequality (79). By Hypothesis 6.1 and the Young inequality, we get

$$
\begin{aligned}
I_1(t) &\le 2\int_0^t \langle y(s) - y_h(s), f(y(s)) - f(y_h(s))\rangle \, ds + \int_0^t \langle y(s) - y_h(s), f(y_h(s)) - \varphi_f(y_h(s))\rangle \, ds \\
&\le 3\int_0^t |y(s) - y_h(s)|^2 ds + D'h^2 \int_0^t 1 + |y_h(s)|^{q'} ds.
\end{aligned}
$$

Since $y_h(t)$ the has bonded moments, there exists a universal constant L which does not depends on h such that

$$
\mathbb{E}\left[I_1(s)\right] \le L\int_0^t \mathbb{E}|e(s)|^2 ds + Lh^2. \qquad (80)
$$

Using Hypotheses 2.1 and 6.1 it is followed

$$
I_2(t) \le 2L_g \int_0^t |y(s) - y_h(s)|^2 ds + 2D'h^2 \int_0^t 1 + |y_h(s)|^q ds,
$$

thus

$$
\mathbb{E}\left[I_2(s)\right] \le L\int_0^t \mathbb{E}|e(s)|^2 ds + Lh^2. \qquad (81)
$$

Note that $\mathbb{E}\left[I_3(t)\right] \le \mathbb{E}\left[\sup_{0\le s\le t}|I_3(s)|\right]$. From the Burkholder-Davis-Gaundy inequality, Hypotheses 2.1 and 6.1 and as $y_h(t)$ has bounded moments, we obtain

$$
\begin{aligned}
\mathbb{E}\left[\sup_{0\le s\le t}|I_3(s)|\right] &\le 2^4\mathbb{E}\left[\sup_{0\le s\le t}|e(s)|^2 \int_0^t |g(y(s)) - g_h(y(s))|^2 ds\right]^{1/2} \\
&\le 2^4\mathbb{E}\left[\frac{1}{2\cdot 2^9}\left(\sup_{0\le s\le t}|e(s)|^2\right) + \frac{2^9}{2}\left(\int_0^t |g(y(s)) - g_h(y_h(s))|^2 ds\right)^2\right] \\
&\le 2L_g\mathbb{E}\left[\int_0^t |y(s) - y_h(s)|^2 ds\right] + D'Th^2 + D'Th^2\int_0^t \mathbb{E}|y_h(s)|^{q'} ds \\
&\le L\int_0^t \mathbb{E}|e(s)|^2 ds + Lh^2. \qquad (82)
\end{aligned}
$$

Substituting inequalities (80), (81) and (82) on equation (79), we deduce that

$$
\mathbb{E}\left[\sup_{0\le s\le t}|e(t)|^2\right] \le L\int_0^t \mathbb{E}|e(s)|^2 ds + Lh^2 \le L\int_0^t \mathbb{E}\left[\sup_{0\le r\le s}|e(s)|^2\right] ds + Lh^2.
$$

By the Gronwall inequality, we conclude that

$$\mathbb{E}\left[\sup_{0\le t\le T}|e(t)|^2\right] \le L\exp(LT)h^2 \le Ch^2.$$

□

We can now obtain the convergence rate of the explicit Linear Steklov method.

Theorem 6.2. *Under Hypotheses 2.1–6.1 and consider the explicit LS method* (10) *for the SDE* (1). *Then there exists a continuous-time extension $\overline{Y}(t)$ of the LS numerical approximation for which*

$$\mathbb{E}\left[\sup_{0\le t\le T}|\overline{Y}(t)-y(t)|^2\right] = \mathcal{O}(h). \tag{83}$$

PROOF. Using bound (68) then by lemma 6.2 and since the LS continuous-time extension (64) is equivalent to the EM continuous-time extension (38), we can use Theorem 6.1 and conclude that the LS has order one-half. □

7. Numerical Experiments

Here we analyze the behavior of the explicit Linear Steklov method (LS) for scalar and vector SDEs. The tests confirm the convergence order 1/2 for stochastic differential systems with locally Lipschitz drift and suggest that the LS scheme reproduces almost surely stability (a.s.). We validate the efficiency of the new method by comparing with other actual methods like the Euler-Maruyama, Backward Euler (BEM) [26] and tamed Euler (TEM) [18]. All simulations are implemented in Python 2.7 and we use the Mersenne random number generator with fixed seed 100.

Example 7.1. *Here we illustrate the stability* **??** *through an numerical example presented in [1, sec 7, pg. 420]. Here, Appleby and Kelly, had proved that the EM fails to preserve the almost sure stability of the test SDE*

$$dy(t) = -\beta y(t)|y(t)|^p dt + \sigma(t)|y(t)|^\rho dW(t). \tag{84}$$

That is, the EM approximation explodes to infinity on finite time when $p+1>2\rho$. But, with the same parameters $\lim_{t\to\infty} y(t) = 0 \quad \text{a.s.}$, *see [1, 2] for more details. The authors consider*

$$dy(t) = -y^3 dt + \frac{1}{[\log(t+1)]^{1.1}} dW_t, \qquad t>0, \tag{85}$$

and deduce conditions for the step-size h and initial condition $y(t_0)=y_0$ in order to claim with high probability when the EM scheme for SDE (85) *is GASA-stable or diverge [1, Cor 7.1 pg. 421]. More specifically, given $h<0.0384$ and the EM for SDE* (85)

$$X_{k+1} = X_k - hX_k^3 + \frac{1}{[\log(n+1)]^{1.1}}\Delta W_k, \qquad X_0 = y(t_0). \tag{86}$$

(i) If $X_0 \in \left(-\sqrt{\frac{2}{h}} + 7\sqrt{h}, \sqrt{\frac{2}{h}} - 7\sqrt{h}\right)$, *then* $\mathbb{P}\left[\lim_{k\to\infty} X_k = 0\right] > 0.95$.

(ii) If $X_0 \in \left(-\infty, -\sqrt{\frac{2}{h}} - 7\sqrt{h}\right) \bigcup \left(\sqrt{\frac{2}{h}} + 7\sqrt{h}, \infty\right)$, *then*

$$\mathbb{P}\left[\limsup_{n\to\infty} X_k = \infty \ \text{or} \ \liminf_{n\to\infty} X_k = -\infty\right] > 0.95 \quad .$$

Thus we perform a simulation with step size $h = 0.2$ *using the EM, Tamed Euler-Maruyama (TEM) and the LS schemes with unstable EM initial conditions. Figure 2 shows how the EM scheme produce spurious solutions. Meanwhile, the TEM ans LS approximations reproduce the asymptotic behavior, also we note a better initial precision for the LS approximation.*

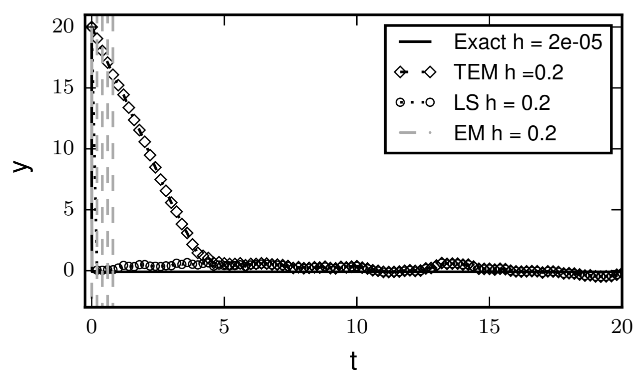

Figure 2: Likening between the EM, TEM and LS approximations with unstable EM conditions. Here "exact" means a BEM solutions with step size $h = 2 \times 10^{-5}$.

Example 7.2. *We examine the LS method using a SDE with super-linear grow diffusion. We consider the SDE reported by Tretyakov and Zhang in [35, Eq. (5.6)]*

$$dy(t) = \left(1 - y^5(t) + y^3(t)\right) dt + y^2(t) dW(t), \qquad y_0 = 0. \tag{87}$$

Tretyakov and Zhang shows via simulation of (87) *that the increment-tamed scheme [16, Eq(1.5)]*

$$X_{k+1} = X_k + \frac{f(X_k)h + g(X_k)\Delta W_k}{\max\left(1, h\left|hf(X_k) + g(X_k)\Delta W_k\right|\right)} \tag{88}$$

produces spurious oscillations. Hutzenthaler and Jentzen prove the convergence of this scheme under linear growth condition over diffusion. So, this suggest us that only certain kind of explicit schemes with convergence under globally Lipschitz and linear growth diffusion conditions can *extended their convergence to a locally Lipschitz diffusion and other kind of growth bound. Using* $a(x) := -x^4 + x^2$, $b := 1$ *and* $E = \{-1, 0, 1\}$, *we construct the LS method*

$$Y_{k+1} = \exp(ha(Y_k))Y_k + \frac{\exp(ha(Y_k)) - 1}{a(Y_k)}\mathbf{1}_{\{E^c\}} + h\mathbf{1}_{\{E\}} + Y_k^2 \Delta W_k. \tag{89}$$

Figure 3 shows the numerical solution of SDE (87) *with the Increment-Tamed (I-TEM)* (88), *LS method* (89), *and the Tamed (TEM) scheme with. We consider the implicit Midpoint scheme [35, Eq.(5.3)] with* $h = 10^{-4}$ *as reference.*

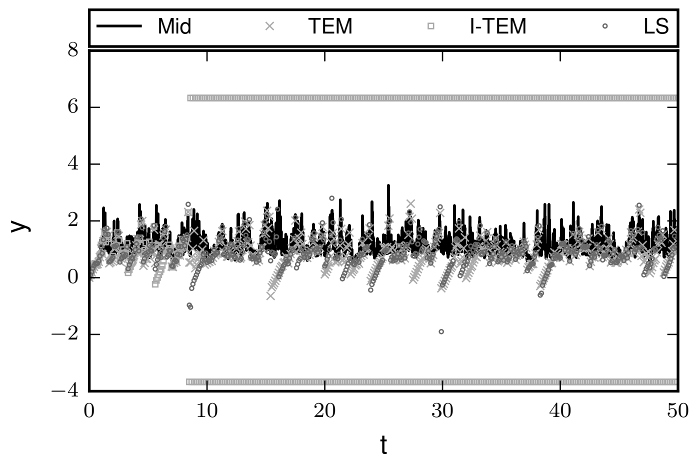

Figure 3: Numerical solution of SDE (87) using the I-TEM (88), LS method (89) and TEM with $h = 0.1$, the reference solutions is a Midpoint rule approximation with $h = 10^{-4}$.

Example 7.3. *Now we compare the order of convergenceand the run time of the LS method withthe TEM scheme as in [18]. That is,we consider a Langevin equation under the d-dimensional potential* $U(x) = \frac{1}{4}|x|^4 - \frac{1}{2}|x|^2$, *and d-dimensional Brownian additive noise. The corresponding SDE reads*

$$dy(t) = (y(t) - |y(t)|{\cdot}y(t))\,dt + dW(t), \qquad y(0) = 0. \tag{90}$$

This model describes the motion of a Brownian particle of unit mass immersed on the potential $U(x)$. *Taking* $a_j(x) := 1 - |x|$ *and* $b_j = 0$, $j \in 1 \dots d$ *we obtain the LS method*

$$Y_{k+1} = \operatorname{diag}\left[e^{ha_1(Y_k)}, \dots, e^{ha_d(Y_k)})\right] Y_k + \Delta W_k. \tag{91}$$

Table 1 shows the root means square errors at a final time $T = 1$, *which is approximated by*

$$\sqrt{\mathbb{E}\left[|Y_N - y(T)|^2\right]} \approx \frac{1}{M}\left(\sum_{i=1}^{M} |y_i(T) - Y_{N,i}|^2\right)^{1/2}, \tag{92}$$

over a sample of M =10 000 *trajectories of the TEM, LS and BEM solutions to SDE* (90) *with dimension* $d = 10$. *We consider the TEM solution with step* $h = 2^{-19}$ *as reference solution. In*

	TEM		LS		BEM	
h	ms-error	ECO	ms-error	ECO	ms-error	ECO
2^{-2}	1.703 88	—	1.553 94	—	1.381 57	—
2^{-3}	1.169 77	0.54	1.107 75	0.48	1.053 09	0.39
2^{-7}	0.278 95	0.48	0.277 95	0.48	0.276 895	0.48
2^{-11}	0.070 10	0.50	0.070 09	0.50	0.070 07	0.50
2^{-15}	0.017 39	0.51	0.017 39	0.51	0.017 39	0.51

Table 1: Mean square errors and the experimental convergence order (ECO) for the SDE (90) with a TEM with $h = 2^{-19}$ as reference solution.

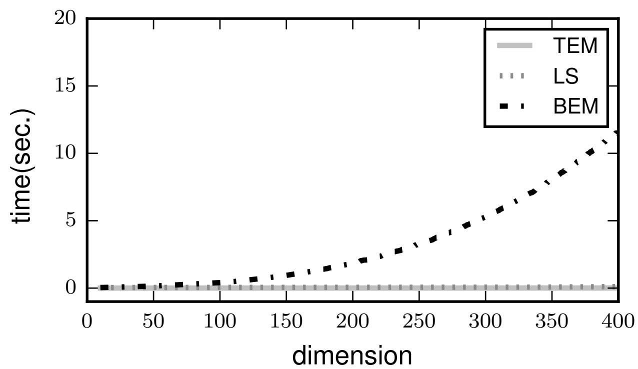

Figure 4: Runtime calculation of Y_N with $h = 2^{-17}$, using the BEM, LS and TEM methods for SDE (90).

this experiment we confirm that the LS method converges with standard order 1/2 and is almost equal accurate than the TEM.

In some application as in Browninan Dynamics Simulations [6], the dimension of a SDE increases considerable the complexity and computational cost — this prohibits the use of implicit methods. Figure 4 supports this (for SDE(90)*): the runtime of BEM depends on dimension in a quadratic way, while the LS and TEM depends on linear form.*

Example 7.4. *Hutzenthaler et al. improve convergence of the Euler method by taming the drift increment term with the factor* $\frac{1}{1+h|f(Y_k)|}$, *as consequence, the norm of* $\frac{hf(Y_k)}{1+h|f(Y_k)|}$, *is bounded by 1, which controls the drift contribution of the TEM method at each step. This idea works very well over SDEs with drift contribution and initial condition that are comparable with this bound. However, we observed that on models where the drift contribution has other scales, the TEM over damps the drift contribution. To fix ideas, we consider the stochastic model reported* *in [7],*

$$
\begin{aligned}
dy_1(t) &= (\lambda - \delta y_1(t) - (1-\gamma)\beta y_1(t) y_3(t))\, dt - \sigma_1 y_1(t) dW_t^{(1)}, \\
dy_2(t) &= ((1-\gamma)\beta y_1(t) y_3(t) - \alpha y_2(t))\, dt - \sigma_1 y_2(t) dW_t^{(1)}, \\
dy_3(t) &= ((1-\eta) N_0 \alpha y_2(t) - \mu y_3(t) - (1-\gamma)\beta y_1(t) y_3(t))\, dt - \sigma_2 y_3(t) dW_t^{(2)}.
\end{aligned} \tag{93}
$$

Taking

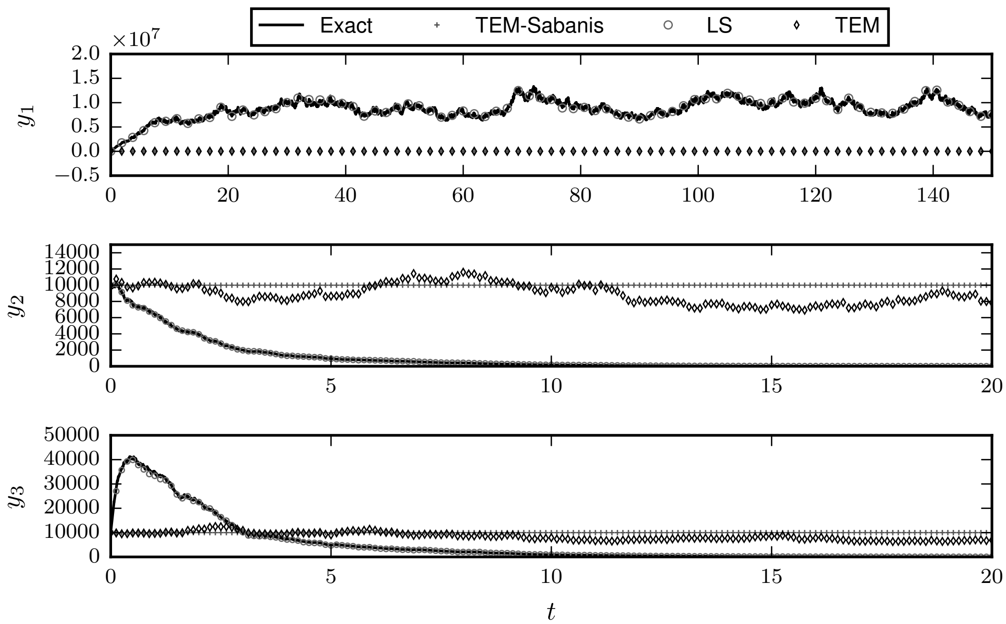

Figure 5: Likening between EM, LS, TEM approximations for SDE (93) with $\gamma = 0.5$, $\eta = 0.5$, $\lambda = 10^6$, $\delta = 0.1$, $\beta = 10^{-8}$, $\alpha = 0.5$, $N_0 = 100$, $\mu = 5$, $\sigma_1 = 0.1$, $\sigma_2 = 0.1$, $y_0 = (10\,000, 10\,000, 10\,000.)^T$, $h = 0.125$. Here the reference solution means a BEM simulation with the same parameters but with a step-size $h = 10^{-5}$.

$$
\begin{aligned}
E_1 &:= \left\{ (x,y,z)^T \in \mathbb{R}^3 : z = 0 \ \ or\ z = 0\frac{-\delta}{\beta(1-\gamma)} \right\}, \quad E_2 := \emptyset, \\
E_3 &:= \left\{ (x,y,z)^T \in \mathbb{R}^3 : x = 0, \ \ or\ x = \frac{-\mu}{\beta(1-\gamma)} \right\}
\end{aligned}
$$

$$
\begin{aligned}
a_1(Y_k)) &:= -\left(\delta + (1-\gamma)\beta Y_k^{(3)}\right), & b_1(Y_k^{(-1)}) &:= \lambda, \\
a_2(Y_k) &:= -\alpha, & b_2(Y_k^{(-2)}) &:= (1-\gamma)\beta Y_k^{(1)} Y_k^{(3)}, \\
a_3(Y_k) &= -\left(\mu + (1-\gamma)\beta Y_k^{(1)}\right), & b_3(Y_k^{(-3)}) &:= (1-\eta)\, N_0 \alpha Y_k^{(2)},
\end{aligned}
$$

the LS method for the stochastic model (93) *reads,*

$$
\begin{aligned}
Y_{k+1} &= A^{(1)}(h, Y_k)\, Y_k + A^{(2)}(h, Y_k) b(Y_k) + g(Y_k)\, \Delta W_k, \qquad \Delta W_k = \left(W_k^{(1)}, W_k^{(2)}\right)^T, \\
A^{(1)}(h, Y_k) &:= \begin{pmatrix} e^{h a_1(Y_k)} & 0 & 0 \\ 0 & e^{h a_2(Y_k))} & 0 \\ 0 & 0 & e^{h a_3(Y_k))} \end{pmatrix}, \\
A^{(2)} &:= \begin{pmatrix} h\Phi_1(Y_k)\mathbf{1}_{\{E_1^c\}} & 0 & 0 \\ 0 & \left(\dfrac{e^{-h\alpha} - 1}{\alpha}\right) & 0 \\ 0 & 0 & h\Phi_3(Y_k)\mathbf{1}_{\{E_3^c\}} \end{pmatrix} + h \begin{pmatrix} \mathbf{1}_{\{E_1\}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathbf{1}_{\{E_3\}} \end{pmatrix}, \\
b(Y_k) &:= \begin{pmatrix} b_1(Y_k^{(-1)}) \\ b_2(Y_k^{(-2)}) \\ b_3(Y_k^{(-3)}) \end{pmatrix}, \qquad g(Y_k) := \begin{pmatrix} -\sigma_1 Y_k^{(1)} & 0 \\ -\sigma_1 Y_k^{(2)} & 0 \\ 0 & -\sigma_2 Y_k^{(3)} \end{pmatrix}.
\end{aligned} \tag{94}
$$

Dalal et al. in [7, Thm 5.1] gives conditions over the parameter of SDE (93)*, which assure a.s. exponential stability — in the sense that the infected cells (y_2) and virus particles (y_3) will tend to their equilibrium value 0 exponentially with probability 1—and verify this asymptotic behavior by simulation with parameters reported in published literature [4, 5, 30, 31]. Figure 5 shows a simulation path with same parameters with the LS and TEM approximations. We observe how the TEM oscillates around of initial condition while the LS reproduce the underlying asymptotic behavior.*

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Appendix A Useful Inequalities

Hölder.

$$\mathbb{E}[X^TY] \leq \left(\mathbb{E}|X|^p\right)^{\frac{1}{p}} \left(\mathbb{E}|X|^q\right)^{\frac{1}{q}}. \tag{A.1}$$

Young.

$$|a||b| \leq \frac{\delta}{p}|a|^p + \frac{\delta}{q\delta^{q/p}}|b|^q. \tag{A.2}$$

Minkowski.

$$\left(\mathbb{E}|X+Y|^p\right)^{\frac{1}{p}} \leq \left(\mathbb{E}|X|^p\right)^{\frac{1}{p}} + \left(\mathbb{E}|Y|^p\right)^{\frac{1}{p}}. \tag{A.3}$$

A standard inequality. Fix $1 < p < \infty$ and consider a sequence of real numbers $\{a_i\}_{i=1}^N$ with $N \in \mathbb{N}$. Then one can formulate this usefully inequality

$$\left(\sum_{j=1}^N a_j\right)^p \leq N^{p-1}\sum_{j=1}^N a_j^p. \tag{A.4}$$

Doob's Martingale Inequality. Let $\{M_t\}_{t\geq 0}$ be a $\mathbb{R}^d$-valued martingale. Let $[a,b]$ be a bounded interval in $\mathbb{R}_+$. If $p > 1$ and $M_t \in L^p(\Omega;\mathbb{R}^d)$ then

$$\mathbb{E}\left(\sup_{a\leq t\leq b}|M_t|^p\right) \leq \left(\frac{p}{p-1}\right)^p \mathbb{E}|M_b|^p. \tag{A.5}$$

Burkholder–Davis–Gundy inequality. Let $g \in \mathcal{L}(\mathbb{R}_+;\mathbb{R}^{d\times m})$. Define for $t \geq 0$

$$x(t) = \int_0^t g(s)dW(s) \quad \text{and} \qquad A(t) = \int_0^t |g(s)|^2 ds. \tag{A.6}$$

Then for all $p > 0$, there exist universal positive constants c_p, C_p such that

$$c_p \mathbb{E}|A(t)|^{\frac{p}{2}} \leq \mathbb{E}\left[\sup_{0\leq s\leq t} |x(s)|^p\right] \leq C_p \mathbb{E}|A(t)|^{\frac{p}{2}}, \tag{A.7}$$

for all $t \geq 0$. In particular, one may take

$$\begin{aligned}
&c_p = (p/2)^p, && C_p = (32/p)^{\frac{p}{2}} && \text{if } 0 < p < 2;\\
&c_p = 1, && C_p = (32/p)^{\frac{p}{2}} && \text{if } p = 2;\\
&c_p = (2p)^{-\frac{p}{2}}, && C_p = \frac{p+1}{2(p-1)^{\frac{p}{2}}} && \text{if } p > 2.
\end{aligned}$$

Gronwall inequality. Let $T > 0$ and $c \geq 0$. Let $u(\cdot)$ be a Borel measurable bounded nonnegative function on $[0, T]$, and let v be a nonnegative integrable function on $[0, T]$ If

$$u(t) \leq c + \int_0^t v(s)u(s)ds \qquad \forall t \in [0, T],$$

then

$$u(t) \leq c \exp\left(\int_0^t v(s)ds\right) \qquad \forall t \in [0, T]. \tag{A.8}$$

Discrete Gronwall Inequality. Let M be a positive integer. Let u_k and v_k be non-negative numbers for $k = 0, 1, \ldots, M$. If

$$u_k \leq u_0 + \sum_{j=0}^{k-1} u_j v_j$$

then

$$u_k \leq u_0 \exp\left(\sum_{j=0}^{k-1} v_j\right). \tag{A.9}$$