JPMC Quant Challenge: Modelling Section

A derivative is a financial security that derives its value from one or more underlying assets or economic risk factors. Examples of some commonly traded derivatives are:

- Forward contract: This gives the holder the obligation to exchange an asset at a fixed price in the future.
- Call option: A call option is defined as the right to sell / buy an asset at a fixed agreed upon price K at a fixed time T (option expiry / maturity date) in the future.

The market standard for quoting prices for call options is in terms of an 'implied volatility', which is the volatility number that equates the Black-Scholes call option price function that equates the market price. Mathematically,

$$\mathrm{C}\ (\mathrm{K}, \mathrm{T}, \mathrm{r}, \sigma) = e^{-rT}\ [\mathrm{F(T)} \times \mathrm{N(d1)} - \mathrm{K}\ \times \mathrm{N(d2)}]$$

where

- $\mathrm{d1}\ = \frac{\ln\left(\frac{\mathrm{F(T)}}{K}\right) + 0.5\sigma^2\ \mathrm{T}}{\sigma\sqrt{T}}$
- $\mathrm{d2}\ = \mathrm{d1} - \sigma\sqrt{T}$
- r is the risk-free interest rate
- $\mathrm{F(T)}$ is the forward price of the asset

To understand this, consider a forward contract with exchange price K maturing at time T. $\mathrm{F(T)}$ is that value of the exchange price which sets the current value of the forward contract to 0. For the purpose of this paper, consider $\mathrm{F(T)} = \mathrm{S}_0\ e^{(r-q)T}$, where S_0 is the current asset price, r is the risk-free rate and q is the given dividend yield. Here K is the option 'strike' price, σ is the 'implied' volatility. The 'implied' volatility σ = ∑ (K,T) is the value of σ that solves,

$$\mathrm{C}\ (\mathrm{K}, \mathrm{T}, \mathrm{r}, \sigma) = \text{ given market price of option with strike K and maturity T}$$

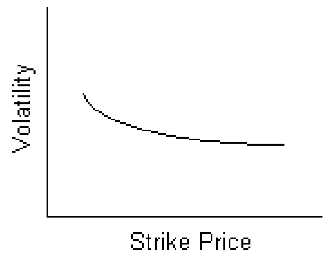
The standard Black-Scholes model assumes the following dynamics of the stock price:

$$\mathrm{dS}_t\ =\ \mu_t\ \mathrm{dt}\ +\ \sigma_t\ \mathrm{dW}_t \qquad \ldots (1)$$

Where we assume $\mu_t = r - q$ for our purposes. If this was true, then all options in the market with a given maturity T would have the same implied volatility given by,

$$\sum(K, T) = \sqrt{\int_0^T \sigma_s^2\ ds} \quad \text{for each strike price K}$$

This is equivalent to assuming a log-normal distribution for the asset price (and equivalent, normal distribution for the log-returns). However, this is rarely the case, and the asset returns, distributions are known to deviate notably from normality (fat tails, skewed etc.) If we plot the implied volatility $\sum(K, T)$, as a function of K (for any fixed T), we often find it shaped like a 'skew' or a 'smile' instead of being a constant function as the Black-Scholes predicts.



Dupire, Derman and Kani came up with a formulation of asset price dynamics that allows consistency with these volatility ‘smile’ observed in the market. The dynamics can be represented as:

$$\frac{dS_t}{S_t} = \mu_t \, dt + \sigma^{LV}(t, S_t) \, dW_t$$

where the term $\sigma^{LV}(t, S_t)$ is called the ‘local volatility’. When contrasted with (1), it can be seen that the instantaneous (at each time t), the noise has a variable volatility which is dependent on the asset price S_t at that time instant.

Dupire shows that, given information of the values of $C(K, T)$ for all K and T, and the existence of an arbitrage free market, this function $\sigma^{LV}(t, S_t)$ can be uniquely determined. Below, we give the final result for the local volatility function, derived using the standard assumptions:

$$\sigma^{LV}(T, K) = \sqrt{\frac{\Sigma^2 + 2\Sigma T\left(\frac{\partial \Sigma}{\partial T} + \mu_T K \frac{\partial \Sigma}{\partial K}\right)}{1 + 2Kx\sqrt{T}\frac{\partial \Sigma}{\partial K} + K^2 T \Sigma \frac{\partial^2 \Sigma}{\partial K^2} + K^2 x(x - \Sigma\sqrt{T})T\left(\frac{\partial \Sigma}{\partial K}\right)^2}}$$

where $\Sigma = \Sigma(K, T)$ is the Black-Scholes implied volatility for an option with strike K and maturity T, and

$$x = \frac{\log\left(\frac{F_0(T)}{K}\right) + \frac{\Sigma^2 T}{2}}{\Sigma\sqrt{T}}$$

Section A

Inputs:

Stock price S_0 = 1000, risk-free rate r = 0.05, dividend yield q = 0.02, $\sum(K,T)$ for a vector of strikes $\{K\} = \{K_1, K_2, \ldots., K_{nK}\}$ (where nK is the number of input strikes) & a vector of expiry times $\{T\} = \{T_1, T_2, \ldots., T_{nT}\}$ (where nT is the number of input expiries). Also,

K_1= 50, K_{1000} = 2750, {K} = 1000 equidistant strikes from 50 to 2750

T_1 = 1/180, T_n = (1/12) * (n-1) for 01 < n < 182

$K_i - K_{i-1} = dK = \text{constant for each i}, (1 \le \text{ i } \le \text{ nK})$
$T_j - T_{j-1} = dT = \text{constant for each j}, (1 \le \text{ i } \le \text{ nT})$

Question 1 (5 marks): Compute the price of the grid of call option prices $C(K,T)$ using the Black-Scholes formula given earlier. Output the notionally-weighted return on a portfolio holding one unit of each of these options, i.e.

$$\frac{\left(\sum_{j=1}^{nT} \sum_{i=1}^{nK} C_{BS}\left(K_{j,}T_i\right)\right)}{S_0 * nK * nT}$$

Next, we compute local volatility. Compute the local volatility function $\sigma^{LV}(T,K)$ with $T \in \{T_1, T_2, \ldots., T_{nT}\}$ and $K \in \{K_1, K_2, \ldots., K_{nK}\}$ using the formula given earlier. For the derivatives, use the following finite difference approximations:

$$\frac{\partial \Sigma(\mathrm{K},\mathrm{T})}{\partial K} |_{K=K_i} = \begin{cases} \dfrac{\Sigma(\mathrm{K}_{i+1},\mathrm{T}) - \Sigma(\mathrm{K}_{i-1},\mathrm{T})}{2 \times dK} & for\ 2 \le i \le nK-1 \\ \dfrac{\Sigma(\mathrm{K}_2,\mathrm{T}) - \Sigma(\mathrm{K}_1,\mathrm{T})}{dK} & for\ i = 1 \\ \dfrac{\Sigma(\mathrm{K}_{nK},\mathrm{T}) - \Sigma(\mathrm{K}_{nK-1},\mathrm{T})}{dK} & for\ i = nK \end{cases}$$

$$\frac{\partial^2 \Sigma(\mathrm{K},\mathrm{T})}{\partial K^2} |_{K=K_i} = \begin{cases} \dfrac{\Sigma(\mathrm{K}_{i+1},\mathrm{T}) + \Sigma(\mathrm{K}_{i-1},\mathrm{T}) - 2\Sigma(\mathrm{K}_i,\mathrm{T})}{dK^2} & for\ 2 \le i \le nK-1 \\ 0 & for\ i = 1, nK \end{cases}$$

Note: The above style of numerically approximating derivatives is known as 'finite differencing' and it plays an important role in the numerical implementation of derivatives' pricing models. $\frac{\partial \Sigma}{\partial T}$ is defined analogously to $\frac{\partial \Sigma}{\partial K}$ with the differencing in the first variable T instead of the second K.

Now that $\sigma^{LV}(T,K)$ has been computed numerically at each $T = \{T_i\},\ 1 \le \mathrm{i} \le \mathrm{nT}$ and $K = \{K_i\},\ 1 \le \mathrm{i} \le \mathrm{nK}$, define an interpolation function to complete the definition of $\hat{\sigma}^{LV}$ as below:

$$\hat{\sigma}^{LV}(t,K) = \begin{cases} \hat{\sigma}^{LV}(T_1,K) & if\ \ t \le T_1 \\ \hat{\sigma}^{LV}(T_{nT},K) & if\ \ t \le T_{nT} \\ \hat{\sigma}^{LV}(T_i,K) & if\ \ T_i \le t \le T_{i+1} \end{cases}$$

and

$$\hat{\sigma}^{LV}(t,k) \begin{cases} = \text{Linearly interpolated between } \hat{\sigma}^{LV}(t,K_i) \text{ and } \hat{\sigma}^{LV}(t,K_{i+1}) \text{ for } K_i \le \mathrm{k} \le K_{i+1} \\ \qquad\qquad \text{and } 2 \le \mathrm{i} \le \mathrm{n}K - 1 \\ = \hat{\sigma}^{LV}(t,K_1)\ \ for\ \ K \le K_1 \\ = \hat{\sigma}^{LV}(t,K_{nK})\ \ for\ \ K \ge K_{nK} \end{cases}$$

i.e. the numerically approximated local volatility function $\sigma^{LV}(T,K)$ is defined first at the benchmark points on the implied volatility grid, then interpolated as piecewise constant in time, and linear in strike. This is a much simplified treatment compared to what is used in real life production situations with higher requirements for robustness, accuracy and speed.

Now we will price call options using a Monte Carlo simulation of the local volatility based asset dynamics and compare to call options priced with Black Scholes. By construction, the local volatility definition used should be consistent with the full implied volatility surface, and any differences from the Black Scholes prices should be due to the numerical approximations used in the definition of $\sigma^{LV}(T,K)$.

To recap,

$$\frac{dS_t}{S_t} = \mu_t dt + \sigma(t,S_t)dW_t$$

$$d(\log S_t) = \left[\mu_t - \frac{1}{2}\sigma(t,S_t)\right]^2 dt + \sigma(t,S_t)dW_t$$

Discretize this in 'N_T' steps as:

$$log(S_{t_i}) - log(S_{t_{i-1}}) = \frac{\mu_t - \frac{1}{2}\hat{\sigma}_{LV}^2\left(t_{i-1},S_{t_{i-1}}\right)}{t_i - t_{i-1}} + \hat{\sigma}_{LV}\ (t_{i-1},S_{t_{i-1}})\ \varepsilon_t$$

where, ε_t is a normal random variable with mean 0 and variance $\Delta t = t_i - t_{i-1}$

Using a single Monte Carlo simulation, price $C_{MC}(K,T)$ for $K \in \{K_1,K_2,\dots.,K_{nK}\}$ and $T \in \{T_1,T_2,\dots.,T_{nT}\}$ where C_{MC} is the call option priced using the Monte Carlo simulation. Insert the dates $\{T_1,T_2,\dots.,T_{nT}\}$ in your Monte Carlo simulation timeline. Note that:

a) The payoff of a call option with strike K at expiry time T can be given as $max(S_t - K,0)$
b) The price of an 'attainable' derivative with payoff given by $f(S_T)$ of an underlying asset at time T is given by :

$$V(0) = e^{-rT} E[f(S_T)]$$

Under the standard assumptions, this expectation can be approximated using Monte Carlo using the Central Limit Theorem. The "pricing error" compared to the Black-Scholes prices will be a combined function of: Bias introduced by our numerical approximations in the definition of $\sigma^{LV}(T, K)$, Monte Carlo error.

Question 2 (10 marks): Number of Monte Carlo simulations N = 10000, number of time steps N_T = 52 steps / year, i.e., weekly or, Δt = 1/52. Output the same metric as in Question 1.

Question 3 (15 Marks)

Now we will price a more complex 'path-dependent' derivative (a double knock-out option).

Contract definition: A 3 years derivative which pays the regular call option payoff $max(S_t - K, 0)$ at expiry time T (given strike K = 1.1*S_0) if and only if the stock price $S_{t_{mon}}$ stays with pre-defined barriers [0.4*S_0, 1.5*S_0] on a set of defined dates t_{mon} = {1/12, 2/12.. 35/12,3} (i.e the barrier condition is monitored monthly). If the barrier is 'hit' before or at the expiry, the contract payoff is 0.

Use a Monte Carlo simulation to price the contract using 80000 simulations and time-step ΔT = 1/48. Use the local volatility function computed earlier.

Section B

Derivative pricing problems can often be solved in 1 of the 2 following ways:

1. As an expectation evaluated using Monte Carlo, the approach we used in the previous section
2. As a solution to a parabolic partial derivative equation (PDE). The solutions often have no closed-form and must be arrived at using numerical finite differencing schemes. We will use this approach in this section.

Consider a Markovian stochastic process defined using the following dynamics:

$$dX(t) = \mu(t,x)dt + \sigma(t,x)dW_t$$

The functions μ and σ are deterministic functions.

Define a function $V(t,x)$ as the price of a derivative contract; we assume that X_t is a Markov process and hence the derivative price can be defined at (t,X_t). It can be shown that under certain assumptions, $V(t,x)$ satisfies the following PDE:

$$\frac{\partial V(t,x)}{\partial t} + \mu(t,x)\frac{\partial V(t,x)}{\partial x} + \frac{1}{2}\sigma(t,x)^2\frac{\partial^2 V(t,x)}{\partial x^2} - rV(t,x) = 0 \qquad \text{(A)}$$

Solution of the above PDE gives $V(t,x)$ for all (t,x) in the relevant domain, and the current derivative price can be read as $V(0,X_0)$

<u>Numerical solution</u>

The PDE in (A) is generally solved by discretizing over a rectangular domain $(t,x) \in [0,T] \times [M_L, M_U]$ where T is the maturity time and M_L and M_U are reasonable bounds for the underlying process values. We introduce two equidistant grids

$\{\mathrm{t}_i\}_{i=0}^{n}$ and $\{\mathrm{x}_j\}_{j=0}^{m+1}$ $where\ t_i = iT/n,\ i = 0,1,\dots,n\ and\ x_j = M_L + \frac{j(M_U - M_L)}{m+1}$

The terminal value $V(t,x) = g(x)$ is imposed at $t_n = T$, and spatial boundary conditions are imposed at x_0 and x_{m+1}.

The derivatives $\frac{\partial V}{\partial x}$ and $\frac{\partial^2 V}{\partial x^2}$ are approximated by finite central differences; this keeps the error to $O(\Delta x^2)$. For example, $\frac{\partial V(t_i,X_j)}{\partial x} \approx \frac{V(t_i,x_{j+1}) - V(t_i,x_{j-1})}{2\Delta x}$

For the time dimension, the approximation of $\frac{\partial V}{\partial t}$ is usually done via the Crank – Nicholson scheme which sets $\frac{\partial V(\theta t_i + (1-\theta)t_{i+1}, x)}{\partial t} = \frac{(V(t_{i+1},x) - V(t_i,x))}{\Delta t}$

The most common choice for the value of θ is ½. θ = 0 or 1 are also used in practice depending on the problem constraints.

We will now solve a PDE for the call option price, with the underlying dynamics defined using the local volatility formulation. We'll be using the following PDE:

$$d\left(\log\frac{X_t}{X_0}\right) = \left[\mu_t - \frac{1}{2}\sigma(t, X_t)\right]^2 dt + \sigma(t, X_t)dW_t$$

By equivalence with the SDE and (A), we define the PDE as:

$$\frac{\partial V}{\partial t} + c\frac{\partial^2 V}{\partial x^2} + a\frac{\partial V}{\partial x} + f = 0$$

with V meaning $V(t, x)$ and where,

$c(t, x) = \frac{1}{2}\sigma(t, x)^2$

$a(t, x) = \mu - \frac{1}{2}\sigma(t, x)^2, \qquad where\ \mu = r - q$

$f = -r$

The discretized PDE can be represented as:

$l_{i,j}V_{i,j-1} + c_{i,j}V_{i,j} + u_{i,j}V_{i,j+1} = L_{i+1,j}V_{i+1,j-1} + C_{i+1,j}V_{i+1,j} + u_{i+1,j}V_{i+1,j+1} + r_{i,i+1,j+1}$ ……….(B)

where $V_{i,j}$ refers to $V(t_i, x_j)$ with t_i and x_j as defined earlier in the specification of the PDE grid construction.

The above coefficients l, c, u, L, C, U are defined as:

$$l_{i,j} = -\theta(\Delta t)\left(-\frac{1}{2}\frac{a(t_i, x_j)}{\Delta x} + \frac{c(t_i, x_j)}{\Delta x^2}\right)$$

$$c_{i,j} = 1 - \theta(\Delta t)\left(\frac{-2c(t_i, x_j)}{\Delta x^2} + f\right)$$

$$u_{i,j} = -\theta(\Delta t)\left(\frac{1}{2}\frac{a(t_i, x_j)}{\Delta x} + \frac{c(t_i, x_j)}{\Delta x^2}\right)$$

$$L_{i,j} = (1-\theta)\Delta t\left(-\frac{1}{2}\frac{a(t_i, x_j)}{\Delta x} + \frac{c(t_i, x_j)}{\Delta x^2}\right)$$

$$C_{i,j} = 1 + (1-\theta)\Delta t\left(\frac{-2c(t_i, x_j)}{\Delta x^2} + f\right)$$

$$U_{i,j} = (1-\theta)\Delta t\left(\frac{1}{2}\frac{a(t_i, x_j)}{\Delta x} + \frac{c(t_i, x_j)}{\Delta x^2}\right)$$

$$r_{i,i+1,j} = \begin{cases} (1-\theta)V_{i+1,0}L_{i+1,1} + \theta V_{i,0}l_{i+1,1} & if\ j = 1 \\ (1-\theta)V_{i+1,m+1}U_{i+1,m} + \theta V_{i,m+1}u_{i+1,m} & if\ j = m \end{cases}$$

where $V_{i,0}$ and $V_{i,m+1}$ are the spatial boundary conditions. All the boundary conditions for a call option with strike K and expiry T are given as:

a) Terminal: $V(t_n, x) = (X_0e^x - K)^+$
b) Spatial: $V_{i,0} = 0$ (very out of the money call)

$$V_{i,m+1} = X_0\, e^{-q(T-t_i)} - Ke^{-r(T-t_i)} \text{ (very in the money call)}$$

Objective: The inputs are maturity time T and strike K.

Set $M_L = \Sigma(S_0, T)\sqrt{T}\,(-5)$ and $M_U = \Sigma(S_0, T)\sqrt{T}\,(+5)$

Here the spatial bounds are set to +/- 5 standard deviation of the implied at-the-money spot volatility. To get the $\Sigma(S_0, T)$, linearly interpolate the given $\Sigma(K, T)$.

m (defining Δx) = 400 and n (defining Δt) = 500

Use the discretized PDE defined earlier to solve for the call option price. Hint: Equation (B) can be written in matrix form and solved iteratively starting from $t_n = T$ (using the terminal and spatial boundary conditions.

<u>Question 4 (20 Marks)</u>

T = 5Y, K = 993.2432 (strike no. 350 in the input grid of strikes). Compute the call option price V(T, K) using FD. Use theta = ½. After solving for $V(0, x)$, compute the derivative price as $V(0, 0)$ by linearly interpolating the $V(0, x)$ output for $x = 0$.

<u>Question 5 (25 Marks)</u>

Price the same double knock-out contract as in the previous section. Use m = 400, n = T / Δt where T = 3 (expiry of the contract) and Δt = 1/60, such that the timeline includes the barrier observation dates spaced 1/12 apart)

Boundary conditions:

a) $\Omega = 0$ (if underlying is too low or too high, it is almost sure to be beyond [L, U] on the current or immediate next barrier monitoring date)

b) $V(T,x) = \begin{cases} 0 & \text{if } \mathrm{S_0} e^x < L \text{ } or \text{ } \mathrm{S_0} e^x > U \\ \max(S_0 e^x - K, 0) & if \text{ } L \leq \mathrm{S_0} e^x \leq U \end{cases}$

c) $V(t_i, x) = 0 \text{ if } \mathrm{S_0} e^x < \mathrm{L} \text{ or } \mathrm{S_0} e^x > \mathrm{U} \text{ if } \mathrm{t}_i$ is a barrier monitoring date, else as given by the solution of (2)

After solving for $V(0,x)$, compute the derivative price as $V(0,0)$ by linearly interpolating the $V(0,x)$ output for $x = 0$.