classmate
Date
Page

UNIT - 1

PARTIAL DIFFERENTIAL EQNS.

$y = f(x)$

$y = \sin x + x^2 + 3$

$\left(\frac{dy}{dx}\right)' = \cos x + 2x + 0$

$\left(\frac{d^2y}{dx^2}\right)'' = -\sin x + 2$

$\left(\frac{d^3y}{dx^3}\right) = -\cos x + 0$

$\frac{d}{dx} \rightarrow$ differential operater

$\frac{dy}{dx} \rightarrow$ differential coeffecient

y → dependent variable

x → independent variable

$\frac{d^3y}{dx^2} = -\cos x$

$\frac{d^2y}{dx^2} = -\sin x + c_1$

$\frac{dy}{dx} = \cos x + c_1 x + c_2$

$y = \sin x + \frac{c_1 x^2}{2} + c_2 x$

c_1, c_2, c_3 are arbitary constants

let $z = f(x,y)$

$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = f(x,y)$

$z = \sin x + \cos y + x + 3$

$p = \frac{\partial z}{\partial x} = \cos x + 1$

$q = \frac{\partial z}{\partial y} = -\sin y$

$r = \frac{\partial^2 z}{\partial x^2} = -\sin x$

$s = \frac{\partial^2 z}{\partial x \partial y} = 0$

$t = \frac{\partial^2 z}{\partial y^2} = -\cos y$

linear Partial differential equation

$$\frac{\partial^2 z}{\partial x^2} = \cos x$$

$$\frac{\partial z}{\partial x} = \sin x + f_1(y)$$

$$\boxed{z = -\cos x + f_1(y)x + f_2(y)}$$

where f_1, f_2 are arbitary function

Q. Form partial differential equations for

i) $z = ax + by + ab$

ii) $z = ax + a^2y^2 + b$

iii) $z = (x+a)(y+b)$

iv) $az + b = a^2x + b$. where a, b are arbitary constants.

Sol^n- (i) $z = ax + by + ab$ — (1)

$$\frac{\partial z}{\partial x} = p = a \quad \text{— (2)}$$

$$\frac{\partial z}{\partial y} = q = b \quad \text{— (3)}$$

from (1)

$$z = px + qy + pq$$

(ii) $z = ax + a^2y^2 + b$ — (1)

$$p = \frac{\partial z}{\partial x} = a \quad \text{— (2)}$$

$$q = \frac{\partial z}{\partial y} = 2a^2y = 2p^2y \quad \text{— (3)}$$

~~from (1)~~

~~$z = px + p^2y + b$~~

$$\boxed{q = 2p^2y}$$

(iii) $z = (x+a)(y+b)$

$$p = \frac{\partial z}{\partial x} = (y+b)$$

$$q = \frac{\partial z}{\partial y} = (x+a)$$

$$\boxed{z = pq}$$

(iv) $az + b = a^2 x + y$

$$az = a^2 x + y - b$$

$$p = a \cdot \frac{\partial z}{\partial x} = a^2 = a = \frac{\partial z}{\partial x}$$

$$q = a \cdot \frac{\partial z}{\partial y} = 1 = 1/a = \frac{\partial z}{\partial y} = \frac{1}{p}$$

$$\cancel{\boxed{\sqrt{q}\, z = px + y - b}}$$

$\therefore$

$$\boxed{qp = 1}$$

Q. Form p.d.e for the following equations -

i) $z = f(x^2 - y^2)$

ii) $z = \phi(x) \cdot \psi(y)$

iii) $z = x + y + f(xy)$

iv) $z = f(x+iy) + g(x-iy)$

where ϕ, ψ, f, g are arbitary functions.

(i) $z = f(x^2 - y^2)$ — ①

$$p = \frac{\partial z}{\partial x} = f'(x^2-y^2)(2x) \quad — ②$$

$$q = \frac{\partial z}{\partial y} = f'(x^2-y^2)(-2y) \quad — ③$$

divide ② and ③

$$\frac{p}{q} = -\frac{x}{y}$$

or $\boxed{py + xq = 0}$

(ii) $z = \phi(x) \cdot \psi(y)$

$$p = \frac{\partial z}{\partial x} = \phi'(x) \cdot \psi(y)$$

$$q = \frac{\partial z}{\partial y} = \phi(x) \cdot \psi'(y)$$

$\frac{2}{5}, \frac{2}{4}$

$$r = \frac{\partial^2 z}{\partial x^2} = \phi''(x) \cdot \psi(y)$$

$$s = \frac{\partial^2 z}{\partial x \partial y} = \phi'(x) \cdot \psi'(y)$$

$$t = \frac{\partial^2 z}{\partial y^2} = \phi(x) \cdot \psi''(y)$$

$$pq = \underline{\phi'(x)\psi'(y)}\ \underline{\phi(x)\psi(y)}$$

$\boxed{pq = sz}$

(iii) $$z = x + y + f(xy)$$

$$p = \frac{\partial z}{\partial x} = 1 + f'(xy) \cdot y \quad — \text{①}$$

$$q = \frac{\partial z}{\partial y} = 1 + f'(xy) \cdot x \quad — \text{②}$$

divide ① & ②

$$\boxed{\frac{p-1}{q-1} = \frac{y}{x}}$$

(iv) $$z = f(x+iy) + g(x-iy)$$

$$p = \frac{\partial z}{\partial x} = f'(x+iy) + g'(x-iy)$$

$$q = \frac{\partial z}{\partial y} = f'(x+iy)i + g'(x-iy)(-i)$$

$$r = \frac{\partial^2 z}{\partial x^2} = f''(x+iy) + g''(x-iy) \quad — \text{①}$$

$$t = \frac{\partial^2 z}{\partial y^2} = f''(x+iy)i^2 + g''(x-iy)i^2$$

$$= -f''(x+iy) - g''(x-iy) \quad — \text{②}$$

$$s = \frac{\partial^2 z}{\partial x \partial y} = f''(x+iy)i + g''(x-iy)(-i)$$

from ① & ②

$$\boxed{t + r = 0}$$

LAGRANGE'S METHOD

To solve first order linear partial differential equation –

Lagrange's equation –
$$Pp + Qq = R,$$
where P, Q, R are the functions of (x, y, z)
and $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$

Lagrange's auxilary equation

$$\boxed{\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}}$$

Q. Solve P.d.e.

(i) $yzp - xzq = xy$

(ii) $y^2p - xyq = x(z - 2y)$

(iii) $x^2p + y^2q = (x+y)z$

Solⁿ- (i) Lagrange's A.E –

$$\frac{dx}{yz} = \frac{dy}{-xz} = \frac{dz}{xy}$$

Taking $\frac{dx}{dz} = \frac{dy}{-xz}$

$$x\,dx = -y\,dy$$

$$\frac{x^2}{2} = -\frac{y^2}{2} + C_1$$

or $\boxed{x^2 + y^2 = 2C_1 = k_1}$

Taking $\frac{dx}{yz} = \frac{dz}{xy}$

$$x \cdot dx = z \cdot dz$$

$$\frac{x^2}{2} = \frac{z^2}{2} + C_2$$

$$\boxed{x^2 - z^2 = 2C_2 = k_2}$$

$\therefore \quad \phi(x^2+y^2,\ x^2-z^2) = 0$

(ii) Lagrange's A.E.

$$\frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{x(z-2y)}$$

Taking $\frac{dx}{y^2} = \frac{dy}{-xy}$

$$x \cdot dx = -y\,dy$$

$$\boxed{x^2 + y^2 = 2C_1 = k_1}$$

Taking $\frac{dy}{-xy} = \frac{dz}{x(z-2y)}$

$$z \cdot dy - 2y \cdot dy = -y \cdot dz$$

or $z \cdot dy + y \cdot dz = 2y \cdot dy$

or $d(yz) = 2y \cdot dy$

$$yz = 2 \cdot \frac{y^2}{2} + C_2$$

$\therefore \phi(x^2+y^2,\ y^2-yz) = 0$ $\boxed{y^2 - yz = C_2 = k_2}$

(iii) Lagrange's A.E.

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{(x+y)z} = t$$

$$\frac{dx}{x^2} = \frac{dy}{y^2} \Rightarrow \frac{-1}{x} + \frac{1}{y} = c_1$$

$$\boxed{\frac{1}{x} - \frac{1}{y} = -c_1 = k_1}$$

taking $dx = x^2 t$

$dy = t y^2$

$dz = t(x+y)z$

$$dx - dy = t(x+y)(x-y)$$

or $$\frac{dx - dy}{(x+y)(x-y)} = t$$

Now $$\frac{dz}{\cancel{(x+y)}z} = \frac{dx - dy}{\cancel{(x+y)}(x-y)}$$

$$\Rightarrow \frac{dz}{z} = \frac{d(x-y)}{(x-y)}$$

$$\Rightarrow \log z + \log c_2 = \log(x-y)$$

$$\Rightarrow \log(x-y) - \log z = \log c_2$$

$$\boxed{\frac{x-y}{z} = c_2}$$

$$\therefore \phi\left(\frac{1}{x} - \frac{1}{y}, \frac{x-y}{z}\right) = 0$$

Q. Solve P.d.e.

(i) $(mz - ny)p + (nx - lz)q = ly - mx$

(ii) $x^2(y-z)p + y^2(z-x)q = z^2(x-y)$

Solⁿ (i) lagrange's A.E. -

$$\frac{dx}{mz-ny} = \frac{dy}{nx-lz} = \frac{dz}{ly-mx} = t$$

$$dx = t(mz-ny)$$
$$dy = t(nx-lz)$$
$$dz = t(ly-mx)$$

→ $l\,dx + m\,dy + n\,dz$

$= t[\cancel{lmz} - \cancel{lny} + \cancel{mnx} - \cancel{mlz} + \cancel{nly} - \cancel{mnx})$

$$l\,dx + m\,dy + n\,dz = 0$$

$$\boxed{lx + my + nz = c_1}$$

→ $x\,dx + y\,dy + z\,dz$

$= t[\cancel{xmz} - \cancel{xny} + \cancel{ynx} - \cancel{lyz} + \cancel{lyz} - \cancel{mxz})$

$$x\,dx + y\,dy + z\,dz = 0$$

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = c_2$$

$$\boxed{x^2 + y^2 + z^2 = 2c_2 = k_2}$$

$$\phi(lx + my + nz,\ x^2 + y^2 + z^2) = 0$$

(ii) $\dfrac{dx}{x^2(y-z)} = \dfrac{dy}{y^2(z-x)} = \dfrac{dz}{z^2(x-y)} = t$

$$dx = t \cdot x^2(y-z)$$
$$dy = t \cdot y^2(z-x)$$
$$dz = t \cdot z^2(x-y)$$

$\rightarrow$ $\dfrac{dx}{x^2} + \dfrac{dy}{y^2} + \dfrac{dz}{z^2}$

$$\Rightarrow t[\cancel{y}-\cancel{z}+\cancel{z}-\cancel{x}+\cancel{x}-\cancel{y}]$$

$$= 0$$

$$-\frac{1}{x} - \frac{1}{y} - \frac{1}{z} = c_1$$

$$\boxed{\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = k_1}$$

$\rightarrow$ $\dfrac{dx}{x} + \dfrac{dy}{y} + \dfrac{dz}{z}$

$$\Rightarrow t[\cancel{xy}-\cancel{xz}+\cancel{yz}-\cancel{xy}+\cancel{xz}-\cancel{yz}]$$

$$= 0$$

$$\log x + \log y + \log z = \log c_2$$

$$\boxed{xyz = k_2}$$

$$\phi\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z},\ xyz\right) = 0$$

classmate
Date
Page

Q. Solve
i) $p + 3q = 5z + \tan(y-3x)$
ii) $(x^2-y^2-yz)p + (x^2-y^2-zx) = z(x-y)$

Sol^n (i) Lagrange's A.E. -

$$\frac{dx}{1} = \frac{dy}{3} = \frac{dz}{5z + \tan(y-3x)}$$

$$dx = \frac{dy}{3} \Rightarrow 3dx = dy \Rightarrow \boxed{y - 3x = c_1}$$

$$dx = \frac{dz}{5z + \tan(y-3x)} = \frac{dz}{5z + \tan(c_1)}$$

$$x = \frac{1}{5}\log(5z + \tan c_1) + c_2$$

$$\boxed{5x = \log[5z + \tan(y-3x)] + c_2}$$

$$\phi\left(y-3x,\ 5x - \log[5z + \tan(y-3x)]\right) = 0$$

(ii) Lagrange's A.E. -

$$\frac{dx}{(x^2-y^2-yz)} = \frac{dy}{(x^2-y^2-zx)} = \frac{dz}{z(x-y)} = t$$

$$dx = t[x^2-y^2-yz]$$
$$dy = t[x^2-y^2-zx]$$
$$dz = t(z(x-y))$$

$\Rightarrow dx - dy - dz$

$\Rightarrow t[\cancel{x^2}-\cancel{y^2}-\cancel{yz} - \cancel{x^2} + \cancel{y^2} + \cancel{zx} - \cancel{zx} + \cancel{zy}] = 0$

$$dx - dy - dz = 0$$
$$\boxed{x - y - z = c_1}$$

$$xdx - ydy = t[x^3 - xy^2 - xyz - yx^2 + y^3 + xyz]$$
$$= t[x^3 - x^2y + y^3 - xy^2]$$
$$= t[x^2(x-y) + y^2(y-x)]$$
$$= t[x^2(x-y) - y^2(x-y)]$$
$$= t(x-y)(x^2-y^2)$$

or $$\frac{xdx - ydy}{(x-y)(x^2-y^2)} = t$$

Now $$\frac{xdx - ydy}{\cancel{(x-y)}(x^2-y^2)} = \frac{dz}{z\cancel{(x-y)}}$$

let $x^2 - y^2 = u$

$\cancel{2xdx - 2ydy = du}$

$xdx - ydy = \frac{du}{2}$

$$\frac{xdx - ydy}{x^2-y^2} = \frac{dz}{z}$$

$$\Rightarrow \frac{1}{2}\log(x^2-y^2) = \log z + \log c_2$$

$$\log(x^2-y^2) - \log z^2 = \log c_2^2 = \log k$$

$$\boxed{\frac{x^2-y^2}{z^2} = \cancel{\log}\, k_2}$$

$$\phi\left(x-y-z, \frac{x^2-y^2}{z^2}\right) = 0$$

classmate

Date
Page

Q. (i) $\left(\frac{y-z}{yz}\right)p + \left(\frac{z-x}{zx}\right)q = \frac{x-y}{xy}$

multiply by xyz.

$x(y-z)p + y(z-x)q = z(x-y)$

$$\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)} = t$$

$\Rightarrow dx + dy + dz$

$\Rightarrow t\left[\cancel{xy} - \cancel{xz} + \cancel{yz} - \cancel{xy} + \cancel{zx} - \cancel{zy}\right]$

$= 0$

$$\boxed{x + y + z = c_1}$$

$$\frac{dx}{x} = t(y-z)$$

$$\frac{dy}{y} = t(z-x)$$

$$\frac{dz}{z} = t(x-y)$$

Now add -

$$\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = t\left[y - z + z - x + x - y\right]$$

$$\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0$$

$$\log x + \log y + \log z = \log c_2$$

$$\boxed{xyz = c_2}$$

$$\phi\left(x+y+z,\ xyz\right) = 0$$

(ii) $(y^2+z^2)p - xyq = -xz$

→ Lagrange's A.E. -

$$\frac{dx}{y^2+z^2} = \frac{dy}{-xy} = \frac{dz}{-xz} = t$$

taking $\frac{dy}{\cancel{-}xy} = \frac{dz}{\cancel{-}xz}$

$$\log y - \log z = \log c_1$$

$$\boxed{\frac{y}{z} = c_1}$$

$x\,dx + y\,dy + z\,dz$

$t\left[xy^2 + xz^2 - xy^2 - xz^2\right] = 0$

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = c_2$$

$$\boxed{x^2+y^2+z^2 = k_2}$$

$$\phi\left(\frac{y}{z},\ x^2+y^2+z^2\right)$$

→ Charpit's Method:

To solve non-linear p.d.e

$$f(x, y, z, p, q) = 0$$

Charpit's A.E.

$$\frac{dp}{\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}} = \frac{dx}{\frac{-\partial f}{\partial p}} = \frac{dy}{\frac{-\partial f}{\partial q}} = \frac{dF}{0}$$

then solution is given by - $\boxed{dz = p\,dx + q\,dy}$

Q. Solve. $2zx - px^2 - 2qxy + pq = 0$

Sol^n^ let $f = 2xz - px^2 - 2qxy + pq = 0$ — (1)

Charpit's A.E.

$$\frac{dp}{\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}} = \frac{dx}{\frac{-\partial f}{\partial p}} = \frac{dy}{\frac{-\partial f}{\partial q}} = \frac{dF}{0}$$

$$\frac{\partial f}{\partial x} = 2z - 2px - 2qy, \quad \frac{\partial f}{\partial p} = -x^2+q$$

$$\frac{\partial f}{\partial y} = -2qx, \quad \frac{\partial f}{\partial z} = 2x, \quad \frac{\partial f}{\partial q} = -2xy + p$$

$$\Rightarrow \frac{dp}{2z-2qy} = \frac{dq}{0} = \frac{dz}{px^2-2pq+2qxy} = \frac{dx}{x^2-q} = \frac{dy}{2xy-p} = \frac{dF}{0}$$

Taking $\frac{dp}{2z-2qy} = \frac{dq}{0} \Rightarrow dq = 0$

$$\boxed{q = C_1}$$

from (1)

$$f = 2xz - px^2 - 2C_1xy + pC_1 = 0$$

$$p(x^2 - C_1) = 2x(z - C_1 y)$$

$$p = \frac{2x(z - C_1 y)}{x^2 - C_1}$$

∴ Solution is given by -

$$dz = p\,dx + q\,dy$$

$$dz = \frac{2x(z - c_1 y)}{(x^2 - c_1)}\,dx + c_1\,dy$$

$$dz - c_1\,dy = \frac{2x(z - c_1 y)\,dx}{x^2 - c_1}$$

$$\frac{dz - c_1\,dy}{(z - c_1 y)} = \frac{2x\,dx}{(x^2 - c_1)}$$

$$\log(z - c_1 y) = \log(x^2 - c_1) + \log c_2$$

$$\frac{z - c_1 y}{x^2 - c_1} = k_1$$

$$\boxed{z - c_1 y = k_1(x^2 - c_1)}$$

Q. Solve $px + qy = pq$

Sol^n $f = px + qy - pq = 0$ —— (1)

$$\frac{\partial f}{\partial x} = p, \quad \frac{\partial f}{\partial y} = q, \quad \frac{\partial f}{\partial z} = 0, \quad \frac{\partial f}{\partial p} = x - q$$

$$\frac{\partial f}{\partial q} = y - p$$

$$\frac{dp}{p+0} = \frac{dq}{q} = \frac{dz}{-px+pq-qy+pq} = \frac{dx}{q-x} = \frac{dy}{p-y} = \frac{dF}{0}$$

Taking $\frac{dp}{p} = \frac{dq}{q}$

$$\log p = \log q + \log c_1$$

$$\boxed{p = q c_1}$$

$$f = px + qy - pq = 0$$

$$qc_1x + qy - q^2c_1 = 0$$
$$q^2c_1 - qy - qc_1x = 0$$
$$qc_1 - y - c_1x = 0$$
$$q = \frac{c_1x+y}{c_1}, \quad p = c_1x+y$$

solution is given by -

$$dz = p\,dx + q\,dy$$

$$dz = (c_1x+y)\,dx + \frac{c_1x+y}{c_1} \cdot dy$$

~~$c_1dz - y\,dy = (c_1x+y)dx$~~

$$c_1 dz = c_1(c_1x+y)\,dx + (c_1x+y)\,dy$$

$$c_1 dz = (c_1x+y)(c_1dx+dy)$$

$$\boxed{c_1 \cdot z = \frac{(c_1x+y)^2}{2} + c_2}$$

→ Cauchy's Method of characterstics

Equation of the form $a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} = f(x,y) + k$

where $u(x,0) = h(x)$

then Cauchy's system of equations

$$\frac{dx}{a} = \frac{dy}{b} = \frac{du}{f(x,y) + ku}$$

Q. Solve by Cauchy's method
$U_x + U_y = U$, given $u(x,0) = 1 + e$

Sol^n Cauchy's system of equations

$$\frac{dx}{1} = \frac{dy}{1} = \frac{du}{u}$$

$dx = dy$

$x - y = c_1$

$dx = \frac{du}{u}$

$\log u = x + \log c_2$

$\frac{u}{c_2} = e^x$

$u = c_2 e^x$

$c_2 = g(c_1)$

$c_2 = g(x-y)$

$\therefore \quad u = g(x-y)e^x$ —— (1)

using $u(x,0) = 1 + e^x$ in (1)

classmate
Date
Page

$$1 + e^x = g(x) e^x$$

$$g(x) = \frac{1+e^x}{e^x} = e^{-x} + 1$$

$$g(x-y) = \frac{1+e^{(x-y)}}{e^{(x-y)}} = e^{-(x-y)} + 1$$

$\therefore$ Solution $u = \left(e^{-(x-y)} + 1\right) e^x$

$$\boxed{u = e^y + e^x}$$

Q. Solve $U_x - yU_y = 0$, given $U(0, y) = 1$

Solⁿ Cauchy's system of equation -

$$\frac{dx}{1} = \frac{dy}{0} = \frac{du}{yu}$$

$dy = 0$

$\boxed{y = c_1}$

$$dx = \frac{du}{yu}$$

$$dx = \frac{du}{c_1 u}$$

$$\frac{du}{u} = c_1 dx$$

$$\log u + \log c_2 = c_1 x$$

$$u c_2 = e^{c_1 x}$$

let $c_2 = g(c_1) = g(y)$

$$u \cdot g(y) = e^{yx}$$

$u(0, y) = 1 \Rightarrow 1 = \frac{e^0}{g(y)} \Rightarrow g(y) = 1$

$$\boxed{u = e^{yx}}$$

classmate

Date
Page

Q. Solve $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = x+y$, given $u(x,0)=0$

→ Cauchy's system of equations -

$$\frac{dx}{1} = \frac{dy}{1} = \frac{du}{x+y}$$

$$\frac{du}{1} = \frac{dy}{1}$$

$$\boxed{x-y=c_1}$$

$$\frac{dx}{1} = \frac{du}{x+y}$$

$$\frac{dx}{1} = \frac{du}{x+x-c_1}$$

$$(2x-c_1)\,dx = du$$

$$x^2 - c_1x + c_2 = u$$

$$\boxed{u = x^2 - c_1x + c_2}$$

let $c_2 = g(c_1) = g(x-y)$

$$u = x^2 - (x-y)x + g(x-y)$$

$$u = \cancel{x^2} - \cancel{x^2} + xy + g(x-y)$$

$$\therefore u = xy + g(x-y)$$

for $u(x,0)=0$

$$0 = 0 + g(x)$$

$$g(x)=0$$

$$g(x-y)=0$$

$$\therefore \boxed{u = xy}$$

Q. $U_x - U_y = 0$, $U(x,0) = x$

Sol^n- $\frac{dx}{1} = \frac{dy}{-1} = \frac{du}{0}$

$\frac{dx}{1} = -\frac{dy}{1}$

$dx + dy = 0$

$\boxed{x + y = c_1}$

$\frac{du}{0} = \frac{dx}{1}$

$u = c_2$

let $c_2 = g(c_1)$

$= g(x+y)$

$u = g(x+y)$

$x = g(x)$

$x+y = g(x+y)$

$\therefore \boxed{u = x+y}$

Q. $U_x - U_y = 0$, $U(x,0) = x$

Sol^n:- $\frac{dx}{1} = \frac{dy}{-1} = \frac{du}{0}$

$\frac{dx}{1} = \frac{-dy}{1}$

$dx + dy = 0$

$\boxed{x + y = c_1}$

$\frac{du}{0} = \frac{dx}{1}$

$u = c_2$

let $c_2 = g(c_1)$
$= g(x+y)$

$u = g(x+y)$
$x = g(x)$
$x+y = g(x+y)$

$\therefore \boxed{u = x+y}$

→ <u>Homogeneous linear p.d.e. with constant coefficients</u>

$$a\frac{\partial^2 z}{\partial x^2} + b\frac{\partial^2 z}{\partial x \partial y} + c\frac{\partial^2 z}{\partial y^2} = f(ax+by)$$

or $(aD^2 + bDD' + cD'^2)z = f(ax+by)$

where $D = \frac{\partial}{\partial x}$, $D' = \frac{\partial}{\partial y}$

$D \to m$
$D' \to 1$

A.E. $am^2 + bm + c = 0$

(i) If roots are distinct, $m = m_1, m_2, \ldots$
then C.F. $= f_1(y + m_1 x) + f_2(y + m_2 x) + \ldots$

classmate

Date ______
Page ______

(ii) If roots are equal, $m = m_1, m_1, \ldots$ then -

$$C.F. = f_1(y + m_1 x) + x \cdot f_2(y + m_1 x) + \ldots$$

$$P.I. = \frac{1}{f(D, D')} f(ax + by)$$

replace $D \to a$
and $D' \to b$

$$= \frac{1}{f(a,b)} \iint f(u)\, du\, du \quad ; \text{ where } f(a,b) \neq 0$$

Q. Solve $\dfrac{\partial^3 u}{\partial x^3} - 3\dfrac{\partial^3 u}{\partial x^2 \partial y} + 4\dfrac{\partial^3 u}{\partial y^3} = e^{x+2y}$

Sol^n -

$$(D^3 - 3D^2D' + 4D'^3)u = e^{x+2y}$$

A.E. $m^3 - 3m^2 + 4 = 0$

$m = -1, 2, 2$

$$C.F. = f_1(y - x) + f_2(y + 2x) + x \cdot f_3(y + 2x)$$

$$P.I. = \frac{1}{D^3 - 3D^2D' + 4D'^3} e^{x+2y}$$

Replace $D \to 1$
and $D' \to 2$

$$= \frac{1}{1 - 6 + 32} e^{x+2y}$$

$$= \frac{1}{27} \iiint e^u\, du\, du\, du$$

$$= \frac{1}{27} \cdot e^u = \frac{e^{x+2y}}{27}$$

$\therefore$ solution -

$$u = C.F. + P.I.$$

$$\boxed{u = f_1(y-x) + f_2(y+2x) + f_3(y+2x) + \frac{1}{27} e^{x+2y}}$$

Q. Solve $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = \sin x$

Sol^n - $(D^2 - 2DD' + D'^2)z = \sin x$

$$A.E. = m^2 - 2m + 1 = 0$$

$$m = 1, 1$$

$$C.F. = f_1(y+x) + x f_2(y+x)$$

$$P.I. = \frac{1}{D^2 - 2DD' + D'^2} \sin(x)$$

$$= \frac{1}{1 - 0 + 0} \iint \sin u \, du \, du$$

$$= \iint \sin u \, du \, du$$

$$= -\sin u = -\sin x$$

$$z = C.F. + P.I.$$

$$\boxed{z = f_1(y+x) + x f_2(y+x) + (-\sin x)}$$

Q. Solve $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x \partial y} = \sin x \cdot \cos 2y$

Sol^n:- $D^2 - 2DD' = \sin x \cdot \cos 2y$

A.E. $m^2 - 2m = 0$

$m = 0, 2$

$$C.F. = f_1(y+0x) + f_2(y+2x)$$

$$P.I. = \frac{1}{D^2 - 2DD'} \cdot \sin x \cdot \cos 2y$$

$$= \frac{1}{D^2-2DD'} \cdot \frac{1}{2}\left[2\sin x \cdot \cos 2y\right]$$

$$= \frac{1}{D^2-2DD'} \cdot \frac{1}{2}\left[\sin(x+2y) + \sin(x-2y)\right]$$

$$= \frac{1}{2(D^2-2DD')}\sin(x+2y) + \frac{1}{2(D^2-2DD')}(\sin(x-2y))$$

$$= \frac{1}{2(1-4)}\iint \sin u \cdot du\, du + \frac{1}{2(1+4)}\iint \sin v \cdot du\, dv$$

$$= -\frac{1}{6} \times -\sin u + \frac{1}{10}(-\sin v)$$

$$= \frac{\sin u}{6} - \frac{\sin v}{10}$$

$Z = C.F. + P.I.$

$$\boxed{z = f_1(y) + f_2(y+2x) + \frac{\sin(x+2y)}{6} - \frac{\sin(x-2y)}{10}}$$

classmate
Date
Page

Q. $4r - 4s + t = 16 \log(x+2y)$

Sol^n $4\dfrac{\partial^2 z}{\partial x^2} - 4\dfrac{\partial^2 z}{\partial x \partial y} + \dfrac{\partial^2 z}{\partial y^2} = 16 \log(x+2y)$

$$(4D^2 - 4DD' + D'^2) = 16 \log(x+2y)$$

$$A.E. \Rightarrow \quad 4m^2 - 4m + 1 = 0$$
$$m = \frac{1}{2}, \frac{1}{2}$$

$$C.F. = f_1\left(y + \frac{1}{2}x\right) + x f_2\left(y + \frac{1}{2}x\right)$$
$$= \frac{f_1}{2}(2y+x) + x\frac{f_2}{2}(2y+x)$$
$$= g_1(x+2y) + x \cdot g_2(x+2y)$$

$$P.I. = 16 \cdot \frac{1}{4D^2 - 4DD' + D'^2} \log(x+2y)$$

$$D \rightarrow 1, \quad D' \rightarrow 2$$
$$= 16 \cdot \frac{1}{4 - 8 + 4} \quad \times$$

$$= 16\, x \cdot \frac{1}{8D - 4D'} \log(x+2y)$$
$$= 16\, x^2 \cdot \frac{1}{8} \log(x+2y)$$

$$Z = C.F. + P.I.$$
$$\boxed{Z = g_1(x+2y) + x g_2(x+2y) + 2x^2 \log(x+2y)}$$

classmate

Date
Page

Q. Solve $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \cos mx \cos ny + 30(2x+y)$

Sol^n- $(D^2 + D'^2)z = \cos mx \cos ny + 30(2x+y)$

A.E. $\Rightarrow m^2+1=0 \Rightarrow m = \pm i$

$$\boxed{C.F. = f_1(y+ix) + f_2(y-ix)}$$

$$P.I. = \frac{1}{(D^2+D'^2)}\left[\frac{2}{2}\cos mx \cos ny + 30(2x+y)\right]$$

$$= \frac{1}{(D^2+D'^2)}\left[\frac{1}{2}\cos(mx+ny) + \frac{1}{2}\cos(mx-ny) + 30(2x+y)\right]$$

$$= \frac{1}{2(D^2+D'^2)}\cos(mx+ny) + \frac{1}{2(D^2+D'^2)}\cos(mx-ny) + 30 \cdot \frac{1}{(D^2+D'^2)}(2x+y)$$

$$= \frac{1}{2(m^2+n^2)}\iint \cos u \, du \, du + \frac{1}{2(m^2+n^2)}\iint \cos v \cdot dv \, dv + \frac{30}{5}\iint w \cdot dw \, dw$$

$$= \frac{-\cos u}{2(m^2+n^2)} - \frac{\cos v}{2(m^2+n^2)} + w^3$$

$$= \frac{-\cos(mx+ny)}{2(m^2+n^2)} - \frac{\cos(mx-ny)}{2(m^2+n^2)} + (2x+y)^3$$

$$z = C.F. + P.I.$$

$$\boxed{z = f_1(y+ix) + f_2(y-ix) - \frac{1}{2(m^2+n^2)}\left[\cos(mx+ny) + \cos(mx-ny)\right] + (2x+y)^3}$$

Q. $(D-D')^2 z = x + \phi(x+y)$

Sol^n $(D^2 - 2DD' + D'^2)z = x + \phi(x+y)$

A.E. $\Rightarrow$ $m^2 - 2m + 1 = 0$

$m = 1, 1$

$$\boxed{C.F. = f_1(y+x) + x f_2(y+x)}$$

$$P.I. = \frac{1}{D^2 - 2DD' + D'^2}(x) + \frac{1}{D^2 - 2DD' + D'^2}\phi(x+y)$$

$$= \frac{1}{1-0+0}\iint u \cdot du \cdot du + \frac{1}{1-2+1}\iint \cancel{\phi} \cdot v \cdot dv \cdot dv$$

$$= \frac{u^3}{6} + \frac{x}{2D - 2D'}\phi(x+y)$$

$$= \frac{x^3}{6} + \frac{x^2}{2}\phi(x+y)$$

$Z = C.F. + P.I.$

$$\boxed{Z = f_1(y+x) + x f_2(y+x) + \frac{x^3}{6} + \frac{x^2}{2}\phi(x+y)}$$

Q. $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 6\frac{\partial^2 z}{\partial y^2} = y \cos x$

Sol^n $(D^2 + DD' - 6D'^2)z = y \cos x$

A.E. $m^2 + m - 6 = 0$

$m = -3, 2$

$I\int II\,dx - \int\left[\frac{d}{dx}I \cdot \int II\,dx\right]dx$

$$\boxed{C.F. = f_1(y-3x) + f_2(y+2x)}$$

$$P.I. = \frac{1}{D^2+DD'-6D'^2}\, y\cos x$$

$$= \frac{1}{(D+3D')(D-2D')}\, y\cos x$$

Put $y = c_1 + 3x$

$$= \frac{1}{(D-2D')}\int \underset{I}{(c_1+3x)}\,\underset{II}{\cos x}\cdot dx$$

$$= \frac{1}{(D-2D')}\left[(c_1+3x)\sin x + 3\cos x\right]$$

$$= \frac{1}{(D-2D')}\left[y\sin x + 3\cos x\right]$$

Put $y = c_2 - 2x$

$$= \int\left[(c_2-2x)\sin x + 3\cos x\right]\cdot dx$$

$$= (c_2-2x)(-\cos x) - \int(-2)\cdot(-\cos x)\cdot dx + 3\sin x$$

$$= -(c_2-2x)\cos x - 2\sin x + 3\sin x$$

Date ______
Page ______

$$z = C.F. + P.I.$$

$$\boxed{z = f_1(y-3x) + f_2(y+2x) - y\cos x + x\sin x}$$

Q. $(D^2 + DD' - 2D'^2) = (y-1)e^x$

Sol^n - $A.E. \Rightarrow$ $m^2 + m - 2 = 0$

$m^2 + 2m - m - 2 = 0$

$m(m+2) - 1(m+2) = 0$

$m = -2, 1$

$$\boxed{C.F. = f_1(y-2x) + f_2(y+x)}$$

$$P.I. = \frac{1}{(D^2 + DD' - 2D'^2)}(y-1)e^x$$

$$= \frac{1}{(D+2D')(D-D')}(y-1)e^x$$

Put $y = c_1 + 2x$

$$= \frac{1}{(D-D')}\int (c_1 + 2x - 1)e^x \cdot dx$$

$$= \frac{1}{(D-D')}\left[(c_1+2x-1)e^x - 2e^x\right]$$

$$= \frac{1}{(D-D')}\left[(y-1)e^x - 2e^x\right]$$

Put $y = c_1 - x$

$$= \int\left[(c_1-x-1)e^x - 2e^x\right] \cdot dx$$

$$= (c_1-x-1)e^x + e^x - 2e^x \quad = (y-1)e^x - e^x$$

$$\boxed{P.I. = ye^x - 2e^x}$$

$$z = C.F. + P.I.$$

$$\boxed{z = f_1(y-2x) + f_2(y+x) + (y-2)e^x}$$

→ **NON HOMOGENEUS P.D.E.**

$$a_1 \frac{\partial^2 z}{\partial x^2} + a_2 \frac{\partial^2 z}{\partial y^2} + a_3 \frac{\partial^2 z}{\partial x \partial y} + a_4 \frac{\partial z}{\partial x} + a_5 z = f(ax+by)$$

$$(a_1 D^2 + a_2 D'^2 + a_3 DD' + a_4 D + a_5) z = f(ax+by)$$

$$(D - m_1 D' - \alpha_1)(D - m_2 D' - \alpha_2) z = f(ax+by)$$

(i) then $CF = e^{\alpha_1 x} f_1(y+m_1 x) + e^{\alpha_2 x} f_2(y+m_2 x) + \ldots$

(ii) If $(D - m_1 D' - \alpha_1)^2 z = f(ax+by)$

then

$$CF = e^{\alpha_1 x}\left[f_1(y+m_1 x) + x f_2(y+m_1 x) + \ldots\right]$$

① $PI = \frac{1}{f(D,D')} e^{ax+by} = \frac{1}{f(a,b)} e^{(ax+by)}$, $f(a,b) \neq 0$

② $PI = \frac{1}{f(D,D')} \sin(ax+by)$ or $\cos(ax+by)$ then put $D^2 = -a^2$, $D'^2 = -b^2$, $DD' = -ab$

Q. Solve $(D^2 - DD' - 2D)z = \sin(3x+4y) - e^{2x+y}$

Sol^n^

$$D(D - D' - 2)z = \sin(3x+4y) - e^{2x+y}$$
$$(D - 0D' - 0)(D - D' - 2)z = \sin(3x+4y) - e^{2x+y}$$

$$C.F. = e^{0x} f_1(y+0x) + e^{2x} f_2(y+x)$$

$$P.I. = \frac{1}{D^2 - DD' - 2D}\sin(3x+4y) - \frac{1}{D^2 - DD' - 2D} e^{(2x+y)}$$

$$= \frac{1}{-9+12-2D}\sin(3x+4y) - \frac{1}{4-2-4} e^{(2x+y)}$$

$$= \frac{1}{3-2D}\sin(3x+4y) + \frac{1}{2} e^{(2x+y)}$$

$$= \frac{(3+2D)}{9-4D^2}\sin(3x+4y) + \frac{1}{2} e^{(2x+y)}$$

$$= \frac{(3+2D)\sin(3x+4y)}{9+36} + \frac{1}{2} e^{(2x+y)}$$

$$= \frac{1}{45}\left[3\sin(3x+4y) + 6\cos(3x+4y)\right] + \frac{1}{2} e^{(2x+y)}$$

$\therefore \quad Z = C.F. + P.I.$

$$\boxed{Z = f_1(y) + e^{2x} f_2(y+x) + \frac{1}{15}\left[\sin(3x+4y) + 2\cos(3x+4y)\right] + \frac{e^{(2x+y)}}{2}}$$

classmate

Date

Page

Q. $(\underbrace{D^2 - DD' - 2D'^2} + \underbrace{2D + 2D'})z = \sin(2x+y)$

Sol^n- $[D^2 - 2DD' + DD' - 2D'^2 + 2(D+D')]z = \sin(2x+y)$

$[D(D-2D') + D'(D-2D') + 2(D+D')]z = \sin(2x+y)$

$[(D-2D')(D+D') + 2(D+D')]z = \sin(2x+y)$

$(D+D')(D-2D'+2)z = \sin(2x+y)$

C.F. $= F_1(y-x) + e^{-2x} f_2(y+2x)$

$$P.I. = \frac{1}{D^2 - DD' - 2D'^2 + 2D + 2D'} \sin(2x+y)$$

$$= \frac{1}{\cancel{-4} + \cancel{2} + \cancel{2} + 2D + 2D'} \sin(2x+y)$$

$$= \frac{1}{2(D+D')} \sin(2x+y)$$

$$= \frac{D-D'}{2(D^2-D'^2)} \sin(2x+y)$$

$$= \frac{D-D'}{2(-4+1)} \sin(2x+y)$$

$$= \frac{D(\sin(2x+y)) - D'(\sin(2x+y))}{2(-3)}$$

$$= \frac{1}{-6}[2\cos(2x+y) - \cos(2x+y)]$$

$$= \frac{-1}{6}\cos(2x+y)$$

Or

$$\frac{1}{2(2+1)}\int \sin u \, du$$

$$\frac{-1}{6}\cos u$$

$$\frac{-1}{6}\cos(2x+y)$$

$$Z = C.F. + P.I.$$

$$\boxed{Z = f_1(y-x) + e^{-2x} f_2(y+2x) - \frac{1}{6}\cos(2x+y)}$$

Q. Solve $(D^2 - 4DD' + 4D'^2 - D + 2D')Z = e^{3x+4y}$

Soln $(D^2 - 2DD' - 2DD' + 4D'^2 - D + 2D')Z = e^{3x+4y}$

$$[D(D-2D') - 2D'(D-2D') - 1(D-2D')]Z = e^{3x+4y}$$

$$[(D-2D')(D-2D') - 1(D-2D')]Z = e^{3x+4y}$$

$$(D-2D')(D-2D'-1)Z = e^{3x+4y}$$

$$\boxed{C.F. = f_1(y+2x) + e^{x} f_2(y+2x)}$$

$$P.I. = \frac{1}{D^2 - 4DD' + 4D'^2 - D + 2D'} e^{(3x+4y)}$$

$$= \frac{1}{9 - 48 + 64 - 3 + 8} e^{(3x+4y)}$$

$$= \frac{1}{30} e^{(3x+4y)}$$

$$\boxed{Z = f_1(y+2x) + e^{x} f_2(y+2x) + \frac{1}{30} e^{(3x+4y)}}$$

Q. Solve $(D^2 - D'^2 - 3D + 3D')z = e^{(x+2y)} + (xy)$

Sol^n. $[(D+D')(D-D') - 3(D-D')]z = e^{x+2y} + xy$

$$(D-D')(D+D'-3)z = e^{x+2y} + xy$$

$$C.F. = f_1(y+x) + e^{3x} f_2(y-x)$$

$$P.I. = \underbrace{\frac{1}{(D-D')(D+D'-3)} e^{x+2y}}_{PI_1} + \underbrace{\frac{1}{(D-D')(D+D'-3)} xy}_{PI_2}$$

$$P.I._1 = \frac{1}{(D-D')(D+D'-3)} e^{x+2y}$$

$$\cancel{= \frac{1}{(1-2)(1+2-3)}}$$

$$\cancel{= \frac{x}{1} e^{x+2y}} \qquad \cancel{\boxed{x e^{x+2y}}}$$

$$\cancel{P.I._2 = \frac{1}{(D-D')(D+D'-3)} xy}$$

$$= \frac{1}{D^2 - D'^2 - 3D + 3D'} \cdot e^{x+2y}$$

$$= x \cdot \frac{1}{2D-3} e^{x+2y} = \frac{x e^{x+2y}}{2-3} = -x e^{x+2y}$$

$D'^2\left(\frac{y+x}{3}\right)$

$\frac{y}{3}+\frac{x}{3}$

$D'\left(\frac{x}{3}\right)$

$$P.I._2 = \frac{1}{(D-D')(D+D'-3)}xy$$

$$= \frac{1}{D\left(1-\frac{D'}{D}\right)(-3)\left[1-\frac{D}{3}-\frac{D'}{3}\right]}xy$$

$$= \frac{-1}{3D}\left[1-\frac{D'}{D}\right]^{-1}\left[1-\left(\frac{D+D'}{3}\right)\right]^{-1}xy$$

$$= \frac{-1}{3D}\left[\left(1+\frac{D'}{D}+\frac{D'^2}{D^2}+\cdots\right)\left(1+\frac{D+D'}{3}+\frac{D^2+D'^2+2DD'}{9}+\cdots\right)xy\right.$$

$$= \frac{-1}{3D}\left[\left(1+\frac{D'}{D}+\frac{D'^2}{D^2}+\cdots\right)\left(xy+\frac{y+x}{3}+\frac{2}{9}\right)\right]$$

$$= \frac{-1}{3D}\left[xy+\frac{y}{3}+\frac{x}{3}+\frac{2}{9}+\frac{1}{D}x+\frac{1}{D}\left(\frac{1}{3}\right)+0\right]$$

$$= \frac{-1}{3D}\left[xy+\frac{y}{3}+\frac{x}{3}+\frac{2}{9}+\frac{x^2}{2}+\frac{x}{3}\right]$$

$$= \frac{-1}{3D}\left[xy+\frac{y}{3}+\frac{2x}{3}+\frac{x^2}{2}+\frac{2}{9}\right]$$

$$= \frac{-1}{3}\left[\frac{x^2y}{2}+\frac{xy}{3}+\frac{x^2}{3}+\frac{x^3}{6}+\frac{2x}{9}\right]$$

$$Z = C.F. + P.I.$$

$$\boxed{Z = f_1(y+x)+e^{3x}f_2(y-x)-xe^{x+2y}-\frac{1}{3}\left[\frac{x^2y}{2}+\frac{xy}{3}+\frac{x^2}{3}+\frac{x^3}{6}+\frac{2x}{9}\right]}$$

classmate

Date
Page

Ex)
$$P.I. = \frac{1}{f(D,D')} e^{(ax+by)} f(x,y)$$

$$D \rightarrow (D+a)$$
$$D' \rightarrow (D'+b)$$

$$\Rightarrow P.I. \Rightarrow e^{(ax+by)} \frac{1}{f(D+a, D'+b)} f(x,y)$$

Q. $(D-3D'-2)^2 z = 2e^{2x} \tan(3x+y)$

Soln: $C.F. = e^{2x}[f_1(y+3x) + x f_2(y+3x)]$

$$P.I. = 2 \frac{1}{(D-3D'-2)^2} e^{(2x+0y)} \tan(3x+y)$$

$$= 2e^{(2x+0y)} \frac{1}{[(D+\cancel{2})-3(D')-\cancel{2}]^2} \tan(3x+y)$$

$$= 2e^{2x} \frac{1}{(D-3D')^2} \tan(3x+y)$$

$$= \cancel{2}e^{2x} \cdot x \cdot \frac{1}{\cancel{2}(D-3D')} \tan(3x+y)$$

$$= e^{2x} \cdot x^2 \cdot \frac{1}{1} \tan(3x+y)$$

$$\boxed{Z = e^{2x}[f_1(y+3x) + x f_2(y+3x)] + e^{2x} \cdot x^2 \cdot \tan(3x+y)}$$

$(D' - mD - \alpha) \xrightarrow{C.F.} e^{\alpha y} f(x + my)$

Q. Solve $s + p - q = z + xy$

Sol^n- $\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} - z = xy$

$\Rightarrow (DD' + D - D' - 1)z = xy$

$\Rightarrow [D(D'+1) - 1(D'+1)]z = xy$

$= (D-1)(D'+1)z = xy$

$C.F. = e^{x} f_1(y) + e^{-y} f_2(x)$

$P.I. = \frac{1}{(DD' + D - D' - 1)} (\cancel{z} + xy)$

$P.I. = \frac{1}{DD' + D - D' - 1} (xy)$

$= -1 \frac{1}{[1 - D + D' - DD')} (xy)$

$= -1 [1 - (D + D' - DD')]^{-1} (xy)$

Q. Solve $(D^3 - 3DD' + D' + 4)z = e^{2x+y}$

Sol^n: When factors of $f(D, D')$ are not possible, then for C.F.

let $z = Ae^{(hx+ky)}$, be the solution

then $CF = \Sigma A e^{(hx+ky)}$

where $(h^3 - 3hk + k + 4)Ae^{(hx+ky)} = 0$

or $h^3 - 3hk + k + 4 = 0$

$$P.I. = \frac{1}{D^3 - 3DD' + D' + 4} e^{2x+y}$$

$$= \frac{1}{7} e^{2x+y}$$

$\therefore$ Solution $z = CF + P.I$

$$z = \Sigma A e^{(hx+ky)} + \frac{1}{7} e^{2x+y}$$

where $h^3 - 3hk + k + 4 = 0$

→ P.d.e. reducible to constant coefficients -

Q. $x^2 \frac{\partial^2 z}{\partial x^2} - 4xy \frac{\partial^2 z}{\partial x \partial y} + 4y^2 \frac{\partial^2 z}{\partial y^2} + 6y \frac{\partial z}{\partial y} = x^3 y^4$

Sol. Put $x = e^X$, $y = e^Y$

$$X = \log x, \quad Y = \log y$$

So that, $$x \frac{\partial z}{\partial x} = Dz$$

$$x^2 \frac{\partial^2 z}{\partial x^2} = D(D-1)z$$

$$y \frac{\partial z}{\partial y} = D'z, \quad y^2 \frac{\partial^2 z}{\partial y^2} = D'(D'-1)z$$

$$xy \frac{\partial^2 z}{\partial x \partial y} = DD'z$$ where $D = \frac{\partial}{\partial X}$, $D' = \frac{\partial}{\partial Y}$

Put these values in given equations

$$\left[D(D-1) - 4DD' + 4D'(D'-1) + 6D'\right] z = e^{3X} e^{4Y}$$

$$\left[D^2 - D - 4DD' + 4D'^2 - 4D' + 6D'\right] z = e^{3X} e^{4Y}$$

$$\left[D^2 - 4DD' - D + 4D'^2 + 2D'\right] z = e^{(3X+4Y)}$$

$$\left[D^2 + 4D'^2 - 4DD' - D + 2D'\right] z = e^{(3X+4Y)}$$

$$\left[(D-2D')^2 - (D-2D')\right] z = e^{(3X+4Y)}$$

$$(D-2D'-1)(D-2D')z = e^{(3X+4Y)}$$

9 + 64 − 48 − 3 + 8
73 − 51 + 13
73 − 43
30

$$C.F. = f_1(Y+2X) + e^{x} f_2(Y+2X)$$

$$P.I. = \frac{1}{D^2+4D'^2-4DD'-D+2D'} e^{(3X+4Y)}$$

$$= \frac{1}{30} e^{(3X+4Y)}$$

$$Z = C.F. + P.I.$$

$$\boxed{Z = f_1(Y+2X) + e^{X} f_2(Y+2X) + \frac{1}{30} e^{(3X+4Y)}}$$

$$Z = f_1(\log y + 2\log x) + \cancel{2x}\, f_2(\log y + 2\log x) + \frac{1}{30} x^3 y^4$$

$$Z = f_1(\log yx^2) + x f_2(\log yx^2) + \frac{1}{30} x^3 y^4$$

$$\boxed{Z = h_1(yx^2) + x h_2(yx^2) + \frac{1}{30} x^3 y^4}$$

Q. Solve $x^2 \frac{\partial^2 z}{\partial x^2} - y^2 \frac{\partial^2 z}{\partial y^2} = xy$

Soln → $x = e^X$, $y = e^Y$

$X = \log x \qquad Y = \log y$

$$\left[D(D-1) - D'(D'-1)\right]z = \cancel{xy}\; e^{x}\cdot e^{y}$$

$$\left[D^2 - D - D'^2 + D'\right]z = \cancel{xy}\; e^{x}\cdot e^{y}$$

$$\cancel{D^2(D-1)z - D'(D'-1)z}$$

$$\left[D^2 - D'^2 - D + D'\right]z = \cancel{xy}\; e^{x}\cdot e^{y}$$

$$\left[(D+D')(D-D') - 1(D-D')\right]z = \cancel{xy}\; e^{(x+y)}$$

$$(D-D')(D+D'-1)z = \cancel{xy}\; e^{(x+y)}$$

$$C.F. = f_1(y+x) + e^{x} f_2(y-x)$$

$$P.I. = \frac{1}{D^2 - D'^2 - D + D'} \cdot e^{(x+y)}$$

$$= x \cdot \frac{1}{2D-1} e^{(x+y)}$$

$$= x \cdot e^{(x+y)}$$

$$z = C.F. + P.I.$$

$$z = f_1(y+x) + e^{x} f_2(y-x) + x\cdot e^{(x+y)}$$

$$z = f_1(\log xy) + x f_2\left(\log \frac{y}{x}\right) + \log x \cdot xy$$

$$\boxed{z = h_1(xy) + x\, h_2(y/x) + xy\cdot \log x}$$

classmate
Date
Page

UNIT - 2

APPLICATION OF P.D.E.

→ Classification of P.d.e.:

Let

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + f(x,y,z,u,p,q)=0$$

then this equation will be

(i) elliptic if $B^2 - 4AC < 0$

(ii) parabolic if $B^2 - 4AC = 0$

(iii) hyperbolic if $B^2 - 4AC > 0$

Q: Classify the operator

(i) $5\frac{\partial^2 u}{\partial x^2} - 9\frac{\partial^2 u}{\partial x \partial t} + 4\frac{\partial^2 u}{\partial t^2}$

(ii) $x\frac{\partial^2 u}{\partial x^2} + t\frac{\partial^2 u}{\partial x \partial t} + \frac{\partial^2 u}{\partial t^2}$

Solⁿ- (i) $B = -9$, $A = 5$, $C = 4$

$B^2 - 4AC = 81 - 80 = 1 > 0$

∴ hyperbolic

(ii) $B^2 - 4AC = t^2 - 4x1 = t^2 - 4x$

eqⁿ will be elliptic if $B^2 - 4AC < 0$

$t^2 - 4x < 0$

$$\boxed{x > \frac{t^2}{4}}$$

classmate
Date
Page

eq^n will be parabolic if $B^2 - 4AC = 0$

$$t^2 - 4x = 0$$

$$\boxed{x = t^2/4}$$

eq^n will be hyperbolic if $B^2 - 4AC > 0$

$$t^2 - 4x = 0$$

$$\boxed{x < t^2/4}$$

Q. Classify $\sqrt{y^2+x^2}\,U_{xx} + 2(x-y)U_{xy} + \sqrt{y^2+x^2}\,U_{yy} = 0$ in second quadrant

Sol^n- $B^2 - 4AC = 4(x-y)^2 - 4(y^2+x^2)$

$$= 4x^2 + \cancel{4y^2} - 8xy - \cancel{4y^2} - 4x^2$$

$$= -8xy > 0$$

$\because$ in second quadrant, $x<0$, $y>0$

Q. Classify $Z_{xx} + 2xZ_{xy} + (1-y^2)Z_{yy} = 0$

Sol^n- $B^2 - 4AC = 4x^2 - 4(1-y^2) \gtrless 0$

$$4x^2 - 4 + 4y^2 \gtrless 0$$

$$x^2 + y^2 - 1 \gtrless 0$$

(i) for elliptic $\rightarrow$ $x^2+y^2-1 < 0$

$x^2+y^2 < 1$ {inside the circle}

(ii) for parabolic $\Rightarrow$ $x^2+y^2-1 = 0$

$x^2+y^2 = 1$ {on the circle}

(iii) for hyperbolic $\Rightarrow$ $x^2+y^2-1 > 0$

$x^2+y^2 > 1$ {outside the circle}

$m = m_1, m_2, \ldots$

$CF = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

$m = \alpha \pm i\beta$

$CF = e^{\alpha x}[c_1 \cos\beta x + c_2 \sin\beta x]$

Q. <u>Solve by method of seperation of variable</u>

$$4\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 3u, \text{ given that}$$

$$u = 3e^{-x} - e^{-5x}, \text{ at } t = 0$$

Solⁿ:- $u = u(x,t)$

let solution $u = XT$, where $X = X(x)$
$T = T(t)$

$$\frac{\partial u}{\partial t} = X\frac{\partial T}{\partial t}, \quad \frac{\partial u}{\partial x} = T\cdot\frac{\partial X}{\partial x}$$

Put these values in given equations-

$$4X\frac{\partial T}{\partial t} + T\frac{\partial X}{\partial x} = 3XT$$

$$\Rightarrow \frac{4}{T}\cdot\frac{\partial T}{\partial t} + \frac{1}{X}\frac{\partial X}{\partial x} = 3$$

$$\Rightarrow \frac{1}{X}\frac{\partial X}{\partial x} = 3 - \frac{4}{T}\frac{\partial T}{\partial t} = -k^2$$

$$\frac{1}{X}\frac{dX}{dx} = -k^2$$

or $\frac{dX}{dx} + k^2 X = 0$

$(D + k^2)X = 0$

$m + k^2 = 0$

$m = -k^2$

$C.F. = c_1 e^{-k^2 x}$

$X = c_1 e^{-k^2 x}$

$$3 - \frac{4}{T}\frac{\partial T}{\partial t} = -k^2$$

$$\frac{4}{T}\frac{\partial T}{\partial t} = (3+k^2)$$

$$\frac{\partial T}{\partial t} - \left(\frac{3+k^2}{4}\right)T = 0$$

$$\left(D - \frac{3+k^2}{4}\right)T = 0$$

$m - \left(\frac{3+k^2}{4}\right) = 0$

$m = \frac{3+k^2}{4}$

$CF = c_2 e^{\left(\frac{3+k^2}{4}\right)t}$

$T = c_2 e^{\left(\frac{3+k^2}{4}\right)t}$

Solution

$$u = XT = c_1 c_2 e^{\left[-k^2 x + \left(\frac{3+k^2}{4}\right) t\right]}$$

let $c_1 c_2 = \Sigma b_n$

Now given, $u = 3e^{-x} - e^{-5x}$ at $t = 0$

$$u = \Sigma b_n e^{\left[-k^2 x + \left(\frac{3+k^2}{4}\right) t\right]} \quad \text{—— (1)}$$

at $t = 0$

$\Rightarrow 3e^{-x} - e^{-5x} = \Sigma b_n e^{-k^2 x}$

$\Rightarrow 3e^{-x} - e^{-5x} = b_1 e^{-k^2 x} + b_2 e^{-k^2 x} + \cdots$

on comparing -

$b_1 = 3$, $b_2 = -1$

$k = 1$, $k = \sqrt{5}$

$\therefore$ Solution from (1)

$$\boxed{u = 3e^{-x+t} - e^{-5x+2t}}$$