Threshold behavior of a epidemic vector plant model: The Tomato Yellow Curl Virus

Asymtotic analysis and simulation.

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Abstract BACKGROUND
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2 Model formulation

2.1 Deterministic base dynamics

3 Deterministic base dynamics

$$
\begin{aligned}
\dot{S_p} &= -\beta_p S_p \frac{I_v}{N_v} + \tilde{r}_1 L_p + \tilde{r}_2 I_p \\
\dot{L_p} &= \beta_p S_p \frac{I_v}{N_v} - b L_p - \tilde{r}_1 L_p \\
\dot{I_p} &= b L_p - \tilde{r}_2 I_p \\
\dot{S_v} &= -\beta_v S_v \frac{I_p}{N_p} - \tilde{\gamma} S_v + (1-\theta)\mu \\
\dot{I_v} &= \beta_v S_v \frac{I_p}{N_p} - \tilde{\gamma} I_v + \theta\mu
\end{aligned}
\tag{1}
$$

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Gabriel: Aqui anexa Los paquetes y contenido de lo que llevas escrito. Si necesitas carpetas agregalas. Tabien sube el archivo bib y las figuras en extencion eps de la simulaciones del modelo determinista que estamos perturbando.

translate this section

Theorem 1 *With the notation of ODE* (1)*, let*

$$N_v(t) := S_v(t) + I_v(t)$$
$$N_v^\infty := \frac{\mu}{\gamma}.$$

Then for any initial condition $(S_p(0), L_p(0), I_p(0), S_v(0), I_v(0))^\top \in (0, \infty) \times (0, N_v^\infty)$, *she plant and vector total populations respectively satisfies*

$$\frac{dN_p}{dt} = \frac{d}{dt}(S_p + L_p + I_p) = 0,$$
$$\lim_{t \to \infty} N_v(t) = N_v^\infty.$$

Following ideas from [referencia], we quantify uncertity in replanting rate of plants, and died rate of vector, r_1, r_2 and γ, to this end, we perturb parameters $r_1 \dots$ whit a Winner process to obtain a stochastic differential equation(SDE). Here, the perturbation describe stochastic environmental noise on each population. In symbols $dB(t) = B(t + dt) - B(t)$ denotes the increment of a standard Wiener process, thus we perturb potentially replanting r_1, r_2, and vector death γ in the infinitiesimal time interval $[t, t + dt)$ by

$$r_1 dt \rightsquigarrow r_1 dt + \sigma_L dB_p(t),$$
$$r_2 dt \rightsquigarrow r_2 dt + \sigma_I dB_p(t), \qquad (2)$$
$$\gamma dt \rightsquigarrow \gamma dt + \sigma_v dB_v(t).$$

Note that right hand side of (2) is a random perturbations of parameters r_1, r_2, γ, with mean $\mathbb{E}[r_1 dt + \sigma_L dB_p(t)]$ and variance $\mathrm{Var}[r_1 dt + \sigma_L dB_p(t)] = \sigma_L^2 dt$, $\mathbb{E}(\tilde{r}_2 dt) = r_2 dt$ and $Var(\tilde{r}_2 dt) = \sigma_I^2 dt$ and $\mathbb{E}(\tilde{\gamma} dt) = \gamma dt$ and $Var(\tilde{\gamma} dt) = \sigma_v^2 dt$. Thus, we establish an stochastic extencion from deterministic tomato model (1) by the Itô SDE

$$dS_p = \left(-\beta_p S_p \frac{I_v}{N_v} + r_1 L_p + r_2 I_p\right) dt + (\sigma_L L_p + \sigma_I I_p) dB_p(t)$$

$$dL_p = \left(\beta_p S_p \frac{I_v}{N_v} - bL_p - r_1 L_p\right) dt - \sigma_L L_p dB_p(t)$$

$$dI_p = (bL_p - r_2 I_p) dt - \sigma_I I_p dB_p(t) \qquad (3)$$

$$dS_v = \left(-\beta_v S_v \frac{I_p}{N_p} - \gamma S_v + (1 - \theta)\mu\right) dt - \sigma_v S_v dB_v(t)$$

$$dI_v = \left(\beta_v S_v \frac{I_p}{N_p} - \gamma I_v + \theta \mu\right) dt - \sigma_v I_v dB_v(t).$$

4 Deterministic fixed points

Fix notation to distinguish between free disease and endemic

Here we compute the determinsitic fixed points of system (1) and show that its unicity. Thus by definition of we solve

$$-\beta_p S_p^* \frac{I_v^*}{N_v} + r(N_p - S_p^*) = 0$$

$$\beta_p S_p^* \frac{I_v^*}{N_v} - bL_p^* - rL_p^* = 0$$

$$bL_p^* - rI_p^* = 0 \tag{4}$$

$$-\beta_v S_v^* \frac{I_p^*}{N_p} - \gamma S_v^* + (1-\theta)\mu = 0$$

$$\beta_v S_v^* \frac{I_p^*}{N_p} - \gamma I_v^* + \theta\mu = 0.$$

to determine our fixed points. There is two fixed points—free disease equilibrium and the endemic equilibrium. We characterize the fist the relation $L_p^* = I_p^* = I_v^* = 0$, wich implies that

$$r(N_p - S_p^*) = 0,$$

and therefore, we obtain $S_p^* = N_p$. For the vector population we have by Theorem (1) that $S_v^* + I_v^* \to \frac{\mu}{\gamma}$ as $\to \infty$, then $S_v^* \to \frac{\mu}{\gamma}$ when we have $I_v^* = 0$. The free disease equilibrium point is $(N_p, 0, 0, \frac{\mu}{\gamma}, 0)^\top$. For the case of endemic equilibrium point, we need suppose that $L_p^*, I_p^*, I_v^* \neq 0$ and solve each right hand side of system (1) in terms of other variable. From $\dot{S}_p$, we can obtain

$$S_p^* = \frac{rN_p N_v}{rN_v + I_v^*\beta_p},$$

and similar for the other equations we obtain

$$L_p^* = \frac{\beta_p S_p^* I_v^*}{N_v(b+r)},$$

$$I_p^* = \frac{bL_p^*}{r},$$

$$S_v^* = \frac{(1-\theta)\mu N_p}{\gamma N_p + I_p^*\beta_v},$$

Expresing the above coordinate in terms of I_v, we obtain

$$S_p^* = \frac{rN_p N_v}{rN_v + I_v^*\beta_p},$$

$$L_p^* = \frac{\beta_p r N_p I_v^*}{(b+r)(rN_v + I_v^*\beta_p)},$$

$$I_p^* = \frac{b\beta_p N_p I_v^*}{(b+r)(rN_v + I_v^*\beta_p)},$$

$$S_v^* = \frac{(1-\theta)\mu(b+r)(rN_v + \beta_p I_v^*)}{\gamma(b+r)(rN_v + \beta_p I_v^*) + b\beta_p\beta_v I_v^*},$$

We only need substituting the above expression into the differential equation of I_v and solve the following quadratic equation

$$-N_p(b\gamma^2 r I_v^* N_v + b\gamma^2 (I_v^*)^2\beta_p - b\gamma\mu r\theta N_v - b\gamma\mu\theta I_v^*\beta_p + b\gamma(I_v^*)^2\beta_p\beta_v + b\mu\theta I_v^*\beta_p^2$$
$$- b\mu\theta I_v^*\beta_p\beta_v + \gamma^2 r^2 I_v^* N_v + \gamma^2 r(I_v^*)^2\beta_p - \gamma\mu r^2\theta N_v - \gamma\mu r\theta I_v^*\beta_p - b\mu I_v^*\beta_p^2) = 0$$

In sake of clearnes we define

$$a_1 := b\gamma^2\beta_p + b\gamma\beta_p\beta_v + \gamma^2 r\beta_p,$$
$$a_2 := -b\gamma\mu\theta\beta_p + b\mu\theta\beta_p^2 - b\mu\theta\beta_p\beta_v + \gamma^2 r^2 N_v - \gamma\mu r\theta\beta_p - b\mu\beta_p^2 + \gamma^2 rN_v,$$
$$a_3 := -b\gamma\mu r\theta N_v - \gamma\mu r^2\theta N_v.$$

and rewrite the above eqution in this new notation as

$$\underbrace{()}_{:=a_1} I_v^{*2} + \underbrace{()}_{:=a_2} I_v + \underbrace{()}_{} \tag{5}$$

We need a positive solution, then according to discriminant, we obtain

$$\Delta = a_2^2 - 4a_1 a_3$$
$$= (-b\gamma\mu\theta\beta_p + b\mu\theta\beta_p^2 - b\mu\theta\beta_p\beta_v + \gamma^2 r^2 N_v - \gamma\mu r\theta\beta_p - b\mu\beta_p^2 + \gamma^2 rN_v)^2$$
$$+ 4(b\gamma^2\beta_p + b\gamma\beta_p\beta_v + \gamma^2 r\beta_p)(b\gamma\mu r\theta N_v + \gamma\mu r^2\theta N_v),$$

which ever is positive, then we have two different real solution, since we require the positive, we deduce that

$$I_v^* = \frac{-a_2 + \sqrt{a_2^2 - 4a_1 a_3}}{2a_1}.$$

5 Existence of a unique positive solution

Thereom *.* of [Mao Book] assures ths existence of unique solution of (3) in a compact interval. Since we study asymptotic behaviour, we have to assure the existence of unique positive invariant solution to SDE (*). To this end, let $\mathbb{R}_+^n$ the first octant of $\mathbb{R}^n$ and consider

$$\mathbf{E} := \left\{ (S_p, L_p, I_p, S_v, I_v)^\top \in \mathbb{R}_+^5 : \quad \le S_p + L_p + I_p \ge N_p, \quad S_v + I_v \le \frac{\mu}{\gamma} \right\},$$

the following result prove that this set is positive invariant.

Theorem 2 *For any initial values* $(S_p(0), L_p(0), I_p(0), S_v(0), I_v(0)) \in \mathbf{E}$, *exists unique invariant global positive solution to SDE* (3) $(S_p(t), L_p(t), I_p(t), S_v(t), I_v(t))^\top$ *with probability one, that is,*

$$\mathbb{P}\left[(L_p(t), I_p(t), S_v(t), I_v(t)) \in \mathbf{E}, \quad \forall t \geq 0\right] = 1.$$

Proof

6 Extinction of the disease

Our analysis needs the following hypotesis.

(H-1) According to SDE (3), replatin rates satisfies $r_1 = r_2 = r$.
(H-2) The replanting noise intesities are equal $\sigma_L = \sigma_I = \sigma$.

We define the repoductive nomber of our stochastic model in SDE (*) by

$$\mathscr{R}_0^s := \frac{\beta_p \beta_v}{\gamma r}. \qquad (6)$$

> Define here the infinitesimal operator $\mathscr{L}$.

As our deterministic base structure this paramenters summarizes the behavior of extinction and persistence according with a threshold.

Theorem 3 *Let* $(S_p(t), L_p(t), I_p(t), I_v(t))$ *be the solution of (3) with initial condition* $(S_p(0), L_p(0), I_p(0), I_v(0)) \in \mathbf{E}$. *If* $0 \leq \mathscr{R}_0^s < 1$ *then, infected individuals in SDE (*) tends to zero exponentially a.s, that is, the disease will extinguishes with probability one.*

Proof The proof consitst verify the hypotheses of Khasminskii Theorem [*] for the Lyapunov function

$$V(S_p, L_p, I_p, S_v, I_v) = \left(S_p - S_p^0 - S_p^0 \ln\left(\frac{S_p}{S_p^0}\right)\right) + L_p + I_p + \frac{\beta_p N_p}{\gamma N_v} I_v. \qquad (7)$$

Let f, g respectively be the dirft and difussion of SDE(*). Applyng the inifinitesimal opreator $\mathscr{L}$ we have

> In the following step apply the operator $\mathscr{L}$

$$V_x f = \left(1 - \frac{S_p^0}{S_p}\right)\left(-\frac{\beta_p}{N_v^\infty}S_p I_v + r N_p - r S_p\right) + \frac{\beta_p}{N_v^\infty}S_p I_v - (b+r)L_p$$

$$+ b L_p - r I_p + \frac{\beta_p N_p}{\gamma N_v^\infty}\left(\frac{\beta_v N_v}{N_p}I_p - \frac{\beta_v}{N_v^\infty}I_v I_p - \gamma I_v\right)$$

$$= -r S_p\left(1 - \frac{S_p^0}{S_p}\right)^2 - \frac{\beta_p}{N_v^\infty}S_p I_v + \frac{\beta_p}{N_v^\infty}I_v S_p^0 + \frac{\beta_p}{N_v^\infty}S_p I_v - r(L_p + I_p)$$

$$+ \frac{\beta_p N_p}{\gamma N_v^\infty}\left(\frac{\beta_v N_v}{N_p}I_p - \frac{\beta_v}{N_v^\infty}I_v I_p - \gamma I_v\right)$$

$$= -r S_p\left(1 - \frac{S_p^0}{S_p}\right)^2 + \frac{\beta_p}{N_v^\infty}I_v S_p^0 - r(L_p + I_p) + \frac{\beta_p N_p}{\gamma N_v^\infty}\frac{\beta_v N_v}{N_p}I_p$$

$$- \frac{\beta_p N_p}{\gamma N_v^\infty}\frac{\beta_v}{N_v^\infty}I_v I_p - \frac{\beta_p N_p}{\gamma N_v^\infty}\gamma I_v.$$

Then,

$$V_x f = -r S_p\left(1 - \frac{S_p^0}{S_p}\right)^2 - \left[\frac{\beta_p N_p}{\gamma N_v^\infty}\gamma - \frac{\beta_p N_p}{N_v^\infty}\right]I_v + \left[\frac{\beta_p N_p}{\gamma N_v^\infty}\beta_v \frac{N_v^\infty}{N_p} - r\right]I_p$$

$$- r L_p - \frac{\beta_p N_p}{\gamma N_v^\infty}\frac{\beta_v}{N_p}I_v I_p$$

$$= -r S_p\left(1 - \frac{S_p^0}{S_p}\right)^2 + \left[\frac{\beta_p N_p}{\gamma N_v^\infty}\beta_v \frac{N_v^\infty}{N_p} - r\right]I_p - r L_p - \frac{\beta_p N_p}{\gamma N_v^\infty}\frac{\beta_v}{N_p}I_v I_p$$

$$= -r S_p\left(1 - \frac{S_p^0}{S_p}\right)^2 + r\left[\frac{\beta_p \beta_v}{\gamma r} - 1\right]I_p - r L_p - \frac{\beta_p \beta_v}{\gamma N_v^\infty}I_v I_p.$$

Expressing the right hand side of above equation in term of the basic reproductive number, $\mathscr{R}_0^s$ we get

$$V_x f = -r S_p\left(1 - \frac{S_p^0}{S_p}\right)^2 + r\left[\mathscr{R}_0^s - 1\right]I_p - r L_p - \frac{\beta_p \beta_v}{\gamma N_v^\infty}I_v I_p.$$

Moreover,

$$\frac{1}{2}trace(g^T V_{xx} g) = \frac{1}{2}\sigma^2 N_p\left(\frac{N_p - S_p}{S_p}\right)^2$$

$$\leq \frac{1}{2}\sigma^2 N_p.$$

The stochastic terms are not neccesary, because they do a martingale procces and therefore, when we use integral and expectation they vanising. Incorporation all terms calculate above, we obtain

$$dV(X) = -rS_p \left(1 - \frac{S_p^0}{S_p}\right)^2 + r\left[\mathscr{R}_0^s - 1\right]I_p - rL_p - \frac{\beta_p\beta_v}{\gamma N_v^\infty}I_vI_p + \frac{1}{2}\sigma^2 N_p \left(\frac{N_p - S_p}{S_p}\right)^2$$

$$\leq -rS_p \left(1 - \frac{S_p^0}{S_p}\right)^2 + r\left[\mathscr{R}_0^s - 1\right]I_p - rL_p - \frac{\beta_p\beta_v}{\gamma N_v^\infty}I_vI_p + \frac{1}{2}\sigma^2 N_p.$$

Define $LV(X)$ as

$$LV(X) = -rS_p \left(1 - \frac{S_p^0}{S_p}\right)^2 + r\left[\mathscr{R}_0^s - 1\right]I_p - rL_p - \frac{\beta_p\beta_v}{\gamma N_v^\infty}I_vI_p + \frac{1}{2}\sigma^2 N_p.$$

Using Itô's formula and integrating dV from 0 to t as well as taking expectation yield the following

$$0 \leq \mathbb{E}V(t) - \mathbb{E}V(0) \leq \mathbb{E}\int_0^t LV(X(s))ds$$

$$\leq -\mathbb{E}\int_0^t \left[rS_p \left(1 - \frac{S_p^0}{S_p}\right)^2 - r\left[\mathscr{R}_0^s - 1\right]I_p + rL_p + \frac{\beta_p\beta_v}{\gamma N_v^\infty}I_vI_p\right]ds + \frac{1}{2}\sigma^2 N_p$$

Therefore,

$$\lim_{t\to\infty} \frac{1}{t}\mathbb{E}\int_0^t \left[-rS_p \left(1 - \frac{S_p^0}{S_p}\right)^2 + r\left[\mathscr{R}_0^s - 1\right]I_p - rL_p - \frac{\beta_p\beta_v}{\gamma N_v^\infty}I_vI_p\right]ds \leq \frac{1}{2}\sigma^2 N_p.$$

$$\lim_{t\to\infty} \frac{1}{t}\mathbb{E}\int_0^t \left[r[\mathscr{R}_0^s - 1]I_p - rS_p \left(1 - \frac{S_p^0}{S_p}\right)^2 - rL_p - \frac{\beta_p\beta_v}{\gamma}I_vI_p\right]dr \leq \frac{1}{2}\sigma^2 N_p, a.s.$$

Theorem 4 *Let $(S_p(t), L_p(t), I_p(t), I_v(t))$ be the solution of SDE (3) with initial values $(S_p(0), L_p(0), I_p(0), I_v(0)) \in (0,N_p) \times (0,N_p) \times (0,N_p) \times (0,N_v)$. If $0 \leq \mathscr{R}_0^s < 1$, then the following conditions holds*

$$\lim_{t\to\infty} \frac{1}{t}\mathbb{E}\int_0^t \left[r[\mathscr{R}_0^s - 1]I_p - rS_p \left(1 - \frac{S_p^0}{S_p}\right)^2 - rL_p - \frac{\beta_p\beta_v}{\gamma}I_vI_p\right]dr \leq \frac{1}{2}\sigma^2 N_p, a.s.,$$

namely, the infected individual tends to zero exponentially a.s, i.e the disease will die out with probability one.

Proof The proof consitst verify the hypotheses of Khasminskii Theorem [*] for the Lyapunov function

$$V(S_p, L_p, I_p, S_v, I_v) = \left(S_p - N_p - N_p \ln \frac{S_p}{N_p} \right) + L_p + I_p + \frac{\beta_p N_p}{\gamma N_v^{\infty}} I_v$$
$$+ \left(S_v - N_v - N_v \ln \frac{S_v}{N_v} \right),$$

Let f, g respectively be the dirft and difussion of SDE (**??**). Applyng the infinitesimal opreator $\mathscr{L}$ we have

$$V_x f = \left(1 - \frac{N_p}{S_p} \right) \left(-\frac{\beta_p}{N_v^{\infty}} S_p I_v + r N_p - r S_p \right) + \frac{\beta_p}{N_v^{\infty}} S_p I_v - (b+r) L_p \qquad (8)$$

$$+ b L_p - r I_p + \left(1 - \frac{N_v}{S_v} \right) \left(-\frac{\beta_v}{N_p} S_v I_p - \gamma S_v + (1-\theta)\mu \right) \qquad (9)$$

$$+ \frac{\beta_p N_p}{\gamma N_v^{\infty}} \left(\frac{\beta_v S_v}{N_p} I_p - \gamma I_v + \theta\mu \right) \qquad (10)$$

$$\qquad (11)$$

Expanded the first term and factoring the term S_p, we obtain

$$\left(1 - \frac{N_p}{S_p} \right) \left(-\frac{\beta_p}{N_v^{\infty}} S_p I_v + r N_p - r S_p \right) = \left(1 - \frac{N_p}{S_p} \right) \left(-r S_p \left(1 - \frac{N_p}{S_p} \right) - \frac{\beta_p}{N_v^{\infty}} S_p I_v \right)$$
$$= -r S_p \left(1 - \frac{N_p}{S_p} \right)^2 - \frac{\beta_p}{N_v^{\infty}} S_p I_v + \frac{\beta_p}{N_v^{\infty}} N_p I_v$$
$$\qquad (12)$$

For the second term, since $(1-\theta)\mu \leq \gamma N_v$ we can bounded by the following

$$\left(1 - \frac{N_v}{S_v} \right) \left(-\frac{\beta_v}{N_p} S_v I_p - \gamma S_v + (1-\theta)\mu \right) \leq \left(1 - \frac{N_v}{S_v} \right) \left(-\frac{\beta_v}{N_p} S_v I_p - \gamma S_v + \gamma N_v \right)$$
$$\leq \left(1 - \frac{N_v}{S_v} \right) \left(-\gamma S_v \left(1 - \frac{N_v}{S_v} \right) - \frac{\beta_v}{N_p} S_v I_p \right)$$
$$\leq -\gamma S_v \left(1 - \frac{N_v}{S_v} \right)^2 - \frac{\beta_v}{N_p} S_v I_p + \frac{\beta_v}{N_p} N_v I_p$$
$$\qquad (13)$$

Same way from above calculation, and since $\theta\mu \leq \theta\gamma N_v$, we obtain

$$\frac{\beta_p N_p}{\gamma N_v^{\infty}} \left(\frac{\beta_v S_v}{N_p} I_p - \gamma I_v + \theta\mu \right) \leq \frac{\beta_p N_p}{\gamma N_v^{\infty}} \left(\frac{\beta_v S_v}{N_p} I_p - \gamma I_v + \theta\gamma N_v \right)$$
$$\leq \frac{\beta_p \beta_v S_v I_p}{\gamma N_v} - \frac{\beta_p N_p}{N_v^{\infty}} I_v + \beta_p \theta N_p \qquad (14)$$

Then, sustituting (12)-(14) into $V_x f$

$$V_x f \leq -rS_p \left(1 - \frac{N_p}{S_p}\right)^2 + \frac{\beta_p}{N_v^\infty} N_p I_v - r(L_p + I_p)$$
$$- \gamma S_v \left(1 - \frac{N_v}{S_v}\right)^2 - \frac{\beta_v}{N_p} S_v I_p + \frac{\beta_v}{N_p} N_v I_p$$
$$+ \frac{\beta_p \beta_v S_v I_p}{\gamma N_v} - \frac{\beta_p N_p}{N_v^\infty} I_v + \beta_p \theta N_p$$

$$V_x f \leq -rS_p \left(1 - \frac{N_p}{S_p}\right)^2 + \left[\frac{\beta_p}{N_v^\infty} N_p - \frac{\beta_p N_p}{N_v^\infty}\right] I_v - r(L_p + I_p)$$
$$- \gamma S_v \left(1 - \frac{N_v}{S_v}\right)^2 - \frac{\beta_v}{N_p} S_v I_p + \frac{\beta_v}{N_p} N_v I_p$$
$$+ \frac{\beta_p \beta_v S_v I_p}{\gamma N_v} + \beta_p \theta N_p$$

Moreover, since $S_v + I_v \leq N_v$, we can obtain the following relation

$$V_x f \leq -rS_p \left(1 - \frac{N_p}{S_p}\right)^2 - r(L_p + I_p)$$
$$- \gamma S_v \left(1 - \frac{N_v}{S_v}\right)^2 + \frac{\beta_v}{N_p} I_v I_p$$
$$+ \frac{\beta_p \beta_v I_p}{\gamma} - \frac{\beta_p \beta_v I_v I_p}{\gamma N_v} + \beta_p \theta N_p$$

Expressing the right hand side of above equation in term of the basic reproductive number, $\mathscr{R}_0^s$ we get

$$V_x f = -rS_p \left(1 - \frac{S_p^0}{S_p}\right)^2 - \gamma S_v \left(1 - \frac{N_v}{S_v}\right)^2 - rL_p - r[1 - \mathscr{R}_0^s] I_p$$
$$- \left[\frac{\beta_p \beta_v}{\gamma N_v^\infty} - \frac{\beta_v}{N_p}\right] I_v I_p - \frac{\beta_v}{N_p} S_v I_p + \beta_p \theta N_p.$$

Moreover,

$$\frac{1}{2} trace(g^T V_{xx} g) = \frac{1}{2} \frac{(\sigma_p (L_p + I_p))^2}{N_p} + \frac{1}{2} \sigma_v^2 N_v$$
$$\leq \frac{1}{2} \sigma_p^2 N_p + \frac{1}{2} \sigma_v^2 N_v.$$

The stochastic terms are not neccesary, because they do a martingale procces and therefore, when we use integral and expectation they vanising.

Incorporation all terms calculate above, we obtain

$$\mathscr{L}V(X) \le -rS_p \left(1 - \frac{S_p^0}{S_p}\right)^2 - \gamma S_v \left(1 - \frac{N_v}{S_v}\right)^2 - rL_p - r\left[1 - \mathscr{R}_0^s\right]I_p$$
$$- \left[\frac{\beta_p \beta_v}{\gamma N_v^\infty} - \frac{\beta_v}{N_p}\right]I_v I_p - \frac{\beta_v}{N_p}S_v I_p + \beta_p \theta N_p + \frac{1}{2}\sigma_p^2 N_p + \frac{1}{2}\sigma_v^2 N_v.$$

Define $\sigma_{p,v} := \beta_p \theta N_p + \frac{1}{2}\sigma_p^2 N_p + \frac{1}{2}\sigma_v^2 N_v$, then

$$\mathscr{L}V(X) \le -rS_p \left(1 - \frac{S_p^0}{S_p}\right)^2 - \gamma S_v \left(1 - \frac{N_v}{S_v}\right)^2 - rL_p - r\left[1 - \mathscr{R}_0^s\right]I_p$$
$$- \left[\frac{\beta_p \beta_v}{\gamma N_v^\infty} - \frac{\beta_v}{N_p}\right]I_v I_p - \frac{\beta_v}{N_p}S_v I_p + \sigma_{p,v}.$$

Since $V(x) \ge 0$, and using Itô's formula and integrating dV from 0 to t as well as taking expectation yield the following

$$0 \le \mathbb{E}V(t) - \mathbb{E}V(0) \le \mathbb{E}\int_0^t \mathscr{L}V(X(s))ds$$
$$\le -\mathbb{E}\int_0^t \left[rS_p \left(1 - \frac{S_p^0}{S_p}\right)^2 + \gamma S_v \left(1 - \frac{N_v}{S_v}\right)^2 + rL_p + r\left[1 - \mathscr{R}_0^s\right]I_p\right.$$
$$\left. + \left[\frac{\beta_p \beta_v}{\gamma N_v^\infty} + \frac{\beta_v}{N_p}\right]I_v I_p + \frac{\beta_v}{N_p}S_v I_p - \sigma_{p,v}\right]ds$$

Therefore,

$$\frac{1}{t}\mathbb{E}\int_0^t \left[rS_p \left(1 - \frac{S_p^0}{S_p}\right)^2 + \gamma S_v \left(1 - \frac{N_v}{S_v}\right)^2 + rL_p + r\left[1 - \mathscr{R}_0^s\right]I_p\right.$$
$$\left. + \left[\frac{\beta_p \beta_v}{\gamma N_v^\infty} + \frac{\beta_v}{N_p}\right]I_v I_p + \frac{\beta_v}{N_p}S_v I_p\right]ds \le \sigma_{p,v}$$

This implies that,

$$\lim_{t \to \infty}\frac{1}{t}\mathbb{E}\int_0^t \left[rS_p \left(1 - \frac{S_p^0}{S_p}\right)^2 + \gamma S_v \left(1 - \frac{N_v}{S_v}\right)^2 + rL_p + r\left[1 - \mathscr{R}_0^s\right]I_p\right.$$
$$\left. + \left[\frac{\beta_p \beta_v}{\gamma N_v^\infty} + \frac{\beta_v}{N_p}\right]I_v I_p + \frac{\beta_v}{N_p}S_v I_p\right]ds \le \sigma_{p,v}$$

Taking θ, σ_p, and σ_v such that $0 < \sigma_{p,v} < 1$, we have

$$\lim_{t \to \infty} \frac{1}{t} \log \mathbb{E} \int_0^t \left[rS_p \left(1 - \frac{S_p^0}{S_p} \right)^2 + \gamma S_v \left(1 - \frac{N_v}{S_v} \right)^2 + rL_p + r\left[1 - \mathscr{R}_0^s \right] I_p \right.$$
$$\left. + \left[\frac{\beta_p \beta_v}{\gamma N_v^\infty} + \frac{\beta_v}{N_p} \right] I_v I_p + \frac{\beta_v}{N_p} S_v I_p \right] ds \leq \log \sigma_{p,v} < 0.$$

Therefore,

$$\lim_{t \to \infty} \mathbb{E} \int_0^t \left[rS_p \left(1 - \frac{S_p^0}{S_p} \right)^2 + \gamma S_v \left(1 - \frac{N_v}{S_v} \right)^2 + rL_p + r\left[1 - \mathscr{R}_0^s \right] I_p \right.$$
$$\left. + \left[\frac{\beta_p \beta_v}{\gamma N_v^\infty} + \frac{\beta_v}{N_p} \right] I_v I_p + \frac{\beta_v}{N_p} S_v I_p \right] ds \leq \lim_{t \to \infty} e^{\sigma_{p,v}t} = 0$$

Thus

$$S_p \to N_p \; L_p \to 0 \; I_p \to 0$$
$$S_v \to N_v \; I_v \to 0.$$

exponentially a.s.

7 Persistence

Theorem 5 *Let $(S_p(t), L_p(t), I_p(t), I_v(t))$ be the solution of (3) with initial values $(S_p(0), L_p(0), I_p(0), I_v(0)) \in (0, N_p) \times (0, N_p) \times (0, N_p) \times (0, N_v)$. If $\mathscr{R}_0^s > 1$, then the system (3) is globally asymptotically stable at endemic equilibrium point if*

$$\lim_{t \to \infty} \frac{1}{t} \mathbb{E} \int_0^t \left[\frac{rS_p^*}{S_p S_p^*} (S_p^* - S_p)^2 + \frac{\beta_p}{N_v} S_p^* I_v^* A_1 + \frac{\beta_v}{N_p} \frac{I_p}{I_v} (I_v - I_v^*)^2 + \gamma I_v^* A_2 \right] dr \leq A_3.$$

namely, the disease will persist with probability one.

Proof Let us define the following Lyapunov function $V : \mathbb{R}_+^4 \to \mathbb{R}_+$

$$V(S_p, L_P, I_p, I_v) = (S_p + L_p + I_p + I_v) - (S_p^* + L_p^* + I_p^* + I_v^*)$$
$$- \left(S_p^* \ln \frac{S_p}{S_p^*} + L_p^* \ln \frac{L_p}{L_p^*} + I_p^* \ln \frac{I_p}{I_p^*} + I_v^* \ln \frac{I_v}{I_v^*} \right).$$

Computing the Itô formula terms as:

$$V_x f = \left(1 - \frac{S_p^*}{S_p}\right)\left(rN_p - \beta_p S_p \frac{I_v}{N_v^\infty} - rS_p\right) + \left(1 - \frac{L_p^*}{L_p}\right)\left(\beta_p S_p \frac{I_v}{N_v^\infty} - (r+b)L_p\right)$$
$$+ \left(1 - \frac{I_p^*}{I_p}\right)(bL_p - rI_p) + \left(1 - \frac{I_v^*}{I_v}\right)\left(\beta_v N_v \frac{I_p}{N_p} - \beta_v \frac{I_p}{N_p} I_v - \gamma I_v\right).$$

The system (3) satisfy the following relations at equilibrium point

$$rN_p = \beta_p S_p^* \frac{I_v^*}{N_v^\infty} + rS_p^*$$

$$(r+b) = \beta_p S_p^* \frac{I_v^*}{L_p^* N_v^\infty}$$

$$r = b\frac{L_p^*}{I_p^*}$$

$$\beta_v \frac{N_v}{N_p} = \frac{\beta_v}{N_p} I_v^* + \gamma \frac{I_v^*}{I_p^*}$$

Moreover,

$$V_x f = \left(1 - \frac{S_p^*}{S_p}\right)\left(\beta_p S_p^* \frac{I_v^*}{N_v^\infty} + rS_p^* - \beta_p S_p \frac{I_v}{N_v^\infty} - rS_p\right)$$

$$+ \left(1 - \frac{L_p^*}{L_p}\right)\left(\beta_p S_p \frac{I_v}{N_v^\infty} - \beta_p S_p^* \frac{I_v^*}{L_p^* N_v^\infty} L_p\right) + \left(1 - \frac{I_p^*}{I_p}\right)\left(bL_p - b\frac{L_p^*}{I_p^*}I_p\right)$$

$$+ \left(1 - \frac{I_v^*}{I_v}\right)\left(\frac{\beta_v}{N_p} I_v^* I_p + \gamma \frac{I_v^*}{I_p^*} I_p - \beta_v \frac{I_p}{N_p} I_v - \gamma I_v\right)$$

$$= rS_p^*\left(1 - \frac{S_p^*}{S_p}\right)\left(1 - \frac{S_p}{S_p^*}\right) + \frac{\beta_p}{N_v^\infty} S_p^* I_v^*\left(1 - \frac{S_p^*}{S_p}\right)\left(1 - \frac{S_p I_v}{S_p^* I_v^*}\right)$$

$$+ \frac{\beta_p}{N_v^\infty} S_p^* I_v^*\left(1 - \frac{L_p^*}{L_p}\right)\left(\frac{S_p I_v}{S_p^* I_v^*} - \frac{L_p}{L_p^*}\right) + bL_p^*\left(1 - \frac{I_p^*}{I_p}\right)\left(\frac{L_p}{L_p^*} - \frac{I_p}{I_p^*}\right)$$

$$+ \left(1 - \frac{I_v^*}{I_v}\right)\left(-\frac{\beta_v}{N_p} I_v I_p\left(1 - \frac{I_v^*}{I_v}\right) + \gamma I_v^*\left(\frac{I_p}{I_p^*} - \frac{I_v}{I_v^*}\right)\right)$$

$$= rS_p^*\left(2 - \frac{S_p}{S_p^*} - \frac{S_p^*}{S_p}\right) + \frac{\beta_p}{N_v^\infty} S_p^* I_v^*\left(1 - \frac{I_v}{I_v^*}\left(\frac{S_p}{S_p^*} - 1\right) - \frac{S_p^*}{S_p}\right)$$

$$+ \frac{\beta_p}{N_v^\infty} S_p^* I_v^*\left(\frac{S_p I_v}{S_p^* I_v^*}\left(1 - \frac{L_p^*}{L_p}\right) - \frac{L_p}{L_p^*}\left(1 - \frac{L_p^*}{L_p}\right)\right) + bL_p^*\left(1 + \frac{L_p}{L_p^*} - \frac{I_p}{I_p^*} - \frac{I_p^* L_p}{I_p L_p^*}\right)$$

$$- \frac{\beta_v}{N_v} I_v I_p\left(1 - \frac{I_v^*}{I_v}\right)^2 + \gamma I_v^*\left(\frac{I_p}{I_p^*} - \frac{I_v^* I_p}{I_v I_p^*} - \frac{I_v}{I_v^*} + 1\right)$$

$$= rS_p^*\left(2 - \frac{S_p}{S_p^*} - \frac{S_p^*}{S_p}\right) + \frac{\beta_p}{N_v^\infty} S_p^* I_v^*\left(1 - \frac{S_p^*}{S_p} - \frac{I_v}{I_v^*}\left(\frac{S_p}{S_p^*} - 1\right)\right)$$

$$+ \frac{\beta_p}{N_v^\infty} S_p^* I_v^*\left(1 - \frac{L_p}{L_p^*} - \frac{S_p I_v}{S_p^* I_v^*}\left(\frac{L_p^*}{L_p} - 1\right)\right) + bL_p^*\left(1 - \frac{I_p}{I_p^*} + \frac{L_p}{L_p^*}\left(1 - \frac{I_p^*}{I_p}\right)\right)$$

$$- \frac{\beta_v}{N_v^\infty} I_v I_p\left(1 - \frac{I_v^*}{I_v}\right)^2 + \gamma I_v^*\left(1 - \frac{I_v}{I_v^*} - \frac{I_p}{I_p^*}\left(\frac{I_v^*}{I_v} - 1\right)\right).$$

Then

$$V_x f = rS_p^*\left(2 - \frac{S_p}{S_p^*} - \frac{S_p^*}{S_p}\right) + \frac{\beta_p}{N_v^\infty} S_p^* I_v^*\left(2 - \frac{S_p^*}{S_p} - \frac{L_p}{L_p^*} - \frac{I_v}{I_v^*}\left(\frac{S_p L_p^*}{S_p^* L_p} - 1\right)\right)$$

$$+ bL_p^*\left(1 - \frac{I_p}{I_p^*} + \frac{L_p}{L_p^*}\left(1 - \frac{I_p^*}{I_p}\right)\right) - \frac{\beta_v}{N_p} I_v I_p\left(1 - \frac{I_v^*}{I_v}\right)^2$$

$$+ \gamma I_v^*\left(1 - \frac{I_v}{I_v^*} - \frac{I_p}{I_p^*}\left(\frac{I_v^*}{I_v} - 1\right)\right).$$

Now we need compute the term $g^T V_{xx} g$,

$$g^T V_{xx} g = \begin{bmatrix} \sigma^2 \left(\frac{N_p - S_p}{S_p} \right)^2 S_p^* + \sigma^2 L_p^* & 0 \\ 0 & I_p^* \sigma^2 + I_v^* \sigma_v^2 \end{bmatrix}$$

therefore,

$$\frac{1}{2} trace(g^T V_{xx} g) = \frac{1}{2} \left(\sigma^2 \left(\frac{N_p - S_p}{S_p} \right)^2 S_p^* + \sigma^2 L_p^* + \sigma^2 I_p^* + \sigma_v^2 I_v^* \right)$$

$$\leq \frac{1}{2} \left(\sigma^2 S_p^* + \sigma^2 L_p^* + \sigma^2 I_p^* + \sigma_v^2 I_v^* \right)$$

The stochastics terms are not neccesary, because the are a martingale and therefore, when we use integrating and expectation they vanishing, obtaining the following $LV(X)$ operator

$$LV(X) = -r S_p^* \frac{(S_p^* - S_p)^2}{S_p S_p^*} - \frac{\beta_p}{N_v^\infty} S_p^* I_v^* A_1 - b L_p^* A_2 - \frac{\beta_v}{N_p} I_v I_p \left(1 - \frac{I_v^*}{I_v} \right)^2$$

$$- \gamma I_v^* A_3 + A_4.$$

where

$$A_1 = \left(\frac{S_p^*}{S_p} + \frac{L_p}{L_p^*} + \frac{I_v}{I_v^*} \left(\frac{S_p L_p^*}{S_p^* L_p} - 1 \right) - 2 \right) > 0,$$

$$A_2 = \left(\frac{I_p}{I_p^*} - \frac{L_p}{L_p^*} \left(1 - \frac{I_p^*}{I_p} \right) - 1 \right) > 0,$$

$$A_3 = \left(\frac{I_v}{I_v^*} + \frac{I_p}{I_p^*} \left(\frac{I_v^*}{I_v} - 1 \right) - 1 \right) > 0,$$

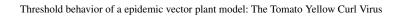$$A_4 = \frac{1}{2} \left(\sigma^2 S_p^* + \sigma^2 L_p^* + \sigma^2 I_p^* + \sigma_v^2 I_v^* \right) > 0.$$

Applying Itô formula, integrating dV from 0 to t and taking expectation gives the following

$$0 \leq \mathbb{E}V(t) - \mathbb{E}V(0) = \mathbb{E} \int_0^t LV(s) ds$$

$$- \mathbb{E} \int_0^t \left(r S_p^* \frac{(S_p^* - S_p)^2}{S_p S_p^*} + \frac{\beta_p}{N_v^\infty} S_p^* I_v^* A_1 + b L_p^* A_2 + \frac{\beta_v}{N_p} I_v I_p \left(1 - \frac{I_v^*}{I_v} \right)^2 + \gamma I_v^* A_3 \right) ds$$

$$+ A_4 t.$$

Therefore,

$$\lim_{t\to\infty}\frac{1}{t}\mathbb{E}\int_0^t \left(rS_p^* \frac{(S_p^* - S_p)^2}{S_p S_p^*} + \frac{\beta_p}{N_v^\infty}S_p^* I_v^* A_1 + bL_p^* A_2 + \frac{\beta_v}{N_p}I_v I_p \left(1 - \frac{I_v^*}{I_v}\right)^2 + \gamma I_v^* A_3 \right) ds$$
$$\leq A_4.$$

8 Numerical Results

9 Conclusions

Referecnce	Priority	Observation
[1]		
[2]	**	See Lyapnov Function.
[3]	**	For persistece def
[4]	*	Dengue
[5]	*	Mobility
[6]		
[7]		
[8]		
[9]		
[10]		
[11]	***	Review
[12]	***	Review
[13]	**	Review
[14]	*	Vaccination
[15]	**	General ideas
[16]	***	For extinction by noise
[17]	***	Threshold behaviour
[18]	***	Good idea for COVID 19
[19]	**	Lie approach
[20]	**	Threshold
[21]	***	Thickbone with CMCM deduction
[22]	***	Permanence
[23]	*	Degenerate Difussion
[24]	*	General force of infection

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A Background