Reinforcement Learning

Universidad de Sonora

Problem set 1

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1. Consider the following *consumption–saving problem* with dynamics

$$x_{k+1} = (1+r)(x_k - a_k), \qquad k = 0, 1, \ldots, N-1,$$

and utility function

$$\beta^N (x_N)^{1-\gamma} + \sum_{k=0}^{N-1} \beta^k (a_k)^{1-\gamma}.$$

Show that the value functions of the DP algorithm take the form

$$J_k(x) = A_k \beta^k x^{1-\gamma},$$

where $A_N = 1$ and for $k = N-1, \ldots, 0$,

$$A_k = [1 + ((1+r)\beta A_{k+1})^{1/\gamma}]^\gamma.$$

Show also that the optimal policies are $h_k(x) = A_k^{-1/\gamma} x$.

2. Consider now the infinite–horizon version of the above consumption–saving problem.

   (a) Write down the associated Bellman equation.

   (b) Argue why a solution to the Bellman equation should be of the form

   $$v(x) = cx^{1-\gamma},$$

   where c is constant. Find the constant c and the stationary optimal policy.

   Hint: Insert $v(x) = cx^{1-\gamma}$ into the Bellman equation and solve the minimization problem.

3. Let $\{\xi_k\}$ be a sequence of iid random variables such that $E[\xi] = 0$ and $E[\xi^2] = d$. Consider the dynamics

$$x_{k+1} = x_k + a_k + \xi_k, \qquad k = 0, 1, 2, \ldots,$$

and the discounted cost

$$E \sum \beta^k (a_k^2 + x_k^2).$$

   (a) Write down the associated Bellman equation.

   (b) Conjecture that the solution to the Bellman equation takes the form $v(x) = ax^2 + b$, where a and b are constant. Determine the constants a and b.