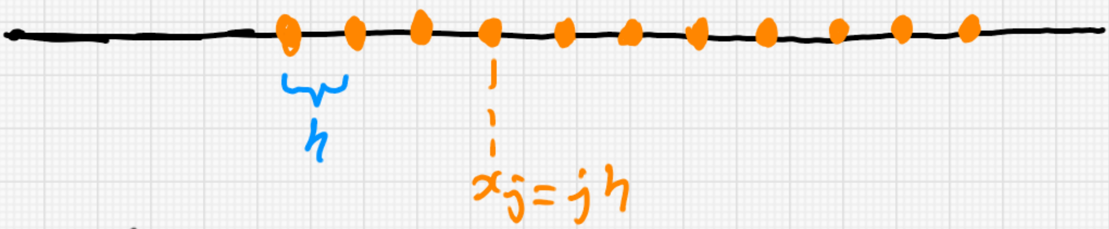
Unbonded Grids: the Semidiscrete Fourier transform

This scheme aplies to a discrete, unbonded domain. So it is not a practical method. However, it does introduce the mathematical ideas needed for the derivation and anlaysis of the practical schemes.

Our infinite grid is denoted by $h\mathbb{Z}$, with $x_j = jh$, $j \in \mathbb{Z}$



We shall derive (1.4) by various methods based on the key ideas of the semidiscrete Fourier transform and band-limited sinc function interpolation. Before discretizing, we review the continuous case

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See

H. Dym and H.P. McKean
Fourier Series and Integrals,
Academic Press, 1986.

Y. Katznelson, An introduction
to Harmonic Analysis, Dover.

T.W. Körner, Fourier Analysis,
Cambridge U. Press, 1990.

The Fourier transform of a function $u(x)$, $x \in \mathbb{R}$ is the function $\hat{u}(k)$ defined by.

$$\hat{u}(k) = \int_{-\infty}^{\infty} e^{-ikx} u(x), \quad k \in \mathbb{R}. \tag{2.1}$$

The number $\hat{u}(k)$ can be interpreted as the amplitude density of u at wavenumber k, and this process decomposing a function into its constitute waves is called Fourier analysis.

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Conversely, we can reconstruct u from $\hat{u}$ by the inverse Fourier

$$(2.2) \quad u(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\,\hat{u}(k)\,dk, \quad x \in \mathbb{R}.$$

This is Fourier synthesis. The variable x is the physical variable and k is the Fourier variable or wavenumber.

We want to consider x ranging over $h\mathbb{Z}$ rather than $\mathbb{R}$. Precise analogues of the Fourier transform and its inverse exist for this case. The crucial point is that because the spatial domain is discrete, the wave number k will no longer range over all $\mathbb{R}$. Instead, the appropriate wave number domain is a bounded interval of length $\frac{2\pi}{h}$, and one suitable choice is

$\left[-\frac{\pi}{h}, \pi/h\right]$. Remember, k is bounded

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Remember, k is bounded because x is discrete:

Physical space: $x \in h\mathbb{Z}$ — discrete, unbounded

Fourier space: $k \in [-\pi/h, \pi/h]$ — continuos, bounded

the reason for these conections is the phenomenon known as <u>aliasing</u>.

Two complex exponentials

if $f(x) = e^{ik_1 x}$ $\quad g(x) = e^{ik_2 x}$

then for $x \in \mathbb{R}$ $f(x) \neq g(x)$, always that $k_1 \neq k_2$.

But if we restrict $x \in h\mathbb{Z}$

$f_j = \exp(ik_1 x_j) = g_j := \exp(ik_2 x_j)$

if $k_1 - k_2$ is an integer multiple of $2\pi/h$.

Consequently it suffices to measure wavenumbers for the grid in the interval of length $\frac{2\pi}{h}$, and for reason of symmetry we chose

$$\left[-\pi/h, \pi/h\right]$$

Let $v_j = v(x_j)$, $x_j \in h\mathbb{Z}$, the semidiscrete Fourier transform is defined by

$$\hat{v}(k) := h \sum_{j=-\infty}^{\infty} e^{-ikx_j} v_j$$

$$k \in \left[-\pi/h, \pi/h\right] \qquad (2.3).$$

The fourier syntesis is valid for $u, \hat{u} \in L^2(\mathbb{R})$.

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and the inverse semidiscrete Fourier tranform* is

$$v_j = \frac{1}{2\pi}\int_{-\pi/h}^{\pi/h} e^{ikx_j}\,\hat{v}(k)\,dk \qquad (2.4)$$

$j \in \mathbb{Z}$.

Note that

$$h\sum_{j=-\infty}^{\infty} e^{-ikx_j} v_j \overset{\text{Trapezoide rule}}{\approx} \int_{-\infty}^{\infty} e^{-ikx}\, v(x)\,dx$$

$k \in \mathbb{R}$

$k \in [-\pi/h, \pi/h]$

$$\frac{1}{2\pi}\int_{-\pi/h}^{\pi/h} e^{ikx_j}\hat{v}(k) \xrightarrow{h\to 0} \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\,\hat{v}(k)\,dk$$

$k \in \mathbb{R}$

The most usual name for this transform or discretization is the Fourier Series. Authors use "semidiscrete" because we want to emphasize that our concern is the inverse problem:

As the "space" variable that is discrete and the Fourier variable that is a bounded interval.

For spectral differentiation, we need an interpolant, and the inverse transform

$$v_j = \frac{1}{2\pi}\int_{-\pi/h}^{\pi/h} e^{ikx_j}\,\hat{v}(k)\,dk$$

$j \in \mathbb{Z}$.

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that is, afther, after determinating $\hat{v}$ we define our interpolant p by

$$p(x) := \frac{1}{2\pi}\int_{-\pi/h}^{\pi/h} e^{ikx}\,\hat{v}(k)\,dk, \qquad (2.5)$$

$x \in \mathbb{R}$.

this is an analytic function of x, with $p(x_j) = v_j$. Moreover by construction, the Fourier transform $\hat{p}$, defined by $\int_{-\infty}^{\infty} e^{-ikx} v(x)\,dx$, $k \in \mathbb{R}$ reads.

$$\hat{p}(k) := \begin{cases} \hat{v}(k), & k \in [-\pi/h, \pi/h] \\ 0 & \text{otherwise} \end{cases}$$

Analytic or holomorphic means that at a pointe $z \in \mathbb{C}$ f it is differentiable in the complex sense, or equivalently, if its Taylor series converges to f.

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We say that p is the band-limit interpolant of v. That is:

$\hat{p}$ has compact support contained on $[-\pi/h, \pi/h]$.

There is only one band-limited interpolant defined in this sense.

See the sampling thm.

HW01 Review bibliography to enunciate this sample theorem.

So the spectral scheme of differentiation reads is:

Given v defined on $h\mathbb{Z}$

- Determine its band-limit interpolant by
$$p(x) = \frac{1}{2\pi}\int_{-\pi/h}^{\pi/h} e^{ikx}\,\hat{v}(k)\,dk$$
- set $w_j = p'(x_j)$,

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An other schem is by using inverse transform:

If u is a differentiable function, then the Fourier transform of u' is

$ik\,\hat{u}(k)$, that is

$$\widehat{u'}(k) = ik\,\hat{u}(k) \qquad (2.6)$$

HW02. Show this result.

Our scheme.

- Given v, compute its semi-discrete Fourier $\hat{v}$ by
$$\hat{v}(k) = h\sum_{j=-\infty}^{\infty} e^{-ikx_j},$$
- Define $\widehat{w'}(k) = ik\,\hat{v}(k)$
- Comput the inverse.
$$w'(x) = \frac{1}{2\pi}\int_{-\pi/h}^{\pi/h} e^{ikx_j}\,\hat{v}(k)\,dk$$

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Now, we deduce the entries of Laurent's operator

$$\begin{bmatrix} & & \vdots & & \\ & & 1/3 & & \\ & & -1/2 & & \\ & \ddots & 1 & & \\ & \ddots & 0 & \ddots & \\ & \ddots & -1 & \ddots & \\ & & 1/2 & \ddots & \\ & & -1/3 & & \\ & & \vdots & & \end{bmatrix}$$

To do this, we can use Fourier transform to go back and get a fuller understanding of the band limit interpolant $p(x)$!

Let δ be the Kronecker delta function,

$$\delta_j := \begin{cases} 1 & j = 0 \\ 0 & j \neq 0 \end{cases} \qquad (2.7)$$

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By the definition

$$\hat{v}(k) = h \sum_{j=-\infty}^{\infty} e^{-ikx_j} v_j,$$

$$k \in [-\pi/h, \pi/h]$$

the semidiscrete fourier transform of δ is constant. That is

$$\hat{\delta}(k) = h \sum_{j=-\infty}^{\infty} e^{-ikhj} \underbrace{\delta_j}_{j=0,\ \forall j\neq 0}$$

$$\hat{\delta}(k) = h, \text{ for all } k \in [-\pi/h, \pi/h]$$

By definition of our interpolant

$$p(x) = \frac{1}{2\pi} \int_{-\pi/h}^{\pi/h} e^{ikx} \underbrace{\hat{v}(k)}_{=h} dk.$$

Ex of class

so

$$p(x) = \frac{h}{2\pi} \int_{-\pi/h}^{\pi/h} e^{ikx} dk = \frac{\sin \pi x/h}{\pi x/h}$$

with the value 1 at $x=0$.
¡¡ this value why !!

So we define

$$S_h(x) = \frac{\operatorname{sen}(\pi x/h)}{\pi x/h}$$

See

E.T. Whittaker, On the functions which are represented by the expantions of the interpolation theory, Pro. Roy. Soc. Edinb 35 (1915), 181—194.

F. Stenger, Numerical Methods Based on Sinc and Analytic Functions.

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Now that we know how to interpolate the delta function, we can interpolate anything. Band-limited interpolation is an translation invariant process in the sense that for any m, the band limited interpolant of δ_{j-m} is $S_h(x-x_m)$.

HW03.

Deduce that

$$\delta_{j-m} = S_h(x-x_m)$$

Therefore a general grid function v can be written as.

$$v_j = \sum_{m=-\infty}^{\infty} v_m \delta_{j-m} \qquad (2.9)$$

Why? (2.9)

to discusse in class

So it follows by linearity of Fourier transform

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that the band-limited interpolation of v is a linear combination of translated sinc functions:

$$p(x) = \sum_{m=-\infty}^{\infty} v_m S_h(x - x_m) \quad (2.10)$$

{Deduce this formula in class.}

The derivative is accordingly

$$w_j = p'(x_j) = \sum_{m=-\infty}^{\infty} v_m S_h'(x_j - x_m)$$

- Calculate $S_h'(x)$ and then give the Laurent's entries.

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