Stochastic-Tomato-Vector-Plant-Disease

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August 4, 2020

Abstract

Gabriel: Aqui anexa Los paquetes y contenido de lo que llevas escrito. Si necesitas carpetas agregalas. Tabien sube el archivo bib y las figuras en extencion eps de la simulaciones del modelo determinista que estamos perturbando.

1 Deterministic base dynamics

translate this section

En esta sección, vamos a definir el modelo básico que trabajaremos, consideraremos que las plantas se dividen en tres tipos: plantas susceptibles, latentes e infectadas. Las moscas blancas, las cuales llamaremos vectores, se dividen en susceptibles e infectadas.

Las plantas susceptibles pasan a ser plantas latentes cuando un vector infectado se alimenta de ella a una tasa de β_p, continuando el proceso cuando las plantas latentes se convierten en plantas infectadas a una tasa de b, en cada uno de estos casos consideraremos que estaremos revisando los cultivos para el cual removeremos plantas latentes e infectadas si se detecta que dicha planta esta infectada a una tasa de r_1 y r_2 respectivamente.

Plants become latent by infected vectors, replanting latent and infected plants, latent plants become infectious plants, vectors become infected by infected plants, vectors die or depart per day, immigration from alternative hosts.

$$
\begin{aligned}
\dot{S_p} &= -\beta_p S_p \frac{I_v}{N_v} + \tilde{r_1} L_p + \tilde{r_2} I_p \\
\dot{L_p} &= \beta_p S_p \frac{I_v}{N_v} - b L_p - \tilde{r_1} L_p \\
\dot{I_p} &= b L_p - \tilde{r_2} I_p \\
\dot{S_v} &= -\beta_v S_v \frac{I_p}{N_p} - \tilde{\gamma} S_v + (1-\theta)\mu \\
\dot{I_v} &= \beta_v S_v \frac{I_p}{N_p} - \tilde{\gamma} I_v + \theta\mu
\end{aligned} \tag{1}
$$

donde β_p: tasa de infección de las plantas susceptibles mediante un vector infectado. r_1: tasa de replanteo de plantas latentes. r_2: tasa de replanteo de plantas infecciosas. b: tasa de latencia (planta latente se convierte en infecciosa). β_v: tasa de infección de los vectores susceptibles mediante una planta infectada.

Make a table for description of all parameters

γ: tasa de muerte o alejamiento de los vectores, μ: migración de los vectores de plantas hospederas alternas, θ: proporción de migración de los vectores.

Redact this conservation law to the entire system (1). Write a introductory paragraph to Thm 1

Theorem 1 *With the notation of ODE* (1)*, let*

$$N_v(t) := S_v(t) + I_v(t)$$
$$N_v^{\infty} := \frac{\mu}{\gamma}.$$

Then for any initial condition $(S_p(0), L_p(0), I_p(0), S_v(0), I_v(0))^{\top} \in (0, \infty) \times (0, N_v^{\infty})$ *she plant and vector total populations respectively satisfies*

$$\frac{dN_p}{dt} = \frac{d}{dt}(S_p + L_p + I_p) = 0,$$
$$\lim_{t \to \infty} N_v(t) = N_v^{\infty}.$$

Why we want to normalize?

We use the following variable change to normalize (1):

$$x = \frac{S_p}{N_p}, \quad y = \frac{L_p}{N_p}, z = \frac{I_p}{N_p}, \quad v = \frac{I_p}{N_v}, w = \frac{I_v}{N_v}. \tag{2}$$

Then, deterministic system (1) becomes

$$\begin{aligned}
\dot{x} &= -\beta_p x w + \tilde{r_1} y + \tilde{r_2} z \\
\dot{y} &= \beta_p x w - (b + \tilde{r_1}) y \\
\dot{z} &= b y - \tilde{r_2} z \\
\dot{v} &= -\beta_v v z + (1 - \theta - v) \frac{\mu}{N_v} \\
\dot{w} &= \beta_v v z + (\theta - w) \frac{\mu}{N_v}
\end{aligned} \tag{3}$$

Following ideas from [referencia], we quantify uncertinity in replanting rate of plants, and died rate of vector, r_1, r_2 and γ, to this end, we perturb parameters $r_1 \ldots$ whit a Winner process to obtain a stochastic differential equation(SDE). Here, the perturbation describe stochastic environmental noise on each population. In symbols $dB(t) = B(t + dt) - B(t)$ denotes the increment of a standard Wiener process, thus we perturb potentially replanting r_1, r_2, and vector death γ in the infinitiesimal time interval $[t, t + dt)$ by

write here the parameters

$$\begin{aligned}
r_1 dt &\rightsquigarrow r_1 dt + \sigma_L dB(t), \\
r_2 dt &\rightsquigarrow r_2 dt + \sigma_I dB(t), \\
\gamma dt &\rightsquigarrow \gamma dt + \sigma_v dB(t).
\end{aligned} \tag{4}$$

I want to explore the perturbation:

$r_1 \rightsquigarrow r_1 dt + \sigma_p \frac{S_p}{N_p} dB_p(t)$

$r_2 \rightsquigarrow r_2 dt + \sigma_p \frac{S_p}{N_p} dB_p(t)$

$\gamma \rightsquigarrow \gamma dt + \sigma_v dB_v(t)$

$\leq$: $r_1 + \sigma_p dB_p$

Is the same Brownian motion for the three equations?

Note that right hand side of (4) is a random perturbations of parameters r_1, r_2, γ, with mean $\mathbb{E}\left[r_1 dt + \sigma_L dB(t)\right]$ and variance $\operatorname{Var}\left[r_1 dt + \sigma_L dB(t)\right] = \sigma_L^2 dt$, $\mathbb{E}(\tilde{r}_2 dt) = r_2 dt$ and $Var(\tilde{r}_2 dt) = \sigma_I^2 dt$ and $\mathbb{E}(\tilde{\gamma} dt) = \gamma dt$ and $Var(\tilde{\gamma} dt) = \sigma_v^2 dt$.

Note that here we will use the latex proba package, plase use the same commands in the re-

Thus, we establish an stochastic extencion from deterministic tomato model (1) by the Itô SDE, with the new perturbation

$$
\begin{aligned}
dS_p &= \left(-\beta_p S_p \frac{I_v}{N_v} + r_1 L_p + r_2 I_p\right) dt + \cancel{(\sigma_L L_p + \sigma_I I_p)dB(t)} \quad \boxed{\sigma_p \frac{S_p}{N_p}} \underbrace{(L_p + I_p)}_{(N_p - S_p)} dB_p(t) \\
dL_p &= \left(\beta_p S_p \frac{I_v}{N_v} - bL_p - r_1 L_p\right) dt - \cancel{\sigma_L L_p dB(t)} \quad \sigma_p \frac{S_p L_p}{N_p} dB_p(t) \\
dI_p &= (bL_p - r_2 I_p)\, dt - \cancel{\sigma_I I_p dB(t)} \quad \sigma_p \frac{S_p}{N_p} I_p dB_p(t) \qquad (5) \\
dS_v &= \left(-\beta_v S_v \frac{I_p}{N_p} - \gamma S_v + \underbrace{(1-\theta)}\mu\right) dt - \sigma_v S_v dB_v(t) \quad \leq \quad \mu - \beta_v S_v I_v / N_v - \gamma S_v \\
dI_v &= \left(\beta_v S_v \frac{I_p}{N_p} - \gamma I_v + \theta\mu\right) dt - \sigma_v I_v dB_v(t). \quad \leq \quad \mu + \beta_v S_v - \gamma I_v \longrightarrow \infty
\end{aligned}
$$

Applyng the change of variable (2) to system (5) results

$$
\begin{aligned}
dx(t) &= (-\beta_p xw + r_1 y + r_2 z)dt + (\sigma_L y + \sigma_I z)dB(t) \\
dy(t) &= (\beta_p xw - (b + r_1)y)dt - \sigma_L y dB(t) \\
dz(t) &= (by - r_2 z)dt - \sigma_I z dB(t) \\
dv(t) &= \left(-\beta_v vz + (1 - \theta - v)\frac{\mu}{N_v}\right) dt \\
dw(t) &= \left(\beta_v vz + (\theta - w)\frac{\mu}{N_v}\right) dt
\end{aligned} \qquad (6)
$$

2 Existence of unique positive solution

Thereom *.* of [Mao Book] assures ths existence of unique solution of (5) in a compact interval. Since we study asymptotic behaviour, we have to assure the existence of unique positive invariant solution to SDE (*). To this end, let $\mathbb{R}^n_+$ the first octant of $\mathbb{R}^n$ and consider

$$
\mathbf{E} := \left\{ (S_p, L_p, I_p, S_v, I_v)^\top \in \mathbb{R}^5_+ : \quad \leq S_p + L_p + I_p \geq N_p, \quad S_v + I_v \leq \frac{\mu}{\gamma} \right\},
$$

the following result prove that this set is positive invariant.

Theorem 2 *For any initial values* $(S_p(0), L_p(0), I_p(0), S_v(0), I_v(0)) \in \mathbf{E}$*, exists unique invariant global positive solution to SDE* (5) $(S_p(t), L_p(t), I_p(t), S_v(t), I_v(t))^\top$ *with probability one, that is,*

$$
\mathbb{P}\left[(L_p(t), I_p(t), S_v(t), I_v(t)) \in \mathbf{E}, \quad \forall t \geq 0\right] = 1.
$$

Proof. ■

3 Extinction of the disease

Our analysis needs the following hypotesis.

(H-1) According to SDE (5), replatin rates satisfies $r_1 = r_2 = r$.

(H-2) The replanting noise intesities are equal $\sigma_L = \sigma_I = \sigma$.

We define the repoductive nomber of our stochastic model in SDE (*) by

Define here the infinitesimal operator $\mathcal{L}$.

$$\mathcal{R}_0^s := \frac{\beta_p \beta_v}{\gamma r}. \tag{7}$$

As our deterministic base structure this paramenters summarizes the behavior of extinction and persistence according with a threshold.

Theorem 3 *Let $(S_p(t), L_p(t), I_p(t), I_v(t))$ be the solution of (5) with initial condition $(S_p(0), L_p(0), I_p(0), I_v(0)) \in \mathbf{E}$. If $0 \leq \mathcal{R}_0^s < 1$ then, infected individuals in SDE (*) tends to zero exponentially a.s, that is, the disease will extinguishes with probability one.*

Proof. The proof consitst verify the hypotheses of Khasminskii Theorem [*] for the Lyapunov function

$$V(S_p, L_p, I_p, S_v, I_v) = \left(S_p - S_p^0 - S_p^0 \ln\left(\frac{S_p}{S_p^0}\right)\right) N_p + L_p + I_p + \frac{\beta_p N_p}{\gamma N_v} I_v. \tag{8}$$

$S_v - N_v^0 - N_v^0 \ln(S_v / N_v^0)$

1 Let f, g respectively be the dirft and difussion of SDE(*). Applyng the inifinitesimal opreator $\mathcal{L}$ we have

In the following step apply the operator $\mathcal{L}$

$$\begin{aligned}
V_x f &= \left(1 - \frac{\cancel{S_p^0}\, N_p}{S_p}\right)\left(-\frac{\beta_p}{N_v^\infty} S_p I_v + r N_p - r S_p\right) + \left[\frac{\beta_p}{N_v^\infty} S_p I_v - (b + r) L_p \right.\\
&\quad \left. + b L_p - r I_p\right] + \frac{\beta_p N_p}{\gamma N_v^\infty}\left(\frac{\beta_v N_v}{N_p} I_p - \frac{\beta_v}{N_v^\infty} I_v I_p - \gamma I_v\right\} + \mu\theta \\
&= -r S_p \left(1 - \frac{S_p^0}{S_p}\right)^2 - \frac{\beta_p}{N_v^\infty} S_p I_v + \frac{\beta_p}{N_v^\infty} I_v S_p^0 + \frac{\beta_p}{N_v^\infty} S_p I_v - r(L_p + I_p) \\
&\quad + \frac{\beta_p N_p}{\gamma N_v^\infty}\left(\frac{\beta_v N_v}{N_p} I_p - \frac{\beta_v}{N_v^\infty} I_v I_p - \gamma I_v\right) \\
&= -r S_p \left(1 - \frac{S_p^0}{S_p}\right)^2 + \frac{\beta_p}{N_v^\infty} I_v S_p^0 - r(L_p + I_p) + \frac{\beta_p N_p}{\gamma N_v^\infty} \frac{\beta_v N_v}{N_p} I_p \\
&\quad - \frac{\beta_p N_p}{\gamma N_v^\infty} \frac{\beta_v}{N_v^\infty} I_v I_p - \frac{\beta_p N_p}{\gamma N_v^\infty} \gamma I_v.
\end{aligned}$$

Then,

① Define explicity f and g:

$$f := \begin{pmatrix} -\beta_p S_p \dfrac{I_v}{N_v^{\circledast}} + r_1 L_p + r_2 I_p \\ \beta_p S_p I_v / N_v^{\circledast} - b L_p - r_1 L_p \\ b L_p - r_2 L_p \\ -\beta_v S_v \dfrac{I_p}{N_p} - \gamma S_v + (1-\theta)\mu \\ \beta_v S_v I_p / N_p - \gamma I_v + \theta \mu \end{pmatrix}, \quad g := \begin{pmatrix} \frac{S_p}{N_p}(\sigma_L L_p + \sigma_I I_p), & 0 \\ -\sigma_p \frac{S_p L_p}{N_p}, & 0 \\ -\sigma_p \frac{S_p}{N_p} I_p & 0 \\ 0 & -\sigma_v S_v \\ 0 & -\sigma_v I_v \end{pmatrix}$$

Thus our model in SDE (⋆) can be rewritten as

$$\underbrace{\begin{pmatrix} dS_p \\ dL_p \\ dI_p \\ dS_v \\ dI_v \end{pmatrix}} = \begin{pmatrix} f_{S_p} \\ f_{L_p} \\ f_{I_p} \\ f_{S_v} \\ f_{I_v} \end{pmatrix} dt + \begin{bmatrix} \frac{S_p}{N_p}(\sigma_p L_p + \sigma_p I_p) & 0 \\ -\frac{\sigma_p S_p L_p}{N_p} & 0 \\ -\frac{\sigma_p S_p I_p}{N_p} & 0 \\ 0 & -\sigma_v S_v \\ 0 & -\sigma_v I_v \end{bmatrix} \begin{pmatrix} dB_p(t) \\ dB_v(t) \end{pmatrix}$$

$$dX_t = f(X_t)dt + g(X_t)dB_t, \qquad B_t = (B_p(t), B_v(t))^T$$

According to the Lyapunov function

$$V(X_t) := \left(S_p - N_p - N_p \ln\left(\frac{S_p}{N_p}\right)\right) + L_p + I_p + \frac{\beta_p N_p I_v}{\gamma N_v^{\circledast}}$$

$$\frac{\partial V}{\partial S_p} = 1 - \frac{N_p}{S_p}, \quad \frac{\partial V}{\partial L_p} = 1, \quad \frac{\partial V}{\partial I_p} = 1$$

$$\frac{\partial V}{\partial S_v} = 0, \quad \frac{\partial V}{\partial I_v} = \frac{\beta_p N_p}{\gamma N_v^{\circledast}}$$

Thus

$$V_x = \left(1 - \frac{N_p}{S_p}, 1, 1, 0, \frac{\beta_p N_p}{\gamma N_v^{\circledast}}\right)$$

$$V_{xx} = \left(\begin{array}{c|cccc} \frac{N_p}{S_p^2}, & 0, & 0, & 0, & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

then

$$g^T V_{xx} g = \begin{pmatrix} \frac{S_p(\sigma_p L_p + \sigma_p I_p)}{N_p} & -\frac{\sigma_L L_p S_p}{N_p} & -\frac{\sigma_p S_p I_p}{N_p} & 0 & 0 \\ 0 & 0 & 0 & -\sigma_v S_v & -\sigma_v I_v \end{pmatrix} \left(\begin{array}{c|c} \frac{N_p}{S_p^2} & 0 \\ \hline 0 & 0 \end{array}\right) \begin{pmatrix} \frac{\sigma_p S_p}{N_p}(L_p + I_p) & 0 \\ -\frac{\sigma_p S_p L_p}{N_p} & 0 \\ -\frac{\sigma_p S_p I_p}{N_p} & 0 \\ 0 & -\sigma_v S_v \\ 0 & -\sigma_v I_v \end{pmatrix}$$

$$= \begin{pmatrix} \frac{S_p}{N_p}(\sigma_L L_p + \sigma_I I_p)\frac{N_p}{S_p^2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{\sigma_p S_p}{N_p}(L_p + I_p) & 0 \\ -\frac{\sigma_p S_p L_p}{N_p} & 0 \\ -\frac{\sigma_p S_p I_p}{N_p} & 0 \\ 0 & -\sigma_v S_v \\ 0 & -\sigma_v I_v \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\cancel{S_p^2}}{\cancel{N_p^2}}(\sigma_L L_p + \sigma_I I_p)^2 \frac{\cancel{N_p}}{\cancel{S_p^2}} & 0 \\ 0 & 0 \end{pmatrix}.$$

Therefore.

$$\operatorname{trace}\left(g^T V_{xx} g\right) = \left(\sigma_p L_p + \sigma_p I_p\right)^2 \left(\frac{1}{N_p}\right)$$

$$\le \sigma_p^2 (N_p)^2 / N_p = \boxed{\sigma_p^2 N_p}.$$

Thus we have taken

$$0 \le \sigma_p \le \sqrt{\frac{1}{N_p}}$$

Calculating $\frac{\partial V}{\partial S_p} = \left(1 - \frac{N_p}{S_p}\right)$, $\frac{\partial V}{\partial L_p} =$

$$L^0 V[X_t] = \frac{\partial}{\partial t} + \sum_{k=1} f^{[k]} \frac{\partial}{\partial x^{[k]}} + \frac{1}{2} \sum_{k,l=1}^{5} \sum_{j=1}^{2} g^{[k,j]} g^{[l,j]} \frac{\partial^2}{\partial x^{[k]} \partial x^{[l]}}$$

we obtain

$$L^0 V(X_t) = \left(-\beta_p \frac{S_v I_p}{N_v^\infty} + r_1 L_p + r_2 I_p\right)\left(1 - \frac{N_p}{S_p}\right) +$$

$$\left(\beta_p \frac{S_v I_v}{N_v^\infty} - \cancel{b L_p} - r_1 L_p\right) + \cancel{b I_p} - r_2 I_p$$

$$\frac{\beta_p N_p}{\gamma N_v}\left(\frac{\beta_v S_v I_v}{N_v} - \gamma I_v + \theta\mu\right) + \frac{1}{2}\frac{\sigma_p (L_p + I_p)^2}{N_p} +$$

$$\le \left[\underbrace{(r_1 + r_2)}_{:= r} \overbrace{(L_p + I_p)}^{= N_p - S_p} - \beta_p \frac{S_v I_p}{N_v^\infty}\right]\left(1 - N_p/S_p\right)$$

$$\left(\beta_p \frac{S_v I_v}{N_v^\infty} - \hat{r}\,(L_p + I_p\ I_p)\right) +$$

$$\frac{\beta_p N_p}{N_v}\left(\frac{\beta_v S_v I_v}{N_v} - \gamma I_v + \theta\mu\right) + \frac{1}{2}\frac{\sigma_p (L_p + I_p)^2}{N_p} +$$

Let $\hat{r} := \min\{r_1, r_2\}$

$$= \underbrace{\left[r N_p - \frac{\beta_p S_p I_v}{N^\infty} - r S_p\right]\left(1 - N_p/S_p\right)}_{:= T_1} +$$

$$\left(\beta_p \frac{S_v I_v}{N_v^\infty} - \hat{r}_1 (L_p + I_p)\right) +$$

$$\frac{\beta_p N_p}{N_v}\left(\frac{\beta_v S_v I_v}{N_v} - \gamma I_v + \theta\mu\right) + \frac{1}{2}\frac{\sigma_p (L_p + I_p)^2}{N_p} \qquad (\star +)$$

On sake of clearness, we work term T_1. First we factor the therm rS_p, thus.

$$T_1 = \underbrace{\left[-r S_p\left(1 - \frac{N_p}{S_p}\right) - \frac{\beta_p S_p I_v}{N_v^\infty}\right]\left(1 - N_p/S_p\right)}_{\text{expanding.}} +$$

$$T_1 = -r S_p \left(1 - N_p/S_p\right)^2 - \frac{\beta_p S_p I_v}{N_{v^\infty}}\left(1 - N_p/S_p\right).$$

substituting in $(\star +)$

$$L^0[V] \le -r S_p \left(1 - N_p/S_p\right)^2 - \frac{\beta_p S_p I_v}{N_{v^\infty}}\left(1 - N_p/S_p\right) +$$

$$\left(\beta_p \frac{S_v I_v}{N_v^\infty} - \hat{r}(L_p + I_p)\right) +$$

$$\frac{\beta_p N_p}{N_v}\left(\frac{\beta_v S_v I_v}{N_v} - \gamma I_v + \theta\mu\right) + \frac{1}{2}\frac{\sigma_p (L_p + I_p)^2}{N_p}$$

$$
\mathcal{L}^0[V] \le -r S_p \left(1 - N_p / S_p\right)^2 - \cancel{\frac{\beta_p S_p I_v}{N_v^\infty}} + \frac{\beta_p \cancel{S_p} I_v}{N_v^\infty} \frac{N_p}{\cancel{S_p}} +
$$

$$
\left(\cancel{\frac{\beta_p S_v I_v}{N_v^\infty}} - \hat{r}\left(L_p + I_p\right) \right) +
$$

$$
\frac{\beta_p N_p}{N_v} \left(\frac{\beta_v S_v I_v}{N_v} - \gamma I_v + \theta\mu \right) + \frac{1}{2} \frac{\sigma_p \overbrace{\left(L_p + I_p\right)^2}^{\le N_p^2}}{N_p}
$$

$$
\le -r S_p \left(1 - N_p / S_p\right)^2 + \beta_p \frac{I_v}{N_v^\infty} N_p - \hat{r}\left(L_p + I_p\right) + \frac{\beta_p N_p}{\gamma N_v} \left(\frac{\beta_v S_v I_v}{N_v} - \gamma I_v \theta\mu \right) + \frac{1}{2} \sigma_p \frac{\left(N_p\right)^2}{\cancel{N_p}}
$$

THM Extinction.

Aug - 10 - 2020

$$\underbrace{V(S_p, L_p, I_v, S_v, I_v)}_{:=X} := S_p - N_p - N_p \log(S_p/N_p) + L_p + I_v + \frac{\beta_p N_p}{\gamma N_v} I_v$$
$$S_v - N_v - N_v \log(S_v/N_v)$$

$$LV^0[X] = \Big(\underbrace{-\frac{\beta_p S_p I_v}{N_v} + r_1 L_p + r_2 I_p}_{:=T_1}\Big)\Big(1 - \frac{N_p}{S_p}\Big) +$$
$$\frac{\beta_p S_p I_v}{N_v} - r_1 L_p - r_2 I_p + \Big(-\frac{\beta_v S_v I_p}{N_p} - \gamma S_v + (1-\theta)\mu\Big)\Big(1 - \frac{N_v}{S_v}\Big) \quad (1)$$
$$\Big(\frac{\beta_v S_v I_p}{N_p} - \gamma I_v + \theta\mu\Big)\frac{\beta_p N_p}{\gamma N_v} + \frac{1}{2}\big(\sigma_p(L_p + I_p)\big)^2 \frac{1}{N_p} + \frac{1}{2}\sigma_v^2 N_v$$

Working with term T_1. Let $\hat{r} := \min\{r_1, r_2\}$, $r := r_1 + r_2$

$$T_1 \leq \Big(-\beta_p \frac{S_p I_v}{N_v} + r\underbrace{(L_p + I_p)}_{= N_p - S_p}\Big)\Big(1 - \frac{N_p}{S_p}\Big)$$
$$= \Big(r N_p - \frac{\beta_p S_p I_v}{N_v} - r S_p\Big)\Big(1 - N_p/S_p\Big)$$

Factorizing,

$$T_1 \leq \Big[-r S_p\Big(1 - \frac{N_p}{S_p}\Big) - \beta_p \frac{S_p I_v}{N_v}\Big]\Big(1 - N_p/S_p\Big)$$
$$= -r S_p (1 - N_p/S_p)^2 - \beta_p \frac{S_p I_v}{N_v}\Big(1 - \frac{N_p}{S_p}\Big)$$
$$= -r S_p (1 - N_p/S_p)^2 - \frac{\beta_p S_p I_v}{N_v} + \frac{\beta_p N_p \cancel{S_p} I_v}{N_v \cancel{S_p}}$$

$T_1 \leq -r S_p (1 - N_p/S_p)^2 - \frac{\beta_p S_p I_v}{N_v} + \beta_p \frac{N_p}{N_v} I_v$. Substituting this relation into (1)

$$LV^0 \leq -r S_p (1 - N_p/S_p)^2 - \frac{\beta_p S_p I_v}{N_v} + \beta_p \frac{N_p}{N_v} I_v +$$
$$\frac{\beta_p S_p I_v}{N_v} - r_1 L_p - r_2 I_p + \underbrace{\Big(-\frac{\beta_v S_v I_p}{N_p} - \gamma S_v + (1-\theta)\mu\Big)\Big(1 - \frac{N_v}{S_v}\Big)}_{:=T_2}$$
$$\Big(\frac{\beta_v S_v I_p}{N_p} - \gamma I_v + \underset{\leq \theta\gamma N_v}{\theta\mu}\Big)\frac{\beta_p N_p}{\gamma N_v} + \frac{1}{2}\big(\sigma_p(L_p + I_p)\big)^2 \frac{1}{N_p} + \frac{1}{2}\sigma_v^2 N_v \quad (3)$$

$$\boxed{\text{since } \underset{\leq}{(1-\theta)\mu} \leq \gamma N_v \quad (2)}$$
$N_v = \mu/\gamma$

Working T_2. Applying (2) we obtain,

$$T_2 \leq \Big(-\frac{\beta_v S_v I_p}{N_p} - \gamma S_v + \gamma N_v\Big)\Big(1 - \frac{N_v}{S_v}\Big)$$

Factorizing.

$$= \Big[-\gamma S_v\Big(1 - N_v/S_v\Big) - \frac{\beta_v S_v I_p}{N_p}\Big]\Big(1 - N_v/S_v\Big)$$

$$= -\gamma S_v (1 - N_v/S_v)^2 - \frac{\beta_v S_v I_p}{N_p} + \frac{\beta_v I_p N_v}{N_p} \quad (4).$$ Substituting (4) into (3) results,

$$LV^0 \le -\hat{r}S_p\left(1-N_p/S_p\right)^2 - \cancel{\beta_p S_p \frac{I_v}{N_v}} + \cancel{\frac{\beta_p N_p I_v}{N_v}} + \cancel{\frac{\beta_p S_p I_v}{N_v}} - \hat{r}\left(L_p + \bar{I}_p\right)$$

$S_v + \bar{I}_v \le N_v^\infty$

$S_v \le N_v^\infty - \bar{I}_v$

5

$$-\gamma S_v\left(1-N_v/S_v\right)^2 - \frac{\beta_v S_v \bar{I}_p}{N_p} + \frac{\beta_v \bar{I}_p N_v}{N_p} +$$

$$\left(\frac{\beta_v S_v \bar{I}_p}{N_p} - \cancel{\gamma \bar{I}_v} + \gamma N_v\right)\frac{\beta_p N_p}{\gamma N_v} + \underbrace{\frac{1}{2}\left(\sigma_p(L_p + \bar{I}_p)\right)^2 \frac{1}{N_p} + \frac{1}{2}\sigma_v^2 N_v}_{\le \hat{\sigma}_{pv}}$$

Factorizing respect to I_v,

$N_v = \frac{\mu}{\gamma}$

$$L_v^0 \le -\hat{r}_p S_p\left(1-N_p/S_p\right)^2 + \left(\frac{\beta_p N_p}{N_v} - \cancel{\frac{\gamma\beta_p N_p}{\gamma N_v}}\right)^{\to 0} I_v + \frac{\beta_v \beta_p N_v}{N_p \gamma N_v} I_p$$

Since $R_0^s := \frac{\beta_p \beta_v}{\gamma \hat{r}}$, define.

$$- I_v \beta_v \frac{I_p}{N_p}\frac{\beta_v N_p}{\gamma N_v} \underbrace{+ \theta_\gamma \frac{\beta_p N_p}{\gamma N_v} + \hat{\sigma}_{pv}}_{= \sigma_{pv}} - \frac{\beta_v S_v}{N_p} I_p$$

$$\sigma_{pv} := \theta \frac{\beta_p N_p}{\gamma N_v} + \hat{\sigma}_{pv}$$

$$-\gamma S_v\left(1-N_v/S_v\right)^2$$

$$= \hat{r} I_p \left(1 - \overbrace{\frac{\beta_v \beta_p}{\gamma \hat{r}}}^{1-R_0^s}\right)$$

$$\le -\hat{r}_p S_p\left(1-N_p/S_p\right)^2 - \gamma S_v\left(1-N_v/S_v\right)^2 - I_p\left(-\frac{\beta_v \beta_p}{\gamma} + \hat{r}\right)$$

$$-\hat{r} L_p - \frac{\beta_p \beta_v}{\gamma N_v} I_p I_v + \sigma_{pv} - \frac{\beta_v S_v I_p}{N_p}$$

Let $0 < \theta << 1$

$$\le -\hat{r}_p S_p\left(1-\frac{N_p}{S_p}\right)^2 - \gamma S_v\left(1-\frac{N_v}{S_v}\right)^2 - \hat{r}\left(1-R_0^s\right) I_p$$

$$-\hat{r} L_p - \frac{\beta_v \beta_p}{\gamma N_v} I_p I_v - \frac{\beta_v S_v I_p}{N_p} + \underbrace{\theta\beta_v N_p + \frac{1}{2}\sigma_p^2 N_p + \frac{1}{2}\sigma_v^2 N_v}_{C_\sigma > 0,}$$

$\beta_v N_p > 1$

$\theta \le \frac{1}{\beta_v N_p}$

Since $V(X) \ge 0$, we see that

$$0 \le \mathbb{E}\, V(X) = V_0 + \mathbb{E}\int_0^t LV(X(s))\,ds$$

$$0 \le \frac{1}{t}\mathbb{E}\left(\hat{\gamma}_p S_p\cancel{\left(1-\frac{N_p}{S_p}\right)^2} + \gamma S_v \cancel{\left(1-\frac{N_v}{S_v}\right)^2} + \gamma\cancel{\left(1-R_0^s\right) I_p}\right.$$ $t > 0$

$$\left.\cancel{\hat{r} L_p} + \cancel{\frac{\beta_v \beta_p}{\gamma N_v} I_p I_v}^{\to 0} + \frac{\beta_v S_p I_v}{N_p}\right) \le C_\sigma,$$

Let θ σ_v, σ_p such that $0 < C_\sigma < 1$

Then

$$\lim_{t\to\infty} \mathbb{E}\left(t \dots\right) \le C_\sigma < 1$$

$$\lim_{t\to\infty} \frac{1}{t}\log\left(\mathbb{E}\left(\dots\right)\right) \le -\hat{C}_\sigma \quad \Rightarrow \lim_{t\to\infty}\mathbb{E}\left(\dots\right) \le \lim_{t\to\infty} e^{-\hat{C}_\sigma t} = 0.$$

Thus. $S_p \to N_p$, $S_v \to N_v$, $I_p \to 0$, $L_p \to 0$, $I_v \to 0$ exponentially a.s.

$= 2 =$

$$
\begin{aligned}
V_x f &= -rS_p\left(1-\frac{S_p^0}{S_p}\right)^2 - \left[\frac{\beta_p N_p}{\gamma N_v^\infty}\gamma - \frac{\beta_p N_p}{N_v^\infty}\right] I_v + \left[\frac{\beta_p N_p}{\gamma N_v^\infty}\beta_v \frac{N_v^\infty}{N_p} - r\right] I_p \\
&\quad - rL_p - \frac{\beta_p N_p}{\gamma N_v^\infty}\frac{\beta_v}{N_p} I_v I_p \\
&= -rS_p\left(1-\frac{S_p^0}{S_p}\right)^2 + \left[\frac{\beta_p N_p}{\gamma N_v^\infty}\beta_v \frac{N_v^\infty}{N_p} - r\right] I_p - rL_p - \frac{\beta_p N_p}{\gamma N_v^\infty}\frac{\beta_v}{N_p} I_v I_p \\
&= -rS_p\left(1-\frac{S_p^0}{S_p}\right)^2 + r\left[\frac{\beta_p\beta_v}{\gamma r} - 1\right] I_p - rL_p - \frac{\beta_p\beta_v}{\gamma N_v^\infty} I_v I_p.
\end{aligned}
$$

Substitutin $\mathcal{R}_0^s$ in right hand side of above relation we get

$$
V_x f = -rS_p\left(1-\frac{S_p^0}{S_p}\right)^2 + r\left[\mathcal{R}_0^s - 1\right] I_p - rL_p - \frac{\beta_p\beta_v}{\gamma N_v^\infty} I_v I_p.
$$

Moreover, This bound is unclear

$$
\begin{aligned}
\frac{1}{2}\operatorname{trace}(g^T V_{xx} g) &= \frac{1}{2}\sigma^2 N_p \boxed{\left(\frac{N_p - S_p}{S_p}\right)^2} = \frac{1}{2}\sigma^2 N_p\left(\frac{N_p}{S_p} - 1\right)^2 \\
&\le \frac{1}{2}\sigma^2 N_p. \qquad \le 1/2\,\sigma^2 N_p\left(\frac{N_p}{S_p}\right)^2 + 1 \le \frac{1}{2}\sigma^2 N_p\left(\frac{N_p}{\varepsilon}\right)^2 + 1
\end{aligned}
$$

If $S_p \geq \varepsilon > 0$

The stochastic terms are not neccesary, because they do a martingale procces and therefore, when we use integral and expectation they vanising. Incorporation all terms calculate above, we obtain

$$
\begin{aligned}
dV(X) &= -rS_p\left(1-\frac{S_p^0}{S_p}\right)^2 + r\left[\mathcal{R}_0^s - 1\right] I_p - rL_p - \frac{\beta_p\beta_v}{\gamma N_v^\infty} I_v I_p + \frac{1}{2}\sigma^2 N_p\left(\frac{N_p - S_p}{S_p}\right)^2 \\
&\le -rS_p\left(1-\frac{S_p^0}{S_p}\right)^2 + r\left[\mathcal{R}_0^s - 1\right] I_p - rL_p - \frac{\beta_p\beta_v}{\gamma N_v^\infty} I_v I_p + \frac{1}{2}\sigma^2 N_p.
\end{aligned}
$$

Define $LV(X)$ as

$$
\mathbb{E}\,LV(X) = \mathbb{E}\underbrace{-rS_p\left(1-\frac{S_p^0}{S_p}\right)^2 + r\left[\mathcal{R}_0^s - 1\right] I_p - rL_p - \frac{\beta_p\beta_v}{\gamma N_v^\infty} I_v I_p}_{LV,\ \le 1/2\,\sigma_p^2 N_p} + \frac{1}{2}\sigma^2 N_p. \geq 0
$$

Using Itô's formula and integrating dV from 0 to t as well as taking expectation yield the following

$S_p \to N_0$
$L_p \to 0$
$I_p \to 0$

$$\mathbb{E}\,V(X) \le V_0 + \mathbb{E}\int LV$$

$$\lim_{t\to\infty} \frac{1}{t}\mathbb{E} - r_p S_p\left(1 - \frac{N_p}{S_p}\right)^2 \quad 5 \quad + r\left[\mathcal{R}_0^s - 1\right] I_p - rL_p$$

$$- \frac{\beta_p\beta_v}{\gamma N_v^\infty} I_v I_p \le \frac{1}{2}\sigma_p^2 N_p$$

$$\lim_{t\to\infty}\frac{1}{t}\log\left(\mathbb{E}\right) - r_p S_p\left(1 - N_p/S_p\right)^2 + r\left[\mathcal{R}_0^s - 1\right]\left(I_p \to 0 - r L_p \to 0 \ldots\right] \le -c$$

$$\lim_{t\to\infty}\left(\cdot \quad \mathbb{E}(f(x))\right) \le \lim_{t\to\infty} e^{-ct} \to 0.$$

Thm 4.1
R. Agarwal

Let $(S_h(t), I_h(t), I_v(t))$ be the solution of system (3) with initial condition $(S_h(0), I_h(0), I_v(0)) \in \Gamma$. If $R_0^s<1$, then

$$\lim_{t\to\infty} \mathbb{E}\int_0^t \left[(R_0^s-1)I_h + \mu_1 S_h\left(1 - S_h^0/S_h\right)^2 + \frac{\gamma\Lambda_h(\alpha\phi+\beta\psi\theta)}{\mu_1\mu_3} I_h I_v\right] dr \le \frac{\sigma_1^2 S_h^0}{2} \text{ a.s}$$

namely, the infected individuals tends to zero exponentially a.s.

Proof.

See [22,31] of this paper.
22: Zhang 2017a
31: Khasminski.

The main idea is verify Thm 2.4 with the Lyapunov function

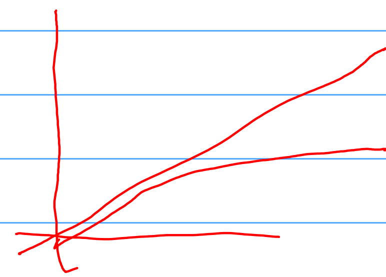
$$V(S_h, I_h, I_v) = \left(S_h - S_h^0 - S_h^0 \ln\frac{S_h}{S_h^0}\right) + I_h + \frac{\Lambda_h(\alpha\phi+\beta\psi\theta)}{\mu_1\mu_3} I_v.$$

Remember that
$$R_0^s = \frac{\Lambda_h\Lambda_v\gamma(\alpha\phi+\beta\psi\theta)}{\mu_1\mu_2\mu_3^2}$$

33: Check for Permanence

The SDE of this paper reads:

$$dS_h = \Big[\overbrace{\Lambda_h - \alpha\phi S_h I_v - \beta\psi\theta S_h I_v - \mu_1 S_h}^{:=f_1}\Big]dt + \sigma_1 S_h dB_1$$
$$dI_h = \Big[\overbrace{\alpha\phi S_h I_v + \beta\psi\theta S_h I_v - \mu_2 I_h}^{:=f_2}\Big]dt + \sigma_2 I_h dB_2$$
$$dI_v = \Big[\underbrace{\gamma I_h\left(\frac{\Lambda_v}{\mu_3} - I_v\right) - \mu_3 I_v}_{:=f_3}\Big]dt.$$



Note that

$$V_x = \left(\frac{\partial V}{\partial S_h}, \frac{\partial V}{\partial I_h}, \frac{\partial V}{\partial I_v}\right), \text{ where}$$

$$\frac{\partial V}{\partial S_h} = 1 - \frac{S_h^0}{S_h}, \quad \frac{\partial V}{\partial I_h} = 1, \quad \frac{\partial V}{\partial I_v} = \frac{\Lambda_h(\alpha\phi+\beta\psi\theta)}{\mu_1\mu_3}.$$

Then

$$V_x f = \left(1 - S_h^0/S_h\right)\left(\Lambda_h - \alpha\phi S_h I_v - \beta\psi\theta S_h I_v - \mu_1 S_h\right) +$$
$$\left(\alpha\phi S_h I_v + \beta\psi\theta S_h I_v - \mu_2 I_h\right) +$$
$$\frac{\Lambda_h(\alpha\phi+\beta\psi\theta)}{\mu_1\mu_3}\left[\gamma I_h\left(\frac{\Lambda_v}{\mu_3} - I_v\right) - \mu_3 I_v\right],$$

(Factorizing)

$$= \left(1-S_h^0/S_h\right)\left(\mu_1\left(\frac{\Lambda_h}{\mu_1} - S_h\right) - (\alpha\phi+\beta\psi\theta)S_h I_v\right) + \left(\alpha\phi S_h I_v + \beta\psi\theta S_h I_v - \mu_2 I_h\right) +$$
$$\frac{\Lambda_h(\alpha\phi+\beta\psi\theta)}{\mu_1\mu_3}\left(\gamma I_h\left(\frac{\Lambda_v}{\mu_3} - I_v\right) - \mu_3 I_v\right) \quad (*.1), \qquad \text{let } S_h^0 := \frac{\Lambda_h}{\mu_1}.$$

Then the factor $\mu_1\left(\frac{\Delta_h}{\mu_1} - S_h\right)$ can be rewritten as.

$$\mu_1\left(\frac{\Delta_h}{\mu_1} - S_h\right) = -S_h\mu_1\left(1 - \frac{\Delta_h}{\mu_1 S_h}\right),$$
$$= -S_h\mu_1\left(1 - \frac{S_h^0}{S_h}\right).$$

Substituting in relation (A-1), we obtain

$$V_x f = (1 - S_h^0/S_h)\left(-\mu_1 S_h\left(1 - \frac{S_h^0}{S_h}\right) - (\alpha\phi + \beta\varphi\theta)S_h I_v\right) + (\alpha\phi\xi I_v + \beta\varphi\theta S_h I_v - \mu_2 I_h) +$$
$$\frac{\Delta_h(\alpha\phi+\beta\varphi\theta)}{\mu_1\mu_3}\left(\gamma I_h\left(\frac{\Delta_v}{\mu_3} - I_v\right) - \mu_3 I_v\right).$$

Then

$$V_x f = -\mu_1 S_h\left(1 - S_h^0/S_h\right)^2 - \underbrace{\left(1 - S_h^0/S_h\right)(\alpha\phi+\beta\varphi\theta)S_h I_v}_{:=\circledast} + (\alpha\phi\xi I_v + \beta\varphi\theta S_h I_v - \mu_2 I_h) +$$
$$\left(\frac{\Delta_h(\alpha\phi+\beta\varphi\theta)}{\mu_1\mu_3}\right)\gamma\frac{\Delta_v}{\mu_3}I_h - \left(\frac{\Delta_h(\alpha\phi+\beta\varphi\theta)}{\mu_1\mu_3}\right)\gamma I_h I_v - \left(\frac{\Delta_h(\alpha\phi+\beta\varphi\theta)}{\mu_1\mu_3}\right)\mu_3 I_v.$$

Expanding $\circledast$ we obtain

$$V_x f = -\mu_1 S_h\left(1 - S_h^0/S_h\right)^2 - \cancel{(\alpha\phi+\beta\varphi\theta)S_h I_v} - S_h^0/\cancel{S_h}(\alpha\phi+\beta\varphi\theta)\cancel{S_h} I_v + (\cancel{\alpha\phi\xi I_v} + \beta\varphi\theta S_h I_v - \mu_2 I_h)$$
$$:= *$$
$$\left(\frac{\Delta_h(\alpha\phi+\beta\varphi\theta)}{\mu_1\mu_3}\right)\gamma\frac{\Delta_v}{\mu_3}I_h - \left(\frac{\Delta_h(\alpha\phi+\beta\varphi\theta)}{\mu_1\mu_3}\right)\gamma I_h I_v - \left(\frac{\Delta_h(\alpha\phi+\beta\varphi\theta)}{\mu_1\mu_3}\right)\mu_3 I_v.$$

$$= \mu_1 S_h\left(1 - S_h^0/S_h\right)^2 - S_h^0(\alpha\phi+\beta\varphi\theta)I_v + \beta\varphi\theta S_h I_v - \mu_2 I_h$$
$$\left(\frac{\Delta_h(\alpha\phi+\beta\varphi\theta)}{\mu_1\mu_3}\right)\gamma\frac{\Delta_v}{\mu_3}I_h - \left(\frac{\Delta_h(\alpha\phi+\beta\varphi\theta)}{\mu_1\mu_3}\right)\gamma I_h I_v - \left(\frac{\Delta_h(\alpha\phi+\beta\varphi\theta)}{\mu_1\mu_3}\right)\mu_3 I_v.$$

Factorizing respect $I_h, I_v, I_h I_v$, we obtain

$$V_x f = -\mu_1 S_h\left(1 - S_h^0/S_h\right)^2 - \left[\underbrace{\cancel{S_h^0(\alpha\phi+\beta\varphi\theta)}}_{=\frac{\Delta_h}{\mu_1}} - \cancel{\left(\frac{\Delta_h(\alpha\phi+\beta\varphi\theta)}{\mu_1\mu_3}\right)\mu_3}\right] I_v +$$
$$\left[\frac{\Delta_h(\alpha\phi+\beta\varphi\theta)}{\mu_1\mu_3}\frac{\gamma\Delta_v}{\mu_3} - \mu_2\right]I_h - \left(\frac{\Delta_h(\alpha\phi+\beta\varphi\theta)}{\mu_1\mu_3}\right)\gamma I_h I_v$$

$$= -\mu_1 S_h\left(1 - S_h^0/S_h\right)^2 + \mu_2\left[\frac{\Delta_h(\alpha\phi+\beta\varphi\theta)}{\mu_1\mu_3}\frac{\gamma\Delta_v}{\mu_2\mu_3} - 1\right]I_h - \left(\frac{\Delta_h(\alpha\phi+\beta\varphi\theta)}{\mu_1\mu_3}\right)\gamma I_h I_v.$$

$$= -\mu_1 S_h\left(1 - S_h^0/S_h\right)^2 + \mu_2\left[\underbrace{\frac{\Delta_h\Delta_v\gamma(\alpha\phi+\beta\varphi\theta)}{\mu_1\mu_2\mu_3^2}}_{=\mathcal{R}_0^s} - 1\right]I_h - \left(\frac{\Delta_h(\alpha\phi+\beta\varphi\theta)}{\mu_1\mu_3}\right)\gamma I_h I_v.$$

Therefore

$$\boxed{V_x f = -\mu_1 S_h^0 \left(1 - S_h^0/S_h\right)^2 + \mu_2 \left[R_0^s - 1\right] I_h - \left(\frac{\Lambda_h(\alpha\phi + \beta\psi\theta)}{\mu_1\mu_3}\right)\gamma I_h I_v \quad (14)}$$

Moreover,

$$\frac{1}{2}\,\mathrm{trace}\left(g^T V_{xx} g\right) = \begin{bmatrix} \sigma_1 S_h & 0 & 0 \\ 0 & \sigma_2 I_h & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{S_h^0}{S_h^2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \sigma_1 S_h & 0 & 0 \\ 0 & \sigma_2 I_h & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \frac{1}{2}\mathrm{trace}\begin{pmatrix} (\sigma_1 S_h)^2\, S_h^0/(S_h)^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \frac{1}{2}\sigma_1^2 S_h^0. \quad (15)$$

Moreover, $$V_x g = \underbrace{\frac{\partial V}{\partial S_h}}_{1 - S_h^0/S_h}\left(\sigma_1 S_h\right) dB_1(t) + \underbrace{\frac{\partial V}{\partial I_h}}_{=1}\left(\sigma_2 I_2\right) dB_2(t)$$

$$= \left(1 - S_h^0/S_h\right)\left(\sigma_1 S_h\right) dB_1 + \sigma_2 I_2\, dB_2(t)$$

$$= \sigma_1 (S_h - S_h^0) dB_1 + \sigma_2 I_2 dB_2 \quad (16).$$

Combining (14)–(16), the Itô formula for $V(\cdot)$ and SDE(*), results

$$dV(x) = \left[\mu_2(R_0^s - 1) I_h - \mu_1 S_h\left(1 - \frac{S_h^0}{S_h}\right)^2 - \frac{\gamma\Lambda_h(\alpha\phi + \beta\psi\theta)}{\mu_1\mu_3} I_h I_v + \frac{\sigma_1^2 S_h^0}{2}\right] dt$$

$$+ \sigma_1 (S_h - S_h^0) dB_1 + \sigma_2 I_h dB_2 \quad (17)$$

$$= \mathcal{L}V(x)\, dt + \sigma_1 (S_h - S_h^0) dB_1 + \sigma_2 I_h dB_2 .$$

Then integrating (17), we obtain

$$0 \le V(x) = V(0) + \int_0^t \mathcal{L}V(x)\, ds + \underbrace{\int_0^t \sigma_1 (S_h(s) - S_h^0) dB_1(s) + \int_0^t \sigma_2 (I_h(s)) dB_2(s)}_{:= M(t)},$$

then

$$0 \le \mathbb{E}\left[V(t)\right] - V(0) = \mathbb{E}\int_0^t \mathcal{L}V(X(s))\, ds + \cancel{\mathbb{E} M(t)}^{\,0}$$

$$= -\mathbb{E}\int_0^t \mu_2(1 - R_0^s) I_h + \mu_1 S_h\left(1 - S_h^0/S_h\right)^2 + \frac{\gamma\Lambda_h(\alpha\phi + \beta\psi\theta)}{\mu_1\mu_3} I_h I_v - \frac{\sigma_1^2}{2} S_h^0\, ds$$

$$= -\mathbb{E}\int_0^t \mu_2(1 - R_0^s) I_h + \mu_1 S_h\left(1 - S_h^0/S_h\right)^2 + \frac{\gamma\Lambda_h(\alpha\phi + \beta\psi\theta)}{\mu_1\mu_3} I_h I_v\, ds + \frac{\sigma_1^2}{2} S_h^0 t.$$

The above holds for all $t \ge 0$, thus

$$\frac{\sigma_1^2 S_h^0}{2} - \frac{1}{t}\mathbb{E}\int_0^t \mu_2 \underbrace{(1 - R_0^s)}_{0 \le} I_h + \mu_1 S_h\left(1 - S_h^0/S_h\right)^2 + \frac{\gamma\Lambda_h(\alpha\phi + \beta\psi\theta)}{\mu_1\mu_3} I_h I_v\, ds \ge 0, \quad \forall t \ge 0$$

that is

$$\lim_{t\to\infty} \frac{1}{t}\mathbb{E}\int_0^t \mu_2(1 - R_0^s) I_h + \mu_1 S_h\left(1 - S_h^0/S_h\right)^2 + \gamma\Lambda_h \frac{(\alpha\phi + \beta\psi\theta)}{\mu_1\mu_3} I_h I_v \le \frac{\sigma_1^2}{2} S_h^0 \quad (\ast\!-)$$

In other words.

$$0 \le \lim_{t\to\infty} \frac{1}{t} \mathbb{E}\left(\int_0^t \mu_2(1-R_0^s) I_h + \mu_1 S_h \left(1 - S_h^0/S_h\right)^2 + \gamma \frac{\Lambda_h(\alpha\phi + \beta\psi\theta)}{\mu_1\mu_3} I_h I_v\right) \le \frac{\sigma_1^2}{2} S_h^0 \quad (\star\text{-}2).$$

Let σ_1 s.t $\frac{\sigma_1^2}{2} S_h^0 < 1$ i.e

$$0 < \sigma_1 \le \sqrt{2/S_h^0}$$

(A) $$\lim_{t\to\infty} \frac{1}{t} \log\left(\mathbb{E} \underbrace{\int_0^t \mu_2(1-R_0^s) I_h + \mu_1 S_h \left(1 - S_h^0/S_h\right)^2 + \gamma \frac{\Lambda_h}{\mu_1\mu_3}(\alpha\phi + \beta\psi\theta) I_h I_v}_{I_1}\right) \le -c < 0$$

$\Rightarrow$

$$\lim_{t\to\infty} \mathbb{E}\left(\int_0^t I_1\right) \le e^{-ct} \to 0$$

since $I_1 \ge 0 \;\forall t \ge 0 \Rightarrow I_1(t) \xrightarrow{t\to\infty} 0$ i.e.

$$I_h(t) \to 0$$
$$S_h\left(1 - S_h^0/S_h\right)^2 \to 0 \Rightarrow S_h \to \frac{\Lambda_h}{\mu_1}$$
$$c_1 I_v S_h^0 \to 0$$

From (A)

$$\mathbb{E}\int_0^t I_1\, dt \le \mathbb{E}\int_0^t \mu_2(1-R_0^s) I_h + \mu_1 S_h\left(1 - S_h^0/S_h\right)^2 + \frac{\gamma\Lambda_h}{\mu_1\mu_3}(\alpha\phi + \beta\psi\theta)\frac{\Lambda_h}{\mu_1} I_v$$

$\le \frac{\sigma_1^2}{2}(S_h^0)^2$, . Take σ_1 such that $\sigma_1^2/2\,(S_h^0)^2 < 1$

$$0 \leq \mathbb{E}V(t) - \mathbb{E}V(0) \leq \mathbb{E}\int_0^t LV(X(s))ds$$
$$\leq -\mathbb{E}\int_0^t \left[rS_p\left(1 - \frac{S_p^0}{S_p}\right)^2 - r\left[\mathcal{R}_0^s - 1\right] I_p + rL_p + \frac{\beta_p\beta_v}{\gamma N_v^\infty} I_v I_p \right] ds + \frac{1}{2}\sigma^2 N_p$$

Therefore,

$$\lim_{t\to\infty} \frac{1}{t}\mathbb{E}\int_0^t \left[-rS_p\left(1 - \frac{S_p^0}{S_p}\right)^2 + r\left[\mathcal{R}_0^s - 1\right] I_p - rL_p - \frac{\beta_p\beta_v}{\gamma N_v^\infty} I_v I_p \right] ds \leq \frac{1}{2}\sigma^2 N_p.$$

$$\lim_{t\to\infty} \frac{1}{t}\mathbb{E}\int_0^t \left[r[\mathcal{R}_0^s - 1]I_p - rS_p\left(1 - \frac{S_p^0}{S_p}\right)^2 - rL_p - \frac{\beta_p\beta_v}{\gamma} I_v I_p \right] dr \leq \frac{1}{2}\sigma^2 N_p, \ a.s.$$

■

Write a paragraph to describe why the limit above exponentially goes to zero.

4 Persistence

Theorem 4 *Let $(S_p(t), L_p(t), I_p(t), I_v(t))$ be the solution of (5) with initial values $(S_p(0), L_p(0), I_p(0), I_v(0)) \in (0, N_p)\times(0, N_p)\times(0, N_p)\times(0, N_v)$. If $\mathcal{R}_0^s > 1$, then the system (5) is globally asymptotically stable at endemic equilibrium point if*

$$\lim_{t\to\infty} \frac{1}{t}\mathbb{E}\int_0^t \left[\frac{rS_p^*}{S_pS_p^*}(S_p^* - S_p)^2 + \frac{\beta_p}{N_v} S_p^* I_v^* A_1 + \frac{\beta_v}{N_p}\frac{I_p}{I_v}(I_v - I_v^*)^2 + \gamma I_v^* A_2 \right] dr \leq A_3.$$

namely, the disease will persist with probability one.

Proof. Let us define the following Lyapunov function $V : \mathbb{R}_+^4 \to \mathbb{R}_+$

$$V(S_p, L_P, I_p, I_v) = (S_p + L_p + I_p + I_v) - (S_p^* + L_p^* + I_p^* + I_v^*)$$
$$- \left(S_p^* \ln \frac{S_p}{S_p^*} + L_p^* \ln \frac{L_p}{L_p^*} + I_p^* \ln \frac{I_p}{I_p^*} + I_v^* \ln \frac{I_v}{I_v^*} \right).$$

Computing the Itô formula terms as:

$$V_x f = \left(1 - \frac{S_p^*}{S_p}\right)\left(rN_p - \beta_p S_p \frac{I_v}{N_v^\infty} - rS_p\right) + \left(1 - \frac{L_p^*}{L_p}\right)\left(\beta_p S_p \frac{I_v}{N_v^\infty} - (r+b)L_p\right)$$
$$+ \left(1 - \frac{I_p^*}{I_p}\right)(bL_p - rI_p) + \left(1 - \frac{I_v^*}{I_v}\right)\left(\beta_v N_v \frac{I_p}{N_p} - \beta_v \frac{I_p}{N_p} I_v - \gamma I_v\right).$$

The system (5) satisfy the following relations at equilibrium point

$$
\begin{aligned}
rN_p &= \beta_p S_p^* \frac{I_v^*}{N_v^\infty} + rS_p^* \\
(r+b) &= \beta_p S_p^* \frac{I_v^*}{L_p^* N_v^\infty} \\
r &= b \frac{L_p^*}{I_p^*} \\
\beta_v \frac{N_v}{N_p} &= \frac{\beta_v}{N_p} I_v^* + \gamma \frac{I_v^*}{I_p^*}
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
V_x f =& \left(1 - \frac{S_p^*}{S_p}\right) \left(\beta_p S_p^* \frac{I_v^*}{N_v^\infty} + rS_p^* - \beta_p S_p \frac{I_v}{N_v^\infty} - rS_p\right) \\
&+ \left(1 - \frac{L_p^*}{L_p}\right) \left(\beta_p S_p \frac{I_v}{N_v^\infty} - \beta_p S_p^* \frac{I_v^*}{L_p^* N_v^\infty} L_p\right) + \left(1 - \frac{I_p^*}{I_p}\right) \left(bL_p - b\frac{L_p^*}{I_p^*} I_p\right) \\
&+ \left(1 - \frac{I_v^*}{I_v}\right) \left(\frac{\beta_v}{N_p} I_v^* I_p + \gamma \frac{I_v^*}{I_p^*} I_p - \beta_v \frac{I_p}{N_p} I_v - \gamma I_v\right) \\
&= rS_p^* \left(1 - \frac{S_p^*}{S_p}\right) \left(1 - \frac{S_p}{S_p^*}\right) + \frac{\beta_p}{N_v^\infty} S_p^* I_v^* \left(1 - \frac{S_p^*}{S_p}\right) \left(1 - \frac{S_p I_v}{S_p^* I_v^*}\right) \\
&+ \frac{\beta_p}{N_v^\infty} S_p^* I_v^* \left(1 - \frac{L_p^*}{L_p}\right) \left(\frac{S_p I_v}{S_p^* I_v^*} - \frac{L_p}{L_p^*}\right) + bL_p^* \left(1 - \frac{I_p^*}{I_p}\right) \left(\frac{L_p}{L_p^*} - \frac{I_p}{I_p^*}\right) \\
&+ \left(1 - \frac{I_v^*}{I_v}\right) \left(-\frac{\beta_v}{N_p} I_v I_p \left(1 - \frac{I_v^*}{I_v}\right) + \gamma I_v^* \left(\frac{I_p}{I_p^*} - \frac{I_v}{I_v^*}\right)\right) \\
&= rS_p^* \left(2 - \frac{S_p}{S_p^*} - \frac{S_p^*}{S_p}\right) + \frac{\beta_p}{N_v^\infty} S_p^* I_v^* \left(1 - \frac{I_v}{I_v^*} \left(\frac{S_p}{S_p^*} - 1\right) - \frac{S_p^*}{S_p}\right) \\
&+ \frac{\beta_p}{N_v^\infty} S_p^* I_v^* \left(\frac{S_p I_v}{S_p^* I_v^*} \left(1 - \frac{L_p^*}{L_p}\right) - \frac{L_p}{L_p^*} \left(1 - \frac{L_p^*}{L_p}\right)\right) + bL_p^* \left(1 + \frac{L_p}{L_p^*} - \frac{I_p}{I_p^*} - \frac{I_p^* L_p}{I_p L_p^*}\right) \\
&- \frac{\beta_v}{N_v} I_v I_p \left(1 - \frac{I_v^*}{I_v}\right)^2 + \gamma I_v^* \left(\frac{I_p}{I_p^*} - \frac{I_v^* I_p}{I_v I_p^*} - \frac{I_v}{I_v^*} + 1\right) \\
&= rS_p^* \left(2 - \frac{S_p}{S_p^*} - \frac{S_p^*}{S_p}\right) + \frac{\beta_p}{N_v^\infty} S_p^* I_v^* \left(1 - \frac{S_p^*}{S_p} - \frac{I_v}{I_v^*} \left(\frac{S_p}{S_p^*} - 1\right)\right) \\
&+ \frac{\beta_p}{N_v^\infty} S_p^* I_v^* \left(1 - \frac{L_p}{L_p^*} - \frac{S_p I_v}{S_p^* I_v^*} \left(\frac{L_p^*}{L_p} - 1\right)\right) + bL_p^* \left(1 - \frac{I_p}{I_p^*} + \frac{L_p}{L_p^*} \left(1 - \frac{I_p^*}{I_p}\right)\right) \\
&- \frac{\beta_v}{N_v^\infty} I_v I_p \left(1 - \frac{I_v^*}{I_v}\right)^2 + \gamma I_v^* \left(1 - \frac{I_v}{I_v^*} - \frac{I_p}{I_p^*} \left(\frac{I_v^*}{I_v} - 1\right)\right).
\end{aligned}
$$

Then

$$
\begin{aligned}
V_x f = & \, rS_p^* \left(2 - \frac{S_p}{S_p^*} - \frac{S_p^*}{S_p}\right) + \frac{\beta_p}{N_v^\infty} S_p^* I_v^* \left(2 - \frac{S_p^*}{S_p} - \frac{L_p}{L_p^*} - \frac{I_v}{I_v^*}\left(\frac{S_p L_p^*}{S_p^* L_p} - 1\right)\right) \\
& + bL_p^* \left(1 - \frac{I_p}{I_p^*} + \frac{L_p}{L_p^*}\left(1 - \frac{I_p^*}{I_p}\right)\right) - \frac{\beta_v}{N_p} I_v I_p \left(1 - \frac{I_v^*}{I_v}\right)^2 \\
& + \gamma I_v^* \left(1 - \frac{I_v}{I_v^*} - \frac{I_p}{I_p^*}\left(\frac{I_v^*}{I_v} - 1\right)\right).
\end{aligned}
$$

Now we need compute the term $g^T V_{xx} g$,

$$
g^T V_{xx} g = \begin{bmatrix} \sigma^2 \left(\frac{N_p - S_p}{S_p}\right)^2 S_p^* + \sigma^2 L_p^* & 0 \\ 0 & I_p^* \sigma^2 + I_v^* \sigma_v^2 \end{bmatrix}
$$

therefore,

$$
\begin{aligned}
\frac{1}{2} trace(g^T V_{xx} g) &= \frac{1}{2}\left(\sigma^2 \left(\frac{N_p - S_p}{S_p}\right)^2 S_p^* + \sigma^2 L_p^* + \sigma^2 I_p^* + \sigma_v^2 I_v^*\right) \\
&\leq \frac{1}{2}\left(\sigma^2 S_p^* + \sigma^2 L_p^* + \sigma^2 I_p^* + \sigma_v^2 I_v^*\right)
\end{aligned}
$$

The stochastics terms are not neccesary, because the are a martingale and therefore, when we use integrating and expectation they vanishing, obtaining the following $LV(X)$ operator

$$
\begin{aligned}
LV(X) = & -rS_p^* \frac{\left(S_p^* - S_p\right)^2}{S_p S_p^*} - \frac{\beta_p}{N_v^\infty} S_p^* I_v^* A_1 - bL_p^* A_2 - \frac{\beta_v}{N_p} I_v I_p \left(1 - \frac{I_v^*}{I_v}\right)^2 \\
& - \gamma I_v^* A_3 + A_4.
\end{aligned}
$$

where

$$
\begin{aligned}
A_1 &= \left(\frac{S_p^*}{S_p} + \frac{L_p}{L_p^*} + \frac{I_v}{I_v^*}\left(\frac{S_p L_p^*}{S_p^* L_p} - 1\right) - 2\right) > 0, \\
A_2 &= \left(\frac{I_p}{I_p^*} - \frac{L_p}{L_p^*}\left(1 - \frac{I_p^*}{I_p}\right) - 1\right) > 0, \\
A_3 &= \left(\frac{I_v}{I_v^*} + \frac{I_p}{I_p^*}\left(\frac{I_v^*}{I_v} - 1\right) - 1\right) > 0, \\
A_4 &= \frac{1}{2}\left(\sigma^2 S_p^* + \sigma^2 L_p^* + \sigma^2 I_p^* + \sigma_v^2 I_v^*\right) > 0.
\end{aligned}
$$

Applying Itô formula, integrating dV from 0 to t and taking expectation gives the following

$$
\begin{aligned}
0 \le \mathbb{E}V(t) - \mathbb{E}V(0) &= \mathbb{E}\int_0^t LV(s)ds \\
&- \mathbb{E}\int_0^t \left(rS_p^* \frac{\left(S_p^* - S_p\right)^2}{S_p S_p^*} + \frac{\beta_p}{N_v^\infty} S_p^* I_v^* A_1 + bL_p^* A_2 + \frac{\beta_v}{N_p} I_v I_p \left(1 - \frac{I_v^*}{I_v}\right)^2 + \gamma I_v^* A_3 \right) ds \\
&+ A_4 t.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\lim_{t\to\infty} \frac{1}{t} \mathbb{E}\int_0^t \left(rS_p^* \frac{\left(S_p^* - S_p\right)^2}{S_p S_p^*} + \frac{\beta_p}{N_v^\infty} S_p^* I_v^* A_1 + bL_p^* A_2 + \frac{\beta_v}{N_p} I_v I_p \left(1 - \frac{I_v^*}{I_v}\right)^2 + \gamma I_v^* A_3 \right) ds \\
\le A_4.
\end{aligned}
$$

■