Stochastic-Tomato-Vector-Plant-Disease

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Abstract

Gabriel: Aqui anexa Los paquetes y contenido de lo que llevas escrito. Si necesitas carpetas agregalas. Tabien sube el archivo bib y las figuras en extencion eps de la simulaciones del modelo determinista que estamos perturbando.

1 Deterministic base dynamics

translate this section

En esta sección, vamos a definir el modelo básico que trabajaremos, consideraremos que las plantas se dividen en tres tipos: plantas susceptibles, latentes e infectadas. Las moscas blancas, las cuales llamaremos vectores, se dividen en susceptibles e infectadas.

Las plantas susceptibles pasan a ser plantas latentes cuando un vector infectado se alimenta de ella a una tasa de β_p, continuando el proceso cuando las plantas latentes se convierten en plantas infectadas a una tasa de b, en cada uno de estos casos consideraremos que estaremos revisando los cultivos para el cual removeremos plantas latentes e infectadas si se detecta que dicha planta esta infectada a una tasa de r_1 y r_2 respectivamente.

Plants become latent by infected vectors, replanting latent and infected plants, latent plants become infectious plants, vectors become infected by infected plants, vectors die or depart per day, immigration from alternative hosts.

$$
\begin{aligned}
\dot{S_p} &= -\beta_p S_p \frac{I_v}{N_v} + \tilde{r_1} L_p + \tilde{r_2} I_p \\
\dot{L_p} &= \beta_p S_p \frac{I_v}{N_v} - b L_p - \tilde{r_1} L_p \\
\dot{I_p} &= b L_p - \tilde{r_2} I_p \\
\dot{S_v} &= -\beta_v S_v \frac{I_p}{N_p} - \tilde{\gamma} S_v + (1-\theta)\mu \\
\dot{I_v} &= \beta_v S_v \frac{I_p}{N_p} - \tilde{\gamma} I_v + \theta\mu
\end{aligned} \tag{1}
$$

donde β_p: tasa de infección de las plantas susceptibles mediante un vector infectado. r_1: tasa de replanteo de plantas latentes. r_2: tasa de replanteo de plantas infecciosas. b: tasa de latencia (planta latente se convierte en infecciosa). β_v: tasa de infección de los vectores susceptibles mediante una planta infectada.

Make a table for all parameters description

γ: tasa de muerte o alejamiento de los vectores, μ: migración de los vectores de plantas hospederas alternas, θ: proporción de migración de los vectores.

Redact this conservation law to the entire system (1). Write a introductory paragraph to Thm 1

Theorem 1 *With the notation of ODE* (1)*, let*

$$N_v(t) := S_v(t) + I_v(t)$$
$$N_v^{\infty} := \frac{\mu}{\gamma}.$$

Then for any initial condition $(S_p(0), L_p(0), I_p(0), S_v(0), I_v(0))^{\top} \in (0,\infty) \times (0, N_v^{\infty})$ *she plant and vector total populations respectively satisfies*

$$\frac{dN_p}{dt} = \frac{d}{dt}(S_p + L_p + I_p) = 0,$$
$$\lim_{t\to\infty} N_v(t) = N_v^{\infty}.$$

Why we want to normalize?

We use the following variable change to normalize (1):

$$x = \frac{S_p}{N_p}, \quad y = \frac{L_p}{N_p}, z = \frac{I_p}{N_p}, \quad v = \frac{I_p}{N_v}, w = \frac{I_v}{N_v}. \tag{2}$$

Then, deterministic system (1) becomes

$$\begin{aligned}
\dot{x} &= -\beta_p x w + \tilde{r_1} y + \tilde{r_2} z \\
\dot{y} &= \beta_p x w - (b + \tilde{r_1}) y \\
\dot{z} &= b y - \tilde{r_2} z \\
\dot{v} &= -\beta_v v z + (1 - \theta - v)\frac{\mu}{N_v} \\
\dot{w} &= \beta_v v z + (\theta - w)\frac{\mu}{N_v}
\end{aligned} \tag{3}$$

Following ideas from [referencia], we quantify uncertinity in replanting rate of plants, and died rate of vector, r_1, r_2 and γ, to this end, we perturb parameters $r_1 \dots$ whit a Winner process to obtain a stochastic differential equation(SDE). Here, the perturbation describe stochastic environmental noise on each population. In symbols $dB(t) = B(t + dt) - B(t)$ denotes the increment of a standard Wiener process, thus we perturb potentially replanting r_1, r_2, and vector death γ in the infinitiesimal time interval $[t, t + dt)$ by

write here the parameters

$$\begin{aligned}
r_1 dt &\rightsquigarrow r_1 dt + \sigma_L dB(t), \\
r_2 dt &\rightsquigarrow r_2 dt + \sigma_I dB(t), \\
\gamma dt &\rightsquigarrow \gamma dt + \sigma_v dB(t).
\end{aligned} \tag{4}$$

Is the same Brownian motion for the three equations?

Note that right hand side of (4) is a random perturbations of parameters r_1, r_2, γ, with mean $\mathbb{E}\left[r_1 dt + \sigma_L dB(t)\right]$ and variance $\mathrm{Var}\left[r_1 dt + \sigma_L dB(t)\right] = \sigma_L^2 dt$, $\mathbb{E}(\tilde{r}_2 dt) = r_2 dt$ and $Var(\tilde{r}_2 dt) = \sigma_I^2 dt$ and $\mathbb{E}(\tilde{\gamma} dt) = \gamma dt$ and $Var(\tilde{\gamma} dt) = \sigma_v^2 dt$.

Note that here we will use the latex proba package, plase use the same commands in the re-

Thus, we establish an stochastic extencion from deterministic tomato model (1) by the Itô SDE

$$\begin{aligned}
dS_p &= \left(-\beta_p S_p \frac{I_v}{N_v} + r_1 L_p + r_2 I_p\right) dt + (\sigma_L L_p + \sigma_I I_p) dB(t) \\
dL_p &= \left(\beta_p S_p \frac{I_v}{N_v} - bL_p - r_1 L_p\right) dt - \sigma_L L_p dB(t) \\
dI_p &= (bL_p - r_2 I_p)\, dt - \sigma_I I_p dB(t) \\
dS_v &= \left(-\beta_v S_v \frac{I_p}{N_p} - \gamma S_v + (1-\theta)\mu\right) dt - \sigma_v S_v dB(t) \\
dI_v &= \left(\beta_v S_v \frac{I_p}{N_p} - \gamma I_v + \theta\mu\right) dt - \sigma_v I_v dB(t).
\end{aligned} \tag{5}$$

Applyng the change of variable (2) to system (5) results

$$\begin{aligned}
dx(t) &= (-\beta_p xw + r_1 y + r_2 z)dt + (\sigma_L y + \sigma_I z)dB(t) \\
dy(t) &= (\beta_p xw - (b + r_1)y)dt - \sigma_L y dB(t) \\
dz(t) &= (by - r_2 z)dt - \sigma_I z dB(t) \\
dv(t) &= \left(-\beta_v vz + (1-\theta-v)\frac{\mu}{N_v}\right) dt \\
dw(t) &= \left(\beta_v vz + (\theta - w)\frac{\mu}{N_v}\right) dt
\end{aligned} \tag{6}$$

2 Existence of unique positive solution

Theorem 2 *For any initial values* $(L_p(0), I_p(0), S_v(0), I_v(0)) \in (0, N_p)\times(0, N_p)$ $\times(0, N_v)\times(0, N_v)$, *the SDE (5) has a unique global positive solution* $(L_p(t), I_p(t)$ $, S_v(t), I_v(t)) \in (0, N_p) \times (0, N_p) \times (0, N_v) \times (0, N_v)$ *for all* $t \geq 0$ *with probability one, namely,*

$$\mathbb{P}[(L_p(t), I_p(t), S_v(t), I_v(t)) \in (0, N_p) \times (0, N_p) \times (0, N_v) \times (0, N_v)\ \forall t \geq 0] = 1.$$

Proof.

■

3 Extinction of the disease

Our analysis needs the following function and conditions.

$$V(S_p, L_p, I_p, S_v, I_v) = \left(S_p - S_p^0 - S_p^0 \ln\left(\frac{S_p}{S_p^0}\right)\right) + L_p + I_p + \frac{\beta_p N_p}{\gamma N_v} I_v \tag{7}$$

We consider in the system (5) that $r = r_1 = r_2$ and $\sigma = \sigma_L = \sigma_I$. These conditions provided us with the necessary factors for the construction of the following stochastic basic reproductive number. (posiblemente umbral),

$$\mathcal{R}_0^s = \frac{\beta_p\beta_v}{\gamma r} \tag{8}$$

Theorem 3 *Let $(S_p(t), L_p(t), I_p(t), I_v(t))$ be the solution of (5) with initial values $(S_p(0), L_p(0), I_p(0), I_v(0)) \in (0, N_p) \times (0, N_p) \times (0, N_p)$ $\times (0, N_v)$. If $\mathcal{R}_0^s < 1$, then the following conditions holds*

$$\lim_{t\to\infty} \frac{1}{t}\mathbb{E}\int_0^t \left[r[\mathcal{R}_0^s - 1]I_p - rS_p\left(1 - \frac{S_p^0}{S_p}\right)^2 - rL_p - \frac{\beta_p\beta_v}{\gamma}I_vI_p \right] dr \leq \frac{1}{2}\sigma^2 N_p, \ a.s.,$$

namely, the infected individual tends to zero exponentially a.s, i.e the disease will die out with probability one.

Proof. The main idea is to propose a Lyapunov function $V : \mathbb{R}^4 \to \mathbb{R}_+$ defined as

$$V(S_p, L_p, I_p, I_v) = \left(S_p - S_p^0 - S_p^0 \ln\frac{S_p}{S_p^0}\right) + L_p + I_p + \frac{\beta_p N_p}{\gamma N_v^\infty} I_v,$$

and to verify the hypotheses of Khasminskii Theorem. To apply Itô formula we proceed as

$$\begin{aligned}
V_x f &= \left(1 - \frac{S_p^0}{S_p}\right)\left(-\frac{\beta_p}{N_v^\infty}S_pI_v + rN_p - rS_p\right) + \frac{\beta_p}{N_v^\infty}S_pI_v - (b+r)L_p \\
&+ bL_p - rI_p + \frac{\beta_p N_p}{\gamma N_v^\infty}\left(\frac{\beta_v N_v}{N_p}I_p - \frac{\beta_v}{N_v^\infty}I_vI_p - \gamma I_v\right) \\
&= -rS_p\left(1 - \frac{S_p^0}{S_p}\right)^2 - \frac{\beta_p}{N_v^\infty}S_pI_v + \frac{\beta_p}{N_v^\infty}I_vS_p^0 + \frac{\beta_p}{N_v^\infty}S_pI_v - r(L_p + I_p) \\
&+ \frac{\beta_p N_p}{\gamma N_v^\infty}\left(\frac{\beta_v N_v}{N_p}I_p - \frac{\beta_v}{N_v^\infty}I_vI_p - \gamma I_v\right) \\
&= -rS_p\left(1 - \frac{S_p^0}{S_p}\right)^2 + \frac{\beta_p}{N_v^\infty}I_vS_p^0 - r(L_p + I_p) + \frac{\beta_p N_p}{\gamma N_v^\infty}\frac{\beta_v N_v}{N_p}I_p \\
&- \frac{\beta_p N_p}{\gamma N_v^\infty}\frac{\beta_v}{N_v^\infty}I_vI_p - \frac{\beta_p N_p}{\gamma N_v^\infty}\gamma I_v
\end{aligned}$$

Then,

$$
\begin{aligned}
V_x f &= -rS_p\left(1-\frac{S_p^0}{S_p}\right)^2 - \left[\frac{\beta_p N_p}{\gamma N_v^\infty}\gamma - \frac{\beta_p N_p}{N_v^\infty}\right] I_v + \left[\frac{\beta_p N_p}{\gamma N_v^\infty}\beta_v \frac{N_v^\infty}{N_p} - r\right] I_p \\
&\quad - rL_p - \frac{\beta_p N_p}{\gamma N_v^\infty}\frac{\beta_v}{N_p} I_v I_p \\
&= -rS_p\left(1-\frac{S_p^0}{S_p}\right)^2 + \left[\frac{\beta_p N_p}{\gamma N_v^\infty}\beta_v \frac{N_v^\infty}{N_p} - r\right] I_p - rL_p - \frac{\beta_p N_p}{\gamma N_v^\infty}\frac{\beta_v}{N_p} I_v I_p \\
&= -rS_p\left(1-\frac{S_p^0}{S_p}\right)^2 + r\left[\frac{\beta_p \beta_v}{\gamma r} - 1\right] I_p - rL_p - \frac{\beta_p \beta_v}{\gamma N_v^\infty} I_v I_p.
\end{aligned}
$$

Expressing the right hand side of above equation in term of the basic reproductive number, $\mathcal{R}_0^s$ we get

$$
V_x f = -rS_p\left(1-\frac{S_p^0}{S_p}\right)^2 + r\left[\mathcal{R}_0^s - 1\right] I_p - rL_p - \frac{\beta_p \beta_v}{\gamma N_v^\infty} I_v I_p.
$$

Moreover,

$$
\begin{aligned}
\frac{1}{2} trace(g^T V_{xx} g) &= \frac{1}{2}\sigma^2 N_p \left(\frac{N_p - S_p}{S_p}\right)^2 \\
&\leq \frac{1}{2}\sigma^2 N_p.
\end{aligned}
$$

The stochastic terms are not neccesary, because they do a martingale procces and therefore, when we use integral and expectation they vanising.

Incorporation all terms calculate above, we obtain

$$
\begin{aligned}
dV(X) &= -rS_p\left(1-\frac{S_p^0}{S_p}\right)^2 + r\left[\mathcal{R}_0^s - 1\right] I_p - rL_p - \frac{\beta_p \beta_v}{\gamma N_v^\infty} I_v I_p + \frac{1}{2}\sigma^2 N_p \left(\frac{N_p - S_p}{S_p}\right)^2 \\
&\leq -rS_p\left(1-\frac{S_p^0}{S_p}\right)^2 + r\left[\mathcal{R}_0^s - 1\right] I_p - rL_p - \frac{\beta_p \beta_v}{\gamma N_v^\infty} I_v I_p + \frac{1}{2}\sigma^2 N_p.
\end{aligned}
$$

Define $LV(X)$ as

$$
LV(X) = -rS_p\left(1-\frac{S_p^0}{S_p}\right)^2 + r\left[\mathcal{R}_0^s - 1\right] I_p - rL_p - \frac{\beta_p \beta_v}{\gamma N_v^\infty} I_v I_p + \frac{1}{2}\sigma^2 N_p.
$$

Using Itô's formula and integrating dV from 0 to t as well as taking expectation yield the following

$$0 \le \mathbb{E}V(t) - \mathbb{E}V(0) \le \mathbb{E}\int_0^t LV(X(s))ds$$
$$\le -\mathbb{E}\int_0^t \left[rS_p \left(1 - \frac{S_p^0}{S_p}\right)^2 - r\left[\mathcal{R}_0^s - 1\right] I_p + rL_p + \frac{\beta_p \beta_v}{\gamma N_v^\infty} I_v I_p \right] ds + \frac{1}{2}\sigma^2 N_p$$

Therefore,

$$\lim_{t\to\infty} \frac{1}{t}\mathbb{E}\int_0^t \left[-rS_p \left(1 - \frac{S_p^0}{S_p}\right)^2 + r\left[\mathcal{R}_0^s - 1\right] I_p - rL_p - \frac{\beta_p \beta_v}{\gamma N_v^\infty} I_v I_p \right] ds \le \frac{1}{2}\sigma^2 N_p.$$

■

4 Persistence

Theorem 4 *Let $(S_p(t), L_p(t), I_p(t), I_v(t))$ be the solution of (5) with initial values $(S_p(0), L_p(0), I_p(0), I_v(0)) \in (0, N_p) \times (0, N_p) \times (0, N_p) \times (0, N_v)$. If $\mathcal{R}_0^s > 1$, then the system (5) is globally asymptotically stable at endemic equilibrium point if*

$$\lim_{t\to\infty} \frac{1}{t}\mathbb{E}\int_0^t \left[\frac{rS_p^*}{S_p S_p^*}(S_p^* - S_p)^2 + \frac{\beta_p}{N_v} S_p^* I_v^* A_1 + \frac{\beta_v}{N_p}\frac{I_p}{I_v}(I_v - I_v^*)^2 + \gamma I_v^* A_2 \right] dr \le A_3.$$

namely, the disease will persist with probability one.

Proof. Let us define the following Lyapunov function $V : \mathbb{R}_+^4 \to \mathbb{R}_+$

$$\begin{aligned} V(S_p, L_P, I_p, I_v) &= (S_p + L_p + I_p + I_v) - (S_p^* + L_p^* + I_p^* + I_v^*) \\ &\quad - \left(S_p^* \ln \frac{S_p}{S_p^*} + L_p^* \ln \frac{L_p}{L_p^*} + I_p^* \ln \frac{I_p}{I_p^*} + I_v^* \ln \frac{I_v}{I_v^*} \right). \end{aligned}$$

Computing the Itô formula terms as:

$$\begin{aligned} V_x f &= \left(1 - \frac{S_p^*}{S_p}\right)\left(rN_p - \beta_p S_p \frac{I_v}{N_v^\infty} - rS_p\right) + \left(1 - \frac{L_p^*}{L_p}\right)\left(\beta_p S_p \frac{I_v}{N_v^\infty} - (r+b)L_p\right) \\ &\quad + \left(1 - \frac{I_p^*}{I_p}\right)(bL_p - rI_p) + \left(1 - \frac{I_v^*}{I_v}\right)\left(\beta_v N_v \frac{I_p}{N_p} - \beta_v \frac{I_p}{N_p} I_v - \gamma I_v\right). \end{aligned}$$

The system (5) satisfy the following relations at equilibrium point

$$\begin{aligned}
rN_p &= \beta_p S_p^* \frac{I_v^*}{N_v^\infty} + rS_p^* \\
(r+b) &= \beta_p S_p^* \frac{I_v^*}{L_p^* N_v^\infty} \\
r &= b\frac{L_p^*}{I_p^*} \\
\beta_v \frac{N_v}{N_p} &= \frac{\beta_v}{N_p} I_v^* + \gamma \frac{I_v^*}{I_p^*}
\end{aligned}$$

Moreover,

$$\begin{aligned}
V_x f = &\left(1 - \frac{S_p^*}{S_p}\right)\left(\beta_p S_p^* \frac{I_v^*}{N_v^\infty} + rS_p^* - \beta_p S_p \frac{I_v}{N_v^\infty} - rS_p\right) \\
&+ \left(1 - \frac{L_p^*}{L_p}\right)\left(\beta_p S_p \frac{I_v}{N_v^\infty} - \beta_p S_p^* \frac{I_v^*}{L_p^* N_v^\infty} L_p\right) + \left(1 - \frac{I_p^*}{I_p}\right)\left(bL_p - b\frac{L_p^*}{I_p^*} I_p\right) \\
&+ \left(1 - \frac{I_v^*}{I_v}\right)\left(\frac{\beta_v}{N_p} I_v^* I_p + \gamma \frac{I_v^*}{I_p^*} I_p - \beta_v \frac{I_p}{N_p} I_v - \gamma I_v\right) \\
&= rS_p^* \left(1 - \frac{S_p^*}{S_p}\right)\left(1 - \frac{S_p}{S_p^*}\right) + \frac{\beta_p}{N_v^\infty} S_p^* I_v^* \left(1 - \frac{S_p^*}{S_p}\right)\left(1 - \frac{S_p I_v}{S_p^* I_v^*}\right) \\
&+ \frac{\beta_p}{N_v^\infty} S_p^* I_v^* \left(1 - \frac{L_p^*}{L_p}\right)\left(\frac{S_p I_v}{S_p^* I_v^*} - \frac{L_p}{L_p^*}\right) + bL_p^* \left(1 - \frac{I_p^*}{I_p}\right)\left(\frac{L_p}{L_p^*} - \frac{I_p}{I_p^*}\right) \\
&+ \left(1 - \frac{I_v^*}{I_v}\right)\left(-\frac{\beta_v}{N_p} I_v I_p \left(1 - \frac{I_v^*}{I_v}\right) + \gamma I_v^* \left(\frac{I_p}{I_p^*} - \frac{I_v}{I_v^*}\right)\right) \\
&= rS_p^* \left(2 - \frac{S_p}{S_p^*} - \frac{S_p^*}{S_p}\right) + \frac{\beta_p}{N_v^\infty} S_p^* I_v^* \left(1 - \frac{I_v}{I_v^*}\left(\frac{S_p}{S_p^*} - 1\right) - \frac{S_p^*}{S_p}\right) \\
&+ \frac{\beta_p}{N_v^\infty} S_p^* I_v^* \left(\frac{S_p I_v}{S_p^* I_v^*}\left(1 - \frac{L_p^*}{L_p}\right) - \frac{L_p}{L_p^*}\left(1 - \frac{L_p^*}{L_p}\right)\right) + bL_p^* \left(1 + \frac{L_p}{L_p^*} - \frac{I_p}{I_p^*} - \frac{I_p^* L_p}{I_p L_p^*}\right) \\
&- \frac{\beta_v}{N_v} I_v I_p \left(1 - \frac{I_v^*}{I_v}\right)^2 + \gamma I_v^* \left(\frac{I_p}{I_p^*} - \frac{I_v^* I_p}{I_v I_p^*} - \frac{I_v}{I_v^*} + 1\right) \\
&= rS_p^* \left(2 - \frac{S_p}{S_p^*} - \frac{S_p^*}{S_p}\right) + \frac{\beta_p}{N_v^\infty} S_p^* I_v^* \left(1 - \frac{S_p^*}{S_p} - \frac{I_v}{I_v^*}\left(\frac{S_p}{S_p^*} - 1\right)\right) \\
&+ \frac{\beta_p}{N_v^\infty} S_p^* I_v^* \left(1 - \frac{L_p}{L_p^*} - \frac{S_p I_v}{S_p^* I_v^*}\left(\frac{L_p^*}{L_p} - 1\right)\right) + bL_p^* \left(1 - \frac{I_p}{I_p^*} + \frac{L_p}{L_p^*}\left(1 - \frac{I_p^*}{I_p}\right)\right) \\
&- \frac{\beta_v}{N_v^\infty} I_v I_p \left(1 - \frac{I_v^*}{I_v}\right)^2 + \gamma I_v^* \left(1 - \frac{I_v}{I_v^*} - \frac{I_p}{I_p^*}\left(\frac{I_v^*}{I_v} - 1\right)\right).
\end{aligned}$$

Then

$$
\begin{aligned}
V_x f &= rS_p^* \left(2 - \frac{S_p}{S_p^*} - \frac{S_p^*}{S_p}\right) + \frac{\beta_p}{N_v^\infty} S_p^* I_v^* \left(2 - \frac{S_p^*}{S_p} - \frac{L_p}{L_p^*} - \frac{I_v}{I_v^*}\left(\frac{S_p L_p^*}{S_p^* L_p} - 1\right)\right) \\
&+ bL_p^* \left(1 - \frac{I_p}{I_p^*} + \frac{L_p}{L_p^*}\left(1 - \frac{I_p^*}{I_p}\right)\right) - \frac{\beta_v}{N_p} I_v I_p \left(1 - \frac{I_v^*}{I_v}\right)^2 \\
&+ \gamma I_v^* \left(1 - \frac{I_v}{I_v^*} - \frac{I_p}{I_p^*}\left(\frac{I_v^*}{I_v} - 1\right)\right).
\end{aligned}
$$

Now we need compute the term $g^T V_{xx} g$,

$$
g^T V_{xx} g = \begin{bmatrix} \sigma^2 \left(\frac{N_p - S_p}{S_p}\right)^2 S_p^* + \sigma^2 L_p^* & 0 \\ 0 & I_p^* \sigma^2 + I_v^* \sigma_v^2 \end{bmatrix}
$$

therefore,

$$
\begin{aligned}
\frac{1}{2} trace(g^T V_{xx} g) &= \frac{1}{2}\left(\sigma^2 \left(\frac{N_p - S_p}{S_p}\right)^2 S_p^* + \sigma^2 L_p^* + \sigma^2 I_p^* + \sigma_v^2 I_v^*\right) \\
&\leq \frac{1}{2}\left(\sigma^2 S_p^* + \sigma^2 L_p^* + \sigma^2 I_p^* + \sigma_v^2 I_v^*\right)
\end{aligned}
$$

The stochastics terms are not neccesary, because the are a martingale and therefore, when we use integrating and expectation they vanishing, obtaining the following $LV(X)$ operator

$$
\begin{aligned}
LV(X) &= -rS_p^* \frac{\left(S_p^* - S_p\right)^2}{S_p S_p^*} - \frac{\beta_p}{N_v^\infty} S_p^* I_v^* A_1 - bL_p^* A_2 - \frac{\beta_v}{N_p} I_v I_p \left(1 - \frac{I_v^*}{I_v}\right)^2 \\
&- \gamma I_v^* A_3 + A_4.
\end{aligned}
$$

where

$$
\begin{aligned}
A_1 &= \left(\frac{S_p^*}{S_p} + \frac{L_p}{L_p^*} + \frac{I_v}{I_v^*}\left(\frac{S_p L_p^*}{S_p^* L_p} - 1\right) - 2\right) > 0, \\
A_2 &= \left(\frac{I_p}{I_p^*} - \frac{L_p}{L_p^*}\left(1 - \frac{I_p^*}{I_p}\right) - 1\right) > 0, \\
A_3 &= \left(\frac{I_v}{I_v^*} + \frac{I_p}{I_p^*}\left(\frac{I_v^*}{I_v} - 1\right) - 1\right) > 0, \\
A_4 &= \frac{1}{2}\left(\sigma^2 S_p^* + \sigma^2 L_p^* + \sigma^2 I_p^* + \sigma_v^2 I_v^*\right) > 0.
\end{aligned}
$$

Applying Itô formula, integrating dV from 0 to t and taking expectation gives the following

$$
\begin{aligned}
0 \le \mathbb{E}V(t) - \mathbb{E}V(0) = \mathbb{E}\int_0^t LV(s)ds \\
- \mathbb{E}\int_0^t \left(rS_p^* \frac{\left(S_p^* - S_p\right)^2}{S_p S_p^*} + \frac{\beta_p}{N_v^\infty} S_p^* I_v^* A_1 + bL_p^* A_2 + \frac{\beta_v}{N_p} I_v I_p \left(1 - \frac{I_v^*}{I_v}\right)^2 + \gamma I_v^* A_3 \right) ds \\
+ A_4 t.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\lim_{t\to\infty} \frac{1}{t}\mathbb{E}\int_0^t \left(rS_p^* \frac{\left(S_p^* - S_p\right)^2}{S_p S_p^*} + \frac{\beta_p}{N_v^\infty} S_p^* I_v^* A_1 + bL_p^* A_2 + \frac{\beta_v}{N_p} I_v I_p \left(1 - \frac{I_v^*}{I_v}\right)^2 + \gamma I_v^* A_3 \right) ds \\
\le A_4.
\end{aligned}
$$

■