Mathematics 1

engineering mathematics -1 (Dr. A.P.J. Abdul Kalam Technical University)

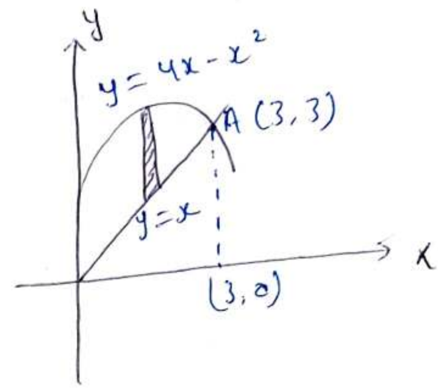
2022-23

Section - A

①(f) Evaluate $\iint y\,dx\,dy$ over the part of the plane bounded by line $y = x$ and the parabola $y = 4x - x^2$.

Sol^n:

$$I = \int_0^3 \int_x^{4x-x^2} y \, dy \, dx$$



$$= \int_0^3 \left[\frac{y^2}{2}\right]_x^{4x-x^2} dx$$

$$= \frac{1}{2}\int_0^3 \left[(4x - x^2)^2 - x^2\right] dx$$

$$= \frac{1}{2}\int_0^3 (16x^2 + x^4 - 8x^3 - x^2)\, dx$$

$$= \frac{1}{2}\int_0^3 (x^4 - 8x^3 + 15x^2)\, dx = \frac{1}{2}\left[\frac{x^5}{5} - \frac{\cancel{8}^{2} x^4}{\cancel{4}} + \frac{\cancel{15}^{5} x^3}{\cancel{3}}\right]_0^3$$

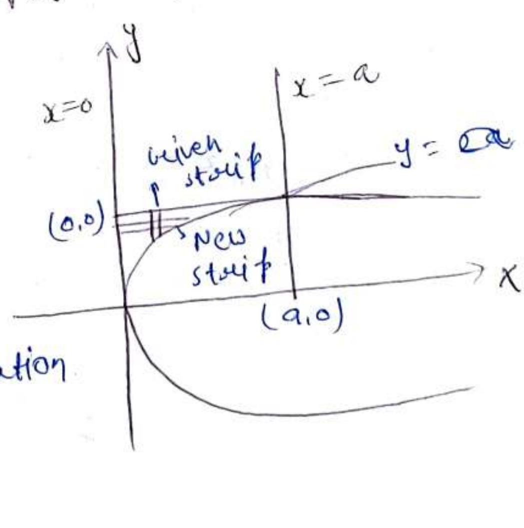
$$\frac{1}{2}\left[\frac{243}{5} - 162 + 135\right] = \frac{1}{2}\left[\frac{243}{5} - 27\right] = \boxed{\frac{108}{10}}$$

Section - B

② (d) Evaluate double integral $\int_0^a \int_{\sqrt{ax}}^a \frac{y^2}{\sqrt{(y^4 - a^2x^2)}}\, dx\, dy$ by changing the order of integration.

Sol^n: Given limit: $x = a$, $x = a$

$y = \sqrt{ax}$, $y = a$



New limit:

$y = 0$, $y = x$

$x = 0$, $x = y^2/a$

By changing order of integration

$$= \int_0^a \int_0^{y^2/a} \frac{y^2}{\sqrt{y}}$$

Section-C

6 (a) Evaluate $\iiint_R (x-2y+z)\,dz\,dy\,dx$, where R is the region determined by $0 \le x \le 1,\ 0 \le y \le x^2,\ 0 \le z \le x+y$.

Sol^n:

$$I = \int_0^1\int_0^{x^2}\int_0^{x+y} (x-2y+z)\,dz\,dy\,dx$$

$$= \int_0^1\int_0^{x^2} \left(xz - 2yz + \frac{z^2}{2}\right)_0^{x+y} dy\,dx$$

$$= \int_0^1\int_0^{x^2} \left[x(x+y) - 2y(x+y) + \frac{(x+y)^2}{2}\right] dy\,dx$$

$$= \int_0^1\int_0^{x^2} \left[x^2 + \cancel{xy} - 2\cancel{x}y - 2y^2 + \frac{x^2}{2} + \frac{y^2}{2} + \cancel{xy}\right] dy\,dx$$

$$= \int_0^1\int_0^{x^2} \left(\frac{3}{2}x^2 - \frac{3}{2}y^2\right) dy\,dx = \int_0^1 \left[\frac{3}{2}x^2 y - \frac{\cancel{3}}{2}\frac{y^3}{\cancel{3}}\right]_0^{x^2} dx$$

$$= \int_0^1 \left(\frac{3}{2}x^2 y - \frac{y^3}{2}\right)_0^{x^2} dx = \int_0^1 \left(\frac{3}{2}x^4 - \frac{x^6}{2}\right) dx$$

$$= \left[\frac{3x^5}{10} - \frac{x^7}{14}\right]_0^1 = \frac{3}{10} - \frac{1}{14} = \frac{21-5}{70} = \frac{16}{70} = \boxed{\frac{8}{35}}$$

(b) Use Dirichlet's integral to evaluate $\iiint xyz\,dx\,dy\,dz$ throughout the volume bounded by $x=0,\ y=0,\ z=0$ & $x+y+z=1$

Sol^n: let $V = \iiint xyz\,dx\,dy\,dz$

$\therefore\ x \ge 0,\ y \ge 0,\ z \ge 0$ and $x+y+z \le 1$

Applying Dirichlet's integral, we get

$$V = \iiint x^{2-1} y^{2-1} z^{2-1}\,dx\,dy\,dz = \frac{\Gamma 2\,\Gamma 2\,\Gamma 2}{\Gamma(2+2+2+1)}$$

$$= \frac{1!\,1!\,1!}{\Gamma 7} = \frac{1}{6!} = \boxed{\frac{1}{720}}$$

2021-22

Section-A

(1)(h) Find the value of $\int_0^1\int_0^x\int_0^{x+y} dx\,dy\,dz$

Sol^n:
$$I = \int_0^1\int_0^x\int_0^{x+y} dz\,dy\,dx = \int_0^1\int_0^x [z]_0^{x+y}\,dy\,dx$$

$$\int_0^1\int_0^x (x+y)\,dy\,dx = \int_0^1 \left[xy + \frac{y^2}{2}\right]_0^x dx$$

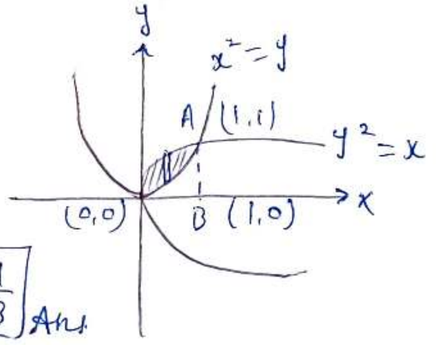
$$\int_0^1 \left[x^2+\frac{x^2}{2}\right]dx = \int_0^1 \frac{3x^2}{2}\,dx = \left[\frac{3x^3}{2\times3}\right]_0^1 = \boxed{\frac{1}{2}}$$

(1)(g) find the area bounded by curve $y^2 = x$ and $x^2 = y$.

Sol^n: Area $= \int_0^1\int_{x^2}^{\sqrt{x}} dy\,dx$

$$= \int_0^1 [y]_{x^2}^{\sqrt{x}}\,dx = \int_0^1 (\sqrt{x} - x^2)\,dx$$

$$= \left[\frac{2}{3}x^{3/2} - \frac{x^3}{3}\right]_0^1 = \frac{2}{3} - \frac{1}{3} = \boxed{\frac{1}{3}} \text{ Ans.}$$



Section-B

(2)(d) Find the volume bounded by the cylinder $x^2+y^2=4$ and the plane $y+z=4$, $z=0$.

Sol^n:

Section-c

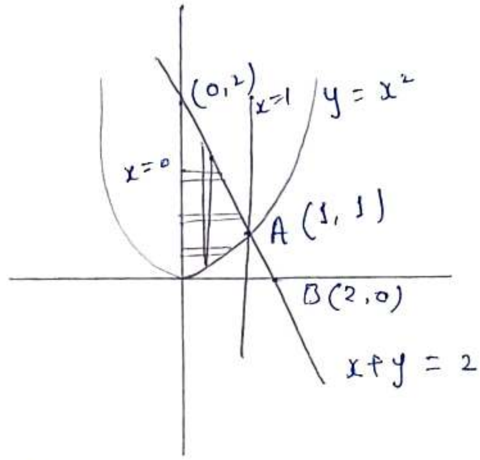
6 (a) Change the order of integration in $I = \int_0^1 \int_{x^2}^{2-x} xy\,dy\,dx$ & hence evaluate the same.

Sol:

Given limit: $x=0$, $x=1$

$y = x^2$, $y = 2-x$

$\boxed{x+y=2}$



New limit

$y=0$, $y=1$

$y=1$, $y=2$

$x=0$, $x=0$

$x=\sqrt{y}$, $x=2-y$

By changing the order of integration;

$$I = \int_0^1 \int_{x^2}^{2-x} xy\,dx\,dy = \int_0^1 \int_0^{\sqrt{y}} xy\,dx\,dy + \int_1^2 \int_0^{2-y} xy\,dx\,dy$$

$$= \int_0^1 y\,dy \left[\frac{x^2}{2}\right]_0^{\sqrt{y}} + \int_1^2 y\,dy \left[\frac{x^2}{2}\right]_0^{2-y}$$

$$= \int_0^1 y\left(\frac{y}{2}\right)dy + \int_1^2 \frac{y}{2}\left[(2-y)^2\right]dy$$

$$= \int_0^1 \frac{y^2}{2}\,dy + \int_1^2 \frac{y}{2}(4+y^2-4y)\,dy$$

$$= \int_0^1 \frac{y^2}{2}\,dy + \int_1^2 \left(2y + \frac{y^3}{2} - 2y^2\right)dy$$

$$= \left[\frac{y^3}{6}\right]_0^1 + \left[\frac{2y^2}{2} + \frac{y^4}{8} - \frac{2y^3}{3}\right]_1^2$$

$$= \frac{1}{6} + \left[\left(4+2-\frac{16}{3}\right) - \left(1+\frac{1}{8}-\frac{1}{3}\right)\right]$$

$$= \frac{1}{6} + \frac{5}{24} = \boxed{\frac{3}{8}}$$

2020-21

Section - A

① (g) Evaluate $\int_0^1 \int_0^{x^2} e^{y/x} \, dy \, dx$

sol^n:

① (h) Calculate volume of solid bounded by surface $x=0$, $y=0$ $x+y+z=1$ and $z=0$

sol^n: $z=0$

$z = 1-x-y \quad \therefore z = 0, \ 1-x-y$

$y=0 \quad , \quad y = 1-x-z$

$z=0, \quad y = 1-x \quad \therefore y = 0, 1-x$

similarly $x=0$

$x = 1-y-z$

$x = 1, 0$

Volume : $\int_0^1 \int_0^{1-x} \int_0^{1-x-y} dz \, dy \, dx$

$= \int_0^1 \int_0^{1-x} (1-x-y) \, dy \, dx = \int_0^1 \left[(1-x)y - \frac{y^2}{2} \right]_0^{1-x} dx$

$= \int_0^1 (1-x)(1-x) - \frac{(1-x)^2}{2} \, dx = \frac{1}{2} \int_0^1 (1-x^2)^2 \, dx$

$= \frac{1}{2} \int_0^1 (1+x^4-2x^2) \, dx = \frac{1}{2} \left[x + \frac{x^5}{5} - \frac{2x^3}{3} \right]_0^1$

$= \frac{1}{2} \left(1 + \frac{1}{5} - \frac{2}{3} \right) = \frac{1}{2} \times \frac{8}{15} = \boxed{\frac{4}{15}}$

Section-B

(2)(b)) Evaluate by changing the variables $\iint_R (x+y)^2\,dx\,dy$ where R is the region bounded by the parallelogram $x+y=0,\ x+y=2,\ 3x-2y=0$ and $3x-2y=3$.

Sol^n: Let $x+y=u,\quad 3x-2y=v$

$\therefore\ x=\frac{1}{5}(2u+v),\quad y=\frac{1}{5}(3u-v)$

$$J=\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}2/5 & 1/5\\ 3/5 & -1/5\end{vmatrix}=-\frac{1}{5}$$

$|J|=1/5$

$dx\,dy=|J|\,du\,dv=\frac{1}{5}du\,dv$

limit: $u=0$ to $u=2$

$v=0$, to $v=3$

Given integral;

$$I=\iint_R (x+y)^2\,dx\,dy=\int_0^3\int_0^2 u^2\frac{1}{5}\,du\,dv$$

$$=\frac{1}{5}\int_0^3\left(\frac{u^3}{3}\right)_0^2 dv=\frac{1}{5}\times\frac{8}{3}\int_0^3 dv=\frac{8}{15}[v]_0^3=\frac{8}{15}\times 3=\frac{8}{5}$$

(6) Que: Evaluate following integral by changing the order of integration

$$\int_0^\infty\int_x^\infty \frac{e^{-y}}{y}\,dy\,dx$$

Sol^n: Given limit: $x=0,\ \infty$

$y=x,\ \infty$

new limit: $y=0,\ \infty$

$x=0,\ y$

$$I=\int_0^\infty\int_x^\infty \frac{e^{-y}}{y}\,dy\,dx=\int_0^\infty\int_0^y \frac{e^{-y}}{y}\,dx\,dy$$

$$=\int_0^\infty \frac{e^{-y}}{y}(x)_0^y\,dy=\int_0^\infty e^{-y}\,dy$$

$$\left(-e^{-y}\right)_0^\infty=1-0=1$$

$\boxed{I=1}$

(5)

Section - A

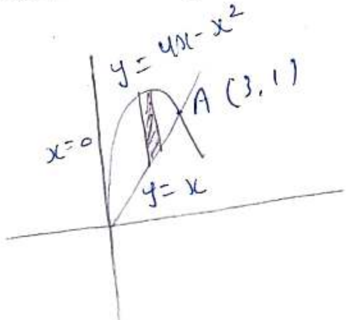
① (i) Evaluate $\int_0^2\int_0^1 (x^2+3y^2)\,dy\,dx$

Sol^n: $= \int_0^2 \left[x^2y + \frac{\cancel{3}y^3}{\cancel{3}}\right]_0^1 dx = \int_0^2 (x^2+1)\,dx = \left[\frac{x^3}{3}+x\right]_0^2$

$= \frac{8}{3}+2 = \boxed{\frac{14}{3}}$

(j) Find the area lying b/w the parabola $y = 4x - x^2$ and above the line $y = x$.

Sol^n: Area $= \iint_R dx\,dy$



Area $= \int_0^3 \int_x^{4x-x^2} dy\,dx$

$= \int_0^3 [y]_x^{4x-x^2} dx = \int_0^3 (4x - x^2 - x)\,dx$

$\int_0^3 (3x - x^2)\,dx = \left[\frac{3x^2}{2} - \frac{x^3}{3}\right]_0^3 = \frac{27}{2} - 9 = \frac{9}{2}$

$\boxed{\text{Area} = 4.5}$

Section - B

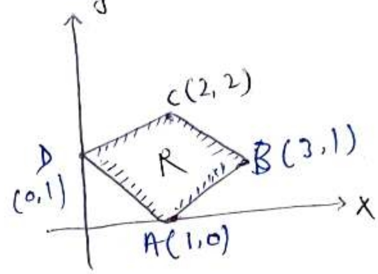
② (d) Evaluate the integral by changing the order of integration

$I = \int_0^1 \int_{x^2}^{2-x} xy\,dy\,dx$

Sol^n: Repeated

Section-C

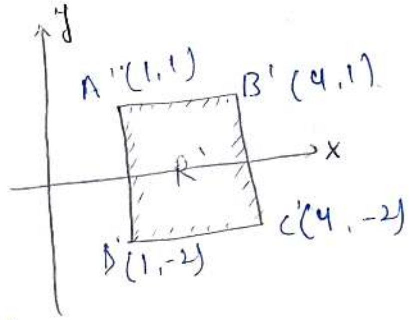
⑥ (i) Evaluate $\iint (x+y)^2\,dx\,dy$, where R is parallelogram in xy-plane with vertices $(1,0)$, $(3,1)$, $(2,2)$, $(0,1)$ using the transformation $u = x+y$, $v = x-2y$.



Sol^n: Let $u = x+y$, $v = x-2y$

region R becomes the region R' by given transformation.

$$x = \frac{1}{3}(2u+v), \quad y = \frac{1}{3}(u-v)$$



$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 2/3 & 1/3 \\ 1/3 & -1/3 \end{vmatrix} = -\frac{1}{3}$$

$$\because \iint_R (x+y)^2\,dx\,dy = \iint_{R'} u^2\,|J|\,du\,dv$$

$$= \int_{-2}^{1}\int_{1}^{4} u^2 \cdot \frac{1}{3} \cdot du\,dv = \frac{1}{3}\int_{-2}^{1}\left[\frac{u^3}{3}\right]_1^4 dv$$

$$= \int_{-2}^{1} 7\,dv = 7\,[v]_{-2}^{1} = 7\,(1-(-2)) = 7\times 3 = 21$$

$$\boxed{I = 21} \text{ Ans.}$$

2018-19

Que find the vol. of region bounded by the surface $y = x^2$, $x = y^2$ and the planes $z=0$, $z=3$.

Sol^n: Repeated

2018-19

Section-A

1 (f) Evaluate $\int_0^1 \int_0^{x^2} e^{y/x} \, dx \, dy$.

putting $\frac{y}{x} = t$

$\frac{1}{x} dy = dt$

$dy = x\,dt$

when $y = 0$, $t = 0$

$y = x^2$, $t = x$

Sol^n: $I = \int_0^1 \int_0^x e^{t} \cdot x\,dt \cdot dx$

$= \int_0^1 x\,dx \int_0^x e^{t}\,dt$

$= \int_0^1 x\,dx \left[e^{t}\right]_0^x = \int_0^1 x\,dx \left[e^{x} - e^{0}\right]$

$= \int_0^1 x e^{x}\,dx - \int_0^1 x\,dx$

$= \left[x e^{x} - \int \frac{dx}{dx} \cdot e^{x}\,dx\right]_0^1 - \left(\frac{x^2}{2}\right)_0^1$

$= \left[x e^{x} - e^{x}\right]_0^1 - \frac{1}{2} = \left[(e^1 - e^1) - (0-1)\right] - \frac{1}{2}$

$1 - \frac{1}{2} = \boxed{\frac{1}{2}}$

Section-C

6(a) change order of integration and evaluate $\int_0^2 \int_{x^2/4}^{3-x} xy\,dy\,dx$

Sol^n: Repeated

(b) calculate volume of the solid bounded by surface $x=0, y=0$, $x+y+z=1$ & $z=0$

Sol^n: Let $V = \iiint dx\,dy\,dz = \iiint x^{1-1} y^{1-1} z^{1-1}\,dx\,dy\,dz$.

$\therefore x \geq 0,\; y \geq 0,\; z \geq 0,\; x+y+z \leq 1$

$V = \frac{\Gamma 1\,\Gamma 1\,\Gamma 1}{\Gamma(1+1+1+1)} = \frac{1}{\Gamma 4}$

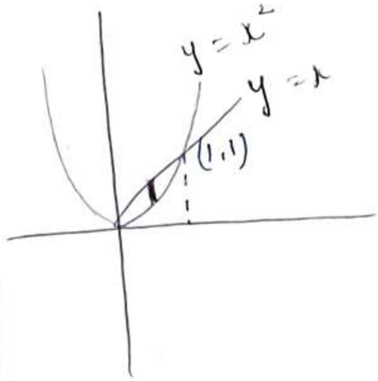
$V = \frac{1}{3!} = \boxed{\frac{1}{6}}$

2017 - 18

Section-A

Q(d) Evaluate area enclosed b/w the parabola $y = x^2$ and straight line $y = x$

Sol^n:



$$= \int_0^1 \int_{x^2}^{x} dy\, dx$$

$$= \int_0^1 [y]_{x^2}^{x}\, dx = \int_0^1 (x - x^2)\, dx$$

$$= \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{2} - \frac{1}{3} = \boxed{\frac{1}{3}}$$

Section- B

(d) i) change order of Integration in $I = \int_0^1 \int_{x^2}^{2-x} f(x,y)\, dy\, dx$

Sol^n: Given limit: $x = 0, 1$
$y = x^2, 2-x$
$x + y = 2$

New limit:

First limit:
$y = 0, 1$
$x = 0, \sqrt{y}$

IInd limit:
$y = 1, 2$
$x = 0, 2-y$

By changing order of integration

$$I = \int_0^1 \int_{x^2}^{2-x} f(x,y)\, dy\, dx = \int_0^1 \int_0^{\sqrt{y}} f(x,y)\, dx\, dy + \int_1^2 \int_0^{2-y} f(x,y)\, dx\, dy$$

ii:) We know that

$$\Gamma n = K^n \int_0^\infty e^{-Kx} x^{n-1}\, dx$$

$$= z^n \int_0^\infty e^{-zx} x^{n-1}\, dx$$

multiplying both side by $e^{-z} z^{m-1}$.

$$\Gamma n = e^{-z} z^{m-1} = \int_0^\infty z^n e^{-zn}\, \cancel{x}^{n-1} .\ e^{-z}\, \cancel{z}^{m-1}\, dx$$

$$\Gamma n \cdot e^{-z} \cdot z^{m-1} = \int_0^\infty z^{n+m-1}\, e^{-x(1+x)} \cdot x^{n-1}\, dx$$

Integrating both sides w.r.t. z from 0 to ∞

$$\Gamma n \int_0^\infty e^{-z} z^{m-1} dz = \int_0^\infty x^{n-1} \left\{ \int_0^\infty e^{-z(1+x)} \cdot z^{m+n-1} dz \right\} dx$$

$$\Gamma n \Gamma m = \int_0^\infty x^{n-1} \left\{ \int_0^\infty e^{-y} \cdot \frac{y^{m+n-1}}{(1+x)^{m+n-1}} \cdot \frac{dy}{(1+x)} \right\} dx$$

$$\Gamma n \Gamma m = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} \left\{ \int_0^\infty e^{-y} y^{m+n-1} dy \right\} dx$$

$$= \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} dx \, \Gamma m+n$$

$$\Gamma n \cdot \Gamma m = \beta(m.n) \cdot \Gamma m+n$$

$$\beta(m.n) = \frac{\Gamma m \Gamma n}{\Gamma m+n}$$

Hence proved

Section-c

Ques: Find the mass of a plate which is formed by co-ordinate planes and plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$, the density is given by $\rho = kxyz$

Sol^n: putting $\frac{x}{a} = u \Rightarrow x = au \Rightarrow dx = a \cdot du$

$\frac{y}{b} = v \Rightarrow y = bv \Rightarrow dy = b \cdot dv$

$\frac{z}{c} = w \Rightarrow z = cw \Rightarrow dw = c\,dw$

$\therefore u \geq 0, \; v \geq 0, \; w \geq 0$

$u+v+w \leq 1$

$$\text{Mass} = \iiint_D \rho \, dx \cdot dy \, dz = \iiint_D kxyz \, dx \, dy \, dz$$

$$M = k \iiint xyz \, dx \, dy \, dz$$

$$= k \iiint (au)(bv)(cw)(du \cdot a)(b\,dv)(c\,dw)$$

$$= k a^2 b^2 c^2 \iiint uvw \, du \, dv \, dw$$

$$= ka^2b^2c^2 \iiint u^{2-1} v^{2-1} w^{2-1} \, du\, dv\, dw$$

$$= ka^2b^2c^2 \frac{\Gamma 2\, \Gamma 2\, \Gamma 2}{\Gamma 2+2+2+1} = ka^2b^2c^2 \cdot \frac{1!\, 1!\, 1!}{6!}$$

$$\Rightarrow M = \frac{ka^2b^2c^2}{720}$$

Ques: Evaluate $I = \int_0^1 \left(\frac{x^3}{(1-x^3)} \right)^{1/2} dx$

Sol^n: $I = \int_0^1 x^{3/2} (1-x^3)^{-1/2} dx$.

put $x^3 = y \Rightarrow x = y^{1/3} \Rightarrow dx = \frac{1}{3} y^{-2/3} dy$

$$I = \int_0^1 y^{1/2} (1-y)^{-1/2} \cdot \frac{1}{3} y^{-2/3} dy$$

$$= \frac{1}{3} \int_0^1 y^{-1/6} (1-y)^{-1/2} dy$$

$$= \frac{1}{3} \int_0^1 y^{5/6-1} \cdot (1-y)^{1/2-1} dy = \frac{1}{3} B\left(\frac{5}{6}, \frac{1}{2}\right)$$

$$I = \frac{1}{3} \frac{\Gamma 5/6\, \Gamma 1/2}{\Gamma 4/3} = \frac{\Gamma 5/6\, \sqrt{\pi}}{3\, \Gamma 4/3}$$

$$\boxed{I = \frac{\Gamma 5/6\, \sqrt{\pi}}{\Gamma 1/3}}$$ Ans.

Ques: Evaluate $\iiint_R (x+y+z)\, dx\, dy\, dz$ when $R: 0 \le x \le 1,\ 1 \le y \le 2,\ 2 \le z \le 3$

Sol^n:

$$I = \int_0^1 \int_1^2 \int_2^3 (x+y+z)\, dz\, dy\, dx = \int_0^1 \int_1^2 \left[xz + yz + \frac{z^2}{2} \right]_2^3 dy\, dx$$

$$= \int_0^1 \int_1^2 \left(x + y + \frac{5}{2}\right) dy\, dx = \int_0^1 \left(xy + \frac{y^2}{2} + \frac{5}{2} y \right)_1^2 dx$$

2020-21

14

$$= \int_0^1 \left[(2x+2+5) - \left(x+\frac{1}{2}+\frac{5}{2}\right)\right] dx$$

$$= \int_0^1 (x+4)\, dx = \left[\frac{x^2}{2} + 4x\right]_0^1$$

$$= \frac{1}{2} + 4 = \frac{9}{2}$$

$$\boxed{I = \frac{9}{2}}$$ Ans

2020-21

①

Section.A

Q1.> Show that the vector $\vec{V} = (x+3y)\hat{i} + (y-3z)\hat{j} + (x-2z)\hat{k}$ is Solenoidal.

Solⁿ: for solenoidal div $\vec{V}$ will be zero

$$div\vec{V} = \nabla\cdot\vec{V} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\left[(x+3y)\hat{i} + (y-3z)\hat{j} + (x-2z)\hat{k}\right]$$

$$\Rightarrow \frac{\partial}{\partial x}(x+3y) + \frac{\partial}{\partial y}(y-3z) + \frac{\partial}{\partial z}(x-2z)$$

$$\Rightarrow 1+1-2 = 0$$

$div\vec{V} = 0$ $\quad$ $\vec{V}$ is solenoidal.

Q.2 State green's theorem

Green's theorem → if C is regular closed curve in xy Plane and R be region bounded by C then

$$\boxed{\int_C \{M\,dx + N\,dy\} = \iint_R \left\{\frac{dN}{dx} - \frac{dM}{dy}\right\} dx\,dy}$$

Section-B

Q-1.> use divergence theorem to evaluate the surface integral $\iint_S (x\,dy\,dz + y\,dz\,dx + z\,dx\,dy)$ where S is the Portion of the Plane $x + 2y + 3z = 6$ which lies in first octant.

Divergence theorem

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \operatorname{div}\vec{F}\,dv$$

$$\vec{F}\cdot\vec{ds} = x\,dy\,dz + y\,dz\,dx + z\,dx\,dy$$

$$[f_1\hat{i} + f_2\hat{j} + f_3\hat{k}]\cdot[dy\,dz\,\hat{i} + dx\,dz\,\hat{j} + dx\,dy\,\hat{k}]$$

$$= [x\,dy\,dz + y\,dz\,dx + z\,dx\,dy]$$

$$\Rightarrow f_1\,dy\,dz + f_2\,dx\,dz + f_3\,dx\,dy = x\,dy\,dz + y\,dz + y\,dz\,dx + z\,dx\,dy$$

$$f_1 = x, \quad f_2 = y, \quad f_3 = z$$

$$\vec{F} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\iint_S (x\,dy\,dz + y\,dz\,dx + z\,dx\,dy) = \iiint_V d\vec{v}\,\vec{F}\cdot dx\,dy\,dz$$

$$= \iiint [\nabla\cdot(x\hat{i} + y\hat{j} + z\hat{k})]\,dx\,dy\,dz.$$

$$\iiint \left[\frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z)\right] dx\,dy\,dz = \iiint (1+1+1)\,dx\,dy\,dz$$

$$= 3\iiint dx\,dy\,dz$$

$$S = x + 2y + 3z = 6$$

$$z = \frac{6 - x - 3y}{3}$$

$$= 3\int_{x=0}^{6}\int_{y=0}^{\frac{6-x}{2}}\int_{z=0}^{\frac{6-x-3y}{3}} dz\,dy\,dx$$

$$3\int_0^6 \int_0^{6-\frac{x}{2}} [z]_0^{\frac{6-x-2y}{3}} \, dy\, dx$$

$$= 3\int_0^6 \int_0^{6-x} \left[\frac{6-x-2y}{3}\right] dy\, dx$$

$$= \int_0^6 \left\{(6-x)y - y^2\right\}_0^{\frac{6-x}{2}} dx.$$

$$= \int_0^6 \left\{\frac{(6-x)^2}{2} - \frac{(6-x)^2}{4}\right\} dx$$

$$= \frac{1}{4} \times \frac{1}{3} \times 6^3 = \underline{\underline{18}} \quad \text{Ans}$$

Section-C

Q-1 A fluid motion is given by $\vec{V} = (y\sin z - \sin x)\hat{i} + (x\sin z + 2yz)\hat{j} + (xy\cos z + y^2)\hat{k}$. Is the motion irrotational? If so, find the velocity potential.

irrotational curl $\vec{V} = 0$

$$\text{curl}\,\vec{V} = \nabla \times \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \partial/\partial y & \partial/\partial z \\ y\sin z - \sin x & x\sin z + 2yz & xy\cos z + y^2 \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(xy\cos z + y^2) - \frac{\partial}{\partial z}(x\sin z + 2yz)\right] - \hat{j}\left[\frac{\partial}{\partial x}(xy\cos z + y^2) - \frac{\partial}{\partial z}(y\sin z - \sin x)\right] + \hat{k}\left[\frac{\partial}{\partial x}(x\sin z + 2yz) - \frac{\partial}{\partial y}(y\sin z - \sin x)\right]$$

$\hat{i}\left[x\cancel{\cos}z + 2\cancel{y} - x\cancel{\cos}z - \cancel{2y}\right] - \hat{j}\left[y\cancel{\cos}z + 0 - y\cancel{\cos}z + 0\right]$

$+\hat{k}\left[\cancel{\sin}z + 0 - \cancel{\sin}z + 0\right] = 0\hat{i} + 0\hat{j} + 0\hat{k} = \vec{0}$

$\boxed{\text{Curl } \vec{V} = \vec{0}}$ motion is irrotational

$\vec{V} = \text{grad } \phi = \nabla\phi$ $\qquad \phi$ = velocity Potential

$$\left[(y\sin z - \sin x)\hat{i} + (x\sin z + 2yz)\hat{j} + (xy\cos z + y^2)\hat{k}\right]$$

$$\left[dx\hat{i} + dy\hat{j} + dz\hat{k}\right] = d\phi$$

$\Rightarrow (y\sin z - \sin x)dx + (x\sin z + 2yz)dy + (xy\cos z + y^2)dz$

$= d\phi$

$\phi = \int \vec{V}\cdot d\vec{r} + c$

$$\boxed{\phi = xy\sin z + y^2 z + \cos x + c}$$

$$\vec{N} = (\nabla s)_{(1,-2,1)} = \hat{i} - 4\hat{j} + 2\hat{k}$$

$$\hat{N} = \frac{\hat{i} - 4\hat{j} + 2\hat{k}}{\sqrt{1+16+4}} = \frac{\hat{i} - 4\hat{j} + 2\hat{k}}{\sqrt{21}}$$

Directional Derivative

$$= (\nabla\phi)_P \cdot \hat{N} = (348\hat{i} - 144\hat{j} + 400\hat{k}) \cdot \frac{(\hat{i} - 4\hat{j} + 2\hat{k})}{\sqrt{21}}$$

$$= \frac{348 + 144(4) + 800}{\sqrt{21}} = \frac{1724}{\sqrt{21}} \text{ Ans}$$

Q.2.) Find the constant a, b, c so that

$\vec{F} = (x + 2y + az)\hat{i} + (bx - 3y - z)\hat{j} + (4x + cy + 2z)\hat{k}$

is irrotational and hence find function ϕ such that

$\vec{F} = \nabla\phi$

Irrotational $\quad$ Curl $\vec{F} = \vec{0}$

$\nabla \times \vec{F} = \vec{0}$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x+2y+az) & (bx-3y-z) & (4x+cy+2z) \end{vmatrix} = \vec{0}$$

$= \hat{i}(c+1) - \hat{j}(4-a) + \hat{k}(b-2) = 0\hat{i} + 0\hat{j} + 0\hat{k}$

$= c+1 = 0 \Rightarrow c = -1$

$4 - a = 0 \Rightarrow a = 4$

$b - 2 = 0 \Rightarrow b = 2$

$$\vec{F} = (x + 2y + 4z)\hat{i} + (2x - 3y - z)\hat{j} + (4x - y + 2z)\hat{k}$$

$$\vec{F} = \nabla\phi \qquad \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\phi = \vec{F} \cdot d\vec{r}$$

$$d\phi = (x + 2y + 4x)dx + (2x - 3y - z)dy + (4x - y + 2z)dz$$

$$\Rightarrow x\,dx + 2y\,dx + 4z\,dx + 2x\,dy - 3y\,dy - z\,dy + 4x\,dz - y\,dz + 2z\,dz$$

$$= (x\,dx - 3y\,dy + 2z\,dz) + d(2xy) + d(4xz) - d(yz)$$

$$\boxed{\phi = \frac{x^2}{2} - \frac{3y^2}{2} + z^2 + 2xy + 4xz - yz + c}$$

$$\nabla \cdot \nabla f = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot \left(6x^2y^2z^4\hat{i} + 4x^3yz\hat{j} + 8x^3y^2z^3\hat{k}\right)$$

$$= \frac{\partial}{\partial x}(6x^2y^2z^4) + \frac{\partial}{\partial y}(4x^3yz^4) + \frac{\partial}{\partial z}(8x^3y^2z^3)$$

$$\phi = 12xy^2z^4 + 4x^3z^4 + 24x^3y^2z^2$$

$$\text{grad}\,\phi = \nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

$$= \left(12y^2z^4 + 48x^2z^4 + 72x^2y^2z^2\right)\hat{i} + \left(24xyz^4 + 48x^3yz^2\right)\hat{j} + \left(48xy^2z^3 + 16x^3z^3 + 48x^3y^2z\right)\hat{k}$$

$(\nabla\phi)$ at Point $= \cancel{348\hat{i}}$

$$= \left[12\times4 + 12 + 72\times4\right]\hat{i} + \left[-24\times2 - 48\times2\right] + \left[48(4) + 16 + 48\times4\right]\hat{k}$$

$$= 348\hat{i} - 144\hat{j} + 400\hat{k}$$

$$S = 2y^2z - 3x - z^2$$

$$\nabla S = \frac{\partial S}{\partial x}\hat{i} + \frac{\partial S}{\partial y}\hat{j} + \frac{\partial S}{\partial z}\hat{k}$$

$$= (y^2z - 3)\hat{i} + (2xyz)\hat{j} + (xy^2 - 2z)\hat{k}$$

$$= 2\left[\int_{-a}^{a} x\sqrt{a^2-x^2}\cdot dx + \int_{-a}^{a} \frac{a^2-x^2}{2} dx\right]$$

$$= 2\left[\int_{-a}^{+a} \cancel{x\sqrt{a^2-x^2}\cdot dx}^{\,0} + \int_{-a}^{a} \frac{a^2-x^2}{2} dx\right]$$

odd even

$f(x) = f(x)$ even

$f(-x) = -f(x)$: odd

$$\cancel{2}\int_{-a}^{a} \frac{a^2-x^2}{\cancel{2}} dx = 2\int_{0}^{a} (a^2-x^2)\, dx$$

$$= 2\left[a^2x - \frac{x^3}{3}\right]_0^a = 2\left[a^3 - \frac{a^3}{3}\right] = 2\left[\frac{2a^3}{3}\right]$$

$$= \frac{4a^3}{3}$$

Section C

Q-1.) find the directional derivative of $\nabla(\nabla f)$ at the Point $(1,-2,1)$ in the direction of the normal to the surface $xy^2z = 3x + z^2$ where $f = 2x^3y^2z^4$

Soln: Directional derivative of ϕ at Point P in direction of a

$$= (\nabla\phi)_P \cdot \hat{a} \qquad \hat{a} = \frac{\vec{a}}{|\vec{a}|}$$

$$\nabla f = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)(2x^3y^2z^4)$$

$$= 6x^2y^2z^4\,\hat{i} + 4x^3yz^4\,\hat{j} + 8x^3y^2z^3\,\hat{k}$$

Q.2:) State Stoke's theorem (9)

Relation b/w line and surface integral

$$\boxed{\oint_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot\hat{N}\,ds}$$

~~Vect~~ $\vec{F} \rightarrow$ Vector Point function

$S \rightarrow$ open surface bounded by curve C

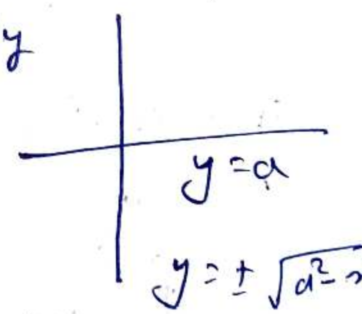
$\hat{N} \rightarrow$ unit normal vector at any Point of S

Section-B

Q-1:) Apply Green's theorem to evaluate $\int_C [(2x^2-y^2)dx + (x^2+y^2)dy]$, where C is the boundary of the area enclosed by the x-axis and the upper half of the circle $x^2+y^2=a^2$.

by Greens theorem

$$\int_C M\,dx + N\,dy = \iint_S \left(\frac{dN}{dx} - \frac{dM}{dy}\right)dx\,dy$$



$$I = \int_C (2x^2-y^2)dx + (x^2+y^2)dy$$

$M = 2x^2-y^2$

$\frac{dM}{dy} = -2y$

$N = x^2+y^2$

$\frac{dN}{dx} = 2x$

$$= \iint_S (2x+2y)\,dx\,dy$$

$$= \int_{x=-a}^{a}\int_{y=0}^{\sqrt{a^2-x^2}} (2x+2y)\,dy\,dx$$

$$= 2\int_{-a}^{a}\left(xy+\frac{y^2}{2}\right)_0^{\sqrt{a^2-x^2}} dx \Rightarrow 2\int_{-a}^{a}\left[x\sqrt{a^2-x^2} + \frac{(a^2-x^2)}{2}\right]$$

2021-22

Section-A

(i) find a unit normal vector to the surface $z^2 = x^2 + y^2$ at the Point $(1, 0, -1)$.

let ϕ be surface $\vec{N} = \nabla \phi$

$$\phi = z^2 - x^2 - y^2$$

$$\vec{N} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k} \right)\left(z^2 - x^2 - y^2 \right)$$

$$= \frac{\partial}{\partial x}(z^2 - x^2 - y^2)\hat{i} + \frac{\partial}{\partial x}(z^2 - x^2 - y^2)\hat{j} + \frac{\partial}{\partial x}(z^2 - x^2 - y^2)\hat{k}$$

$$\nabla \phi = -2x\hat{i} - 2y\hat{j} + 2z\hat{k} = \vec{N}$$

$$(\vec{N}) \text{ at } (1, 0, -1) = -2\hat{i} - 2\hat{k}$$

$$\hat{N} = \frac{-2\hat{i} - 2\hat{k}}{\sqrt{4+4}} = \frac{-2(\hat{i} + \hat{k})}{\sqrt{8}}$$

if S is open surface bounded by closed current C and $\vec{F}$ is any vector point function then

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl}\, \vec{F} \cdot \hat{n}\, ds$$

where $\hat{n}$ is unit normal to surface S.

2022-23

Section-A

Ques: Find curl of a vector field given by $\vec{F} = (x^2 + xy^2)\hat{i} + (y^2 + x^2y)\hat{j}$.

Sol^n:

$$\text{curl}\,\vec{F} = \nabla \times \vec{F}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2+xy^2 & y^2+x^2y & 0 \end{vmatrix}$$

$$\text{curl}\,\vec{F} = \hat{i}\left[\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(y^2+x^2y)\right] - \hat{j}\left[\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial z}(x^2+xy^2)\right] + \hat{k}\,(0)$$

$$= 0\hat{i} - 0\hat{j} = \vec{0}$$

$$\text{curl}\,\vec{F} = \vec{0}$$

$\vec{F}$ is irrotational

(B)

Ques: Find the directional derivative of scalar function $f(x,y,z) = xyz$ at point $P(1,1,3)$ in the direction of outward drawn normal to the sphere $x^2+y^2+z^2=11$ through the point P.

Sol^n: let $\phi_1 = xyz$

$$\text{grad}\,\phi_1 = \nabla\phi_1 = \hat{i}\frac{\partial}{\partial x}(xyz) + \hat{j}\frac{\partial}{\partial y}(xyz) + \hat{k}\frac{\partial}{\partial z}(xyz)$$

$$\text{grad}\,\phi_1 = yz\hat{i} + xz\hat{j} + xy\hat{k}$$

at $P(1,1,3)$

$$\text{grad}\,\phi_1 = 3\hat{i} + 3\hat{j} + \hat{k}$$

Section-C

let a unit vector on the surface (12)

$$\phi_2 = x^2+y^2+z^2-11$$

$$\text{grad } \phi_2 = \hat{i}\frac{\partial}{\partial x}(x^2+y^2+z^2-11) + \hat{j}\frac{\partial}{\partial y}(x^2+y^2+z^2-11) + \hat{k}\frac{\partial}{\partial z}(x^2+y^2+z^2-11)$$

$$\text{grad } \phi_2 = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

at point $(1,1,3)$

$$\text{grad } \phi_2 = 2\hat{i} + 2\hat{j} + 6\hat{k}$$

$$\vec{a} = 2\hat{i} + 2\hat{j} + 6\hat{k}$$

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{2\hat{i}+2\hat{j}+6\hat{k}}{\sqrt{4+4+36}} = \frac{\cancel{2}(\hat{i}+\hat{j}+3\hat{k})}{\cancel{2}\sqrt{11}}$$

$$\hat{a} = \frac{\hat{i}+\hat{j}+3\hat{k}}{\sqrt{11}}$$

Directional derivative $= (3\hat{i}+3\hat{j}+\hat{k}) \cdot \frac{(\hat{i}+\hat{j}+3\hat{k})}{\sqrt{11}}$

$$= \frac{3+3+3}{\sqrt{11}} = \boxed{\frac{9}{\sqrt{11}}} \text{ Ans.}$$