UNIT I

Rules for finding the Complementary function (CF)

1. When the roots of Auxiliary eq^n are real and distinct $m = m_1, m_2, \dots m_n$.

$$\boxed{CF = C_1 e^{m_1 x} + C_2 e^{m_2 x} + \dots + C_n e^{m_n x}}$$

2. When the roots of auxiliary eq^n are equal. $m = m_1, m_1$

$$\boxed{CF = (C_1 + C_2 x) e^{m_1 x}}$$

If $m = m_1, m_1, m_1$

$$\boxed{CF = (C_1 + C_2 x + C_3 x^2) e^{m_1 x}}$$

3. When roots of auxiliary eq^n are imaginary $(m = \alpha \pm i\beta)$

$$\boxed{CF = e^{\alpha x}(C_1 \cos\beta x + C_2 \sin\beta x)}$$

4. When roots of auxiliary eq^n are irrational $(m = \alpha \pm \sqrt{\beta})$

$$\boxed{CF = e^{\alpha x}(C_1 \cosh\sqrt{\beta}x + C_2 \sinh\sqrt{\beta}x)}$$

Rules for finding the particular integral (PI)

$$PI = \frac{1}{f(D)} Q$$

1. When $Q = e^{ax}$.
   Put $D = a$ in $f(D)$ provided $f(a) \neq 0$
   If $f(a) = 0$, Case fail
   Multiply N^r by x & Differentiate D^r wrt D then put $D = a$ provided $f'(a) \neq 0$ & so on.

2. When $Q = \sin(ax+b)$ or $\cos(ax+b)$
   Put $D^2 = -a^2$. provided $f(-a^2) \neq 0$
   If $f(-a^2) = 0$, Case fail
   Multiply N^r by x & Differentiate D^r wrt D then put $D^2 = -a^2$ & so on.

3. When $Q = x^m$, m being a positive integer
   * Take out lowest degree term from $f(D)$ to make the first term unity
     The remaining factor will be of form $[1 \pm f(D)]$

4. When $Q = e^{ax}.V$, where V is a function of x.

Bring e^{ax} before the operator and then put $D+a$ in place of D

ie $\frac{1}{f(D)} e^{ax} V = e^{ax} \frac{1}{f(D+a)} V$.

5 When Q is any other function of x.

Use formulae

(1) $\frac{1}{D-\alpha} Q = e^{\alpha x} \int e^{-\alpha x} Q\, dx$

(2) $\frac{1}{D+\alpha} Q = e^{-\alpha x} \int e^{\alpha x} Q\, dx$

BAS 203: ENGINEERING MATHEMATICS-

UNIT I

Ordinary Differential Equation of Higher Order

Homogeneous Linear differential Equations (Euler & Cauchy Equations)

An equation of the form

$$x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + a_2 x^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \cdots + a_n y = Q \quad \text{--- (1)}$$

where $a_1, a_2, \cdots a_n$ are constants and Q is a function of x, is called Homogeneous Linear equation.

Working Rule

1. Put $x = e^z$ so that $z = \log x$, $D \equiv \frac{d}{dz}$.
2. Replace $x\frac{d}{dx}$ by D, $x^2 \frac{d^2}{dx^2}$ by $D(D-1)$, $x^3 \frac{d^3}{dx^3}$ by $D(D-1)(D-2)$ & so on.
3. By doing so, this type of equation reduces to Linear differential equation with constant Coefficients which can be solved by the earlier method.

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Q1 Solve

$$x^2\frac{d^2y}{dx^2}+4x\frac{dy}{dx}+2y=e^x$$

Sol. Put $x=e^z$ $\therefore z=\log x$, $D\equiv\frac{d}{dz}$, Substitute $x\frac{d}{dx}=D$, $x^2\frac{d^2}{dx^2}=D(D-1)$

$\therefore$ given eq^n becomes

$$[D(D-1)+4D+2]y=e^{e^z}$$

$$[D^2+3D+2]y=e^{e^z}$$

A.E $m^2+3m+2=0$

$\Rightarrow (m+1)(m+2)=0 \Rightarrow m=-1,-2.$

$$\therefore \boxed{CF=c_1e^{-z}+c_2e^{-2z}}$$

$$PI=\frac{1}{D^2+3D+2}e^{e^z}=\left(\frac{1}{D+1}-\frac{1}{D+2}\right)e^{e^z}$$

$$=\frac{1}{D+1}e^{e^z}-\frac{1}{D+2}e^{e^z}$$

$$=e^{-z}\int e^z.e^{e^z}dz-e^{-2z}\int e^{2z}e^{e^z}dz$$

$\left(\text{using } \frac{1}{D+\alpha}Q=e^{-\alpha x}\int e^{\alpha x}Q\,dx\right.$, $\left.\frac{1}{D-\alpha}Q=e^{\alpha x}\int e^{-\alpha x}Q\,dx\right)$

$$=e^{-z}e^{e^z}-e^{-2z}(e^z-1)e^{e^z}$$

$$\boxed{PI=e^{-2z}e^{e^z}}$$

Hence Complete sol

$$y=CF+PI$$

$$\boxed{y=c_1e^{-z}+c_2e^{-2z}+e^{-2z}e^{e^z}}$$

$$\therefore \boxed{y=\frac{c_1}{x}+\frac{c_2}{x^2}+\frac{e^x}{x^2}}$$

Q2 Solve the homogeneous differential equation

$$x^3\frac{d^3y}{dx^3}+3x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}=24x^2$$

Sol: Put $x=e^z$, $\therefore z=\log x$, $D\equiv\frac{d}{dz}$

Then the given eq^n reduces to

$$[D(D-1)(D-2)+3D(D-1)+D]y=24e^{2z}$$

$$(D^3-3D^2+2D+3D^2-3D+D)y=24e^{2z}$$

$\Rightarrow D^3y=24e^{2z}$

$\therefore$ AE $m^3=0 \Rightarrow m=0,0,0$

$\therefore CF=(c_1+c_2z+c_3z^2)e^{0z}$

$$\boxed{CF=c_1+c_2z+c_3z^2}$$

$$PI=\frac{1}{D^3}(24e^{2z})=24\frac{e^{2z}}{8}=3e^{2z} \quad (\text{Put } D=a=2)$$

The Complete Sol is

$$y=CF+PI=c_1+c_2z+c_3z^2+3e^{2z}$$

$$\Rightarrow \boxed{y=c_1+c_2\log x+c_3(\log x)^2+3x^2}$$

Q3 Solve the eq^n

$$x^2\frac{d^2y}{dx^2}-3x\frac{dy}{dx}+5y=x^2\sin(\log x)$$

Sol Put $x=e^z$, $z=\log x$, $D\equiv\frac{d}{dz}$

The the eq^n reduces to

$$[D(D-1)-3D+5]y=e^{2z}\sin z$$

$$(D^2-4D+5)y=e^{2z}\sin z$$

$\therefore CF = e^{2z}(c_1\cos z + c_2 \sin z)$

$$PI = \frac{1}{D^2-4D+5} e^{2z}\sin z$$

$$= e^{2z}\frac{1}{(D+2)^2-4(D+2)+5}\sin z$$

$$= e^{2z}\frac{1}{D^2+1}\sin z \quad = e^{2z}\cdot z\frac{1}{2D}\sin z \qquad \text{(Case fail)}$$

Put $D^2 = -1^2 = -1$

$$= -\frac{z}{2}e^{2z}\cos z$$

$\therefore$ Complete solⁿ

$$y = CF + PI = e^{2z}(c_1\cos z + c_2\sin z) - \frac{z}{2}e^{2z}\cos z$$

$$\therefore y = x^2[c_1\cos(\log x) + c_2\sin(\log x)] - \frac{x^2}{2}\log x \cos(\log x)$$

Linear differential eqⁿ of II Order

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R.$$

where P, Q, R are function of x

Method I — By Changing Independent Variable

To solve $y'' + Py' + Qy = R$.

Working Rule

1. Make the Coefficient of $\frac{d^2y}{dx^2}$ as 1 if it is not so.
2. Compare the equation with standard form. $y'' + Py' + Qy = R$ and get P, Q and R.
3. Choose z such that
$$\left(\frac{dz}{dx}\right)^2 = Q.$$
Here Q is taken in such a way that it remains the whole square of a function without surd and its negative sign is ignored.
4. Find $\frac{dz}{dx}$ hence obtain z (on integration) and $\frac{d^2z}{dx^2}$ (on differentiation)
5. Find P_1, Q_1 & R_1 by the formulae
$$P_1 = \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}, \quad Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2}, \quad R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2}$$
6. Reduced eqⁿ is $\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1 y = R_1$ which we solve for y in terms of z [illegible] terms of x by replacing

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Q1 Solve by Changing independent Variable

$$(1+x)^2\frac{d^2y}{dx^2}+(1+x)\frac{dy}{dx}+y=4\cos\log(1+x)$$

Sol $\frac{d^2y}{dx^2}+\frac{1}{1+x}\frac{dy}{dx}+\frac{y}{(1+x)^2}=\frac{4}{(1+x)^2}\cos\log(1+x)$

Choose z such that

$$\left(\frac{dz}{dx}\right)^2=Q=\frac{1}{(1+x)^2}$$

$$\Rightarrow \frac{dz}{dx}=\frac{1}{1+x}$$

On Integrating $\boxed{z=\log(1+x)}$

on differentiation $\frac{d^2z}{dx^2}=-\frac{1}{(1+x)^2}$

$$\therefore P_1=\frac{\frac{d^2z}{dx^2}+P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}=\frac{\frac{-1}{(1+x)^2}+\frac{1}{1+x}\cdot\frac{1}{1+x}}{\left(\frac{1}{1+x}\right)^2}=0$$

$$Q_1=\frac{Q}{\left(\frac{dz}{dx}\right)^2}=\frac{\frac{1}{(1+x)^2}}{\frac{1}{(1+x)^2}}=1$$

$$R_1=\frac{R}{\left(\frac{dz}{dx}\right)^2}=\frac{\frac{4}{(1+x)^2}\cos\log(1+x)}{\frac{1}{(1+x)^2}}=4\cos z$$

Reduced eqⁿ $\Rightarrow \frac{d^2y}{dz^2}+P_1\frac{dy}{dz}+Q_1y=R_1$

$$\frac{d^2y}{dz^2}+y=4\cos z$$

AE $m^2+1=0 \Rightarrow \boxed{m=\pm i}$

$$PI=\frac{1}{D^2+1}(4\cos z)=4\cdot\frac{z}{2D}\cos z$$

$$=2z\sin z$$

Complete sol

$$y=CF+PI$$

$$\therefore y=C_1\cos\log(1+x)+C_2\sin\log(1+x)+2\log(1+x)\sin\log(1+x)$$

Q2 Solve by Changing the independent Variable

$$\frac{d^2y}{dx^2}+(3\sin x-\cot x)\frac{dy}{dx}+2y\sin^2x=e^{-\cos x}\sin^2x$$

Sol. $P=3\sin x-\cot x$, $Q=2\sin^2x$, $R=e^{-\cos x}\sin^2x$

$$\left(\frac{dz}{dx}\right)^2=\sin^2x \qquad \text{(neglect 2)}$$

$$\Rightarrow \frac{dz}{dx}=\sin x$$

$$\Rightarrow \boxed{z=-\cos x} \quad \& \quad \frac{d^2z}{dx^2}=\cos x$$

$$P_1=\frac{\frac{d^2z}{dx^2}+P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}=\frac{\cos x+\sin x(3\sin x-\cot x)}{(\sin x)^2}=3$$

$$Q_1=\frac{Q}{\left(\frac{dz}{dx}\right)^2}=\frac{2\sin^2x}{\sin^2x}=2$$

$$R_1=\frac{R}{\left(\frac{dz}{dx}\right)^2}=\frac{e^{-\cos x}\sin^2x}{\sin^2x}=e^{-\cos x}=e^{z}$$

Reduced eq, $\frac{d^2y}{dz^2}+P_1\frac{dy}{dz}+Q_1y=R_1$

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AE $m^2+3m+2=0$

$\boxed{m=-1,-2}$

$\boxed{CF = C_1 e^{-z} + C_2 e^{-2z}}$

$PI = \frac{1}{D^2+3D+2}(e^z) = \frac{e^z}{6}$

Complete sol $y = C_1 e^{-z} + C_2 e^{-2z} + \frac{z^2}{6}$

$\Rightarrow \boxed{y = C_1 e^{\cos x} + C_2 e^{2\cos x} + \frac{e^{\cos x}}{6}}$

Q3 Solve by Changing independent Variable

$$(1+x^2)^2 \frac{d^2y}{dx^2} + 2x(1+x^2)\frac{dy}{dx} + 4y = 0$$

Sol:- $\frac{d^2y}{dx^2} + \frac{2x}{1+x^2}\frac{dy}{dx} + \frac{4}{(1+x^2)^2} y = 0$ on comparing with $\left(\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = \right.$

$\therefore P = \frac{2x}{1+x^2}, \; Q_1 = \frac{4}{(1+x^2)^2}, \; R=0$

Choose z, such that

$$\left(\frac{dz}{dx}\right)^2 = Q = \frac{4}{(1+x^2)^2}$$

$\Rightarrow \boxed{\frac{dz}{dx} = \frac{2}{1+x^2}}$

on Integration $\boxed{z = 2\tan^{-1}x}$

on differentiation $\boxed{\frac{d^2z}{dx^2} = \frac{-4x}{(1+x^2)^2}}$

$\therefore P_1 = \dfrac{\frac{-4x}{(1+x^2)^2} + \frac{2x}{1+x^2}\cdot\frac{2}{1+x^2}}{\frac{4}{(1+x^2)^2}} = 0$

$Q_1 = \dfrac{\frac{4}{(1+x^2)^2}}{\frac{4}{(1+x^2)^2}} = 1$

$R_1 = \dfrac{0}{\left(\frac{4}{(1+x^2)^2}\right)} = 0$

$\therefore$ Reduced eq^n $\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1 y = R_1$

$\Rightarrow \frac{d^2y}{dz^2} + y = 0$

$\therefore$ AE $m^2+1=0$

$\boxed{m = \pm i}$

$CF = C_1\cos z + C_2 \sin z$, $PI = 0$

Complete sol^n

$\therefore y = CF + PI$

$y = C_1\cos z + C_2\sin z$

$\Rightarrow \boxed{y = C_1\cos(2\tan^{-1}x) + C_2\sin(2\tan^{-1}x)}$

Method 2 Variation of Parameter

Working Rule To solve $y'' + Py' + Qy = R$.

1. Find out the parts of C.F
2. Let them be u and v
3. Consider $y = Au + Bv$ as the Complete solution of given eq^n.
4. A and B are determined by the formulae.

$$A = \int -\frac{Rv}{W} dx + c_1 \quad \text{and} \quad B = \int \frac{Ru}{W} dx + c_2$$

where c_1, c_2 are constants of integration

& $W = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = uv' - vu'$

5. At last Write $\boxed{y = Au + Bv}$ as the Complete Solution

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Q1. Solve by the method of Variation of parameters.

$$\frac{d^2y}{dx^2} - y = \frac{2}{1+e^x}$$

Sol $(D^2-1)y = \frac{2}{1+e^x}$

AE $m^2 - 1 = 0 \Rightarrow m = \pm 1$ (distinct roots)

$CF = c_1 e^{-x} + c_2 e^{x}$.

$\therefore$ parts of CF are $u = e^x$, $v = e^{-x}$

$R = \frac{2}{1+e^x}$.

Let $y = Au + Bv \Rightarrow \boxed{y = Ae^x + Be^{-x}}$ be. Complete sol^n

where $A = \int -\frac{Rv}{W} dx + c_1$

$$W = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix}$$

$= -e^x e^{-x} - e^x e^{-x}$

$\boxed{W = -2.}$

$\Rightarrow A = \int - \frac{2\, e^{-x}}{1+e^x (-2)} dx + c_1$

$= \int \frac{e^{-x}}{1+e^x} dx + c_1$

$= \int \frac{e^{-2x}}{e^{-x}+1} dx + c_1$

$$\boxed{A = \log\left(\frac{1+e^x}{e^x}\right) - e^{-x} + c_1}$$

$B = \int \frac{Ru}{W} dx + c_2$

$= - \int \frac{e^x}{1+e^x} dx + c_2$

$\boxed{B = -\log(1+e^x) + c_2}$

$\therefore$ Complete sol^n is

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Q2 Solve $\frac{d^2y}{dx^2} + y = \operatorname{cosec} x$ by the method of variation of parameter.

Sol CF is $C_1 \cos x + C_2 \sin x$

$\therefore u = \cos x, \ v = \sin x, \quad R = \operatorname{cosec} x$

Complete sol^n $\boxed{y = Au + Bv}$ —— (1)

$$W = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x$$

$$\boxed{W = 1}$$

$$A = \int \frac{-Rv}{W}\,dx + C_1$$

$$= \int (-1)\,dx + C_1$$

$$\boxed{A = -x + C_1}$$

$$B = \int \frac{Ru}{W}\,dx + C_2$$

$$= \int \cot x\,dx + C_2 = \log \sin x + C_2$$

$$\therefore \boxed{B = \log \sin x + C_2}$$

Complete sol^n is

$$y = Au + Bv$$

$$y = (-x + C_1)\cos x + (\log \sin x + C_2)\sin x$$

$$\therefore \boxed{y = C_1 \cos x + C_2 \sin x - x\cos x + \sin x \log \sin x}$$

Q3 Using variation of parameter, solve

$$x^2 \frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - 12y = x^3 \log x \quad \text{—— (1)}$$

Sol Put $x = e^z$, $z = \log x$, $D \equiv \frac{d}{dz}$ (homog. eq)

eq (1) reduces to

$$(D(D-1) + 2D - 12)\,y = z e^{3z}$$

AE $m^2 + m - 12 = 0 \Rightarrow m = 3, -4$

$$CF = C_1 e^{3z} + C_2 e^{-4z}$$

$$= C_1 x^3 + C_2 x^{-4}$$

$\therefore$ parts of CF $\boxed{u = x^3, \ v = x^{-4}}$, $R = x \log x$

Complete sol^n $\boxed{y = Au + Bv}$

$$W = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = uv' - vu'$$

$$A = -\int \frac{Rv}{W}\,dx + C_1$$

$$= -\int \frac{x \log x \cdot x^{-4}}{x^3(-4x^{-5}) - 3x^2(x^{-4})}\,dx + C_1$$

$$= -\int \frac{x^{-3}\log x}{-7x^{-2}}\,dx + C_1$$

$$= \frac{1}{7}\int \frac{\log x}{x}\,dx + C_1 = \frac{1}{7}\frac{(\log x)^2}{2} + C_1$$

$$\boxed{A = \frac{1}{14}(\log x)^2 + C_1}$$

$$B = \int \frac{Ru}{W}\,dx + C_2$$

$$= \int \frac{x \log x \cdot x^3}{-7x^{-2}}\,dx + C_2 = -\frac{1}{7}\int x^6 \log x\,dx + C_2$$

$$= -\frac{1}{7}\left[\log \cdot \frac{x^7}{7} - \int \frac{1}{x} \cdot \frac{x^7}{7}\,dx\right] + C_2$$

$$= -\frac{1}{7}\left[\frac{x^7 \log x}{7} - \frac{1}{7}\left(\frac{x^7}{7}\right)\right] + C_2 = \frac{x^7}{\ldots}\left(\frac{1}{\ldots} - \log x\right)$$

$\therefore$ Complete solⁿ

$$y = Au + Bv = Ax^3 + Bx^{-4}$$

$$\therefore y = \left[\frac{1}{14}(\log x)^2 + c_1\right]x^3 + \left[\frac{x^7}{49}\left(\frac{1}{7} - \log x\right) + c_2\right]x^{-4}$$

where c_1 & c_2 are arbitrary Constants.

Unit-2 (BAS203)

Laplace Transform of Periodic Function:

Statement: If $F(t)$ is a periodic function with period T i.e. $F(t+T) = F(t+2T) = F(t+3T) =$

$F(t + nT) = F(t)$, then

$$L\{F(t)\} = \frac{1}{1-e^{-sT}}\int_0^T e^{-st}F(t)dt$$

Example: Find Laplace Transform of the rectified semi-wave function defined by

$$f(t) = \begin{cases} \sin\omega t, & 0 < t \le \frac{\pi}{\omega} \\ 0 & \frac{\pi}{\omega} < t < \frac{2\pi}{\omega} \end{cases}$$

Solution: Here $f(t)$ is a Periodic function with period $\frac{2\pi}{\omega}$

$$L\{F(t)\} = \frac{1}{1-e^{\frac{-2\pi s}{\omega}}}\int_0^{2\pi/\omega} e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{\frac{-2\pi s}{\omega}}}\left[\int_0^{\pi/\omega} e^{-st}\sin\omega t + \int_{\pi/\omega}^{2\pi/\omega} e^{-st}.0dt\right]$$

$$= \frac{1}{1-e^{\frac{-2\pi s}{\omega}}}\left[\frac{e^{-st}(-s\sin\omega t - \omega\cos\omega t)}{s^2+\omega^2}\right]_0^{\pi/\omega}$$

$$= \frac{\omega}{(1-e^{\frac{-\pi s}{\omega}})(s^2+\omega^2)}$$

Example: Find Laplace Transform of the square-wave function of period a defined as

$$f(t) = \begin{cases} 1 & 0 \le t < a/2 \\ -1 & a/2 < t < a \end{cases}$$

Solution: $L\{f(t)\} = \frac{1}{1-e^{-as}}\int_0^a e^{-st} f(t)\,dt$

$$= \frac{1}{1-e^{-as}}\left[\int_0^{a/2} e^{-st}.1\,dt + \int_{a/2}^a e^{-st}(-1)\,dt\right]$$

$$= \frac{1}{1-e^{-as}}\left[\left(\frac{e^{-st}}{-s}\right)_0^{a/2} + \left(\frac{e^{-st}}{s}\right)_{a/2}^a\right]$$

$$= \frac{1}{s(1-e^{-as})}\left[1-e^{-\frac{as}{2}}\right]^2$$

$$= \frac{1}{s}\left[\frac{e^{\frac{as}{4}}-e^{-\frac{as}{4}}}{e^{\frac{as}{4}}+e^{\frac{-as}{4}}}\right]=\frac{1}{s}\tanh\left(\frac{as}{4}\right)$$

Convolution theorem:

Statement: If $L^{-1}\{f(s)\}=F(t)$ and $L^{-1}\{g(s)\}=G(t)$ then

$$L^{-1}\{f(s)g(s)\} = F * G = \int_0^t F(u)G(t-u)du$$

Example. Using Convolution theorem to evaluate: $L^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right\}$

Solution: Here, $\frac{s^2}{(s^2+a^2)(s^2+b^2)}=\frac{s}{s^2+a^2}.\frac{s}{s^2+b^2}$

Let, $f(s)=\frac{s}{s^2+a^2}$ and $g(s)=\frac{s}{s^2+b^2}$

Such that, $F(t)=L^{-1}\{f(s)\} \Rightarrow L^{-1}\left(\frac{s}{s^2+a^2}\right) = \boldsymbol{cosat}$

and $G(t)=L^{-1}\{g(s)\}\Rightarrow L^{-1}\left(\frac{s}{s^2+b^2}\right) = \boldsymbol{cosbt}$

So using convolution theorem

$$L^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right\} = \int_0^t Cosau.Cosb(t-u)du$$

$$= \frac{1}{2}\int_0^t Cos\{(a-b)u+bt\}+Cos\{(a+b)u-bt\}du$$

$$= \frac{1}{2}\left[\frac{Sin\{(a-b)u+bt}{a-b}+\frac{Sin\{(a+b)u-bt}{a+b}\right]_0^t$$

$$= \frac{a\sin at-b\sin bt}{a^2-b^2}$$

Example 4: Find inverse Laplace Transform of $\frac{s}{(s^2+a^2)^2}$

Solution: $L^{-1}\left(\frac{a}{s^2+a^2}\right)=Sin\,at$

$$L^{-1}\left[\frac{d}{ds}\left\{\frac{a}{s^2+a^2}\right\}\right] = -t\ Sinat$$ (Using formula of Inverse Laplace Transform of Derivation)

$$\boldsymbol{L^{-1}\left\{\frac{-2as}{(s^2+a^2)^2}\right\} = -t\ Sinat}$$

$$L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}=\frac{t}{2a}Sin\,at\,.$$

Application of Laplace Transform to solution of differential Equation:

Example: Using Laplace transform, solve the following differential equation

$\frac{d^2x}{dt^2}+2\frac{dx}{dt}+5x=e^t\sin t$, where x (0) = 0 and x' (0) = 1. [P.T.U. 2007, 2009, 2010]

Solution: The given equation is $\frac{d^2x}{dt^2}+2\frac{dx}{dt}+5x=e^t\sin t$

Taking Laplace Transform of both sides, we get

$$L(x'')+2L(x')+5L(x)=L(e^t\sin t)$$

$$[s^2\bar{x}-s\,x(0)-x'(0)]+2\,[s(\bar{x})-x(0)]+5\bar{x}=\frac{1}{(s+1)^2+1}$$

Using given conditions, it reduces to

$$(s^2+2s+5)\bar{x}-1=\frac{1}{s^2+2s+2}$$

$$\bar{x}=\frac{1}{(s^2+2s+2)(s^2+2s+5)}+\frac{1}{s^2+2s+2}$$

$$\bar{x}=\frac{1}{3}\left[\frac{1}{s^2+2s+2}-\frac{1}{s^2+2s+s}\right]+\frac{1}{s^2+2s+2}$$

$$\bar{x}=\frac{1}{3}\left[\frac{1}{s^2+2s+2}+\frac{2}{s^2+2s+5}\right]$$

$$\bar{x}=\frac{1}{3}\left[\frac{1}{(s+1)^2}+\frac{2}{(s+1)^2+2^2}\right]$$

Taking Inverse Laplace transform of both sides, we get.

$$x=\frac{1}{3}L^{-1}\left\{\frac{1}{(s+1)^2+1}++2.\frac{1}{(s+1)^2+2^2}\right]$$

$$x=\frac{1}{3}\{e^{-t}\sin t+2e^{-t}\sin 2t\}$$

Example: Using Laplace Transformation, solve the differential equation

$$\frac{d^2x}{dt^2}+9x=\cos 2t \qquad \text{If} \qquad x(0)=1,\ x\left(\frac{\pi}{2}\right)=-1$$

Solution: The given equation is $x''+9x=\cos 2t$

Taking the Laplace transform of both sides, we get

$L(x'')+9L(x)=L(\cos 2t)$

$[s^2\bar{x}-sx(0)-x'(0)]+9\bar{x}=\frac{s}{s^2+4}$

$(s^2+9)\bar{x}-s-A=\frac{s}{s^2+4}$, Where $x'(0)=A$ Say

$$\bar{x}=\frac{s}{(s^2+4)(s^2+9)}+\frac{s}{s^2+9}+\frac{A}{s^2+9}$$

$$\bar{x}=\frac{1}{s}\left(\frac{s}{s^2+4}-\frac{s}{s^2+9}\right)+\frac{s}{s^2+9}+\frac{A}{s^2+9}$$

Taking the inverse Laplace transform of both sides, we get

$$x=\frac{1}{5}L^{-1}\left(\frac{s}{s^2+4}\right)-\frac{1}{5}L^{-1}\left(\frac{s}{s^2+9}\right)+L^{-1}\left(\frac{s}{s^2+9}\right)+A\,L^{-1}\left(\frac{1}{s^2+9}\right)$$

$$x=\frac{1}{5}Cos2t-\frac{1}{5}\cos 3t+\cos 3t+\frac{A}{3}Sin3t$$

But $\quad x\left(\frac{\pi}{2}\right)=-1$

$$-1=\quad \frac{1}{5}(-1)-\frac{1}{5}(0)+0+\frac{A}{3}(-1)$$

Or $A=\frac{12}{5}$

Therefore required solution will be,

$$x=\frac{1}{5}[Cos2t-Cos3t]+Cos3t+\frac{12}{5}Sin3t$$

Dirichlet's Condition:—

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}[a_n\cos nx+b_n\sin nx]$$

:Provided (i) $f(x)$ is single valued, Periodic and finite.

(ii) $f(x)$ has finite number of max and min.

(iii) $f(x)$ has finite number of finite discontinuites.

When above conditions satisfy, then Fourier Series will converges to $f(x)$ at every pt of continuity.

$F(x)=\frac{1}{2}[f(x+0)+f(x-0)]$ (at pt of discontinuity)

Note: If $f(x)$ is an even function in $-\pi<x<\pi$

Then $\boxed{b_n=0}$ $\qquad a_0=\frac{2}{\pi}\int_0^{\pi}f(x)\,dx$

$$a_n=\frac{2}{\pi}\int_0^{\pi}f(x).\cos nx\,dx$$

Note: If $f(x)$ is an odd function in $-\pi<x<\pi$

$\boxed{a_0=0}$ $\qquad \boxed{a_n=0}$ $\qquad b_n=\frac{2}{\pi}\int_0^{\pi}f(x)\sin nx\,dx$

2

Q: obtain the Fourier series for function $f(x)=x^2$ $-\pi \le x \le \pi$. hence show that

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots+\frac{1}{n^2}+\cdots=\frac{\pi^2}{6}$$

Sol: $\therefore f(x)=f(x)$

$\therefore f(-x)=(-x)^2=x^2=f(x)$

$\therefore f(x)$ is an even function and hence $\boxed{b_n=0}$

Let $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos nx$ ——— (1)

Now $a_0=\frac{2}{\pi}\int_0^{\pi} f(x)\,dx=\frac{2}{\pi}\int_0^{\pi} x^2\,dx=\frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi}=\frac{2\pi^2}{3}$

$$\boxed{a_0=\frac{2\pi^2}{3}}$$

$a_n=\frac{2}{\pi}\int_0^{\pi} f(x).\cos nx\,dx=\frac{2}{\pi}\int_0^{\pi} x^2\cos nx\,dx$

$=\frac{2}{\pi}\left[\left(x^2\frac{\sin nx}{n}\right)_0^{\pi}-\left(\frac{2x\cos nx}{-n^2}\right)_0^{\pi}+\left(\frac{2\sin nx}{-n^3}\right)_0^{\pi}\right]$

$=\frac{2}{\pi}\left[0+\frac{2}{n^2}(\pi\cos n\pi-0)-0\right]$ $\left(\because \begin{array}{l}\sin n\pi=0\\ \sin 0=0\end{array}\right)$

$\boxed{a_n=\frac{4}{n^2}(-1)^n}$ $\left(\cos n\pi=(-1)^n\right)$

From eqn (1) Put these values in eqn (1)

$f(x)=\frac{\pi^2}{3}+\sum_{n=1}^{\infty}\frac{4(-1)^n}{n^2}\cos nx$

$=\frac{\pi^2}{3}-4\left[\frac{\cos x}{1^2}-\frac{\cos 2x}{2^2}+\frac{\cos 3x}{3^2}\cdots\right]$ ——— (2)

3

hence $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\cdots=\frac{\pi^2}{6}$

H.P

Q: Express $F(x)=\frac{(\pi-x)}{2}$ in a Fourier series in the interval $0<x<2\pi$

Sol^n: we have Fourier series

$F(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left[a_n\cos nx+b_n\sin nx\right]$ ——— (1)

$a_0=\frac{1}{\pi}\int_0^{2\pi} f(x)\,dx=\frac{1}{2\pi}\int_0^{2\pi}(\pi-x)\,dx=\frac{1}{2\pi}\left[(\pi x)_0^{2\pi}-\left(\frac{x^2}{2}\right)_0^{2\pi}\right]$

$a_0=\frac{1}{\pi}\left[\pi(2\pi-0)-\frac{\cancel{4}^2\pi^2}{\cancel{2}}\right]=\frac{1}{\pi}\left[2\pi^2-2\pi^2\right]$

$$\boxed{a_0=0}$$

$a_n=\frac{1}{\pi}\int_0^{2\pi} f(x).\cos nx\,dx=\frac{1}{2\pi}\int_0^{2\pi}(\pi-x)\cos nx\,dx$

$=\frac{1}{2\pi}\left[(\pi-x)\cdot\frac{\sin nx}{n}-(-1)\frac{\cos nx}{-n^2}\right]_0^{2\pi}$

$=\frac{1}{2\pi}\left[\frac{(\pi-x)}{n}\sin nx\right]_0^{2\pi}-\frac{1}{2\pi n^2}\left[\cos nx\right]_0^{2\pi}$

$\boxed{a_n=0}$ $\left(\because \sin 2\pi=0,\ \sin 0=0,\ \begin{array}{l}\cos 2\pi=1\\ \cos 0=1\end{array}\right)$

$b_n=\frac{1}{\pi}\int_0^{2\pi} f(x).\sin nx\,dx=\frac{1}{2\pi}\int_0^{2\pi}(\pi-x)\sin nx\,dx$

$=\frac{1}{2\pi}\left[\left((\pi-x)\frac{\cos nx}{-n}\right)_0^{2\pi}-\left(-1.\frac{\sin nx}{-n^2}\right)_0^{2\pi}\right]$

$$\boxed{b_n = \frac{1}{n}}$$

Put these values in equation (1)

$$f(x) = \sum_{n=1}^{\infty} \frac{\sin nx}{n}$$

Q: Find the half range cosine series for the function $f(x) = x(\pi - x) \quad 0 < x < \pi$

Sol^n: Fourier half Range cosine series

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \text{——— (1)}$$

we have

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} (x\pi - x^2)\,dx$$

$$a_0 = \frac{2}{\pi}\left[\left(\frac{\pi x^2}{2}\right)_0^{\pi} - \left(\frac{x^3}{3}\right)_0^{\pi}\right]$$

$$a_0 = \frac{2}{\pi}\left[\frac{\pi^3}{2} - \frac{\pi^3}{3}\right] = \frac{\pi^2}{3} \qquad \boxed{a_0 = \frac{\pi^2}{3}}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x).\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} (\pi x - x^2)\cos nx\,dx$$

$$a_n = \frac{2}{\pi}\left[(\pi x - x^2)\frac{\sin nx}{n}\right]_0^{\pi} - \frac{2}{\pi}\left[(\pi - 2x)\frac{\cos nx}{-n^2}\right]_0^{\pi}$$

$$+ \frac{2}{\pi}\left[-2.\frac{\sin nx}{-n^3}\right]_0^{\pi} \qquad \left(\begin{array}{l} \because \sin 0 = 0 \\ \sin \pi = 0 \\ \cos n\pi = (-1)^n \\ \cos 0 = 1 \end{array}\right)$$

$$a_n = -\frac{2}{\pi}\left[\frac{\pi \cos n\pi}{n^2} + \frac{\pi}{n^2}\right]$$

Put these values in eq^n (1)

$$F(x) = \frac{\pi^2}{6} - \sum_{n=1}^{\infty} \frac{2}{n^2}\left[(-1)^n + 1\right]\cos nx$$

$$F(x) = \frac{\pi^2}{6} - 4\left[\frac{\cos 2x}{2^2} + \frac{\cos 4x}{4^2} + \cdots\cdots\right] \text{ Ans.}$$

Q: Find Fourier Sine series for $f(t) = t - t^2$ in the interval $0 < t < 1$

Sol^n: Fourier Sine series is

$$F(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi t}{l} \quad \text{——— (1)}$$

$$\boxed{l = 1}$$

$$b_n = \frac{2}{l}\int_0^{l} f(t).\sin\frac{n\pi t}{l}\,dt$$

$$b_n = 2\int_0^{1} (t - t^2)\sin n\pi t\,dt$$

$$b_n = 2\left[(t - t^2)\frac{\cos n\pi t}{-n\pi} - (1 - 2t)\frac{\sin n\pi t}{-n^2\pi^2} + (-2)\frac{\cos n\pi t}{n^3\pi^3}\right]_0^1$$

$$b_n = 2\left[(t - t^2)\frac{\cos n\pi t}{-n\pi}\right]_0^1 + \frac{2}{n^2\pi^2}\left[(1 - 2t)\sin n\pi t\right]_0^1 - \frac{4}{n^3\pi^3}\left[\cos n\pi t\right]_0^1$$

$$b_n = -\frac{4}{n^3\pi^3}\left[\cos n\pi t\right]_0^1 \qquad \left(\begin{array}{l} \sin n\pi = 0 \\ \sin 0 = 0 \\ \cos n\pi = (-1)^n \\ \cos 0 = 1 \end{array}\right)$$

$$\boxed{b_n = \frac{-4\left[(-1)^n - 1\right]}{n^3\pi^3}}$$

Put these values in eq^n (1)

ex $\langle a_n \rangle = \langle (-1)^n . n \rangle$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (-1)^n . n = \infty \text{ or } -\infty$$

oscillate Infinitely

Series:

Limit u_n Test

For a series Σu_n to be convergent, it is necessary but not sufficient that

$$\lim_{n\to\infty} u_n = 0$$

ex Test the convergence of series

$$\sqrt{\frac{1}{2}} + \frac{2}{\sqrt{5}} + \frac{4}{\sqrt{17}} + \frac{8}{\sqrt{65}} + \cdots + \frac{2^n}{\sqrt{4^n+1}}$$

$$u_n = \frac{2^n}{\sqrt{4^n+1}}$$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{2^n}{\sqrt{4^n+1}} = \lim_{n\to\infty} \frac{\cancel{2^n}}{\cancel{2^n}\sqrt{1+\frac{1}{4^n}}} = 1 \neq 0$$

hence series is not convergent.

P- Series Test :- An infinite series

$$\Sigma \frac{1}{n^p} \text{ i.e } \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots$$

is convergent if $P > 1$ and divergent if $P \leq 1$

Comparision Test :- Let Σu_n be the series of positive terms and $\Sigma V_n = \frac{1}{n^p}$ be an auxillary series then if $\lim_{n\to\infty} \frac{u_n}{V_n} =$ ~~Posi~~ fixed, finite and non zero.

Then both series will converge or diverge together according to $\Sigma V_n = \frac{1}{\ldots}$

D' Alembert Test or Ratio Test :-

Statement: An infinite series Σu_n of positive terms is convergent if $\lim_{n\to\infty} \frac{u_n}{u_{n+1}} > 1$

divergent if $\lim_{n\to\infty} \frac{u_n}{u_{n+1}} < 1$

and Ratio Test fails if $\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = 1$

Raabe's Test: Let Σu_n be the infinite series of positive terms

Convergent if $\lim_{n\to\infty} n \cdot \left[\frac{u_n}{u_{n+1}} - 1\right] > 1$

divergent if $\lim_{n\to\infty} n \cdot \left[\frac{u_n}{u_{n+1}} - 1\right] < 1$

This Test fails if $\lim_{n\to\infty} n \cdot \left[\frac{u_n}{u_{n+1}} - 1\right] = 1$

ex Test the Series

$$\frac{1}{1\cdot2\cdot3} + \frac{x}{4\cdot5\cdot6} + \frac{x^2}{7\cdot8\cdot9} + \cdots$$

soln: Let $u_n = \frac{x^{n-1}}{(3n-2)(3n-1)3n}$

$$u_{n+1} = \frac{x^n}{(3n+1)(3n+2)(3n+3)}$$

$$\frac{u_n}{u_{n+1}} = \frac{(3n+1)(3n+2)(3n+3)}{(3n-2)(3n-1)3n} \cdot \frac{1}{x}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{(3+\frac{1}{n})(3+\frac{2}{n})(1+\frac{1}{n})}{(3-\frac{2}{n})(3-\frac{1}{n})} \cdot \frac{1}{x}$$

$$\lim \frac{u_n}{u_{n+1}} = \frac{1}{x}$$

hence by Ratio Test Σu_n is

Convergent if $\frac{1}{x} > 1 \Rightarrow x < 1$

Divergent if $\frac{1}{x} < 1 \Rightarrow x > 1$

Ratio Test fails if $x = 1$

when $x=1$

$$u_n = \frac{1}{(3n-2)(3n-1)(3n)}$$

Let $\Sigma V_n = \frac{1}{n^3}$ be an auxiliary series

$$\frac{u_n}{V_n} = \frac{n^3}{(3n-2)(3n-1)(3n)}$$

$$\lim_{n\to\infty} \frac{u_n}{V_n} = \lim_{n\to\infty} \frac{\cancel{n^3}}{27 \cancel{n^3} (1-\frac{2}{3n})(1-\frac{1}{3n})} = \frac{1}{27}$$

which is fixed, finite and non-zero.

hence comparision Test can be applied.

but $\Sigma V_n = \frac{1}{n^3}$ is convergent as $P = 3 > 1$ by P-Test

hence Σu_n is also convergent

finally $\boxed{\begin{array}{l}\text{divergent for } x>1 \\ \text{convergent for } x \le 1\end{array}}$

ex Test the series.

$$1 + \frac{3}{7}x + \frac{3\cdot 6}{7\cdot 10}x^2 + \frac{3\cdot 6\cdot 9}{7\cdot 10\cdot 13}x^3 + \cdots\cdots, x>0$$

$$u_n = \frac{3n}{(3n+4)} x^n$$

$$u_{n+1} = \frac{(3n+3)}{(3n+7)} x^{n+1}$$

$$\frac{u_n}{u_{n+1}} = \frac{3n(3n+7)}{(3n+4)(3n+3)} \cdot \frac{1}{x}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{\cancel{3n} \times \cancel{3n}(1+\frac{7}{3n})}{\cancel{3n} \times \cancel{3n}(1+\frac{4}{3n})(1+\frac{1}{n})} \times \frac{1}{x}$$

$$= \frac{1}{x}$$

By Ratio Test Σu_n is

Convergent if $\frac{1}{x} > 1 \Rightarrow x < 1$

Divergent if $\frac{1}{x} < 1 \Rightarrow x > 1$

Ratio Test fails if $x = 1$

if $x=1$

$$\frac{u_n}{u_{n+1}} = \frac{3n(3n+7)}{(3n+4)(3n+3)}$$

$$n\left[\frac{u_n}{u_{n+1}} - 1\right] = n\left[\frac{9n^2+21n}{9n^2+21n+12} - 1\right]$$

$$\lim_{n\to\infty}\left[\frac{u_n}{u_{n+1}} - 1\right] = \lim_{n\to\infty} \cancel{n}\left[\frac{\cancel{9n^2}+\cancel{21n}-\cancel{9n^2}-\cancel{21n}-12}{9n^{\cancel{2}}(1+\frac{21}{9n}+\frac{12}{9n^2})}\right]$$

$$= 0 < 1$$

By Raabe's Test Σu_n is Divergent

finally convergent if $x<1$

divergent $x \geq 1$

UNIT – 4 ①
COMPLEX VARIABLE – DIFFERENTIATION

Analytic function : A function $f(z)$ is said to be analytic at a point z_0 if it is one valued and differentiable not only at z_0 but at every point of some neighbourhood of z_0.

C-R equation in Cartesian coordinates :

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

where $f(z) = u(x,y) + iv(x,y)$

C-R equation in Polar coordinates :

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

where $f(z) = u + iv = f(re^{i\theta})$

Necessary and Sufficient conditions for $f(z)$ to be analytic : WORKING RULE :

(i) Necessary condition : CR equations must be satisfied

(ii) Sufficient condition : $f'(z)$ exists at every point of the region R.

Note 1 : $f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

or $f'(z) = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$

Question 1 : Examine the nature of the function

$$f(z) = \frac{x^3y(y-ix)}{x^6+y^2}, \quad z \neq 0$$
$$= 0, \quad z = 0$$

Prove that $\frac{f(z)-f(0)}{z} \to 0$ as $z \to 0$ along any radius vector but not as $z \to 0$ in any manner and also that $f(z)$ is not analytic at $z=0$.

Solution : $f(z) = u + iv = \frac{x^3y(y-ix)}{x^6+y^2}, \quad z \neq 0$

$$\Rightarrow u = \frac{x^3y^2}{x^6+y^2} \quad \& \quad v = -\frac{x^4y}{x^6+y^2}$$

$$\frac{\partial u}{\partial x} = \lim_{x \to 0} \frac{u(x,0) - u(0,0)}{x} = 0$$

$$\frac{\partial u}{\partial y} = \lim_{y \to 0} \frac{u(0,y) - u(0,0)}{y} = 0$$

$$\frac{\partial v}{\partial x} = \lim_{x \to 0} \frac{v(x,0) - v(0,0)}{x} = 0$$

$$\frac{\partial v}{\partial y} = \lim_{y \to 0} \frac{v(0,y) - v(0,0)}{y} = 0$$

∴ It is clear from above results

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

∴ CR equations are satisfied at origin.

Now $\frac{f(z)-f(0)}{z} = \left[\frac{x^3y(y-ix)}{x^6+y^2} - 0\right] \cdot \frac{1}{x+iy}$

$$= -\frac{ix^3y(x+iy)}{x^6+y^2} \cdot \frac{1}{(x+iy)}$$

$$= -i\frac{x^3y}{x^6+y^2}$$

Let $z \to 0$ along radius vector $y = mx$ then

$$\lim_{z \to 0} \frac{f(z)-f(0)}{z} = \lim_{x \to 0} -\frac{ix^3 \cdot mx}{x^6+m^2x^2} = \lim_{x \to 0} \frac{-imx^2}{x^4+m^2}$$

$$= 0 \quad \text{——— (1)}$$

Hence

Now

Hence $\frac{f(z)-f(0)}{z}$ as $z \to 0$ along any radius vector.

Now let $z \to 0$ along a curve $y = x^3$ then

$$\lim_{z \to 0} \frac{f(z)-f(0)}{z} = \lim_{x \to 0} \frac{-i x^3 \cdot x^3}{x^6 + x^6} = -\frac{i}{2} \quad \text{—(2)}$$

Hence $f'(0) = \lim_{z \to 0} \frac{f(z)-f(0)}{z}$ does not exist as the limits of (1) & (2) are different.

So $f(z)$ is not analytic at origin.

HARMONIC FUNCTION: A function of x, y which possesses continuous partial derivatives of the first and second orders and satisfies following Laplace equation, is called Harmonic function.

Let $u = u(x, y)$

Then $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ (Laplace equation)

Question 2: Show that the function $u = \frac{1}{2}\log(x^2+y^2)$ is harmonic. Find its harmonic conjugate.

(Note 2: u & v are called harmonic conjugate of each other.)

Solution: $u = \frac{1}{2}\log(x^2+y^2)$

$$\frac{\partial u}{\partial x} = \frac{1}{2} \frac{1}{x^2+y^2} \cdot 2x$$

$$= \frac{x}{x^2+y^2}$$

Similarly, $\frac{\partial u}{\partial y} = \frac{y}{x^2+y^2}$

(3)

Hence $\frac{f(z)-f(0)}{z}$ as $z \to 0$ along any radius vector.

Now let $z \to 0$ along a curve $y = x^3$ then

$$\lim_{z \to 0} \frac{f(z)-f(0)}{z} = \lim_{x \to 0} \frac{-i x^3 \cdot x^3}{x^6 + x^6} = -\frac{i}{2} \quad \text{—(2)}$$

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$$= \frac{x}{x^2+y^2}$$

Similarly, $\frac{\partial u}{\partial y} = \frac{y}{x^2+y^2}$

$$\frac{\partial^2 u}{\partial x^2} = \frac{(x^2+y^2)\cdot 1 - x(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2} \quad \text{--- (1)}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{(x^2+y^2)\cdot 1 - y(2y)}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2} \quad \text{--- (2)}$$

Adding eq (1) & (2) we have

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \cancel{\frac{y^2}{(x^2+y^2)^2}} - \cancel{\frac{x^2}{(x^2+y^2)^2}} + \cancel{\frac{x^2}{(x^2+y^2)^2}} - \cancel{\frac{y^2}{(x^2+y^2)^2}}$$

$$= 0$$

Hence u is a harmonic function.

Let v be harmonic conjugate of u.

$$\therefore \; dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$$

$$= -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy \quad \text{(by CR eqn)}$$

$$= -\frac{y}{x^2+y^2} dx + \frac{x}{x^2+y^2} dy$$

$$dv = \frac{x\,dy - y\,dx}{x^2+y^2}$$

$$= d\left(\tan^{-1}\frac{y}{x}\right)$$

Integrating both sides, we get

$$v = \tan^{-1}\frac{y}{x} + C, \text{ where } C \text{ is integral constt.}$$

MILNE'S THOMSON METHOD: This method is used to construct $f(z)$ in terms of z without first finding out v when u is given and vice versa.

Case 1: When u is given

(i) Find $\phi_1(x,y) = \frac{\partial u}{\partial x}$

(ii) Find $\phi_2(x,y) = \frac{\partial u}{\partial y}$

(iii) Find $\phi_1(z,0)$ and $\phi_2(z,0)$ by replacing x by z and y by 0.

(iv) Obtain $f(z)$ by following formula

$$f(z) = \int \{\phi_1(z,0) - i\,\phi_2(z,0)\}\, dz + C \text{ (constt.)}$$

Case 2: When v is given

(i) Find $\psi_1(x,y) = \frac{\partial v}{\partial y}$

(ii) Find $\psi_2(x,y) = \frac{\partial v}{\partial x}$

(iii) Find $\psi_1(z,0)$ and $\psi_2(z,0)$ by replacing x by z and y by 0.

(iv) Obtain $f(z)$ by following formula

$$f(z) = \int \{\psi_1(z,0) + i\,\psi_2(z,0)\}\, dz + C \text{ (constt.)}$$

Case 3: when $u-v$ is given

(i) Consider $U = u - v$

(ii) Find $\phi_1(x,y) = \frac{\partial U}{\partial x}$

(iii) Find $\phi_2(x,y) = \frac{\partial U}{\partial y}$

(iv) Find $\phi_1(z,0)$ and $\phi_2(z,0)$

(v) Obtain $f(z)$ by following formula

$$f(z) = \frac{1}{1+i}\int \{\phi_1(z,0) - i\,\phi_2(z,0)\}\,dz + \frac{C}{1+i}$$

where C is integral constt.

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Case 4 : When $u+v$ is given

(i) consider V as $u+v$

(ii) Find $\psi_1(x,y) = \frac{\partial V}{\partial y}$

(iii) Find $\psi_2(x,y) = \frac{\partial V}{\partial x}$

(iv) Find $\psi_1(z,0)$ & $\psi_2(z,0)$

(v) Obtain $f(z)$ by following formula

$$f(z) = \frac{1}{1+i}\int \{\psi_1(z,0) + i\,\psi_2(z,0)\}\,dz + \frac{c}{1+i}$$

Where c is integral constt.

Question 3 : Determine an analytic function $f(z)$ in terms of z whose real part is $e^{-x}(x\sin y - y\cos y)$

Solution : We have $u = e^{-x}(x\sin y - y\cos y)$.

$$\phi_1(x,y) = \frac{\partial u}{\partial x} = e^{-x}\sin y - e^{-x}(x\sin y - y\cos y)$$

$$\phi_2(x,y) = \frac{\partial u}{\partial y} = e^{-x}(x\cos y - \cos y + y\sin y)$$

$\therefore\ \phi_1(z,0) = 0$ & $\phi_2(z,0) = e^{-z}(z-1)$

By Milne's Thomson method

$$f(z) = \int\{\phi_1(z,0) - i\,\phi_2(z,0)\}\,dz + c$$

$$= \int\{0 - ie^{-z}(z-1)\}\,dz + c$$

$$= -i\int e^{-z}(z-1)\,dz + c$$

$$= -i\left[(z-1)(-e^{-z}) - e^{-z}\right] + c$$

$$= -i\left[-ze^{-z} + \cancel{e^{-z}} - \cancel{e^{-z}}\right] + c$$

$$= ize^{-z} + c \quad (\text{where } c = \text{constt.})$$

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Question 4 : If $f(z)$ is a regular function of z Prove that $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$

Solution : Let $f(z) = u + iv$

so that $|f(z)| = \sqrt{u^2+v^2}$

$\Rightarrow |f(z)|^2 = u^2+v^2 = \phi(x,y)$ (say)

$$\therefore\ \frac{\partial\phi}{\partial x} = 2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x}$$

$$\frac{\partial^2\phi}{\partial x^2} = 2\left[u\frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial u}{\partial x}\right)^2 + v\frac{\partial^2 v}{\partial x^2} + \left(\frac{\partial v}{\partial x}\right)^2\right] \quad \text{— (1)}$$

Similarly, $$\frac{\partial^2\phi}{\partial y^2} = 2\left[u\frac{\partial^2 u}{\partial y^2} + \left(\frac{\partial u}{\partial y}\right)^2 + v\frac{\partial^2 v}{\partial y^2} + \left(\frac{\partial v}{\partial y}\right)^2\right] \quad \text{— (2)}$$

Adding eq (1) & (2)

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 2\left[u\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + v\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right] \quad \text{— (3)}$$

$\because f(z) = u + iv$ is a regular function of z, u, v satisfy C-R eqn and laplace equation

$$\left.\begin{aligned}\therefore\ \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}\\ \text{and}\ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0\end{aligned}\right\} \text{— (4)}$$

Using eq (3) & (4) we get

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 2\left[0 + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + 0 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial x}\right)^2\right]$$

$$= 4\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right] \quad \text{— (5)}$$

(7)

Question 4 : If $f(z)$ is a regular function of z

Prove that $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$

Solution : let $f(z) = u+iv$

so that $|f(z)| = \sqrt{u^2+v^2}$

$\Rightarrow |f(z)|^2 = u^2+v^2 = \phi(x,y)$ (say)

$$\therefore \frac{\partial \phi}{\partial x} = 2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x}$$

$$\frac{\partial^2\phi}{\partial x^2} = 2\left[u\frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial u}{\partial x}\right)^2 + v\frac{\partial^2 v}{\partial x^2} + \left(\frac{\partial v}{\partial x}\right)^2\right] \quad \text{— (1)}$$

Similarly, $$\frac{\partial^2\phi}{\partial y^2} = 2\left[u\frac{\partial^2 u}{\partial y^2} + \left(\frac{\partial u}{\partial y}\right)^2 + v\frac{\partial^2 v}{\partial y^2} + \left(\frac{\partial v}{\partial y}\right)^2\right] \quad \text{— (2)}$$

Adding eq (1) & (2)

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 2\left[u\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + v\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right) + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right] \quad \text{— (3)}$$

$\because$ $f(z) = u+iv$ is a regular function of z, u, v satisfy C-R eqn and laplace equation

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

and $$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = 0, \quad \frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2} = 0 \quad \text{— (4)}$$

Using eq (3) & (4) we get

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 2\left[0 + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + 0 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial x}\right)^2\right]$$

$$= 4\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right] \quad \text{— (5)}$$

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$\because f(z) = u+iv$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$\therefore |f'(z)|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 \quad \text{— (6)}$$

$\therefore$ Using (5) and (6) we get

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\phi = 4|f'(z)|^2$$

or $$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$$

CONFORMAL MAPPING : A transformation which preserves angles both in magnitude and sense between every pair of curves through a point is said to be ~~conform~~ conformal at the point.

STANDARD TRANSFORMATIONS :

(i) Translation : $w = z + c$, where c is complex constt.

eg. $w = z + (1+i)$

(ii) Rotation : $w = ze^{i\theta_0}$ ($\theta_0 > 0 \Rightarrow$ anticlockwise rotation & $\theta_0 < 0 \Rightarrow$ clockwise rotation)

eg. $w = ze^{i\pi/4}$

(iii) Magnification : $w = az$ (where a is real)

eg. $w = 2z$

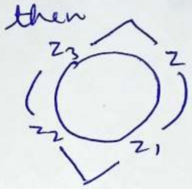
(iv) Inversion : $w = \frac{1}{z}$

BILINEAR TRANSFORMATION : $w = \frac{az+b}{cz+d}$

where a, b, c, d = constt. & $ad - bc \neq 0$

Also called as Mobius or Fractional transformation

Note: If BLT transforms z_1, z_2, z_3 in z-plane to w_1, w_2, w_3 in w-plane then



$$\frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)} = \frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)}$$

As BLT preserves cross ratio of four points.

Question 5: Find Mobius transformation which maps $z = 1, i, -1$ into $w = i, 0, -i$ in w plane. Hence find image of $|z|<1$

Solution:

	z	w	
z_1	1	i	w_1
z_2	i	0	w_2
z_3	-1	$-i$	w_3

$$\because \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)} = \frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)}$$

$$\Rightarrow \frac{(z-1)(i+1)}{(1-i)(-1-z)} = \frac{(w-i)(0+i)}{(i-0)(-i-w)}$$

$$\Rightarrow \cancel{-}\frac{(z-1)(i+1)}{(1-i)(1+z)} = \cancel{-}\frac{(w-i)\cancel{i}}{\cancel{i}(i+w)}$$

$$\Rightarrow \frac{w-i}{w+i} = \frac{iz+z-i-1}{1+z-i-iz}$$

Using componendo Dividendo

$$\frac{w-\cancel{i}+w+\cancel{i}}{\cancel{w}-i-\cancel{w}-i} = \frac{\cancel{iz}+z-i-\cancel{1}+\cancel{1}+z-i-\cancel{iz}}{iz+\cancel{z}-\cancel{i}-1-1-\cancel{z}+\cancel{i}+iz}$$

$$\frac{2w}{-2i} = \frac{2(z-i)}{2(iz-1)}$$

$$w = \frac{-i(z-i)}{iz-1} = \frac{-i(z-i)}{iz+i^2}$$

$$= \frac{\cancel{i}(i-z)}{\cancel{i}(z+i)}$$

$w = \frac{i-z}{i+z}$ which is reqd BLT

which can be rewritten as

$$z = i\left(\frac{1-w}{1+w}\right)$$

$\therefore$ $|z|<1$ is mapped into the region

$$\left|i\left(\frac{1-w}{1+w}\right)\right| < 1$$

$$|i| \frac{|(1-w)|}{|1+w|} < 1$$

$$\Rightarrow |i|^2 |1-w|^2 < |1+w|^2$$

$$(i \cdot \bar{i}) |1-u-iv|^2 < |1+u+iv|^2$$

$$1 \cdot [(1-u)^2+v^2] < (1+u)^2+v^2$$

$$1+u^2-2u+v^2 < 1+u^2+2u+v^2$$

$$\Rightarrow -4u < 0$$

$$\Rightarrow 4u > 0$$

or $u > 0$

$\therefore$ image in w-plane for $|z|<1$ is entire half of w-plane to the right of img. axis.

INVARIANT OR FIXED POINTS: The points which coincide with their transformation are called fixed points i.e. $w = f(z) = z$.

Question 6: Find the fixed points of $w = \frac{3+2z}{2+z}$

Solution: For fixed points $w = z$ i.e. $\frac{3+2z}{2+z} = z$

$$\Rightarrow 3+2z = 2z+z^2$$

$$\Rightarrow z^2 = 3$$

$$\Rightarrow z = \pm\sqrt{3}$$

Hence fixed points are $z = -\sqrt{3}, \sqrt{3}$.

Unit-5 (complex Integration) (1)

Cauchy integral theorem:

If a function $f(z)$ be analytic and $f'(z)$ continuous at all points inside and on a simple closed curve C, then $\boxed{\oint_C f(z)\,dz = 0.}$

Note: 1. $\sin z$, $\cos z$, e^z, $P_n(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$ all are analytic function in finite complex plane i.e, for all finite values of z.

2. Any function $f(z) = \frac{f_1(z)}{f_2(z)}$, where $f_1(z)$ & $f_2(z)$ both are analytic is again analytic for where $f_2(z)$ vanishes. Values of z for which $f_2(z) = 0$ are called singular point of $f(z)$.

Q. Find value of integral $\int_C \frac{3z^2+7z+1}{z+1}\,dz$, where c is the circle $|z| = \frac{1}{2}$.

Solⁿ: Being ratio of two polynomial functions $f(z) = \frac{3z^2+7z+1}{z+1}$ will be analytic except for those value of z for which is a singular point of $f(z)$.

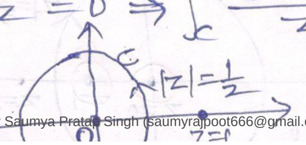
$z+1 = 0 \Rightarrow z = -1$ is a singular point of $f(z)$.

Also $z = -1$ will lie outside the circle $|z| = \frac{1}{2}$

$\therefore f(z)$ is analytic inside and on C

So by cauchy integral theorem

$$\oint_C f(z)\,dz = 0 \Rightarrow \int_C \frac{3z^2+7z+1}{z+1}\,dz = 0$$



(2)

Cauchy integral formula:

If $f(z)$ is analytic inside and on simple closed curve C and 'a' be any point within C, then

$$\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}\,dz = f(a)$$

or $$\boxed{\oint_C \frac{f(z)}{z-a}\,dz = 2\pi i \times f(a)}$$

Cauchy integral formula for n^{th} derivative:

If a function $f(z)$ is analytic in a region R then its derivative at any point $z = a$ of R of n^{th} order is also analytic and its value is given by

$$f^{(n)}(a) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{n+1}}\,dz$$

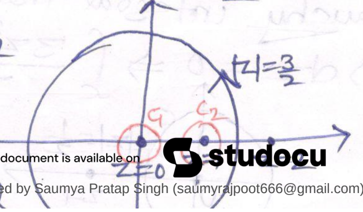
or $$\boxed{\oint_C \frac{f(z)}{(z-a)^{n+1}}\,dz = \frac{2\pi i}{n!} f^{n}(a)}$$

Q. Evaluate $\int_C \frac{4-3z}{z(z-1)(z-2)}\,dz$ where $C: |z| = \frac{3}{2}$.

Solⁿ: The singular point of $\frac{4-3z}{z(z-1)(z-2)}$ are given by $z(z-1)(z-2) = 0 \Rightarrow z = 0, 1, 2$

Also given circle $|z| = \frac{3}{2}$ is a below:

$\therefore$ only two singular points $z=0$, $z=1$ lies inside C.



So we draw two non overlapping small

(3)

$$\int_C \frac{4-3z}{z(z-1)(z-2)}\,dz = \int_{C_1} \frac{\frac{4-3z}{(z-1)(z-2)}}{(z-0)}\,dz + \int_{C_2} \frac{\frac{4-3z}{z(z-2)}}{z-1}\,dz$$

clearly $\frac{4-3z}{(z-1)(z-2)}$ will be analytic over C_1 & $\frac{4-3z}{z(z-2)}$ will be analytic over C_2.

∴ By cauchy integral formula

$$\int_C \frac{f(z)}{z-a}\,dz = 2\pi i \times f(a)$$

will imply that

$$\int_C \frac{4-3z}{z(z-1)(z-2)}\,dz = 2\pi i \times \left[\frac{4-3z}{(z-1)(z-2)}\right]_{z=0} + 2\pi i \left[\frac{4-3z}{z(z-2)}\right]_{z=1}$$

$$= 2\pi i \times \left[\frac{4-0}{(0-1)(0-2)}\right] + 2\pi i \left[\frac{4-3}{1(1-2)}\right]$$

$$= 2\pi i \times \left(\frac{4}{2}\right) + 2\pi i\left(\frac{1}{-1}\right)$$

$$= 4\pi i - 2\pi i$$

$$= 2\pi i$$

Hence $\boxed{\int_C \frac{4-3z}{z(z-1)(z-2)}\,dz = 2\pi i}$

Q. Evaluate $\int_C \frac{z^2-2z}{(z+1)^2(z^2+4)}\,dz$, where c is the circle $|z| = 10$.

Solⁿ: The poles or singular points of $\frac{z^2-2z}{(z+1)^2(z^2+4)}$ are given by $(z+1)^2(z^2+4) = 0$

$\Rightarrow (z+1)^2 = 0 \Rightarrow z = -1, -1$ a pole of order 2

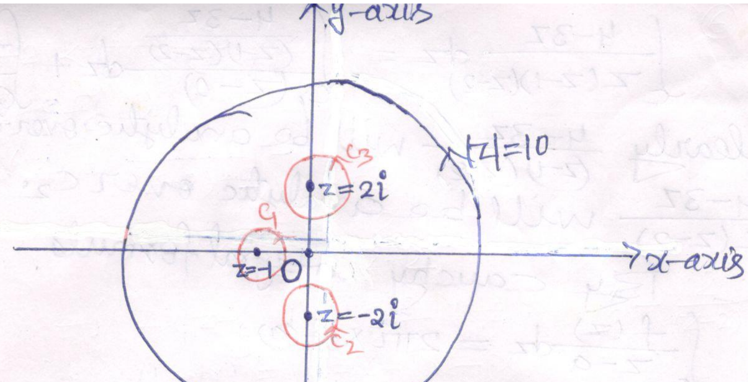
& $z^2+4 = 0 \Rightarrow z^2 = -4 \Rightarrow z = \pm 2i$

i,e, $z = -2i$ & $z = 2i$ are simple pole

∴ All the poles will lie inside $c: |z| = 10$

So we draw three circle C_1, C_2, C_3 with centre at

(4)



$$\int_C \frac{z^2-2z}{(z+1)^2(z^2+4)}\,dz = \int_C \frac{z^2-2z}{(z+1)^2(z+2i)(z-2i)}\,dz$$

$$= \int_{C_1} \frac{\frac{z^2-2z}{(z+2i)(z-2i)}}{\{z-(-1)\}^2}\,dz + \int_{C_2} \frac{\frac{z^2-2z}{(z+1)^2(z-2i)}}{(z+2i)}\,dz$$

$$+ \int_{C_3} \frac{\frac{z^2-2z}{(z+1)^2(z+2i)}}{(z-2i)}\,dz$$

$$= 2\pi i \times \left[\frac{z^2-2z}{(z+2i)(z-2i)}\right]_{z=-1} + 2\pi i \left[\frac{z^2-2z}{(z+1)^2(z-2i)}\right]_{z=-2i}$$

$$+ 2\pi i \left[\frac{z^2-2z}{(z+1)^2(z+2i)}\right]_{z=2i}$$

$$= 2\pi i \left[\frac{(-1)^2-2(-1)}{(-1+2i)(-1-2i)}\right] + 2\pi i \left[\frac{(-2i)^2-2(-2i)}{(-2i+1)^2(-4i)}\right]$$

$$+ 2\pi i \left[\frac{(2i)^2-2(2i)}{(2i+1)^2[4i]}\right]$$

$$= 0$$

Hence $\boxed{\int_C \frac{z^2-2z}{(z+1)^2(z^2+4)}\,dz = 0}$

Laurent theorem:- If $f(z)$ is analytic on C_1 & C_2, and the annular region R bounded by the two concentric circles C_1 & C_2 of radii r_1 and r_2 $(r_2<r_1)$ and with centre at $z=a$, then for all z in R

$$f(z)=\sum_{n=0}^{\infty} a_n(z-a)^n+\sum_{n=1}^{\infty} b_n(z-a)^{-n} \quad \text{---(1)}$$

where $a_n=\frac{1}{2\pi i}\int_{C_1}\frac{f(z)}{(z-a)^{n+1}}dz$ & $b_n=\frac{1}{2\pi i}\int_{C_2}\frac{f(z)}{(z-a)^{-n+1}}dz$

The series represented by eqn (1) is called Laurent series expansion of $f(z)$ about $z=a$.

Q. Expand $f(z)=\frac{z}{(z-1)(2-z)}$ in Laurent series valid for
(i) $|z-1|>1$ and (ii) $0<|z-2|<1$

Solⁿ:- Let $f(z)=\frac{z}{(z-1)(2-z)}$ we after making partial fractions of $f(z)$ we get

$$f(z)=\frac{1}{z-1}+\frac{2}{2-z} \quad \text{---(1)}$$

(i) for $|z-1|>1$

$$f(z)=\frac{1}{z-1}-\frac{2}{(z-1)-1}$$

$$=\frac{1}{z-1}-\frac{2}{z-1}\cdot\left\{1-\frac{1}{z-1}\right\}^{-1}$$

$$=\frac{1}{z-1}-\frac{2}{z-1}\left\{1+\left(\frac{1}{z-1}\right)+\left(\frac{1}{z-1}\right)^2+\left(\frac{1}{z-1}\right)^3+\cdots\right\}$$

$$\left\{\because (1-x)^{-1}=1+x+x^2+x^3+\cdots\right\}$$

$$f(z)=\frac{1}{z-1}-\frac{2}{z-1}\sum_{n=0}^{\infty}\left(\frac{1}{z-1}\right)^n$$

Which is Laurent series for $|z-1|>1$.

Note:- $f(z)=\sum_{n=0}^{\infty} a_n(z-a)^n$ represents Taylor's series

(ii) $0<|z-2|<1$,
from eqn (1)

$$f(z)=\frac{1}{(z-2)+1}-\frac{2}{z-2}$$

$$=\{1+(z-2)\}^{-1}-\frac{2}{(z-2)}$$

$$=\{1-(z-2)+(z-2)^2-(z-2)^3+\cdots\}-\frac{2}{(z-2)}$$

$$\{\because (1+x)^{-1}=1-x+x^2-x^3+\cdots\}$$

$$f(z)=\sum_{n=0}^{\infty}(-1)^n(z-2)^n-\frac{2}{(z-2)}$$

Which is required Laurent series.

Q Find Taylor's and Laurent series which represent the function $\frac{z^2-1}{(z+2)(z+3)}$ when $|z|<2$, (ii) $2<|z|<3$.

Solⁿ:- Let $f(z)=\frac{z^2-1}{(z+2)(z+3)}$

∵ Degree of Neominator & Denominator are equal so we first convert it into proper form by dividing as

$$f(z)=1-\frac{5z+7}{(z+2)(z+3)}$$

Now on making partial fractions we get

$$f(z)=1+\frac{3}{z+2}-\frac{8}{z+3} \quad \text{---(1)}$$

(i) For $|z|<2$,

$$f(z)=1+\frac{3}{2}\left\{1+\frac{z}{2}\right\}^{-1}-\frac{8}{3}\cdot\left\{1+\frac{z}{3}\right\}^{-1}$$

$$=1+\frac{3}{2}\left\{1-\left(\frac{z}{2}\right)+\left(\frac{z}{2}\right)^2-\left(\frac{z}{2}\right)^3+\cdots\right\}$$

$$-\frac{8}{3}\left\{1-\left(\frac{z}{3}\right)+\left(\frac{z}{3}\right)^2-\left(\frac{z}{3}\right)^3+\cdots\right\}$$

$$f(z) = 1 + \frac{3}{2}\sum(-1)^n\left(\frac{z}{2}\right)^n - \frac{8}{3}\sum(-1)^n\left(\frac{z}{3}\right)^n$$

which represent Taylor series for $f(z)$ in $|z|<2$.

(II) $2<|z|<3$,

$$f(z) = 1 + \frac{3}{z}\left\{1+\frac{2}{z}\right\}^{-1} - \frac{8}{3}\left\{1+\frac{z}{3}\right\}^{-1}$$

$$= 1 + \frac{3}{z}\left\{1-\left(\frac{2}{z}\right)+\left(\frac{2}{z}\right)^2-\left(\frac{2}{z}\right)^3+\cdots\right\}$$

$$-\frac{8}{3}\left\{1-\left(\frac{z}{3}\right)+\left(\frac{z}{3}\right)^2-\left(\frac{z}{3}\right)^3+\cdots\right\}$$

$$= 1 + \frac{3}{z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{2}{z}\right)^n - \frac{8}{3}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{3}\right)^n$$

which represents Laurent series of $f(z)$ in $2<|z|<3$.

Residue:

Residue of a function $f(z)$ at a pole $z=a$ is defined as the coefficient of $\frac{1}{z-a}$ in Laurent series expansion of $f(z)$ about $z=a$ given by

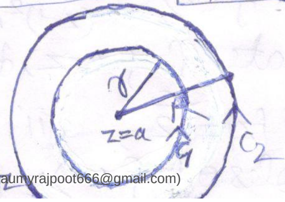
$$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n + \sum_{n=1}^{\infty} b_n(z-a)^{-n}$$

$$= \sum_{n=0}^{\infty} a_n(z-a)^n + \frac{b_1}{z-a} + \frac{b_2}{(z-a)^2} + \cdots$$

$\therefore$ Residue at $(z=a)$ = coeff. of $\frac{1}{z-a}$

$$= b_1$$

Also $\boxed{\text{Res (at } z=a) = b_1 = \frac{1}{2\pi i}\int_{C_1} f(z)\,dz}$



Note:

Res(at $z=\infty$) $= -b_1$

$$= -\frac{1}{2\pi i}\oint f(z)\,dz$$

Methods of finding residues:

(i) When $z=a$ is a simple pole

$$\text{Res}(z=a) = \lim_{z\to a}\{(z-a)\cdot f(z)\}$$

(ii) When $z=a$ is a pole of order 'k'

$$\text{Res(at } z=a) = \frac{1}{(k-1)!}\lim_{z\to a}\left[\frac{d^{k-1}}{dz^{k-1}}\left\{(z-a)^k\cdot f(z)\right\}\right]$$

(iii) If $f(z) = \frac{\phi(z)}{\psi(z)}$ s.t $\phi(a)\neq 0$ & $\psi(a)=0$

then

$$\text{Res (at } z=a) = \frac{\phi(a)}{\psi'(a)}$$

Cauchy Residue theorem:

If $f(z)$ is analytic in a closed curve C except at a finite number of poles within C, then

$$\oint_C f(z)\,dz = 2\pi i\,[\text{Sum of residues at the poles within } C]$$

Q. Find poles, residue at each pole of the function $f(z) = \frac{12z-7}{(z-1)^2(2z+3)}$ and hence evaluate $\int_C f(z)\,dz$ where C is the circle (i) $|z|=2$, (ii) $|z+i|=\sqrt{3}$.

Sol^n: the poles of $f(z) = \frac{12z-7}{(z-1)^2(2z+3)}$ are given by clearly $z=-1$ will be a pole of order 2 while $z=-\frac{3}{2}$ a simple pole.

$$R_1 = \text{Res}\,(\text{at } z=-1) = \frac{1}{(2-1)!}\left[\frac{d}{dz}\left\{(z-1)^2\cdot\frac{12z-7}{(z-1)^2(2z+3)}\right\}\right]_{z=1}$$

$$= \frac{d}{dz}\left[\frac{12z-7}{2z+3}\right] = \left[\frac{(2z+3)12-(12z-7)\cdot 2}{(2z+3)^2}\right]_{z=1}$$

$$= \frac{5\times 12-(12-7)\times 2}{(5)^2}$$

$$= 2$$

$$R_2 = \text{Res}(\text{at } z=-\tfrac{3}{2}) = \lim_{z\to -\frac{3}{2}} \left(z+\tfrac{3}{2}\right)\cdot\frac{12z-7}{(z-1)^2(2z+3)}$$

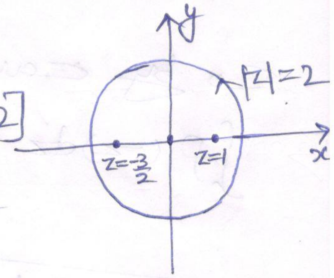
$$R_2 = \lim_{z\to(-\frac{3}{2})} \frac{1}{2}\cdot\frac{12z-7}{(z-1)^2} = -2$$

(i) For $|z|=2$ both poles $z=1$ and $z=-\frac{3}{2}$ will lie inside C

$$\therefore \int_C f(z)\,dz = 2\pi i\,[R_1+R_2]$$

$$\int_C \frac{12z-7}{(z-1)^2(2z+3)}\,dz = 2\pi i\,[2-2]$$

$$= 0. \quad \text{Ans}$$

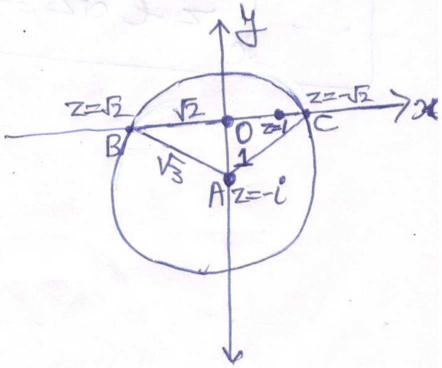


(ii) For $|z+i|=\sqrt{3} \Rightarrow |z-(-i)|=\sqrt{3}$

$$\int_C f(z)\,dz = 2\pi i\,R_1$$

$$\int_C \frac{12z-7}{(z-1)^2(2z+3)}\,dz = 2\pi i\times 2$$

$$= 4\pi i \quad \text{Ans}$$



Q. Apply Cauchy Residue theorem to evaluate $\oint_C z\,e^{\frac{1}{z}}\,dz$ around the unit circle.

Solⁿ: Let $f(z) = z\,e^{\frac{1}{z}}$

$$= z\left[1+\frac{1}{1!}\left(\frac{1}{z}\right)+\frac{1}{2!}\left(\frac{1}{z}\right)^2+\frac{1}{3!}\left(\frac{1}{z}\right)^3+\cdots\right]$$

$$\left\{\because e^x = 1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right.$$

$$= z+1+\frac{1}{2}\cdot\left(\frac{1}{z}\right)+\frac{1}{6}\left(\frac{1}{z^2}\right)+\cdots$$

Here $z=0$ is the singular point of $f(z)$

$\therefore$ Res(at $z=0$) = coeff of $\frac{1}{z}$ in Laurent series of $f(z)$ about $z=0$

$$= \frac{1}{2}$$

So by Cauchy Residue theorem

$$\int_C f(z)\,dz = 2\pi i\times \text{Res}\,(\text{at } z=0)$$

$$= 2\pi i\times\frac{1}{2}$$

$$\boxed{\therefore \int_C z\,e^{\frac{1}{z}}\,dz = \pi i}$$