# Modelado con Ecuaciones Diferenciales Estocásticas via Perturbación de Parámetros

y una Invitación a la solución numérica de EDEs

Saúl Díaz Infante Velasco

Agosto 21, 2019

CONACYT-Universidad de Sonora

# Introducción

# Por que EDEs?

**En ocaciones**

$$EDO + ruido = Mejor\ modelo$$

# Por que EDEs?

**En ocaciones**

$$EDO + ruido = Mejor\ modelo$$

## Crecimiento de Poblaciones

$$\frac{dN}{dt} = a(t)N(t) \qquad N_0 = N(0) = cte.$$

# Por que EDEs?

En ocaciones

$$EDO + ruido = Mejor\ modelo$$

**Crecimiento de Poblaciones**

$$\frac{dN}{dt} = a(t)N(t) \qquad N_0 = N(0) = cte.$$

$$\boxed{a(t) = r(t) + ''ruido''}$$

# Por que EDEs?

En ocaciones

$$EDO + ruido = Mejor\ modelo$$

## Circuitos Eléctricos

$$L \cdot Q''(t) + R \cdot Q'(t) + \frac{1}{C} \cdot Q(t) = F(t)$$

$$Q(0) = Q_0$$

$$Q'(0) = I_0$$

# Por que EDEs?

**En ocaciones**

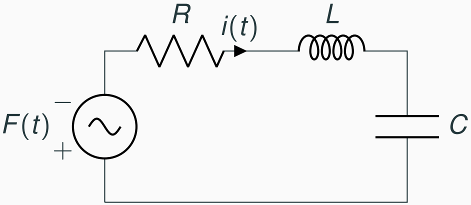
$$EDO + ruido = Mejor\ modelo$$

### Circuitos Eléctricos

$$L \cdot Q''(t) + R \cdot Q'(t) + \frac{1}{C} \cdot Q(t) = F(t)$$

$$Q(0) = Q_0$$

$$Q'(0) = I_0$$



# Por que EDEs?

**En ocaciones**

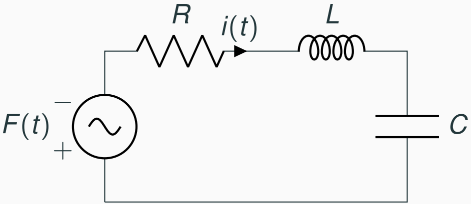
$$EDO + ruido = Mejor\ modelo$$

**Circuitos Eléctricos**

$$L \cdot Q''(t) + R \cdot Q'(t) + \frac{1}{C} \cdot Q(t) = F(t)$$

$$Q(0) = Q_0$$

$$Q'(0) = I_0$$




$$\boxed{F(t) = G(t) + "\,ruido"}$$

# Para fijar ideas

**Ejemplo**

$$dN(t) = aN(t)dt$$

# Para fijar ideas

**Ejemplo**

$$dN(t) = aN(t)dt$$

**Perturba sobre** $[t, t+dt)$

# Para fijar ideas

**Ejemplo**

$$dN(t) = aN(t)dt$$

**Perturba sobre $[t, t + dt)$**

$$adt \rightsquigarrow adt + \sigma dB(t)$$

# Para fijar ideas

**Ejemplo**

$$dN(t) = aN(t)dt$$

**Perturba sobre $[t, t+dt)$**

$$adt \rightsquigarrow adt + \sigma dB(t)$$

**obten una EDE**

$$dN(t) = aN(t)dt + \sigma N(t)dB(t)$$

# Para fijar ideas

**Ejemplo**

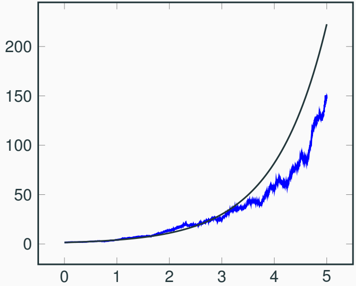
$$dN(t) = aN(t)dt$$

**Perturba sobre $[t, t+dt)$**

$$adt \rightsquigarrow adt + \sigma dB(t)$$

**obten una EDE**

$$dN(t) = aN(t)dt + \sigma N(t)dB(t)$$



# Esquema de Charla

1. Construcción de Métodos Numéricos

2. Aproximación Fuerte vs. Débil

3. Ejemplo: Reconstrucción de masa osea

4. Comentarios Finales

# Construcción de Métodos Numéricos

# Notación

**EDE**

$$dx(t) = \underbrace{f(x(t),t)dt}_{\text{deriva}} + \underbrace{g(x(t),t)dB(t)}_{\text{difusión}},$$

$$f : \mathbb{R}^d \times [0,T] \to \mathbb{R}^d, \qquad g : \mathbb{R}^d \times [0,T] \to \mathbb{R}^{d\times m}$$

$$B(t) = (B_1(t), \ldots, B_m(t))^T, \quad (\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$$

# Notación

**EDE**

$$dx(t) = \underbrace{f(x(t),t)dt}_{\text{deriva}} + \underbrace{g(x(t),t)dB(t)}_{\text{difusión}},$$

$$x_0 = x(0), \quad t \in [0,T].$$

$$f : \mathbb{R}^d \times [0,T] \to \mathbb{R}^d, \qquad g : \mathbb{R}^d \times [0,T] \to \mathbb{R}^{d\times m}$$

$$B(t) = (B_1(t), \ldots, B_m(t))^T, \quad (\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$$

# Notación

EDE

$$dx(t) = \underbrace{f(x(t),t)dt}_{\text{deriva}} + \underbrace{g(x(t),t)dB(t)}_{\text{difusión}},$$

$$x_0 = x(0), \quad t \in [0,T].$$

$$f : \mathbb{R}^d \times [0,T] \to \mathbb{R}^d, \qquad g : \mathbb{R}^d \times [0,T] \to \mathbb{R}^{d\times m}$$

$$B(t) = (B_1(t), \ldots, B_m(t))^T, \quad (\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$$

# Notación

**EDE**

$$x(t) = x_0 + \int_0^t f(x(s), s)ds + \int_0^t g(x(s), s)dB(s)$$

$$f : \mathbb{R}^d \times [0, T] \to \mathbb{R}^d, \qquad g : \mathbb{R}^d \times [0, T] \to \mathbb{R}^{d \times m}$$

$$B(t) = (B_1(t), \ldots, B_m(t))^T, \quad (\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t \geq 0}, \mathbb{P})$$

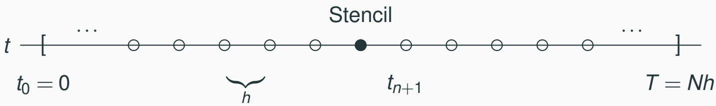
# Idea general de la construcción

EDE

$$x(t) = x_0 + \int_0^t f(x(s), s)ds + \int_0^t g(x(s), s)dB(s)$$

EDE

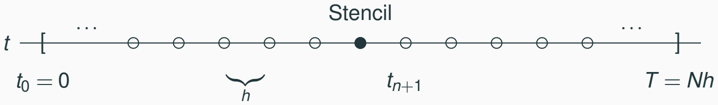
$$x(t) = x_0 + \int_0^t f(x(s), s)ds + \int_0^t g(x(s), s)dB(s)$$



# Idea general de la construcción

**EDE**

$$x(t) = x_0 + \int_0^t f(x(s), s)ds + \int_0^t g(x(s), s)dB(s)$$

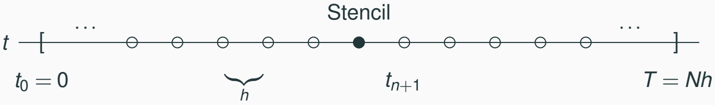



$$x(t_{n+1}) = x(t_n) + \int_{t_n}^{t_{n+1}} f(x(s), s)ds + \int_{t_n}^{t_{n+1}} g(x(s), s)dB(s)$$

# Idea general de la construcción

**EDE**

$$\boxed{x(t) = x_0 + \int_0^t f(x(s), s)ds + \int_0^t g(x(s), s)dB(s)}$$

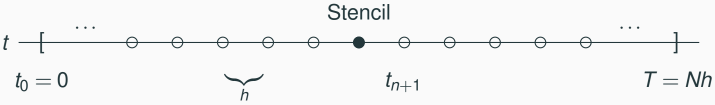



$$x(t_{n+1}) = x(t_n) + \underbrace{\int_{t_n}^{t_{n+1}} f(x(s), s)ds}_{\approx \text{det}} + \underbrace{\int_{t_n}^{t_{n+1}} g(x(s), s)dB(s)}_{\approx}$$

# Idea general de la construcción

EDE

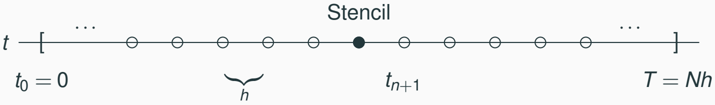
$$x(t) = x_0 + \int_0^t f(x(s),s)ds + \int_0^t g(x(s),s)dB(s)$$




$$x(t_{n+1}) = x(t_n) + \underbrace{\int_{t_n}^{t_{n+1}} f(x(s),s)ds}_{\approx \text{det}} + \underbrace{\int_{t_n}^{t_{n+1}} g(x(s),s)dB(s)}_{\approx}$$

**EDE**

$$\boxed{x(t) = x_0 + \int_0^t f(x(s), s)ds + \int_0^t g(x(s), s)dB(s)}$$




$$x(t_{n+1}) = x(t_n) + \underbrace{\int_{t_n}^{t_{n+1}} f(x(s), s)ds}_{\approx \text{det}} + \underbrace{\int_{t_n}^{t_{n+1}} g(x(s), s)dB(s)}_{\approx}$$

# Idea general de la construcción

**EDE**

$$x(t) = x_0 + \int_0^t f(x(s), s)ds + \int_0^t g(x(s), s)dB(s)$$


$t$ $[\ \cdots$ ○ ○ ○ ○ ○ ○ ○ $\cdots\ ]$

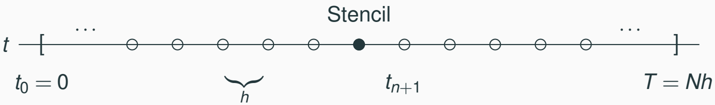
$t_0 = 0$ $T = Nh$

$$x(t_{n+1}) = x(t_n) + \underbrace{\int_{t_n}^{t_{n+1}} f(x(s), s)ds}_{\approx f(x_n)h} + \underbrace{\int_{t_n}^{t_{n+1}} g(x(s), s)dB(s)}_{\substack{\approx g(x_n)\Delta B_n \\ \Delta B_n = B_{t_{n+1}} - B_{t_n}}}$$

# Idea general de la construcción

**EDE**

$$x(t) = x_0 + \int_0^t f(x(s),s)ds + \int_0^t g(x(s),s)dB(s)$$



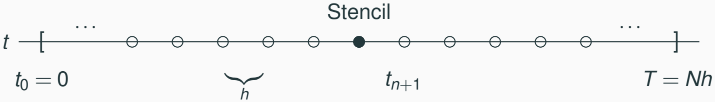

$$x(t_{n+1}) = x(t_n) + \underbrace{\int_{t_n}^{t_{n+1}} f(x(s),s)ds}_{\approx f(x_n)h} + \underbrace{\int_{t_n}^{t_{n+1}} g(x(s),s)dB(s)}_{\substack{\approx g(x_n)\Delta B_n \\ \Delta B_n = B_{t_{n+1}} - B_{t_n}}}$$

$$X_0 = x_0, \qquad X_n \approx x(t_n), \qquad n = 1\ldots,N-1$$
$$X_{n+1} = X_n + f(X_n)h + g(X_n)\Delta B_n,$$

# Idea general de la construcción

$$\boxed{\text{EDE}}$$

$$x(t) = x_0 + \int_0^t f(x(s),s)ds + \int_0^t g(x(s),s)dB(s)$$

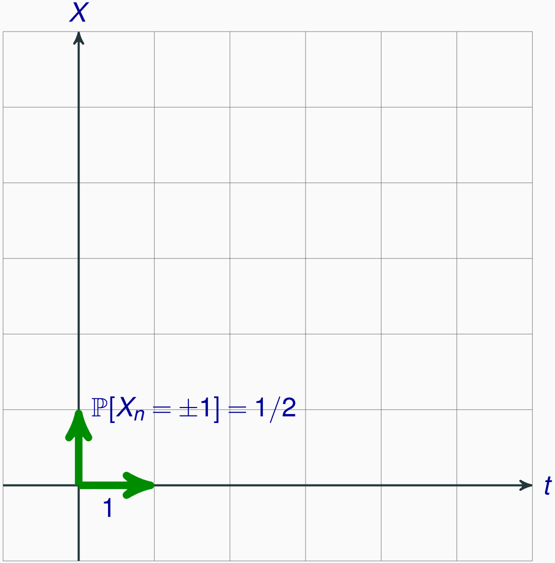



$$x(t_{n+1}) = x(t_n) + \underbrace{\int_{t_n}^{t_{n+1}} f(x(s),s)ds}_{\approx f(x_n)h} + \underbrace{\int_{t_n}^{t_{n+1}} g(x(s),s)dB(s)}_{\substack{\approx g(x_n)\Delta B_n \\ \Delta B_n = B_{t_{n+1}} - B_{t_n}}}$$

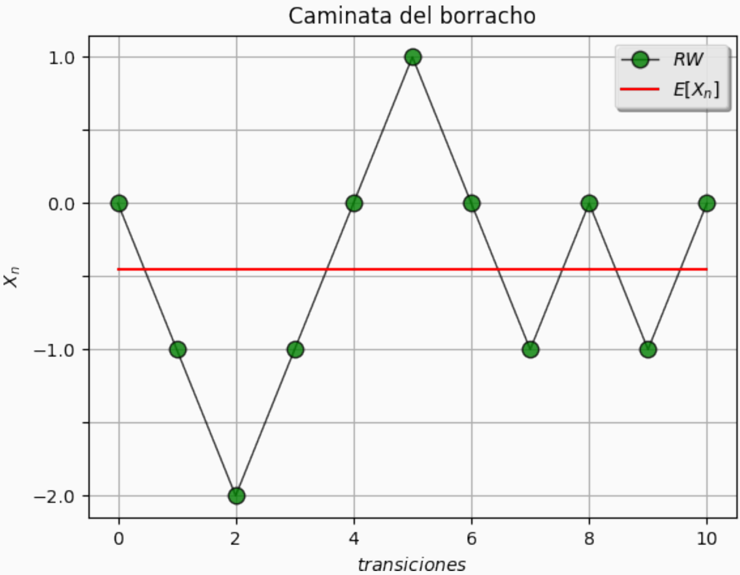
$$X_0 = x_0, \qquad X_n \approx x(t_n), \qquad n = 1\ldots, N-1$$

$$X_{n+1} = X_n + f(X_n)h + g(X_n)\underbrace{\Delta B_n}_{\approx}$$

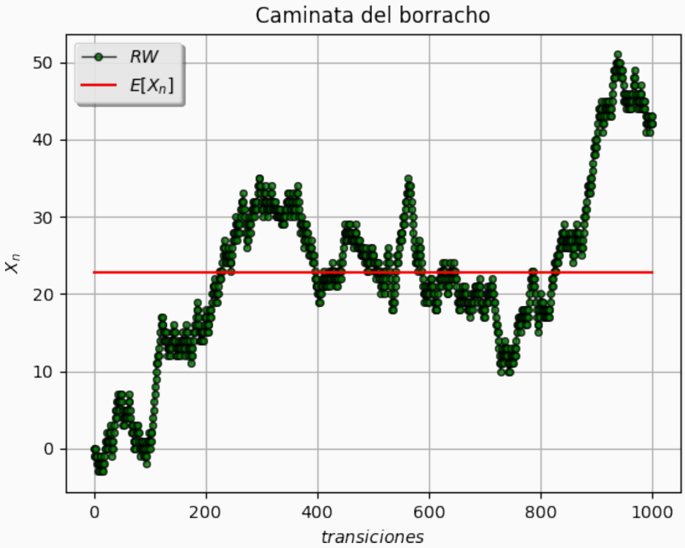
# Caminata del borracho



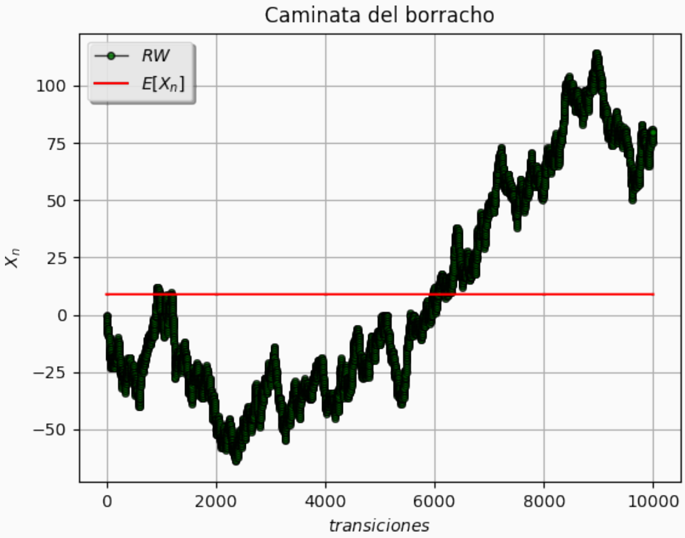
# Caminata Aleatoria



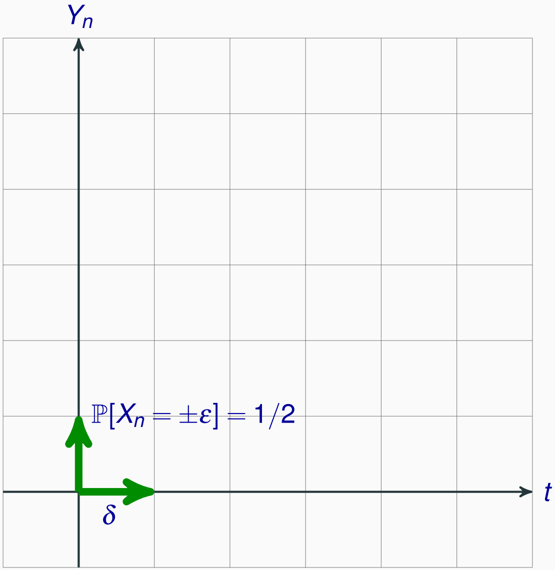
# Caminata Aleatoria



# Caminata Aleatoria



# Caminata del borracho



# Construcción

# Construcción

$\{X_n\}_{n=1}^{\infty} \quad i.i.d$

$\mathbb{P}\left[X_j = \pm\varepsilon\right] = \dfrac{1}{2}.$

# Construcción

$$\{X_n\}_{n=1}^{\infty} \quad i.i.d$$

$$\mathbb{P}\left[X_j = \pm\varepsilon\right] = \frac{1}{2}.$$

$$\boxed{\begin{aligned} Y_{\delta,\varepsilon}(0) &= 0 \\ Y_{\delta,\varepsilon}(n\delta) &= X_1 + X_2 + \cdots + X_n. \end{aligned}}$$

# Construcción

$$\{X_n\}_{n=1}^{\infty} \quad i.i.d$$

$$\mathbb{P}\left[X_j = \pm\varepsilon\right] = \frac{1}{2}.$$

$$\boxed{\begin{aligned} Y_{\delta,\varepsilon}(0) =& 0 \\ Y_{\delta,\varepsilon}(n\delta) =& X_1 + X_2 + \cdots + X_n. \end{aligned}}$$

Interpola linealmente

$$\begin{aligned} Y_{\delta,\varepsilon}(t) =& \frac{(n+1)\delta - t}{\delta} Y_{\delta,\varepsilon}(n\delta) \\ &+ \frac{t - n\delta}{\delta} Y_{\delta,\varepsilon}\left((n+1)\delta\right). \\ & n\delta < t < (n+1)\delta. \end{aligned}$$

# Construcción

$$\{X_n\}_{n=1}^{\infty} \quad i.i.d$$

$$\mathbb{P}\left[X_j = \pm\varepsilon\right] = \frac{1}{2}.$$

$$\boxed{\begin{aligned} Y_{\delta,\varepsilon}(0) &= 0 \\ Y_{\delta,\varepsilon}(n\delta) &= X_1 + X_2 + \cdots + X_n. \end{aligned}}$$

Interpola linealmente

$$\begin{aligned} Y_{\delta,\varepsilon}(t) = &\frac{(n+1)\delta - t}{\delta} Y_{\delta,\varepsilon}(n\delta) \\ &+ \frac{t - n\delta}{\delta} Y_{\delta,\varepsilon}\left((n+1)\delta\right). \\ &n\delta < t < (n+1)\delta. \end{aligned}$$

**Queremos**

$$\lim_{\substack{\delta\to 0\\ \varepsilon\to 0}} Y_{\delta,\varepsilon}$$

# Construcción

$$\{X_n\}_{n=1}^{\infty} \quad i.i.d$$

$$\mathbb{P}\left[X_j = \pm\varepsilon\right] = \frac{1}{2}.$$

$$\boxed{\begin{aligned} Y_{\delta,\varepsilon}(0) &= 0 \\ Y_{\delta,\varepsilon}(n\delta) &= X_1 + X_2 + \cdots + X_n. \end{aligned}}$$

Interpola linealmente

$$\begin{aligned} Y_{\delta,\varepsilon}(t) = & \frac{(n+1)\delta - t}{\delta} Y_{\delta,\varepsilon}(n\delta) \\ & + \frac{t - n\delta}{\delta} Y_{\delta,\varepsilon}((n+1)\delta). \\ & n\delta < t < (n+1)\delta. \end{aligned}$$

**Queremos**

$$\lim_{\substack{\delta\to 0\\ \varepsilon\to 0}} Y_{\delta,\varepsilon}$$

Tomate $\lambda \in \mathbb{R}$ fijo. Calcula

▸ caracteristica $\lim_{\delta,\varepsilon\to 0} \mathbb{E}\left[e^{i\lambda Y_{\delta,\varepsilon}(t)}\right].$

# Construcción

$$\{X_n\}_{n=1}^{\infty} \quad i.i.d$$

$$\mathbb{P}\left[X_j = \pm\varepsilon\right] = \frac{1}{2}.$$

$$\boxed{\begin{aligned} Y_{\delta,\varepsilon}(0) =& 0 \\ Y_{\delta,\varepsilon}(n\delta) =& X_1 + X_2 + \cdots + X_n. \end{aligned}}$$

Interpola linealmente

$$\begin{aligned} Y_{\delta,\varepsilon}(t) =& \frac{(n+1)\delta - t}{\delta} Y_{\delta,\varepsilon}(n\delta) \\ &+ \frac{t - n\delta}{\delta} Y_{\delta,\varepsilon}((n+1)\delta). \end{aligned}$$

$$n\delta < t < (n+1)\delta.$$

**Queremos**

$$\lim_{\substack{\delta\to 0 \\ \varepsilon\to 0}} Y_{\delta,\varepsilon}$$

$t = n\delta$,

$$\begin{aligned} \mathbb{E}\left[e^{i\lambda Y_{\delta,\varepsilon}(t)}\right] &= \prod_{j=1}^{n} \mathbb{E}\left[e^{i\lambda X_j}\right] \\ &= \left(\mathbb{E}\left[e^{i\lambda X_j}\right]\right)^n \\ &= \left(\frac{1}{2}e^{i\lambda\varepsilon} + \frac{1}{2}e^{-i\lambda\varepsilon}\right)^n \\ &= (\cos(\lambda\varepsilon))^n \\ &= (\cos(\lambda\varepsilon))^{\frac{t}{\delta}}. \end{aligned}$$

# Construcción

$$\{X_n\}_{n=1}^{\infty} \quad i.i.d$$

$$\mathbb{P}\left[X_j = \pm\varepsilon\right] = \frac{1}{2}.$$

$$\boxed{\begin{aligned} Y_{\delta,\varepsilon}(0) &= 0 \\ Y_{\delta,\varepsilon}(n\delta) &= X_1 + X_2 + \cdots + X_n. \end{aligned}}$$

Interpola linealmente

$$\begin{aligned} Y_{\delta,\varepsilon}(t) =& \frac{(n+1)\delta - t}{\delta} Y_{\delta,\varepsilon}(n\delta) \\ &+ \frac{t - n\delta}{\delta} Y_{\delta,\varepsilon}((n+1)\delta). \end{aligned}$$

$$n\delta < t < (n+1)\delta.$$

**Queremos**

$$\lim_{\substack{\delta\to 0\\ \varepsilon\to 0}} Y_{\delta,\varepsilon}$$

$$t = n\delta, \quad u = (cos(\lambda\varepsilon))^{\frac{1}{\delta}}$$

# Construcción

$$\{X_n\}_{n=1}^{\infty} \quad i.i.d$$

$$\mathbb{P}\left[X_j = \pm\varepsilon\right] = \frac{1}{2}.$$

$$\boxed{\begin{aligned} Y_{\delta,\varepsilon}(0) =& 0 \\ Y_{\delta,\varepsilon}(n\delta) =& X_1 + X_2 + \cdots + X_n. \end{aligned}}$$

Interpola linealmente

$$\begin{aligned} Y_{\delta,\varepsilon}(t) =& \frac{(n+1)\delta - t}{\delta} Y_{\delta,\varepsilon}(n\delta) \\ &+ \frac{t - n\delta}{\delta} Y_{\delta,\varepsilon}((n+1)\delta). \end{aligned}$$

$$n\delta < t < (n+1)\delta.$$

**Queremos**

$$\lim_{\substack{\delta\to 0\\ \varepsilon\to 0}} Y_{\delta,\varepsilon}$$

$t = n\delta, \quad u = (cos(\lambda\varepsilon))^{\frac{1}{\delta}}$

$\ln(u) = \frac{1}{\delta}\ln(cos(\lambda\varepsilon))$

Para $x$ chirris!!! $\ln(1+x) \approx x$

Para $\varepsilon$ chirris!!! $\cos(\lambda\varepsilon) \approx 1 - \frac{1}{2}\lambda^2\varepsilon^2$.

Entonces

# Construcción

$$\{X_n\}_{n=1}^{\infty} \quad i.i.d$$

$$\mathbb{P}\left[X_j = \pm\varepsilon\right] = \frac{1}{2}.$$

$$\boxed{\begin{aligned} Y_{\delta,\varepsilon}(0) =& 0 \\ Y_{\delta,\varepsilon}(n\delta) =& X_1 + X_2 + \cdots + X_n. \end{aligned}}$$

Interpola linealmente

$$\begin{aligned} Y_{\delta,\varepsilon}(t) =& \frac{(n+1)\delta - t}{\delta} Y_{\delta,\varepsilon}(n\delta) \\ &+ \frac{t - n\delta}{\delta} Y_{\delta,\varepsilon}((n+1)\delta). \end{aligned}$$

$$n\delta < t < (n+1)\delta.$$

**Queremos**

$$\lim_{\substack{\delta\to 0\\ \varepsilon\to 0}} Y_{\delta,\varepsilon}$$

$t = n\delta, \quad u = (cos(\lambda\varepsilon))^{\frac{1}{\delta}}$

$\ln(u) = \frac{1}{\delta}\ln(cos(\lambda\varepsilon))$

Para $x$ chirris!!! $\ln(1+x) \approx x$

Para $\varepsilon$ chirris!!! $\cos(\lambda\varepsilon) \approx 1 - \frac{1}{2}\lambda^2\varepsilon^2$.

Entonces

# Construcción

$$\{X_n\}_{n=1}^{\infty} \quad i.i.d$$

$$\mathbb{P}\left[X_j = \pm\varepsilon\right] = \frac{1}{2}.$$

$$\boxed{\begin{aligned} Y_{\delta,\varepsilon}(0) =& 0 \\ Y_{\delta,\varepsilon}(n\delta) =& X_1 + X_2 + \cdots + X_n. \end{aligned}}$$

Interpola linealmente

$$\begin{aligned} Y_{\delta,\varepsilon}(t) =& \frac{(n+1)\delta - t}{\delta} Y_{\delta,\varepsilon}(n\delta) \\ &+ \frac{t - n\delta}{\delta} Y_{\delta,\varepsilon}((n+1)\delta). \end{aligned}$$

$$n\delta < t < (n+1)\delta.$$

## Queremos

$$\lim_{\substack{\delta\to 0 \\ \varepsilon \to 0}} Y_{\delta,\varepsilon}$$

$t = n\delta, \quad u = (cos(\lambda\varepsilon))^{\frac{1}{\delta}}$

$$u \approx e^{-\frac{1}{2\delta}\lambda^2\varepsilon^2}$$

$$\mathbb{E}\left[e^{i\lambda Y_{\delta,\varepsilon}(t)}\right] \approx e^{-\frac{1}{2\delta}t\lambda^2\varepsilon^2}.$$

$\varepsilon^2 = \delta$

$$\lim_{\delta\to 0} \mathbb{E}\left[e^{i\lambda Y_{\delta,\sqrt{\delta}}(t)}\right] = e^{-\frac{1}{2}t\lambda^2}, \qquad \lambda \in \mathbb{R}.$$

# Construcción

$$\{X_n\}_{n=1}^{\infty} \quad i.i.d$$

$$\mathbb{P}\left[X_j = \pm\varepsilon\right] = \frac{1}{2}.$$

$$\boxed{\begin{aligned} Y_{\delta,\varepsilon}(0) =& 0 \\ Y_{\delta,\varepsilon}(n\delta) =& X_1 + X_2 + \cdots + X_n. \end{aligned}}$$

Interpola linealmente

$$\begin{aligned} Y_{\delta,\varepsilon}(t) =& \frac{(n+1)\delta - t}{\delta} Y_{\delta,\varepsilon}(n\delta) \\ &+ \frac{t - n\delta}{\delta} Y_{\delta,\varepsilon}((n+1)\delta). \end{aligned}$$

$$n\delta < t < (n+1)\delta.$$

**Queremos**

$$\lim_{\substack{\delta\to 0 \\ \varepsilon\to 0}} Y_{\delta,\varepsilon}$$

$$t = n\delta, \quad u = (cos(\lambda\varepsilon))^{\frac{1}{\delta}}$$

$$u \approx e^{-\frac{1}{2\delta}\lambda^2\varepsilon^2}$$

$$\mathbb{E}\left[e^{i\lambda Y_{\delta,\varepsilon}(t)}\right] \approx e^{-\frac{1}{2\delta}t\lambda^2\varepsilon^2}.$$

$$\varepsilon^2 = \delta$$

$$\lim_{\delta\to 0} \mathbb{E}\left[e^{i\lambda Y_{\delta,\sqrt{\delta}}(t)}\right] = e^{-\frac{1}{2}t\lambda^2}, \qquad \lambda \in \mathbb{R}.$$

$$\boxed{\therefore B(t) \overset{\mathscr{D}}{=} \lim_{\delta\to 0} Y_{\delta,\sqrt{\delta}}(t)}$$

# Construcción

**Teorema**

*Sea $Y_{\delta,\varepsilon}(t)$ una caminata aleatoria que inicia en 0 de saltos $\varepsilon$ y $-\varepsilon$ con igual probabilidad en los tiempos $\delta, 2\delta, 3\delta, \ldots$. Supongamos que $\varepsilon^2 = \delta$. Entonces para cada $t \geq 0$, el limite*

$$B(t) = \lim_{\delta \to 0} Y_{\delta,\sqrt{\delta}}(t),$$

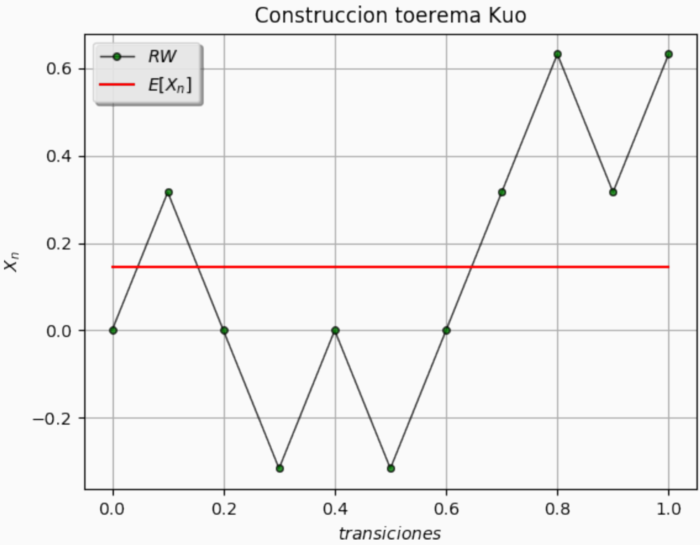
*existe en distribución. Además,*

$$\mathbb{E}\left[e^{i\lambda B(t)}\right] = e^{-\frac{1}{2}t\lambda^2}, \qquad \lambda \in \mathbb{R}.$$

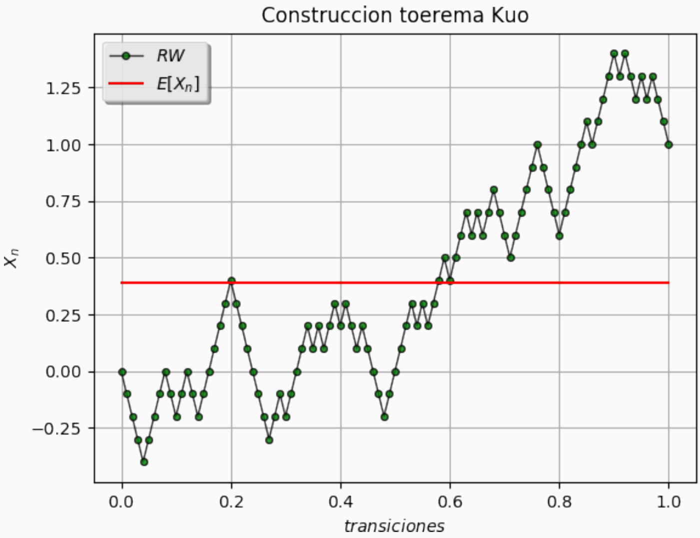
# Código

```
N = 10
T = 1.0
delta = T/np.float(N)
eps = 1.0/np.sqrt(np.float(N))
t = np.linspace(0,T,N+1)
b = np.random.binomial(1,.5, N) # bernulli 0,1
omega = 2.0 * b - 1         # bernulli -1,1
Xn = eps * (omega.cumsum())  # bernulli -h,h
Xn = np.concatenate(([0], Xn))
```

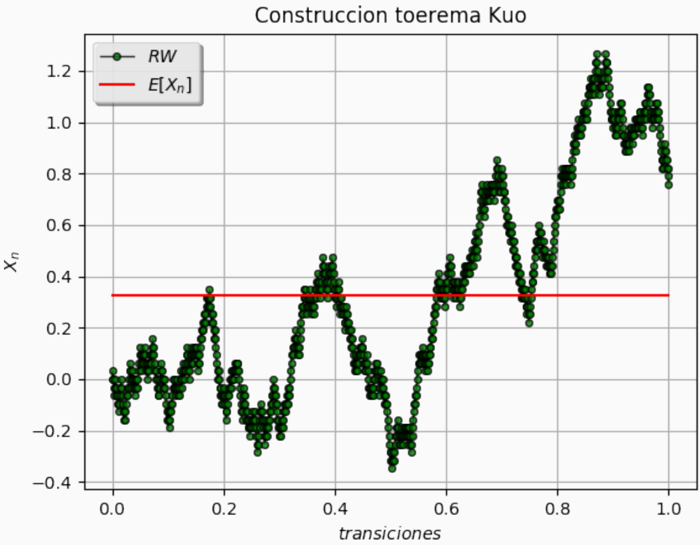
# Caminata Aleatoria de *n* transiciones



# Caminata Aleatoria de $n$ transiciones



# Caminata Aleatoria de $n$ transiciones



# Construcción

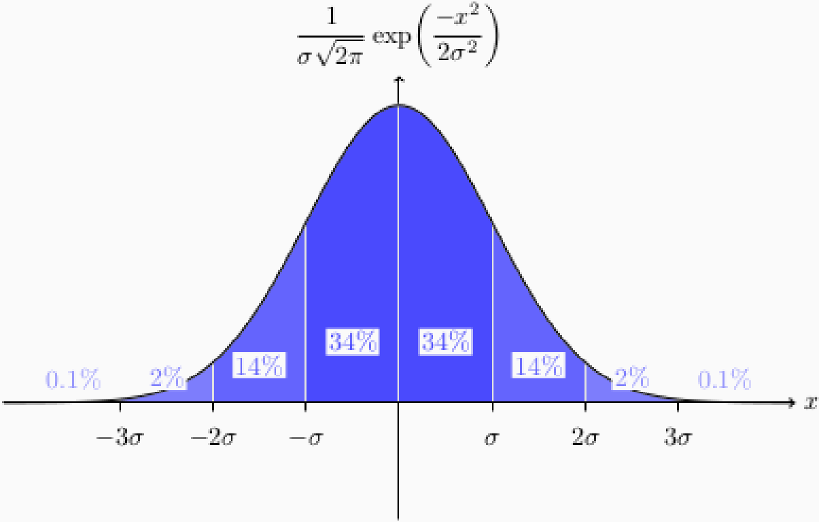
**Construcción**

$$\varepsilon^2 = \delta$$

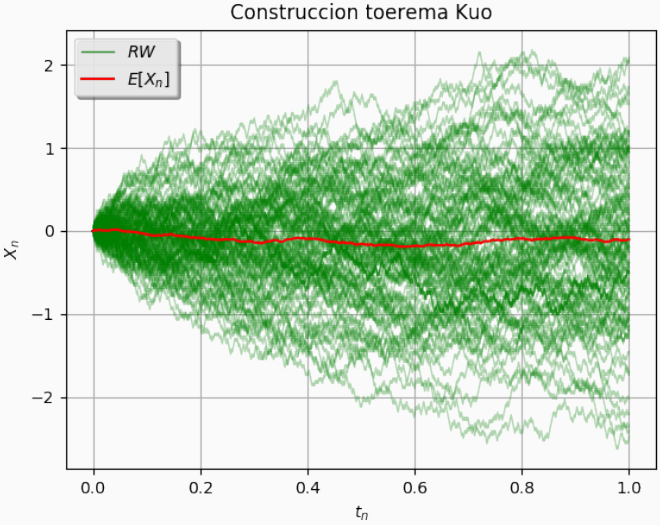
$$Y_{\delta,\varepsilon}(t) \xrightarrow[\delta,\varepsilon\to 0]{\mathscr{D}} B(t) \qquad \forall t \geq 0$$

$$\mathbb{E}\left[e^{i\lambda B(t)}\right] \xrightarrow{\delta,\varepsilon\to 0} e^{-\frac{1}{2}t\lambda^2}, \quad \lambda \in \mathbb{R}.$$

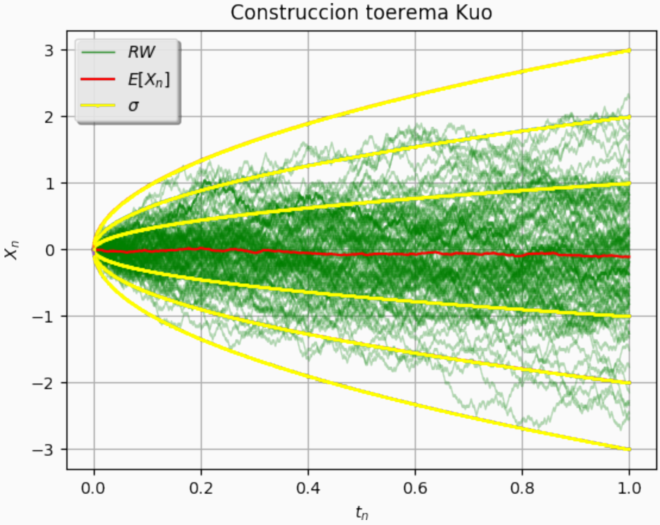
# Distribución Gaussiana



# Caminata Aleatoria en $[0, 1]$



# Caminata Aleatoria en $[0, 1]$



# Aproximación del MB en sentido Fuerte

**Definición**

El movimiento Browniano B(t) es el único proceso que satisface:

(I) $B(0) = 0$ c.s.

(II) Para $0 \leq s \leq t$, $B(t) - B(s) \sim \sqrt{t-s}N(0,1)$.

(III) Para culquier $t_0 \leq t_1 \leq \cdots \leq t_n \in [0, T]$, las v.a $B(t_i) - B(t_j)$ son independientes

# Aproximación del MB en sentido Fuerte

**Definición**

El movimiento Browniano B(t) es el único proceso que satisface:

(I) $B(0) = 0$ c.s.

(II) Para $0 \leq s \leq t$, $B(t) - B(s) \sim \sqrt{t-s}N(0,1)$.

(III) Para culquier $t_0 \leq t_1 \leq \cdots \leq t_n \in [0,T]$, las v.a $B(t_i) - B(t_j)$ son independientes

Entonces, dados $t \in [0,T]$, y un stencil

$$0 = t_0 \leq t_1 \leq \cdots \leq t_{M-1} \leq t_M = t$$

# Aproximación del MB en sentido Fuerte

**Definición**

El movimiento Browniano B(t) es el único proceso que satisface:

(I) $B(0) = 0$ c.s.

(II) Para $0 \leq s \leq t$, $B(t) - B(s) \sim \sqrt{t-s}N(0,1)$.

(III) Para culquier $t_0 \leq t_1 \leq \cdots \leq t_n \in [0,T]$, las v.a $B(t_i) - B(t_j)$ son independientes

Entonces, dados $t \in [0,T]$, y un stencil

$$0 = t_0 \leq t_1 \leq \cdots \leq t_{M-1} \leq t_M = t$$

$$B(t) = \sum_{j=1}^{M} B(t_j) - B(t_{j-1}).$$

# Aproximación del MB en sentido Fuerte

**Definición**

El movimiento Browniano B(t) es el único proceso que satisface:

(I) $B(0) = 0$ c.s.

(II) Para $0 \leq s \leq t$, $B(t) - B(s) \sim \sqrt{t-s}N(0,1)$.

(III) Para culquier $t_0 \leq t_1 \leq \cdots \leq t_n \in [0,T]$, las v.a $B(t_i) - B(t_j)$ son independientes

Entonces, dados $t \in [0,T]$, y un stencil

$$0 = t_0 \leq t_1 \leq \cdots \leq t_{M-1} \leq t_M = t$$

$$B(t) = \sum_{j=1}^{M} \underbrace{B(t_j) - B(t_{j-1})}_{:=\Delta B_j}.$$

# Aproximación del MB en sentido Fuerte

**Definición**

El movimiento Browniano B(t) es el único proceso que satisface:

(I) $B(0) = 0$ c.s.

(II) Para $0 \leq s \leq t$, $B(t) - B(s) \sim \sqrt{t-s}N(0,1)$.

(III) Para culquier $t_0 \leq t_1 \leq \cdots \leq t_n \in [0,T]$, las v.a $B(t_i) - B(t_j)$ son independientes

Tomando $\{t_n\}_{n=0}^{N}$, $t_n = nh$, entonces

$$B(t_n) \approx \sum_{j=0}^{n} \Delta B_j, \qquad \Delta B_0 := 0,$$

# Aproximación del MB en sentido Fuerte

**Definición**

El movimiento Browniano B(t) es el único proceso que satisface:

(I) $B(0) = 0$ c.s.

(II) Para $0 \leq s \leq t$, $B(t) - B(s) \sim \sqrt{t-s}N(0,1)$.

(III) Para culquier $t_0 \leq t_1 \leq \cdots \leq t_n \in [0, T]$, las v.a $B(t_i) - B(t_j)$ son independientes

Tomando $\{t_n\}_{n=0}^{N}$, $t_n = nh$, entonces

$$B(t_n) \approx \sum_{j=0}^{n} \Delta B_j, \qquad \Delta B_0 := 0, \qquad \Delta B_j \sim \sqrt{h}N(0,1).$$

# Aproximación Fuerte vs. Débil

# Debil vs Fuerte

dada

$$dx(t) = f(x(t))dt + g(x(t))dB(t),$$
$$x(0) = x_0, \quad t \in [0, T]$$

## Debil

$$X_{n+1} = X_n + f(X_n)h + g(X_n) \underbrace{\Delta B_n}_{\substack{\approx\sqrt{h}\varepsilon_n \\ \mathbb{P}[\varepsilon_n=\pm 1]=1/2}}$$

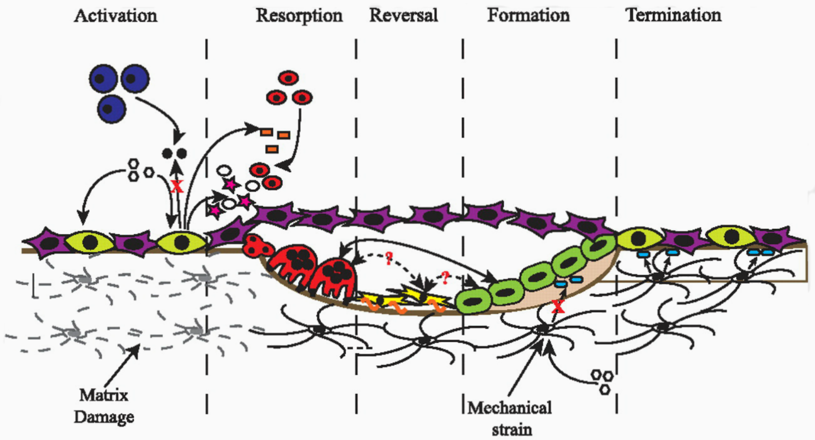
## Fuerte

$$X_{n+1} = X_n + f(X_n)h + g(X_n) \underbrace{\Delta B_n}_{\substack{\approx\sqrt{h}\varepsilon_n \\ \varepsilon_n\sim N(0,1)}}$$

# Ejemplo: Reconstrucción de masa osea

# Proceso de Remodelación en BMU

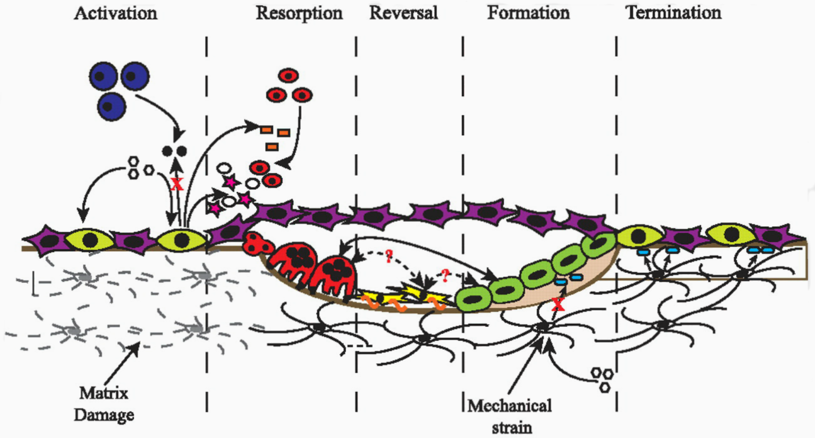
# Fases de remodelación




## Fases del proceso de remodelación

Raggatt, L. J. and Partridge, N. C. (2010).
**Cellular and Molecular Mechanisms of Bone Remodeling.**
*Journal of Biological Chemistry*, 285(33):25103–25108.

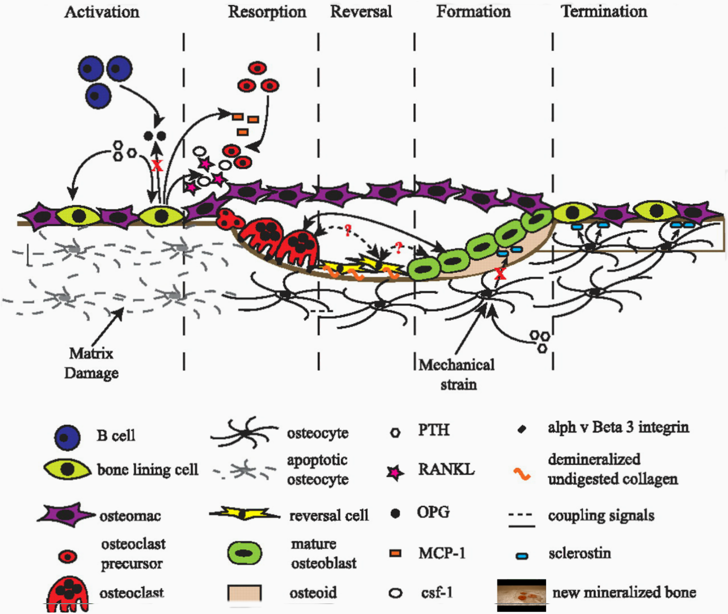
# Fases de remodelación



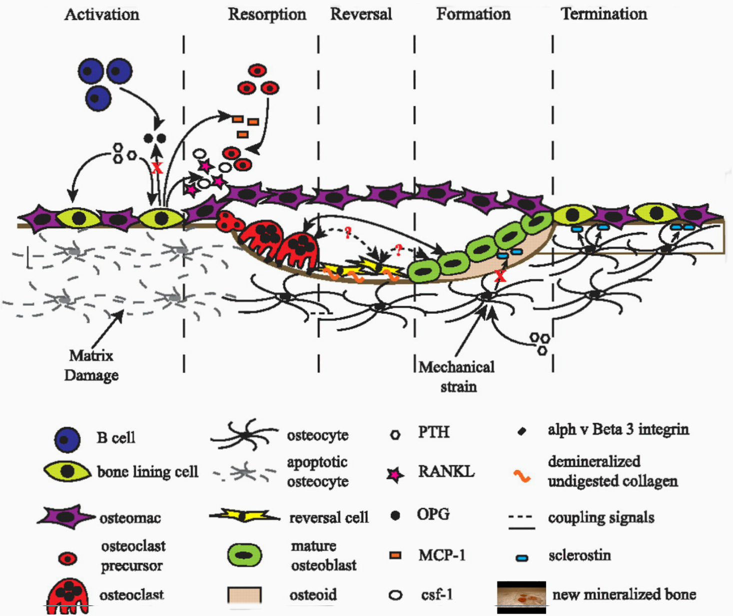

## Fases del proceso de remodelación

Raggatt, L. J. and Partridge, N. C. (2010).
**Cellular and Molecular Mechanisms of Bone Remodeling.**
*Journal of Biological Chemistry*, 285(33):25103–25108.

# Fases de remodelación



# Fases de remodelación



# El Modelo de Komarova

$$
\begin{aligned}
\frac{du}{dt} &= u^{\kappa_1}\left(\alpha_1 v^{\gamma_1} - \beta_1\right)\\
\frac{dv}{dt} &= v^{\kappa_2}\left(\alpha_2 u^{\gamma_2} - \beta_2\right)
\end{aligned}
$$

# El Modelo de Komarova

$$\frac{du}{dt} = u^{\kappa_1} \left( \alpha_1 v^{\gamma_1} - \beta_1 \right)$$
$$\frac{dv}{dt} = v^{\kappa_2} \left( \alpha_2 u^{\gamma_2} - \beta_2 \right)$$

Svetlana V. Komarova, Robert J. Smith, S.Jeffrey Dixon, Stephen M. Sims, and Lindi M. Wahl.
**Mathematical model predicts a critical role for osteoclast autocrine regulation in the control of bone remodeling.**
*Bone*, 33(2):206–215, aug 2003.

# El Modelo de Komarova

$$
\begin{aligned}
\frac{du}{dt} &= u^{\kappa_1}\left(\alpha_1 v^{\gamma_1} - \beta_1\right) \\
\frac{dv}{dt} &= v^{\kappa_2}\left(\alpha_2 u^{\gamma_2} - \beta_2\right) \\
\frac{dz}{dt} &= -k_1 \max\{u - \widetilde{u}, 0\} \\
&\quad + k_1 \max\{v - \widetilde{v}, 0\}
\end{aligned}
$$

Svetlana V. Komarova, Robert J. Smith, S.Jeffrey Dixon, Stephen M. Sims, and Lindi M. Wahl.
**Mathematical model predicts a critical role for osteoclast autocrine regulation in the control of bone remodeling.**
*Bone*, 33(2):206–215, aug 2003.

# El Modelo de Komarova

## Descripción

osteoclast regulación Crecimiento Muerte

$$\dot{u} = u^{\kappa_1}\left(\alpha_1 v^{\gamma_1} - \beta_1\right)$$

$$\dot{v} = v^{\kappa_2}\left(\alpha_2 u^{\gamma_2} - \beta_2\right)$$

osteoblast Interacción


Svetlana V. Komarova, Robert J. Smith, S.Jeffrey Dixon, Stephen M. Sims, and Lindi M. Wahl.
**Mathematical model predicts a critical role for osteoclast autocrine regulation in the control of bone remodeling.**
*Bone*, 33(2):206–215, aug 2003.

# El Modelo de Komarova

## Descripción

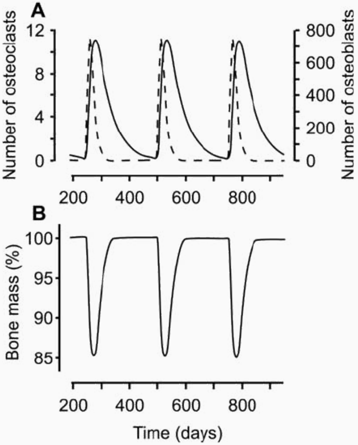
Punto fijo

$$\frac{dz}{dt} = -k_1 \max\{u - \tilde{u}, 0\} + k_2 \max\{v - \tilde{v}, 0\},$$

Centro
de masa

# El Modelo de Komarova

$$\frac{du}{dt} = u^{\kappa_1}\left(\alpha_1 v^{\gamma_1} - \beta_1\right)$$
$$\frac{dv}{dt} = v^{\kappa_2}\left(\alpha_2 u^{\gamma_2} - \beta_2\right)$$
$$\frac{dz}{dt} = -k_1 \max\{u - \widetilde{u}, 0\} + k_1 \max\{v - \widetilde{v}, 0\}$$



# Simplificación

$$\frac{du}{dt} = u^{\kappa_1}\left(\alpha_1 v^{\gamma_1} - \beta_1\right)$$
$$\frac{dv}{dt} = v^{\kappa_2}\left(\alpha_2 u^{\gamma_2} - \beta_2\right)$$

# Simplificación

$$\frac{du}{dt} = u^{\cancel{\kappa_1}}(\alpha_1 v^{\gamma_1} - \beta_1)$$
$$\frac{dv}{dt} = v^{\cancel{\kappa_2}}(\alpha_2 u^{\gamma_2} - \beta_2)$$

# Simplificación

$$\frac{du}{dt} = u(\alpha_1 v^{\gamma_1} - \beta_1)$$
$$\frac{dv}{dt} = v(\alpha_2 u^{\gamma_2} - \beta_2)$$

Silvia Jerez and B Chen.
**Stability analysis of a Komarova type model for the interactions of osteoblast and osteoclast cells during bone remodeling.**
*Mathematical biosciences*, 264:29–37, jun 2015.

**osteoclast** Crecimiento Muerte

$$
\begin{aligned}
\dot{u} &= u\left(\alpha_1 v^{\gamma_1} - \beta_1\right) \\
\dot{v} &= v\left(\alpha_2 u^{\gamma_2} - \beta_2\right),
\end{aligned}
$$

**osteoblast** Interacción

Silvia Jerez and B Chen.
**Stability analysis of a Komarova type model for the interactions of osteoblast and osteoclast cells during bone remodeling.**
*Mathematical biosciences*, 264:29–37, jun 2015.

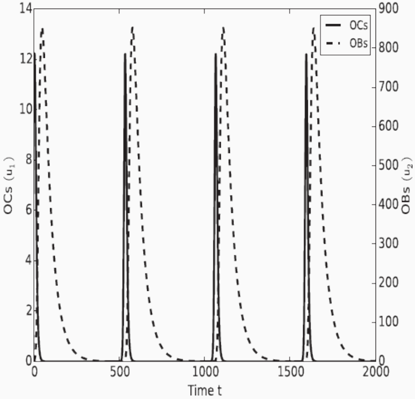
# Simplificación

$$\frac{du}{dt} = u(\alpha_1 v^{\gamma_1} - \beta_1)$$
$$\frac{dv}{dt} = v(\alpha_2 u^{\gamma_2} - \beta_2)$$
$$\frac{dz}{dt} = -k_1 \operatorname{máx}\{u - \widetilde{u}, 0\} + k_1 \operatorname{máx}\{v - \widetilde{v}, 0\}$$


Silvia Jerez and B Chen.
**Stability analysis of a Komarova type model for the interactions of osteoblast and osteoclast cells during bone remodeling.**
*Mathematical biosciences*, 264:29–37, jun 2015.

# Simplificación

$$\frac{du}{dt} = u(\alpha_1 v^{\gamma_1} - \beta_1)$$
$$\frac{dv}{dt} = v(\alpha_2 u^{\gamma_2} - \beta_2)$$
$$\frac{dz}{dt} = -k_1 \max\{u - \widetilde{u}, 0\} + k_1 \max\{v - \widetilde{v}, 0\}$$

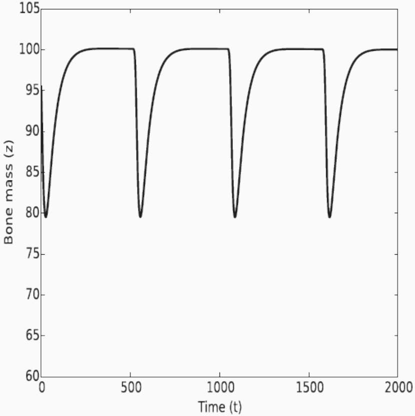




Silvia Jerez and B Chen.
**Stability analysis of a Komarova type model for the interactions of osteoblast and osteoclast cells during bone remodeling.**
*Mathematical biosciences*, 264:29–37, jun 2015.

# Simplificación

$$\frac{du}{dt} = u(\alpha_1 v^{\gamma_1} - \beta_1)$$
$$\frac{dv}{dt} = v(\alpha_2 u^{\gamma_2} - \beta_2)$$
$$\frac{dz}{dt} = -k_1 \max\{u - \widetilde{u}, 0\} + k_1 \max\{v - \widetilde{v}, 0\}$$





Silvia Jerez and B Chen.
**Stability analysis of a Komarova type model for the interactions of osteoblast and osteoclast cells during bone remodeling.**
*Mathematical biosciences*, 264:29–37, jun 2015.

# ¿Por qué incorporar incertidumbre a un modelo?

**Efectos Ambientales**

- Extinción
- Epidemias

# ¿Por qué incorporar incertidumbre a un modelo?

**Efectos Ambientales**

- Extinción
- Epidemias

**Ruido ambiental suprime extinción**

Mao, X., Marion, G., and Renshaw, E. (2002).
**Environmental Brownian noise suppresses explosions in population dynamics.**
*Stochastic Processes and their Applications*, 97(1):95–110.

# ¿Por qué incorporar incertidumbre a un modelo?

## Efectos Ambientales

- Extinción
- Epidemias

## Color (correlación) induce extinción


Ripa, J. and Lundberg, P. (1996).
**Noise Colour and the Risk of Population Extinctions.**
*Proceedings of the Royal Society B: Biological Sciences*, 263(1377):1751–1753.

# ¿Por qué incorporar incertidumbre a un modelo?

**Efectos Ambientales**

- Extinción
- Epidemias

**$\mathscr{R}_0$: endémico g.a.e $\rightarrow$ osc. per**

Allen, L. and van den Driessche, P. (2013). **Relations between deterministic and stochastic thresholds for disease extinction in continuous- and discrete-time infectious disease models.** *Mathematical Biosciences*, 243(1):99–108.

# Alternativas

## Efectos Ambientales

- Extinción
- Epidemias

## En biología

- CTMCs
- Perturbación de parámetros
- Procesos reversibles en media

# Alternativas

## Efectos Ambientales

- Extinción
- Epidemias

## En biología

- CTMCs
- Perturbación de parámetros
- Procesos reversibles en media

## MC + ME $\rightarrow$ *SDE*


Allen, L. J. (2017).
**A primer on stochastic epidemic models: Formulation, numerical simulation, and analysis.**
*Infectious Disease Modelling*, 2(2):128–142.

# Alternativas

## Efectos Ambientales

- Extinción
- Epidemias

## En biología

- CTMCs
- Perturbación de parámetros
- Procesos reversibles en media

## $\varphi dt \rightsquigarrow \varphi dt + \sigma dB_t$


Gray, A., Greenhalgh, D., Hu, L., Mao, X., and Pan, J. (2011).
**A Stochastic Differential Equation SIS Epidemic Model.**
*SIAM Journal on Applied Mathematics*, 71(3):876–902.

# Alternativas

## Efectos Ambientales

- Extinción
- Epidemias

## En biología

- CTMCs
- Perturbación de parámetros
- Procesos reversibles en media

## $\varphi dt \rightsquigarrow \varphi dt + F(x)dB_t$

Schurz, H. and Tosun, K. (2015). **Stochastic Asymptotic Stability of SIR Model with Variable Diffusion Rates.** *Journal of Dynamics and Differential Equations*, 27(1):69–82.

# Alternativas

**Efectos Ambientales**

- Extinción
- Epidemias

**En biología**

- CTMCs
- Perturbación de parámetros
- Procesos reversibles en media

$d\varphi_t = (\varphi_e - \varphi_t)dt + \sigma_\varphi dBt$


Allen, E. (2016).
**Environmental variability and mean-reverting processes.**
*Discrete and Continuous Dynamical Systems - Series B*, 21(7):2073–2089.

# Alternativas

### Efectos Ambientales

- Extinción
- Epidemias

### En biología

- CTMCs
- **Perturbación de parámetros**
- Procesos reversibles en media

$\varphi dt \rightsquigarrow \varphi dt + \sigma dB_t$

# Pertrubación

$$\frac{du}{dt} = u(\alpha_1 v^{\gamma_1} - \beta_1)$$
$$\frac{dv}{dt} = v(\alpha_2 u^{\gamma_2} - \beta_2)$$

# Pertrubación

$$\frac{du}{dt} = u\left(\alpha_1 v^{\gamma_1} - \beta_1\right)$$
$$\frac{dv}{dt} = v\left(\alpha_2 u^{\gamma_2} - \beta_2\right)$$

$$du = \alpha_1 u v^{\gamma_1} dt - u\beta_1 dt$$
$$dv = \alpha_2 u^{\gamma_2} v dt - v\beta_2 dt$$

# Pertrubación

$$\frac{du}{dt} = u(\alpha_1 v^{\gamma_1} - \beta_1)$$
$$\frac{dv}{dt} = v(\alpha_2 u^{\gamma_2} - \beta_2)$$

$$du = \alpha_1 u v^{\gamma_1} dt - u\beta_1 dt$$
$$dv = \alpha_2 u^{\gamma_2} v dt - v\beta_2 dt$$

$$\beta_i dt \rightsquigarrow \beta_i dt + \sigma_i dB_i(t)$$

# Pertrubación

$$\frac{du}{dt} = u\left(\alpha_1 v^{\gamma_1} - \beta_1\right)$$
$$\frac{dv}{dt} = v\left(\alpha_2 u^{\gamma_2} - \beta_2\right)$$

$$du = \alpha_1 u v^{\gamma_1} dt - u\beta_1 dt$$
$$dv = \alpha_2 u^{\gamma_2} v dt - v\beta_2 dt$$

$$\beta_i dt \rightsquigarrow \beta_i dt + \sigma_i dB_i(t)$$

**Nuevo Modelo**

$$du_t = u_t\left(\alpha_1 v_t^{\gamma_1} - \beta_1\right) dt + \sigma_1 u_t dB_1(t)$$
$$dv_t = v_t\left(\alpha_2 u_t^{\gamma_2} - \beta_2\right) dt + \sigma_2 v_t dB_2(t)$$

# Existe, única y positiva

(H-1) $\gamma_1 < 0$, $\gamma_2 > 0$,

(H-2) $|\gamma_1| \leq \gamma_2$ ,

(H-3) $\alpha_1\gamma_2 \leq \alpha_2|\gamma_1|$,

(H-4) $-1 < \gamma_1 < 0$ and
$0 < \gamma_2 < 1$,

(H-5) $\exists p > 1$ t.q
$\beta_i > \frac{1}{2}p(p-1)\sigma_i$

# Existe, única y positiva

(H-1) $\gamma_1 < 0$, $\gamma_2 > 0$,

(H-2) $|\gamma_1| \leq \gamma_2$ ,

(H-3) $\alpha_1 \gamma_2 \leq \alpha_2 |\gamma_1|$,

(H-4) $-1 < \gamma_1 < 0$ and
$0 < \gamma_2 < 1$,

(H-5) $\exists p > 1$ t.q
$\beta_i > \frac{1}{2} p(p-1)\sigma_i$

# Existe, única y positiva

(H-1) $\gamma_1 < 0$, $\gamma_2 > 0$,

(H-2) $|\gamma_1| \leq \gamma_2$ ,

(H-3) $\alpha_1 \gamma_2 \leq \alpha_2 |\gamma_1|$,

(H-4) $-1 < \gamma_1 < 0$ and $0 < \gamma_2 < 1$,

(H-5) $\exists p > 1$ t.q $\beta_i > \frac{1}{2} p(p-1)\sigma_i$

$$
\begin{aligned}
du_t &= u_t \left( \alpha_1 v_t^{\gamma_1} - \beta_1 \right) dt + \sigma_1 u_t dB_1(t) \\
dv_t &= v_t \left( \alpha_2 u_t^{\gamma_2} - \beta_2 \right) dt + \sigma_2 v_t dB_2(t) \\
x_t &= (u_t, v_t)
\end{aligned}
$$

# Existe, única y positiva

(H-1) $\gamma_1 < 0$, $\gamma_2 > 0$,

(H-2) $|\gamma_1| \leq \gamma_2$ ,

(H-3) $\alpha_1 \gamma_2 \leq \alpha_2 |\gamma_1|$,

(H-4) $-1 < \gamma_1 < 0$ and $0 < \gamma_2 < 1$,

(H-5) $\exists p > 1$ t.q $\beta_i > \frac{1}{2} p(p-1)\sigma_i$

$$du_t = u_t \left(\alpha_1 v_t^{\gamma_1} - \beta_1\right) dt + \sigma_1 u_t dB_1(t)$$
$$dv_t = v_t \left(\alpha_2 u_t^{\gamma_2} - \beta_2\right) dt + \sigma_2 v_t dB_2(t)$$
$$x_t = (u_t, v_t)$$

**Teorema**

*$\forall$ $(u_0, v_0)$ positivos, $\exists!$ $(u_t, v_t)$ continua e invariante $\in \mathbb{R}^2_+$ (c.p.1.).*

# Existe, única y positiva

(H-1) $\gamma_1 < 0$, $\gamma_2 > 0$,

(H-2) $|\gamma_1| \leq \gamma_2$ ,

(H-3) $\alpha_1 \gamma_2 \leq \alpha_2 |\gamma_1|$,

(H-4) $-1 < \gamma_1 < 0$ and $0 < \gamma_2 < 1$,

(H-5) $\exists p > 1$ t.q $\beta_i > \frac{1}{2} p(p-1)\sigma_i$

$$
\begin{aligned}
du_t &= u_t \left(\alpha_1 v_t^{\gamma_1} - \beta_1\right) dt + \sigma_1 u_t dB_1(t) \\
dv_t &= v_t \left(\alpha_2 u_t^{\gamma_2} - \beta_2\right) dt + \sigma_2 v_t dB_2(t) \\
x_t &= (u_t, v_t)
\end{aligned}
$$

**Teorema**

*$\forall$ $(u_0, v_0)$ positivos, $\exists!$ $(u_t, v_t)$ continua e invariante $\in \mathbb{R}^2_+$ (c.p.1.).*

# Existe, única y positiva

(H-1) $\gamma_1 < 0$, $\gamma_2 > 0$,

(H-2) $|\gamma_1| \leq \gamma_2$ ,

(H-3) $\alpha_1 \gamma_2 \leq \alpha_2 |\gamma_1|$,

(H-4) $-1 < \gamma_1 < 0$ and
$0 < \gamma_2 < 1$,

(H-5) $\exists p > 1$ t.q
$\beta_i > \frac{1}{2} p(p-1)\sigma_i$

$$du_t = u_t \left( \alpha_1 v_t^{\gamma_1} - \beta_1 \right) dt + \sigma_1 u_t dB_1(t)$$
$$dv_t = v_t \left( \alpha_2 u_t^{\gamma_2} - \beta_2 \right) dt + \sigma_2 v_t dB_2(t)$$
$$x_t = (u_t, v_t)$$

**Teorema**

*$\forall$ $(u_0, v_0)$ positivos, $\exists!$ $(u_t, v_t)$ continua e invariante $\in \mathbb{R}_+^2$ (c.p.1.).*

**Teorema (a.l.p.)**

*$\forall \varepsilon > 0$, $\exists K(\varepsilon) < \infty$ t.q.*
$\limsup_{t\to\infty} \mathbb{P}[|x_t| \geq K] \leq \varepsilon$.

# Existe, única y positiva

(H-1) $\gamma_1 < 0$, $\gamma_2 > 0$,

(H-2) $|\gamma_1| \leq \gamma_2$ ,

(H-3) $\alpha_1 \gamma_2 \leq \alpha_2 |\gamma_1|$,

(H-4) $-1 < \gamma_1 < 0$ and

$0 < \gamma_2 < 1$

(H

$$
\begin{aligned}
du_t &= u_t \left(\alpha_1 v_t^{\gamma_1} - \beta_1\right) dt + \sigma_1 u_t dB_1(t) \\
dv_t &= v_t \left(\alpha_2 u_t^{\gamma_2} - \beta_2\right) dt + \sigma_2 v_t dB_2(t) \\
x_t &= (u_t, v_t)
\end{aligned}
$$

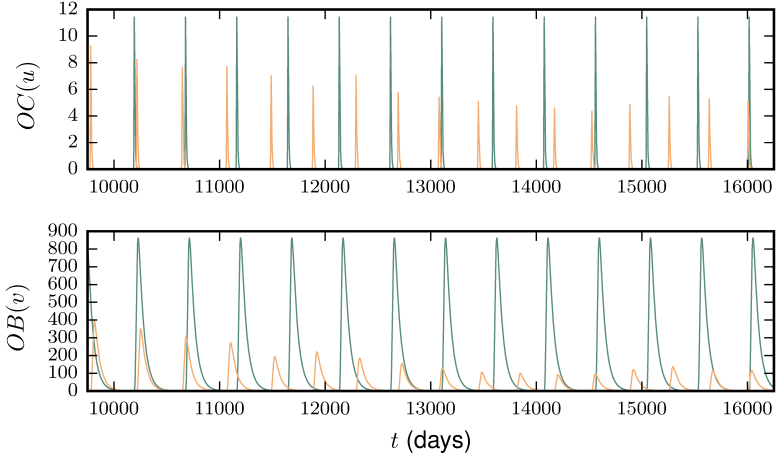
**Teorema (oscilaciones)**

$$
\limsup_{t\to\infty} u_t \geq \xi_2, \qquad \limsup_{t\to\infty} v_t \geq \xi_1, \qquad c.s.
$$

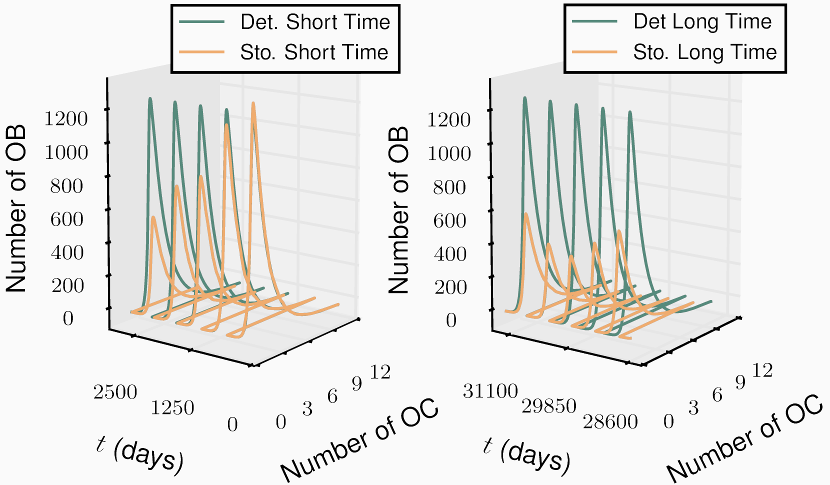
$$
\liminf_{t\to\infty} u_t \leq \xi_2, \qquad \liminf_{t\to\infty} v_t \leq \xi_1, \qquad c.s.
$$

$$
\xi_1 = \left(\frac{\beta_1 + \frac{1}{2}\sigma_1^2}{\alpha_1}\right)^{\frac{1}{\gamma_1}}, \qquad \xi_2 = \left(\frac{\beta_2 + \frac{1}{2}, \sigma_2^2}{\alpha_2}\right)^{\frac{1}{\gamma_2}}.
$$

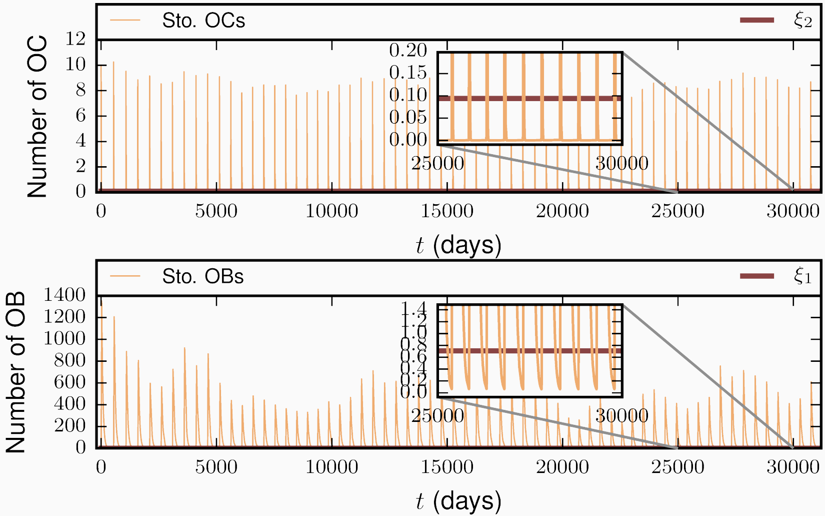
# Comparación de Fases



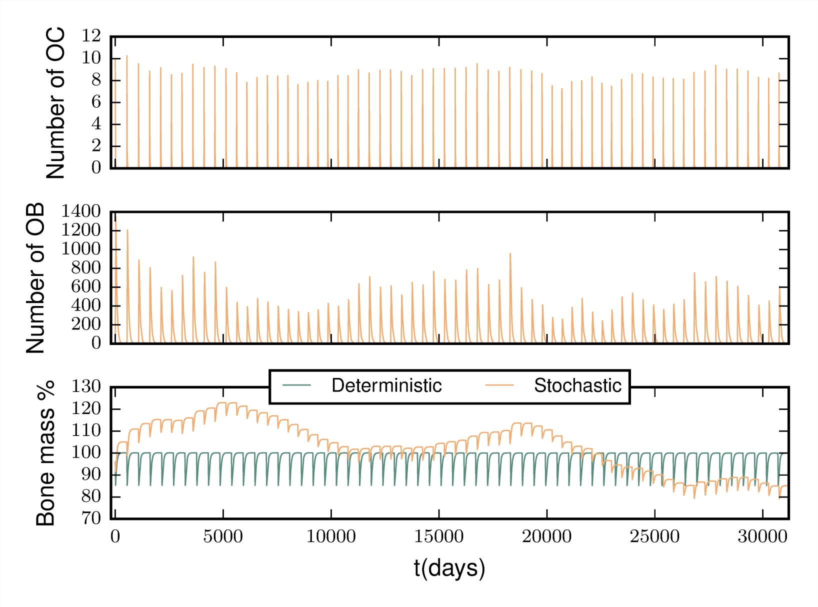
# PF tiempo corto (7 años) vs timpo largo (80-90 años)



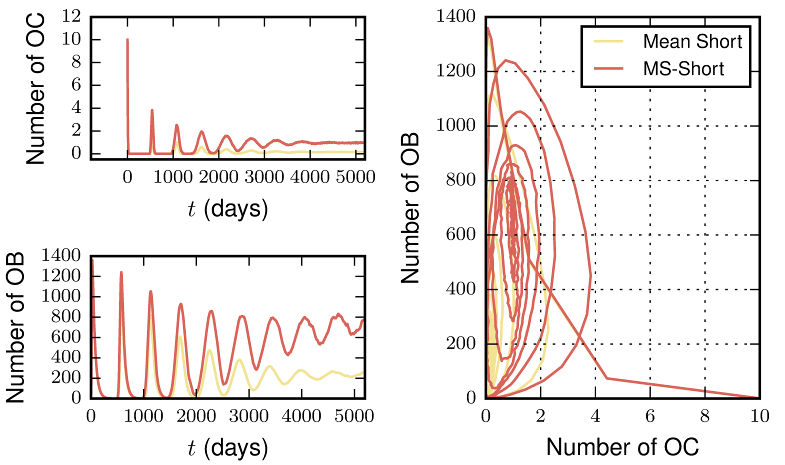
# Oscilaciones en torno a $\xi_i$



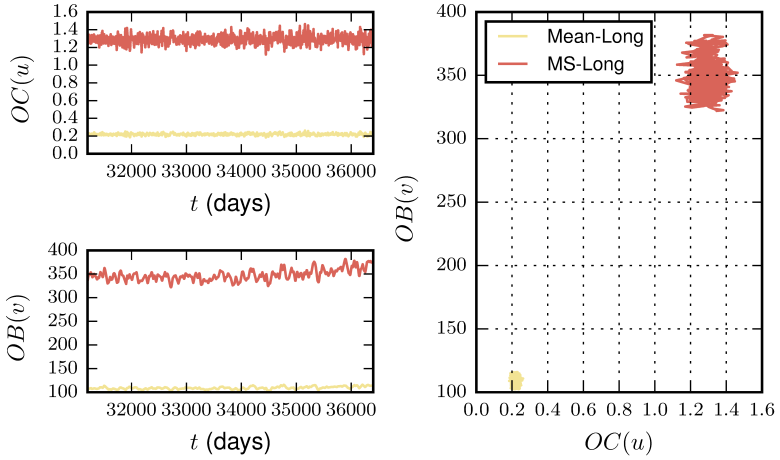
# Trayectoria larga y masa osea



# Momentos a tiempo corto (13 años)



# Momentos a tiempo largo



# Comentarios Finales

S. Jerez, S. Díaz-Infante, and B. Chen.
***Mathematical Biosciences*, 299:153 – 164, 2018.**

# Gracias!!!

S. Jerez, S. Díaz-Infante, and B. Chen.
***Mathematical Biosciences*, 299:153 – 164, 2018.**

S. Jerez, S. Díaz-Infante, and B. Chen.
***Mathematical Biosciences*, 299:153 – 164, 2018.**

Git-Hub

# Función característica

**Definición (Función característica)**

Sea $X$ v. a., entonces,

$$\phi_X(t) = \mathbb{E}\left[e^{itX}\right], \qquad t \in \mathbb{R},$$

es la función característica de $X$.

**Teorema de continuidad**

Sea $\{X_n\}_{n=1}^{\infty}$ $v.a.$, entonces

$$X_n \xrightarrow{\mathscr{D}} X \Leftrightarrow \phi_{X_n}(t) \to \phi_X(t).$$

◂ Const

# Función característica

**Definición (Función característica)**

Sea $X$ v. a., entonces,

$$\phi_X(t) = \mathbb{E}\left[e^{itX}\right], \qquad t \in \mathbb{R},$$

es la función característica de $X$.

**Teorema de continuidad**

Sea $\{X_n\}_{n=1}^{\infty} \quad v.a.$, entonces

$$X_n \xrightarrow{\mathscr{D}} X \Leftrightarrow \phi_{X_n}(t) \to \phi_X(t).$$

◂ Const

# Integral Estocástica

**Integral**

$$\int_0^T f(\cdot)d(\cdot)$$

# Integral Estocástica

**Integral**

$$\int_0^T f(\cdot) d(\cdot)$$

$$f : [0, T] \to \mathbb{R}$$

Determinista:

$$\int_0^T f(\cdot) d(\cdot) \approx \sum_{j=0}^{N-1} f(t_j)(t_{j+1} - t_j)$$

# Integral Estocástica

**Integral**

$$\int_0^T f(\cdot)dB(\cdot)$$

$$f : [0, T] \times \Omega \to \mathbb{R}$$

Determinista:

$$\int_0^T f(\cdot)d(\cdot) \approx \sum_{j=0}^{N-1} f(t_j)(t_{j+1} - t_j)$$

**Itô**

$$\approx \sum_{j=0}^{N-1} f(t_j)(B_{t_{j+1}} - B_{t_j})$$

# Integral Estocástica

**Integral**

$$\int_0^T f(\cdot)dB(\cdot)$$

$$f : [0, T] \times \Omega \to \mathbb{R}$$

Determinista:

$$\int_0^T f(\cdot)d(\cdot) \approx \sum_{j=0}^{N-1} f(t_j)(t_{j+1} - t_j)$$

**Itô**

$$\approx \sum_{j=0}^{N-1} f(t_j)(B_{t_{j+1}} - B_{t_j})$$

**Stratonovich**

$$\approx \sum_{j=0}^{N-1} f\left(\frac{t_j + t_{j+1}}{2}\right)(B_{t_{j+1}} - B_{t_j})$$

# Existencia y unicidad de soluciones fuertes para EDEs

**Sea $dX_t = f(t, X_t)dt + f(t, X_t)dB_t$ en el sentido de Itô, t.q.**

(EU1) (Medibles): $f, g$ son $\mathscr{L}^2-$medibles en $(t, x) \in [t_0, T] \times \mathbb{R}^d$.

(EU2) (Lipschitz): $\exists K > 0$ t.q.
$\forall t \in [t_0, T]$, $\forall x, y \in \mathbb{R}^d$

$$|f(t,x) - f(t,y)| \leq K|x-y|, \quad |g(t,x) - g(t,y)| \leq K|x-y|$$

(EU3) (De crecimiento lineal): $\exists K > 0$, t.q. $\forall t \in [t_0, T]$, $\forall x \in \mathbb{R}^d$

$$|f(t,x)|^2 \leq K^2(1+|x|^2), \quad |g(t,x)|^2 \leq K^2(1+|x|^2)$$

(EU4) (Condición inicial): $X_{t_0}$ es $\mathscr{F}_{t_0}-$medible con $\mathbb{E}\left[|X_{t_0}|\right] < \infty$.

Entonces, $\exists ! X_t$ en $[t_0, T]$ con $\sup_{t_0 \leq t \leq T} \mathbb{E}(|X_t|^2) < \infty$.

◂ Construccion

# Existencia y unicidad de soluciones fuertes para EDEs

**Sea $dX_t = f(t, X_t)dt + f(t, X_t)dB_t$ en el sentido de Itô, t.q.**

(EU1) (Medibles): $f, g$ son $\mathscr{L}^2-$medibles en $(t, x) \in [t_0, T] \times \mathbb{R}^d$.

(EU2) (Local Lipschitz): $\exists K_n > 0$ t.q.
$\forall t \in [t_0, T]$, $\forall x, y \in \mathbb{R}^d$ t.q. $|x - y| \leq n$

$$|f(t,x) - f(t,y)| \leq K_n|x - y|, \quad |g(t,x) - g(t,y)| \leq K_n|x - y|$$

(EU3) (De crecimiento lineal): $\exists K > 0$, t.q. $\forall t \in [t_0, T]$, $\quad \forall x \in \mathbb{R}^d$

$$|f(t,x)|^2 \leq K^2(1 + |x|^2), \quad |g(t,x)|^2 \leq K^2(1 + |x|^2)$$

(EU4) (Condición inicial): $X_{t_0}$ es $\mathscr{F}_{t_0}-$medible con $\mathbb{E}\left[|X_{t_0}|\right] < \infty$.

Entonces, $\exists! X_t$ en $[t_0, T]$ con $\sup\limits_{t_0 \leq t \leq T} \mathbb{E}(|X_t|^2) < \infty$.

Construccion

# Existencia y unicidad de soluciones fuertes para EDEs

**Sea $dX_t = f(t,X_t)dt + f(t,X_t)dB_t$ en el sentido de Itô, t.q.**

(EU1) (Medibles): $f, g$ son $\mathscr{L}^2$−medibles en $(t,x) \in [t_0,T] \times \mathbb{R}^d$.

(EU2) (Local Lipschitz): $\exists K_n > 0$ t.q.
$\forall t \in [t_0,T]$, $\forall x,y \in \mathbb{R}^d$ t.q. $|x-y| \leq n$

$$|f(t,x) - f(t,y)| \leq K_n|x-y|, \quad |g(t,x) - g(t,y)| \leq K_n|x-y|$$

(EU3) (Monotonia) $\exists K > 0$, t.q. $\forall t \in [t_0,T]$, $\quad \forall x \in \mathbb{R}^d$

$$\langle x, f(t,x)\rangle + |g(x)|^2 \leq K(1+|x|^2)$$

(EU4) (Condición inicial): $X_{t_0}$ es $\mathscr{F}_{t_0}$−medible con $\mathbb{E}\left[|X_{t_0}|\right] < \infty$.

Entonces, $\exists! X_t$ en $[t_0,T]$ con $\sup_{t_0 \leq t \leq T} \mathbb{E}(|X_t|^2) < \infty$.

Construccion

# Existencia y unicidad de soluciones fuertes para EDEs

**Sea $dX_t = f(t, X_t)dt + f(t, X_t)dB_t$ en el sentido de Itô, t.q.**

(EU1) (Medibles): $f, g$ son $\mathscr{L}^2-$medibles en $(t,x) \in [t_0, T] \times \mathbb{R}^d$.

(EU2) (Local Lipschitz): $\exists K_n > 0$ t.q.
$\forall t \in [t_0, T], \forall x, y \in \mathbb{R}^d$ t.q. $|x-y| \leq n$

$$|f(t,x) - f(t,y)| \leq K_n|x-y|, \quad |g(t,x) - g(t,y)| \leq K_n|x-y|$$

(EU3) (Monotonia) $\exists K > 0$, t.q. $\forall t \in [t_0, T], \quad \forall x \in \mathbb{R}^d$

$$\langle x, f(t,x)\rangle + |g(x)|^2 \leq K(1+|x|^2)$$

(EU4) (Condición inicial): $X_{t_0}$ es $\mathscr{F}_{t_0}-$medible con $\mathbb{E}\left[|X_{t_0}|\right] < \infty$.

Entonces, $\exists! X_t$ en $[t_0, T]$ con $\sup_{t_0 \leq t \leq T} \mathbb{E}(|X_t|^2) < \infty$.

Construccion

# Lema de Gronwall

**Lema (de Gronwall)**

Sean $\alpha, \beta : [t_0, T] \to \mathbb{R}$ funciones integrables t.q.

$$0 \leq \alpha(t) \leq \beta(t) + L \int_{t_0}^{t} \alpha(s) ds \qquad t \in [t_0, T].$$

Entonces

$$\alpha(t) \leq \beta(t) + L \int_{t_0}^{t} e^{L(t-s)} \beta(s) ds$$

◂ Prueba

◂ idea

# Desigualdad de LyapunovI

Sea $X$ una v.a integrable y $0 < q \leq p$ entonces

**Sea $X$ una v.a integrable y $0 < q \leq p$ entonces**

$$\mathbb{E}\left(|X|^q\right) \leq \mathbb{E}\left(|X|^p\right)^{\frac{q}{p}}$$

Prueba

# Isometría de Itô

## Propiedades Integral de Itô

1. $\mathbb{E}\left[\int_0^T g(r)dB_r\right] = 0$
2. (Isometría) $\mathbb{E}\left[\left(\int_0^T g(r)dB_r\right)^2\right] = \int_0^T g^2(r)dr$

◂ Prueba

## Apendice A

$$A^{(1)}(h,u) := \begin{pmatrix} e^{ha_1(u)} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & e^{ha_d(u)} \end{pmatrix},$$

$$A^{(2)}(h,u) := \begin{pmatrix} \left(\frac{e^{ha_1(u)}-1}{a_1(u)}\right)\mathbf{1}_{\{E_1^c\}} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \left(\frac{e^{ha_d(u)}-1}{a_d(u)}\right)\mathbf{1}_{\{E_d^c\}} \end{pmatrix} + h \begin{pmatrix} \mathbf{1}_{\{E_1\}} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \mathbf{1}_{\{E_d\}} \end{pmatrix},$$

$$E_j := \{x \in \mathbb{R}^d : a_j(x) = 0\}, \qquad b(u) := \Big(b_1(u^{(-1)}), \dots, b_d(u^{(-d)})\Big)^T.$$

◂ LS

# Apendice B: Resultado para ceros aislados

**Definición (DD respecto a $p$)**

$u, \mathbf{p} \in \mathbb{R}^2$, $\alpha$ angulo positivo respecto a eje-$x$ segmento $\overline{u\mathbf{p}}$.

$$f_\alpha(u) = \frac{\langle q-u, \nabla f(u)\rangle}{|u-q|}$$

*derivada direccional respecto $\mathbf{p}$ en $u$.*

**Definición (Star-like set)**

$S \subset \mathbb{R}^2$ es *star-like* respecto $\mathbf{p}$, $\forall\, s \in S$ el segmento abierto $\overline{s\mathbf{p}}$ esta en $S$.

**Teorema**

- *$\mathbf{p} \in \mathbb{R}^2$, $S \subset \mathbb{R}^2$ star-like respecto $\mathbf{p}$ en el dominio de $f, g$.*
- *En $S$, $f, g$ diferenciables , $g_\alpha(s) \neq 0$,*
- $f(\mathbf{p}) = g(\mathbf{p}) = 0, \quad \lim_{x \to \mathbf{p}} \frac{f_\alpha(x)}{g_\alpha(x)} = L,$

*Entonces* $\lim_{x \to \mathbf{p}} \frac{f(x)}{g(x)} = L.$

# Apéndice B: Condiciones para ceros de $a_j(\cdot)$

$E_j := \{x \in \mathbb{R}^d : a_j(x) = 0\}$ satisface alguno de los puntos:

(I) $p \in E_j$ es un cero no aislado de $a_j(\cdot)$ y:

- $D := \{u : e^{ha_j(u)} - 1 = a_j(u) = 0\}$, es una curva suave que pasa por $p$.
- El vector canónico $e_j$ es no tangente a $D$.
- Para cada $p \in E_j$, existe una bola $B_r(p)$ t.q.

$$a_j \neq 0, \qquad \frac{\partial a_j(u)}{\partial u^{(j)}} \neq 0, \qquad \forall u \in D \setminus B_r(p).$$

(II) $p \in E_j$ es un cero aislado de $a_j(\cdot)$ y:

- Para cada $q \in E_j$, $p$ no es punto limite de $E_\alpha := \{x \in \mathbb{R}^d : (a_j)_\alpha(x) = 0\}$.
- Para cada $p \in E_j$ existe $B_r(p)$, t.q. la derivada direccional respecto a $p$ satiface

$$(a_j)_\alpha(x) \neq 0, \qquad \forall x \in B_r(p).$$

LSHyp

**Temporary page!**

LaTeX was unable to guess the total number of pages correctly. As there was some unprocessed data that should have been added to the final page this extra page has been added to receive it.

If you rerun the document (without altering it) this surplus page wil away, because LaTeX now knows how many pages to expect for this document.