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Threshold behavior of a epidemic vector plant model: The Tomato Yellow Curl Virus

Asymtotic analysis and simulation.

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Abstract BACKGROUND
PROBLEM SETUP
FINDINGS
IMPLICATIONS

Gabriel: Aqui anexa Los paquetes y contenido de lo que llevas escrito. Si necesitas carpetas agregalas. Tabien sube el archivo bib y las figuras en extencion eps de la simulaciones del modelo determinista que estamos perturbando.

1 Introduction

2 Deterministic base dynamics

3 Model formulation

4 Deterministic base dynamics

translate this section

$$
\begin{aligned}
\dot{S}_p &= -\beta_p S_p \frac{I_v}{N_v} + \tilde{r}_1 L_p + \tilde{r}_2 I_p \\
\dot{L}_p &= \beta_p S_p \frac{I_v}{N_v} - b L_p - \tilde{r}_1 L_p \\
\dot{I}_p &= b L_p - \tilde{r}_2 I_p \\
\dot{S}_v &= -\beta_v S_v \frac{I_p}{N_p} - \tilde{\gamma} S_v + (1-\theta)\mu \\
\dot{I}_v &= \beta_v S_v \frac{I_p}{N_p} - \tilde{\gamma} I_v + \theta\mu
\end{aligned}
\tag{1}
$$

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Make a table for description of all parameters

Redact this conservation law to the entire system (1). Write a introductory paragraph to Thm 1

Theorem 1 *With the notation of ODE* (1)*, let*

$$
\begin{aligned}
N_v(t) &:= S_v(t) + I_v(t)\\
N_v^{\infty} &:= \frac{\mu}{\gamma}.
\end{aligned}
$$

Then for any initial condition $(S_p(0), L_p(0), I_p(0), S_v(0), I_v(0))^{\top} \in (0,\infty) \times (0, N_v^{\infty})$, *she plant and vector total populations respectively satisfies*

$$
\begin{aligned}
&\frac{dN_p}{dt} = \frac{d}{dt}(S_p + L_p + I_p) = 0,\\
&\lim_{t\to\infty} N_v(t) = N_v^{\infty}.
\end{aligned}
$$

Why we want to normalize?

write here the parameters

Following ideas from [referencia], we quantify uncertinity in replanting rate of plants, and died rate of vector, r_1, r_2 and γ, to this end, we perturb parameters $r_1 \ldots$ whit a Winner process to obtain a stochastic differential equation(SDE). Here, the perturbation describe stochastic environmental noise on each population. In symbols $dB(t) = B(t + dt) - B(t)$ denotes the increment of a standard Wiener process, thus we perturb potentially replanting r_1, r_2, and vector death γ in the infinitiesimal time interval $[t, t + dt)$ by

$$
\begin{aligned}
r_1 dt &\rightsquigarrow r_1 dt + \sigma_L dB_p(t),\\
r_2 dt &\rightsquigarrow r_2 dt + \sigma_I dB_p(t),\\
\gamma dt &\rightsquigarrow \gamma dt + \sigma_v dB_v(t).
\end{aligned}
\tag{2}
$$

Note that here we will use the latex proba package, plase use the same commands in the remain of the manuscript

Note that right hand side of (2) is a random perturbations of parameters r_1, r_2, γ, with mean $\mathbb{E}\left[r_1 dt + \sigma_L dB_p(t)\right]$ and variance $\operatorname{Var}\left[r_1 dt + \sigma_L dB_p(t)\right] = \sigma_L^2 dt$, $\mathbb{E}(\tilde{r}_2 dt) = r_2 dt$ and $Var(\tilde{r}_2 dt) = \sigma_I^2 dt$ and $\mathbb{E}(\tilde{\gamma} dt) = \gamma dt$ and $Var(\tilde{\gamma} dt) = \sigma_v^2 dt$. Thus, we establish an stochastic extencion from deterministic tomato model (1) by the Itô SDE

$$
\begin{aligned}
dS_p &= \left(-\beta_p S_p \frac{I_v}{N_v} + r_1 L_p + r_2 I_p\right) dt + (\sigma_L L_p + \sigma_I I_p) dB_p(t)\\
dL_p &= \left(\beta_p S_p \frac{I_v}{N_v} - b L_p - r_1 L_p\right) dt - \sigma_L L_p dB_p(t)\\
dI_p &= (b L_p - r_2 I_p)\, dt - \sigma_I I_p dB_p(t)\\
dS_v &= \left(-\beta_v S_v \frac{I_p}{N_p} - \gamma S_v + (1-\theta)\mu\right) dt - \sigma_v S_v dB_v(t)\\
dI_v &= \left(\beta_v S_v \frac{I_p}{N_p} - \gamma I_v + \theta\mu\right) dt - \sigma_v I_v dB_v(t).
\end{aligned}
\tag{3}
$$

4.1 Deterministic fixed points

Fix notation to distinguish between free disease and endemic

Here we compute the determinsitic fixed points of system (1) and show that its unicity. Thus by definition of we solve

$$
\begin{aligned}
-\beta_p S_p^* \frac{I_v^*}{N_v} + r(N_p - S_p^*) &= 0 \\
\beta_p S_p^* \frac{I_v^*}{N_v} - bL_p^* - rL_p^* &= 0 \\
bL_p^* - rI_p^* &= 0 \\
-\beta_v S_v^* \frac{I_p^*}{N_p} - \gamma S_v^* + (1-\theta)\mu &= 0 \\
\beta_v S_v^* \frac{I_p^*}{N_p} - \gamma I_v^* + \theta\mu &= 0.
\end{aligned} \tag{4}
$$

to determine our fixed points. There is two fixed points—free disease equilibrium and the endemic equilibrium. We characterize the fist the relation $L_p^* = I_p^* = I_v^* = 0$, wich implies that

$$
r(N_p - S_p^*) = 0,
$$

and therefore, we obtain $S_p^* = N_p$.F or the vector population we have by Theorem (1) that $S_v^* + I_v^* \to \frac{\mu}{\gamma}$ as $\to \infty$, then $S_v^* \to \frac{\mu}{\gamma}$ when we have $I_v^* = 0$. The free disease equilibrium point is $(N_p, 0, 0, \frac{\mu}{\gamma}, 0)^\top$. For the case of endemic equilibrium point, we need suppose that $L_p^*, I_p^*, I_v^* \neq 0$ and solve each right hand side of system (1) in terms of other variable. From $\dot{S_p}$, we can obtain

$$
S_p^* = \frac{rN_pN_v}{rN_v + I_v^*\beta_p},
$$

and similar for the other equations we obtain

$$
L_p^* = \frac{\beta_p S_p^* I_v^*}{N_v\,(b+r)},
$$

$$
I_p^* = \frac{bL_p^*}{r},
$$

$$
S_v^* = \frac{(1-\theta)\,\mu\, N_p}{\gamma\, N_p + I_p^*\beta_v},
$$

Expresing the above coordinate in terms of I_v, we obtain

$$
S_p^* = \frac{rN_pN_v}{rN_v + I_v^*\beta_p},
$$

$$
L_p^* = \frac{\beta_p r N_p I_v^*}{(b+r)\,(rN_v + I_v^*\beta_p)},
$$

$$I_p^* = \frac{b\beta_p N_p I_v^*}{(b+r)\,(rN_v + I_v^*\beta_p)},$$
$$S_v^* = \frac{(1-\theta)\,\mu(b+r)(rN_v+\beta_p I_v^*)}{\gamma(b+r)(rN_v+\beta_p I_v^*)+b\beta_p\beta_v I_v^*},$$

We only need substituting the above expression into the differential equation of I_v and solve the following quadratic equation

$$-N_p(b\gamma^2 r I_v^* N_v + b\gamma^2 (I_v^*)^2\beta_p - b\gamma\mu r\theta N_v - b\gamma\mu\theta I_v^*\beta_p + b\gamma(I_v^*)^2\beta_p\beta_v + b\mu\theta I_v^*\beta_p^2$$
$$- b\mu\theta I_v^*\beta_p\beta_v + \gamma^2 r^2 I_v^* N_v + \gamma^2 r (I_v^*)^2\beta_p - \gamma\mu r^2\theta N_v - \gamma\mu r\theta I_v^*\beta_p - b\mu I_v^*\beta_p^2) = 0$$

In sake of clearnes we define

$$a_1 := b\gamma^2\beta_p + b\gamma\beta_p\beta_v + \gamma^2 r\beta_p,$$
$$a_2 := -b\gamma\mu\theta\beta_p + b\mu\theta\beta_p^2 - b\mu\theta\beta_p\beta_v + \gamma^2 r^2 N_v - \gamma\mu r\theta\beta_p - b\mu\beta_p^2 + \gamma^2 r N_v,$$
$$a_3 := -b\gamma\mu r\theta N_v - \gamma\mu r^2\theta N_v.$$

and rewrite the above eqution in this new notation as

$$\underbrace{()}_{:=a_1} I_v^{*2} + \underbrace{()}_{:=a_2} I_v + \underbrace{()} \tag{5}$$

Fill according to each term

We need a positive solution, then according to discriminant, we obtain

$$\begin{aligned}\Delta &= a_2^2 - 4a_1a_3\\ &= (-b\gamma\mu\theta\beta_p + b\mu\theta\beta_p^2 - b\mu\theta\beta_p\beta_v + \gamma^2 r^2 N_v - \gamma\mu r\theta\beta_p - b\mu\beta_p^2 + \gamma^2 r N_v)^2\\ &+ 4(b\gamma^2\beta_p + b\gamma\beta_p\beta_v + \gamma^2 r\beta_p)(b\gamma\mu r\theta N_v + \gamma\mu r^2\theta N_v),\end{aligned}$$

which ever is positive, then we have two different real solution, since we require the positive, we deduce that

$$I_v^* = \frac{-a_2 + \sqrt{a_2^2 - 4a_1a_3}}{2a_1}.$$

5 Existence of a unique positive solution

Thereom *.* of [Mao Book] assures ths existence of unique solution of (3) in a compact interval. Since we study asymptotic behaviour, we have to assure the existence of unique positive invariant solution to SDE (*). To this end, let $\mathbb{R}_+^n$ the first octant of $\mathbb{R}^n$ and consider

$$\mathbf{E} := \left\{ (S_p, L_p, I_p, S_v, I_v)^\top \in \mathbb{R}_+^5 : \quad \leq S_p + L_p + I_p \geq N_p, \quad S_v + I_v \leq \frac{\mu}{\gamma} \right\},$$

the following result prove that this set is positive invariant.

Theorem 2 *For any initial values* $(S_p(0), L_p(0), I_p(0), S_v(0), I_v(0)) \in \mathbf{E}$, *exists unique invariant global positive solution to SDE* (3) $(S_p(t), L_p(t), I_p(t), S_v(t), I_v(t))^\top$ *with probability one, that is,*

$$\mathbb{P}\left[(L_p(t), I_p(t), S_v(t), I_v(t)) \in \mathbf{E}, \quad \forall t \geq 0\right] = 1.$$

Proof

6 Extinction of the disease

7 Existence of unique positive solution

Thereom *.* of [Mao Book] assures ths existence of unique solution of (3) in a compact interval. Since we study asymptotic behaviour, we have to assure the existence of unique-globally-positive invariant solution of SDE (*). To this end, let $\mathbb{R}^n_+$ the first octant of $\mathbb{R}^n$ and consider

$$\mathbf{E} := \left\{ (S_p, L_p, I_p, S_v, I_v)^\top \in \mathbb{R}^5_+ : \quad 0 \leq S_p + L_p + I_p \leq N_p, \quad S_v + I_v \leq \frac{\mu}{\gamma} \right\},$$

the following result prove that this set is positive invariant.

Theorem 3 *For any initial values* $(S_p(0), L_p(0), I_p(0), S_v(0), I_v(0))^\top \in \mathbf{E}$, *exists unique a.s. invariant global positive solution to SDE* (3) *in* $\mathbf{E}$, *that is,*

$$\mathbb{P}\left[(L_p(t), I_p(t), S_v(t), I_v(t)) \in \mathbf{E}, \quad \forall t \geq 0 \right] = 1.$$

Proof Since the right hand side of system (3) are quadratic, linear and constans terms, this imply that they are locally Lipschitz. We know by [ref Mao], that for any initial condition $(S_p(0), L_p(0), I_p(0), S_v(0), I_v(0))^\top \in \mathbf{E}$ there is a unique maximal local solution $(S_p(t), L_p(t), I_p(t), S_v(t), I_v(t))^\top$ at $t \in [0, \tau_e)$, where τ_e is the explosion time. Let $k_0 > 0$ be sufficiently large, and define the stopping time

$$\begin{aligned}\tau_k = \inf \Bigg\{ t \in [0, \tau_e) : L_p(t) \notin \left(\frac{1}{k_0}, N_p - \frac{1}{k_0} \right) \bigcup I_p(t) \notin \left(\frac{1}{k_0}, N_p - \frac{1}{k_0} \right) \\ \bigcup I_v(t) \notin \left(\frac{1}{k_0}, N_v - \frac{1}{k_0} \right) \Bigg\}, \end{aligned} \tag{6}$$

We know that $\tau_k \nearrow \tau_\infty$. In other words, $\tau_\infty = \infty$ a.s. implies

Give an argument

$$(S_p(t), L_p(t), I_p(t), S_v(t), I_v(t))^\top \in \mathbf{E} \tag{7}$$

a.s. for all $t \geq 0$. Thus, we show that $\tau_\infty = \infty$ a.s. To this end, we proced by contradiction. Suppose that the above statement is false for a given time t, then there is a pair of constants $T > 0$ and $\epsilon \in (0, 1)$ such that some component from L_p, I_p, I_v, or L_p, get-outs from its corresponding interval

$$\left(\frac{1}{k_0}, N_\bullet - \frac{1}{k_0} \right),$$

then $\mathbb{P}[\tau_\infty \leq T] > \epsilon$. Hence, there is an integer $k_1 \geq k_0$ such that

$$\mathbb{P}[\tau_k \leq T] > \epsilon, \quad \forall k \geq k_1. \tag{8}$$

Define a function $V_p : (0, N_p) \to \mathbb{R}_+$ by

$$V_p(x) := \frac{1}{x} + \frac{1}{N_p - x}$$

According to the inifinitesimal operation $\mathcal{L}$ see APPENDIX By diffusion operator, we have, for any $t \in [0, T]$ and $k \geq k_1$

Write auxiliar results in a fucking appendix

$$\begin{aligned}\mathcal{L}[V_p(L_p)] = &\left[-\frac{1}{L_p^2} + \frac{1}{(N_p - L_p)^2}\right]\left[\beta_p S_p \frac{I_v}{N_v} - (b + r_1)L_p\right] \\ &+ \frac{1}{2}\left[\frac{2}{L_p^3} + \frac{2}{(N_p - L_p)^3}\right]\sigma_p^2 \frac{L_p^2 S_p^2}{N_p^2}.\end{aligned}$$

Expanding each term, we have

$$\begin{aligned}\mathcal{L}[V_p(L_p)] = &-\beta_p \frac{S_p I_v}{L_p^2 N_v} + \beta_p \frac{S_p I_v}{(N_p - L_p)^2 N_v} + \frac{(b + r_1)}{L_p} - \frac{(b + r_1)L_p}{(N_p - L_p)^2} \\ &+ \left[\frac{1}{L_p^3} + \frac{1}{(N_p - L_p)^3}\right]\sigma_p^2 \frac{L_p^2 S_p^2}{N_p^2}.\end{aligned}$$

Since each term is positive, we can bound the above by

$$\mathcal{L}[V_p(L_p)] \leq \beta_p \frac{S_p}{(N_p - L_p)^2} + \frac{(b + r_1)}{L_p} + \left[\frac{1}{L_p^3} + \frac{1}{(N_p - L_p)^3}\right]\sigma_p^2 \frac{L_p^2 S_p^2}{N_p^2}.$$

Moreover, by definition of N_p, we can bound $S_p \leq N_p - L_p = S_p + I_p$ to obtain

$$\mathcal{L}[V_p(L_p)] \leq \frac{\beta_p}{N_p - L_p} + \frac{(b + r_1)}{L_p} + \sigma_p^2 \left[\frac{1}{L_p} + \frac{L_p^2}{N_p^2} \frac{1}{N_p - L_p}\right].$$

And this implies that

$$\mathcal{L}[V_p(L_p)] \leq \frac{b + r_1}{L_p} + \frac{\beta_p}{N_p - L_p} + \sigma_p^2 \left[\frac{1}{L_p} + \frac{1}{N_p - L_p}\right].$$

Now define $C := (b + r_1) \vee \beta_p + \sigma_p^2$, we obtain the following inequality

$$\mathcal{L}[V(L_p)] \leq C V_p(L_p). \tag{9}$$

By Itô's formula and applying expectation, we have, for any $t \in [0, T]$ and $k \geq k_1$

$$\mathbb{E}V(L_p(t \wedge \tau_k)) = V(L_p(0)) + \mathbb{E}\int_0^{t \wedge \tau_k} \mathcal{L}[V(L_p(s))]ds.$$

By equation (9) and Fubini's Theorem, we have

$$\mathbb{E}V(L_p(t\wedge\tau_k)) \le V(L_p(0)) + C\int_0^t \mathbb{E}V(L_p(s\wedge\tau_k))ds.$$

Applying the Gronwall inequality yields that

$$\mathbb{E}V(L_p(t\wedge\tau_k)) \le V(L_p(0))e^{CT} \tag{10}$$

Set $\Omega_k = \{\omega : \tau_k \le T\}$ for $k \ge k_1$, note that by equation (8), $\mathbb{P}(\Omega_k) > \epsilon$. For every $\omega \in \Omega_k$, we have $L_p(t,\omega) \in \left(\frac{1}{k_0}, N_p - \frac{1}{k_0}\right)^{\mathtt{C}}$, and hence

$$\begin{aligned} V_p(L_p(t,\omega)) &= \frac{1}{L_p} + \frac{1}{N_p - L_p} \\ &\ge k + \frac{1}{N_p - \frac{1}{k}} \\ &\ge k. \end{aligned}$$

It follows from equation (10), that

$$V_p(L_p(0))e^{CT} \ge \mathbb{E}\left[1_{\Omega_k}(\omega)V_p(L_p(\tau_k,\omega))\right] \ge k\mathbb{P}(\Omega_k) \ge \epsilon k.$$

Thus, letting $k\to\infty$ leads to the contradiction

$$\infty > V_p(L_p(0))e^{CT} \ge \infty.$$

Therfore we have $\tau_\infty = \infty$ a.s., and the proof is complete.

Theorem 4 *Let $(S_p(t), L_p(t), I_p(t), I_v(t))$ be the solution of SDE (3) with initial values $(S_p(0), L_p(0), I_p(0), I_v(0)) \in (0,N_p)\times(0,N_p)\times(0,N_p)$ $\times(0,N_v)$. If $0 \le \mathcal{R}_0^s < 1$, then the following conditions holds*

$$\lim_{t\to\infty}\frac{1}{t}\mathbb{E}\int_0^t\left[r[\mathcal{R}_0^s - 1]I_p - rS_p\left(1 - \frac{S_p^0}{S_p}\right)^2 - rL_p - \frac{\beta_p\beta_v}{\gamma}I_vI_p\right]dr \le \frac{1}{2}\sigma^2 N_p, \; a.s.,$$

namely, the infected individual tends to zero exponentially a.s, i.e the disease will die out with probability one.

Proof The proof consitst verify the hypotheses of Khasminskii Theorem [*] for the Lyapunov function

$$\begin{aligned} V(S_p, L_p, I_p, S_v, I_v) = &\left(S_p - N_p - N_p \ln\frac{S_p}{N_p}\right) + L_p + I_p + \frac{\beta_p N_p}{\gamma N_v^\infty}I_v \\ &+ \left(S_v - N_v - N_v \ln\frac{S_v}{N_v}\right), \end{aligned}$$

Let f, g respectively be the dirft and difussion of SDE (10). Applyng the inifinitesimal opreator $\mathcal{L}$ we have

$$V_x f = \left(1 - \frac{N_p}{S_p}\right)\left(-\frac{\beta_p}{N_v^\infty} S_p I_v + rN_p - rS_p\right) + \frac{\beta_p}{N_v^\infty} S_p I_v - (b+r)L_p \tag{11}$$

$$+ bL_p - rI_p + \left(1 - \frac{N_v}{S_v}\right)\left(-\frac{\beta_v}{N_p} S_v I_p - \gamma S_v + (1-\theta)\mu\right) \tag{12}$$

$$+ \frac{\beta_p N_p}{\gamma N_v^\infty}\left(\frac{\beta_v S_v}{N_p} I_p - \gamma I_v + \theta\mu\right) \tag{13}$$

$$\tag{14}$$

Expanded the first term and factoring the term S_p, we obtain

$$\begin{aligned}\left(1 - \frac{N_p}{S_p}\right)\left(-\frac{\beta_p}{N_v^\infty} S_p I_v + rN_p - rS_p\right) &= \left(1 - \frac{N_p}{S_p}\right)\left(-rS_p\left(1 - \frac{N_p}{S_p}\right) - \frac{\beta_p}{N_v^\infty} S_p I_v\right) \\ &= -rS_p\left(1 - \frac{N_p}{S_p}\right)^2 - \frac{\beta_p}{N_v^\infty} S_p I_v + \frac{\beta_p}{N_v^\infty} N_p I_v\end{aligned} \tag{15}$$

For the second term, since $(1-\theta)\mu \leq \gamma N_v$ we can bounded by the following

$$\begin{aligned}\left(1 - \frac{N_v}{S_v}\right)\left(-\frac{\beta_v}{N_p} S_v I_p - \gamma S_v + (1-\theta)\mu\right) &\leq \left(1 - \frac{N_v}{S_v}\right)\left(-\frac{\beta_v}{N_p} S_v I_p - \gamma S_v + \gamma N_v\right) \\ &\leq \left(1 - \frac{N_v}{S_v}\right)\left(-\gamma S_v\left(1 - \frac{N_v}{S_v}\right) - \frac{\beta_v}{N_p} S_v I_p\right) \\ &\leq -\gamma S_v\left(1 - \frac{N_v}{S_v}\right)^2 - \frac{\beta_v}{N_p} S_v I_p + \frac{\beta_v}{N_p} N_v I_p\end{aligned} \tag{16}$$

Same way from above calculation, and since $\theta\mu \leq \theta\gamma N_v$, we obtain

$$\begin{aligned}\frac{\beta_p N_p}{\gamma N_v^\infty}\left(\frac{\beta_v S_v}{N_p} I_p - \gamma I_v + \theta\mu\right) &\leq \frac{\beta_p N_p}{\gamma N_v^\infty}\left(\frac{\beta_v S_v}{N_p} I_p - \gamma I_v + \theta\gamma N_v\right) \\ &\leq \frac{\beta_p \beta_v S_v I_p}{\gamma N_v} - \frac{\beta_p N_p}{N_v^\infty} I_v + \beta_p \theta N_p\end{aligned} \tag{17}$$

Then, sustituting (15)-(17) into $V_x f$

$$\begin{aligned}V_x f \leq &-rS_p\left(1 - \frac{N_p}{S_p}\right)^2 + \frac{\beta_p}{N_v^\infty} N_p I_v - r(L_p + I_p) \\ &- \gamma S_v\left(1 - \frac{N_v}{S_v}\right)^2 - \frac{\beta_v}{N_p} S_v I_p + \frac{\beta_v}{N_p} N_v I_p \\ &+ \frac{\beta_p \beta_v S_v I_p}{\gamma N_v} - \frac{\beta_p N_p}{N_v^\infty} I_v + \beta_p \theta N_p\end{aligned}$$

$$
\begin{aligned}
V_x f \leq & -rS_p\left(1-\frac{N_p}{S_p}\right)^2+\left[\frac{\beta_p}{N_v^{\infty}}N_p-\frac{\beta_p N_p}{N_v^{\infty}}\right]I_v-r(L_p+I_p)\\
& -\gamma S_v\left(1-\frac{N_v}{S_v}\right)^2-\frac{\beta_v}{N_p}S_vI_p+\frac{\beta_v}{N_p}N_vI_p\\
& +\frac{\beta_p\beta_vS_vI_p}{\gamma N_v}+\beta_p\theta N_p
\end{aligned}
$$

Moreover, since $S_v+I_v\leq N_v$, we can obtain the following relation

$$
\begin{aligned}
V_x f \leq & -rS_p\left(1-\frac{N_p}{S_p}\right)^2-r(L_p+I_p)\\
& -\gamma S_v\left(1-\frac{N_v}{S_v}\right)^2+\frac{\beta_v}{N_p}I_vI_p\\
& +\frac{\beta_p\beta_vI_p}{\gamma}-\frac{\beta_p\beta_vI_vI_p}{\gamma N_v}+\beta_p\theta N_p
\end{aligned}
$$

Expressing the right hand side of above equation in term of the basic reproductive number, $\mathcal{R}_0^s$ we get

$$
\begin{aligned}
V_x f = & -rS_p\left(1-\frac{S_p^0}{S_p}\right)^2-\gamma S_v\left(1-\frac{N_v}{S_v}\right)^2-rL_p-r\left[1-\mathcal{R}_0^s\right]I_p\\
& -\left[\frac{\beta_p\beta_v}{\gamma N_v^{\infty}}-\frac{\beta_v}{N_p}\right]I_vI_p-\frac{\beta_v}{N_p}S_vI_p+\beta_p\theta N_p.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
\frac{1}{2}trace(g^TV_{xx}g) &= \frac{1}{2}\frac{(\sigma_p(L_p+I_p))^2}{N_p}+\frac{1}{2}\sigma_v^2N_v\\
&\leq \frac{1}{2}\sigma_p^2N_p+\frac{1}{2}\sigma_v^2N_v.
\end{aligned}
$$

The stochastic terms are not neccesary, because they do a martingale procces and therefore, when we use integral and expectation they vanising.

Incorporation all terms calculate above, we obtain

$$
\begin{aligned}
\mathcal{L}V(X) \leq & -rS_p\left(1-\frac{S_p^0}{S_p}\right)^2-\gamma S_v\left(1-\frac{N_v}{S_v}\right)^2-rL_p-r\left[1-\mathcal{R}_0^s\right]I_p\\
& -\left[\frac{\beta_p\beta_v}{\gamma N_v^{\infty}}-\frac{\beta_v}{N_p}\right]I_vI_p-\frac{\beta_v}{N_p}S_vI_p+\beta_p\theta N_p+\frac{1}{2}\sigma_p^2N_p+\frac{1}{2}\sigma_v^2N_v.
\end{aligned}
$$

Define $\sigma_{p,v} := \beta_p \theta N_p + \frac{1}{2}\sigma_p^2 N_p + \frac{1}{2}\sigma_v^2 N_v$, then

$$\begin{aligned}\mathcal{L}V(X) \leq & -rS_p\left(1-\frac{S_p^0}{S_p}\right)^2 - \gamma S_v\left(1-\frac{N_v}{S_v}\right)^2 - rL_p - r\left[1-\mathcal{R}_0^s\right]I_p \\ & - \left[\frac{\beta_p\beta_v}{\gamma N_v^\infty} - \frac{\beta_v}{N_p}\right]I_vI_p - \frac{\beta_v}{N_p}S_vI_p + \sigma_{p,v}.\end{aligned}$$

Since $V(x) \geq 0$, and using Itô's formula and integrating dV from 0 to t as well as taking expectation yield the following

$$\begin{aligned}0 &\leq \mathbb{E}V(t) - \mathbb{E}V(0) \leq \mathbb{E}\int_0^t \mathcal{L}V(X(s))ds \\ &\leq -\mathbb{E}\int_0^t \left[rS_p\left(1-\frac{S_p^0}{S_p}\right)^2 + \gamma S_v\left(1-\frac{N_v}{S_v}\right)^2 + rL_p + r\left[1-\mathcal{R}_0^s\right]I_p \right. \\ &\left. + \left[\frac{\beta_p\beta_v}{\gamma N_v^\infty} + \frac{\beta_v}{N_p}\right]I_vI_p + \frac{\beta_v}{N_p}S_vI_p - \sigma_{p,v}\right]ds\end{aligned}$$

Therefore,

$$\begin{aligned}&\frac{1}{t}\mathbb{E}\int_0^t \left[rS_p\left(1-\frac{S_p^0}{S_p}\right)^2 + \gamma S_v\left(1-\frac{N_v}{S_v}\right)^2 + rL_p + r\left[1-\mathcal{R}_0^s\right]I_p \right. \\ &\qquad \left. + \left[\frac{\beta_p\beta_v}{\gamma N_v^\infty} + \frac{\beta_v}{N_p}\right]I_vI_p + \frac{\beta_v}{N_p}S_vI_p\right]ds \leq \sigma_{p,v}\end{aligned}$$

This implies that,

$$\begin{aligned}&\lim_{t\to\infty}\frac{1}{t}\mathbb{E}\int_0^t \left[rS_p\left(1-\frac{S_p^0}{S_p}\right)^2 + \gamma S_v\left(1-\frac{N_v}{S_v}\right)^2 + rL_p + r\left[1-\mathcal{R}_0^s\right]I_p \right. \\ &\qquad \left. + \left[\frac{\beta_p\beta_v}{\gamma N_v^\infty} + \frac{\beta_v}{N_p}\right]I_vI_p + \frac{\beta_v}{N_p}S_vI_p\right]ds \leq \sigma_{p,v}\end{aligned}$$

Taking θ, σ_p, and σ_v such that $0 < \sigma_{p,v} < 1$, we have

$$\begin{aligned}&\lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}\int_0^t \left[rS_p\left(1-\frac{S_p^0}{S_p}\right)^2 + \gamma S_v\left(1-\frac{N_v}{S_v}\right)^2 + rL_p + r\left[1-\mathcal{R}_0^s\right]I_p \right. \\ &\qquad \left. + \left[\frac{\beta_p\beta_v}{\gamma N_v^\infty} + \frac{\beta_v}{N_p}\right]I_vI_p + \frac{\beta_v}{N_p}S_vI_p\right]ds \leq \log\sigma_{p,v} < 0.\end{aligned}$$

Therefore,

$$\lim_{t\to\infty} \mathbb{E} \int_0^t \left[rS_p \left(1 - \frac{S_p^0}{S_p}\right)^2 + \gamma S_v \left(1 - \frac{N_v}{S_v}\right)^2 + rL_p + r\left[1 - \mathcal{R}_0^s\right] I_p \right.$$
$$\left. + \left[\frac{\beta_p \beta_v}{\gamma N_v^\infty} + \frac{\beta_v}{N_p}\right] I_v I_p + \frac{\beta_v}{N_p} S_v I_p \right] ds \le \lim_{t\to\infty} e^{\sigma_{p,v} t} = 0$$

Thus

$$\begin{aligned} &S_p \to N_p \;\; L_p \to 0 \;\; I_p \to 0 \\ &S_v \to N_v \;\; I_v \to 0. \end{aligned}$$

exponentially a.s.

8 Persistence

Theorem 5 *Let $(S_p(t), L_p(t), I_p(t), I_v(t))$ be the solution of (3) with initial values $(S_p(0), L_p(0), I_p(0), I_v(0)) \in (0, N_p)\times(0, N_p)\times(0, N_p)\times(0, N_v)$. If $\mathcal{R}_0^s > 1$, then the system (3) is globally asymptotically stable at endemic equilibrium point if*

$$\lim_{t\to\infty} \frac{1}{t} \mathbb{E} \int_0^t \left[\frac{rS_p^*}{S_p S_p^*}(S_p^* - S_p)^2 + \frac{\beta_p}{N_v} S_p^* I_v^* A_1 + \frac{\beta_v}{N_p}\frac{I_p}{I_v}(I_v - I_v^*)^2 + \gamma I_v^* A_2 \right] dr \le A_3.$$

namely, the disease will persist with probability one.

Proof Let us define the following Lyapunov function $V : \mathbb{R}_+^4 \to \mathbb{R}_+$

$$\begin{aligned} V(S_p, L_P, I_p, I_v) &= (S_p + L_p + I_p + I_v) - (S_p^* + L_p^* + I_p^* + I_v^*) \\ &\quad - \left(S_p^* \ln \frac{S_p}{S_p^*} + L_p^* \ln \frac{L_p}{L_p^*} + I_p^* \ln \frac{I_p}{I_p^*} + I_v^* \ln \frac{I_v}{I_v^*} \right). \end{aligned}$$

Computing the Itô formula terms as:

$$\begin{aligned} V_x f &= \left(1 - \frac{S_p^*}{S_p}\right)\left(rN_p - \beta_p S_p \frac{I_v}{N_v^\infty} - rS_p\right) + \left(1 - \frac{L_p^*}{L_p}\right)\left(\beta_p S_p \frac{I_v}{N_v^\infty} - (r+b)L_p\right) \\ &\quad + \left(1 - \frac{I_p^*}{I_p}\right)(bL_p - rI_p) + \left(1 - \frac{I_v^*}{I_v}\right)\left(\beta_v N_v \frac{I_p}{N_p} - \beta_v \frac{I_p}{N_p} I_v - \gamma I_v\right). \end{aligned}$$

The system (3) satisfy the following relations at equilibrium point

$$
\begin{aligned}
rN_p &= \beta_p S_p^* \frac{I_v^*}{N_v^\infty} + rS_p^* \\
(r+b) &= \beta_p S_p^* \frac{I_v^*}{L_p^* N_v^\infty} \\
r &= b\frac{L_p^*}{I_p^*} \\
\beta_v \frac{N_v}{N_p} &= \frac{\beta_v}{N_p} I_v^* + \gamma \frac{I_v^*}{I_p^*}
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
V_x f = &\left(1 - \frac{S_p^*}{S_p}\right)\left(\beta_p S_p^* \frac{I_v^*}{N_v^\infty} + rS_p^* - \beta_p S_p \frac{I_v}{N_v^\infty} - rS_p\right) \\
&+ \left(1 - \frac{L_p^*}{L_p}\right)\left(\beta_p S_p \frac{I_v}{N_v^\infty} - \beta_p S_p^* \frac{I_v^*}{L_p^* N_v^\infty} L_p\right) + \left(1 - \frac{I_p^*}{I_p}\right)\left(bL_p - b\frac{L_p^*}{I_p^*} I_p\right) \\
&+ \left(1 - \frac{I_v^*}{I_v}\right)\left(\frac{\beta_v}{N_p} I_v^* I_p + \gamma \frac{I_v^*}{I_p^*} I_p - \beta_v \frac{I_p}{N_p} I_v - \gamma I_v\right) \\
&= rS_p^*\left(1 - \frac{S_p^*}{S_p}\right)\left(1 - \frac{S_p}{S_p^*}\right) + \frac{\beta_p}{N_v^\infty} S_p^* I_v^* \left(1 - \frac{S_p^*}{S_p}\right)\left(1 - \frac{S_p I_v}{S_p^* I_v^*}\right) \\
&+ \frac{\beta_p}{N_v^\infty} S_p^* I_v^* \left(1 - \frac{L_p^*}{L_p}\right)\left(\frac{S_p I_v}{S_p^* I_v^*} - \frac{L_p}{L_p^*}\right) + bL_p^*\left(1 - \frac{I_p^*}{I_p}\right)\left(\frac{L_p}{L_p^*} - \frac{I_p}{I_p^*}\right) \\
&+ \left(1 - \frac{I_v^*}{I_v}\right)\left(-\frac{\beta_v}{N_p} I_v I_p \left(1 - \frac{I_v^*}{I_v}\right) + \gamma I_v^* \left(\frac{I_p}{I_p^*} - \frac{I_v}{I_v^*}\right)\right) \\
&= rS_p^*\left(2 - \frac{S_p}{S_p^*} - \frac{S_p^*}{S_p}\right) + \frac{\beta_p}{N_v^\infty} S_p^* I_v^* \left(1 - \frac{I_v}{I_v^*}\left(\frac{S_p}{S_p^*} - 1\right) - \frac{S_p^*}{S_p}\right) \\
&+ \frac{\beta_p}{N_v^\infty} S_p^* I_v^* \left(\frac{S_p I_v}{S_p^* I_v^*}\left(1 - \frac{L_p^*}{L_p}\right) - \frac{L_p}{L_p^*}\left(1 - \frac{L_p^*}{L_p}\right)\right) + bL_p^*\left(1 + \frac{L_p}{L_p^*} - \frac{I_p}{I_p^*} - \frac{I_p^* L_p}{I_p L_p^*}\right) \\
&- \frac{\beta_v}{N_v} I_v I_p \left(1 - \frac{I_v^*}{I_v}\right)^2 + \gamma I_v^* \left(\frac{I_p}{I_p^*} - \frac{I_v^* I_p}{I_v I_p^*} - \frac{I_v}{I_v^*} + 1\right) \\
&= rS_p^*\left(2 - \frac{S_p}{S_p^*} - \frac{S_p^*}{S_p}\right) + \frac{\beta_p}{N_v^\infty} S_p^* I_v^* \left(1 - \frac{S_p^*}{S_p} - \frac{I_v}{I_v^*}\left(\frac{S_p}{S_p^*} - 1\right)\right) \\
&+ \frac{\beta_p}{N_v^\infty} S_p^* I_v^* \left(1 - \frac{L_p}{L_p^*} - \frac{S_p I_v}{S_p^* I_v^*}\left(\frac{L_p^*}{L_p} - 1\right)\right) + bL_p^*\left(1 - \frac{I_p}{I_p^*} + \frac{L_p}{L_p^*}\left(1 - \frac{I_p^*}{I_p}\right)\right) \\
&- \frac{\beta_v}{N_v^\infty} I_v I_p \left(1 - \frac{I_v^*}{I_v}\right)^2 + \gamma I_v^* \left(1 - \frac{I_v}{I_v^*} - \frac{I_p}{I_p^*}\left(\frac{I_v^*}{I_v} - 1\right)\right).
\end{aligned}
$$

Then

$$
\begin{aligned}
V_x f &= rS_p^*\left(2-\frac{S_p}{S_p^*}-\frac{S_p^*}{S_p}\right)+\frac{\beta_p}{N_v^\infty}S_p^*I_v^*\left(2-\frac{S_p^*}{S_p}-\frac{L_p}{L_p^*}-\frac{I_v}{I_v^*}\left(\frac{S_pL_p^*}{S_p^*L_p}-1\right)\right)\\
&+bL_p^*\left(1-\frac{I_p}{I_p^*}+\frac{L_p}{L_p^*}\left(1-\frac{I_p^*}{I_p}\right)\right)-\frac{\beta_v}{N_p}I_vI_p\left(1-\frac{I_v^*}{I_v}\right)^2\\
&+\gamma I_v^*\left(1-\frac{I_v}{I_v^*}-\frac{I_p}{I_p^*}\left(\frac{I_v^*}{I_v}-1\right)\right).
\end{aligned}
$$

Now we need compute the term $g^TV_{xx}g$,

$$
g^TV_{xx}g=\begin{bmatrix}\sigma^2\left(\frac{N_p-S_p}{S_p}\right)^2S_p^*+\sigma^2L_p^* & 0\\ 0 & I_p^*\sigma^2+I_v^*\sigma_v^2\end{bmatrix}
$$

therefore,

$$
\begin{aligned}
\frac{1}{2}trace(g^TV_{xx}g)&=\frac{1}{2}\left(\sigma^2\left(\frac{N_p-S_p}{S_p}\right)^2S_p^*+\sigma^2L_p^*+\sigma^2I_p^*+\sigma_v^2I_v^*\right)\\
&\le\frac{1}{2}\left(\sigma^2S_p^*+\sigma^2L_p^*+\sigma^2I_p^*+\sigma_v^2I_v^*\right)
\end{aligned}
$$

The stochastics terms are not neccesary, because the are a martingale and therefore, when we use integrating and expectation they vanishing, obtaining the following $LV(X)$ operator

$$
\begin{aligned}
LV(X)&=-rS_p^*\frac{(S_p^*-S_p)^2}{S_pS_p^*}-\frac{\beta_p}{N_v^\infty}S_p^*I_v^*A_1-bL_p^*A_2-\frac{\beta_v}{N_p}I_vI_p\left(1-\frac{I_v^*}{I_v}\right)^2\\
&-\gamma I_v^*A_3+A_4.
\end{aligned}
$$

where

$$
\begin{aligned}
A_1&=\left(\frac{S_p^*}{S_p}+\frac{L_p}{L_p^*}+\frac{I_v}{I_v^*}\left(\frac{S_pL_p^*}{S_p^*L_p}-1\right)-2\right)>0,\\
A_2&=\left(\frac{I_p}{I_p^*}-\frac{L_p}{L_p^*}\left(1-\frac{I_p^*}{I_p}\right)-1\right)>0,\\
A_3&=\left(\frac{I_v}{I_v^*}+\frac{I_p}{I_p^*}\left(\frac{I_v^*}{I_v}-1\right)-1\right)>0,\\
A_4&=\frac{1}{2}\left(\sigma^2S_p^*+\sigma^2L_p^*+\sigma^2I_p^*+\sigma_v^2I_v^*\right)>0.
\end{aligned}
$$

Applying Itô formula, integrating dV from 0 to t and taking expectation gives the following

$$
\begin{aligned}
0 \leq \mathbb{E}V(t) - \mathbb{E}V(0) = \mathbb{E}\int_0^t LV(s)ds \\
&- \mathbb{E}\int_0^t \left(rS_p^* \frac{\left(S_p^* - S_p\right)^2}{S_p S_p^*} + \frac{\beta_p}{N_v^\infty} S_p^* I_v^* A_1 + bL_p^* A_2 + \frac{\beta_v}{N_p} I_v I_p \left(1 - \frac{I_v^*}{I_v}\right)^2 + \gamma I_v^* A_3 \right) ds \\
&+ A_4 t.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\lim_{t\to\infty} \frac{1}{t}\mathbb{E}\int_0^t \left(rS_p^* \frac{\left(S_p^* - S_p\right)^2}{S_p S_p^*} + \frac{\beta_p}{N_v^\infty} S_p^* I_v^* A_1 + bL_p^* A_2 + \frac{\beta_v}{N_p} I_v I_p \left(1 - \frac{I_v^*}{I_v}\right)^2 + \gamma I_v^* A_3 \right) ds \\
\leq A_4.
\end{aligned}
$$

9 Numerical Results

10 Conclusion

Referecnce	Priority	Observation
[1]		
[2]	**	See Lyapnov Function.
[3]	**	For persistece def
[4]	*	Dengue
[5]	*	Mobility
[6]		
[7]		
[8]		
[9]		
[10]		
[11]	***	Review
[12]	***	Review
[13]	**	Review
[14]	*	Vaccination
[15]	**	General ideas
[16]	***	For extinction by noise
[17]	***	Threshold behaviour
[18]	***	Good idea for COVID 19
[19]	**	Lie approach
[20]	**	Threshold
[21]	***	Thickbone with CMCM deduction
[22]	***	Permanence
[23]	*	Degenerate Difussion
[24]	*	General force of infection

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A Background