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Threshold behavior of a epidemic vector-plant model: The Yellow Curl Virus

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1 Introduction

2 Model formulation

2.1 Deterministic base dynamics

Model description and parameters meaning

Hold citation and figure with data

$$
\begin{aligned}
\dot{S_p} &= -\beta_p S_p \frac{I_v}{N_v} + \tilde{r}_1 L_p + \tilde{r}_2 I_p \\
\dot{L_p} &= \beta_p S_p \frac{I_v}{N_v} - bL_p - \tilde{r}_1 L_p \\
\dot{I_p} &= bL_p - \tilde{r}_2 I_p \\
\dot{S_v} &= -\beta_v S_v \frac{I_p}{N_p} - \tilde{\gamma} S_v + (1-\theta)\mu \\
\dot{I_v} &= \beta_v S_v \frac{I_p}{N_p} - \tilde{\gamma} I_v + \theta\mu
\end{aligned}
\tag{1}
$$

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Gabriel: Aqui anexa Los paquetes y contenido de lo que llevas escrito. Si necesitas carpetas agregalas. Tabien sube el archivo bib y las figuras en extencion eps de la simulaciones del modelo determinista que estamos perturbando.

translate this section

Make a table for description of all parameters

Redact this conservation law to the entire system (1). Write

Free disease and endemiq fixed points

Abstract Theorem 1 *With the notation of ODE* (1)*, let*

$$
\begin{aligned}
N_v(t) &:= S_v(t) + I_v(t) \\
N_v^{\infty} &:= \frac{\mu}{\gamma}.
\end{aligned}
$$

Then for any initial condition $(S_p(0), L_p(0), I_p(0), S_v(0), I_v(0))^{\top} \in (0,\infty) \times (0, N_v^{\infty})$*, she plant and vector total populations respectively satisfies*

$$
\begin{aligned}
\frac{dN_p}{dt} &= \frac{d}{dt}(S_p + L_p + I_p) = 0, \\
\lim_{t\to\infty} N_v(t) &= N_v^{\infty}.
\end{aligned}
$$

ways of uncertinity quantification

write here the parameters

Following ideas from [HOLD], we quantify uncertinity in replanting rate of plants, and died rate of vector, r_1, r_2 and γ, to this end, we perturb parameters $r_1 \ldots$ whit a Winner process to obtain a stochastic differential equation(SDE). Here, the perturbation describe stochastic environmental noise on each population. In symbols $dB(t) = B(t+dt) - B(t)$ denotes the increment of a standard Wiener process, thus we perturb potentially replanting r_1, r_2, and vector death γ in the infinitiesimal time interval $[t, t+dt)$ by

$$
\begin{aligned}
r_1 dt &\rightsquigarrow r_1 dt + \sigma_p S_p \, dB_p(t), \\
r_2 dt &\rightsquigarrow r_2 dt + \sigma_p S_p \, dB_p(t), \\
\gamma dt &\rightsquigarrow \gamma dt + \sigma_v \, dB_v(t).
\end{aligned}
\tag{2}
$$

Is the same Brownian motion for the three equations?

Note that here we will use the latex proba package, plase use the same commands in the remain of the manuscript

Note that right hand side of (2) is a random perturbations of parameters r_1, r_2, γ, with mean $\mathbb{E}[r_1 dt + \sigma_p dB(t)]$ and variance $\mathrm{Var}[r_1 dt + \sigma_L dB(t)] = \sigma_L^2 dt$, $\mathbb{E}(\tilde{r}_2 dt) = r_2 dt$ and $Var(\tilde{r}_2 dt) = \sigma_I^2 dt$ and $\mathbb{E}(\tilde{\gamma} dt) = \gamma dt$ and $Var(\tilde{\gamma} dt) = \sigma_v^2 dt$. Thus, we establish an stochastic extencion from deterministic tomato model (1) by the Itô SDE

$$
\begin{aligned}
dS_p &= \left(-\beta_p S_p \frac{I_v}{N_v} + r_1 L_p + r_2 I_p \right) dt + (\sigma_L L_p + \sigma_I I_p) dB(t) \\
dL_p &= \left(\beta_p S_p \frac{I_v}{N_v} - b L_p - r_1 L_p \right) dt - \sigma_L L_p dB(t) \\
dI_p &= (b L_p - r_2 I_p) \, dt - \sigma_I I_p dB(t) \\
dS_v &= \left(-\beta_v S_v \frac{I_p}{N_p} - \gamma S_v + (1-\theta)\mu \right) dt - \sigma_v S_v dB(t) \\
dI_v &= \left(\beta_v S_v \frac{I_p}{N_p} - \gamma I_v + \theta \mu \right) dt - \sigma_v I_v dB(t).
\end{aligned}
\tag{3}
$$

3 Existence of a unique positive solution

Thereom *.* of [Mao Book] assures ths existence of unique solution of (3) in a compact interval. Since we study asymptotic behaviour, we have to assure the existence of unique positive invariant solution to SDE (*). To this end, let $\mathbb{R}^n_+$ the first octant of $\mathbb{R}^n$ and consider

$$\mathbf{E} := \left\{ (S_p, L_p, I_p, S_v, I_v)^\top \in \mathbb{R}^5_+ : \quad \leq S_p + L_p + I_p \geq N_p, \quad S_v + I_v \leq \frac{\mu}{\gamma} \right\},$$

the following result prove that this set is positive invariant.

Theorem 2 *For any initial values* $(S_p(0), L_p(0), I_p(0), S_v(0), I_v(0)) \in \mathbf{E}$*, exists unique invariant global positive solution to SDE* (3) $(S_p(t), L_p(t), I_p(t), S_v(t), I_v(t))^\top$ *with probability one, that is,*

$$\mathbb{P}\left[(L_p(t), I_p(t), S_v(t), I_v(t)) \in \mathbf{E}, \quad \forall t \geq 0\right] = 1.$$

Proof

4 Extinction of the disease

Our analysis needs the following hypotesis.

(H-1) According to SDE (3), replatin rates satisfies $r_1 = r_2 = r$.
(H-2) The replanting noise intesities are equal $\sigma_L = \sigma_I = \sigma$.

We define the repoductive nomber of our stochastic model in SDE (*) by

Define here the infinitesimal operator $\mathscr{L}$.

$$\mathscr{R}^s_0 := \frac{\beta_p \beta_v}{\gamma r}. \tag{4}$$

As our deterministic base structure this paramenters summarizes the behavior of extinction and persistence according with a threshold.

Theorem 3 *Let* $(S_p(t), L_p(t), I_p(t), I_v(t))$ *be the solution of (3) with initial condition* $(S_p(0), L_p(0), I_p(0), I_v(0)) \in \mathbf{E}$*. If* $0 \leq \mathscr{R}^s_0 < 1$ *then, infected individuals in SDE (*) tends to zero exponentially a.s, that is, the disease will extinguishes with probability one.*

Proof The proof consitst verify the hypotheses of Khasminskii Theorem [*] for the Lyapunov function

$$V(S_p, L_p, I_p, S_v, I_v) = \left(S_p - S^0_p - S^0_p \ln\left(\frac{S_p}{S^0_p}\right)\right) + L_p + I_p + \frac{\beta_p N_p}{\gamma N_v} I_v. \tag{5}$$

Let f, g respectively be the dirft and difussion of SDE(*). Applyng the inifinitesimal opreator $\mathscr{L}$ we have

In the following step apply the operator $\mathscr{L}$

$$
\begin{aligned}
V_x f &= \left(1 - \frac{S_p^0}{S_p}\right)\left(-\frac{\beta_p}{N_v^\infty} S_p I_v + rN_p - rS_p\right) + \frac{\beta_p}{N_v^\infty} S_p I_v - (b+r)L_p \\
&+ bL_p - rI_p + \frac{\beta_p N_p}{\gamma N_v^\infty}\left(\frac{\beta_v N_v}{N_p} I_p - \frac{\beta_v}{N_v^\infty} I_v I_p - \gamma I_v\right) \\
&= -rS_p\left(1 - \frac{S_p^0}{S_p}\right)^2 - \frac{\beta_p}{N_v^\infty} S_p I_v + \frac{\beta_p}{N_v^\infty} I_v S_p^0 + \frac{\beta_p}{N_v^\infty} S_p I_v - r(L_p + I_p) \\
&+ \frac{\beta_p N_p}{\gamma N_v^\infty}\left(\frac{\beta_v N_v}{N_p} I_p - \frac{\beta_v}{N_v^\infty} I_v I_p - \gamma I_v\right) \\
&= -rS_p\left(1 - \frac{S_p^0}{S_p}\right)^2 + \frac{\beta_p}{N_v^\infty} I_v S_p^0 - r(L_p + I_p) + \frac{\beta_p N_p}{\gamma N_v^\infty}\frac{\beta_v N_v}{N_p} I_p \\
&- \frac{\beta_p N_p}{\gamma N_v^\infty}\frac{\beta_v}{N_v^\infty} I_v I_p - \frac{\beta_p N_p}{\gamma N_v^\infty}\gamma I_v.
\end{aligned}
$$

Then,

$$
\begin{aligned}
V_x f &= -rS_p\left(1 - \frac{S_p^0}{S_p}\right)^2 - \left[\frac{\beta_p N_p}{\gamma N_v^\infty}\gamma - \frac{\beta_p N_p}{N_v^\infty}\right] I_v + \left[\frac{\beta_p N_p}{\gamma N_v^\infty}\beta_v \frac{N_v^\infty}{N_p} - r\right] I_p \\
&- rL_p - \frac{\beta_p N_p}{\gamma N_v^\infty}\frac{\beta_v}{N_p} I_v I_p \\
&= -rS_p\left(1 - \frac{S_p^0}{S_p}\right)^2 + \left[\frac{\beta_p N_p}{\gamma N_v^\infty}\beta_v \frac{N_v^\infty}{N_p} - r\right] I_p - rL_p - \frac{\beta_p N_p}{\gamma N_v^\infty}\frac{\beta_v}{N_p} I_v I_p \\
&= -rS_p\left(1 - \frac{S_p^0}{S_p}\right)^2 + r\left[\frac{\beta_p \beta_v}{\gamma r} - 1\right] I_p - rL_p - \frac{\beta_p \beta_v}{\gamma N_v^\infty} I_v I_p.
\end{aligned}
$$

Expressing the right hand side of above equation in term of the basic reproductive number, $\mathscr{R}_0^s$ we get

$$
V_x f = -rS_p\left(1 - \frac{S_p^0}{S_p}\right)^2 + r\left[\mathscr{R}_0^s - 1\right] I_p - rL_p - \frac{\beta_p \beta_v}{\gamma N_v^\infty} I_v I_p.
$$

Moreover,

$$
\begin{aligned}
\frac{1}{2} trace(g^T V_{xx} g) &= \frac{1}{2}\sigma^2 N_p \left(\frac{N_p - S_p}{S_p}\right)^2 \\
&\leq \frac{1}{2}\sigma^2 N_p.
\end{aligned}
$$

The stochastic terms are not neccesary, because they do a martingale procces and therefore, when we use integral and expectation they vanising. Incorporation all terms calculate above, we obtain

$$dV(X) = -rS_p\left(1-\frac{S_p^0}{S_p}\right)^2 + r[\mathscr{R}_0^s - 1]I_p - rL_p - \frac{\beta_p\beta_v}{\gamma N_v^\infty}I_vI_p + \frac{1}{2}\sigma^2 N_p\left(\frac{N_p-S_p}{S_p}\right)^2$$
$$\leq -rS_p\left(1-\frac{S_p^0}{S_p}\right)^2 + r[\mathscr{R}_0^s - 1]I_p - rL_p - \frac{\beta_p\beta_v}{\gamma N_v^\infty}I_vI_p + \frac{1}{2}\sigma^2 N_p.$$

Define $LV(X)$ as

$$LV(X) = -rS_p\left(1-\frac{S_p^0}{S_p}\right)^2 + r[\mathscr{R}_0^s - 1]I_p - rL_p - \frac{\beta_p\beta_v}{\gamma N_v^\infty}I_vI_p + \frac{1}{2}\sigma^2 N_p.$$

Using Itô's formula and integrating dV from 0 to t as well as taking expectation yield the following

$$0 \leq \mathbb{E}V(t) - \mathbb{E}V(0) \leq \mathbb{E}\int_0^t LV(X(s))ds$$
$$\leq -\mathbb{E}\int_0^t \left[rS_p\left(1-\frac{S_p^0}{S_p}\right)^2 - r[\mathscr{R}_0^s - 1]I_p + rL_p + \frac{\beta_p\beta_v}{\gamma N_v^\infty}I_vI_p\right]ds + \frac{1}{2}\sigma^2 N_p$$

Therefore,

$$\lim_{t\to\infty}\frac{1}{t}\mathbb{E}\int_0^t \left[-rS_p\left(1-\frac{S_p^0}{S_p}\right)^2 + r[\mathscr{R}_0^s - 1]I_p - rL_p - \frac{\beta_p\beta_v}{\gamma N_v^\infty}I_vI_p\right]ds \leq \frac{1}{2}\sigma^2 N_p.$$

$$\lim_{t\to\infty}\frac{1}{t}\mathbb{E}\int_0^t \left[r[\mathscr{R}_0^s - 1]I_p - rS_p\left(1-\frac{S_p^0}{S_p}\right)^2 - rL_p - \frac{\beta_p\beta_v}{\gamma}I_vI_p\right]dr \leq \frac{1}{2}\sigma^2 N_p,\ a.s.$$

Write a paragraph to describe why the limit above exponentially goes to zero.

Theorem 4 *Let $(S_p(t), L_p(t), I_p(t), I_v(t))$ be the solution of SDE (3) with initial values $(S_p(0), L_p(0), I_p(0), I_v(0)) \in (0,N_p)\times(0,N_p)\times(0,N_p)$ $\times(0,N_v)$. If $0 \leq \mathscr{R}_0^s < 1$, then the following conditions holds*

$$\lim_{t\to\infty}\frac{1}{t}\mathbb{E}\int_0^t \left[r[\mathscr{R}_0^s - 1]I_p - rS_p\left(1-\frac{S_p^0}{S_p}\right)^2 - rL_p - \frac{\beta_p\beta_v}{\gamma}I_vI_p\right]dr \leq \frac{1}{2}\sigma^2 N_p,\ a.s.,$$

namely, the infected individual tends to zero exponentially a.s, i.e the disease will die out with probability one.

Proof The proof consitst verify the hypotheses of Khasminskii Theorem [*] for the Lyapunov function

$$
\begin{aligned}
V(S_p, L_p, I_p, S_v, I_v) = &\left(S_p - N_p - N_p \ln \frac{S_p}{N_p}\right) + L_p + I_p + \frac{\beta_p N_p}{\gamma N_v^\infty} I_v \\
&+ \left(S_v - N_v - N_v \ln \frac{S_v}{N_v}\right),
\end{aligned}
$$

Let f, g respectively be the dirft and difussion of SDE (**??**). Applyng the inifinitesimal opreator $\mathscr{L}$ we have

$$
\begin{aligned}
V_x f = &\left(1 - \frac{N_p}{S_p}\right)\left(-\frac{\beta_p}{N_v^\infty} S_p I_v + rN_p - rS_p\right) + \frac{\beta_p}{N_v^\infty} S_p I_v - (b+r) L_p && (6)\\
&+ bL_p - rI_p + \left(1 - \frac{N_v}{S_v}\right)\left(-\frac{\beta_v}{N_p} S_v I_p - \gamma S_v + (1-\theta)\mu\right) && (7)\\
&+ \frac{\beta_p N_p}{\gamma N_v^\infty}\left(\frac{\beta_v S_v}{N_p} I_p - \gamma I_v + \theta\mu\right) && (8)\\
& && (9)
\end{aligned}
$$

Expanded the first term and factoring the term S_p, we obtain

$$
\begin{aligned}
\left(1 - \frac{N_p}{S_p}\right)\left(-\frac{\beta_p}{N_v^\infty} S_p I_v + rN_p - rS_p\right) &= \left(1 - \frac{N_p}{S_p}\right)\left(-rS_p\left(1 - \frac{N_p}{S_p}\right) - \frac{\beta_p}{N_v^\infty} S_p I_v\right) \\
&= -rS_p\left(1 - \frac{N_p}{S_p}\right)^2 - \frac{\beta_p}{N_v^\infty} S_p I_v + \frac{\beta_p}{N_v^\infty} N_p I_v
\end{aligned}
\tag{10}
$$

For the second term, since $(1-\theta)\mu \leq \gamma N_v$ we can bounded by the following

$$
\begin{aligned}
\left(1 - \frac{N_v}{S_v}\right)\left(-\frac{\beta_v}{N_p} S_v I_p - \gamma S_v + (1-\theta)\mu\right) &\leq \left(1 - \frac{N_v}{S_v}\right)\left(-\frac{\beta_v}{N_p} S_v I_p - \gamma S_v + \gamma N_v\right) \\
&\leq \left(1 - \frac{N_v}{S_v}\right)\left(-\gamma S_v\left(1 - \frac{N_v}{S_v}\right) - \frac{\beta_v}{N_p} S_v I_p\right) \\
&\leq -\gamma S_v\left(1 - \frac{N_v}{S_v}\right)^2 - \frac{\beta_v}{N_p} S_v I_p + \frac{\beta_v}{N_p} N_v I_p
\end{aligned}
\tag{11}
$$

Same way from above calculation, and since $\theta\mu \leq \theta\gamma N_v$, we obtain

$$
\begin{aligned}
\frac{\beta_p N_p}{\gamma N_v^\infty}\left(\frac{\beta_v S_v}{N_p} I_p - \gamma I_v + \theta\mu\right) &\leq \frac{\beta_p N_p}{\gamma N_v^\infty}\left(\frac{\beta_v S_v}{N_p} I_p - \gamma I_v + \theta\gamma N_v\right) \\
&\leq \frac{\beta_p \beta_v S_v I_p}{\gamma N_v} - \frac{\beta_p N_p}{N_v^\infty} I_v + \beta_p \theta N_p
\end{aligned}
\tag{12}
$$

Then, sustituting (10)-(12) into $V_x f$

$$\begin{aligned}
V_x f \leq & -rS_p\left(1-\frac{N_p}{S_p}\right)^2+\frac{\beta_p}{N_v^\infty}N_pI_v-r(L_p+I_p)\\
& -\gamma S_v\left(1-\frac{N_v}{S_v}\right)^2-\frac{\beta_v}{N_p}S_vI_p+\frac{\beta_v}{N_p}N_vI_p\\
& +\frac{\beta_p\beta_vS_vI_p}{\gamma N_v}-\frac{\beta_pN_p}{N_v^\infty}I_v+\beta_p\theta N_p
\end{aligned}$$

$$\begin{aligned}
V_x f \leq & -rS_p\left(1-\frac{N_p}{S_p}\right)^2+\left[\frac{\beta_p}{N_v^\infty}N_p-\frac{\beta_pN_p}{N_v^\infty}\right]I_v-r(L_p+I_p)\\
& -\gamma S_v\left(1-\frac{N_v}{S_v}\right)^2-\frac{\beta_v}{N_p}S_vI_p+\frac{\beta_v}{N_p}N_vI_p\\
& +\frac{\beta_p\beta_vS_vI_p}{\gamma N_v}+\beta_p\theta N_p
\end{aligned}$$

Moreover, since $S_v+I_v\leq N_v$, we can obtain the following relation

$$\begin{aligned}
V_x f \leq & -rS_p\left(1-\frac{N_p}{S_p}\right)^2-r(L_p+I_p)\\
& -\gamma S_v\left(1-\frac{N_v}{S_v}\right)^2+\frac{\beta_v}{N_p}I_vI_p\\
& +\frac{\beta_p\beta_vI_p}{\gamma}-\frac{\beta_p\beta_vI_vI_p}{\gamma N_v}+\beta_p\theta N_p
\end{aligned}$$

Expressing the right hand side of above equation in term of the basic reproductive number, $\mathscr{R}_0^s$ we get

$$\begin{aligned}
V_x f = & -rS_p\left(1-\frac{S_p^0}{S_p}\right)^2-\gamma S_v\left(1-\frac{N_v}{S_v}\right)^2-rL_p-r\left[1-\mathscr{R}_0^s\right]I_p\\
& -\left[\frac{\beta_p\beta_v}{\gamma N_v^\infty}-\frac{\beta_v}{N_p}\right]I_vI_p-\frac{\beta_v}{N_p}S_vI_p+\beta_p\theta N_p.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\frac{1}{2}trace(g^TV_{xx}g) &= \frac{1}{2}\frac{(\sigma_p(L_p+I_p))^2}{N_p}+\frac{1}{2}\sigma_v^2N_v\\
&\leq \frac{1}{2}\sigma_p^2N_p+\frac{1}{2}\sigma_v^2N_v.
\end{aligned}$$

The stochastic terms are not neccesary, because they do a martingale procces and therefore, when we use integral and expectation they vanising.

Incorporation all terms calculate above, we obtain

$$
\begin{aligned}
\mathscr{L}V(X) \leq & -rS_p\left(1-\frac{S_p^0}{S_p}\right)^2 - \gamma S_v\left(1-\frac{N_v}{S_v}\right)^2 - rL_p - r\left[1-\mathscr{R}_0^s\right]I_p \\
& - \left[\frac{\beta_p\beta_v}{\gamma N_v^\infty} - \frac{\beta_v}{N_p}\right]I_vI_p - \frac{\beta_v}{N_p}S_vI_p + \beta_p\theta N_p + \frac{1}{2}\sigma_p^2N_p + \frac{1}{2}\sigma_v^2N_v.
\end{aligned}
$$

Define $\sigma_{p,v} := \beta_p\theta N_p + \frac{1}{2}\sigma_p^2N_p + \frac{1}{2}\sigma_v^2N_v$, then

$$
\begin{aligned}
\mathscr{L}V(X) \leq & -rS_p\left(1-\frac{S_p^0}{S_p}\right)^2 - \gamma S_v\left(1-\frac{N_v}{S_v}\right)^2 - rL_p - r\left[1-\mathscr{R}_0^s\right]I_p \\
& - \left[\frac{\beta_p\beta_v}{\gamma N_v^\infty} - \frac{\beta_v}{N_p}\right]I_vI_p - \frac{\beta_v}{N_p}S_vI_p + \sigma_{p,v}.
\end{aligned}
$$

Since $V(x) \geq 0$, and using Itô's formula and integrating dV from 0 to t as well as taking expectation yield the following

$$
\begin{aligned}
0 \leq & \, \mathbb{E}V(t) - \mathbb{E}V(0) \leq \mathbb{E}\int_0^t \mathscr{L}V(X(s))ds \\
\leq & -\mathbb{E}\int_0^t \left[rS_p\left(1-\frac{S_p^0}{S_p}\right)^2 + \gamma S_v\left(1-\frac{N_v}{S_v}\right)^2 + rL_p + r\left[1-\mathscr{R}_0^s\right]I_p \right.\\
+ & \left. \left[\frac{\beta_p\beta_v}{\gamma N_v^\infty} + \frac{\beta_v}{N_p}\right]I_vI_p + \frac{\beta_v}{N_p}S_vI_p - \sigma_{p,v}\right] ds
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\frac{1}{t}\mathbb{E}\int_0^t & \left[rS_p\left(1-\frac{S_p^0}{S_p}\right)^2 + \gamma S_v\left(1-\frac{N_v}{S_v}\right)^2 + rL_p + r\left[1-\mathscr{R}_0^s\right]I_p \right.\\
& + \left. \left[\frac{\beta_p\beta_v}{\gamma N_v^\infty} + \frac{\beta_v}{N_p}\right]I_vI_p + \frac{\beta_v}{N_p}S_vI_p\right] ds \leq \sigma_{p,v}
\end{aligned}
$$

This implies that,

$$
\begin{aligned}
\lim_{t\to\infty}\frac{1}{t}\mathbb{E}\int_0^t & \left[rS_p\left(1-\frac{S_p^0}{S_p}\right)^2 + \gamma S_v\left(1-\frac{N_v}{S_v}\right)^2 + rL_p + r\left[1-\mathscr{R}_0^s\right]I_p \right.\\
& + \left. \left[\frac{\beta_p\beta_v}{\gamma N_v^\infty} + \frac{\beta_v}{N_p}\right]I_vI_p + \frac{\beta_v}{N_p}S_vI_p\right] ds \leq \sigma_{p,v}
\end{aligned}
$$

Taking θ, σ_p, and σ_v such that $0 < \sigma_{p,v} < 1$, we have

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E} \int_0^t \left[rS_p \left(1 - \frac{S_p^0}{S_p}\right)^2 + \gamma S_v \left(1 - \frac{N_v}{S_v}\right)^2 + rL_p + r\left[1 - \mathscr{R}_0^s\right] I_p \right.$$
$$\left. + \left[\frac{\beta_p \beta_v}{\gamma N_v^\infty} + \frac{\beta_v}{N_p}\right] I_v I_p + \frac{\beta_v}{N_p} S_v I_p \right] ds \le \log \sigma_{p,v} < 0.$$

Therefore,

$$\lim_{t\to\infty} \mathbb{E} \int_0^t \left[rS_p \left(1 - \frac{S_p^0}{S_p}\right)^2 + \gamma S_v \left(1 - \frac{N_v}{S_v}\right)^2 + rL_p + r\left[1 - \mathscr{R}_0^s\right] I_p \right.$$
$$\left. + \left[\frac{\beta_p \beta_v}{\gamma N_v^\infty} + \frac{\beta_v}{N_p}\right] I_v I_p + \frac{\beta_v}{N_p} S_v I_p \right] ds \le \lim_{t\to\infty} e^{\sigma_{p,v} t} = 0$$

Thus

$$\begin{aligned} &S_p \to N_p \ L_p \to 0 \ I_p \to 0 \\ &S_v \to N_v \ I_v \to 0. \end{aligned}$$

exponentially a.s.

5 Persistence

Theorem 5 *Let $(S_p(t), L_p(t), I_p(t), I_v(t))$ be the solution of (3) with initial values $(S_p(0), L_p(0), I_p(0), I_v(0)) \in (0, N_p) \times (0, N_p) \times (0, N_p) \times (0, N_v)$. If $\mathscr{R}_0^s > 1$, then the system (3) is globally asymptotically stable at endemic equilibrium point if*

$$\lim_{t\to\infty} \frac{1}{t} \mathbb{E} \int_0^t \left[\frac{rS_p^*}{S_p S_p^*} (S_p^* - S_p)^2 + \frac{\beta_p}{N_v} S_p^* I_v^* A_1 + \frac{\beta_v}{N_p} \frac{I_p}{I_v} (I_v - I_v^*)^2 + \gamma I_v^* A_2 \right] dr \le A_3.$$

namely, the disease will persist with probability one.

Proof Let us define the following Lyapunov function $V : \mathbb{R}_+^4 \to \mathbb{R}_+$

$$\begin{aligned} V(S_p, L_P, I_p, I_v) &= (S_p + L_p + I_p + I_v) - (S_p^* + L_p^* + I_p^* + I_v^*) \\ &\quad - \left(S_p^* \ln \frac{S_p}{S_p^*} + L_p^* \ln \frac{L_p}{L_p^*} + I_p^* \ln \frac{I_p}{I_p^*} + I_v^* \ln \frac{I_v}{I_v^*} \right). \end{aligned}$$

Computing the Itô formula terms as:

$$
\begin{aligned}
V_x f = &\left(1 - \frac{S_p^*}{S_p}\right)\left(rN_p - \beta_p S_p \frac{I_v}{N_v^\infty} - rS_p\right) + \left(1 - \frac{L_p^*}{L_p}\right)\left(\beta_p S_p \frac{I_v}{N_v^\infty} - (r+b)L_p\right) \\
&+ \left(1 - \frac{I_p^*}{I_p}\right)(bL_p - rI_p) + \left(1 - \frac{I_v^*}{I_v}\right)\left(\beta_v N_v \frac{I_p}{N_p} - \beta_v \frac{I_p}{N_p} I_v - \gamma I_v\right).
\end{aligned}
$$

The system (3) satisfy the following relations at equilibrium point

$$
\begin{aligned}
rN_p &= \beta_p S_p^* \frac{I_v^*}{N_v^\infty} + rS_p^* \\
(r+b) &= \beta_p S_p^* \frac{I_v^*}{L_p^* N_v^\infty} \\
r &= b \frac{L_p^*}{I_p^*} \\
\beta_v \frac{N_v}{N_p} &= \frac{\beta_v}{N_p} I_v^* + \gamma \frac{I_v^*}{I_p^*}
\end{aligned}
$$

Moreover,

$$\begin{aligned}
V_x f &= \left(1-\frac{S_p^*}{S_p}\right)\left(\beta_p S_p^* \frac{I_v^*}{N_v^\infty} + rS_p^* - \beta_p S_p \frac{I_v}{N_v^\infty} - rS_p\right) \\
&+ \left(1-\frac{L_p^*}{L_p}\right)\left(\beta_p S_p \frac{I_v}{N_v^\infty} - \beta_p S_p^* \frac{I_v^*}{L_p^* N_v^\infty} L_p\right) + \left(1-\frac{I_p^*}{I_p}\right)\left(bL_p - b\frac{L_p^*}{I_p^*} I_p\right) \\
&+ \left(1-\frac{I_v^*}{I_v}\right)\left(\frac{\beta_v}{N_p} I_v^* I_p + \gamma \frac{I_v^*}{I_p^*} I_p - \beta_v \frac{I_p}{N_p} I_v - \gamma I_v\right) \\
&= rS_p^*\left(1-\frac{S_p^*}{S_p}\right)\left(1-\frac{S_p}{S_p^*}\right) + \frac{\beta_p}{N_v^\infty} S_p^* I_v^* \left(1-\frac{S_p^*}{S_p}\right)\left(1-\frac{S_p I_v}{S_p^* I_v^*}\right) \\
&+ \frac{\beta_p}{N_v^\infty} S_p^* I_v^* \left(1-\frac{L_p^*}{L_p}\right)\left(\frac{S_p I_v}{S_p^* I_v^*} - \frac{L_p}{L_p^*}\right) + bL_p^*\left(1-\frac{I_p^*}{I_p}\right)\left(\frac{L_p}{L_p^*} - \frac{I_p}{I_p^*}\right) \\
&+ \left(1-\frac{I_v^*}{I_v}\right)\left(-\frac{\beta_v}{N_p} I_v I_p \left(1-\frac{I_v^*}{I_v}\right) + \gamma I_v^* \left(\frac{I_p}{I_p^*} - \frac{I_v}{I_v^*}\right)\right) \\
&= rS_p^*\left(2-\frac{S_p}{S_p^*}-\frac{S_p^*}{S_p}\right) + \frac{\beta_p}{N_v^\infty} S_p^* I_v^* \left(1-\frac{I_v}{I_v^*}\left(\frac{S_p}{S_p^*}-1\right)-\frac{S_p^*}{S_p}\right) \\
&+ \frac{\beta_p}{N_v^\infty} S_p^* I_v^* \left(\frac{S_p I_v}{S_p^* I_v^*}\left(1-\frac{L_p^*}{L_p}\right) - \frac{L_p}{L_p^*}\left(1-\frac{L_p^*}{L_p}\right)\right) + bL_p^*\left(1+\frac{L_p}{L_p^*}-\frac{I_p}{I_p^*}-\frac{I_p^* L_p}{I_p L_p^*}\right) \\
&- \frac{\beta_v}{N_v} I_v I_p \left(1-\frac{I_v^*}{I_v}\right)^2 + \gamma I_v^* \left(\frac{I_p}{I_p^*} - \frac{I_v^* I_p}{I_v I_p^*} - \frac{I_v}{I_v^*} + 1\right) \\
&= rS_p^*\left(2-\frac{S_p}{S_p^*}-\frac{S_p^*}{S_p}\right) + \frac{\beta_p}{N_v^\infty} S_p^* I_v^* \left(1-\frac{S_p^*}{S_p}-\frac{I_v}{I_v^*}\left(\frac{S_p}{S_p^*}-1\right)\right) \\
&+ \frac{\beta_p}{N_v^\infty} S_p^* I_v^* \left(1-\frac{L_p}{L_p^*}-\frac{S_p I_v}{S_p^* I_v^*}\left(\frac{L_p^*}{L_p}-1\right)\right) + bL_p^*\left(1-\frac{I_p}{I_p^*}+\frac{L_p}{L_p^*}\left(1-\frac{I_p^*}{I_p}\right)\right) \\
&- \frac{\beta_v}{N_v^\infty} I_v I_p \left(1-\frac{I_v^*}{I_v}\right)^2 + \gamma I_v^* \left(1-\frac{I_v}{I_v^*}-\frac{I_p}{I_p^*}\left(\frac{I_v^*}{I_v}-1\right)\right).
\end{aligned}$$

Then

$$\begin{aligned}
V_x f = rS_p^* &\left(2-\frac{S_p}{S_p^*}-\frac{S_p^*}{S_p}\right) + \frac{\beta_p}{N_v^\infty} S_p^* I_v^* \left(2-\frac{S_p^*}{S_p}-\frac{L_p}{L_p^*}-\frac{I_v}{I_v^*}\left(\frac{S_p L_p^*}{S_p^* L_p}-1\right)\right) \\
&+ bL_p^*\left(1-\frac{I_p}{I_p^*}+\frac{L_p}{L_p^*}\left(1-\frac{I_p^*}{I_p}\right)\right) - \frac{\beta_v}{N_p} I_v I_p \left(1-\frac{I_v^*}{I_v}\right)^2 \\
&+ \gamma I_v^* \left(1-\frac{I_v}{I_v^*}-\frac{I_p}{I_p^*}\left(\frac{I_v^*}{I_v}-1\right)\right).
\end{aligned}$$

Now we need compute the term $g^T V_{xx} g$,

$$g^T V_{xx} g = \begin{bmatrix} \sigma^2 \left(\frac{N_p - S_p}{S_p}\right)^2 S_p^* + \sigma^2 L_p^* & 0 \\ 0 & I_p^* \sigma^2 + I_v^* \sigma_v^2 \end{bmatrix}$$

therefore,

$$\begin{aligned} \frac{1}{2} trace(g^T V_{xx} g) &= \frac{1}{2}\left(\sigma^2 \left(\frac{N_p - S_p}{S_p}\right)^2 S_p^* + \sigma^2 L_p^* + \sigma^2 I_p^* + \sigma_v^2 I_v^*\right) \\ &\leq \frac{1}{2}\left(\sigma^2 S_p^* + \sigma^2 L_p^* + \sigma^2 I_p^* + \sigma_v^2 I_v^*\right) \end{aligned}$$

The stochastics terms are not neccesary, because the are a martingale and therefore, when we use integrating and expectation they vanishing, obtaining the following $LV(X)$ operator

$$\begin{aligned} LV(X) = &-rS_p^* \frac{(S_p^* - S_p)^2}{S_p S_p^*} - \frac{\beta_p}{N_v^\infty} S_p^* I_v^* A_1 - bL_p^* A_2 - \frac{\beta_v}{N_p} I_v I_p \left(1 - \frac{I_v^*}{I_v}\right)^2 \\ &- \gamma I_v^* A_3 + A_4. \end{aligned}$$

where

$$\begin{aligned} A_1 &= \left(\frac{S_p^*}{S_p} + \frac{L_p}{L_p^*} + \frac{I_v}{I_v^*}\left(\frac{S_p L_p^*}{S_p^* L_p} - 1\right) - 2\right) > 0, \\ A_2 &= \left(\frac{I_p}{I_p^*} - \frac{L_p}{L_p^*}\left(1 - \frac{I_p^*}{I_p}\right) - 1\right) > 0, \\ A_3 &= \left(\frac{I_v}{I_v^*} + \frac{I_p}{I_p^*}\left(\frac{I_v^*}{I_v} - 1\right) - 1\right) > 0, \\ A_4 &= \frac{1}{2}\left(\sigma^2 S_p^* + \sigma^2 L_p^* + \sigma^2 I_p^* + \sigma_v^2 I_v^*\right) > 0. \end{aligned}$$

Applying Itô formula, integrating dV from 0 to t and taking expectation gives the following

$$\begin{aligned} &0 \leq \mathbb{E}V(t) - \mathbb{E}V(0) = \mathbb{E}\int_0^t LV(s)ds \\ &-\mathbb{E}\int_0^t \left(rS_p^* \frac{(S_p^* - S_p)^2}{S_p S_p^*} + \frac{\beta_p}{N_v^\infty} S_p^* I_v^* A_1 + bL_p^* A_2 + \frac{\beta_v}{N_p} I_v I_p \left(1 - \frac{I_v^*}{I_v}\right)^2 + \gamma I_v^* A_3\right) ds \\ &+A_4 t. \end{aligned}$$

Therefore,

$$\lim_{t\to\infty}\frac{1}{t}\mathbb{E}\int_0^t\left(rS_p^*\frac{\left(S_p^*-S_p\right)^2}{S_pS_p^*}+\frac{\beta_p}{N_v^\infty}S_p^*I_v^*A_1+bL_p^*A_2+\frac{\beta_v}{N_p}I_vI_p\left(1-\frac{I_v^*}{I_v}\right)^2+\gamma I_v^*A_3\right)ds$$
$$\leq A_4.$$

6 Numerical Results

7 Conclusions

Referecnce	Priority	Observation
[1]		
[2]	**	See Lyapnov Function.
[3]	**	For persistece def
[4]	*	Dengue
[5]	*	Mobility
[6]		
[7]		
[8]		
[9]		
[10]		
[11]	***	Review
[12]	***	Review
[13]	**	Review
[14]	*	Vaccination
[15]	**	General ideas
[16]	***	For extinction by noise
[17]	***	Threshold behaviour
[18]	***	Good idea for COVID 19
[19]	**	Lie approach
[20]	**	Threshold
[21]	***	Thickbone with CMCM deduction
[22]	***	Permanence
[23]	*	Degenerate Difussion
[24]	*	General force of infection

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A Background