Steklov Methods for non Linear Stochastic Differential Equations.

Ph.D. Dissertation

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Abstract

In this paper, we develop a new numerical method with asymptotic stability properties for solving stochastic differential equations (SDEs). Foundations for the new solver are the Steklov mean and an exact discretization for the deterministic version of the SDEs. Strong consistency and convergence properties are demonstrated for the proposed method. Moreover, a rigorous linear and nonlinear asymptotic stability analysis is carried out for the multiplicative case in a mean-square sense and for the additive case in a path-wise sense using the pullback limit. In order to emphasize characteristics of the Steklov discretization we use as benchmarks the stochastic logistic equation and the Langevin equation with a nonlinear potential of the Brownian dynamics. We show that the Steklov method has mild stability requirements and allows long-time simulations in several applications.

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Thesis Details

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The main body of this thesis consist of the following papers.

[3] Saúl Díaz-Infante, Silvia Jerez "Convergence and asymptotic stability of the explicit Steklov method for stochastic differential equations," *Journal of Computational and Applied Mathematics* DOI: 10.1016/j.cam.2015.01.016 14-FEB-2015 vol. 291, pp. 36-47, 2015.

[4] Saúl Díaz-Infante, Silvia Jerez, The Linear Steklov Method for SDEs with non-globally Lipschitz Coefficients: Strong convergence and simulation. Paper submitted at *Journal of Computational and Applied Mathematics.*

This thesis has been submitted for assessment in partial fulfillment of the PhD degree. The thesis is based on the submitted or published scientific papers which are listed above. Parts of the papers are used directly or indirectly in the extended summary of the thesis. As part of the assessment, co-author statements have been made available to the assessment committee and are also available at the Faculty.

Chapter 1

Introduction and main results

1.1 Introduction

Stochastic differential modeling becomes a rapidly-growing research area. It appears as extension of deterministic models with over-idealized fluctuations, but, actually, modeling with stochastic differential equations (SDE) permits to capture uncertainty in almost all phenomena — from price in markets until motion of particles [1, 25, 70].

However, we can obtain the explicit solution of only few SDEs. So, developing accurate stochastic numerical approximations, represents the first option to study and confirm (by simulation) the nature of a stochastic model. Stochastic numerics allows the analysis of some properties that are difficult or impossible to measure experimentally in laboratories, for example, long-time behavior. In this cases, we require that a numerical solvers be able to reproduce asymptotic behavior like mean square stability [30, 31, 63], usually, a linear analysis can be considered as the first step for understanding a method, but it is not an indicator of qualitative behavior on the nonlinear case [40]. Thus, some theoretical work on asymptotic stability has appeared for nonlinear SDEs Bokor [9] Buckwar et al. [12].

The first methods for solving SDEs were stochastic extensions of deterministic algorithms, for example schemes as the Eler-Maruyama, Taylor and Runge-Kutta [9, 14, 39]. Lamentably, sometimes their asymptotic stability conditions are very restrictive, for example in Brownian Dynamics Simulations, the EM discretization, is known as the Conventional method for Brownian Dynamics (CBD), because is the standard method to solve the Langevin equation that describe the motion of particles [11, 15, 22]. However, the operation time step size of this scheme has to be pint-size, otherwise the scheme becomes unstable. Now, the construction of methods focuses on structural or dynamic properties of a specific SDE. Some examples are the balanced methods for stiff SDE [55] the quasi-symplectic schemes for stochastic Hamiltonian systems [53] and SDEs with small noise [13].

Even numerical convergence and stability are well understood on SDE with globally Lipschitz continuous coefficients, this setup discard many important models from applications. Moreover, Hutzenthaler, Jentzen, and Kloeden reports in [35] that if a SDE has drift or diffusion, which grows faster that a linear function, then the EM diverges in strong and weak sense. This result opens a new chapter on the design of numerical methods — stochastic models in applications as Finance, Biology and Physics use SDEs with locally Lipschitz coefficients. In addition, Giles propose in [26] a new variance reducing technique that relies in strong numerical convergence, which optimize the traditional Monte Carlo simulation. Thus, developing explicit schemes, which converges in strong sense under super-linear coefficients attracts the right now attention.

In this line, recently research has been focused on modifying the EM method to obtain strong convergence under these conditions keeping its simple structure and its low computational cost. Several methods have been developed in this direction: the family of Tamed schemes [34, 36, 71, 73], a special type of balanced method [69], the stopped scheme [44] . For SDE with super-linear diffusion, Mao and Szpruch provide results for the strong convergence of Implicit methods as the BEM. However, the convergence of explicit schemes for SDEs with super-linear growth is still under development, in this line appears the works of Mao [47] with the truncated Euler method and [61] with a new kind of tamed scheme. In these works, the strong convergence of the proposed method is proved using the theory developed by in Higham, Mao, and Stuart [32] or by means of the new approach given by Hutzenthaler and Jentzen [34]. Both techniques prove strong convergence by verifying boundedness moments of the numerical and analytical solution of the underlying SDE. In spite of the recent work in this subject, it is still necessary to get more accurate numerical methods for SDE under super-linear growth and non-globally Lipschitz coefficients.

1.2 Main Results

Our main contribution follows two principal lines. The first is to design a explicit numerical scheme with good stability properties. We focus on explicit methods because we are interested on applications of Brownian Dynamics, so we want a simple complex and fast numerical solver. Also we want a stable scheme because we need simulation at long-time. For example in Brownian Dynamics, the self-diffusion coefficient is an asymptotic property. In this line we propose the Steklov method, which is the stochastic extension of an exact deterministic numerical scheme.

The second line consist in generalize the above scheme to a multidimensional setup and under more general coefficients. In this direction, we propose the Linear Steklov scheme (LS). This explicit method puts together ideas of implicit split schemes and a linear version of the Steklov average. We prove for this scheme a one-half order of convergence under a one sided Lipschitz condition and polynomial growth on the drift; and a globally Lipschitz condition on the diffusion. Also we provide numerical evidence that this method works of diffusion with super-linear growth, and in a specific example improves the Tamed Euler family of schemes.

1.3 Outline

After this introduction we present an overview of results that we will need in order to discuss our results. So, in chapter 3 we deals with the construction of the Steklov method and prove that have competitive stability properties. This new solver put together ideas from exact numerical approximation [52], implicit schemes and the EM method. Since in deterministic context we know that this schemes solves exactly the linear problem $\frac{dx}{dt} = \lambda x$, we expect to produce a scheme with acceptable linear stability, in fact, we confirm this for the scalar case and prove its strong convergence under classic globally Lipschitz conditions. In chapter 4 we propose an extension of the Steklov method in two directions: a multidimensional setting and more general coefficient. Here we prove convergence rates and provide a numerical evidence that the scheme works for super-linear growth diffusions. Finally in Chapter 5 we discuss a possible future direction for our research.

Chapter 2

Preliminaries

Here we present some results from stochastic analysis. This chapter focus on provide the basic information and tools to understand the nature of a SDE and its numerical approximation. For reference see [4, 39, 54, 59, 72].

2.1 Probability theory and Stochastic Processes

Probability theory is the field that studies the random phenomena. A random event is the set of outcomes from an experiment conducted under the same conditions with a variability in results . Probability theory aims to describe this variability. We denote by Ω the set of observable outcomes, ω, from a experiment or phenomenon. However, not every observable event is measurable, so for the purpose of probability theory, a family of subsets from Ω with particular properties a —-σ-algebra is needed. In the following, we formalize these concepts.

Definition 2.1.1 (σ-algebra)**.** Let Ω a set and $\mathcal{F}$ a family of subset of Ω, we call $\mathcal{F}$ a σ- algebra if the following properties hold:

(i) $\emptyset \in \mathcal{F}$,

(ii) if $F \in \mathcal{F}$ then $F^c \in \mathcal{F}$ where $F^c = \Omega \setminus F$,

(iii) if $\{F_i\}_{i=1}^{\infty} \in \mathcal{F}$ then $\bigcup\limits_{i \geq 1} F_i \in \mathcal{F}$.

Let $\mathcal{C}$ a collection of subsets of Ω. The σ-algebra generated by $\mathcal{C}$ denoted by $\sigma(\mathcal{C})$, is the smallest σ-algebra which contains the collection $\mathcal{C}$, that is $\sigma(\mathcal{C}) \supset \mathcal{C}$, and if $\mathcal{B}$ is an other σ-algebra containing $\mathcal{C}$, then $\mathcal{B} \supset \sigma(\mathcal{C})$.

Definition 2.1.2 (The Borel σ-algebra)**.** The σ-algebra generated by the collection of all open sets $U \subset \Omega$

A probability space is a triple $(\Omega, \mathcal{F}, \mathbb{P})$ where

- Ω is the set of all possible outcomes of an experiment.

- $\mathcal{F}$ is a conveniently σ-algebra of subsets of Ω.
- $\mathbb{P}$ is a probability measure; that is a function $\mathbb{P} : \mathcal{F} \to [0, 1]$ such that
  (i) $\mathbb{P}(A) \geq 0$ for all $A \in \mathcal{F}$.
  (ii) $\mathbb{P}$ is σ-additive, that is: If $\{A_n, n \geq 1\}$ is a collection of disjoint events, then
  $$\mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mathbb{P}(A_n).$$
  (iii) $\mathbb{P}(\Omega) = 1$.

Definition 2.1.3 (Random Variable)**.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\mathcal{B}(\mathbb{R}^d)$ the Borel's σ-algebra. A function $X : \Omega \to \mathbb{R}^n$ is said to be a random variable if X is $(\mathcal{F}, \mathcal{B}(\mathbb{R}^d))$-measurable, that is $X^{-1}(\mathcal{B}(\mathbb{R}^d)) \subset \mathcal{F}$.

Every random variable X induces a probability measure μ_X on $\mathbb{R}^d$ by

$$\mu_X(B) = \mathbb{P}(X^{-1}(B)), \qquad B \in \mathcal{B}(\mathbb{R}^d).$$

Having two different measures $\mathbb{Q}$, $\mathbb{P}$, on a measurable space we can transform one measure into the other via Radon-Nikodym theorem (see for example [72, Thm. 10.1.2]).

Theorem 2.1.1 (Radon-Nikodym)**.** *Let $\mathbb{P}$ and $\mathbb{Q}$ probability measures on the measurable space $(\Omega, \mathcal{F})$. Suppose that for all $B \in \mathcal{F}$ $\mathbb{Q}(B) = 0$ implies $\mathbb{P}(B) = 0$. Then there exist a integrable random variable X such that*

$$\mathbb{Q}(E) = \int_E X d\mathbb{P}, \qquad \forall E \in \mathcal{F}.$$

X is $\mathbb{P}$-a.s. unique and is written as $X = \dfrac{d\mathbb{Q}}{d\mathbb{P}}$.

This important result describes the density p of a random variable X as the $\mathbb{P}$-a.s. unique Radon-Nikodym derivative of the induced distribution μ_X w.r.t. Lebesgue measure, in other words

$$\mu_X(B) = \int_B p(x)dx.$$

Definition 2.1.4 (Expectation)**.** Let X be a integrable random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then the expectation of X is defined by

$$\mathbb{E}\left[X\right] = \int_{\Omega} X(\omega)\mathbb{P}(d\omega).$$

We mean by a stochastic process X a system which could stay at each moment on any state of a given set S

Definition 2.1.1 (Stochastic Process)**.** A stochastic process is a collection of random variables $X = \{X_t : t \in T\}$ on $(\Omega, \mathcal{F})$, which takes values in a measurable space $(S, \mathcal{S})$, and where the index $t \in [0, \infty)$, conveniently receive an interpretation as time. Thus for a fixed $\omega \in \Omega$, the function $X_t(\omega)$, $t \geq 0$ is a sample path of the process X associated with ω, and for any fixed t , $X_t(\omega)$, $\omega \in \Omega$ is a random variable.

The main purpose of this thesis deals with the numerical approximation of sample paths.

Definition 2.1.5 (Measurable Process)**.** A stochastic process X is measurable if the mapping

$$(t,\omega) \to X_t(\omega) : ([0,\infty) \times \Omega, \mathcal{B}([0,\infty)) \otimes \mathcal{F}) \to \left(\mathbb{R}^d, \mathcal{B}\left(\mathbb{R}^d\right)\right)$$

is measurable.

We equip the underlying sample space $(\Omega, \mathcal{F})$ with a filtration $\{\mathcal{F}\}_{t\geq 0}$ in order to keep track information about the past, present and future of a stochastic process. Formally, a filtration is a nondecreasing family of sub-σ-algebras of $\mathcal{F}$ such that $\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}$ for $0 \leq s \leq t < \infty$ and is called right continuous if $\mathcal{F}_t = \bigcap_{r>t} \mathcal{F}_r$ for all $t \geq 0$. Thus if the underlying probability space is complete, right continuous and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets, then we say that the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfies the usual conditions. In the following, we will work only on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ which verifies the usual conditions.

Given a stochastic process, the simplest choice of a filtration is that generated by the process itself, i.e. $\mathcal{F}_t^X := \sigma(X_s; 0 \leq s \leq t)$ the smallest σ-algebra with respect to which X_s is measurable for every $s \in [0,t]$. The introduction of this concept gives sense to the following.

Definition 2.1.6 (Adapted Process)**.** We call a process *adapted to the filtration* $\{\mathcal{F}\}_{t\geq 0}$ *if, for each* $t > 0$ *fixed* X_t *is a* $\mathcal{F}_t$*-measurable random variable.*

Clearly, every process X is adapted to $\{\mathcal{F}_t^X\}$.

Definition 2.1.7 (Progressively Measurable Process)**.** The stochastic process X is progressively measurable if the mapping

$$(s,\omega) \to X_s(\omega) : ([0,t] \times \Omega, \mathcal{B}([0,t]) \otimes \mathcal{F}_t) \to \left(\mathbb{R}, \mathcal{B}\left(\mathbb{R}^d\right)\right)$$

is measurable for each $t \geq 0$, that is, if, for each $t > 0$ and $A \in \mathcal{B}\left(\mathbb{R}^d\right)$, the set

$$\{(\omega, s) : 0 \leq s \leq t, \omega \in \Omega, X_s(\omega) \in A\}$$

belongs to the product σ-algebra $\mathcal{B}([0,t]) \otimes \mathcal{F}_t$.

2.1.1 Stopping time

Definition 2.1.8 (Stopping Time)**.** A random variable $\tau : \Omega \to [0,\infty]$ is called an $\{\mathcal{F}_t\}$-stopping time if $\{\omega : \tau(\omega) \leq t\} \in \mathcal{F}_t$ for any $t \geq 0$.

Theorem 2.1.2. *If* $\{X_t\}_{t\geq 0}$ *is a progressively measurable process and* τ *is a stopping time, then* $X_t \mathbf{1}_{\{\tau<\infty\}}$ *is* $\{\mathcal{F}_t\}$*-measurable.*

Theorem 2.1.3. *Let* $\{X_t\}_{t\geq 0}$ *be and* $\mathbb{R}^d$*-valued cádlág* $\{\mathcal{F}_t\}$*-adapted process and* $D \subset \mathbb{R}^d$ *an open set. Then* $\tau := \inf\{t \geq 0 : X_t \notin D\}$ *is an* $\{\mathcal{F}_t\}$*-stopping time.*

Lemma 2.1.1 (Fatou)**.** *For any non negative measurable functions* $\{X_k\}_{k\geq 1}$ *on* $(\Omega, \mathcal{F}, \mathbb{P})$*, we have*

$$\mathbb{E}\left[\liminf_{k\to\infty} X_k\right] \leq \liminf_{k\to\infty} \mathbb{E}\left[X_k\right].$$

2.1.2 Conditional Expectation

Conditional expectation plays a very important role in the modern probability theory. It gives sense and foundation for the definitions of martingales and Markov processes. In fact, other areas of probability as stochastic dynamics, conditioning permits to describe and to analyze dynamical systems with randomness. Roughly speaking, the conditional expectation is an average that considers only a portion of information. Assume $X \in L_1(\Omega, \mathcal{F}, \mathbb{P})$ and let $\mathcal{G} \subset \mathcal{F}$ a sub-σ-algebra. Then the conditional expectation of the random variable X given $\mathcal{G}$ is the new random variable $Y = \mathbb{E}\left[X|\mathcal{G}\right]$ such that

(i) $\mathbb{E}\left[X|G\right]$ is $\mathcal{G}$-measurable and integrable.

(ii) For all event $G \in \mathcal{G}$ we have $\displaystyle\int_G X d\mathbb{P} = \int_G \mathbb{E}\left[X|\mathcal{G}\right] d\mathbb{P}$.

This new random variable is unique in the sense that if there is an other $\tilde{Y}$ satisfying the same two above properties, then $\mathbb{P}[Y \neq \tilde{Y}] = 0$. In this case $\tilde{Y}$ is said to be a version of the conditional expectation $\mathbb{E}\left[X|\mathcal{G}\right]$. Now we list some standard properties of the conditional expectation [72]. Here X_1, X_2, Z are integrable random variables, a_1,$a_2 \in \mathbb{R}$, and $\mathcal{G}$, $\mathcal{H}$ are sub-σ-algebras of $\mathcal{F}$.

(E1) If Y is any version of $\mathbb{E}\left[X|\mathcal{G}\right]$, then $E[X] = E[Y]$.

(E2) If X is $\mathcal{G}$ measurable, then $\mathbb{E}\left[X|\mathcal{G}\right] = X$, $\mathbb{P}$-a.s.

(E3) (Linearity) $\mathbb{E}\left[a_1 X_1 + a_2 X_2|\mathcal{G}\right] = a_1\mathbb{E}\left[X_1|\mathcal{G}\right] + a_2\mathbb{E}\left[X_2|\mathcal{G}\right]$ $\mathbb{P}$-a.s.
Clarification: if Y_1 is a version of $\mathbb{E}\left[X_1|\mathcal{G}\right]$ and Y_2 is a version of $\mathbb{E}\left[X_2|\mathcal{G}\right]$, then $a_1 Y_1 + a_2 Y2$ is a version of $\mathbb{E}\left[a_1 X_1 + a_2 X_2|\mathcal{G}\right]$.

(E4) (Positivity) If $X \geq 0$, then $\mathbb{E}\left[X|\mathcal{G}\right] \geq 0$, $\mathbb{P}$-a.s.

(E5) (Conditional Monotone Convergence) If $0 \leq X_n \uparrow X$, then $\mathbb{E}\left[X_n|\mathcal{G}\right] \uparrow \mathbb{E}\left[X|\mathcal{G}\right]$, $\mathbb{P}$-a.s.

(E6) (Conditional Fatou) If $X_n \geq 0$, then $\mathbb{E}\left[\liminf_{n\to\infty} X_n|\mathcal{G}\right] \leq \liminf_{n\to\infty} \mathbb{E}\left[X_n|\mathcal{G}\right]$.

(E7) (Conditional Dominated Convergence) If $|X_n(\omega)| \leq V(\omega)$ for all n, $\mathbb{E}\left[V\right] < \infty$, and $X_n \to X$ $\mathbb{P}$-a.s., then $\mathbb{E}\left[X_n|\mathcal{G}\right] \to \mathbb{E}\left[X|\mathcal{G}\right]$, $\mathbb{P}$-a.s.

(E8) (Conditional Jensen) If f is a real-valued convex function, then

$$f\left(\mathbb{E}\left[X|\mathcal{G}\right]\right) \leq \mathbb{E}\left[f(X)|\mathcal{G}\right].$$

(E9) (Tower Property) If $\mathcal{H}$ is a sub-σ-algebra of $\mathcal{G}$, then

$$\mathbb{E}\left[\mathbb{E}\left[X|\mathcal{G}\right]|\mathcal{H}\right] = \mathbb{E}\left[X|\mathcal{H}\right], \qquad \mathbb{P}\text{-a.s.}.$$

(E10) If Z is $\mathcal{G}$-measurable and bounded, then $\mathbb{E}\left[ZX|\mathcal{G}\right] = Z\mathbb{E}\left[X|\mathcal{G}\right]$.

(E11) If X is independent from $\mathcal{H}$, then $\mathbb{E}\left[X|\mathcal{H}\right] = \mathbb{E}\left[X\right]$, $\qquad \mathbb{P}$-a.s.

Now consider a filtered space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ and define

$$\mathcal{F}_\infty := \sigma\left(\bigcup_{t\geq 0} \mathcal{F}_t\right) \subset \mathcal{F}.$$

Definition 2.1.9 (Martingale)**.** A process $\{M_t\}_{t\geq 0}$ is called a martingale (relative to $(\{F_t\}_{t\geq 0}, \mathbb{P})$ if

(i) M is adapted,

(ii) $\mathbb{E}\left[|M_t|\right] < \infty \qquad$ for all $t \geq 0$,

(iii) $\mathbb{E}\left[M_t|\mathcal{F}_s\right] = M_s, \qquad \mathbb{P}\text{-a.s.}, \qquad 0 \leq s \leq t.$

In this way, a *supermartingale* (relative to $(\{F_t\}_{t\geq 0}, \mathbb{P})$ is defined similarly, except that (iii) is replaced by

$$\mathbb{E}\left[M_t|\mathcal{F}_s\right] \leq M_s \qquad \mathbb{P}\text{-a.s.}, \qquad 0 \leq s \leq t,$$

and a *submartingale* is defined with (iii) replaced by

$$\mathbb{E}\left[M_t|\mathcal{F}_s\right] \leq M_s \qquad \mathbb{P}\text{-a.s.}, \qquad 0 \leq s \leq t.$$

Theorem 2.1.4. *Let $\{M_t\}_{t\geq 0}$ be and $\mathbb{R}^d$-valued martingale with respect to $\{\mathcal{F}_t\}$, and let θ, ρ two finite stopping times. Then*

$$\mathbb{E}\left[M_\theta\right]\mathcal{F}_\rho = M_{\theta\wedge\rho}.$$

Definition 2.1.10 (Local Martingale)**.** An $\mathbb{R}^d$-valued $\{F_t\}$-adapted integrable process $\{M_t\}_{t\geq 0}$ is said to be a *local martingale* if there exists a nondecreasing sequence $\{\tau_k\}_{k\geq 1}$ of stopping times with $\tau_k \uparrow \infty$ $\mathbb{P}$-a.s. such that $\{M_{\tau\wedge t} - M_0\}$ is a martingale.

A fundamental process in this thesis is the Brownian Motion.

Definition 2.1.11 (Brownian Motion)**.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with filtration $\{\mathcal{F}_t\}_{t\geq 0}$. A standard unidimensional Brownian motion is a real-valued continuous adapted process $\{W_t\}_{t\geq 0}$ which satisfies:

(i) $W_0 = 0, \qquad \mathbb{P} - \text{a. s.};$

(ii) the increments $W_t - W_s$ are normally distributed with mean zero and variance $t - s$ for $0 \leq s \leq t < \infty$;

(iii) $W_t - W_s$ is independent of $\mathcal{F}_s$.

Consider a Brownian motion $\{W_t\}_{t\geq 0}$ and a sequence of times $0 \leq t_0 < t_1 < \ldots < t_k < \infty$. Results that $\{W_t\}_{t\geq 0}$ has independent increments, that is, the random variables $W_{t_i} - W_{t_{i-1}}$ $1 \leq i \leq k$ are independent. Moreover, the distribution of $W_{t_i} - Wt_{i-1}$ depends only on the difference $t_i - t_{i-1}$, in this sense, we say that the Brownian motion has stationary distribution. With this in mind, also we can say that this process is a martingale. As we will see, the above is fundamental for the numerical approximations of SDEs.

2.2 Stochastic Calculus and SDEs

In this section, we recall some basic results ot the Itô integral

$$\int_0^t f(s)dW(s).$$

with respect an m-dimensional Brownian Motion, $\{W(t)\}$, for a class of $d \times m$-matrix-valued processes $\{f(t)\}$.

Definition 2.2.1. Let $0 \leq a < b < \infty$. We denote by $\mathcal{M}^2([a,b];\mathbb{R})$ the space of all real-valued measurable $\{\mathcal{F}\}$-adapted processes $f = \{f(t)\}_{a \leq t \leq b}$ such that

$$\|f\|^2_{a,b} = \mathbb{E}\left[\int_a^b |f(t)|^2 dt\right] < \infty.$$

Theorem 2.2.1. *Assume $f \in \mathcal{M}\left([a,b];\mathbb{R}^{d\times m}\right)$ and let ρ, τ be two stopping times such that $0 \leq \rho \leq \tau \leq T$. Then*

$$\mathbb{E}\left[\int_\rho^\tau f(t)dW(t)|\mathcal{F}_\rho\right] = 0,$$
$$\mathbb{E}\left[\left|\int_\rho^\tau f(t)dW(t)\right|^2 |\mathcal{F}_\rho\right] = \mathbb{E}\left[\int_\rho^\tau |f(t)|^2 dt|\mathcal{F}_\rho\right].$$

Definition 2.2.2 (Itô process)**.** A d-dimensional Itô process is a $\mathbb{R}^d$-valued continuous adapted process $X(t) = (X_1(t), \dots, X_d(t))^T$ on $t \geq 0$ of the form

$$X(t) = X(0) + \int_0^t f(s)ds + \int_0^t g(s)dW(s),$$

where $f = (f_1, \dots, f_d)^T \in \mathcal{L}_1(\mathbb{R}_+;\mathbb{R}^d)$ and $g = (g_{ij})^{d\times m} \in \mathcal{L}_2(\mathbb{R}_+;\mathbb{R}^{d\times m})$. We will say that $X(t)$ has stochastic differential $dX(t)$ on $t \geq 0$ given by

$$dX(t) = f(t)dt + g(t)dW(t).$$

Theorem 2.2.2 (The multi-dimensional Itô formula)**.** *Let $X(t)$ be a d-dimensional Itô process on $t \geq 0$ and differential*

$$dX(t) = f(t)dt + g(t)dW(t),$$

where $f = (f_1, \dots, f_d)^T \in \mathcal{L}_1(\mathbb{R}_+;\mathbb{R}^d)$ and $g = (g_{ij})^{d\times m} \in \mathcal{L}_2(\mathbb{R}_+;\mathbb{R}^{d\times m})$. Let $V \in \mathcal{C}^{2,1}(\mathbb{R}^d \times \mathbb{R}_+;\mathbb{R})$. Then $V(X(t),t)$ is again an Itô process with stochastic differential given by

$$dV(X(t),t) = \left[V_t(X(t),t) + V_x(X(t),t)f(t) + \frac{1}{2}\operatorname{Tr}\left(g^T(t)V_{xx}(X(t),t)g(t)\right)\right]dt + V_x(X(t),t)g(t)dW(t) \qquad \mathbb{P} - \text{a. s.}$$

For simplicity of notation and under the same meaning as above, we define a diffusion generator L as

$$LV(X(t),t) = V_t(X(t),t) + V_x(X(t),t)f(t) + \frac{1}{2}\operatorname{Tr}\left(g^T(X)V_{xx}(X(t),t)g(t)\right). \tag{2.1}$$

2.3 Numerical Methods of SDEs

The topic of this thesis is the development of new numerical solutions for stochastic differential equations (SDEs)

$$dy(t) = f(y(t))dt + g(y(t))dW(t), \qquad t \in [0,T], \qquad y(0) = y_0. \tag{2.1}$$

Generally, we know the analytical solution for a few SDEs. So, we need numerical schemes in order to approximate the solutions of eq. (2.1). In this section present the most popular numerical methods for SDEs and also give some fundamental results in numerical analysis of stochastic differential equations. We star describing the solutions of SDE (2.1).

Throughout this section, we consider the following setup. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ a filtered and complete probability space with the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ generated by the m-dimensional Brownian process $B_t = (B_t^{(1)} \dots B_t^{(m)})^T$. Now we start by establishing the definition of strong solution and the main theorems of existence and uniqueness.

Definition 2.3.1 (Strong Solution)**.** The strong solution of SDE (2.1) on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, respect to a fixed Brownian motion B and initial condition y_0, is a continuous stochastic process $y = \{y(t) : 0 \leq t < \infty\}$ with the following properties:

(SS-1) y is adapted to the filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$,

(SS-2) $\mathbb{P}[y(0) = y_0] = 1$,

(SS-3) $\mathbb{P}\left[\int_0^t f(s, y(s)) + g(s, y(s))ds < \infty\right] = 1$,

(SS-4) $y(t) = y_0 + \int_0^t f(y(s))ds + \int_0^t g(y(s))dB_s \qquad \text{a. s.}$

Now, consider SDE (2.1) where y_0 is a constant, f is a measurable $d-$vector valued function, g is a measurable $d \times m$-matrix-valued measurable function. In order to assure a unique solution we suppose the following.

Assumption 2.3.1. The coefficients of SDE (2.1) f, g satisfy:

(EU1) *Global Lipschitz condition.* There is a positive constant L such that

$$|f(x) - f(z)| \vee |g(x) - g(z)| \leq L|x - z|, \qquad \forall x, z \in \mathbb{R}^d.$$

(EU2) *Linear Growth condition*. There is a positive constant L such that

$$|f(x)|^2 \vee |g(x)|^2 \leq L(1 + |x|^2), \qquad \forall x \in \mathbb{R}^d.$$

Theorem 2.3.1 (Existence and uniqueness of solutions)**.** *If Assumption 2.3.1 holds, then exists a path-wise unique strong solution of the SDE* (2.1) *with initial condition* y_0 *on the time-interval* $[0,T]$ *and*

$$\sup_{t\in[0,T]} \mathbb{E}\left[|y(t)|^2\right] < \infty. \tag{2.2}$$

Here, path-wise uniqueness means that if $x(t)$ and $y(t)$ are two solutions of SDE (2.1), then

$$\mathbb{P}\left[\sup_{t\in[0,T]} |x(t)-y(t)|=0\right]=1.$$

It is worth mentioning that there exist a unique solution even when the linear growth conditions are removed, in [32], the authors have derived a existence and uniqueness result that depend on a weaker continuity condition on f and g than the Lipschitz condition. In chapter 4 we well explore this ideas. Notice that we need to assure existence and uniqueness of the solution of SDE (2.1) in order to justify the development of a numerical approximation. Assuming that, we now propose several numerical approximations for this SDE.

2.3.1 Explicit and implicit schemes

Consider SDE (2.1) on time interval $[0,T]$, we define a time partition of the time interval $\mathcal{P}^N$ as a finite equidistant sequence of N points $t_k := kh$, for $0 \leq k \leq N$, taking the step size as $h = T/N$.

Definition 2.3.2 (discrete approximation)**.** We call a cádlag process $Y = \{Y(t), t \geq 0\}$, a discrete approximation of the solution of SDE (2.1) with step-size h over a partition $\mathcal{P}^N_{[0,T]} = \{0, h, 2h, \ldots, Nh\}$ if $Y(t_k)$ is $\mathcal{F}_{t_k}$-measurable and $Y(t_{k+1})$ can be expressed as a function of

$$Y(t_0)\ldots Y(t_k), 0, t_1, \ldots, t_k, t_{k+1}$$

and a finite number l of measurable random variables $Z_{k+1,j}$, $j = 1\ldots l$.

We present some of the most known numerical schemes which will be useful to show the efficiency of our method. Here, and in the next Chapter we will suppose Assumption 2.3.1.

Euler-Maruyama

The most easy implementable, popular and studied method is the *Euler-Maruyama* (EM) scheme. Given the SDE (2.1) and a time step-size h it is defined by taking

$$Y_{k+1} = Y_k + hf(Y_k) + g(Y_k)\Delta W_k, \qquad Y_0 = y_0, \tag{2.3}$$

where $\Delta W_k = W_{t_{k+1}} - W_{t_k}$. If we consider a implicit approximation for the drift coefficient, we obtain the *backward Euler-Maruyama* (BEM) [48], under the same notation as above, it has the recurrence:

$$Y_{k+1} = Y_k + hf(Y_{k+1}) + g(Y_k)\Delta W_k. \tag{2.4}$$

The θ-Maruyama scheme

This scheme generalizes the Euler-Maruyama algorithm in the sense that is based on using the parameter θ to weight contributions of the explicit and implicit approximations to the drift coefficient. Its recurrence is

$$Y_{k+1} = Y_k + h(1-\theta)f(Y_k) + \theta f(Y_{k+1}) + g(Y_k)\Delta W_k \qquad \theta \in [0,1]. \tag{2.5}$$

Note that if $\theta = 0$ we recover the explicit EM and if $\theta = 1$ we obtain the BEM.

Split Step Backward Euler

Also we will apply the split-step backward Euler method proposal by the authors in [32]. This scheme is defined by

$$Y_k^{\star} = Y_k + hf(Y_k^{\star}), \qquad Y_0 = y_0, \tag{2.6}$$

$$Y_{k+1} = Y_k^{\star} + g(Y_k^{\star})\Delta W_k. \tag{2.7}$$

2.4 Theorycal Properties of Numerical Methods

It is always important in the construction of new algorithms to study the global discretization error and give an estimation of the speed of convergence. Here, they are carried out with the analysis of the properties of consistency and convergence, see [39]. Also we will study the stability of our numerical schemes. While convergence give us information about behavior of a scheme on a fixed time interval letting the time-step small, the stability analysis allow us to understand behavior of the approximation for a fixed step size when the time interval expands to infinity. For simplicity, we study these properties for a one-dimensional autonomous SDE

$$dy(t) = fy(t))dt + g(y(t))dW(t). \tag{2.1}$$

As a fist step, we suppose that Assumption 2.3.1 are fulfill. But in Chapter 5 we will work under more general setting. Let see the classic definitions of these concepts (see e.g. [39]).

2.4.1 Strong consistency and convergence

It is always important in the construction of new algorithms to study the global discretization error and give an estimation of the speed of convergence. Here, they are carried out with the analysis of the properties of consistency and convergence, see [39]. For simplicity, we study these properties for a one-dimensional autonomous SDE

$$dy(t) = f(y(t))dt + g(y(t))dW(t), \tag{2.2}$$

satisfying the necessary conditions of existence and uniqueness of solution. We give a general setting of these concepts.

Definition 2.4.1. A time discrete approximation Y_n is strongly consistent if there is a nonnegative function $c = c(h)$ such that the following conditions hold for all fixed values $Y_n = y$, and $n = 0, 1, \ldots, N$,

1. $\lim\limits_{h \to 0} c(h) = 0,$
2. $\mathbb{E}\left(\left|\mathbb{E}\left(\dfrac{Y_{n+1} - Y_n}{h}\,|\mathcal{F}_{\tau_n}\right) - F\left(Y_n\right)\right|^2\right) \leq c(h),$
3. $\mathbb{E}\left(\dfrac{1}{h}\,\left|Y_{n+1} - Y_n - \mathbb{E}\left(Y_{n+1} - Y_n\,|\mathcal{F}_{\tau_n}\right) - G\left(Y_n\right)\Delta B_n\right|^2\right) \leq c(h).$

Put a paragraph abaout the importans of the convergence order

On sake of clearness we define $n_t := \max_{n=1\ldots N}\{n : t_n \leq t\}$.

Definition 2.4.2. A time discrete approximation Y_n is strongly convergent if for the end time T is verified

$$\lim_{h\to 0} \mathbb{E}\,|y(T) - Y_{n_T}| = 0.$$

Now, we state a theorem that connects both concepts.

Theorem 2.4.1 ([39, Thm. 9.6.2])**.** *If Y_n is a strongly consistent time discrete approximation maximum step h of the solution of the SDE (3.1) with $Y_0 = y_0$. Then Y_n converges strongly to the solution y.*

Definition 2.4.1 (order)**.** A discrete approximation Y_k *converges strongly with order δ* at time T if there exist a positive constant C independent of the step size h, such that

$$\mathbb{E}\left[|y(T) - Y_{n_T}|\right] \leq Ch^{\delta}. \tag{2.3}$$

In addition, we say that a discrete approximation *converges strongly* with order δ *uniformly* on time if

$$\mathbb{E}\left[\sup_{1\leq k\leq N} |y(t_k) - Y_k|\right] \leq Ch^{\delta}. \tag{2.4}$$

2.4.2 Higham-Mao-Stuart proof convergence technique

Now we discuss a technique reported by Higham, Mao, and Stuart [32] to prove strong convergence of stochastic numerical methods under non-globally Lipschitz conditions. This kind of analysis is useful whenever moment bounds can be established for both, the EM scheme and other method that can be shown to be "close" to it. A vast amount of literature has been used this technique, some of these works are [8, 28, 33, 34, 36, 41, 48, 69], among others.

To review this technique, we recall two conveniently versions for the continuous extension of the EM scheme,

$$\begin{aligned}\overline{Y}(t) &:= Y_{\eta(t)} + (t - t_{\eta(t)})f(Y_{\eta(t)}) + g(Y_{\eta(t)})(W(t) - W_{\eta(t)}), \\ \eta(t) &:= k, \text{ for } t \in [t_k, t_{k+1}),\end{aligned} \tag{2.5}$$

and

$$\overline{Y}(t) := Y_0 + \int_0^t f(Y_{\eta(s)})ds + \int_0^t g(Y_{\eta(s)})dW(s).$$

So, with this notation we have $\overline{Y}(t_k) = Y_k$, see Figure 2.1. Using the continuous extension (2.5) and the uniform mean square norm, the authors work with a stronger version of the ms-error

$$\mathbb{E}\left[\sup_{0\leq t\leq t} |y(t) - \overline{Y}(t)|^2\right].$$

Then, in order to prove strong convergence of the EM method, the authors require the following assumptions.

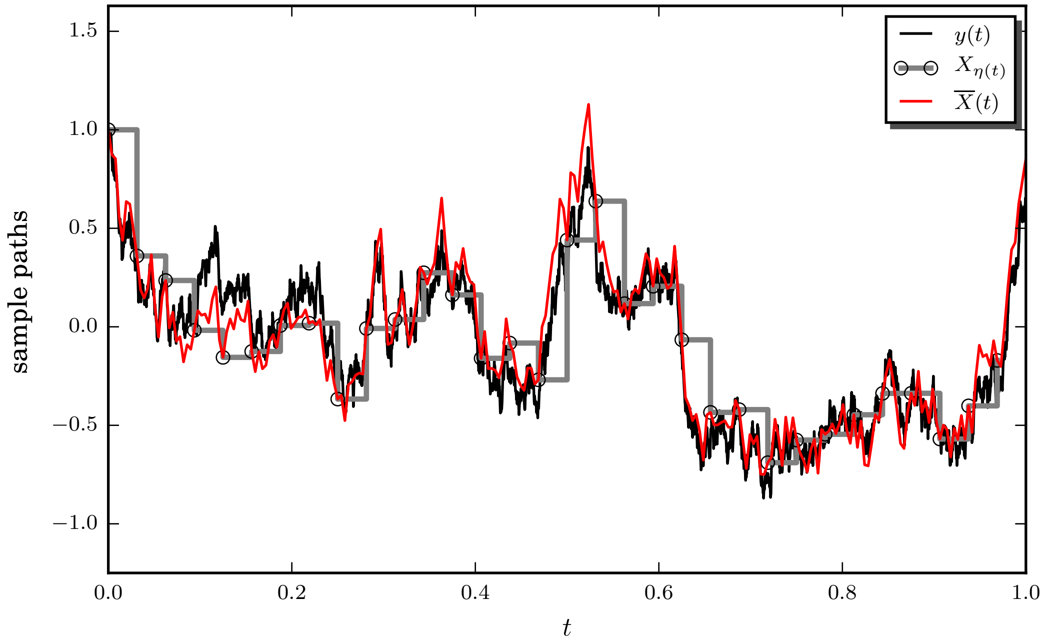

Figure 2.1: The red line represents the continuous extension of the EM scheme. The continuous gray line is the $Y_{\eta(t)}$ process defined in (2.3).

Assumption 2.4.1. For each $R > 0$ there is a positive constant C_R, depending only on R, such that

$$|f(x) - f(y)|^2 \vee |g(x) - g(y)|^2 \leq C_R |x - y|^2, \quad \forall x, y \in \mathbb{R}^d \text{ with } |x| \vee |y| \leq R. \tag{2.6}$$

And for some $p > 2$, there is a constant A such that

$$\mathbb{E}\left[\sup_{0\leq t\leq T} |\overline{Y}(t)|^p\right] \vee \mathbb{E}\left[\sup_{0\leq t\leq T} |y(t)|^p\right] \leq A. \tag{2.7}$$

In [32], the authors prove that the Assumption 2.4.1 is sufficient to ensure strong convergence for the EM scheme, namely

Theorem 2.4.2 ([32, Thm 2.2]). *Under Assumption 2.4.1, the EM scheme* (2.3) *with continuous extension* (2.5) *satisfies*

$$\lim_{h\to 0} \mathbb{E}\left[\sup_{0\leq t\leq T} |\overline{Y}(t) - y(t)|^2\right] = 0. \tag{2.8}$$

Applying this result, the authors prove the strong convergence of an implicit split-step variant of the EM, the SSEM method. Their technique consist in proving each assertion of the following steps.

Step 1: The SSEM for SDE (4.1) is equivalent to the EM for the following conveniently SDE

$$dy_h(t) = f_h(y_h(t))dt + g_h(y_h(t))dW(t). \tag{2.9}$$

Step 2: The solution of the modified SDE (2.9) has bounded moments and it is "close" to y the sense of the uniform mean square norm $\mathbb{E}\left[\sup_{0\leq t\leq T}|\cdot|^2\right]$.

Step 3: Show that the SSEM method for the SDE (4.1) has bounded moments.

Step 4: There is a continuous extension of the SSEM, $\overline{Z}(t)$, with bounded moments.

Step 5: Use the above steps and Theorem 2.4.2 to conclude that

$$\lim_{h\to 0}\left\{\mathbb{E}\left[\sup_{0\leq t\leq T}|y_h(t)-y(t)|^2\right]+\mathbb{E}\left[\sup_{0\leq t\leq T}|\overline{Z}(t)-y_h(t)|^2\right]\right\}=0. \tag{2.10}$$

In the next section, using Theorem 2.4.2 and this technique, we will prove the strong convergence o the LS method (4.2)–(4.3). However, if we are interested in simulating the solution of the SDE (2.1) for large periods of time, we need to use stable methods. We can interpret the stability of a numerical method, in some sense, as its capacity to preserve the dynamical structure of the solution in that sense. Here we recall the topics that we will work in the next chapters.

2.4.3 Numerical Stability

Roughly speaking the aim of a numerical stability analysis answer the following. For what choices of step-size does the a numerical method reproduce behavior of the underlying SDE? The answer of this question is linked of course, whit the kind o behavior, for example in problems where know a priori that each solution path tends to fixed point, or stay on a bounded set or reach an absorbent process. Usually the first step in this direction is a linear stability analysis. This study mimics the deterministic context, which is based in the following steps:

Step 1: Expand in Taylor series around a fixed point the right hand side of a nonlinear ordinary differential equation $x'(t)=f(t,x)$.

Step 2: Form a linear system with the Jacobian of f evaluated at the equilibrium $x'(t)=Ax(t)$.

Step 3: Diagonalize to decouples the linear system and study equations of the form $x'(t)=\lambda x(t), \lambda\in\mathbb{C}$.

If all eigenvalues of A are different of zero, then the theorem of Hartman justify the use of this last equation for study behavior around sufficient small neighborhood. So, ones seek conditions for assure that the numerical methods preserves the dynamics of underlying test.

In stochastic numerics, the history runs similar. But here, the usual test is a linear SDE with multiplicative noise. The advantage of this model is that has the same unique fixed point as its deterministic analogous, the origin [30] . The other test equation is the linear SDE with additive noise. However, for these model the concepts of numerical stability was unclear. The first works with this test [6, 29, 54], differs about the meaning of fixed point and stability. Recently, the works of De la Cruz Cancino, Biscay, Jimenez, Carbonell, and Ozaki [20] and Buckwar, Riedler, and Kloeden [12] analyze these model using the theory of random dynamical systems, which in our opinion clarifies this issue.

Naturally the nonlinear case, is even more complex. Although Lyapunov theory is the usual approach in applications [38], a more general novel approach based on the theory of random dynamical systems [5] attracts the now days attention. In the following we provide this notions.

Linear Stability

Multiplicative noise

Consider the scalar linear SDE

$$dy(t) = \lambda y(t)dt + \xi y(t)dW(t), \quad X_0 = x_0, \quad \lambda, \xi \in \mathbb{C}. \tag{2.11}$$

The solutions of this SDE have the following property

$$\lim_{t\to\infty} \mathbb{E}\left[|y(t)|^2\right] = 0 \Leftrightarrow \mathrm{Re}(\lambda) + \frac{1}{2}|\xi|^2 < 0. \tag{2.12}$$

A solution that satisfies the previous limit is a *mean-square stable* solution. Note that for $\xi = 0$ we have, $\mathrm{Re}(\lambda) < 0$, which is the stability condition for the deterministic case.

Applying the EM method (2.3) to test (2.11), we obtain

$$Y_{k+1} = \left(1 + h\lambda + \sqrt{h}\xi V_k\right) Y_k, \tag{2.13}$$

where each V_k is an independent $\mathcal{N}(0,1)$ random variable. In order to study the stability properties of the EM, we must therefore study the long time behavior of random variables of the form (2.13). Analogously, we will say that sequence (2.13) is mean-square stable if $\lim_{k\to\infty} \mathbb{E}\left[|Y_k|^2\right] = 0$. Note that the EM depends upon the problem parameters λ and ξ, and the method parameter h. Then for a particular choice of parameters, we will say that the EM is mean-square stable if it produces a mean-square stable sequence. Our interest lies in finding the parameter values for which the EM method is stable, and comparing results with the region $\mathrm{Re}(\lambda) + \frac{1}{2}|\xi|^2 < 0$ in (2.12) for the underlying SDE (see Figure 2.2). In this line we have the following result.

Theorem 2.4.3. *Consider the EM method for the linear scalar SDE* (2.11)*. If the parameters λ, ξ, and the step size h satisfies*

$$\mathrm{Re}(\lambda) + \frac{1}{2}\left(|\xi|^2 + h|\lambda|^2\right) < 0.$$

Then the EM solution is mean square stable.

Additive noise

Here we study the additive linear SDE:

$$dy(t) = \lambda y(t)dt + \xi dW(t), \qquad y_0 = y_(t_0), \qquad \lambda, \xi \in \mathbb{R}. \tag{2.14}$$

where λ, $\xi \in \mathbb{C}$ and X_{t_0} is the initial value of the process at time t_0. Equation (2.14) has the following exact solution:

$$y(t) = \exp(\lambda(t-t_0))y(t_0) + \xi \exp(\lambda t)\int_{t_0}^{t} \exp(-\lambda s)dW(s), \qquad t \geq t_0. \tag{2.15}$$

The stochastic process $y(t)$ defined in (3.6) is known as the *Ornstein-Uhlenbeck*'s (OU) process. According to [29], the OU process is *asymptotically mean stable* if $\lim_{t\to\infty} \mathbb{E}y(t) = 0$ and is *asymptotically mean square*

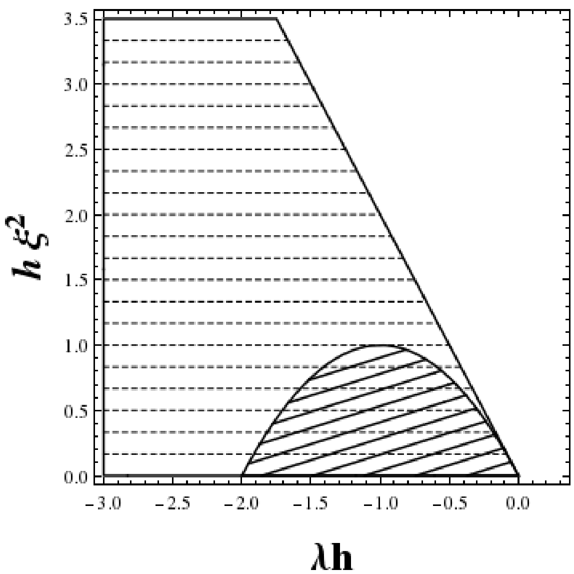

Figure 2.2: Mean square regions of stability. The horizontal lines represents the stability region of SDE (2.11) and diagonal lines for the EM solution.

stable if $\lim_{t\to\infty} \mathbb{E}\,|y(t)|^2 = -\xi/2Re(\lambda)$. Both limits are verified if $\lambda < 0$. Analogous stability properties are given for stochastic difference equations with additive noise [62]. Now, if we consider $\lambda < 0$ then the OU solution (3.6) does not convergence as t tends to infinity but has the following pullback limit:

$$\lim_{t_0\to-\infty} y(t) = \widehat{O}_t := \exp(\lambda t)\int_{-\infty}^{t} \exp(-\lambda s)dW(s), \tag{2.16}$$

$W(t)$ is now defined for all $t \in \mathbb{R}$, see [5, 40]. Furthermore, the process (3.8) is a stationary solution of the additive linear SDE which attracts all other solutions in forward time and path-wise sense. Moreover, it is a finite process for all $t \geq T_{D(\omega)}$ ($\omega \in \Omega$) for appropriate families $D(\omega)$ of bounded sets of initial conditions, see [60]. Therefore, we can evaluate the numerical stability of a given stochastic method by examine if this scheme reproduce the pullback asymptotic behavior. For example, the explicit EM scheme for (2.14)

$$Y_{k+1} = (1+\lambda h)Y_n + \xi\Delta W_n,$$

given a initial value Y_{k_0}, has the form

$$Y_{k+1} = (1+\lambda h)^{k-k_0}Y_{k_0} + \xi\sum_{j=k_0}^{k-1}(1+\lambda h)^{k-1-j}\Delta W_j.$$

So, the path-wise pullback limit (taking $k_0 \to \infty$ with k held fixed and $Y_{k_0} = Y_0$ for all Y_{k_0} and constant time step h) exists, provided that $0 < h < 2/(-\lambda)$, $\lambda < 0$, and is given by

$$\widehat{O}_k^{(h)} := \xi \sum_{j=-\infty}^{k} (1+\lambda h)^{1-k-j} \Delta W_j,$$

for more details see the work of Buckwar, Riedler, and Kloeden [12].

Non-Linear Stabbility

Now we discuss the nonlinear case for multiplicative and additive noise.

Multiplicative Noise

We start with a notion of stability which emulates the continuity respect to initial conditions of deterministic ODEs.

Definition 2.4.3 (Baker and Buckwar [7]). Let Y_n and $\widehat{Y}_n$ two different numerical recurrences with corresponding initial process Y_0 and $\widehat{Y}_0$. We shall say that a discrete time, Y is *numerically zero-stable in quadratic mean-square sense* if given $\epsilon > 0$, there are positive constants h_0 and $\delta = \delta(\epsilon, h_0)$ such that for all $h \in (0, h_0)$ and positive integers $n \leq T/h$ whenever $\mathbb{E}\left|Y_0 - \widehat{Y}_0\right|^2 < \delta$ then

$$\rho_n := \mathbb{E}\left|Y_n - \widehat{Y}_n\right|^2 < \epsilon. \tag{2.17}$$

If the method is stable and $\rho_n \to 0$ when $n \to \infty$, then the method is *asymptotically zero-stable in the quadratic mean-square sense*.

Also, in [7] provides a result to characterizes this type of stability. Here we enunciated it for the EM.

Theorem 2.4.4 ([7, Thm. 4]). *Let C_1, C_2 and C_3 generic positive constants which not depends on h and V a $\mathcal{N}(0,1)$ random variable. If the coefficients of SDE (2.1) satisfies the estimates*

$$\begin{aligned} \left|\mathbb{E}\left[f(x)h + g(x)\sqrt{h}V - \left(f(x')h + g(x')\sqrt{h}V\right)\right]\right| &\leq C_1 h\left(|x - x'|\right), \\ \mathbb{E}\left[\left|hf(x) + g(x)\sqrt{h}V - \left(f(x')h + g(x')\sqrt{h}V\right)\right|^2\right] &\leq C_2 h\left(|x - x'|\right), \end{aligned}$$

then the EM method (2.3) for (2.1) is zero-stable in the quadratic mean-square sense.

Additive noise

Nonlinear differential equations have more complex dynamics than the linear case and the same occurs for the finite difference equations. So, Caraballo and Kloeden in [17] extend the nonlinear stability theory of the deterministic numerical analysis given in [40] to the stochastic case. They propose and justify the use of the following SDE as a test equation with additive noise:

$$dy(t) = (Ay(y) + f(y(t)))\,dt + \xi dW(t), \tag{2.18}$$

where A is a $d \times d$ stiff matrix and function $f : \mathbb{R}^d \to \mathbb{R}^d$ is a nonlinear and non-stiff function that satisfies, a *contractive one-sided Lipschitz* condition with constant $L_1 > 0$

$$\langle u - v, f(u) - f(v) \rangle \leq -L_1 |u - v|^2 \qquad \forall u, v \in \mathbb{R}^d. \tag{2.19}$$

Also the authors give sufficient conditions to assure an asymptotically stable stochastic stationary solution of (2.18). In this context establish the following result for the stability of θ-EM scheme.

Theorem 2.4.5 ([12, Thm. 3.1])**.** *Suppose that the drift coefficient satisfies a contractive one-sided Lipschitz condition, and that the vector field f satisfies a globally Lipschitz condition . Then the θ-EM scheme has a unique stochastic stationary solution which is pathwise asymptotically stable for all step sizes $h > 0$ if*

$$(1-\theta)(|A|+L) < -\theta(\mu[A] - L_1), \qquad \mu[A] = \lim_{\delta \to 0^+} \frac{(|Id + \delta A|)}{\delta},$$

where L refers to the Lipschitz condition and L_1 the to contractive one-sided Lipschitz condition.

The following chapter shows adaptations of these results for the construction of a new method, the Steklov method.

Chapter 3

Steklov method for scalar SDEs with Globally Lipschitz coefficients

In this chapter, we focus on the following scalar stochastic differential equation

$$dy(t) = f(t, y(t))dt + g(t, y(t))dW(t), \qquad y_0 = y(0), \tag{3.1}$$

considering the drift term as $f(t, y(t)) = f_1(t)f_2(y(t))$. Given this functional form of f, we propose an exact explicit algorithm for solving the deterministic equation linked to (3.1); details of this exact differentiation are given in [52]. So, the main characteristic of this new method is that it preserves qualitative features of the deterministic solution associated to the SDE. Next, we prove strong consistency, convergence and study the linear stability of the proposed method using properties of the *Steklov mean* [67]. Moreover, we analyze the nonlinear stability of the Steklov stochastic approximation specifically the asymptotic mean-square stability in the multiplicative case and the path-wise stability in the additive case. Finally, we show the efficiency of the new scheme in numerical problems with harsh requirements of stability like the logistic equation for the multiplicative case and the Langevin equation with a particular potential for the additive case.

In section 3.1, we construct the explicit Steklov method for the SDE (3.1) and show its development with some examples. In the next section, we prove strong consistency and convergence of the new explicit method. In section 3.3, sufficient conditions for the asymptotic mean and mean-square stability are given for both additive and multiplicative cases. A nonlinear stability analysis is carried out in section 3.4, where we prove that the explicit Steklov approximation is asymptoticly stable in square mean sense in the multiplicative case and it is path-wise stable under certain conditions in the additive case. In section 3.5, we test the Steklov method for the stochastic logistic equation in the multiplicative case and and for the Langevin equation in Brownian dynamics. Also, we show numerical evidence that the Steklov method is successful with step sizes significantly large reaching larger time scales of simulation. Finally, we give some conclusions.

3.1 Steklov Method

Under these considerations we construct the Steklov numerical scheme for the SDE (3.1) based on its integral formulation:

$$y(t) = X_0 + \int_0^t f(s, y(s))ds + \int_0^t g(s, y(s))dW(s), \quad t \in [0, T], \quad Y_0 = y_0, \tag{3.1}$$

where $y(t)$ denotes the value of the process at time t with initial value X_0. First we discretize the time domain with a uniform step size h such that $t_n = nh$ for $n = 0, 1, 2, \ldots, N$ and denote by Y_n the numerical solution at t_n. Now we approximate the stochastic integral of (3.1) with the usual form:

$$\int_{t_n}^{t_{n+1}} f(s, y(s))dW(s) \approx g(t_n, Y_n)\Delta W_n, \qquad \Delta W_n := (W(t_{n+1}) - W(t_n)) = \sqrt{h}V_n, \tag{3.2}$$

where $W(t_{n+1}) - W(t_n)$ is a discrete standard Brownian motion such that $V_n \sim \mathcal{N}(0, 1)$. We can obtain different schemes depending on the numerical integration used for the first integral of (3.1). For example, if we choose the Euler's approximation:

$$\int_{t_n}^{t_{n+1}} f(s, y(s))ds \approx f(t_n, Y_n)(t_{n+1} - t_n), \tag{3.3}$$

then we obtain the Euler-Maruyama scheme as follows:

$$Y_{n+1} = Y_n + f(t_n, Y_n)h + g(t_n, Y_n)\Delta W_n, \quad n = 1, \dots, N-1, \qquad Y_0 = x_0. \tag{3.4}$$

Assuming that we can rewrite the function f as $f(t, y(t)) = f_1(t)f_2(y(t))$, we propose an alternative approach to (3.3) based on the construction of an exact discretization for the deterministic differential equation associated to (3.1):

$$\frac{dx}{dt} = f_1(t)f_2(x), \qquad x(0) = x_0. \tag{3.5}$$

Integrating this equation in the interval $[t_n, t_{n+1})$ and using the Steklov mean [52], we have

$$\int_{t_n}^{t_{n+1}} f_1(s)f_2(x)ds \approx \phi_1(t_n)\phi_2(y_n, y_{n+1})(t_{n+1} - t_n), \tag{3.6}$$

where

$$\phi_1(t_n) = \frac{1}{t_{n+1} - t_n}\int_{t_n}^{t_{n+1}} f_1(s)ds \qquad \text{and} \qquad \phi_2(y_n, y_{n+1}) = \left(\frac{1}{y_{n+1} - y_n}\int_{y_n}^{y_{n+1}} \frac{du}{f_2(u)}\right)^{-1}.$$

Thus, the exact scheme for (3.5) is given as:

$$y_{n+1} - y_n = \phi_1(t_n)\phi_2(y_n, y_{n+1})h, \qquad y_0 = x_0. \tag{3.7}$$

Notice that it is an implicit algorithm, so in order to get an explicit formulation we define the following function:

$$H(x) := \int_0^x \frac{du}{f_2(u)}, \tag{3.8}$$

and the exact scheme (3.7) is written as follows:

$$y_{n+1} - y_n = \phi_1(t_n)\,\frac{(y_{n+1} - y_n)}{H(y_{n+1}) - H(y_n)}\,h.$$

Now assuming the existence of the function H^{-1}, we can give the following compact formulation of the scheme (3.7):

$$y_{n+1} = \Psi_h(t_n, y_n), \qquad \Psi_h(t_n, y_n) := H^{-1}[H(y_n) + h\phi_1(t_n)]. \tag{3.9}$$

Finally, the numerical method for the SDE (3.1) is proposed as follows:

$$Y_{n+1} = \Psi_h(t_n, Y_n) + g(t_n, Y_n)\Delta W_n, \quad n = 1, \dots, N-1, \quad Y_0 = x_0, \tag{3.10}$$

and we named it *Steklov* scheme due to the origin of its construction. An important feature of this new stochastic scheme (3.10) is that it preserves qualitative properties of the deterministic solution if the noise term does not become dominant. Notice that the main step to develop Steklov approximations is to obtain the function Ψ_h, so forthcoming examples show the procedure to construct this function. We choose as examples some SDEs which appear in important applications and for which harsh conditions of stability are required for their numerical approximations.

Example 3.1.1. We consider the linear Itô equation

$$dy(t) = \lambda y(t)dt + \xi y(t)dW(t), \qquad Y_0 = y_0, \tag{3.11}$$

where λ, $\xi \in \mathbb{C}$ and $x_0 \neq 0$ with probability one. We construct the function Ψ_h for (3.11) using its integral form and approximating the deterministic integral by (3.6) as:

$$\int_{y_n}^{y_{n+1}} \lambda u du \approx \left(\frac{1}{\lambda(y_{n+1} - y_n)} \ln\left(\frac{y_{n+1}}{y_n}\right)\right)^{-1} h, \quad n = 1, \ldots, N-1.$$

In order to obtain a explicit Steklov approximation, we consider the exact finite difference algorithm associated to $dx/dt = \lambda x$:

$$y_{n+1} - y_n = \lambda h \frac{(y_{n+1} - y_n)}{\ln\left(\frac{y_{n+1}}{y_n}\right)}.$$

By algebraic manipulations, the previous equation is equivalent to the equation

$$y_{n+1} = \exp(\lambda h) y_n$$

and the explicit function Ψ_h for the linear SDE is

$$\Psi_h(y) = \exp(\lambda h) y. \tag{3.12}$$

Notice that we obtain the same function Ψ_h that for an additive linear SDE.

Example 3.1.2. Now we consider the logistic growth SDE proposed by Schurz in [65]:

$$dy(t) = \lambda y(t)(K - y(t))dt + \xi y(t)^{\alpha} |K - y(t)|^{\beta} dW(t), \tag{3.13}$$

where λ, K, α, β and ξ are nonnegative real coefficients. So using (3.6) we approximate the deterministic integral of the integral form of (3.13) as:

$$\int_{y_n}^{y_{n+1}} \lambda u(K-u) du \approx \frac{y_{n+1} - y_n}{\frac{1}{\lambda K} \ln\left(\frac{y_{n+1}(K-y_n)}{y_n(K-y_{n+1})}\right)} h, \quad n = 1, \ldots, N-1.$$

Analogously to the previous example, we develop the Steklov function from the exact finite difference equation associated to the deterministic counterpart of (3.13), obtaining:

$$\Psi_h(y) = \frac{Ky}{K - y + \exp(\lambda K h)}. \tag{3.14}$$

Example 3.1.3. As a final example, we consider the following SDE with additive noise:

$$dy(t) = -y(t)^3 dt + \xi dW(t), \tag{3.15}$$

where ξ is a positive coefficient. Using (3.6), we get

$$\int_{y_n}^{y_{n+1}} -u^3 du \approx 2 \frac{(y_{n+1} y_n)^2}{y_{n+1} + y_n} h, \quad n = 1, ..N-1.$$

By algebraic manipulations on the associated deterministic exact algorithm, we obtain the following Steklov function

$$\Psi_h(y) = \frac{y}{\sqrt{1+2y^2h}}. \tag{3.16}$$

In the section of numerical results, we will show the behavior of the new scheme (3.10) in these three examples and compare it with standard methods. As a next step, we prove important qualitative properties of the explicit Steklov method.

3.2 Strong consistency and convergence

It is always important in the construction of new algorithms to study the global discretization error and give an estimation of the speed of convergence. Here, they are carried out with the analysis of the properties of consistency and convergence, see [39]. For simplicity, we study these properties for a one-dimensional autonomous SDE

$$dy(t) = f(y(t))dt + g(y(t))dW(t), \tag{3.1}$$

satisfying the necessary conditions of existence and uniqueness of solution.

So, considering Definition 2.4.1 and Theorem 2.4.1 we prove convergence of the explicit Steklov approximation via strong consistency.

Theorem 3.2.1. *A time discrete approximation of SDE* (3.1) *generated with the explicit Steklov method* (3.10) *is strongly convergent.*

Proof. We substitute the Steklov recurrence (3.10) in the left hand side of the inequality (2). Given that F, G and Ψ_h are continuous functions adapted to the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ and using standard conditional expectation properties [72], it follows that:

$$\begin{aligned}
\mathbb{E}\left(\left|\mathbb{E}\left(\frac{Y_{n+1}-Y_n}{h}\,|\mathcal{F}_{t_n}\right) - F\left(Y_n\right)\right|^2\right) &= \mathbb{E}\left(\left|\frac{\Psi_h(Y_n)-Y_n}{h} - F(Y_n)\right|^2\right) \\
&= \mathbb{E}\left(\left|\frac{H^{-1}(H(Y_n)+h) - H^{-1}(H(Y_n))}{h} - F(Y_n)\right|^2\right).
\end{aligned}$$

Since the functions F and Ψ_h are Lipschitz we can apply the *Inverse Function* theorem and from (3.8), we have that

$$(H^{-1})'(H(Y_n)) = F(Y_n),$$

then given any $\epsilon > 0$ there exists $\delta = \delta(\epsilon)$ such that whenever $0 < h < \delta(\epsilon)$ then

$$\left|\frac{H^{-1}(H(Y_n)+h) - H^{-1}(H(Y_n))}{h} - F(Y_n)\right| < \epsilon.$$

So, taking $\epsilon = \sqrt{h}$ and $c(h) = h(\delta(\sqrt{h}))^2$, the condition (ii) is satisfied. With an analogous procedure, the condition (iii) is verified and the condition (i) follows straightforward from the definition of $c(h)$. □ □

Thus, we can ensure that the explicit Steklov scheme converges on bounded time intervals [60]. However, if we are interested in simulating the solution of the SDE (3.1) for large periods of time, we need to use stable methods. We can interpret the stability of a numerical method, in some sense, as its capacity to preserve the dynamical structure of the solution in that sense. In the next two sections, we study the stability of the explicit Steklov method (3.10) in mean and mean square sense and extend this study in a path-wise sense for the additive case.

3.3 Linear Stability

We start the stability analysis for the linear case since the stability conditions for the solution of the linear SDE in both additive and multiplicative cases are well known. So, we first recall these conditions for the continuous case and later obtain sufficient conditions to ensure stability and asymptotic stability in mean and mean square for the explicit Steklov method (3.10). Moreover in the additive case, we analyze the stability in a path-wise sense based on the work of Buckwar et al. [12].

3.3.1 Multiplicative noise

For the linear multiplicative SDE (3.11), its zero equilibrium solution is called *mean stable* if $\lim_{t\to\infty} \mathbb{E}y(t) = 0$, and it is said to be *mean square stable* if $\lim_{t\to\infty} \mathbb{E}\,|y(t)|^2 = 0$. Then the zero solution of (3.11) is mean stable if $\lambda < 0$ and it is mean square stable if $Re(\lambda) + \frac{1}{2}|\xi|^2 < 0$, see [30]. In order to obtain the explicit Steklov approximation (3.10) for equation (3.11), we use the function Ψ_h defined in (3.12) so the linear Steklov discretization is written as follows:

$$Y_{n+1} = \exp(\lambda h) Y_n + \xi Y_n \Delta W_n. \tag{3.1}$$

Similarly, we say that the method (3.1) is *mean stable* if $\lim_{n\to\infty} \mathbb{E}Y_n = 0$, and we called it *mean square stable* if $\lim_{n\to\infty} \mathbb{E}\,|Y_n|^2 = 0$. Moreover, a stochastic numerical method is $\mathcal{A}$*-stable* in some sense, if it is stable for all step size h when its associated continuous SDE is stable in the same sense.

Proposition 3.3.1. *Let $\lambda < 0$, then the explicit Steklov method (3.1) for the SDE (3.11) is $\mathcal{A}$-stable in mean. Moreover, it is mean square stable if*

$$\exp\left(2Re(\lambda h)\right) + |\xi|^2 h < 1. \tag{3.2}$$

Proof. Denoting by $p = \exp(\lambda h)$ and $q = \xi\sqrt{h}$, we can rewrite the Steklov method (3.1) as

$$Y_{n+1} = (p + qV_n)Y_n. \tag{3.3}$$

Taking expectation in (3.3) and iterating this recurrence until the initial step, we obtain

$$\mathbb{E}Y_n = (p)^{n+1}\mathbb{E}Y_0, \tag{3.4}$$

thus the limit of the sequence (3.4) as n approaches infinity is zero for $\lambda < 0$. Now, applying square modulus to (3.3) and carrying out an analogous procedure, it follows that:

$$\mathbb{E}\left|Y_n^h\right|^2 = (|p|^2 + |q|^2)^{n+1}\mathbb{E}\left|Y_0^h\right|^2.$$

Therefore the sequence $\mathbb{E}\left|Y_n^h\right|^2$ approaches to zero as n tends to infinity if and only if $|p|^2+|q|^2< 1$.
□ □

In Figure 3.1, we show a comparison between the mean square stability region of the zero solution for the linear SDE and the associated explicit Steklov and Euler-Maruyama approximations [31].

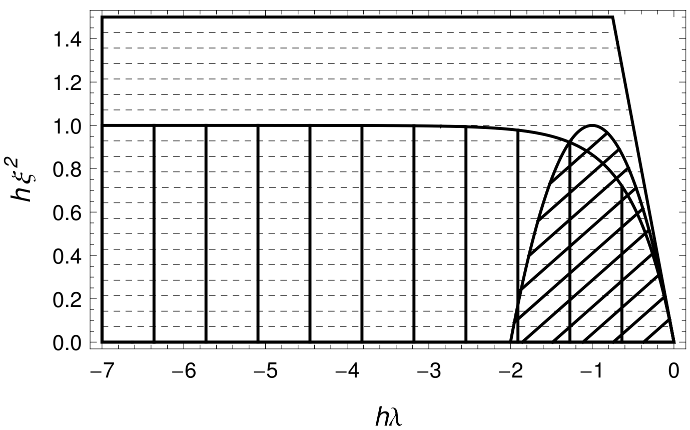

Figure 3.1: Mean square stability regions: horizontal lines represent the region for the linear SDE (3.11), the vertical lines form the explicit Steklov region and the diagonal lines draw the Euler-Maruyama region.

3.3.2 Additive noise

Here we study the additive linear SDE:

$$dy(t) = \lambda y(t)dt + \xi dW(t), \qquad X_{t_0} = x_{t_0}. \tag{3.5}$$

where λ, $\xi \in \mathbb{C}$ and X_{t_0} is the initial value of the process at time t_0. Equation (3.5) has the following exact solution:

$$y(t) = \exp(\lambda(t - t_0))y(t_0) + \xi \exp(\lambda t) \int_{t_0}^{t} \exp(-\lambda s)dW(s), \qquad t \geq t_0. \tag{3.6}$$

The stochastic process $y(t)$ defined in (3.6) is known as the *Ornstein-Uhlenbeck*'s (OU) process. According to [29], the OU process is *asymptotically mean stable* if $\lim_{t\to\infty} \mathbb{E}y(t) = 0$ and is *asymptotically mean square stable* if $\lim\limits_{t\to\infty} \mathbb{E}\,|y(t)|^2 = -\xi/2Re(\lambda)$. Both limits are verified if $\lambda < 0$. Now, the explicit Steklov recurrence to solve additive linear SDE is

$$Y_{n+1} = \exp(\lambda h)Y_n + \xi\Delta W_n. \tag{3.7}$$

Analogous stability properties are given for stochastic difference equations with additive noise [62]. Next, we prove *mean-square consistency* for the explicit Steklov, that is,

$$\lim_{h\to 0}\left(\lim_{n\to\infty} \mathbb{E}\,|Y_n|^2\right) = -\xi/2Re(\lambda).$$

Proposition 3.3.2. *Let $\lambda < 0$, the explicit Steklov method (3.7) for the additive linear SDE (3.5) is $\mathcal{A}$-stable in mean and mean-square consistent.*

Proof. Taking the expected value of (3.7) and iterating backwards this recurrence we obtain the identity (3.4), so the $\mathcal{A}$-stability in mean is verified for $\lambda < 0$. Now, taking the mean square of the recurrence (3.7) and after some algebraic manipulations we get

$$\begin{aligned}
\mathbb{E}\,|Y_{n+1}|^2 &= \exp(2Re(\lambda)h)\mathbb{E}\,|Y_n|^2 + |\xi|^2 h \\
&= \mathbb{E}\,|Y_0|^2 + |\xi|^2 h\{1 + \cdots + \exp(2nRe(\lambda)h)\}\mathbb{E}\,|Y_{n+1}|^2 \\
&= \exp(2nRe(\lambda)h)\mathbb{E}\,|Y_0|^2 + \xi^2 h \frac{[\exp(2Re(\lambda)h)]^{n+1} - 1}{\exp(2Re(\lambda)h) - 1}.
\end{aligned}$$

Given that $\lambda < 0$

$$\lim_{\substack{n\to\infty \\ h\to 0}} \mathbb{E}\,|Y_{n+1}|^2 = \lim_{\substack{n\to\infty \\ h\to 0}} \frac{-|\xi|^2 h}{\exp(2Re(\lambda)h) - 1} = -\frac{|\xi|^2}{2Re(\lambda)}. \quad \square$$

$\square$

So far we have analyzed the asymptotic behavior of the forward motion for the explicit Steklov method (3.7). Now, if we consider $\lambda < 0$ then the OU solution (3.6) does not convergence as t tends to infinity but has the following pullback limit:

$$\lim_{t_0\to-\infty} y(t) = \widehat{O}_t := \exp(\lambda t) \int_{-\infty}^{t} \exp(-\lambda s) dW(s), \tag{3.8}$$

B_t is now defined for all $t \in \mathbb{R}$, see [5, 40]. Furthermore, the process (3.8) is a stationary solution of the additive linear SDE which attracts all other solutions in forward time and path-wise sense. Moreover, it is a finite process for all $t \geq T_{D(\omega)}$ ($\omega \in \Omega$) for appropriate families $D(\omega)$ of bounded sets of initial conditions, see [60]. Therefore, a study of the pullback asymptotic behavior for the Steklov stochastic method (3.7) is important in the additive case and in the next subsection we carry it out based on Caraballo and Kloeden's work [17].

Path-wise linear stability

Here we obtain a stationary discrete process $\widehat{O}_n^{(h)}$ for the linear explicit Steklov and prove that converges to the continuous process (3.8).

Proposition 3.3.3. *Let $\lambda < 0$, the explicit Steklov method (3.7) for the additive linear SDE (3.5) has the following attractor:*

$$\widehat{O}_n^{(h)} := \xi \sum_{j=-\infty}^{n-1} \exp(\lambda h(n-1-j))\Delta W_j, \tag{3.9}$$

for any positive step size h. Moreover, it converges to the Ornstein-Uhlenbeck's process (3.8).

Proof. We consider a recurrence given by the Steklov method (3.7) and iterate it backwards, obtaining the explicit numerical solution

$$Y_n = \exp(\lambda h(n-n_0)) + \xi \sum_{j=n_0}^{n-1} \exp(\lambda h(n-1-j))\Delta W_j, \tag{3.10}$$

where n_0 is the initial point of this recurrence. Taking the path-wise pullback limit of Y_n given in (3.10), i.e. $n_0 \to -\infty$ for each n fixed, we get

$$\begin{aligned} \widehat{O}_n^{(h)} &:= \lim_{n_0 \to -\infty} Y_n \\ &= \xi \sum_{j=-\infty}^{n-1} \exp(\lambda h(n-1-j))\Delta W_j. \end{aligned}$$

Now, we take other explicit Steklov recurrence $\widehat{Y}_n$ and subtract it from the recurrence (3.10). It follows that

$$Y_n - \widehat{Y}_n = \exp(\lambda h(n-n_0))(Y_{n_0} - \widehat{Y}_{n_0}).$$

For any fixed n_0 letting $n \to \infty$ we deduce that $Y_n - \widehat{Y}_n \to Y_{n_0} - \widehat{Y}_{n_0}$. So replacing $\widehat{Y}_n$ by the discrete process (3.9), we have that this process attracts all explicit Steklov approximations forwards in time in the path-wise sense. Furthermore, notice that as $h \to 0$ then the series $\widehat{O}_0^h$ approaches $\widehat{O}_0$ and hence, for each n. □ □

3.4 Nonlinear Stability

To continue the stability analysis of the explicit Steklov method, we now discuss the nonlinear case since a linear stable numerical stochastic method does not imply that is stable under same conditions for any nonlinear problem. So, we study sufficient conditions for the nonlinear stability of the explicit stochastic method (3.10) applied on the autonomous SDE (3.1) in both multiplicative and additive cases.

3.4.1 Multiplicative Noise

Here we prove the nonlinear asymptotic stability in a quadratic mean-square sense for the Steklov approximation.

Definition 3.4.1 (Baker and Buckwar [7])**.** Let Y_n and $\widehat{Y}_n$ two different numerical recurrences with corresponding initial process Y_0 and $\widehat{Y}_0$. We shall say that a discrete time, Y is numerically zero-stable in quadratic mean-square sense if given $\epsilon > 0$, there are positive constants h_0 and $\delta = \delta(\epsilon, h_0)$ such that for all $h \in (0, h_0)$ and positive integers $n \leq T/h$ whenever $\mathbb{E}\left|Y_0 - \widehat{Y}_0\right|^2 < \delta$ then

$$\rho_n := \mathbb{E}\left|Y_n - \widehat{Y}_n\right|^2 < \epsilon. \tag{3.1}$$

If the method is stable and $\rho_n \to 0$ when $n \to \infty$, then the method is asymptotically zero-stable in the quadratic mean-square sense.

In order to prove that the Steklov method satisfies the definition 3.4.1, we will follow the idea of the proof given in [7, Thm. 4].

Theorem 3.4.1. *If the functions Ψ_h and G of the Steklov method (3.10) are Lipschitz with constant L, then the Steklov method for the multiplicative SDE (3.1) is zero-stable in quadratic mean square sense. In addition, if $L < 1$ then the Steklov method is asymptotically zero-stable stable in quadratic mean-square sense.*

Proof. Given two Steklov sequences Y_n and $\widehat{Y}_n$ we have

$$\begin{aligned}\left(Y_{n+1} - \widehat{Y}_{n+1}\right)^2 \leq & \left(\Psi_h(Y_n) - \Psi_h(\widehat{Y}_n)\right)^2 \\ & + 2\left(\Psi_h(Y_n) - \Psi_h(\widehat{Y}_n)\right)\left(G(Y_n) - G(\widehat{Y}_n)\right)\Delta W_n \\ & + (G(Y_n) - G(\widehat{Y}_n))^2(\Delta W_n)^2,\end{aligned}$$

for $0 < n < N$ with $T = Nh$. Now, taking expected values conditioned on the σ-algebra $\mathcal{F}_{t_0}$ of the above inequality and applying properties of the conditional expectation we get

$$\begin{aligned}\mathbb{E}\left|Y_{n+1} - \widehat{Y}_{n+1}\right|^2 \leq & \ \mathbb{E}\left[\left|\Psi_h(Y_n) - \Psi_h(\widehat{Y}_n)\right|^2 |\mathcal{F}_{t_0}\right] \\ & + 2\left|\mathbb{E}\left[\left(\Psi_h(Y_n) - \Psi_h(\widehat{Y}_n)\right)\left(G(Y_n) - G(\widehat{Y}_n)\right)\Delta W_n|\mathcal{F}_{t_0}\right]\right| \\ & + \mathbb{E}\left[|G(Y_n) - G(\widehat{Y}_n)|^2|\mathcal{F}_{t_0}\right]\mathbb{E}\left[|\Delta W_n|^2|\mathcal{F}_{t_0}\right].\end{aligned}$$

The second term in this expression is zero due to the independence properties of Brownian motion. Next, using the Lipschitz condition for Ψ_h and G, we obtain:

$$\mathbb{E}\left[|Y_{n+1} - \widehat{Y}_{n+1}|^2|\mathcal{F}_{t_0}\right] \leq L(1+h)\mathbb{E}\left[|Y_n - \widehat{Y}_n|^2|\mathcal{F}_{t_0}\right]. \tag{3.2}$$

The sequence $\{R_n\}_{n\geq 0}$ defined by

$$R_n = \max_{0\leq r\leq n} \mathbb{E}\left[|Y_r - \widehat{Y}_r|^2|\mathcal{F}_{t_0}\right],$$

is monotonically non-decreasing. Furthermore, by (3.2) we have

$$R_n \leq L(1+h)R_{n-1}. \tag{3.3}$$

First suppose $0 < L < 1$, since $1 + h \leq \exp(h)$ it follows that

$$R_n \leq L\exp(T)R_0, \qquad n = 0, \ldots, N. \tag{3.4}$$

Hence, given $\epsilon > 0$ if we take $\delta = \epsilon L^{-1}\exp(-T)$ then for all $0 < h < h_0 \leq T$ and any integer n such that $0 \leq n \leq N$

$$\mathbb{E}\left|Y_0 - \widehat{Y}_0\right|^2 \leq \delta \Rightarrow \mathbb{E}\left|Y_n - \widehat{Y}_n\right|^2 \leq \epsilon.$$

On the other hand, if $1 < L < +\infty$ and with $h_0 := \frac{L-1}{L}$ then for $0 < h < h_0$ we get

$$L(1+h) < 1 + 2Lh_0.$$

Thus, it follows that

$$R_n \leq \exp(2LNh_0)R_0 = \exp(2LT)R_0.$$

Hence, given $\epsilon > 0$ if we take $h \in (0, (L-1)/L)$, and $\delta = \epsilon \exp(-2LT)$ then for all integers n such that $0 \leq n \leq N$ we obtain

$$\mathbb{E}\left|Y_0 - \widehat{Y}_0\right|^2 \leq \delta \Rightarrow \mathbb{E}\left|Y_n - \widehat{Y}_n\right|^2 \leq \epsilon.$$

So far we have proved the quadratic mean square stability for the explicit Steklov method. Notice that the asymptotic mean-square stability for the method (3.10) is verified for any $h \in (0, T]$ if $0 < L < 1$. □ □

3.4.2 Additive noise

Nonlinear differential equations have more complex dynamics than the linear case and the same occurs for the finite difference equations. So, Caraballo and Kloeden in [17] extend the nonlinear stability theory of the deterministic numerical analysis given in [40] to the stochastic numerical case. Following their work, we consider the non-autonomous additive SDE:

$$dy(t) = f(y(t))dt + \xi dW_t, \tag{3.5}$$

where f satisfies a *contractive one-sided Lipschitz* condition with constant $L_1 > 0$ as follows

$$\langle x - z, f(x) - f(z)\rangle \leq -L_1|x - z|^2 \qquad \forall x, z \in \mathbb{R}, \tag{3.6}$$

and study the path-wise stability for the Steklov method (3.10) for the SDE (3.5).

Theorem 3.4.2. *If the Steklov function Ψ_h satisfies*

*(A1) (**Contractive Lipschitz condition**) There exists a constant $K_1 \in (0,1)$ such that*

$$|\Psi_h(x) - \Psi_h(z)| \leq K_1|x - z| \qquad \forall x, z \in \mathbb{R},$$

*(A2) (**Contractive one sided Lipschitz condition**) There exists a constant K_2 such that*

$$\langle \Psi_h(x) - \Psi_h(z), x - z\rangle \leq -K_2|x - z|^2 \qquad \forall x, z \in \mathbb{R},$$

*(A3) (**Linear growth bound**) There exists a constant K_3 such that*

$$|\Psi_h(x)| \leq K_3(1 + h + |x|) \qquad \forall x \in \mathbb{R},$$

and the condition

$$\frac{K_3}{1 + K_2 - K_3} < 1, \tag{3.7}$$

is verified. Then there exists $h^* > 0$ such that for all $0 < h < h^*$ the Steklov method (3.10) has a unique stochastic stationary solution which is path-wise asymptotically stable for an additive SDE (3.5).

Proof. In order to obtain the path-wise asymptotic stability for the explicit Steklov method we will show: *(i)* the path-wise contractive Lipschitz property for the Steklov numerical solution and *(ii)* the existence of a random attractor for the Steklov approximations.

(i) Let Y_{n+1} and $\widehat{Y}_{n+1}$ two different solutions of the Steklov method (3.10) for the additive SDE (3.5) and using the Lipschitz condition (A1) we get the following upper bound:

$$
\begin{aligned}
|Y_{n+1} - \widehat{Y}_{n+1}|^2 &= \left\langle Y_n - \widehat{Y}_n, \Psi_h(Y_n) - \Psi_h(\widehat{Y}_n) \right\rangle \\
&\leq K_1 |Y_{n+1} - \widehat{Y}_{n+1}||Y_n - \widehat{Y}_n|.
\end{aligned}
$$

From this, we deduce that

$$
|Y_n - \widehat{Y}_n| \leq K_1^{n-n_0} |Y_{n_0} - \widehat{Y}_{n_0}|. \tag{3.8}
$$

then for $0 < K_1 < 1$ the path-wise contractivity. Moreover taking the limit of (3.8) as $n_0 \to -\infty$ for fixed n we have that $|Y_n - \widehat{Y}_n| \to 0$.

(ii) Defining a new variable by $Z_n := Y_n - \widehat{O}_n^{(h)}$ where Y_n is the Steklov approximation and $\widehat{O}_n^{(h)}$ is the Steklov OU process (3.9) we obtain the numerical scheme

$$
Z_{n+1} = \Psi_h(Z_n + \widehat{O}_n^{(h)}) - \exp(\lambda h)\widehat{O}_n^{(h)}. \tag{3.9}
$$

Taking the inner product with Z_{n+1} in (3.9) and adding convenient terms we get

$$
\begin{aligned}
|Z_{n+1}|^2 &= \left\langle Z_n + \widehat{O}_n^{(h)} - (Z_n + \widehat{O}_n^{(h)} + Z_{n+1}), \Psi_h(Z_n + \widehat{O}_n^{(h)}) - \Psi_h(Z_n + \widehat{O}_n^{(h)} + Z_{n+1}) \right\rangle \\
&+ \left\langle Z_{n+1}, \Psi_h(Z_n + \widehat{O}_n^{(h)} + Z_{n+1}) \right\rangle + \left\langle Z_{n+1}, \exp\{(\lambda h)\}\widehat{O}_n^{h} \right\rangle \\
&\leq -K_2 |Z_{n+1}|^2 + |Z_{n+1}| \left| \Psi_h(Z_n + \widehat{O}_n^{(h)} + Z_{n+1}) \right| + \exp\{(\lambda h)\} |Z_{n+1}| \left| \widehat{O}_n^{h} \right|.
\end{aligned}
$$

From the linear growth condition (A3) we deduce that

$$
\begin{aligned}
|Z_{n+1}|^2 &\leq (K_3 - K_2)|Z_{n+1}|^2 + K_3 |Z_n||Z_{n+1}| \\
&\quad + K_3(1+h)|Z_{n+1}| + (K_3 + \exp(\lambda h))|Z_{n+1}||\widehat{O}_n^{(h)}|.
\end{aligned}
$$

Thus, we obtain

$$
|Z_{n+1}| \leq \frac{K_3}{1 + K_2 - K_3}|Z_n| + \frac{K_3(1+h)}{1 + K_2 - K_3} + \frac{(K_3 + \exp(\lambda h))}{1 + K_2 - K_3}|\widehat{O}_n^{(h)}|. \tag{3.10}
$$

Taking

$$
\alpha := \frac{K_3}{1 + K_2 - K_3} \qquad \text{and} \qquad \beta := \frac{(K_3 + \exp(\lambda h))}{1 + K_2 - K_3},
$$

we can rewrite (3.10) as

$$
|Z_n| \leq \alpha^{n-n_0}|Z_{n_0}| + (1+h)\alpha \sum_{j=n_0}^{n-1} \alpha^{n-1-j} + \beta \sum_{j=n_0}^{n-1} \alpha^{n-1-j} |\widehat{O}_n^{(h)}|. \tag{3.11}
$$

Then taking the limit as $n_0 \to -\infty$ for n fixed and assuming the condition (3.7) the first series of (3.11) converges. From [60] we have that for h small enough and considering the set of the bounded initial conditions $D(\omega)$ for the continuous OU process (3.8), the iterates Z_n remain in a ball with center the origin and random radius:

$$R_h(\omega) = C + \beta \sum_{j=n_0}^{n-1} \alpha^{n-1-j} |\widehat{O}_n^{(h)}|,$$

where C is a bound for the first terms of the right hand of the inequality (3.11). Thus, from theory of random numerical dynamical systems [40] and since Z_n inherits the contractivity from Y_n we conclude the existence of a random attractor for the sequence (3.9) defined by a unique stationary stochastic process. So, transforming back to the original variables we can assure that the explicit Steklov method for the SDE (3.5) has a stationary stochastic process $\widehat{Y}_n = \widehat{Z}_n + \widehat{O}_n$, which is a pathwise-attractor for all Steklov approximations in both pullback and forward senses. □

□

3.5 Numerical Results

Here, we analyze the efficiency of the explicit Steklov method (3.10) for SDEs for which a step size of the usual stochastic algorithms has to be small enough to preserve numerical stability. In particular, we consider as benchmarks the examples given in section 3.1 to show the behavior of the Steklov scheme and compare it with the EM approximation, the CBD method [11] and a balanced implicit method [65]. Moreover, long-time simulations of the new method are carried out in order to evidence its good asymptotic dynamical properties. But before, we start by evaluating the accuracy of the Steklov method for the linear SDE where the analytical solution is known.

3.5.1 Linear SDE

We apply the explicit Steklov approximation to the multiplicative (3.11) and additive (3.5) linear SDEs and study its accuracy showing its strong error which is determined by

$$\varepsilon = \mathbb{E}\left(|y(T) - Y_{n_T}|\right), \tag{3.1}$$

where $y(t)$ is the exact solution and Y_n is a time discretization approximation for the linear SDEs. Moreover, we also present numerical results for the EM scheme for the same equations. Numerical results for the Steklov and Euler-Maruyama (EM) approximations for both additive and multiplicative cases are shown in tables 3.1 and 3.2 respectively. The confidence interval for the strong error is obtained for 20 samples of 100 trajectories each. We also estimate the mean square error at a discrete time $t_n = T$ as follows:

$$\varepsilon_{MS}(T) = \left(\frac{1}{N} \sum_{k=1}^{N} \left(y^{[k]}(T) - Y^h_{n_T,k} \right)^2 \right)^{\frac{1}{2}}, \tag{3.2}$$

for $N = 100\,000$ paths. Table 3.3 shows the results for both Steklov and Euler-Maruyama schemes. Notice that the Steklov method maintains its accuracy even when the step size is close to one while the Euler-Maruyama approximation is no longer stable from $h = 0.5$.

h	EM	Steklov
0.250 00	$4.1463 \times 10^{-2} \pm 2.9553 \times 10^{-3}$	$4.1076 \times 10^{-2} \pm 2.5145 \times 10^{-3}$
0.500 00	$1.2815 \times 10^{2} \pm 1.3437 \times 10^{-1}$	$5.5109 \times 10^{-2} \pm 3.6455 \times 10^{-3}$
0.750 00	$7.8644 \times 10^{2} \pm 5.9516 \times 10^{-1}$	$6.8446 \times 10^{-2} \pm 3.7039 \times 10^{-3}$
1.000 00	$1.2800 \times 10^{3} \pm 5.7282 \times 10^{-1}$	$7.8523 \times 10^{-2} \pm 6.0528 \times 10^{-3}$

Table 3.1: Intervals at 95% of confidence of the strong error for the additive linear SDE with $\lambda = -5$, $\xi = 0.1$ and initial condition $x_0 = 5$.

h	EM	Steklov
0.125 00	$1.8376 \times 10^{-2} \pm 8.3217 \times 10^{-4}$	$1.8376 \times 10^{-2} \pm 8.3217 \times 10^{-4}$
0.250 00	$1.7452 \times 10^{-2} \pm 1.3495 \times 10^{-3}$	$1.7452 \times 10^{-2} \pm 1.3495 \times 10^{-3}$
0.500 00	$1.2824 \times 10^{2} \pm 1.4210$	$1.7774 \times 10^{-2} \pm 1.3205 \times 10^{-3}$

Table 3.2: Intervals at 95% of confidence of the strong error for the multiplicative linear SDE with $\lambda = -5.0$, $\xi = 0.1$ and initial condition $x_0 = 5$.

	Additive noise		Multiplicative noise	
h	EM	Steklov	EM	Steklov
0.2500	2.1300×10^{-1}	2.0367×10^{-1}	5.4261×10^{-3}	9.4396×10^{-7}
0.5000	3.5206×10^{2}	3.0370×10^{-1}	2.7560×10^{2}	1.0752×10^{-3}
0.7500	8.1368×10^{2}	3.9055×10^{-1}	8.5490×10^{2}	7.1843×10^{-2}
1.0000	1.2930×10^{3}	4.5875×10^{-1}	1.3337×10^{3}	2.8987×10^{-1}

Table 3.3: MS-Error at time $T = 4.0$ for a linear SDE with $\lambda = -5$, $\xi = 0.1$ and initial condition $x_0 = 5$.

3.5.2 Logistic equation

Here we reconsider the stochastic logistic equation (3.13)

$$dy(t) = \lambda y(t)(K - y(t))dt + \xi y(t)^{\alpha} |K - y(t)|^{\beta} dW(t),$$

where $y(t)$ represents the number of individuals of certain specie with growth rate λ into an environment with limited natural resources and K is the maximum capacity population; α, β and ξ are nonnegative coefficients linked with the random contribution that models the influence of the environmental fluctuations or measurement errors [58, 65, 66]. The analytical solution of this equation in general is unknown. Thus it is necessary to obtain numerical solutions. In order to get an accuracy approximation it is desirable that the stochastic numerical method preserves the dynamic properties of the solution of (3.13). We choose this example to emphasize the structural dynamical consistency between the explicit Steklov defined by the function Ψ_h (3.14) and the SDE (3.13). In figure 3.2, we show the numerical results of the Steklov and Euler-Maruyama schemes and a balanced implicit method (BIM) developed to solve the equation (3.13) in [65]. For step sizes greater than 0.01, we observe that the Euler scheme is outside of its stability region and the BIM method has a slow convergence. On the other hand, the Steklov preserves the deterministic solution profile which is consistent with its structural foundation.

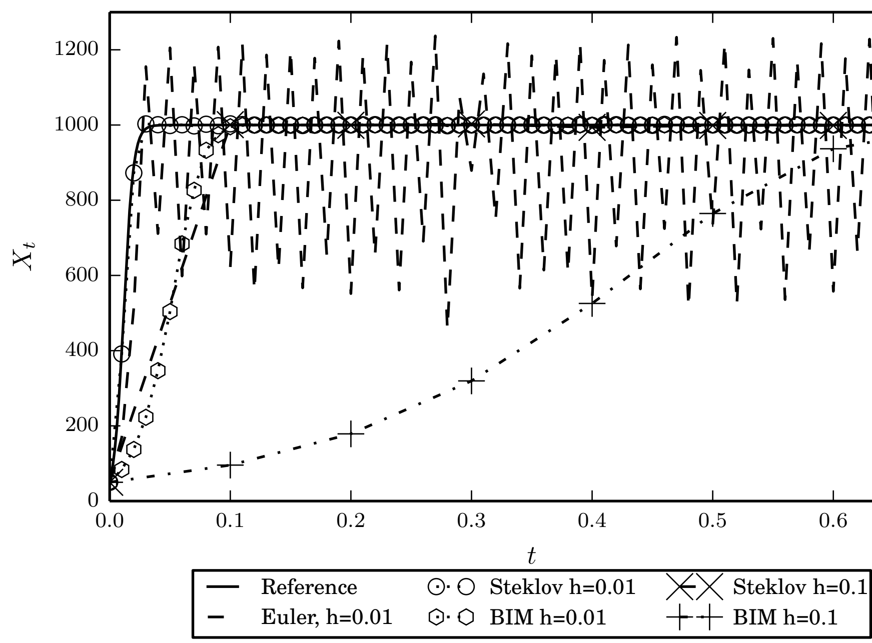

Figure 3.2: Paths obtained with the Euler, Steklov and BIM methods for the logistic SDE (3.13) with $X_0 = 50$ and taking $K = 1000$, $\alpha = 1$, β=0.5, λ=0.25, $\rho = 0$ and σ=0.05.

3.5.3 Langevin equation in Brownian dynamics

Finally, we study the Langevin equation (LE)

$$dy(t) = -y(t)^3 dt + \xi dW(t),$$

where $y(t)$ is the position of a particle at time t which is exposed to deterministic and random forces. This equation is used in Brownian dynamics like a benchmark test, see [11]. As in the logistic SDE, the analytical solution for the Langevin equation is only obtained under special conditions. The most common Brownian dynamics algorithm is the CBD method of Ermak and a. McCammon [22] which is based on the Euler discretization of the LE. Although this method is easy to implement, a small time step size is required, therefore this algorithm runs in relatively small temporal windows. So, to study the asymptotic behavior of the solution of the LE it is convenient to apply methods with good asymptotic stability properties and simple structure. Therefore we show the behavior of the Steklov method defined by the function (3.16) for short-time and long-time dynamics by computing the *self-diffusion* coefficient D/D_0 associated to the LE, for details of the derivation of this coefficient see [11, 45]. In figure3.3, we compare the profiles of the Steklov and CBD approximations for several step sizes. According to the notation in Brownian dynamics, we take Dta=0.00001 as time step size and use 10 000 sample paths to calculate the self-diffusion coefficient. The Steklov and CBD methods have the same behavior at short time with small step sizes. However, for step sizes greater than 1 000 Dta the Euler method diverges and the Steklov method preserves its numerical stability. Thus, it can be used for long-time dynamics with big step sizes as it is shown in figure 3.3.

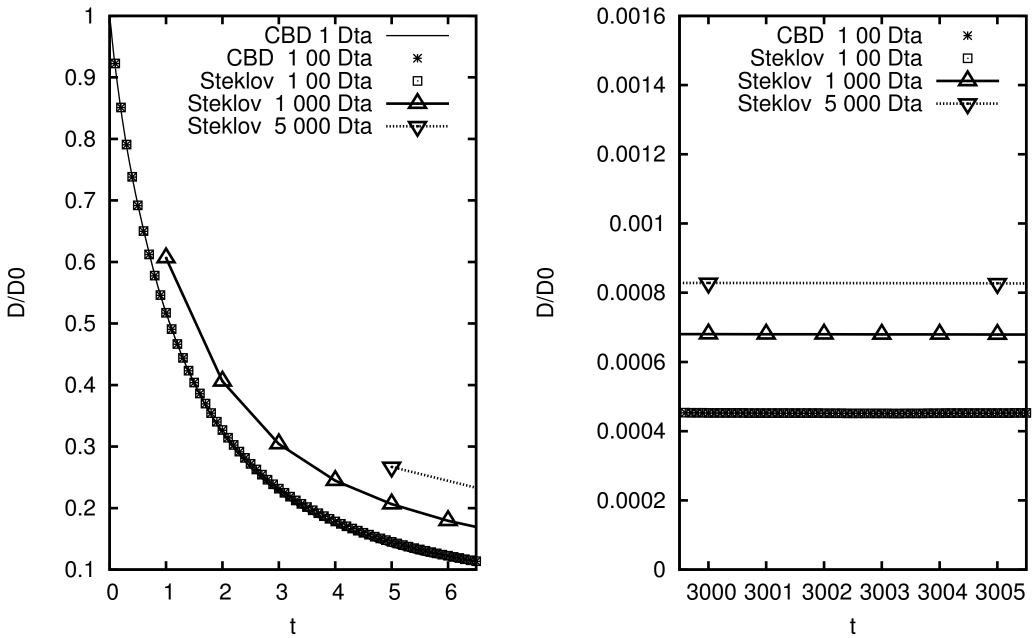

Figure 3.3: Numerical results of the Steklov and CBD methods for the self-diffusion coefficient of the LE with $\xi = 1$: the graph to the left shows short-time simulations and the graph to the right shows long-time simulations.

Chapter 4

Steklov Method for SDEs with Non-Globally Lipschitz Continuous Drift

4.1 Introduction

Applications of Monte Carlo type simulations [26, 27] as Brownian Dynamics [18] require fast numerical methods with low computational cost — excluding the use of implicit schemes in the majority of cases. The Euler-Maruyama (EM) method is the most popular in such simulations due to its simple algebraic structure, cheap computational cost and acceptable convergence rate under global Lipschitz conditions. However, if the drift or diffusion coefficients of stochastic differential equations (SDEs) are super-linear, then the EM approximation diverges in the mean square sense [35, 37]. In most applications, the coefficients of the stochastic models in finances, biology or physics have locally Lipschitz coefficients with super-linear growth. Therefore, recent research has been focused on modifying the EM method to obtain strong convergence under these conditions keeping its simple structure and its low computational cost. In the last years, several methods have been developed in this direction: the family of Tamed schemes [34, 36, 61, 71, 73], a special type of balanced method [69], the stopped scheme [44] and a truncated Euler method [47]. In these works, the strong convergence of the proposed method is proved using the theory developed by Higham, Stuart and Mao in [32] or by means of the new approach given by Hutzenthaler and Jentzen in [34]. Both techniques obtain the property of strong convergence by proving boundedness moments of the numerical and analytical solution of the underlying SDE. In spite of the recent work in this subject, it is still necessary to get more accurate numerical methods for SDE under super-linear growth and non-globally Lipschitz coefficients.

In this chapter, we develop an explicit method based on a linear version of the Steklov method proposed in [21] for the vector Itô stochastic differential equation:

$$dy(t) = f(y(t))dt + g(y(t))dW(t), \quad 0 \leq t \leq T, \quad y(0) = y_0, \tag{4.1}$$

where $(f^{(1)}, \ldots, f^{(d)}) : \mathbb{R}^d \to \mathbb{R}^d$ is one sided Lipschitz and $g = (g^{(j,i)})_{j\in\{1,\ldots,d\}, i\in\{1,\ldots,m\}} : \mathbb{R}^d \to \mathbb{R}^{d\times m}$ is global Lipschitz. Also we assume that each component function $f^{(j)}$ can be written of the form

$$f^{(j)}(x) = a_j(x)x^{(j)} + b_j(x^{(-j)}), \tag{4.2}$$

where a_j and b_j are two scalar functions in $\mathbb{R}^d$ and $x^{(-j)} = \left(x^{(1)}, \ldots, x^{(j-1)}, x^{(j+1)}, \ldots x^{(d)}\right)$. The paper is organized as follows: In section 2, we give known results that are essential for our purposes. In section 3, we construct the new explicit method and prove the always existence of a succession of the Linear Steklov approximation as well as local Lipschitz conditions for its coefficients. In section 4, we prove the strong convergence of the LS method with one-half order using the Higham, Mao and Mao (HMS) technique and in section 5, its convergence rate is obtained. In section 6, we analyze numerically the accuracy and efficiency of the proposed method applied to stochastic differential equations with super-linear growth and locally Lipschitz coefficients. Finally we give some conclusions.

4.2 General Settings

Throughout this paper, we work with a standard setup, that is, $y(t) \in \mathbb{R}^d$ for each t and $W(t)$ is a m-dimensional standard Brownian motion on a filtered and complete probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$, with the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ generated by the Brownian process. Moreover, we denote the norm of a vector $y \in \mathbb{R}^d$ and the Frobenious norm of a matrix $G \in \mathbb{R}^{d\times m}$ by $|y|$ and $|G|$ respectively. The usual scalar product of two vectors $x, y \in \mathbb{R}^d$ is denoted by $\langle x, y\rangle$. The complement of a set E is denoted by E^c and the indicator function of the set E is denoted by $\mathbf{1}_{\{E\}}$. In the following, we recall some classical results about the moment boundedness, existence and uniqueness of the solution of the stochastic differential system (4.1), see [32, 46, 48]. We also state some theorems about the strong convergence of the Euler-Maruyama method given by Higham et al. in [32] which will be useful to prove the strong convergence of the Linear Steklov method.

Let us start by assuming the following:

Hypothesis 4.2.1. The coefficients of SDE (4.1) satisfy the conditions:

(H-1) The functions f, g are in the class $C^1(\mathbb{R}^d)$.

(H-2) **Local, global Lipschitz condition**. For each integer n, there is a positive constant $L_f = L_f(n)$ such that

$$|f(u) - f(v)|^2 \leq L_f|u-v|^2 \qquad \forall u, v \in \mathbb{R}^d, \qquad |u|\vee|v| \leq n,$$

and there is a positive constant L_g such that

$$|g(u) - g(v)|^2 \leq L_g|u-v|^2, \qquad \forall u, v \in \mathbb{R}^d.$$

(H-3) **Monotone condition.** There exist two positive constants α and β such that

$$\langle u, f(u)\rangle + \frac{1}{2}|g(u)|^2 \leq \alpha + \beta|u|^2, \qquad \forall u \in \mathbb{R}^d. \tag{4.1}$$

Under Hypothesis 4.2.1 we can assure existence and uniqueness of the solution of continuous system (4.1). Next we state the bounds on the moments of the solution of (4.1).

Theorem 4.2.1. *Assume Hypothesis 4.2.1 then for all $y(0) = y_0 \in \mathbb{R}^d$ there exists a unique global solution $\{y(t)\}_{t\geq 0}$ to SDE (4.1). Moreover, the solution has the following properties for any $T > 0$,*

$$\mathbb{E}\,|y(T)|^2 < \left(|y_0|^2 + 2\alpha T\right) e^{2\beta T},$$

and

$$\mathbb{P}\left[\tau_n \leq T\right] \leq \frac{\left(|y_0|^2 + 2\alpha T\right) e^{2\beta T}}{n},$$

where n is any positive integer and $\tau_n := \inf\{t \geq 0 : |y(t)| > n\}$.

Theorem 4.2.2. *Let $p \geq 2$ and $x_0 \in L^p(\Omega, \mathbb{R}^d)$. Assume that there exits a constant $C > 0$ such that for all $(x, t) \in \mathbb{R}^d \times [t_0, T]$,*

$$\langle x, f(x,t)\rangle + \frac{p-1}{2}|g(x,t)|^2 \leq C(1 + |x|^2).$$

Then

$$\begin{aligned}
\mathbb{E}\left[\sup_{0\leq t\leq T} |y(t)|^p\right] &\leq C\left(1 + \mathbb{E}|y_0|^p\right), \\
\mathbb{E}|y(t)|^p &\leq 2^{\frac{p-2}{2}}\left(1 + \mathbb{E}|y_0|^p\right) e^{Cpt}, \qquad \forall t \in [0, T].
\end{aligned}$$

Hypothesis 4.2.2. The SDE (4.1) the EM solution (2.3) and its continuous extension (2.5) satisfies

$$\mathbb{E}\left[\sup_{0\leq t\leq T}|y(t)|^p\right] < \infty, \quad \mathbb{E}\left[\sup_{0\leq k\leq N}|Y_k|^p\right] < \infty, \quad \mathbb{E}\left[\sup_{0\leq t\leq T}|\overline{Y}(t)|^p\right] < \infty, \qquad \forall p \geq 1.$$

Theorem 4.2.3 ([48, Thm. 2.2]). *Let Hypothesis 4.2.1 holds. Then for all $y(0) = y_0 \in \mathbb{R}^d$ given, there exist a unique global solution $\{y(t)\}_{t\geq 0}$ to SDE(4.1). Moreover, the solution has the following properties for any $T > 0$,*

$$\mathbb{E}\,|y(T)|^2 < \left(|y_0|^2{+}2\alpha T\right)\exp(2\beta T),$$

and

$$\mathbb{P}\left[\tau_n \leq T\right] \leq \frac{\left(|y_0|^2{+}2\alpha T\right)\exp(2\beta T)}{n},$$

where n is any positive integer and $\tau_n := \inf\{t \geq 0 : |y(t)|> n\}$.

Theorem 4.2.4 ([46, Thm. 2.4.1]). *Let $p \geq 2$ and $x_0 \in L^p(\Omega, \mathbb{R}^d)$. Assume that there exits a constant $C > 0$ such that for all $(x, t) \in \mathbb{R}^d \times [t_0, T]$,*

$$\langle x, f(x,t)\rangle + \frac{p-1}{2}|g(x,t)|^2 \leq C(1+|x|^2).$$

Then

$$\mathbb{E}|y(t)|^p \leq 2^{\frac{p-2}{2}}\,(1+\mathbb{E}|y_0|^p)\exp(Cpt) \quad \text{ for all } t \in [0,T].$$

Lemma 4.2.1 ([32, Lem 3.2]). *Under Hypothesis 4.2.1, for each $p \geq 2$, there is a $C = C(p, T)$ such that*

$$\mathbb{E}\left[\sup_{0\leq t\leq T}|y(t)|^p\right] \leq C\left(1+\mathbb{E}|y_0|^p\right).$$

4.3 Construction of the LS Method

For simplicity, we begin the construction of the Linear Steklov method (LS) considering the scalar case of SDE (4.1), that is, when $d = m = 1$, also, to shorten notation we use a, b instead a_j, b_j. Tough this ideas, we will generalize to higher dimensions. Let $0 = t_0 < t_1 < \cdots < t_N = T$ a partition of the interval $[0, T]$ with constant step-size $h = T/N$ and such that $t_k = kh$ for $k = 0, \ldots, N$. The main idea of the LS approximation consists in estimating the drift coefficient of (4.1) by

$$f(y(t)) \approx \varphi_f(y(t_{\eta_+(t)})) = \left(\frac{1}{y(t_{\eta_+(t)}) - y(t_{\eta(t)})}\int_{y(t_{\eta(t)})}^{y(t_{\eta_+(t)})}\frac{du}{a(y(t_{\eta(t)}))u+b}\right)^{-1}, \qquad t \in [0,T], \tag{4.1}$$

where

$$\eta(t) := k \text{ for } t \in [t_k, t_{k+1}), \quad k \geq 0,$$
$$\eta_+(t) := k+1 \text{ for } t \in [t_k, t_{k+1}), \quad k \geq 0.$$

So we define the LS method for the scalar version of the SDE (4.1) using a split-step formulation as follows

$$Y_k^\star = Y_k + h\varphi_f(Y_k^\star), \tag{4.2}$$
$$Y_{k+1} = Y_k^\star + g(Y_k^\star)\Delta W_k, \tag{4.3}$$

with $Y_0 = y_0$ and $\varphi_f\left(Y_k^\star\right)$ defined by

$$\varphi_f\left(Y_k^\star\right) = \left(\frac{1}{Y_k^\star - Y_k}\int_{Y_k}^{Y_k^\star}\frac{du}{a(Y_k)u+b}\right)^{-1} \tag{4.4}$$

This scheme combines a split-step technique with a linear version of an exact deterministic method see [21, 52]. In detail, first we compute the discrete value $Y_k^\star$ using the Linear Steklov approximation (4.2), and next, Y_{k+1} is obtained by adding the stochastic increment $g(Y_k^\star)\Delta W_k$.

To higher dimensions, we adapt the same split step scheme (4.2)–(4.3) as follows. For each component equation $j \in \{1,\ldots,d\}$, on the iteration $k \in \{1,\ldots,N\}$ take

$$a_{j,k} = a_j\left(Y_k^{(1)},\ldots,Y_k^{(d)}\right), \qquad b_{j,k} = b_j\left(Y_k^{(-j)}\right). \tag{4.5}$$

So, define $\varphi_f(Y_k^\star) = \left(\varphi_{f^{(1)}}(Y_k^\star),\ldots,\varphi_{f^{(d)}}(Y_k^\star)\right)$ by

$$\varphi_{f^{(j)}}\left(Y_k^\star\right) = \left(\frac{1}{Y_k^{\star(j)} - Y_k^{(j)}}\int_{Y_k^{(j)}}^{Y_k^{\star(j)}}\frac{du}{a_{j,k}u+b_{j,k}}\right)^{-1}. \tag{4.6}$$

It is worth mentioning that even this formulation is semi implicit, we always can derive a explicit version. The next result deals with this issue. To simplify notation, we define $A^{(1)} = A^{(1)}(h,u)$, $A^{(2)} = A^{(2)}(h,u)$ and $b = b(u)$ by

$$\begin{aligned}
A^{(1)} &:= \begin{pmatrix} e^{ha_1(u)} & & 0 \\ & \ddots & \\ 0 & & e^{ha_d(u)} \end{pmatrix}, \\
A^{(2)} &:= \begin{pmatrix} \left(\dfrac{e^{ha_1(u)}-1}{a_1(u)}\right)\mathbf{1}_{\{E_1^c\}} & & 0 \\ & \ddots & \\ 0 & & \left(\dfrac{e^{ha_d(u)}-1}{a_d(u)}\right)\mathbf{1}_{\{E_d^c\}} \end{pmatrix} + h\begin{pmatrix} \mathbf{1}_{\{E_1\}} & & 0 \\ & \ddots & \\ 0 & & \mathbf{1}_{\{E_d\}} \end{pmatrix}, \\
E_j &:= \{x \in \mathbb{R}^d : a_j(x) = 0\}, \qquad b(u) := \left(b_1(u^{(-1)}),\ldots,b_d(u^{(-d)})\right)^T.
\end{aligned} \tag{4.7}$$

Also we will need the following results from [42, Thm 2.1], [23, Thm. 1]. The first theorem will help us with the singularities of set E_j in the case where all elements of this set are limit points.

Theorem 4.3.1 (Multivariate L'hôpital's Rule)**.** *Let $\mathcal{N}$ be a neighborhood in $\mathbb{R}^2$ containing a point $\mathbf{q}$ at which two differentiable functions $f : \mathcal{N} \to \mathbb{R}$ and $g : \mathcal{N} \to \mathbb{R}$ are zero. Set*

$$C = \{x \in \mathcal{N} : f(x) = g(x) = 0\},$$

and suppose that C is a smooth curve through $\mathbf{q}$.

Suppose there exist a vector $\mathbf{v}$ not tangent to C at $\mathbf{q}$ such that the directional derivative $D_{\mathbf{v}}g$ of g in the direction of $\mathbf{v}$ is never zero within $\mathcal{N}$. Also we assume that $\mathbf{q}$ is a limit point of $\mathcal{N} \setminus C$. Then

$$\lim_{(x,y)\to\mathbf{q}} \frac{f(x,y)}{g(x,y)} = \lim_{\substack{(x,y)\to\mathbf{q}\\(x,y)\in\mathcal{N}\setminus C}} \frac{D_{\mathbf{v}}f}{D_{\mathbf{v}}g}$$

if the latter limit exist.

For the second auxiliary we will need the following concepts.

Definition 4.3.1 (Directional derivative referred at a point)**.** Let $u, \mathbf{q} \in \mathbb{R}^2$ and α the positive angle respect to the x-axis and the segment $\overline{u\mathbf{q}}$. We denote by

$$f_\alpha(u) = \cos(\alpha)\frac{\partial f}{\partial u^{(1)}}(u) + \sin(\alpha)\frac{\partial f}{\partial u^{(2)}}(u) = \frac{\langle q-u, \nabla f(u)\rangle}{|u-q|}$$

the *directional derivative respect to the point $\mathbf{q}$ on u.*

Definition 4.3.2 (Star-like set)**.** A set $S \subset \mathbb{R}^2$ is *star-like* with respect a point $\mathbf{q}$, if for each point $s \in S$ the open segment $\overline{s\mathbf{q}}$ is in S.

Whit this in mine, second theorem give us a way to analyze isolated singularities.

Theorem 4.3.2. *Let $\mathbf{q} \in \mathbb{R}^2$ and let f,g be functions whose domains include a set $S \subset \mathbb{R}^2$ which is star-like with respect to the point $\mathbf{q}$. Suppose that on S the functions are differentiable and that the directional derivative of g with respect to $\mathbf{q}$ is never zero. With the understanding that all limits are taken from within on S at $\mathbf{q}$ and if*

(i) $f(\mathbf{q}) = g(\mathbf{q}) = 0$,

(ii) $\lim_{x\to\mathbf{q}} \frac{f_\alpha(x)}{g_\alpha(x)} = L$,

then

$$\lim_{x\to\mathbf{q}} \frac{f(x)}{g(x)} = L.$$

With this on mind, we additionally require the following.

Hypothesis 4.3.1. For each component function $f^{(j)} : \mathbb{R}^d :\to \mathbb{R}$ with $j \in \{1, \ldots, d\}$:

(A-1) There are two locally Lipschitz functions of class $C^1(\mathbb{R}^d)$ denoted by $a_j : \mathbb{R}^d \to \mathbb{R}$ and $b_j : \mathbb{R}^{d-1} \to \mathbb{R}$ such that the j-component of the drift function can be rewritten as in (4.2).

(A-2) There is a positive constant L_a such that

$$a_j(x) \leq L_a, \qquad \forall x \in \mathbb{R}^d.$$

(A-3) Each function $b_j(\cdot)$ satisfies the linear growth condition

$$|b_j(x^{(-j)})|^2 \leq L_b(1+|x|^2), \qquad \forall x \in \mathbb{R}^d.$$

Hypothesis 4.3.2. The set $E_j := \{x \in \mathbb{R}^d : a_j(x) = 0\}$ satisfies either:

(i) All point $q \in E_j$ is a non isolated zero of a_j and:

- the set
$$D := \{u \in B_r(q) : e^{ha_j(u)} - 1 = a_j(u) = 0\},$$
is a smooth curve through q.
- The canonical vector e_j is not tangent to D.
- For each $q \in E_j$, there is an open ball with center on q and radio r $B_r(q)$, such that and
$$a_j \neq 0, \qquad \frac{\partial a_j(u)}{\partial u^{(j)}} \neq 0, \qquad \forall u \in B_r(q) \setminus D.$$

(ii) All point $q \in E_j$ is a isolated zero of a_j and:

- For each $q \in E_j$, q is not a limit point of the set $E_\alpha := \{x \in \mathbb{R}^d : (a_j)_\alpha(x) = 0\}$.
- For each $q \in E_j$ there is a star-like set respect to q E_q, such that the directional derivative respect to q satisfies
$$(a_j)_\alpha(x) \neq 0, \qquad \forall x \in E_q.$$

Lemma 4.3.1. *Let Hypotheses 4.2.1, 4.3.1 and 4.3.2 holds, and $A^{(1)}$, $A^{(2)}$, b defined by (4.7). Then given $u \in \mathbb{R}^d$, the equation*

$$v = u + h\varphi_f(v), \tag{4.8}$$

has a unique solution

$$v = A^{(1)}(h,u)u + A^{(2)}(h,u)b(u). \tag{4.9}$$

If we define the functions $F_h(\cdot)$, $\varphi_{f_h}(\cdot)$ and $g_h(\cdot)$ by

$$F_h(u) = v, \qquad \varphi_{f_h}(u) = \varphi_f(F_h(u)), \qquad g_h(u) = g(F_h(u)), \tag{4.10}$$

then $F_h(\cdot)$, $\varphi_{f_h}(\cdot)$, $g_h(\cdot)$ are local Lipschitz functions and for all $u \in \mathbb{R}^d$ and each h fixed, there is a positive constant L_Φ such that

$$|\varphi_{f_h}(u)| \leq L_\Phi |f(u)|. \tag{4.11}$$

Moreover, for each h fixed, there are positive constants α^* and β^* such that

$$\left\langle \varphi_{f_h}(u), u \right\rangle \vee |g_h(u)|^2 \leq \alpha^* + \beta^*|u|^2, \qquad \forall u \in \mathbb{R}^d. \tag{4.12}$$

Proof. Let us first, prove that (4.9) is solution of Equation (4.8). To this end note that for each $j \in \{1, \ldots, d\}$

$$v^{(j)} = u^{(j)} + h\varphi_{f^{(j)}}(v)$$

So, using in Equation (4.9) that

$$\varphi_{f^{(j)}}(v) = \left(\frac{1}{v^{(j)} - u^{(j)}} \int_{u^{(j)}}^{v^{(j)}} \frac{dz}{a_j(u)z + b_j(u^{(-j)})} \right)^{-1}. \tag{4.13}$$

After some algebraic manipulations we arrive at

$$v^{(j)} = \exp\big(ha_j(u)\big)u^{(j)} + \left[\left(\frac{\exp\big(ha_j(u)\big) - 1}{a_j(u)} \right) \mathbf{1}_{\{E_j^c\}} + h\mathbf{1}_{\{E_j\}} \right] b_j(u^{(-j)}), \tag{4.14}$$

for each $j \in \{1, \ldots, d\}$, which is the j-component of the vector $A^{(1)}(h,u)u + A^{(2)}(h,u)b(u)$. Next, we will prove the inequality (4.11). Since

$$\varphi_{f_h}^{(j)}(u) = \frac{F_h^{(j)}(u) - u^{(j)}}{\displaystyle\int_{u^{(j)}}^{F_h^{(j)}(u)} \frac{dz}{a_j(u)z + b_j(u^{(-j)})}} \qquad .$$

We first see that if $u \in E_j$, then $\varphi_{f_h}^{(j)}(u) = b_j(u^{(-j)}) = f(u)$. Then taking $C \geq 1$ we have the conclusion. On the other hand, if $u \in E_j^c$, then

$$\varphi_{f_h}^{(j)}(u) = \frac{(F_h^{(j)}(u) - u^{(j)})a_j(u)}{\underbrace{\ln\left(a_j(u)F_h^{(j)}(u) + b_j(u^{(-j)})\right)}_{:=Ter1} - \ln\left(a_j(u)u^{(j)} + b_j(u^{(-j)})\right)} \tag{4.15}$$

Now consider in Equation (4.15) the term labeled $Ter1$, and observe that

$$\begin{aligned} Ter1 &= \ln\left\{ a_j(u) \left[e^{ha_j(u)}u^{(j)} + \left(\frac{e^{ha_j(u)} - 1}{a_j(u)} \right) b_j(u^{(-j)}) \right] + b_j\left(u^{(-j)}\right) \right\} \\ &= ha_j(u) + \ln\left(f^{(j)}(u)\right). \end{aligned} \tag{4.16}$$

Combining the relations (4.15) and (4.16), after algebraic manipulations we arrive at

$$\varphi_{f_h}^{(j)}(u) = \left(\frac{e^{ha_j(u)} - 1}{ha_j(u)} \right) f^{(j)}(u), \quad \forall u \in E_j^c. \tag{4.17}$$

Hence, it remains to prove that

$$\Phi(h, a_j)(u) := \frac{e^{ha_j(u)} - 1}{ha_j(u)}, \tag{4.18}$$

is bounded on $\mathbb{R}^d$ for each $j \in \{1, \dots, d\}$. First wee see that under the Hypothesis 4.2.1 the operator Φ is continuous on E_j^c. Furthermore, for each fixed $u \in E_j^c$

$$\lim_{h \to 0} \frac{e^{ha_j(u)} - 1}{ha_j} = 1.$$

On the other hand, for each fixed h, by Hypothesis 4.3.2 we obtain one of the following cases:

CASE I:

$$\lim_{\substack{u \to u^* \\ u \in E_j^c}} \Phi(h, a_j)(u) = \lim_{\substack{u \to u^* \\ u \in E_j^c}} \frac{\frac{\partial a_j(u)}{\partial u^{(j)}} h e^{ha_j(u)}}{h \frac{\partial a_j(u)}{\partial u^{(j)}}} = 1. \tag{4.19}$$

CASE II:

$$\lim_{\substack{u \to u^* \\ u \in E_j^c}} \Phi(h, a_j)(u) = \lim_{\substack{u \to u^* \\ u \in E_j^c}} \frac{\left(e^{ha_j(u)} - 1\right)_\alpha}{\left(ha_j(u)\right)_\alpha} = 1, \qquad \alpha = 0, \pi, 2\pi, \dots \tag{4.20}$$

So, under this situation, we can say for each j that the function

$$f^{(j)}(u)\mathbf{1}_{\{E_j\}} + \left(\frac{e^{ha_j(u)} - 1}{ha_j(u)}\right) f^{(j)}(u)\mathbf{1}_{\{E_j^c\}}$$

is continuous on $\mathbb{R}^d$ and bounded on E_j. Now, let

$$a_j^* := \inf_{u \in E_j^c} \{|a_j(u)|\}.$$

So, a_j^* satisfy one of the two following cases:

CASE a: $0 < a_j^* \leq L_a$.

CASE b: $a_j^* = 0$.

In the first case we see that

$$\frac{e^{ha_j(u)} - 1}{ha_j(u)} \leq \frac{e^{hL_a} - 1}{ha_j^*(u)} < \infty, \qquad \forall h \in (0, \infty).$$

For CASE II, we can apply an argument similar as in (4.19)–(4.20). Then there is $L_\Phi > 0$ such that

$$\left| \frac{e^{ha_j(u)} - 1}{ha_j(u)} \right| < L_\Phi, \qquad \forall u \in \mathbb{R}^d. \tag{4.21}$$

Combining this fact with Equation (4.17), we obtain

$$|\varphi_{f_h^{(j)}}(u)| \leq \left| \frac{e^{ha_j(u)} - 1}{ha_j(u)} \right| |f^{(j)}(u)| < L_\Phi |f^{(j)}(u)|, \qquad \forall u \in \mathbb{R}^d,$$

which prove inequality (4.11).

No, we prove the g_h is a locally Lipschitz function. By (H-1) in Hypothesis 4.2.1, g is a globally Lipschitz function, so

$$g_h(x) = g\left(F_h(x)\right),$$

is the composition of a continuous bounded function and a globally Lipschitz function, furthermore, note that

$$\begin{aligned}|g_h(u) - g_h(v)|^2 &\le L_g|F_h(u) - F_h(v)|^2 \\ &\le 2L_g \underbrace{|A^{(1)}(h,u)u - A^{(1)}(h,v)v|^2}_{:=Ter_1} + 2L_g \underbrace{|A^{(2)}(h,u)b(u) - A^{(2)}(h,v)b(v)|^2}_{:=Ter_2}. \end{aligned} \tag{4.22}$$

Now, we work with each term of the right hand of inequality (4.22). First note that $A^{(1)}$ is a continuous differentiable function on all $\mathbb{R}^d$ so by the mean value Theorem we have

$$|A^{(1)}(h,u)u - A^{(1)}(h,v)v|^2 \le \sup_{0\le t\le 1} |\partial A^{(1)}(h, u + t(v-u))|^2 |u-v|^2,$$

then, there is a positive constant $L_{A^{(1)}} = L_{A^{(1)}}(h,n)$ such that

$$|A^{(1)}(h,u)u - A^{(1)}(h,v)v|^2 \le L_{A^{(1)}}|u-v|^2, \quad u,v \in \mathbb{R}^d, \quad |u|\vee|v| \le n. \tag{4.23}$$

In the other hand,

$$\begin{aligned} Ter_2 &= \sum_{j=1}^{d}\left[\mathbf{1}_{\{E_j^c\}}(u)\Phi(h,a_j)(u)b_j(u^{(-j)}) + h\mathbf{1}_{\{E_j\}}(u)b_j(u^{(-j)}) - \mathbf{1}_{\{E_j^c\}}(v)\Phi(h,a_j)(v)b_j(u^{(-j)}) - h\mathbf{1}_{\{E_j\}}(v)b_j(v^{(-j)})\right]^2 \\ &\le 4\sum_{j=1}^{d}\left[\left(\mathbf{1}_{\{E_j^c\}}(u)\Phi(h,a_j)(u)b_j(u^{(-j)})\right)^2 + \left(h\mathbf{1}_{\{E_j\}}(u)b_j(u^{(-j)})\right)^2 + \left(\mathbf{1}_{\{E_j^c\}}(v)\Phi(h,a_j)(v)b_j(v^{(-j)})\right)^2 \right. \\ &\qquad \left. + \left(h\mathbf{1}_{\{E_j\}}(v)b_j(v^{(-j)})\right)^2\right] \\ &\le 4\sum_{j=1}^{d}\left[\left(\mathbf{1}_{\{E_j^c\}}(u)L_\Phi b_j(u^{(-j)})\right)^2 + \left(h\mathbf{1}_{\{E_j\}}(u)b_j(u^{(-j)})\right)^2 + \left(\mathbf{1}_{\{E_j^c\}}(v)L_\Phi b_j(v^{(-j)})\right)^2 + \left(h\mathbf{1}_{\{E_j\}}(v)b_j(v^{(-j)}\right)^2\right]. \end{aligned} \tag{4.24}$$

Since each $b_j^2(\cdot)$ function is of class $C^1(\mathbb{R}^d)$, there is a constant $M_b = M_b(n)$ such that

$$|b_j(u)|^2 \le M_b, \quad \forall u \in \mathbb{R}^d, \quad |u|\vee|v| \le n, \quad j \in \{1,\cdots,d\}. \tag{4.25}$$

Putting this bound in inequality (4.24), we deduce that

$$\begin{aligned} Ter_2 &\le 4\sum_{j=1}^{d}\left[2L_\Phi M_b + 2h^2 M_b\right] \\ &\le \underbrace{8dM_b(n)(L_\Phi + h^2)}_{:=L_0}, \quad \forall u,v \in \mathbb{R}^d, \quad |u|\vee|v| \le n. \end{aligned} \tag{4.26}$$

Then, combining inequalities (4.23) and (4.26) we arrive at

$$|g_h(u) - g_h(v)|^2 \leq L_{A^{(1)}}|u - v|^2 + L_0.$$

So, taking $L_{g_h} = L_{g_h}(h, n) \geq n^2 + 1 + L_0 + L_{A^{(1)}}$, we see that

$$|g_h(u) - g_h(v)|^2 \leq L_{g_h}(n)|u - v|^2, \quad \forall u, v \in \mathbb{R}^d, \quad |u| \vee |v| \leq n.$$

Then $g_h(\cdot)$ is a locally Lipschitz function. Furthermore, note that under some modifications, this argument also prove that $F_h(\cdot)$ is a Locally Lipschitz function, which also implies that φ_{f_h} is a locally Lipschitz function. Finally, we prove the inequality (4.12). We first observe that by Hypotheses 4.2.1 and 4.3.1

$$\begin{aligned}\langle f(u), u\rangle &= \sum_{j=1}^{d} a_j(u)\left(u^{(j)}\right)^2 + \sum_{j=1}^{d} b_j(u)u^{(j)}\\ &\leq \alpha + \beta|u|^2,\end{aligned}$$

then

$$\langle b(u), u\rangle \leq \alpha + (\beta + L_a)|u|^2. \tag{4.27}$$

With this on mind, and using the inequality (4.11), we deduce that

$$\begin{aligned}\left\langle \varphi_{f_h}(u), u\right\rangle &= \sum_{j=1}^{d} \Phi(h, aj)(u) f^{(j)}(u)u^{(j)}\\ &\leq \sum_{j=1}^{d} L_\Phi L_a|u^{(j)}| + L_\Phi \langle b(u), u\rangle\\ &\leq L_\Phi L_a|u| + L_\Phi(\alpha + (L_a + \beta)|u|^2).\end{aligned}$$

So, taking $L_{\varphi_{f_h}} = L_{\varphi_{f_h}}(h) \geq 2L_\Phi \cdot \max\{L_a, \alpha, \beta\} + 1$, we obtain

$$\left\langle \varphi_{f_h}(u), u\right\rangle \leq L_{\varphi_{f_h}}(1 + |u|^2), \quad \forall u \in \mathbb{R}^d. \tag{4.28}$$

On the other hand, since g is globally Lipschitz it follows that

$$\begin{aligned}|g_h(u)|^2 &\leq 2|g(F_h(u)) - g(F_h(0)|^2 + 2|g(F_h(0)|^2\\ &\leq 2L_g|F_h(u) - F_h(0)|^2 + 4|g(F_h(0)) - g(0)|^2 + 4|g(0)|^2\\ &\leq 4L_g|F_h(u)|^2 + 8L_g|F_h(0)|^2 + 4|g(0)|^2.\end{aligned} \tag{4.29}$$

Now we bound each term in the right hand side of inequality (4.29). Let us first observe that by the monotone condition in Hypothesis 4.2.1

$$|g(0)|^2 \leq 2\alpha. \tag{4.30}$$

On the other hand

$$F_h^{(j)}(0) = \frac{e^{ha_j(0)} - 1}{a_j(0)} b_j(0)\mathbf{1}_{\{E_j^c\}}(0) + hb_j(0)\mathbf{1}_{\{E_j\}}(0).$$

So, taking

$$a_0^* := \min_{\substack{j\in\{1,\cdots,d\}\\ a_j(0)\neq 0}} \left\{|a_j(0)|\right\}, \qquad b_0^* := \max_{j\in 1,\cdots,d} \left\{|b_j(0)|\right\}$$

we can deduce that

$$|F_h^{(j)}(0)|\leq \frac{b_0^*}{a_0^*} e^{hL_a}(1+h), \quad \forall j \in \{1,\cdots,d\}.$$

Then

$$|F_h(0)|^2 \leq d\left(\frac{b_0^*}{a_0^*}\right)^2 e^{2hL_a}(1+h)^2. \tag{4.31}$$

In this line, since the operator $\Phi(h,a_j)$ is bounded, it follows that

$$\begin{aligned} F_h^{(j)}(u) &= e^{ha_j(u)}u^{(j)} + h\Phi(h,a_j)(u)b_j(u)\mathbf{1}_{\{E_j^c\}}(u) + hb_j(u)\mathbf{1}_{\{E_j\}}(u) \\ &\leq e^{ha_j(u)}|u^{(j)}| + hL_\Phi|b_j(u)|\mathbf{1}_{\{E_j^c\}}(u) + h|b_j(u)|\mathbf{1}_{\{E_j\}}(u). \end{aligned}$$

Then we deduce by Hypothesis 4.3.1 that

$$\begin{aligned} |F_h^{(j)}(u)|^2 &\leq 3e^{2hL_a}|u|^2 + (3h^2L_\Phi^2L_b + 3h^2L_b)(1+|u|^2) \\ &\leq 3h^2L_b(1+L_\Phi^2) + 3\left(e^{2hL_a} + h^2L_b(L_\Phi^2+1)\right)|u|^2. \end{aligned}$$

So, taking $L_F \geq 3d\max\{\exp(2hL_a), h^2L_b(L_\Phi^2+1)\}$, we obtain that

$$|F_h^{(j)}(u)|^2 \leq L_F(1+|u|^2). \tag{4.32}$$

Then, combining bounds (4.30),(4.31) and (4.32) we arrive at

$$|g_h(u)|^2 \leq 4L_gL_F(1+|u|^2) + 8L_gd\left(\frac{b_0^*}{a_0^*}\right)^2 e^{2hL_a}(1+h)^2 + 8\alpha.$$

Therefore, if $L_{g_h} \geq 4L_gL_F + 8L_gd\left(\frac{b_0^*}{a_0^*}\right)^2 e^{2hL_a}(1+h)^2 + 8\alpha$ wee see that

$$|g_h(u)|^2 \leq L_{g_h}(1+|u|^2) \tag{4.33}$$

Hence, from the inequalities (4.28) and (4.33) and taking for each fixed $h > 0$

$$\alpha^* := L_{\varphi_{f_h}} \vee L_{g_h}, \qquad \beta^* := 2\alpha^*$$

we have the desired conclusion. □ □

Remark 4.3.1. Note that if $b_j = 0$ in (A-1) then Hypotheses 4.3.1 and 4.3.2 are unnecessary to prove Lemma 4.3.1. Several applications as some stochastic Lotka-Volterra systems [50, 51], the Ginzburg-Landau SDE [39] or the damped Langevin Equations where the potential lacks of a constant term [36] have this form. By other hand, if $bj \neq 0$ then SDE as the stochastic SIR [68], the noisy Duffing-Van der Pol Oscillator [64], the stochastic Lorenz equation [24], among others follow this structure.

Remark 4.3.2. Note that by Lemma 4.3.1, we have that $\lim_{h\to 0}|f(x) - \varphi_{f_h}(x)|= 0$. Hence it is convenient to consider the following modified SDE

$$dy_h(t) = \varphi_{f_h}(y_h(t))dt + g_h(y_h(t))dW(t), \qquad y_h(0) = y_0, \qquad t \in [0, T],$$

as a perturbation of SDE (4.1). Moreover, the functions $\varphi_{f_h}(\cdot)$ and $g_h(\cdot)$ in (4.10) are respectively defined as the functions φ_f and g, but evaluated in the solution of $c = d + h\varphi(c)$, then we can rewrite the LS method (4.2)–(4.3) as

$$\begin{aligned} Y_k^{\star} &= Y_k + h\varphi_{f_h}(Y_k), \\ Y_{k+1} &= Y_k^{\star} + g_h(Y_k)\Delta W_k. \end{aligned}$$

We formalize these ideas in the following sections.

4.4 Strong Convergence of LS

Here, we state and prove the main result of this chapter, the strong convergence of the LS method (4.2)–(4.3) for the solution of SDE (4.1). The main idea of the proof consist in applying the technique discussed in the previous section. We begin establishing the underlying convergence theorem.

Theorem 4.4.1. *Let Hypotheses 4.2.1 and 4.3.1 hold, consider the LS method (4.2)–(4.3) for the SDE* (4.1). *Then there is a continuous-time extension $\overline{Y}(t)$ of the LS solution $\{Y_k\}$ for which $\overline{Y}(t_k) = Y_k$ and*

$$\lim_{h\to 0} \mathbb{E}\left[\sup_{0\leq t\leq T} |\overline{Y}(t) - y(t)|^2\right] = 0.$$

To proof this result, we initiate with the first step of the HMS technique, that is, we will show that the LS method for SDE (4.1) is equivalent to the EM scheme applied to the conveniently modified SDE

$$dy_h(t) = \varphi_{f_h}(y_h(t))dt + g_h(y_h(t))dW(t), \qquad y_h(0) = y_0, \qquad t \in [0, T]. \tag{4.1}$$

We formalize this as a Corollary of Lemma 4.3.1.

Corollary 4.4.1. *Let Hypotheses 4.2.1 and 4.3.1 holds, then the LS method for SDE* (4.1) *is equivalent to the EM scheme applied to the modified SDE* (4.1).

Proof 1. Using the functions $\varphi_{f_h}(\cdot)$ and $g_h(\cdot)$ defined in (4.10) of Lemma 4.3.1, we can rewrite the LS method (4.2)–(4.3) as

$$Y_{k+1} = Y_k + h\varphi_{f_h}(Y_k) + g_h(Y_k)\Delta W_k,$$

which is the EM approximation for the modified SDE (4.1). □

Now we proceed with the Step 2, that is, we will prove that the solution of the modified SDE (4.1) has bounded moments and is close in uniform mean square norm to the solution of the SDE (4.1). In what follows we denote by C universal constant, that is, a positive constant independent on h which value could change in occurrences.

Lemma 4.4.1. *Let Hypotheses 4.2.1 and 4.3.1 holds. Then there is a constant $C = C(p, T) > 0$ and a sufficiently small step size h, such that for all $p > 2$*

$$\mathbb{E}\left[\sup_{0\leq t\leq T}|y_h(t)|^p\right] \leq C\left(1+\mathbb{E}|y_0|^p\right). \tag{4.2}$$

Moreover

$$\lim_{h\to 0}\mathbb{E}\left[\sup_{0\leq t\leq T}|y(t)-y_h(t)|^2\right] = 0. \tag{4.3}$$

Proof 2. The bound (4.2) follows from inequality (4.12) in Lemma 4.3.1 and Theorem 4.2.4. On the other hand, to prove (4.3) we will use the properties of φ_{f_h} and the Higham's stopping time technique employed in [32, Thm 2.2]. Note that by Lemma 4.3.1

$$\varphi_{f_h}(x) = \Phi(h, a_j)(u) f^{(j)}(u)\mathbf{1}_{\{E_j^c\}}(u) + f^{(j)}(u)\mathbf{1}_{\{E_j\}}(u).$$

Furthermore, by Hypothesis 4.3.2 and since $f \in C^1(\mathbb{R}^d)$, $\Phi(h, a_j)(\cdot)$ is bounded, hence, there is a positive constant $K_n$which depends only n such that for each $j \in \{1, \ldots, d\}$

$$\begin{aligned}
|\varphi_{f_h}^{(j)}(u) - f^{(j)}(u)| &\leq \mathbf{1}_{\{E_j^c\}}(u)|f^{(j)}(u)|\left|\Phi(h, a_j)(u) - 1\right| \\
&\leq \mathbf{1}_{\{E_j^c\}}(u)\left(L_\Phi + 1\right)|f(u)| \\
&\leq \mathbf{1}_{\{E_j^c\}}(u) K_n (L_\Phi + 1), \quad \forall u \in \mathbb{R}^d, \quad |u|\leq n, \quad \forall j \in \{1, \ldots, d\}.
\end{aligned}$$

Moreover, we know by the proof of Lemma 4.3.1 that

$$\lim_{\substack{h\to 0\\ u\in E_j^c}} \Phi(h, a_j)(u) = 1.$$

Also, we note that for each $j \in \{1, \ldots, d\}$

$$\lim_{h\to 0} F_h^{(j)}(u) = \lim_{h\to 0} e^{ha_j(u)}u^{(j)} + \lim_{h\to 0}\left(\frac{e^{ha_j(u)}-1}{a_j(u)}\mathbf{1}_{\{E_j^c\}}(u) + h\mathbf{1}_{\{E_j\}}(u)\right) b_j(u^{(j)}) = u^{(j)},$$

hence $\lim_{h\to 0} F_h(u) = u$. Consequently, given $n > 0$ there is a function $K_n(\cdot) : (0, \infty) \to (0, \infty)$, such that $K_n(h) \to 0$ when $h \to 0$ and

$$|\varphi_{f_h}(u) - f(u)|^2 \vee |g_h(u) - g(u)|^2 \leq K_n(h) \qquad \forall u \in \mathbb{R}^d, \quad |u|\leq n. \tag{4.4}$$

Now, using that both f, g are C^1, there is a constant $H_n > 0$ such that

$$|f(u) - f(v)|^2 \vee |g(u) - g(v)|^2 \leq H_n|u - v|^2 \qquad \forall u, v \in \mathbb{R}^d, |u|\vee|v|\leq n. \tag{4.5}$$

On the other hand, by Lemma 4.2.1 and inequality (4.2) we obtain

$$\mathbb{E}\left[\sup_{0\leq t\leq T}|y(t)|^p\right] \vee \mathbb{E}\left[\sup_{0\leq t\leq T}|y_h(t)|^p\right] \leq K := C\left(1+\mathbb{E}|y_0|^p\right).$$

Now, we define the stopping times

$$\tau_n := \inf\{t \geq 0 : |y(t)| \geq n\}, \qquad \rho_n := \inf\{t \geq 0 : |y_h(t)| \geq n\}, \qquad \theta_n := \tau_n \wedge \rho_n, \tag{4.6}$$

and the difference function

$$e_h(t) := y(t) - y_h(t).$$

From the Young's inequality (A.2), we deduce that for any $\delta > 0$

$$\begin{aligned}
\mathbb{E}\left[\sup_{0\leq t\leq T}|e_h(t)|^2\right] &= \mathbb{E}\left[\sup_{0\leq t\leq T}|e_h(t)|^2\mathbf{1}_{\{\tau_n>T,\rho_n>T\}}\right] + \mathbb{E}\left[\sup_{0\leq t\leq T}|e_h(t)|^2\mathbf{1}_{\{\tau_n\leq T \text{ or } \rho_n\leq T\}}\right] \\
&\leq \mathbb{E}\left[\sup_{0\leq t\leq T}|e_h(t\wedge\theta_n)|^2\mathbf{1}_{\{\theta_n\geq T\}}\right] + \frac{2\delta}{p}\mathbb{E}\left[\sup_{0\leq t\leq T}|e_h(t)|^p\right] \\
&\quad + \frac{1-2/p}{\delta^{2/(p-2)}}\mathbb{P}\left[\tau_n \leq T \text{ or } \rho_n \leq T\right].
\end{aligned} \tag{4.7}$$

We proceed to bound each term on the right-hand side of inequality (4.7). By Lemma 4.2.1, $y(t)$ has bounded moments, hence there is a positive constant A such that

$$\mathbb{P}\left[\tau_n \leq T\right] = \mathbb{E}\left[\mathbf{1}_{\{\tau_n<T\}}\frac{|y(\tau_n)|^p}{n^p}\right] \leq \frac{1}{n^p}\mathbb{E}\left[\sup_{0\leq t\leq T}|y(t)|^p\right] \leq \frac{A}{n^p}, \qquad \text{for } p \geq 2. \tag{4.8}$$

The same conclusion can be drawn for ρ_n, then

$$\mathbb{P}\left[\tau_n \leq T \text{ or } \rho_n \leq T\right] \leq \mathbb{P}\left[\tau_n \leq T\right] + \mathbb{P}\left[\rho_n \leq T\right] \leq \frac{2A}{n^p}. \tag{4.9}$$

Now, using the inequality (A.4) and Lemma 4.2.1 we have

$$\mathbb{E}\left[\sup_{0\leq t\leq T}|e_h(t)|^p\right] \leq 2^{p-1}\mathbb{E}\left[\sup_{0\leq t\leq T}\left(|y(t)|^p + |y_h(t)|^p\right)\right] \leq 2^pA. \tag{4.10}$$

So, combining the bound (4.9) with (4.10) in inequality (4.7) we obtain

$$\mathbb{E}\left[\sup_{0\leq t\leq T}|e_h(t)|^2\right] \leq \mathbb{E}\left[\sup_{0\leq t\leq T}|e_h(t\wedge\theta_n)|^2\mathbf{1}_{\{\theta_n\geq T\}}\right] + \frac{2^{p+1}\delta A}{p} + \frac{2(p-2)A}{p\delta^{2/(p-2)}n^p}. \tag{4.11}$$

Next, we show that the first term of (4.11) is bounded. Adding conveniently terms yields

$$\begin{aligned}
e_h(t\wedge\theta_n) &= \int_0^{t\wedge\theta_n}\left[f(y(s)) - f(y_h(s)) + f(y_h(s)) - \varphi_{f_h}(y_h(s))\right]ds \\
&\quad + \int_0^{t\wedge\theta_n}\left[g(y(s)) - g(y_h(s)) + g(y_h(s)) - g_h(y_h(s))\right]dW(s).
\end{aligned}$$

Using the bounds (4.4) and (4.5), the Cauchy-Schwarz, and Doob martingale inequalities, we get

$$\mathbb{E}\left[\sup_{0\leq t\leq\tau}|e_h(t\wedge\theta_n)|^2\right] \leq 4H_n(T+4)\int_0^{\tau}\mathbb{E}\left[\sup_{0\leq t\leq\tau}|e_h(t\wedge\theta_n)|^2\right]ds + 4T(T+4)K_n(h).$$

The Gronwall inequality now yields

$$\mathbb{E}\left[\sup_{0\leq t\leq T}|e_h(t\wedge\theta_R)|^2\right]\leq 4T(T+4)K_n(h)\exp(4H_n(T+4)T).$$

Hence, given $\epsilon>0$ for any $\delta>0$ such that $2^{p+1}\delta A/p<\epsilon/3$, we can take $n>0$ verifying $(p-2)2A/(p\delta^{2/(p-2)}n^p)<\epsilon/3$. Moreover, we can take h sufficiently small such that $4T(T+4)K_n(h)\exp(4H_n(T+4)T)<\epsilon/3$. It follows immediately that

$$\mathbb{E}\left[\sup_{0\leq t\leq T}|e_h(t)|^2\right]<\epsilon/3+\epsilon/3+\epsilon/3=\epsilon,$$

which is the desired conclusion. □

Next, we proceed with Step 3, in which we establish that LS method has bounded moments.

Lemma 4.4.2. *Let Hypotheses 4.2.1, 4.3.1 and 4.3.2 holds. Then for each $p\geq 2$ there is a universal positive constant $C=C(p,T)$ such that for the LS method*

$$\mathbb{E}\left[\sup_{kh\in[0,T]}|Y_k|^{2p}\right]\leq C.$$

Proof 3. Using the split formulation of the LS scheme (4.2)–(4.3) we get

$$|Y_k^\star|^2\leq|A^{(1)}(h,Y_k)|^2|Y_k|^2+2\left\langle A^{(1)}(h,Y_k)Y_k,A^{(2)}(h,Y_k)Y_kb(Y_k)\right\rangle+|A^{(2)}(h,Y_k)|^2|b(Y_k)|^2. \tag{4.12}$$

Then, applying the Cauchy-Schwartz inequality and Hypothesis 4.3.1 we arrive at

$$|Y_k^\star|^2\leq|A^{(1)}(h,Y_k)|^2|Y_k|^2+2\sqrt{L_b}d|A^{(1)}(h,Y_k)||A^{(2)}(h,Y_k)||Y_k|(1+|Y_k|)+L_b|A^{(2)}(h,Y_k)|^2(1+|(Y_k)|^2).$$

Using (4.3.1) we see that

$$|A^{(1)}(h,Y_k)|^2=\left|\operatorname{diag}\left(e^{ha_1(Y_k)},\ldots,e^{ha_d(Y_k)}\right)\right|^2\leq\underbrace{d\exp(2TL_a)}_{:=L_{A^{(1)}}}. \tag{4.13}$$

In similar way, we deduce from the bound (4.21) that

$$\begin{aligned}|A^{(2)}(h,Y_k)|^2&=\left|h\operatorname{diag}\left(\mathbf{1}_{\{E_1\}}(Y_k)+\mathbf{1}_{\{E_1^c\}}(Y_k)\Phi(h,a_1)(Y_k),\ldots,\mathbf{1}_{\{E_d\}}(Y_k)+\mathbf{1}_{\{E_d^c\}}(Y_k)\Phi(h,a_d)(Y_k)\right)\right|^2\\&\leq\sum_{j=1}^{d}\left(\mathbf{1}_{\{E_j^c\}}|h\Phi(h,a_1)(Y_k)|^2+h^2\right)\leq\underbrace{2\exp(2L_aT)\sum_{j=1}^{d}\frac{1}{a_j^*}+dT^2}_{:=L_{A^{(2)}}}.\end{aligned} \tag{4.14}$$

Combining (3) of Hypothesis 4.3.1 with bounds (4.13) and (4.14) yields

$$|Y_k^\star|^2\leq L_{A^{(1)}}|Y_k|^2+2d\sqrt{L_{A^{(1)}}L_{A^{(2)}}L_b}|Y_k|(1+|Y_k|)+L_{A^{(2)}}L_b(1+|Y_k|^2).$$

So, taking $C \geq L_{A(1)} + 2d\sqrt{L_{A(1)}L_{A(2)}L_b} + L_{A(2)}L_b$ we can assert that

$$|Y_k^{\star}|^2 \leq C(3|Y_k|^2 + |Y_k| + 1) \leq 6C\left(|Y_k|^2 + 1\right) \leq C(1 + |Y_k|^2). \tag{4.15}$$

Then, applying bound (4.15) in eq. (4.3) we arrive at

$$|Y_{k+1}|^2 \leq C\left(|Y_k|^2 + 1\right) + 2\left\langle Y_k^{\star}, g(Y_k^{\star})\Delta W_k\right\rangle + |g(Y_k^{\star})\Delta W_k|^2$$

Now, we choose two integers N, M such that $Nh \leq Mh \leq T$. So, adding backwards we get

$$|Y_N|^2 \leq S_N\left(\sum_{j=0}^{N-1}(1 + |Y_j|^2) + 2\sum_{j=0}^{N-1}\left\langle Y_j^{\star}, g(Y_j^{\star})\Delta W_j\right\rangle + \sum_{j=0}^{N-1}\left|g(Y_j^{\star})\Delta W_j\right|^2\right)$$

$$S_N := \sum_{j=0}^{N-1} C^{N-j}$$

Raising both sides to the power p and using the standard inequality (A.4) we obtain

$$|Y_N|^{2p} \leq 6^p S_N^p\left(N^{p-1}\sum_{j=0}^{N-1}(1 + |Y_j|^{2p}) + \left|\sum_{j=0}^{N-1}\left\langle Y_j^{\star}, g(Y_j^{\star})\Delta W_j\right\rangle\right|^p + N^{p-1}\sum_{j=0}^{N-1}\left|g(Y_j^{\star})\Delta W_j\right|^{2p}\right) \tag{4.16}$$

Now we will show that the second and third terms of the inequality (4.16) are bounded. Next, using the Bunkholder-Davis-Gundy inequality [46, Thm 7.3 pg. 40], (A.6) we see that

$$\begin{aligned}
\mathbb{E}\left[\sup_{0\leq N\leq M}\left|\sum_{j=0}^{N-1}\left\langle Y_j^{\star}, g(Y_j^{\star})\Delta W_j\right\rangle\right|^p\right] &\leq C\mathbb{E}\left[\sum_{j=0}^{N-1}|Y_j^{\star}|^2|g(Y_j^{\star})|^2 h\right]^{p/2} \\
&\leq Ch^{p/2}M^{p/2-1}\mathbb{E}\sum_{j=0}^{M-1}|Y_j^{\star}|^p(\alpha + \beta|Y_j^{\star}|^2)^{p/2} \\
&\leq 2^{p/2-1}CT^{p/2-1}h\mathbb{E}\sum_{j=0}^{M-1}(\alpha^{p/2}|Y_j^{\star}|^p + \beta^{p/2}|Y_j^{\star}|^{2p}) \\
&\leq Ch\mathbb{E}\sum_{j=0}^{M-1}(1 + 2|Y_j^{\star}|^p + |Y_j^{\star}|^{2p}) \\
&\leq Ch\sum_{j=0}^{M-1}\left[1 + \mathbb{E}|Y_j^{\star}|^{2p}\right] \\
&\leq C + Ch\sum_{j=0}^{M-1}\mathbb{E}|Y_j|^{2p}.
\end{aligned} \tag{4.17}$$

Now, note that

$$\mathbb{E}\left[\sup_{0\leq N\leq M}\sum_{j=0}^{N-1}|Y_j|^{2p}\right] = \sum_{j=0}^{M-1}\mathbb{E}|Y_j|^{2p}. \tag{4.18}$$

Hence, using Cauchy-Schwartz inequality, the monotone condition (3), bound (4.15) and the standard inequality (A.4), we obtain

$$
\begin{aligned}
\mathbb{E}\left[\sup_{0\le N\le M}\sum_{j=0}^{N-1}\left|g(Y_j^\star)\Delta W_j\right|^{2p}\right] &= \mathbb{E}\sum_{j=0}^{M-1}\left|g(Y_j^\star)\Delta W_j\right|^{2p} \\
&\le \sum_{j=0}^{M-1}\mathbb{E}\left|g(Y_j^\star)\right|^{2p}\mathbb{E}\left|\Delta W_j\right|^{2p} \\
&\le Ch^p\sum_{j=0}^{M-1}\mathbb{E}\left[\alpha+\beta|Y_j^\star|^2\right]^p \\
&\le Ch^p\sum_{j=0}^{M-1}\mathbb{E}\left[\alpha^p+\beta^p|Y_j^\star|^{2p}\right] \\
&\le Ch^{p-1}+Ch^p\sum_{j=0}^{M-1}\mathbb{E}|Y_j|^{2p}. \qquad (4.19)
\end{aligned}
$$

Thus, combining the bounds (4.17) and (4.19) with the inequality (4.16), we can assert that

$$
\begin{aligned}
\mathbb{E}\left[\sup_{0\le N\le M}|Y_N|^{2p}\right] &\le C(M,T)+C(M,T)(1+h)\sum_{j=0}^{M-1}\mathbb{E}|Y_j|^{2p} \\
&\le C+C(1+h)\sum_{j=0}^{M-1}\mathbb{E}\left[\sup_{0\le N\le j}|Y_N|^{2p}\right]. \qquad (4.20)
\end{aligned}
$$

Finally, using the discrete-type Gronwall inequality (A.9), we conclude that

$$
\mathbb{E}\left[\sup_{0\le N\le M}|Y_N|^{2p}\right]\le C\exp(C(1+h)M)\le C\exp(C(1+T))<C,
$$

since the constant C does not depend on h, the proof is complete. □

As the LS scheme has bounded moments, we now proceed whit Step 4, that is, we will obtain a convenient continuous extension of the LS method with bounded moments. Let $\{Y_k\}$ denote the LS solution of SDE (4.1). By Corollary 4.4.1, we conveniently made a continuous extension for the LS approximation, from the time continuous extension of the EM (2.5). Moreover, we prove that the moments of this extension remains bounded.

Corollary 4.4.2. *Let Hypotheses 4.2.1, 4.3.1 and 4.3.2 holds and suppose $0<h<1$ and $p\ge 2$. Then there is a continuous extension $\overline{Y}(t)$ of $\{Y_k\}$ and a positive constant $C=C(T,p)$ such that*

$$
\mathbb{E}\left[\sup_{0\le t\le T}|\overline{Y}(t)|^{2p}\right]\le C.
$$

Proof 4. We take $t=s+t_k$ in $[0,T]$, $\Delta W_k(s):=W(t_k+s)-W(t_k)$ and $0\le s<h$. Then we define

$$
\overline{Y}(t_k+s):=Y_k+s\varphi_{f_h}(Y_k)+g_h(Y_k)\Delta W_k(s), \qquad (4.21)
$$

as a continuous extension of the LS scheme. We proceed to show that $\overline{Y}(t)$ has bounded moments. By Lemma 4.3.1, we have $Y_k^\star = Y_k + h\varphi_{f_h}(Y_k)$. Then for $\gamma = s/h$, it follows that

$$\begin{aligned} Y_k + s\varphi_{f_h}(Y_k) &= \gamma(Y_k + h\varphi_{f_h}(Y_k)) + (1-\gamma)Y_k \\ &= \gamma Y_k^\star + (1-\gamma)Y_k. \end{aligned}$$

Hence, we can rewrite the continuous extension (4.21) as

$$\overline{Y}(t) = \gamma Y_k^\star + (1-\gamma)Y_k + g_h(Y_k)\Delta W_k(s).$$

Combining this relation with the inequalities (4.15) and (A.4), we arrive at

$$\begin{aligned} |\overline{Y}(t_k+s)|^2 &\le 3\left[\gamma C + (\gamma C + 1 - \gamma)\,|Y_k|^2 + |g_h(Y_k)\Delta W_k(s)|^2\right] \\ &\le C + C\left(|Y_k|^2 + |g_h(Y_k)\Delta W_k(s)|^2\right). \end{aligned}$$

Thus,

$$\begin{aligned} \sup_{0\le t\le T}|\overline{Y}(t)|^{2p} &\le \sup_{0\le kh\le T}\left[\sup_{0\le s\le h}|\overline{Y}(t_k+s)|^{2p}\right] \\ &\le \sup_{0\le kh\le T}\left[\sup_{0\le s\le h} C\left(1 + |Y_k|^{2p} + |g_h(Y_k)\Delta W_k(s)|^{2p}\right)\right], \end{aligned} \tag{4.22}$$

for $t \in [0,T]$. Now taking a non negative integer $0 \le k \le N$ such that $0 \le Nh \le T$. From the bond (4.22), we get

$$\sup_{0\le t\le T}|\overline{Y}(t)|^{2p} \le C\left(1 + \sup_{0\le kh\le T}|Y_k|^{2p} + \sup_{0\le s\le h}\sum_{j=0}^{N}|g_h(Y_j)\Delta W_j(s)|^{2p}\right). \tag{4.23}$$

So, using the Doob's Martingale inequality (A.5), Lemma 4.4.2 and that g_h is a locally Lipschitz function, we can bound each term of the inequality (4.23), as follows

$$\begin{aligned} \mathbb{E}\left[\sup_{0\le s\le h}|g(Y_j)\Delta W_j(s)|^{2p}\right] &\le \left(\frac{2p}{2p-1}\right)^{2p}\mathbb{E}|g_h(Y_j)\Delta W_j(h)|^{2p} \\ &\le C\mathbb{E}|g_h(Y_j)|^{2p}\mathbb{E}|\Delta W_j(h)|^{2p} \\ &\le Ch^p\left(1 + \mathbb{E}|Y_j|^{2p}\right) \\ &\le Ch, \end{aligned} \tag{4.24}$$

for each $j \in \{0,\ldots,N\}$. Since $Nh \le T$, combining the bounds (4.23) and (4.24) we get the desired conclusion. □

Once we have carried out all the previous steps, we can prove the Theorem 4.4.1 by Step 5.

Proof 5 (of Theorem 4.4.1). First, note that by inequality (A.4), we have

$$\mathbb{E}\left[\sup_{0\le t\le T}|\overline{Y}(t) - y(t)|^2\right] \le 2\mathbb{E}\left[\sup_{0\le t\le T}|\overline{Y}(t) - y_h(t)|^2\right] + 2\mathbb{E}\left[\sup_{0\le t\le T}|y_h(t) - y(t)|^2\right]. \tag{4.25}$$

Using Lemma 4.4.1, which was established in the Step 2, yields

$$\lim_{h\to 0} \mathbb{E}\left[\sup_{0\le t\le T} |y_h(t) - y(t)|^2\right] = 0. \tag{4.26}$$

It remains to prove that the first term of the right hand side in inequality (4.25) decreases to zero when h tends to zero. Recalling that:

i) By Lemma 4.4.1, the solution of the modified SDE (4.1), y_h, has p-bounded moments ($p \ge 2$).

ii) By Corollary 4.4.2, the LS continuous extension for the SDE (4.1), $\overline{Y}(t)$, has bounded moments and it is equivalent to the EM extension for the modified SDE (4.1).

Hence, we can apply Theorem 2.4.2 to conclude that

$$\lim_{h\to 0} \mathbb{E}\left[\sup_{0\le t\le T} |\overline{Y}(t) - y_h(t)|^2\right] = 0. \tag{4.27}$$

Finally, combining the limits (4.26) and (4.27) with inequality (4.25) gives

$$\begin{aligned}\lim_{h\to 0} \mathbb{E}\left[\sup_{0\le t\le T} |\overline{Y}(t) - y(t)|^2\right] &\le 2\lim_{h\to 0} \mathbb{E}\left[\sup_{0\le t\le T} |\overline{Y}(t) - y_h(t)|^2\right] \\ &\quad + 2\lim_{h\to 0} \mathbb{E}\left[\sup_{0\le t\le T} |y_h(t) - y(t)|^2\right] = 0,\end{aligned}$$

which proves the theorem. □

4.5 Convergence Rate

In this section we show that the explicit Linear Steklov method eqs. (4.2) and (4.3) converges with a standard order of one-half. For that, we use a similar procedure as in [32]. In addition to Hypotheses 4.2.1, 4.3.1 and 4.3.2 we also require the following.

Hypothesis 4.5.1. There exist constants $L_f, D \in \mathbb{R}$ and $q \in \mathbb{Z}^+$ such that $\forall u, v \in \mathbb{R}^d$

$$\langle u - v, f(u) - f(v)\rangle \le L_f |u - v|^2, \tag{4.1}$$

$$|f(u) - f(v)|^2 \le D(1 + |u|^q + |v|^q)|u - v|^2. \tag{4.2}$$

Hypothesis 4.5.2. The SDE (4.1) the EM solution and its continuous extension satisfy

$$\mathbb{E}\left[\sup_{0\le t\le T} |y(t)|^p\right], \quad \mathbb{E}\left[\sup_{0\le t\le T} |Y(t)|^p\right], \quad \mathbb{E}\left[\sup_{0\le t\le T} |\overline{Y}(t)|^p\right] < \infty, \qquad \forall p \ge 1. \tag{4.3}$$

Theorem 4.5.1. *[Higham et al. [32, Thm 4.4]] Under Hypotheses 4.3.1–4.5.1 the EM solution with continuous extension $\overline{X}$ satisfies*

$$\mathbb{E}\left[\sup_{0\le t\le T} |\overline{Y}(t) - y(t)|^2\right] = \mathcal{O}(h^2). \tag{4.4}$$

Lemma 4.5.1. *Under Hypotheses 4.5.1 and 4.5.2 and sufficiently small h, there exist constants $D' \in \mathbb{R}$ and $q' \in \mathbb{Z}$ such that for all $u, v \in \mathbb{R}^d$*

$$|\varphi_{f_h}(u) - \varphi_{f_h}(v)|^2 \leq D'\left(1 + |u|^{q'} + |v|^{q'}\right)|u - v|^2, \tag{4.5}$$

$$|f(u) - \varphi_{f_h}(u)|^2 \leq D'\left(1 + |u|^{q'}\right)h^2, \tag{4.6}$$

$$|g(u) - g_h(u)|^2 \leq D'\left(1 + |u|^{q'}\right)h^2. \tag{4.7}$$

Proof 6. From inequality (4.11), we have

$$|\varphi_{f_h}(u) - \varphi_{f_h}(v)|^2 \leq (2 + L_\Phi)|f(u) - f(v)|^2 \leq (2 + L_\Phi)D(1 + |u|^q + |v|^q).$$

Moreover, if $u \in E_j$ then $\varphi_{f_h}(u) = f^{(j)}(u)$. On the other hand, if $u \in E_j^c$ then

$$|f(u) - \varphi_{f_h}(u)|^2 = \sum_{j=1}^{d} |1 - \Phi(h, a_j)(u)|^2 |f^{(j)}(u)|^2,$$

By the L'Hôpital theorem, we get

$$\lim_{h \to 0} |1 - \Phi(h, a_j)(u)| = \left|1 - \lim_{h \to 0} \frac{e^{ha_j(u)} - 1}{ha_j(u)}\right| \leq \left|1 - \lim_{h \to 0} e^{hL_a}\right| = 0.$$

Thus, there is a sufficiently small $h > 0$ such that $|1 - \Phi_j(u)| < Ch$ for all $u \in E_j^c$ and

$$|f(u) - \varphi_{f_h}(u)|^2 \leq Ch^2|f(u)|^2 \leq D'(1 + |u|^q)h^2,$$

as we require. Given that $g_h(u) = g(F_h(u))$ from theorem 4.3.1 we get

$$|g(u) - g_h(u)|^2 \leq L_g|u - u + h\varphi f_h(u)|^2 \leq 2(1 + L_\Phi)h^2|f(u)|^2 \leq 2(1 + L_\Phi)D(1 + |u|^q)h^2.$$

□

Lemma 4.5.2. *Assume Hypotheses 4.5.1 and 4.5.2 hold then the solution $y_h(t)$ of the modified SDE (2.9) satisfies*

$$\mathbb{E}\left[\sup_{0 \leq t \leq T} |y_h(t) - y(t)|^2\right] = \mathcal{O}(h^2). \tag{4.8}$$

Proof 7. We define $e(t) := y(t) - y_h(t)$ where

$$\begin{aligned} y(t) &= y_0 + \int_0^t f(y(s))ds + \int_0^t g(y(s))dW(s), \\ y_h(t) &= y_0 + \int_0^t \varphi_{f_h}(y_h(s))ds + \int_0^t g_h(y_h(s))dW(s). \end{aligned}$$

Using Itô's formula over the function $V(t, x, y) = |x - y|^2$ for all $x, y \in \mathbb{R}^d$, we obtain

$$de(t) = \left(f(y(t)) - \varphi_{f_h}(y(t))dt\right) + \left(g(y(t)) - g_h(y_h(t))\right)dW(t),$$

Thus,

$$
\begin{aligned}
|e(t)|^2 = 2 & \underbrace{\int_0^t \left\langle e(s), f(y(s)) - \varphi_{f_h}(y_h(s)) \right\rangle ds}_{:=I_1} + \underbrace{\int_0^t |g(y(s)) - g_h(y_h(s))|^2 ds}_{:=I_2} \\
& + 2 \underbrace{\int_0^t \langle e(s), [g(y(s)) - g_h(y_h(s))]\, dW(s) \rangle}_{:=I_3} . \qquad (4.9)
\end{aligned}
$$

Now we proceed to bound each integral of inequality (4.9). By Hypothesis 4.5.1 and the Young inequality, we get

$$
\begin{aligned}
I_1(t) &\leq 2 \int_0^t \langle y(s) - y_h(s), f(y(s)) - f(y_h(s)) \rangle \, ds + \int_0^t \left\langle y(s) - y_h(s), f(y_h(s)) - \varphi_f(y_h(s)) \right\rangle ds \\
&\leq 3 \int_0^t |y(s) - y_h(s)|^2 ds + D' h^2 \int_0^t 1 + |y_h(s)|^{q'} ds.
\end{aligned}
$$

Since $y_h(t)$ the has bonded moments, there exists a universal constant L which does not depends on h such that

$$
\mathbb{E}\left[I_1(s)\right] \leq L \int_0^t \mathbb{E}|e(s)|^2 ds + Lh^2. \qquad (4.10)
$$

Using Hypotheses 4.2.1 and 4.5.1 it is followed

$$
I_2(t) \leq 2L_g \int_0^t |y(s) - y_h(s)|^2 ds + 2D' h^2 \int_0^t 1 + |y_h(s)|^q ds,
$$

thus

$$
\mathbb{E}\left[I_2(s)\right] \leq L \int_0^t \mathbb{E}|e(s)|^2 ds + Lh^2. \qquad (4.11)
$$

Note that $\mathbb{E}\left[I_3(t)\right] \leq \mathbb{E}\left[\sup_{0 \leq s \leq t} |I_3(s)|\right]$. From the Burkholder-Davis-Gaundy inequality, Hypotheses 4.2.1 and 4.5.1 and as $y_h(t)$ has bounded moments, we obtain

$$
\begin{aligned}
\mathbb{E}\left[\sup_{0 \leq s \leq t} |I_3(s)|\right] &\leq 2^4 \mathbb{E}\left[\sup_{0 \leq s \leq t} |e(s)|^2 \int_0^t |g(y(s)) - g_h(y(s))|^2 ds\right]^{1/2} \\
&\leq 2^4 \mathbb{E}\left[\frac{1}{2 \cdot 2^9} \left(\sup_{0 \leq s \leq t} |e(s)|^2\right) + \frac{2^9}{2} \left(\int_0^t |g(y(s)) - g_h(y_h(s))|^2 ds\right)^2\right] \\
&\leq 2L_g \mathbb{E}\left[\int_0^t |y(s) - y_h(s)|^2 ds\right] + D' T h^2 + D' T h^2 \int_0^t \mathbb{E}|y_h(s)|^{q'} ds \\
&\leq L \int_0^t \mathbb{E}|e(s)|^2 ds + Lh^2. \qquad (4.12)
\end{aligned}
$$

Substituting inequalities (4.10), (4.11) and (4.12) on equation (4.9), we deduce that

$$
\mathbb{E}\left[\sup_{0 \leq s \leq t} |e(t)|^2\right] \leq L \int_0^t \mathbb{E}|e(s)|^2 ds + Lh^2 \leq L \int_0^t \mathbb{E}\left[\sup_{0 \leq r \leq s} |e(s)|^2\right] ds + Lh^2.
$$

By the Gronwall inequality, we conclude that

$$\mathbb{E}\left[\sup_{0\leq t\leq T}|e(t)|^2\right] \leq L\exp(LT)h^2 \leq Ch^2.$$

□

We can now obtain the convergence rate of the explicit Linear Steklov method.

Theorem 4.5.2. *Under Hypotheses 4.2.1–4.5.1 and consider the explicit LS method (4.7) for the SDE (4.1). Then there exists a continuous-time extension $\overline{Y}(t)$ of the LS numerical approximation for which*

$$\mathbb{E}\left[\sup_{0\leq t\leq T}|\overline{Y}(t)-y(t)|^2\right] = \mathcal{O}(h). \tag{4.13}$$

Proof 8. Using bound (4.25) then by lemma 4.5.2 and since the LS continuous-time extension (4.21) is equivalent to the EM continuous-time extension (2.5), we can use Theorem 4.5.1 and conclude that the LS has order one-half. □

4.6 Almost Sure Stability

In this section we study the globally almost surely asymptotic stability (as-stability) of the Linear Steklov method (4.2)–(4.3), in the scalar case. For simplicity we assume that

$$f(x) = a(x)x,$$

for some suitable nonlinear function $a : \mathbb{R} \to \mathbb{R}$. Here, we will follow the same technique reported by Mao and Szpruch in [48]. First, we need sufficient conditions to characterize when the solution of the SDE (4.1) is as-stable. The following result deals with it.

Theorem 4.6.1 (Mao and Szpruch [48, Thm. 2.2])**.** *Let hypothesis 4.2.1 hold and suppose that there exist a function $z \in \mathcal{C}(\mathbb{R}^d, \mathbb{R}_+)$ such that*

$$\langle x, f(x)\rangle + \frac{1}{2}|g(x)|^2 \leq -z(x), \qquad \forall x \in \mathbb{R}^d,$$

then

(i) For any $y_0 \in \mathbb{R}^n$ the solution of the SDE (4.1), $y(t)$, satisfies

$$\limsup_{t\to\infty}|y(t)|^2 \leq \infty \qquad \text{a.s.} \qquad \textit{and} \qquad \lim_{t\to\infty} z(y(t)) = 0 \qquad \text{a.s.}$$

(ii) additionally, if $z(x) = 0$ only when it is evaluated at $x = 0$, then

$$\lim_{t\to\infty} y(t) = 0 \qquad \text{a.s.} \qquad \forall y_0 \in \mathbb{R}^d.$$

Next, we prove that LS method emulates the as-stability. The proof of this result hinges on the Lemma 4.6.1 see for instance [43, Th. 7, pg. 139]. We will denote by $\{Z \to\}$ the set of all $\omega \in \Omega$ for which the scalar process Z has the property that $\lim_{k\to\infty} Z_k$ exists and is finite.

Lemma 4.6.1 ([43, Thm. 7, pg. 139])**.** *Let $Z = \{Z_k\}$ be a nonnegative semimartingale with $\mathbb{E}|Z| < \infty$ and Doob decomposition*

$$Z = Z_0 + A^{(1)} - A^{(2)} + M,$$

where $A^{(1)} := \{A_k^{(1)}\}_{k\in\mathbb{N}}$ and $A^{(2)} := \{A_k^{(2)}\}_{k\in\mathbb{N}}$ are a. s. *nondecreasing predictable processes with $A_0^{(1)} = A_0^{(2)} = 0$ and $M := \{M_k\}_{k\in\mathbb{N}}$ is a local $\{\mathcal{F}_k\}$-martingale with $M_0 = 0$. Then*

$$\left\{A^{(1)} \rightarrow\right\} \subseteq \left\{A^{(2)} \rightarrow\right\} \cap \{Z \rightarrow\} \qquad \text{a. s.}$$

Theorem 4.6.2. *Let Hypothesis 4.2.1 holds. Suppose that there is a function $z \in \mathcal{C}(\mathbb{R}^n, \mathbb{R}_+)$ and a step size $h^* > 0$ such that for all $x \in \mathbb{R}$ and for all h in $(0, h^*)$,*

$$\langle x, f(x)\rangle + \frac{1}{2}|g(x)|^2 \leq -z(x), \tag{4.1}$$

$$|x|^2 \frac{(\exp(2ha(x)) - 1)}{h} + |g_h(x)|^2 \leq -z(x), \tag{4.2}$$

Then the LS method defined by (4.2)–(4.3) satisfies

$$\limsup_{k\to\infty} |Y_k|^2 < \infty \quad \text{and} \quad \lim_{k\to\infty} w(Y_k) = 0.$$

In addition, if $z(x) = 0$ only when $x = 0$, then $\lim_{k\to\infty} Y_k = 0$.

Proof. Taking advantage of Lemma 4.6.1, we proceed to construct a conveniently semimartingale. To

$$\begin{aligned}|Y_{k+1}|^2 = |Y_k|^2 &+ h^2|\varphi_{f_h}(Y_k)|^2 + |g_h(Y_k)\Delta W_k|^2 + 2h\left\langle Y_k, \varphi_{f_h}(Y_k)\right\rangle \\ &+ 2\left\langle Y_k, g_h(Y_k)\Delta W_k\right\rangle + 2h\left\langle \varphi_{f_h}(Y_k), g_h(Y_k)\Delta W_k\right\rangle.\end{aligned} \tag{4.3}$$

Let

$$\begin{aligned}\Delta M_{k+1} := &|g_h(Y_k)\Delta W_{k+1}|^2 - |g_h(Y_k)|^2 h \\ &+ 2\left\langle Y_k, g_h(Y_k)\Delta W_{k+1}\right\rangle + 2h\left\langle \varphi_{f_h}(Y_k), g_h(Y_k)\Delta W_{k+1}\right\rangle,\end{aligned}$$

which is a local martingale. Taking $B_j := -\left[2\left\langle Y_j, \varphi_{f_h}(Y_j)\right\rangle + |g_h(Y_j)|^2 + h|\varphi_{f_h}(Y_j)|^2\right]$, and fixing $N \in \mathbb{N}$, we can rewrite (4.3) as

$$|Y_{N+1}|^2 = |Y_0|^2 - \sum_{j=0}^{N} B_j h + \sum_{j=0}^{N} \Delta M_{j+1}. \tag{4.4}$$

To prove that (4.4) is the required decomposition to apply Lemma 4.6.1, we use that

$$\varphi_{f_h}(x) = x\frac{(\exp(ha(x)) - 1)}{h}. \tag{4.5}$$

By algebraic manipulations, we obtain

$$B_j = -\left[|Y_j|^2 \frac{\left(\exp\left(2ha(Y_j)\right) - 1\right)}{h} + |g_h(Y_j)|^2\right], \qquad j = 0, \ldots, N.$$

Given that the inequality (4.2) holds, we can deduce that

$$B_j \geq z(Y_j) \geq 0, \quad j = 0, \ldots N.$$

Consequently, $A_k^{(2)} := \sum_{j=0}^{k} B_j h$ is a non decreasing process. Finally, taking $A^{(1)} = 0$, $Z = |Y_k|^2$ and $M_k = \sum_{j=0}^{k} \Delta M_{j+1}$. We can deduce by Lemma 4.6.1 that $\{A^{(1)} \rightarrow\} = \Omega$, thus

$$\limsup_{k\to\infty} |Y_k|^2 < \infty \quad \text{a. s.}, \quad \text{and} \quad \sum_{j=0}^{\infty} z(Y_k) \leq \sum_{j=0}^{\infty} B_j h < \infty.$$

Consequently $\lim_{k\to\infty} z(Y_k) = 0$, and the theorem follows. □ □

4.7 Numerical Simulations

Here we analyze the behavior of the explicit Linear Steklov method (LS) for scalar and vector SDEs. The tests confirm the convergence order 1/2 for stochastic differential systems with locally Lipschitz drift and suggest that the LS scheme reproduces almost surely stability (a.s.). We validate the efficiency of the new method by comparing with other actual methods like the Euler-Maruyama, Backward Euler (BEM) [48] and tamed Euler (TEM) [36]. All simulations are implemented in Python 2.7 and we use the Mersenne random number generator with fixed seed 100.

Example 4.7.1. Here we illustrate the stability Theorem 4.6.2 through an numerical example presented in [2, sec 7, pg. 420]. Here, Appleby and Kelly, had proved that the EM fails to preserve the almost sure stability of the test SDE

$$dy(t) = -\beta y(t)|y(t)|^p dt + \sigma(t)|y(t)|^\rho dW(t). \tag{4.1}$$

That is, the EM approximation explodes to infinity on finite time when $p+1 > 2\rho$. But, with the same parameters $\lim_{t\to\infty} y(t) = 0$ a. s., see [2, 3] for more details. The authors consider

$$dy(t) = -y^3 dt + \frac{1}{[\log(t+1)]^{1.1}} dW_t, \qquad t > 0, \tag{4.2}$$

and deduce conditions for the step-size h and initial condition $y(t_0) = y_0$ in order to claim with high probability when the EM scheme for SDE (4.2) is GASA-stable or diverge [2, Cor 7.1 pg. 421]. More specifically, given $h < 0.0384$ and the EM for SDE (4.2)

$$X_{k+1} = X_k - hX_k^3 + \frac{1}{[\log(n+1)]^{1.1}} \Delta W_k, \qquad X_0 = y(t_0). \tag{4.3}$$

(i) If $X_0 \in \left(-\sqrt{\frac{2}{h}} + 7\sqrt{h}, \sqrt{\frac{2}{h}} - 7\sqrt{h}\right)$, then $\mathbb{P}\left[\lim_{k\to\infty} X_k = 0\right] > 0.95$.

(ii) If $X_0 \in \left(-\infty, -\sqrt{\frac{2}{h}} - 7\sqrt{h}\right) \bigcup \left(\sqrt{\frac{2}{h}} + 7\sqrt{h}, \infty\right)$, then

$$\mathbb{P}\left[\limsup_{n\to\infty} X_k = \infty \text{ or } \liminf_{n\to\infty} X_k = -\infty\right] > 0.95 \quad .$$

Thus we perform a simulation with step size $h = 0.2$ using the EM, Tamed Euler-Maruyama (TEM) and the LS schemes with unstable EM initial conditions. Figure 4.1 shows how the EM scheme produce spurious solutions. Meanwhile, the TEM ans LS approximations reproduce the asymptotic behavior, also we note a better initial precision for the LS approximation.

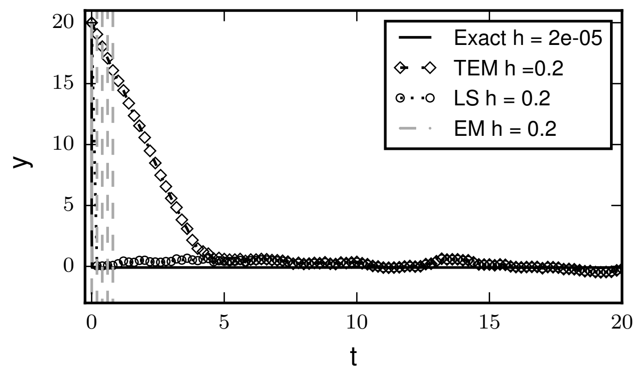

Figure 4.1: Likening between the EM, TEM and LS approximations with unstable EM conditions. Here "exact" means a BEM solutions with step size $h = 2 \times 10^{-5}$.

Example 4.7.2. We examine the LS method using a SDE with super-linear grow diffusion. We consider the SDE reported by Tretyakov and Zhang in [69, Eq. (5.6)]

$$dy(t) = \left(1 - y^5(t) + y^3(t)\right) dt + y^2(t)dW(t), \qquad y_0 = 0. \tag{4.4}$$

Tretyakov and Zhang shows via simulation of (4.4) that the increment-tamed scheme [34, Eq(1.5)]

$$X_{k+1} = X_k + \frac{f(X_k)h + g(X_k)\Delta W_k}{\max\left(1, h\left|hf(X_k) + g(X_k)\Delta W_k\right|\right)} \tag{4.5}$$

produces spurious oscillations. Hutzenthaler and Jentzen prove the convergence of this scheme under linear growth condition over diffusion. So, this suggest us that only certain kind of explicit schemes with convergence under globally Lipschitz and linear growth diffusion conditions can extended their convergence to a locally Lipschitz diffusion and other kind of growth bound. Using $a(x) := -x^4 + x^2$, $b := 1$ and $E = \{-1, 0, 1\}$, we construct the LS method

$$Y_{k+1} = \exp(ha(Y_k))Y_k + \frac{\exp(ha(Y_k)) - 1}{a(Y_k)}\mathbf{1}_{\{E^c\}} + h\mathbf{1}_{\{E\}} + Y_k^2\Delta W_k. \tag{4.6}$$

Figure 4.2 shows the numerical solution of SDE (4.4) with the Increment-Tamed (I-TEM) (4.5), LS method (4.6), and the Tamed (TEM) scheme with. We consider the implicit Midpoint scheme [69, Eq.(5.3)] with $h = 10^{-4}$ as reference.

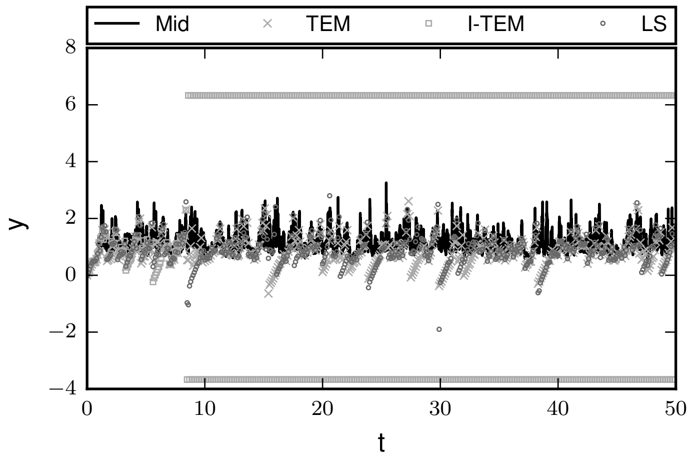

Figure 4.2: Numerical solution of SDE (4.4) using the I-TEM (4.5), LS method (4.6) and TEM with $h = 0.1$, the reference solutions is a Midpoint rule approximation with $h = 10^{-4}$.

Example 4.7.3. Now we compare the order of convergenceand the run time of the LS method withthe TEM scheme as in [36]. That is,we consider a Langevin equation under the d-dimensional potential $U(x) = \frac{1}{4}|x|^4 - \frac{1}{2}|x|^2$, and d-dimensional Brownian additive noise. The corresponding SDE reads

$$dy(t) = (y(t) - |y(t)| \cdot y(t))\, dt + dW(t), \qquad y(0) = 0. \tag{4.7}$$

This model describes the motion of a Brownian particle of unit mass immersed on the potential $U(x)$. Taking $a_j(x) := 1 - |x|$ and $b_j = 0$, $j \in 1 \ldots d$ we obtain the LS method

$$Y_{k+1} = \operatorname{diag}\left[e^{ha_1(Y_k)}, \ldots, e^{ha_d(Y_k)}\right] Y_k + \Delta W_k. \tag{4.8}$$

Table 4.1 shows the root means square errors at a final time $T = 1$, which is approximated by

$$\sqrt{\mathbb{E}\left[|Y_N - y(T)|^2\right]} \approx \frac{1}{M}\left(\sum_{i=1}^{M} |y_i(T) - Y_{N,i}|^2\right)^{1/2}, \tag{4.9}$$

over a sample of M =10 000 trajectories of the TEM, LS and BEM solutions to SDE (4.7) with dimension $d = 10$. We consider the TEM solution with step $h = 2^{-19}$ as reference solution. In this experiment we confirm that the LS method converges with standard order $1/2$ and is almost equal accurate than the TEM.

In some application as in Browninan Dynamics Simulations [18], the dimension of a SDE increases considerable the complexity and computational cost — this prohibits the use of implicit methods. Figure 4.3 supports this (for SDE(4.7)): the runtime of BEM depends on dimension in a quadratic way, while the LS and TEM depends on linear form.

Example 4.7.4. Hutzenthaler et al. improve convergence of the Euler method by taming the drift increment term with the factor $\frac{1}{1+h|f(Y_k)|}$, as consequence, the norm of $\frac{hf(Y_k)}{1+h|f(Y_k)|}$, is bounded by 1, which controls the

	TEM		LS		BEM	
h	ms-error	ECO	ms-error	ECO	ms-error	ECO
2^{-2}	1.703 88	—	1.553 94	—	1.381 57	—
2^{-3}	1.169 77	0.54	1.107 75	0.48	1.053 09	0.39
2^{-7}	0.278 95	0.48	0.277 95	0.48	0.276 895	0.48
2^{-11}	0.070 10	0.50	0.070 09	0.50	0.070 07	0.50
2^{-15}	0.017 39	0.51	0.017 39	0.51	0.017 39	0.51

Table 4.1: Mean square errors and the experimental convergence order (ECO) for the SDE (4.7) with a TEM with $h = 2^{-19}$ as reference solution.

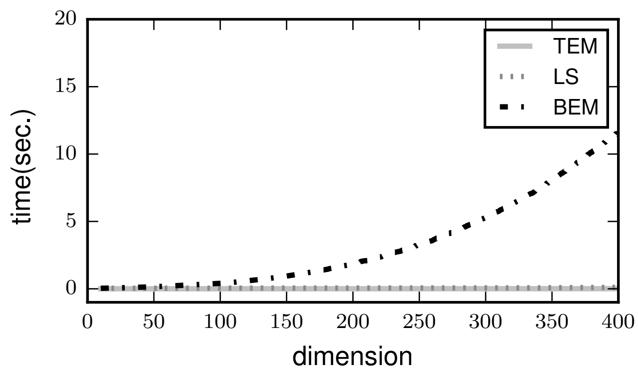

Figure 4.3: Runtime calculation of Y_N with $h = 2^{-17}$, using the BEM, LS and TEM methods for SDE (4.7).

drift contribution of the TEM method at each step. This idea works very well over SDEs with drift contribution and initial condition that are comparable with this bound. However, we observed that on models where the drift contribution has other scales, the TEM over damps the drift contribution. To fix ideas, we consider the stochastic model reported in [19],

$$\begin{aligned} dy_1(t) &= (\lambda - \delta y_1(t) - (1-\gamma)\beta y_1(t) y_3(t))\, dt - \sigma_1 y_1(t) dW_t^{(1)}, \\ dy_2(t) &= ((1-\gamma)\beta y_1(t) y_3(t) - \alpha y_2(t))\, dt - \sigma_1 y_2(t) dW_t^{(1)}, \\ dy_3(t) &= ((1-\eta) N_0 \alpha y_2(t) - \mu y_3(t) - (1-\gamma)\beta y_1(t) y_3(t))\, dt - \sigma_2 y_3(t) dW_t^{(2)}. \end{aligned} \tag{4.10}$$

Taking

$$\begin{aligned} E_1 &:= \left\{ (x,y,z)^T \in \mathbb{R}^3 : z = 0 \text{ or } z = 0 \frac{-\delta}{\beta(1-\gamma)} \right\}, \quad E_2 := \emptyset, \\ E_3 &:= \left\{ (x,y,z)^T \in \mathbb{R}^3 : x = 0, \text{ or } x = \frac{-\mu}{\beta(1-\gamma)} \right\} \end{aligned}$$

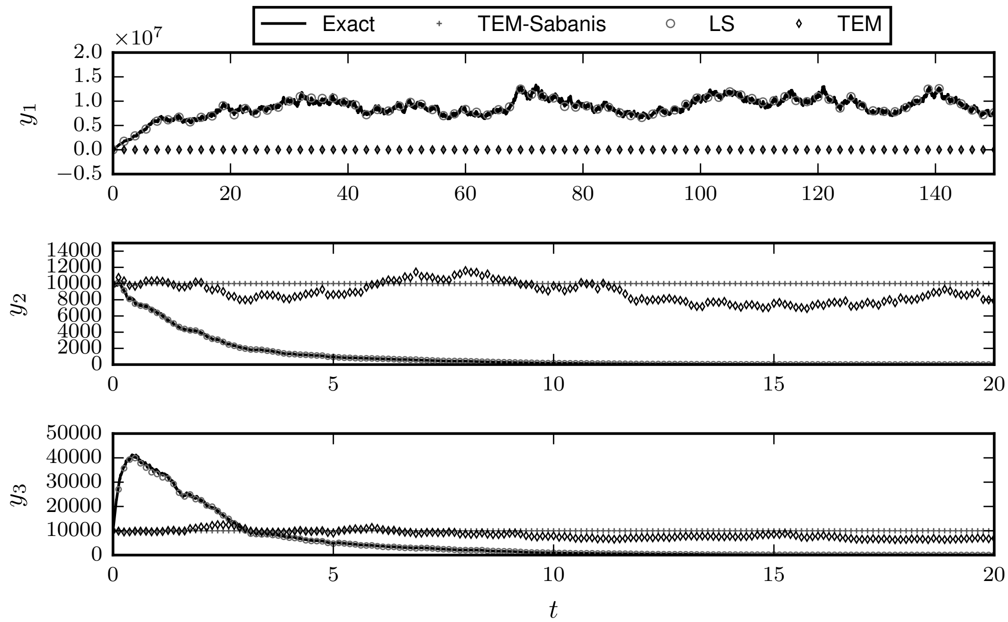

Figure 4.4: Likening between EM, LS , TEM approximations for SDE (4.10) with $\gamma = 0.5$, $\eta = 0.5$, $\lambda = 10^6$, $\delta = 0.1$, $\beta = 10^{-8}$, $\alpha = 0.5$, $N_0 = 100$, $\mu = 5$, $\sigma_1 = 0.1$, $\sigma_2 = 0.1$, $y_0 = (10\,000, 10\,000, 10\,000.)^T$, $h = 0.125$. Here the reference solution means a BEM simulation with the same parameters but with a step-size $h = 10^{-5}$.

$$a_1(Y_k)) := -\left(\delta + (1-\gamma)\beta Y_k^{(3)}\right), \qquad b_1(Y_k^{(-1)}) := \lambda,$$

$$a_2(Y_k) := -\alpha, \qquad b_2(Y_k^{(-2)}) := (1-\gamma)\beta Y_k^{(1)} Y_k^{(3)},$$

$$a_3(Y_k) = -\left(\mu + (1-\gamma)\beta Y_k^{(1)}\right), \qquad b_3(Y_k^{(-3)}) := (1-\eta)\, N_0 \alpha Y_k^{(2)},$$

the LS method for the stochastic model (4.10) reads,

$$
\begin{aligned}
Y_{k+1} &= A^{(1)}(h, Y_k)\, Y_k + A^{(2)}(h, Y_k) b(Y_k) + g(Y_k)\, \Delta W_k, \qquad \Delta W_k = \left(W_k^{(1)}, W_k^{(2)}\right)^T, \\
A^{(1)}(h, Y_k) &:= \begin{pmatrix} e^{ha_1(Y_k)} & 0 & 0 \\ 0 & e^{ha_2(Y_k))} & 0 \\ 0 & 0 & e^{ha_3(Y_k))} \end{pmatrix}, \\
A^{(2)} &:= \begin{pmatrix} h\Phi_1(Y_k)\mathbf{1}_{\{E_1^c\}} & 0 & 0 \\ 0 & \left(\dfrac{e^{-h\alpha} - 1}{\alpha}\right) & 0 \\ 0 & 0 & h\Phi_3(Y_k)\mathbf{1}_{\{E_3^c\}} \end{pmatrix} + h \begin{pmatrix} \mathbf{1}_{\{E_1\}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathbf{1}_{\{E_3\}} \end{pmatrix}, \\
b(Y_k) &:= \begin{pmatrix} b_1(Y_k^{(-1)}) \\ b_2(Y_k^{(-2)}) \\ b_3(Y_k^{(-3)}) \end{pmatrix}, \qquad g(Y_k) := \begin{pmatrix} -\sigma_1 Y_k^{(1)} & 0 \\ -\sigma_1 Y_k^{(2)} & 0 \\ 0 & -\sigma_2 Y_k^{(3)} \end{pmatrix}.
\end{aligned}
\tag{4.11}
$$

Dalal et al. in [19, Thm 5.1] gives conditions over the parameter of SDE (4.10), which assure a.s. exponential stability — in the sense that the infected cells (y_2) and virus particles (y_3) will tend to their equilibrium value 0 exponentially with probability 1—and verify this asymptotic behavior by simulation with parameters reported in published literature [10, 16, 56, 57]. Figure 4.4 shows a simulation path with same parameters with the LS and TEM approximations. We observe how the TEM oscillates around of initial condition while the LS reproduce the underlying asymptotic behavior.

Chapter 5

Conclusions and future work

5.1 Conclusions

We have been proposed a new way to design numerical methods for SDEs based on the Steklov average. Two Steklov type methods were constructed to evidence our new approach. First we presented a scalar scheme with good stability properties —the Steklov method. We verify its convergence and stability over a standard globally Lipschitz setup and compare its performance with a competitive solvers. With the Linear Steklov scheme we obtain a extension over multidimensional and locally Lipschitz context. Also we proved its one-half convergence order and evidence its accuracy by simulation, even for SDEs with super-linear diffusions. However, we see a world of problems that should be consider. We mention just a few of them.

- The numerical evidence suggest that the scalar Steklov and the LS methods works under diffusions with super-linear growth. So, one of the possible future directions points to prove this claim.
- Since we have been prove strong convergence, we would to apply the Multilevel Monte Carlo approach to the Brownian Dynamics Simulation using Steklov type schemes.
- The schemes presented here follow structure of the Euler-Maruyama family, so we see that is possible formulate schemes of type Tamed, Milstein, Balanced, Theta, Runge-Kutta and other kinds of methods, by replacing evaluations of the drift f by its Steklov average.
- Also, we see viable to study stability of the LS method trough the theory of random dynamical systems.
- Furthermore, a natural extension would be to design Steklov type schemes for more general SDEs, that is, SDEs with delay, Poisson jumps or partial derivatives.

Appendices

Appendix A

Useful Inequalities

Hölder.

$$\mathbb{E}[X^T Y] \leq \left(\mathbb{E}|X|^p\right)^{\frac{1}{p}} \left(\mathbb{E}|X|^q\right)^{\frac{1}{q}}. \tag{A.1}$$

Young.

$$|a||b| \leq \frac{\delta}{p}|a|^p + \frac{\delta}{q\delta^{q/p}}|b|^q. \tag{A.2}$$

Minkowski.

$$\left(\mathbb{E}|X+Y|^p\right)^{\frac{1}{p}} \leq \left(\mathbb{E}|X|^p\right)^{\frac{1}{p}} + \left(\mathbb{E}|Y|^p\right)^{\frac{1}{p}}. \tag{A.3}$$

A standard inequality. Fix $1 < p < \infty$ and consider a sequence of real numbers $\{a_i\}_{i=1}^N$ with $N \in \mathbb{N}$. Then one can formulate this usefully inequality

$$\left(\sum_{j=1}^{N} a_j\right)^p \leq N^{p-1} \sum_{j=1}^{N} a_j^p. \tag{A.4}$$

Doob's Martingale Inequality. Let $\{M_t\}_{t\geq 0}$ be a $\mathbb{R}^d$-valued martingale. Let $[a,b]$ be a bounded interval in $\mathbb{R}_+$. If $p > 1$ and $M_t \in L^p(\Omega;\mathbb{R}^d)$ then

$$\mathbb{E}\left(\sup_{a\leq t\leq b} |M_t|^p\right) \leq \left(\frac{p}{p-1}\right)^p \mathbb{E}|M_b|^p. \tag{A.5}$$

Burkholder–Davis–Gundy inequality. Let $g \in \mathcal{L}(\mathbb{R}_+;\mathbb{R}^{d\times m})$. Define for $t \geq 0$

$$x(t) = \int_0^t g(s)dW(s) \quad \text{and} \qquad A(t) = \int_0^t |g(s)|^2 ds. \tag{A.6}$$

Then for all $p > 0$, there exist universal positive constants c_p, C_p such that

$$c_p\mathbb{E}|A(t)|^{\frac{p}{2}} \leq \mathbb{E}\left[\sup_{0\leq s\leq t} |x(s)|^p\right] \leq C_p\mathbb{E}|A(t)|^{\frac{p}{2}}, \tag{A.7}$$

for all $t \geq 0$. In particular, one may take

$$
\begin{array}{lll}
c_p = (p/2)^p, & C_p = (32/p)^{\frac{p}{2}} & \text{if } 0 < p < 2; \\
c_p = 1, & C_p = (32/p)^{\frac{p}{2}} & \text{if } p = 2; \\
c_p = (2p)^{-\frac{p}{2}}, & C_p = \dfrac{p+1}{2(p-1)^{\frac{p}{2}}} & \text{if } p > 2.
\end{array}
$$

Gronwall inequality. Let $T > 0$ and $c \geq 0$. Let $u(\cdot)$ be a Borel measurable bounded nonnegative function on $[0, T]$, and let v be a nonnegative integrable function on $[0, T]$ If

$$u(t) \leq c + \int_0^t v(s)u(s)ds \qquad \forall t \in [0, T],$$

then

$$u(t) \leq c \exp\left(\int_0^t v(s)ds\right) \qquad \forall t \in [0, T]. \tag{A.8}$$

Discrete Gronwall Inequality. Let M be a positive integer. Let u_k and v_k be non-negative numbers for $k = 0, 1, \ldots, M$. If

$$u_k \leq u_0 + \sum_{j=0}^{k-1} u_j v_j$$

then

$$u_k \leq u_0 \exp\left(\sum_{j=0}^{k-1} v_j\right). \tag{A.9}$$

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