Engineering Physics Notes

Engineering Physics (Dr. A.P.J. Abdul Kalam Technical University)

UNIT I

RELATIVISTIC MECHANICS

Lecture 1:

Frame of reference, Inertial & non-inertial frames, Galilean transformations

INTRODUCTION:

The old theory of classical mechanics was based on Newton's law. Initially it was believed that second law is universally applicable but with new experimental evidences in the beginning of 20^{th} century, it was revealed that this law is applicable only for the particle moving with low velocity and fails when applied to the particle moving with velocity comparable to velocity of light.

This failure of old theory, led to the development of **THEORY OF RELATIVITY** by **Albert Einstein** in **1905**. According to this theory, everything in universe is relative and nothing is absolute. Einstein told that old theory is the limiting case of new theory.

This new theory of relativity is further divided into two categories as **General Theory** and **Special Theory** respectively. General theory is applied to the cases of relative motion and to accelerated systems with respect to one another. Whereas the Special theory is applicable with observers which are either in relative motion in a straight line at a constant speed or at rest.

Ques: What is frame of reference? Define inertial and non-inertial frames of reference.

Ans: Frame of Reference

The frame of reference is a coordinate system with respect to which we can predict the position and motion of any object at different instants of time.

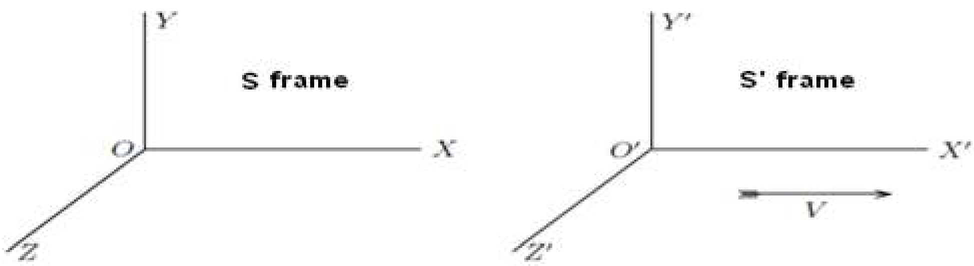


Fig Frame of reference

Frames of reference are of two types-

1. **Inertial Frames of Reference:** Inertial frames of reference are those frames of reference in which Newton's law of inertia and other laws of motion hold good. These frames are either at rest or move with a constant velocity with respect to each other. Any object with net force zero on it, will be un-accelerated with respect to these frames of reference.
   Example: A train is in rest or in uniform motion.
2. **Non-inertial Frames of Reference:** Non-inertial frames of reference are those frames of reference in which Newton's law of inertia and other laws of motion do not hold. These frames are accelerated. Any object with net force zero on it, will be accelerated with respect to non-inertial frames of reference.
   Example: A train is accelerated or retarded motion, Rotating merry-go-round.

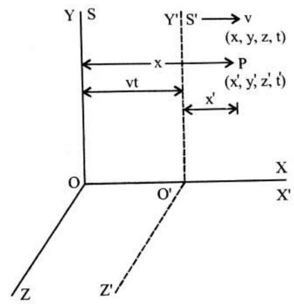
Ques: Is Earth an inertial or non-inertial frame of reference? Discuss.

Ans: Earth is a non-inertial frame of reference because it is an accelerated frame of reference. Earth is rotating not only about its own axis but also orbiting around the sun. Due to its orbital motion it continuously changes its velocity and direction of motion. Due to change in velocity, it is an accelerated frame of reference.

The value of acceleration is very small (0.006 m/s^2) and can be neglected. So, for practical purposes, earth is regarded as an inertial frame of reference.

Ques: Describe Galilean transformation equations. What were the limitations of these transformation equations?

Ans : Galilean-Newtonian transformation equations:



Let S & S' be two frames of reference such that frame S' is moving along positive X axis with v velocity relative to frame S. If (x, y, z, t) and (x', y', z',t') are the coordinates of the same event in frame S and S'. According to classical mechanics, the measurement in x direction made in S' will be smaller than that made in S by the amount vt, which is the distance moved by S' in direction x, i.e.

$$x' = x - vt \qquad \ldots\ldots(1)$$

there is no relative motion in y and z directions, so

$$y' = y \qquad \ldots\ldots(2)$$

and

$$z' = z \qquad \ldots\ldots(3)$$

in classical mechanics,

$$t' = t \qquad \ldots\ldots(4)$$

eqns. (1) to (4) are Galilean –Newtonian transformation equations.

Limitation of Galilean –Newtonian transformation equations

If we apply the concept of relativity (i.e. $v = c$) in equation (1) of Galilean equations, then in frame S' the observed velocity would be $c' = c - v$. *which is the violation of the idea of relativity*. So this requires the modification in Galilean Transformation equations.

Lecture 2:

Ques: What is ether hypothesis?

Ans: Ether hypothesis:

The physicists of 19th century assumed that entire space of universe including vacuum is filled by a hypothetical light transmitting medium called ether: which is rigid, invisible, massless, perfectly transparent, high elasticity and negligible density. All bodies including earth move freely through this hypothetical medium without disturbing it.

Ques: What was the objective of Michelson-Morley experiment? Discuss in detail.

Ans: **Objective** of Michelson-Morley experiment was to detect the motion of earth with respect to ether.

In 1887 Michelson & Morley performed an experiment. A simplified plan of experiment is shown below containing a monochromatic source S. A partially silvered glass plate P is inclined at 45^o to the incident beam. Two mirrors M_1 and $M_2$are placed at equal distance l from glass plate. Telescope T is used to see interference.

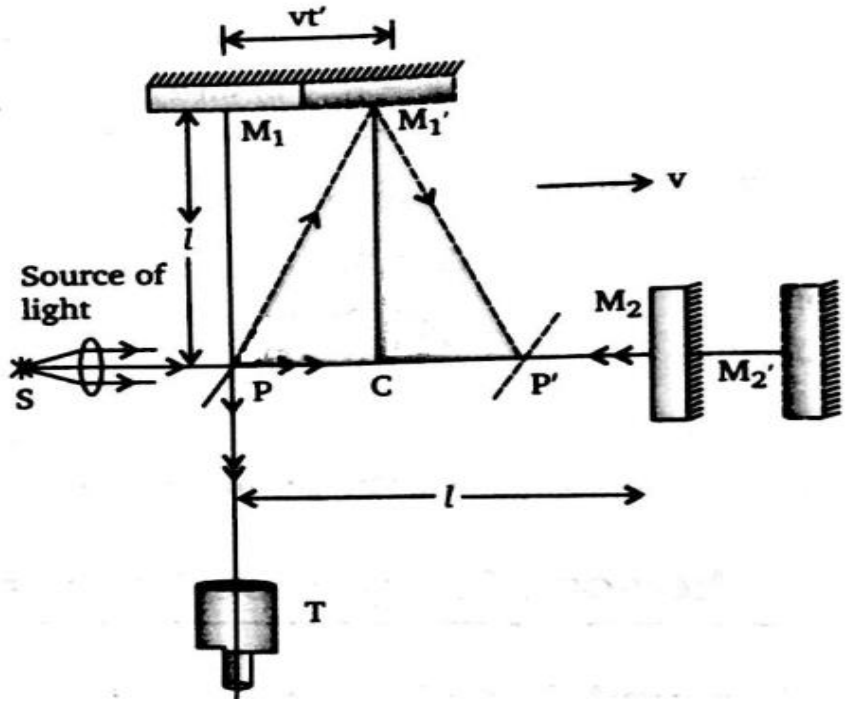


Figure: Michelson Morley Setup

Two mirrors are at equal distance *l* from glass plate. If apparatus is at rest then the two rays will take equal time to return to plate P but apparatus is moving with earth with respect to ether with velocity *v* (say). Let earth is moving in direction (PM_2) of incident light then the time difference between two rays will be calculated as:

if *c* is the velocity of light in direction of arm PM_2, then the relative velocity will be (*c* - *v*) while on returning it will become (*c* + *v*). If t_1 is the time taken in complete journey then

$$t_1 = \frac{l}{(c-v)} + \frac{l}{(c+v)} = \frac{2lc}{(c^2-v^2)} = \frac{2l}{c}\left(\frac{1}{1-\frac{v^2}{c^2}}\right) \qquad \text{....... ...... (1)}$$

The second part of the beam moving towards arm PM_1 with velocity *c*. If t'is the time taken by the beam in going from plate to mirror then the distance travelled will be ct'. Also in time t' the mirror travel the distance vt'. In right angle triangle PM_1M_1' ,

Considering figure, $\{PM_1 = l,\ M_1M_1' = vt',\ PM_1' = ct'\}$

$$(PM_1')^2 = (PM_1)^2 + (M_1M_1')^2,$$

$$\text{i.e. } (ct')^2 = l^2 + (vt')^2$$

$$c^2t'^2 - v^2t'^2 = l^2$$

$$t'^2(c^2 - v^2) = l^2$$

$$t' = \frac{l}{(c^2-v^2)^{1/2}} = \frac{l}{c\left(1-\frac{v^2}{c^2}\right)^{1/2}}$$

Now if t_2 is the total time in journey $PM_1'P'$ then

$$t_2 = 2t' = \frac{2l}{c}\frac{1}{\sqrt{\left(1-\frac{v^2}{c^2}\right)}} \qquad \text{......}(\mathbf{2})$$

If Δt is the time difference of two beams interfering at point P', then

$$\Delta t = t_1 - t_2 = \frac{2l}{c\left(1-\frac{v^2}{c^2}\right)} - \frac{2l}{c\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}}} = \frac{2l}{c}\left[\left(1-\frac{v^2}{c^2}\right)^{-1} - \left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}}\right]$$

Using binomial expansion and neglecting higher terms

$$\Delta t = \frac{2l}{c}\left[\left(1+\frac{v^2}{c^2}+\ \dots\dots\right) - \left(1+\frac{v^2}{2c^2}\dots\dots\right)\right]$$

$$\Delta t = \frac{2l}{c}\left[\frac{1}{2}\frac{v^2}{c^2}\right]$$

$$\Delta t = \frac{lv^2}{c^3} \text{ ----------}(3)$$

The corresponding path difference will be

$$\delta = c\,\Delta t = c\frac{lv^2}{c^3}$$

$$\delta = \frac{lv^2}{c^2} \text{ ----------}(4)$$

If v is made zero, then δ would become zero.

We know that if path difference between two interfering waves changes by λ, then a shift of one fringe appears. Thus, if N fringes are shifted then

$\Delta N = \frac{\delta}{\lambda} = \frac{lv^2}{c^2\lambda}$.

In actual experiment, the apparatus was placed on a block of stone floated on mercury, was rotated by 90°, to introduce a path difference of same amount in opposite direction. Hence a shift of $\frac{2lv^2}{c^2\lambda}$ was expected.

To get the observable shift, experimental values are:

l = 11meter, v (velocity of earth) = $3\text{x}10^4$ *m/sec*, $\lambda = 5.5\text{x}10^{-7}$ *m and c* = $3\text{x}10^8$ *m/sec,*

the expected shift is

$$\Delta N = \frac{2 \text{ X } 11 \text{ X } (3 \text{ X } 10^4)^2}{(3 \text{ X } 10^8)^2 \text{ X } 5.5 \text{ X } 10^{-7}} = 0.4$$

The physicists were surprised to see that there was **no shift** in the fringes. The experiment was repeated various times and at various places but no shift was observed. Thus, **the motion of earth through the ether could not be experimentally detected.**

Ques: How were the negative results of Michelson- Morley experiment interpreted?

Ans: Explaination of negative result: Many explanations were given to explain negative result. Few of them are:

1. **Ether-Drag Hypothesis**: According to this hypothesis, the moving earth drags ether with it. So there is no relative motion between earth and ether. Hence no fringe shift is obtained.
2. **Fitzgerald-Lorentz Contraction Hypothesis:** Fitzgerald and Lorentz gives an hypothesis that all material bodies moving through ether are contracted in direction of motion by a factor $\sqrt{\left(1-\frac{v^2}{c^2}\right)}$. This contraction equalizes time t_1 and t_2 and no shift was observed.
3. **Light Velocity Hypothesis (or Constancy of velocity of light):** According to Einstein, velocity of light is invariant in every inertial frame of reference. In other words velocity of light is constant in free space.

Lecture 3:
Postulates of special theory of relativity, Lorentz transformations

Ques: What are the postulates of special theory of relativity?

Ans: Postulates of special theory of relativity: The special theory of relativity is based upon two postulates:

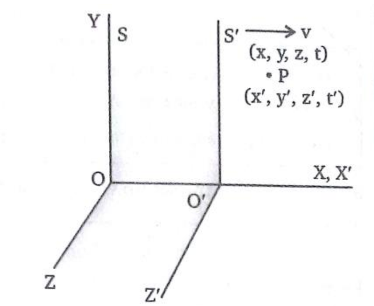
1. Principle of equivalence- The laws of physics have the same form in all inertial frames of reference moving with a constant velocity relative to one another.

2. Principle of constancy of speed of light -The speed of light in free space is the same in all inertial frames of reference.

Ques: Derive Lorentz transformation equation.

Ans: Transformation equations: The equations relating two set of observations of an event in two different frames of reference are called transformation equations.

Lorentz transformations:



Consider two frames of reference S & S′ having origin O and O′ respectively such that S′ is moving with velocity v with respect to S in +ve X direction. At $t = t'$= 0, both frames coincide. Let(x, y, z, t)&(x', y', z', t') are the space-time coordinates for a same event (P) in two frames. Now let us assume that a measurement in x direction made in S frame is proportional to that made in S′ frame, i.e.

$$x' \;\alpha\; (x - vt)$$

Or $$x' = \gamma(x - vt) \quad ......(1) \quad (\gamma \text{ is a Lorentz factor})$$

According to first postulate, laws of physics are same in all frames. Thus the corresponding equation of x will be

$$x = \gamma\,(x' + vt') \quad ......(2)$$

Putting the values of x' from eq. (1) to eq. (2),

$$x = \gamma\,[\gamma(x - vt) + vt']$$

$$\frac{x}{\gamma} = \gamma x - \gamma vt + vt'$$

$$vt' = \frac{x}{\gamma} - \gamma x + \gamma vt$$

$$t' = \frac{x}{\gamma v} - \frac{\gamma x}{v} + \gamma t$$

$$t' = \gamma t - \frac{\gamma x}{v}\left(1 - \frac{1}{\gamma^2}\right) \quad ......(3)$$

similarly, $$t = \gamma t' + \frac{\gamma x'}{v}\left(1 - \frac{1}{\gamma^2}\right) ......(4)$$

Now γ can be evaluated from **second postulate**. Let a light signal is given at origin at time $t = t' = 0$. It travels with velocity c which is same for frames S & S′. Its position will be

$$x = ct \quad \text{and } x' = ct'$$

Substituting these values of x and x' in eqn. (1) & (2)

$$ct' = \gamma(ct - vt)$$
$$ct' = \gamma t(c - v) \qquad \text{......(5)}$$
$$ct = \gamma t'(c + v) \qquad \text{......(6)}$$
multiplying eqn. (5)&(6) we get
$$c^2tt' = \gamma^2tt'(c^2 - v^2)$$
$$\gamma^2 = \frac{c^2}{(c^2 - v^2)}$$
$$\gamma = \frac{1}{\sqrt{\left(1 - \frac{v^2}{c^2}\right)}} \qquad \text{......(7)}$$
putting this value of γ in eq. (1)&(2)
$$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}} \qquad \text{......(8)}$$
solving for t',
$$\gamma^2 = \frac{1}{1 - \frac{v^2}{c^2}} \quad \text{or} \quad \frac{1}{\gamma^2} = \frac{1 - \frac{v^2}{c^2}}{1}$$
$1 - \frac{1}{\gamma^2} = \frac{v^2}{c^2}$, putting this in equation (3)&(4) we get
$$t' = \gamma t - \frac{\gamma x}{v}\left(\frac{v^2}{c^2}\right)$$
$$t' = \frac{t - \frac{xv}{c^2}}{\sqrt{\left(1 - \frac{v^2}{c^2}\right)}} \qquad \text{......(9)}$$
$$y' = y \qquad \text{......(10)}$$
$$z' = z \qquad \text{......(11)}$$
Eqn. (8), (9), (10)&(11) are known as L**orent'z transformation equations.**

Ques: Show that Lorentz transformation are converted in to Galilean transformation at low velocities?

Ans: Reduction of Lorentz transformation in to Galilean transformation:

Lorentz transformation equations are,
$$\left.\begin{aligned} x' &= \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}} \\ y' &= y \\ z' &= z \\ t' &= \frac{t - \frac{xv}{c^2}}{\sqrt{\left(1 - \frac{v^2}{c^2}\right)}} \end{aligned}\right\} \text{....(1)}$$
For low velocity, $v << c$,
$$\frac{v}{c} << 1$$
$$\Rightarrow \frac{v^2}{c^2} \ll 1$$
So at low velocity Lorentz transformation equations become
$$\left.\begin{aligned} x' &= x - vt \\ y' &= y \\ z' &= z \\ t' &= t \end{aligned}\right\}$$

which are Galilean transformation equations.

Ques: What are Inverse Lorentz transformation equations?
Ans: Inverse Lorentz transformation equations

$$x = \frac{x' + vt'}{\sqrt{\left(1 - \frac{v^2}{c^2}\right)}}$$

$$y = y'$$

$$z = z'$$

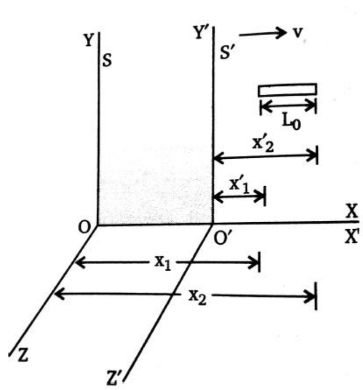
$$t = \frac{t' + \frac{x'v}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Lecture 4:
Length contraction

Ques: What do you mean by length contraction? Derive a mathematical expression for it.

Ans: Length Contraction:

A body moving with a velocity v relative to an observer appears to the observer to be contracted in length in the direction of motion by a factor $\sqrt{\left(1-\frac{v^2}{c^2}\right)}$, whereas its dimensions perpendicular to the direction of motion are unaffected.



Let us consider a rod placed along the x' axis of a moving frame S'. Let $x_1'$and x_2' be the coordinates of the ends of the rod, as seen by an observer in S'. Then the observed length from S' is, ($L_o = x_2' - x_1'$), $L_o$is the rod's length in rest frame. Now another observer measures the length of rod in stationary frame S, relative to which rod is moving with velocity v. if x_1 and x_2 are position coordinates of rod at time t then appeared length L will be $L = x_2 - x_1$.

From Lorentz transformation eqn.

$$x_2' = \frac{x_2 - vt}{\sqrt{1-\frac{v^2}{c^2}}} \text{ and } x_1' = \frac{x_1 - vt}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$\text{So, } L_0 = x_2' - x_1' = \frac{x_2 - x_1}{\sqrt{1-\left(\frac{v^2}{c^2}\right)}} = \frac{L}{\sqrt{1-\left(\frac{v^2}{c^2}\right)}}$$

$$L = L_o\sqrt{1-\left(\frac{v^2}{c^2}\right)}$$

The length L_0 measured in frame in which rod is at rest is called proper length.

CASES

(1) If $v = c$ then from $L = L_o\sqrt{1-\left(\frac{v^2}{c^2}\right)}$

$L = 0$ i.e a rod moving with velocity c, will appear as a point to stationary observer.

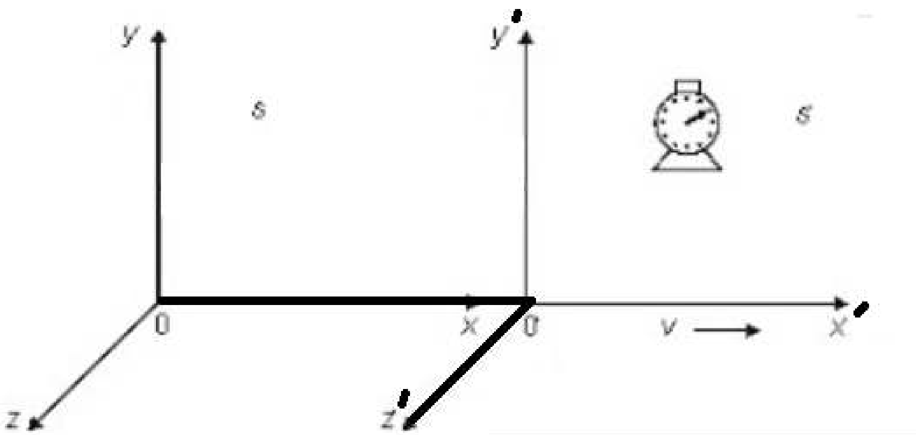
(2) Each observer in S and S' system finds that the other rod is shorter than the rod of his own frame. If the rod is available in each frame.

(3) If $v \ll c$ then $L = L_o$, due to which we cannot observe length contraction in daily life.

Ques: What do you mean by time dilation? Derive mathematical expression for it. Show that it is a real effect. What is proper time?

Ans: Time Dilation:

Time-intervals also affected by relative motion. **A clock in a moving frame S' measures the longer time interval between two events that occurs in a stationary frame of reference than the same time interval measured by the clock in stationary frame S. This is known as time dilation.**



Let a clock be placed at x' in frame S'. An observer in S' finds the clock gives two ticks at time t_1' & t_2' so that the time interval t_0 in frame S' will be

$$t_o = t_2' - t_1'$$

Now an another observer measures the time interval between same two ticks from a stationary frame S, relative to which the S' is moving with velocity v. if he records the ticks at $time$ t_1& $t_2$then the time interval will be $t = t_2 - t_1$.

From Inverse Lorentz transformation equation, we have

$$t_2 = \frac{t_2' + \left(\frac{x'v}{c^2}\right)}{\sqrt{1 - \left(\frac{v^2}{c^2}\right)}} \text{ and } t_1 = \frac{t_1' + \left(\frac{x'v}{c^2}\right)}{\sqrt{1 - \left(\frac{v^2}{c^2}\right)}}$$

$$\text{So, } t = t_2 - t_1 = \frac{t_2' - t_1'}{\sqrt{1-\left(\frac{v^2}{c^2}\right)}}$$

$$t = \frac{t_0}{\sqrt{1-\left(\frac{v^2}{c^2}\right)}}.$$

It shows that to the stationary observer in S, the time interval appears to be lengthened by a factor $\frac{1}{\sqrt{1-\left(\frac{v^2}{c^2}\right)}}$. This phenomenon is known as "time dilation".

If $v = c, \text{then } t = \infty$. it means that clock moving with speed of light will appear to be completely stopped to a stationary observer.

Proper time: The time interval between two events which occurs at same position recorded by a clock in the frame in which the events occurs is called proper time (t_o).

Experimental verification (μ-meson decay):

μ-meson are created by cosmic ray particles at a height of about $10\ km$ from earth and reaches the earth in very large number.

Their speed is $2.994 \times 10^8 m/sec$ which is $0.998\,c$, also their life time is $2 \times 10^{-6} sec$ after which it decays into an electron.
Hence in its lifetime it can travel a distance only of ($2.994 \times 10^8 \times 2 \times 10^{-6}$) ≈ 600 meters.
Then how they reach on earth?
The answer lies in time dilation. Their life time of $meson\ is\ 2 \times 10^{-6}\ sec$. in its own frame of reference (t_0).
In observer's frame (on Earth) it is lengthened due to time dilation as

$$t = \frac{t_0}{\sqrt{1-\left(\frac{v^2}{c^2}\right)}} = \frac{2\times10^{-6}}{\sqrt{1-(0.998)^2}} = \frac{2\times10^{-6}}{0.63} = 3.17 \times 10^{-5}\text{sec}.$$

In this relative life time it can travel a distance

$$= (2.994 \times 10^8) \times (3.17 \times 10^{-5}) \approx 9500 \text{ meters } 9.5 \text{ kms}.$$

Now it is possible for mesons to reach on earth.

Ques: Derive relativistic velocity addition theorem. Show that it is consistant with Einstein's second postulate.

Ans: V<u>elocity addition theorem or relativistic addition of velocities:</u>

The relativistic addition of velocities is been derived by using Lorentz Transformation Equation. Let a frame S' is moving with uniform velocity v with respect to stationary frame S in +ve X direction. Let a particle is also moving in same direction. If the particle moves through a distance dx in time interval dt in frame S, then observed velocity by observer in stationary frame S is given by, $u = \frac{dx}{dt}$. Similarly for an observer in S' the distance covered & time taken will different, let it be dx'& dt'. Then the velocity of particle in S' frame will be $u' = \frac{dx'}{dt'}$.

From Lorentz transformation equations, $x' = \frac{x-vt}{\sqrt{1-\frac{v^2}{c^2}}}$ & $t' = \frac{t-\frac{xv}{c^2}}{\sqrt{1-\left(\frac{v^2}{c^2}\right)}}$.

On the partial Differentiation of the above equations, we get

$$dx' = \frac{dx - vdt}{\sqrt{1 - \frac{v^2}{c^2}}} \text{and } dt' = \frac{dt - \frac{vdx}{c^2}}{\sqrt{\left(1 - \frac{v^2}{c^2}\right)}}$$

Using these values for u', we get

$$u' = \frac{dx'}{dt'}$$

$$u' = \frac{dx - vdt}{dt - \frac{vdx}{c^2}} = \frac{\left(\frac{dx}{dt}\right) - v}{\left(1 - \frac{v}{c^2}\frac{dx}{dt}\right)}$$

Or $$u' = \frac{u-v}{1-\frac{uv}{c^2}}$$

This equation represents the relativistic addition formula. Similarly for u we have, $u = \frac{u'+v}{1+\frac{u'v}{c^2}}$.

Consistency with Einstein's second postulate:

If $u' = c$, then $u = \frac{c+v}{1+\left(\frac{cv}{c^2}\right)} = \frac{c(c+v)}{c+v} = c$,

Thus observer in S' & S recorded the same value for velocity of photon, c is same in all frames which is the second postulate.

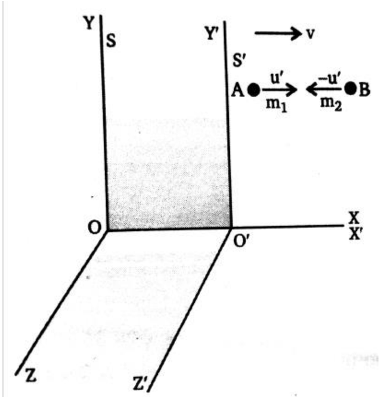
We can also say that any velocity added to c simply reproduces c or c is the maximum attainable velocity in nature or no signal can travel faster than light in vacuum.

Lecture 6:
Variation of mass with velocity

Ques: Show that conservation of linear momentum leads to variation of mass with velocity.
Ans: Variation of mass with velocity
In relativistic mechanics the mass of a body varies with velocity. The mass of a body moving at very high speed (≈ c) relative to an observer is larger than its mass when it is at rest by a factor $\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$. It is given by

$$m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}, m_0 \text{ is the rest mass.}$$

Derivation:
let us consider two frames S & S' such that S' is moving with velocity v in +ve X direction. Let two identical particles moving with velocity u' and –u' in frame S' approaches each other.



The velocity of two particles (mass m_1 and m_2), seen from S frame will be given by relativistic addition of velocities as,

$$u_1 = \frac{u' + v}{1 + \frac{u'v}{c^2}} \qquad \ldots\ldots(1)$$

$$u_2 = \frac{-u' + v}{1 - \frac{u'v}{c^2}} \qquad \ldots\ldots(2)$$

At collision, two particles comes at rest with respect to frame S' but from S they appear to be moving with velocity v.
From the principle of conservation of momentum,

$$m_1u_1 + m_2u_2 = (m_1 + m_2)v$$

$$m_1\left(\frac{u' + v}{1 + \frac{u'v}{c^2}}\right) + m_2\left(\frac{-u' + v}{1 - \frac{u'v}{c^2}}\right) = (m_1 + m_2)v$$

Rearranging

$$m_1\left(\frac{u'+v}{1+\frac{u'v}{c^2}} - v\right) = m_2\left(v - \frac{-u'+v}{1-\frac{u'v}{c^2}}\right)$$

$$m_1\left(\frac{u' + v - v - \frac{u'v^2}{c^2}}{1+\frac{u'v}{c^2}}\right) = m_2\left(\frac{v - \frac{u'v^2}{c^2} + u' - v}{1-\frac{u'v}{c^2}}\right)$$

$$m_1\left(\frac{u' - \frac{u'v^2}{c^2}}{1+\frac{u'v}{c^2}}\right) = m_2\left(\frac{u' - \frac{u'v^2}{c^2}}{1-\frac{u'v}{c^2}}\right)$$

$$\frac{m_1}{m_2} = \frac{\left(1+\frac{u'v}{c^2}\right)}{\left(1-\frac{u'v}{c^2}\right)} \qquad \dots\dots(3)$$

From equation (1)

$$u_1^2 = \left(\frac{u'+v}{1+\frac{u'v}{c^2}}\right)^2$$

$$\left(1-\frac{u_1^2}{c^2}\right) = 1 - \frac{1}{c^2}\left(\frac{u'+v}{1+\frac{u'v}{c^2}}\right)^2$$

$$1-\frac{u_1^2}{c^2} = \frac{\left(1+\frac{u'v}{c^2}\right)^2 - \left(\frac{u'+v}{c}\right)^2}{\left(1+\frac{u'v}{c^2}\right)^2}$$

$$1-\frac{u_1^2}{c^2} = \left\{\frac{\left(1+\frac{u'^2v^2}{c^4}+2\,\frac{u'v}{c^2}\right)-\left(\frac{u'^2}{c^2}+\frac{v^2}{c^2}+2\,\frac{u'v}{c^2}\right)}{\left(1+\frac{u'v}{c^2}\right)^2}\right\}$$

$$1-\frac{u_1^2}{c^2} = \left\{\frac{1+\frac{u'^2v^2}{c^4}+2\,\frac{u'v}{c^2} - \frac{u'^2}{c^2} - \frac{v^2}{c^2} - 2\,\frac{u'v}{c^2}}{\left(1+\frac{u'v}{c^2}\right)^2}\right\}$$

$$1-\frac{u_1^2}{c^2} = \left\{\frac{1+\frac{u'^2v^2}{c^4} - \frac{u'^2}{c^2} - \frac{v^2}{c^2}}{\left(1+\frac{u'v}{c^2}\right)^2}\right\}$$

$$1-\frac{u_1^2}{c^2} = \frac{\left(1-\frac{v^2}{c^2}\right) - \frac{u'^2}{c^2}\left(1-\frac{v^2}{c^2}\right)}{\left(1+\frac{u'v}{c^2}\right)^2}$$

$$1-\frac{u_1^2}{c^2} = \frac{\left(1-\frac{v^2}{c^2}\right)\left(1-\frac{u'^2}{c^2}\right)}{\left(1+\frac{u'v}{c^2}\right)^2}$$

$$\left(1+\frac{u'v}{c^2}\right)^2 = \frac{\left(1-\frac{v^2}{c^2}\right)\left(1-\frac{u'^2}{c^2}\right)}{1-\frac{u_1^2}{c^2}}$$

$$\left(1+\frac{u'v}{c^2}\right) = \sqrt{\frac{\left(1-\frac{v^2}{c^2}\right)\left(1-\frac{u'^2}{c^2}\right)}{1-\frac{u_1^2}{c^2}}}$$

Similarly, $$\left(1-\frac{u'v}{c^2}\right)=\sqrt{\frac{\left(1-\frac{v^2}{c^2}\right)\left(1-\frac{u'^2}{c^2}\right)}{1-\frac{u_2^2}{c^2}}}$$

Putting these values in equation (3),

$$\frac{m_1}{m_2}=\sqrt{\frac{1-\frac{u_2^2}{c^2}}{1-\frac{u_1^2}{c^2}}}$$

Now if second particle is moving with zero velocity in frame S before collision

then $u_2=0$ & $m_2=m_0(\text{Rest Mass})$

Then, $$m_1=\frac{m_0}{\sqrt{1-\frac{u_1^2}{c^2}}}$$

Since two particles are exactly identical therefore replacing m_1 by m & u_1 by v i.e. $m=\frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$. This is the relativistic formula for variation of mass with velocity.

Cases:

(1) $m=\frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$, if $v=c$, $m=\infty$, which is not possible.

It means no material particle can travel with velocity of light.

(2) $v \ll c$, $\frac{v^2}{c^2} \ll 1$, $m=m_o$, Hence in daily life we do not observe mass variation.

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Lecture 7:
Einstein's mass energy relation

Ques: Derive Einstein's mass energy relation.
Ans: Einstein mass energy equivalence
According to Einstein mass can be converted into energy and energy into mass. This is called as mass energy equivalence and can be given by the relation

$$\mathrm{E} = \mathrm{mc}^2$$

Derivation:
According to variation of mass with velocity we have,

$$\mathrm{m} = \frac{\mathrm{m}_0}{\sqrt{1 - \frac{\mathrm{v}^2}{\mathrm{c}^2}}} \qquad \dots\dots(1)$$

We knows that force is the rate of change of momentum

$$F = \frac{dP}{dt} = \frac{d(mv)}{dt} = m\frac{dv}{dt} + v\frac{dm}{dt}$$

According to theory of relativity, mass of the particle varies with velocity, hence m and v both are variable.
If a force F displaces the particle the particle through a small distance ds, then work done, is stored by the particle as its kinetic energy dK. Therefore,

$$\begin{aligned} dW = dK &= F.ds \\ &= \left(m\frac{dv}{dt} + v\frac{dm}{dt}\right).ds \\ &= m.\frac{ds}{dt}.dv + v.\frac{ds}{dt}.dm \\ &= mvdv + v^2dm \qquad \dots\dots(2) \end{aligned}$$

Differentiating equation (1) we get

$$\begin{aligned} \mathrm{dm} &= \mathrm{m}_0\left(-\frac{1}{2}\right)\left(1 - \frac{\mathrm{v}^2}{\mathrm{c}^2}\right)^{-\frac{3}{2}}\left(-\frac{2\mathrm{v}}{\mathrm{c}^2}\mathrm{dv}\right) \\ &= \frac{\mathrm{m}_0}{\mathrm{c}^2}\frac{\mathrm{vdv}}{\left(1 - \frac{\mathrm{v}^2}{\mathrm{c}^2}\right)^{\frac{3}{2}}} \\ &= \frac{\mathrm{m}\left(1 - \frac{\mathrm{v}^2}{\mathrm{c}^2}\right)^{\frac{1}{2}}\mathrm{vdv}}{\mathrm{c}^2\left(1 - \frac{\mathrm{v}^2}{\mathrm{c}^2}\right)^{\frac{3}{2}}} \\ &= \frac{\mathrm{mvdvc}^2}{\mathrm{c}^2(\mathrm{c}^2 - \mathrm{v}^2)} \\ &= \frac{\mathrm{mvdv}}{(\mathrm{c}^2 - \mathrm{v}^2)} \end{aligned}$$

$$mvdv = (c^2 - v^2)dm,$$

putting in eq. 2

$$dK = (c^2 - v^2)dm + v^2dm = c^2dm$$

If the change in kinetic energy of the particle be K, when its mass changes from rest mass m_0 to mass m then

$$K = \int dK = \int_{m_0}^{m} c^2 dm$$

$$K = (m - m_0)c^2 \quad \ldots\ldots..(3)$$

Now the total energy will be

$$E = K + E_0$$

$$E = (m - m_0)c^2 + m_0c^2$$

$$\mathbf{E = mc^2}$$

This is Einstein's mass energy relation.

Lecture 8:

Relativistic relation between energy and momentum, Massless particle

Ques: Derive relativistic relation between energy and momentum.

Ans: We know that

$$\mathrm{E = mc^2 = \frac{m_0c^2}{\sqrt{1-\frac{v^2}{c^2}}}} \quad \ldots\ldots(1)$$

And relativistic momentum $\mathrm{p = mv = \frac{m_0 v}{\sqrt{1-\frac{v^2}{c^2}}}}$ ......(2)

So, $\mathrm{pc = \frac{m_0vc}{\sqrt{1-\frac{v^2}{c^2}}}}$ ......(3)

$$\mathrm{E^2 - p^2c^2 = \frac{m_0^2c^4 - m_0^2v^2c^2}{\left(1-\frac{v^2}{c^2}\right)} = m_0^2c^4}$$

$$\mathrm{E^2 = c^2p^2 + m_0^2c^4}$$

This is relativistic relation between energy and momentum of particle.

Ques: What are massless particles?

Ans: Massless particles:

Particles with zero rest mass are called mass less particles.

Classically such particle can't exist, however in relativity it's possible.

From relativity,

$$\mathrm{E = mc^2 = \frac{m_0c^2}{\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}}}} \quad \ldots\ldots(1)$$

& $\mathrm{p = mv = \frac{m_0v}{\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}}}}$ .....(2)

<u>**Case of photon.**</u>

We know that $\mathrm{E^2 = c^2p^2 + m_0^2c^4}$

For massless particles, here photon, Rest mass $m_o = 0$, mass of photon = $\frac{E}{c^2}$

So, Energy of massless particle $\mathrm{E^2 = c^2p^2 + 0}$; => $\mathrm{E = pc}$

Momentum of massless particle p = $\frac{E}{c} = \frac{h\nu}{c}$

Q1. What will be the fringe-shift according to the ether theory in the Michelson-Morley experiment, if the effective path length of each path is 7 meters and light has 7000 Å wavelength? The velocity of earth is 3×10^4 m/sec.

Solution:

From fringe shift expression $\Delta n = \frac{2lv^2}{c^2} . \frac{1}{\lambda}$

$$\Delta n = \frac{2 \times 7 \times \left(3 \times 10^4\right)^2}{(3 \times 10^8)^2 \times (7 \times 10^{-7})} = 0.2$$

Q2. Calculate the expected fringe shift in Michelson-Morley experiment if the distance of each path is 11 meter and the wavelength of light is $5.6 \times 10^{-7} m$ & the experimental setup was not rotated through $90°$. The linear velocity of earth may be taken as $30 kms^{-1}$.

Solution: if the set up was not rotated by 90°, then from fringe shift expression $\Delta n = \frac{lv^2}{c^2} . \frac{1}{\lambda}$

$$\Delta n = \frac{11 \times (30 \times 10^3)^2}{(3 \times 10^8)^2 \times (5.6 \times 10^{-7})} = 0.196$$

Q3 Use Galilean transformation to prove that distance between two points (x_1, y_1, z_1) and (x_2, y_2, z_2) is invariant in two inertial frames.

Solution: Suppose a frame of reference S' is moving with velocity v relative to frame s at rest, such that $\vec{v} = iv_x + jv_y + kv_z$. Let the co-ordinates of two points in frame S be (x_1, y_1, z_1) and $\left(x_2, y_2, z_2\right)$; While those in frame S' be (x'_1, y'_1, z'_1) and (x'_2, y'_2, z'_2)

From Galilean transformations, we have

$$x'_1 = x_1 - v_x t \text{ ------(1)}$$
$$y'_1 = y_1 - v_y t \text{ ------(2)}$$
$$z'_1 = z_1 - v_z t \text{ ------ (3)}$$

and

$$x'_2 = x_2 - v_x t \text{ ------ (4)}$$
$$y'_2 = y_2 - v_y t \text{ ------ (5)}$$
$$z'_2 = z_2 - v_z t \text{ -------(6)}$$

The distance between two points in moving frame S'

$$d = \sqrt{(x'_2 - x'_1)^2 + (y'_2 - y'_1)^2 + (z'_2 - z'_1)^2} \text{ ------------------(7)}$$

Now using equations (1),(2),(3),(4),(5) and (6) in equation (7) we get

$$d = \sqrt{\{[(x_2 - v_x t) - (x_1 - v_x t)]^2 + [(y_2 - v_y t) - (y_1 - v_y t)]^2 + [(z_2 - v_z t) - (z_1 - v_z t)]^2\}}$$

$$d = \sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]}$$

= The distance between two points in stationary frame S.

Hence, the distance between any two points is invariant under Galilean transformation.

Q4. Show that acceleration is invariant under Galilean transformation equation:

According to **Galilean transformation equation**

$x' = x - vt \;\&\; t' = t$

Differentiating above equation keeping *v* constant, we get

$dx' = dx - vdt$ . . .1

$dt' = dt$. . .2

Dividing, we get $\frac{dx\prime}{dt\prime} = \frac{dx}{dt} - v$. . .3

But $\frac{dx}{dt} and \frac{dx\prime}{dt\prime}$ are the velocities of the particle relative to system S and S', respectively

i.e. $\frac{dx}{dt} = u, \frac{dx\prime}{dt\prime} = u'$.

Hence from 3, the Galilean transformation for velocity may be expressed as $u = u - v$. . . 4

Equation 4 represents Galilean transformation equation for velocity of the particle. This clearly indicates that the velocity is not invariant under Galilean transformation. Differentiating above equation , we get

$$\frac{du'}{dt} = \frac{du}{dt} or \ \frac{du'}{dt'} = \frac{du}{dt} \quad (since\ dt' = dt)$$

But $\frac{du}{dt} = \frac{du'}{dt}$ are acceleration of particle in system S and S', respectively i.e., $a' = a$ i.e. *acceleration observed by observers in different inertial frames is the same.* Thus Newton's laws are valid in every inertial frame or acceleration is invariant under Galilean transformations.

Q5. Derive inverse Lorentz transformation.

Solution.

$$x' = \frac{x - vt}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$y' = y$$

$$z' = z$$

$$t' = \frac{t - \frac{xv}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}$$

Solving for x and t, we find

$$x = x'\sqrt{1 - \frac{v^2}{c^2}} + vt \qquad \text{.........(1)}$$

$$t = t'\sqrt{1 - \frac{v^2}{c^2}} + \frac{xv}{c^2} \qquad \text{..........(2)}$$

Substituting the value of t from equation (2) in equation (1) and solving, we get

$$x = x'\sqrt{1 - \frac{v^2}{c^2}} + v\left[t'\sqrt{1 - \frac{v^2}{c^2}} + \frac{xv}{c^2}\right]$$

$$= (x' + vt')\sqrt{1 - \frac{v^2}{c^2}} + \frac{xv^2}{c^2}$$

$$\left(x - \frac{xv^2}{c^2}\right) = (x' + vt')\sqrt{1 - \frac{v^2}{c^2}}$$

$$x = \frac{x' + vt'}{\sqrt{1-\frac{v^2}{c^2}}} \qquad \text{......... (3)}$$

Similarly, substituting the value of x from equ. (3) in equ. (2), we have

$$t = t'\sqrt{1 - \frac{v^2}{c^2}} + \frac{v}{c^2}\left[\frac{x' + vt'}{\sqrt{1-\frac{v^2}{c^2}}}\right] = t'\sqrt{1 - \frac{v^2}{c^2}} + \frac{\frac{vx'}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} + \frac{\left(\frac{v^2}{c^2}\right)t'}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$t = t'\sqrt{1 - \frac{v^2}{c^2}}\left[1 + \frac{\frac{v^2}{c^2}}{1-\frac{v^2}{c^2}}\right] + \frac{\frac{vx'}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$t = \frac{t + \frac{vx'}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} \qquad \text{......... (4)}$$

Hence Lorentz inverse transformation equations are

$$x = \frac{x' + vt'}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$y' = y$$

$$z' = z$$

$$t = \frac{t + \frac{vx'}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}$$

Q6. Show that space-time interval, $x^2 + y^2 + z^2 - c^2t^2$,
is invariant under Lorentz transformation.
[B.Tech. I Sem (C.O.) 2003, II Sem. 2005, II Sem. 2006]

Solution:

we have to prove that $x^2 + y^2 + z^2 - c^2t^2 = x'^2 + y'^2 + z'^2 - c^2t'^2$ ...1

The Lorentz transformation equations are

$$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}}, y' = y, z' = z, and\ t' = \frac{t - \frac{xv}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Putting these values in eq. 1 we get

$$x^2 + y^2 + z^2 - c^2t^2 = \frac{(x - vt)^2}{\left(1 - \frac{v^2}{c^2}\right)} + y^2 + z^2 - \frac{c^2\left[t - \left(\frac{xv}{c^2}\right)\right]^2}{1 - \frac{v^2}{c^2}}$$

$$= \frac{1}{\left(1 - \frac{v^2}{c^2}\right)}\left[x^2 + v^2t^2 - 2xvt - c^2\left(t^2 + \frac{x^2v^2}{c^4} - \frac{2txv}{c^2}\right)\right] + y^2 + z^2$$

$$= \frac{1}{\left(1 - \frac{v^2}{c^2}\right)}\left[x^2 + v^2t^2 - 2xvt - c^2t^2 - \frac{x^2v^2}{c^2} + 2txv\right] + y^2 + z^2$$

$$= \frac{1}{\left(1 - \frac{v^2}{c^2}\right)}\left[x^2\left(1 - \frac{v^2}{c^2}\right) - c^2t^2\left(1 - \frac{v^2}{c^2}\right)\right] + y^2 + z^2$$

$$= [x^2 - c^2t^2] + +y^2 + z^2$$

$$x^2 + y^2 + z^2 - c^2t^2 = x'^2 + y'^2 + z'^2 - c^2t'^2$$

Hence $x^2 + y^2 + z^2 - c^2t^2$, is invariant under Lorentz transformation.

Q7. Discuss the concept of simultaneity in Relativity.

Ans. The simultaneity of two events means their occurrence at exactly the same time. Let us see whether two events occurring simultaneously in a stationary frame S also appear to be so in frame S', moving with constant velocity *v* with respect to S along positive direction of x.

Let A and B be two points along the x-axis in the stationary frame S, at distance x_1 and x_2 from the origin, with P as their mid-point. Then a flash of a light emitted at P will arrive at A and B at exactly the same time t, because the speed of light is same in all directions. The two events i.e. the arrival of the light signals thus occurs simultaneously at A and B in frame S.

To an observer in the moving frame S', the corresponding value of **t** for the arrival of the signal at A and B will be accordance with Lorentz transformation equation to be,

$$t_1' = \frac{t - \frac{x_1v}{c^2}}{\sqrt{1 - \left(\frac{v^2}{c^2}\right)}}.$$

$$t_2' = \frac{t - \frac{x_2v}{c^2}}{\sqrt{1 - \left(\frac{v^2}{c^2}\right)}}.$$

Since $x_1 \neq x_2$ and therefore $t_1' \neq t_2'$

Therefore the two events will not appear to be simultaneous to the observer in S'. In fact the time interval between the two events will appear to be.

$$\Delta t = t_2' - t_1' = \frac{v(x_1 - x_2)/c^2}{\sqrt{1-\left(\frac{v^2}{c^2}\right)}}$$

In exactly same manner, the two events occurring simultaneously in moving frame S' will not be appear to be so to an observer in the stationary frame S.

Q8. A train whose length is 150m when at rest has to pass through a tunnel of length 125m. The train is moving with uniform speed of $2.4 \times 10^8 m/sec$ towards the tunnel. Find the length of the train and that of the tunnel as observed by an observer (i) at the tunnel, (ii) at the train. [UPTU, B.Tech. I Sem.2007]

OR

A train whose length is 150 m at rest has to pass through a tunnel of length 125m. The train is moving towards the tunnel of length 125m. The train is moving towards the tunnel with a uniform speed of 0.8 time the speed of light (i.e. 2.4$\times 10^8 m/sec$). Find the length of the train and the tunnel as seen by an observer at the train.

Solution:

(i) from length contraction formula $L = L_0\sqrt{\left(1 - \frac{v^2}{c^2}\right)}$

Here $L = L_1'$(length of the train as appeared to an observer at the tunnel), $L_0 = 150m$ $and\ v = 2.4 \times 10^8 m/sec.$

$$L_1' = 150\sqrt{\left(1 - \left(\frac{2.4 \times 10^8}{3 \times 10^8}\right)^2\right)} = 150 \times 0.6 = 90\ m$$

As the tunnel is at rest, the length of the tunnel appeared to the observer at the tunnel will remain the same, i.e. 125m.

(ii) observer in train will observe the tunnel coming towards him with velocity $2.4 \times 10^8 m/sec.$ Therefore the length L_2' of the tunnel as appeared to the observer in the train will be

$L_2' = L_1'\sqrt{\left(1 - \frac{v^2}{c^2}\right)}$ $=125\sqrt{\left(1 - \left(\frac{2.4\times10^8}{3\times10^8}\right)^2\right)} = 125 \times 0.6 = 75\ m.$

Since the train is at rest with respect to the observer, the length of the train appeared to the observer in the train will remain the same, i.e. 150 m.

Q9. A rod has length 100 cm. When the rod is in a satellite with a velocity that is one half of the velocity of light relative to laboratory. What is the length of the rod as determined by an observer (a) in the satellite, and (b) in the laboratory?

Solution. (a) The rod is at rest relative to an observer in the satellite.

The length of the rod as determined by an observer in the satellite is 100 cm.

(b) For an observer in the laboratory, i.e. in the stationary frame of reference, the rod is in motion. Hence, according to length contraction

$$L = L_o\sqrt{1 - \left(\frac{v^2}{c^2}\right)}$$

Given that $L_o = 100\ cm$ and $v = 0.5\ c$

$$L = 100\sqrt{1-(\frac{0.5c}{c})^2}$$
$$L = 100\sqrt{1-0.25}$$
$$L = 86.6\ cm$$

Q10. What will the length of a meter rod appear to be for a person travelling parallel to the length of the rod at a speed of 0.8c relative to the rod?

Solution. According to length contraction formula,

$$L = L_o\sqrt{1-\left(\frac{v^2}{c^2}\right)}$$

Given that $v\ =\ 0.8c$ and $L_o = 1\ m$

$L = 1\sqrt{1-(\frac{0.8c}{c})^2}$ $=1\sqrt{0.36} = 1 \times 0.6 = 0.6\ m\ =\ 60\ cm$

Q11. Calculate the percentage contraction in the length of a rod in a frame of reference, moving with velocity 0.8c in a direction parallel to its length.

Solution.

According to length contraction,

$$L = L_o\sqrt{1-\left(\frac{v^2}{c^2}\right)}$$

Here $v = 0.8c$ and $L_o = L_o$

$L = L_o\sqrt{1-(\frac{0.8c}{c})^2}$ $= =L_o\sqrt{0.36} = 0.6L_o$

Percentage contraction in length = $\frac{L_o-L}{L_o} \times 100$

Hence % Contraction = $\frac{L_o-0.6L_o}{L_o} \times 100\ =\ 40\%.$

Q12. Show that the circle, $x^2+y^2=a^2$ in frame S appears to be an ellipse in frame s' which is moving with velocity v relative to S.

Solution.

The equation of a circle in stationary frame S is given by: $x^2+y^2=a^2$, where a is constant.

According to length contraction

$x = x'\left(1-\left(\frac{v^2}{c^2}\right)\right)^{1/2}$ And $y = y'$

Putting the values of x and y in the equation of circle, we have

$x'^2\left(1-\frac{v^2}{c^2}\right)+y'^2 = a^2$

or

$$\frac{x'^2}{a^2/\left(1-\frac{v^2}{c^2}\right)}+\frac{y'^2}{a^2} = 1$$

Let$\left(a^2/\left(1-\frac{v^2}{c^2}\right)\right) = b^2,$

so

$$\frac{x'^2}{b^2}+\frac{y'^2}{a^2} = 1$$

Hence, in the moving frame s' the circle will appear as an ellipse.

Q13.A circular lamina moves with its plane parallel to the X-Y plane of a reference frame S at rest. Assuming its motion to be along the axis of x (or y), calculate the velocity, at which its surface area would appear to be reduced to half to an observer in frame S at rest.

Solution.

Let D be the diameter of a circular lamina, then its surface area in stationary frame S is

Surface area = $\pi r^2 = \pi(D/2)^2$

As we know that in a moving frame, the circle will appear an ellipse. Here the circular lamina is moving with its plane parallel to the x-y plane of a stationary frame S along the axis of x (or y). It will appear an elliptical lamina, because the length of its side along which it is moving is contracted.

i.e.

$$D_x = D\sqrt{1-\left(\frac{v^2}{c^2}\right)}$$

Area of an elliptical lamina = $\pi\left(\frac{D}{2}\right)\left(\frac{D}{2}\sqrt{1-\left(\frac{v^2}{c^2}\right)}\right) = \left(\frac{\pi D^2}{4}\sqrt{1-\left(\frac{v^2}{c^2}\right)}\right)$

For the given problem.

$$Surface\ area\ of\ elliptical\ lamina = \frac{1}{2}\ Circular\ lamina$$

i.e. $\left(\frac{\pi D^2}{4}\sqrt{1-\left(\frac{v^2}{c^2}\right)}\right)=\frac{1}{2}\left(\frac{\pi D^2}{4}\right)$

$\left(\sqrt{1-\left(\frac{v^2}{c^2}\right)}\right)=\frac{1}{2}$

$1-\frac{v^2}{c^2} = \frac{1}{4}$

$\frac{v^2}{c^2} = 1-\frac{1}{4}$

$v^2 = \frac{3}{4}c^2$

$v = \sqrt{\left(\frac{3}{4}c^2\right)} = \sqrt{\left(\frac{3}{4}\right)} \times 3 \times 10^8 = 2.6 \times 10^8$ m/sec

Q14. Calculate the percentage contraction of a rod moving with a velocity of 0.8c in a direction inclined at 60^o to its own length.

Solution.

Let L_o be the length of the rod in the frame S in which it is at rest. Let S' be the frame moving with a speed of 0.8c in a direction making angle 60^o with X-axis. The components of L along and perpendicular to the direction of motion are $L_o\cos 60^o$ and $L_o\sin 60^o$ respectively.

The apparent length of the rod along the direction of motion

$= L_o\cos 60^o\sqrt{[1-(0.8)^2]}$

$=\frac{L_o}{2}\sqrt{[1-0.64]} = \frac{L_o}{2} \times 0.6 = 0.3L_o$

The apparent length of the rod perpendicular to the direction of motion

$= L_o\sin 60^o = \frac{\sqrt{3}}{2}L_o$ ($\because$ No change in perpendicular direction)

$\therefore$ Length of the moving rod in the moving frame.

$$L = \left[(0.3L_o)^2 + \left(\frac{L_o\sqrt{3}}{2}\right)^2\right]^{1/2} = 0.916L_o$$

Percentage contraction in length = $\frac{L_o - L}{L_o} \times 100$

Hence % Contraction = $\frac{L_o - 0.916L_o}{L_o} \times 100 \; = \; 8.4\%.$

Question 15. Calculate the length and orientation of a rod of length 2 meter in a frame of reference which is moving with 0.6 c velocity in a direction making an angle of 30^o with the rod.

Solution.

The length of the rod along the direction of the moving frame of reference = $L_o \cos 30^o$

The apparent length of the rod along the direction of motion.

$L_x = L_o \cos 30^o \sqrt{\left(1 - \frac{v^2}{c^2}\right)}$ = $\frac{2\sqrt{3}}{2}\sqrt{\left(1 - \frac{(0.6c)^2}{c^2}\right)}$

= $\sqrt{3}(\sqrt{1-(0.6)^2}$) = 1.73× 0.8 = 1.38 m

Apparent length in a direction perpendicular to the direction,

$L_y = L_o \sin 30^o$ = 2× $\frac{1}{2} = 1m$

∴ The length of the rod in a moving frame in a direction making an angle of 30^o with rod.

$$L = \sqrt{L_x^2 + L_y^2} = \sqrt{(0.38)^2 + (1)^2} = 1.704\, m$$

If the rod makes an angle θ with X-axis in the moving frame, then

$$\tan\theta = \frac{L_y}{L_x} = \frac{1}{1.38} = 0.72$$

$\theta = \tan^{-1}(0.72)$ =35.8^0

Q16.- At what speed should a clock be moved so that it may appear to lose 1 min in each hour?

Solution: $t = \frac{t_o}{\sqrt{1-\frac{v^2}{c^2}}}$

t_0 = 59min and t = 60 min

$$v = c\sqrt{1 - \frac{t_0^2}{t^2}} = 3 \times 10^8 \times \sqrt{1 - \left(\frac{59}{60}\right)^2}$$

$$v = 3 \times 10^8 \times 0.1816 = 5.45 \times 10^7 m/s$$

Q17.- A clock measures the proper time. With what velocity it should travel relative to an observer so that it appears to go slow by 30 sec in a day?

Solution: $t = \frac{t_o}{\sqrt{1-\frac{v^2}{c^2}}}$

t = 24 hours = 86400 sec

t_0 = 23 hours 59 min 30 sec = 86370 sec

$$v = c\sqrt{1 - \frac{t_0^2}{t^2}} = 3 \times 10^8 \times \sqrt{1 - \left(\frac{86370}{86400}\right)^2}$$

$$v = 3 \times 10^8 \times 0.0264 = 7.92 \times 10^6 m/s$$

Q18.- The mean life of a meson is 2× 10^{-8} sec. Calculate the mean life of a meson moving with a velocity 0.8c.

Solution: $t = \frac{t_o}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{2\times10^{-8}}{\sqrt{1-\left(\frac{0.8c}{c}\right)^2}} = 3.33 \times 10^{-8} sec$

Q.- A clock keeps correct time. With what speed should it be moved relative to an observer so that it may appear to lose 4 min in 24 hours?

Solution: $t = \frac{t_o}{\sqrt{1-\frac{v^2}{c^2}}}$

$$1444 = \frac{1440}{\sqrt{1-\frac{v^2}{c^2}}} = 1440\left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}}$$

$$1444 = 1440\left(1+\frac{v^2}{2c^2}\right) \qquad \text{(neglecting higher order terms)}$$

$$\frac{v^2}{2c^2} = 0.003$$

$$v^2 = 0.003 \times 2 \times (3 \times 10^8)^2$$

$$v = 2.32 \times 10^7 m/s$$

Q19 At what speed will the mass of a body be 2.25 times its rest mass?

Solution: Suppose at speed *v*, the mass of an object is 2.25 times to its value at rest. Then according to the variation of mass with velocity formula,

$$m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$$

Here $m = 2.25m_0$

Therefore $2.25m_0 = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$

$$1-\frac{v^2}{c^2} = \frac{1}{5.06} \quad \text{or} \quad \frac{v^2}{c^2} = 1 - \frac{1}{5.06} = \frac{4.06}{5.06}$$

$$v = c \times \sqrt{\frac{4.06}{5.06}} = \frac{3\times10^8}{1}\frac{\times\sqrt{4.06}}{\sqrt{5.06}} = 2.68 \times 10^8 \text{m/sec}$$

Q20.A particle has a velocity u'=3i+4j+12k m/sec in a coordinate system moving with velocity .8c relative to laboratory along +ve direction of x-axis. Find u in laboratory frame.

Solution: In the given problem velocity u'= 3i+4j+12k m/sec, therefore $u'_x=3, u'_y = 4, u'_z = 12 \text{ and } v = .8c$.According to the law of addition of velocities, the X,Y and Z components of u in laboratory frame is

$$u_x = \frac{u'_x + v}{1+\frac{vu'_x}{c^2}} = \frac{3+.8c}{1+\frac{(.8c)(3)}{c^2}} = \frac{(3+.8c)c}{c+2.4} \cong .8c = 2.4\times10^{-8} \text{ m/sec}$$

$$u_y = \frac{u'_y\sqrt{1-v^2/c^2}}{1+\frac{vu'_x}{c^2}}$$

$$= \frac{4\sqrt{1-(\frac{.8c}{c})^2}}{1+\frac{(.8c)(3)}{c^2}} = \frac{4\times.6c}{c+2.4} \cong \frac{2.4\,c}{c} = 2.4 \text{ m/sec}$$

$$u_z = \frac{u'_z\sqrt{1-v^2/c^2}}{1+\frac{vu'_x}{c^2}}$$

$$= \frac{12\sqrt{1-(\frac{.8c}{c})^2}}{1+\frac{(.8c)(3)}{c^2}} = \frac{12\times.6c}{c+2.4} \cong \frac{7.2c}{c} = 7.2 \text{m/sec}$$

Therefore u in the laboratory frame is given by

$$u = u_x i + u_y \text{ j } + u_z \text{ k}$$

$$=(2.4 \times 10^8 \text{I} + 2.4\text{j} + 7.2\text{k})\text{m/sec}$$

Q21 Derive relativistic velocity transformation.

Solution: Lorentz transformation equations are

$$x'=\frac{x-vt}{\sqrt{1-\frac{v^2}{c^2}}}\ ,\qquad y'=y,\quad z'=z \quad \text{and } t'=\frac{t-\frac{xv}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} \text{ ------------------------(1)}$$

$$u'_x=\frac{dx'}{dt'}\ ,u'_y=\frac{dy'}{dt} \text{ and } u'_z=\frac{dz'}{dt'}$$

Differentiations of equation(1) give

$$dx' = dx\ ,\quad dy'=dy\ ,\ dz'=dz \text{ and } dt' = \frac{dt-\left(\frac{v}{c^2}\right)dx}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$u'_x=\frac{dx'}{dt'}=\frac{dx-vdt}{\sqrt{1-\frac{v^2}{c^2}}}/\frac{dt-\left(\frac{v}{c^2}\right)dx}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$u'_x=\frac{\frac{dx}{dt}-v}{1-\frac{vdx}{c^2dt}}=\frac{u_x-v}{1-\frac{vu_x}{c^2}}$$

$$u'_y=\frac{dy'}{dt'}=dy/\frac{dt-\left(\frac{v}{c^2}\right)dx}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$=\frac{dy/dt}{1-\frac{v}{c^2}}\frac{\sqrt{1-\frac{v^2}{c^2}}}{dx/dt}$$

$$u'_y=\frac{\sqrt{1-\frac{v^2}{c^2}}\ u_y}{1-\frac{vu_x}{c^2}}$$

$$\text{And } u'_z=\frac{dz'}{dt'}=dz/\frac{dt-\left(\frac{v}{c^2}\right)dx}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$=\frac{dz/dt}{1-\frac{v}{c^2}}\frac{\sqrt{1-\frac{v^2}{c^2}}}{dx/dt}$$

$$u'_z=\frac{\sqrt{1-\frac{v^2}{c^2}}\ u_z}{1-\frac{vu_x}{c^2}}$$

Q22- A man weighs 50kg on the earth. When he is in rocket ship in flight his mass is 50.5 kg as measured by an observer on earth. What is the speed of rocket?

Solution: According to the variation of relativistic mass with velocity formula

$m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$ or $\sqrt{1-\frac{v^2}{c^2}} = \frac{m_0}{m}$

Or $v = c\sqrt{1-\frac{{m_0}^2}{m^2}}$

$V = 3\times10^8\sqrt{1-\frac{(50)^2}{(50.5)^2}} = 3\times10^8\times0.141$

$V= 4.23\times10^7$ m/s

Q23- What is the length of a meter stick moving parallel to its length when its mass is 3/2 times of its rest mass .

Solution: In the given problem, the mass of the rod is 3/2 times its rest mess, that is,

$m=\frac{3}{2}m_0$ or $\frac{3}{2}m_0 = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$

$$\sqrt{1-\frac{v^2}{c^2}} = \frac{2}{3} = 0.667$$

The length of a meter stick moving parallel to its length, according to length contraction formula

$L_0\sqrt{1-\frac{v^2}{c^2}} = L$, Here $L_0 = 1$ m

$L = 1 \times 0.667$

$L = 0.667m.$

Q24- Calculate the rest mass, relativistic mass and momentum of a photon of energy 5eV.

Solution: Energy of the photon, E= 5eV= 5×1.6× 10^{-19} J= 8×10^{-19} J

Momentum of the photon, $p = \frac{E}{c} = \frac{8\times10^{-19}}{3\times10^{8}} = 2.67 \times 10^{-27}$kg ms^{-1}

Relativistic mass of the photon,$m = \frac{p}{c} = \frac{2.67\times10^{-27}}{3\times10^{8}} = 8.9 \times 10^{-36}$kg

According to the variation of relativistic mass with velocity formula

$m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$ or $\sqrt{1-\frac{v^2}{c^2}} = \frac{m_0}{m}$

Velocity of the photon v=c

Rest mass of the photon, $m_0 = 0.$

Q25- If the total energy of a particle is exactly is thrice its rest energy, what is the velocity of the particle?

Solution: According to the Einstein mass energy relation,

$$E = mc^2$$

Here total energy =3×rest energy, therefore,

$E = 3m_0c^2$ or $mc^2 = 3m_0c^2$ or $m = 3m_0$

According to the variation of mass with velocity relation

$$\sqrt{1-\frac{v^2}{c^2}} = \frac{m_0}{m} = 3m_0$$

$\sqrt{1-\frac{v^2}{c^2}} = \frac{1}{3}$ or $1-\frac{v^2}{c^2} = \frac{1}{9}$

$v^2 = \frac{8}{9} \times c^2$ or $v = \frac{2\sqrt{2}}{3}c$

$v = \frac{2\sqrt{2}\times3\times10^8}{3} = 2.828 \times 10^8 ms^{-1}.$

Q26. If the kinetic energy of a body is twice its rest mass energy, find its velocity. [UPTU, B.Tech. I Sem 2003, II Sem. (C.O.) 2004, GBTU B.Tech. I Sem. (old) 2010]

Solution: Relativistic kinetic energy can be expressed as

$$K = (m - m_0)c^2 \text{ or } mc^2 = K + m_0c^2 \qquad ...(1)$$

Given that, $K= 2m_0c^2$

From (1), $$mc^2 = 3m_0c^2$$

$$m = 3m_0 \qquad ...(2)$$

According to the variation of mass with velocity, $m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$

$$1-\frac{v^2}{c^2} = \left(\frac{m_0}{m}\right)^2$$

$$v = c\sqrt{\left[1-\left(\frac{m_0}{m}\right)^2\right]} = 3\times10^8\sqrt{1-\left(\frac{1}{3}\right)^2}$$

using (2)

$$v = 2.829 \times 10^8 \text{ m/s}$$

Q27. A particle of rest mass m_0 moves with speed c⁄√2. Calculate its mass, momentum, total energy and kinetic energy. [UPTU, B.Tech. I Sem. 2006]

Solution: According to the variation of mass with velocity, $m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{m_0}{\sqrt{1-\frac{c^2}{2c^2}}} = \sqrt{2}m_0 = 1.41m_0$

Momentum of particle, $p = mv = \sqrt{2}m_0 \times c/\sqrt{2} = m_0 c$

Total energy of the particle is given by,

$$K = E - m_0c^2 = 1.41m_0c^2 - m_0c^2 = 0.41m_0c^2$$

Q28. The mass of a moving electron is 11 times of its rest mass. Find the kinetic energy and momentum. [UPTU, B.Tech. I Sem and II Sem 2002, II Sem. 2007, GBTU B.Tech. I Sem. 2010, I Sem. 2012]

Solution: Given that, $m = 11\ m_0$ ....(1)

Relativistic kinetic energy can be expressed as

$$K = (m - m_0)c^2 = 11m_0c^2 - m_0c^2 = 10\ m_0c^2$$
$$= 10 \times 9.1 \times 10^{-31}(3 \times 10^8)^2$$
$$= 8.2 \times 10^{-13} joule = \frac{8.2 \times 10^{-13}}{1.6 \times 10^{-19}} eV = 5.12 \times 10^6 eV$$

According to the variation of mass with velocity, $m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$

$$1 - \frac{v^2}{c^2} = \left(\frac{m_0}{m}\right)^2$$
$$v = c\sqrt{\left[1 - \left(\frac{m_0}{m}\right)^2\right]} = 3 \times 10^8\sqrt{1 - \left(\frac{1}{11}\right)^2}$$
$$v = 2.99 \times 10^8 \text{ m/s}$$

Now momentum, $p = mv = 11m_0 \times 2.99 \times 10^8 = 11 \times 9.1 \times 10^{-31} \times 2.99 \times 10^8 = 2.99 \times 10^{-21}$ kg ms^{-1}

Q29. Show that the relativistic form of Newton's second law, when $\vec{F}$ *is parallel to* $\vec{v}$ is $\vec{F} = m.\frac{d\vec{v}}{dt}\left(1 - \frac{v^2}{c^2}\right)^{-\frac{3}{2}}$

Solution: Force acting on a particle is defined as the time rate of change of its momentum,

$$\vec{F} = \frac{d\vec{p}}{dt} = \frac{d(m\,\vec{v})}{dt}$$
$$\vec{F} = \frac{d}{dt}\left(\frac{m_0\,v}{\sqrt{(1-\frac{v^2}{c^2})}}\right)$$
$$\vec{F} = m_0\frac{d}{dt}\left[\vec{v}\left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}}\right]$$
$$= m_0\left\{\frac{d\vec{v}}{dt}\left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} + \vec{v}\frac{d}{dt}\left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}}\right\}$$
$$= m_0\left\{\frac{d\vec{v}}{dt}\left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} + \vec{v}\left(-\frac{1}{2}\right)\left(1 - \frac{v^2}{c^2}\right)^{-\frac{3}{2}}\left(-\frac{2v}{c^2}\frac{d\vec{v}}{dt}\right)\right\}$$
$$= m_0\frac{d\vec{v}}{dt}\left(1 - \frac{v^2}{c^2}\right)^{-\frac{3}{2}}\left[1 - \frac{v^2}{c^2} + \frac{v^2}{c^2}\right]$$
$$= m_0\frac{d\vec{v}}{dt}\left(1 - \frac{v^2}{c^2}\right)^{-\frac{3}{2}}$$

This is required relativistic form of Newton's second law.

Q30. Show that the momentum of a particle of rest mass m_0 *and kinetic energy* k_E is given by the expression.

$$\mathbf{P} = \sqrt{\left(\frac{K_E^2}{c^2} + 2m_0K_E\right)}$$

Solution: The relativistic kinetic energy is given by the relation

$$K_E = E - m_0c^2, \text{ Where } E=\sqrt{m_0^2\,c^4 + p^2c^2}$$

$$= \sqrt{m_0^2 c^4 + p^2c^2} - m_0c^2$$

$$K_E + m_0c^2 = \sqrt{m_0^2 c^4 + p^2c^2}$$

Squaring on both side, we get

$$(K_E + m_0c^2)^2 = m_0^2 c^4 + p^2c^2$$

$$K_E^2 + m_0^2 c^4 + 2\,K_E\, m_0c^2 = m_0^2 c^4 + p^2c^2$$

$$p^2c^2 = K_E^2 + 2\,K_E\, m_0c^2$$

$p^2 = \frac{K_E^2}{c^2} + 2\,K_E\, m_0$ or $p = \sqrt{\frac{K_E^2}{c^2} + 2\,K_E\, m_0}$

Q31. Given the kinetic energy of a relativistic particle, how will you calculate its linear momentum? Only give essential steps. A photon in S′ frame is moving in x-y plane such that its direction of motion makes an angle of θ with the x- axis. Calculate its speed in the S frame which moves with a constant velocity v with respect to S frame and along the x axis of S frame.

Solution: (i) $E^2 = p^2c^2 + m_o^2c^4$

$$(K + m_oc^2)^2 = p^2c^2 + m_o^2c^4$$

$$p = \frac{\sqrt{K^2 + 2m_oc^2K}}{c}$$

(ii) $v_x' = c\, cos\emptyset \quad and \;\; v_y' = c\, sin\emptyset$

$$v_x = \frac{v_x' + V}{1 + v_x'\frac{V}{c^2}} = \frac{c\, cos\emptyset + V}{1 + \frac{V}{c}cos\emptyset} \quad \text{and} \quad v_y = \frac{v_y'\left(1 - \frac{V^2}{c^2}\right)^2}{1 + v_x'\frac{V}{c^2}} = \frac{c\, sin\emptyset\left(1 - \frac{V^2}{c^2}\right)^2}{1 + \frac{V}{c}cos\emptyset}$$

Squaring and adding, we find

$$v_x^2 + v_y^2 = \frac{1}{\left(1 + \frac{V}{c}cos\emptyset\right)^2}\left[(ccos\emptyset + V)^2 + c^2sin^2\emptyset\left(1 - \frac{V^2}{c^2}\right)\right]$$

$$= \frac{1}{\left(1 + \frac{V}{c}cos\emptyset\right)^2}\left[c^2cos^2\emptyset + V^2 + 2cVcos\emptyset + c^2sin^2\emptyset - V^2sin^2\emptyset\right]$$

$$= \frac{1}{\left(1 + \frac{V}{c}cos\emptyset\right)^2}\left[c^2 + V^2 + 2cVcos\emptyset - V^2sin^2\emptyset\right]$$

$$= \frac{c^2\left(1 + \frac{V}{c}cos\emptyset\right)^2}{\left(1 + \frac{V}{c}cos\emptyset\right)^2}$$

$$= c^2$$

Q.32. Calculate the relativistic energy and momentum of a proton which is moving with a speed of 2.4 $\times$ 10^8 m/sec.

Solution: According to mass- energy relation, the relativistic energy E of a proton is given by,

$$E = mc^2 = \frac{m_0\, c^2}{\sqrt{(1 - \frac{v^2}{c^2})}},$$

Given that $v = 2.4 \times 10^8\; m/\sec\, and\; mass\; of\; proton, m_0 = 1.673 \times 10^{-27}\; Kg$

$$E = \frac{1.673\times 10^{-27} \times 9\times 10^{16}}{\sqrt{(1 - \frac{(2.4\times10^8)^2}{(3\times10^8)^2})}} = \frac{1.673\times 10^{-27} \times 9\times 10^{16}}{\sqrt{(1 - 0.64)}} = \frac{1.673\times 10^{-27} \times 9\times 10^{16}}{0.6}$$

= 25.095 $\times$ 10^{-11} = 2.5095 $\times$ 10^{-10} Joule

Momentum of proton is given by,

$$p = mv = \frac{m_0\, v}{\sqrt{(1 - \frac{v^2}{c^2})}} = \frac{1.673\times 10^{-27} \times 2.4\times 10^8}{\sqrt{(1 - \frac{(2.4\times10^8)^2}{(3\times10^8)^2})}}$$

$= \frac{1.673\times 10^{-27} \times 2.4\times 10^8}{0.6}$ = 6.692 $\times$ 10^{-19} $Kg - ms^{-1}$

UNIT 2 ELECTROMAGNETIC THEORY

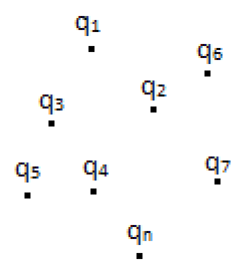
Introduction

Initially electricity and magnetism were studied separately. In 1820 Oersted showed that magnetic field can be produced by electric current, later on Faraday invented phenomenon of electromagnetic induction and showed that electric current can be produced by a time varying magnetic field. In 1864 Maxwell unified both electric and magnetic field and showed that an accelerated charge particle generate electromagnetic waves.

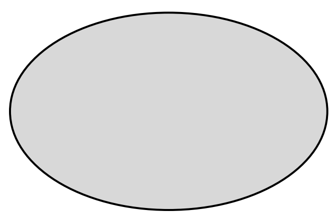
Some important terms

Point Charge – When whole charge is centered at a point

Discrete Charge distribution – The system of point charges separated by a finite distance.



Continuous Charge distribution – The system of charge in which the distance between two successive point charge is zero



Linear charge density – In a continuous line charge distribution, the charge per unit length is known as linear charge density. It is given by

$$\lambda = \frac{q}{l}\ Coulomb/meter$$

Surface charge density – In a continuous surface charge distribution, the charge per unit surface area is known as surface charge density. It is given by

$$\sigma = \frac{q}{S}\ Coulomb/meter^2$$

Volume charge density – In a continuous volume charge distribution the charge per unit volume is known as volume charge density. It is given by

$$\rho = \frac{q}{V}\ Coulomb/meter^3$$

Dell Operator

$$\nabla = \frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}$$

This is a vector

Laplacian Operator

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Laplacian Operator is a scalar

$$Gradiant\ of\ \phi = grad\ \phi = \nabla\phi$$

$$Divergence\ of\ \boldsymbol{A} = div\ \boldsymbol{A} = \nabla.\boldsymbol{A}$$

$$Curl\ of\ \boldsymbol{A} = Curl\ \boldsymbol{A} = \nabla \times \boldsymbol{A}$$

Two Important Tools

1) **Gauss Divergence theorem** – Surface integral of a vector over a closed surface area is equal to volume integral of divergence of the same vector over the volume enclosed by that surface area i.e

$$\oint \boldsymbol{A}.\boldsymbol{dS} = \int (\nabla.\boldsymbol{A})dV$$

2) **Stoke`s theorem -** Line integral of a vector over a closed loop is equal to surface integral of curl of the same vector over the surface area enclosed by that loop i.e

$$\oint \boldsymbol{A}.\boldsymbol{dl} = \int (\nabla \times \boldsymbol{A})\boldsymbol{dS}$$

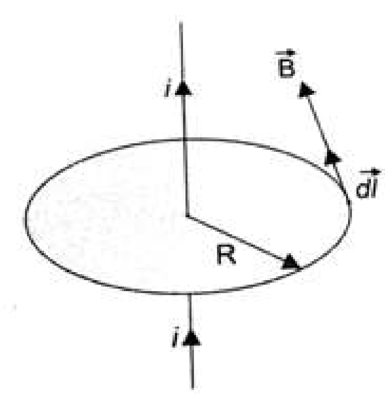
Ampere's Circuital Law

The line integral of the magnetic induction vector around a closed path is equal to μ_0 times of net current crossing the area enclosed by that path

$$\oint \boldsymbol{B}\,.\,\boldsymbol{dl} = \mu_0 I$$

Or

$$\oint \boldsymbol{B}\,.\,\boldsymbol{dl} = \mu_0 \int \boldsymbol{J}.\boldsymbol{ds}$$



Where **J** is current density

Equation of Continuity

Electric current through a surface area is given by

$$I = \int \boldsymbol{J}.\boldsymbol{ds}$$

Where **J** is current density

If the area is closed then

$$I = \oint \boldsymbol{J}\,.\,\boldsymbol{ds} \;\; \textbf{..........................(1)}$$

Consider, a closed surface S enclosing a volume V. If ρ is the volume density inside the volume

Then

$$I = -\frac{dq}{dt}$$

$$I = -\frac{d}{dt}\int \rho dV \text{..........................(2)}$$

-ve sign indicate that charge decreases inside the volume with time

If there is neither source nor sink inside the volume then according to the law of conservation of charge value of electric current given by equations (1) and (2) will be same

Therefore

$$\oint \boldsymbol{J}\,.\,\boldsymbol{ds} = -\frac{d}{dt}\int \rho dV \text{.......................(3)}$$

Applying Gauss divergence theorem on Left Hand Side we get

$$\int \nabla.\boldsymbol{J}\, dV = -\frac{d}{dt}\int \rho dV$$

$$\int \nabla.\boldsymbol{J}\, dV = -\int \frac{\partial \rho}{\partial t} dV$$

Or $$\int \left(\nabla.\boldsymbol{J} + \frac{\partial \rho}{\partial t}\right) dV = 0$$

The integral must be zero for any arbitrary volume element dV. Hence the integrand must be vanish everywhere

$$\nabla.\boldsymbol{J} + \frac{\partial \rho}{\partial t} = 0$$

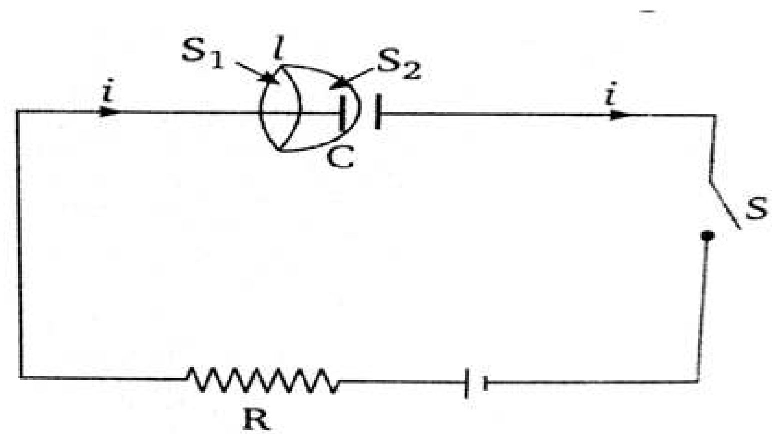
This is called equation of continuity and is a mathematical expression for law of conservation of charge. It states that the total current flowing out of some volume must be equal to the rate of decrease of charge within the volume, assuming that charge can neither be created nor be destroyed.

For steady state

$\frac{\partial \rho}{\partial t} = 0$, therefore $\nabla.\boldsymbol{J} = 0$

Displacement Current

The concept of displacement current was first introduced by Maxwell purely on theoretical ground. The displacement current does not have significance like current in which the charges actually move. To demonstrate the displacement current, Consider an electrical circuit in which a capacitor is charged with a battery of emf E. The current is going to decreasing as the capacitor is charging. When the capacitor is charged up to the voltage equal to emf of battery the current is stopped.



Consider two surfaces S_1 (plane) and S_2 (hemispherical) bounded by common closed path l. let an instant of time t the current is *i*. S_2

Applying Ampere law for surface S_1 we get,

$$\oint \boldsymbol{B}\,.\,\boldsymbol{dl} = \mu_0 I\text{......................(1)}$$

Ampere law for surface S_2 is given by,

$$\oint \boldsymbol{B}\,.\,\boldsymbol{dl} = 0\text{.........................(2)}$$

Because inside the capacitor the dielectric current is zero.

We see that equation (1) & (2) are contradict to each other which is impossible.

Maxwell`s removed this controversy by adding a new factor $\varepsilon_0 \frac{d\phi_E}{dt}$ instead of i in equation (i). This new factor is known as displacement current. Now equation (i) may be written as,

$$\oint \boldsymbol{B}\,.\,\boldsymbol{dl} = \mu_0\left(i + \varepsilon_0 \frac{d\phi_E}{dt}\right)$$

$$\oint \boldsymbol{B}\,.\,\boldsymbol{dl} = \boldsymbol{\mu_0}(\boldsymbol{i} + \boldsymbol{i_d}) \text{ ... ... } \textbf{(3)}$$

The displacement current is equal to the conduction current in the connecting wires.

Let any instant t the charge on the capacitor is q. Then the electric field between the plates of capacitor is given by,

$$E = \frac{\sigma}{\varepsilon_0} = \frac{q}{\varepsilon_0 A} \ldots\ldots\ldots\ldots\ldots\ldots.(4)$$

Where, A→ the surface area of each plate and σ→ surface charge density.

$$\frac{dE}{dt} = \frac{1}{\varepsilon_0 A}\frac{dq}{dt} = \frac{i}{\varepsilon_0 A} \ldots\ldots\ldots\ldots\ldots\ldots(5)$$

$$i = \varepsilon_0 A \frac{dE}{dt}$$

$$= \varepsilon_0 \frac{d(EA)}{dt}$$

$$= \varepsilon_0 \frac{d\phi_E}{dt}$$

$$= \boldsymbol{i_d} \quad \ldots\ldots\ldots\ldots.. (\mathbf{6})$$

Thus the displacement current in the gap is identical with the conduction current in the connecting wire.

<u>Maxwell Equations</u>

In 1864 James Maxwell`s unified electricity and magnetism on the basis of four equations which are given below:

S.No.	Differential form	S.No.	Integral form
1.	$\nabla . \boldsymbol{D} = \rho$	1.	$\oint \boldsymbol{E}.\boldsymbol{dS} = \frac{1}{\varepsilon_0}\int \rho dV$
2.	$\nabla . \boldsymbol{B} = 0$	2.	$\oint \boldsymbol{B}.\boldsymbol{dS} = 0$
3.	$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$	3.	$\oint \boldsymbol{E}.\boldsymbol{dl} = -\frac{d}{dt}\int \boldsymbol{B}.\boldsymbol{dS}$
4.	$\nabla \times \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t}$	4.	$\oint \boldsymbol{H}.\boldsymbol{dl} = i + \varepsilon\frac{d}{dt}\int \boldsymbol{E}.\boldsymbol{dS}$

Physical Significance:

1. **First Equation:**
   It is Gauss law in electrostatics. It states that the surface integral of electric field over any closed surface area is equal to $\frac{1}{\varepsilon_0}$ times of net charge enclosed by that surface.
2. **Second Equation:**
   It is Gauss law in magneto statics. It states that there is no existence of magnetic monopoles or the net magnetic flux through any closed surface area is zero. It also signifies that magnetic field lines are closed curves.

3. **Third Equation:**
   It is Faraday`s law in electromagnetic induction. It states that induced emf around any closed path is equal to the negative rate of change of magnetic flux bounded by the surface w.r.to time. i.e. any changing magnetic field produces a electric field.
4. **Fourth Equation:**
   It is modified Ampere`s law. It states that any current carrying conductor as well as time varying electric field produces a magnetic field.

Derivation of Maxwell Equations in differential form

1. According to Gauss law in electrostatics

$$\oint \boldsymbol{E}.\boldsymbol{dS} = \frac{q}{\varepsilon_0}$$

$$\oint \boldsymbol{E}.\boldsymbol{dS} = \frac{1}{\varepsilon_0}\int \rho dV$$

$$\oint \varepsilon_0 \boldsymbol{E}.\boldsymbol{dS} = \int \rho dV$$

$$\oint D.\boldsymbol{dS} = \int \rho dV \qquad (Because\ D = \varepsilon_0 \boldsymbol{E})$$

Where **D** is called electric displacement vector

Using Gauss divergence theorem, we get

$$\oint \boldsymbol{D}\,.\,\boldsymbol{dS} = \int (\nabla.\boldsymbol{D})dV$$

Therefore

$$\int (\nabla.\boldsymbol{D})dv = \int \rho dV$$

$$\int (\nabla.\boldsymbol{D} - \rho)dV = 0$$

Since the equation is true for any volume the integrand must vanish, thus

$$(\boldsymbol{\nabla}.\boldsymbol{D} - \rho) = 0$$

$$\boldsymbol{\nabla}.\boldsymbol{D} = \rho$$

This is Maxwell's first equation.

2. According to Gauss law in magnetostatics

$$\oint \boldsymbol{B}\,.\,\boldsymbol{dS} = 0$$

Using Gauss divergence theorem, we get

$$\oint \boldsymbol{B}\,.\,\boldsymbol{dS} = \int \boldsymbol{\nabla}.\boldsymbol{B}dV$$

Therefore

$$\int \boldsymbol{\nabla}.\boldsymbol{B}dV = 0$$

The integrand should vanish for the surface boundary as the volume is arbitrary

i.e. $$\boldsymbol{\nabla}.\boldsymbol{B} = 0$$

This is Maxwell's second equation.

3. According to Faraday's law of electromagnetic induction

$$emf = -\frac{d\emptyset}{dt}$$

By the definition of emf we know

$$emf = \oint \boldsymbol{E}\,.\,\boldsymbol{dl}$$

And by the definition of magnetic flux we know

$$\emptyset = \int \boldsymbol{B}.\boldsymbol{dS}$$

Where S is the surface bounded by the circuit thus we have

$$\oint \boldsymbol{E}\,.\,\boldsymbol{dl} = -\frac{d}{dt}\int \boldsymbol{B}.\boldsymbol{dS}$$

Since the surface S does not change its shape or position with time, we can above equation as

$$\oint \boldsymbol{E}\,.\,\boldsymbol{dl} = -\int \frac{\partial \boldsymbol{B}}{\partial t}.\boldsymbol{dS}$$

Using Stokes's theorem we get

$$\oint \boldsymbol{E}\,.\,\boldsymbol{dl} = \int (\boldsymbol{\nabla} \times \boldsymbol{E}).\boldsymbol{dS}$$

Therefore

$$\int (\nabla \times \boldsymbol{E}).\boldsymbol{dS} = -\int \frac{\partial \boldsymbol{B}}{\partial t}.\boldsymbol{dS}$$

$$\int (\nabla \times \boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t}).\boldsymbol{dS} = \boldsymbol{0}$$

This equation must hold for any arbitrary surface, thus the integrand should vanish i.e

$$\nabla \times \boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t} = 0$$

Or

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

This is Maxwell's third equation.

4. According to Ampere's circuital law

$$\oint \boldsymbol{H}.\boldsymbol{dl} = I$$

We also know

$$I = \int \boldsymbol{J}.\boldsymbol{dS}$$

So

$$\oint \boldsymbol{H}.\boldsymbol{dl} = \int \boldsymbol{J}.\boldsymbol{dS}$$

Using Stoke's theorem we get

$$\oint \boldsymbol{H}.\boldsymbol{dl} = \int (\nabla \times \boldsymbol{H}).\boldsymbol{dS}$$

So

$$\int (\nabla \times \boldsymbol{H}).\boldsymbol{dS} = \int \boldsymbol{J}.\boldsymbol{dS}$$

$$\int (\nabla \times \boldsymbol{H} - \boldsymbol{J}).\boldsymbol{dS} = 0$$

This equation must hold for any arbitrary surface, thus the integrand should vanish

$$\nabla \times \boldsymbol{H} - \boldsymbol{J} = 0$$

$$\nabla \times \boldsymbol{H} = \boldsymbol{J} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

This equation derived on the basis of ampere's law stands only for steady state current but for time varying fields the current density should be modified. Thus taking divergence of both sides of above equation we get

$$\boldsymbol{\nabla}.(\boldsymbol{\nabla} \times \boldsymbol{H}) = \boldsymbol{\nabla}.\boldsymbol{J}$$

Since $\quad \boldsymbol{\nabla}.(\boldsymbol{\nabla} \times \boldsymbol{H}) = 0$

Therefore $\quad \boldsymbol{\nabla}.\boldsymbol{J} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$

But by equation of continuity we have

$$\boldsymbol{\nabla}.\boldsymbol{J} = -\frac{\partial \rho}{\partial t} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Maxwell realized the situation and suggested that the definition of total current density is incomplete and suggested to add something to J, such that equation 1 becomes

$$\nabla \times \boldsymbol{H} = \boldsymbol{J} + \boldsymbol{J}' \ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

Now taking divergence of both the sides we get

$$\nabla.(\nabla \times \boldsymbol{H}) = \nabla.(\boldsymbol{J} + \boldsymbol{J}')$$

Or $\quad \boldsymbol{\nabla}.(\boldsymbol{J} + \boldsymbol{J}') = 0$

Or $\quad \nabla.\boldsymbol{J} = -\nabla.\boldsymbol{J}'$

Putting the value of $\nabla.\boldsymbol{J}$ from equation (3) we get

Or $\quad \nabla.\boldsymbol{J}' = \frac{\partial \rho}{\partial t}$

But we know that $\quad \boldsymbol{\nabla}.\boldsymbol{D} = \rho$

Therefore $\quad \boldsymbol{\nabla}.\boldsymbol{J}' = \frac{\partial}{\partial t}(\boldsymbol{\nabla}.\boldsymbol{D})$

$$\boldsymbol{\nabla}.\boldsymbol{J}' = \boldsymbol{\nabla}.\left(\frac{\partial \boldsymbol{D}}{\partial t}\right)$$

Hence

$$\boldsymbol{J}' = \frac{\partial \boldsymbol{D}}{\partial t}$$

Substituting the value of J' in equation 4 we get

$$\nabla \times \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t}$$

This is Maxwell's fourth equation. The term arises when the electric displacement **D** is changing with time and is called displacement current density. J is called conduction current density.

Derivation of Maxwell Equations in integral form

First Equation:-

We know Maxwell's first equation in differential form

$$\nabla . \boldsymbol{D} = \rho$$

Integrating over an entire volume we get

$$\int (\nabla . \boldsymbol{D}) dV = \int \rho \, dV$$

Applying Gauss divergence theorem on LHS we get

$$\oint \boldsymbol{D} . \boldsymbol{dS} = \int \rho \, dV$$

$$\oint (\varepsilon_0 \mathbf{E}) . \boldsymbol{dS} = \int \rho \, dV$$

$$\varepsilon_0 \oint \mathbf{E} . \boldsymbol{dS} = \int \rho \, dV$$

$$\oint \mathbf{E} . \boldsymbol{dS} = \frac{1}{\varepsilon_0} \int \rho \, dV$$

Second Equation:

We know Maxwell's second equation in differential form

$$\nabla . \boldsymbol{B} = 0$$

Integrating over an entire volume we get

$$\int (\nabla . \boldsymbol{B}) dV = 0$$

Applying Gauss divergence theorem on LHS we get

$$\oint \boldsymbol{B} \, . \, \boldsymbol{dS} = 0$$

Third Equation

We know Maxwell's third equation in differential form

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

Integrating over an open surface area we get

$$\int (\nabla \times \boldsymbol{E}) . \boldsymbol{dS} = -\int \frac{\partial \boldsymbol{B}}{\partial t} . \boldsymbol{dS}$$

Applying Stoke`s theorem on LHS we get

$$\oint \boldsymbol{E} \, . \, \boldsymbol{dl} = -\int \frac{\partial \boldsymbol{B}}{\partial t} . \boldsymbol{dS}$$

$$\oint \boldsymbol{E} \, . \, \boldsymbol{dl} = -\frac{d}{dt} \int \boldsymbol{B} . \boldsymbol{dS}$$

<u>Fourth Equation</u>:

We know

$$\nabla \times \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t}$$

Integrating over an open surface area we get

$$\int (\nabla \times \boldsymbol{H}) . \boldsymbol{dS} = \int \left(\boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t} \right) . \boldsymbol{dS}$$

$$\int (\nabla \times \boldsymbol{H}) . \boldsymbol{dS} = \int \boldsymbol{J} . \boldsymbol{dS} + \int \frac{\partial \boldsymbol{D}}{\partial t} . \boldsymbol{dS}$$

Applying Stoke`s theorem on LHS we get

$$\oint \boldsymbol{H} \, . \, \boldsymbol{dl} = I + \int \frac{\partial \boldsymbol{D}}{\partial t} . \boldsymbol{dS}$$

$$\oint \boldsymbol{H} \, . \, \boldsymbol{dl} = I + \frac{d}{dt} \int \boldsymbol{D} . \boldsymbol{dS}$$

Poynting theorem

Poynting theorem describes the flow of energy or power in an electromagnetic field during the propagation of uniform plane wave. Maxwell's third and fourth equations are given

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \qquad \text{....(1)}$$

$$\boldsymbol{\nabla} \times \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t} \qquad \text{....(2)}$$

Taking dot product of equation (1) with **H** and equation (2) with **E** we get

$$\mathbf{H}.(\boldsymbol{\nabla} \times \mathbf{E}) = -\mathbf{H}.\frac{\partial \mathbf{B}}{\partial \mathrm{t}} \qquad \text{....(3)}$$

$$\mathbf{E}.(\boldsymbol{\nabla} \times \mathbf{H}) = \mathbf{E}.(\boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t}) \qquad \text{....(4)}$$

Subtracting equation (4) from equation (3) we get

$$\mathbf{H}.(\boldsymbol{\nabla} \times \mathbf{E}) - \mathbf{E}.(\boldsymbol{\nabla} \times \mathbf{H}) = -\mathbf{H}.\frac{\partial \mathbf{B}}{\partial \mathrm{t}} - \mathbf{E}.\mathbf{J} - \mathbf{E}.\frac{\partial \mathbf{D}}{\partial \mathrm{t}} \qquad \text{...(5)}$$

Using vector identity;

$$\boldsymbol{\nabla}.(\boldsymbol{E} \times \boldsymbol{H}) = \boldsymbol{H}.(\boldsymbol{\nabla} \times \boldsymbol{E}) - \boldsymbol{E}.(\boldsymbol{\nabla} \times \boldsymbol{H}) \qquad \text{...(6)}$$

From equations (5) and (6) we get

$$\boldsymbol{\nabla}.(\boldsymbol{E} \times \boldsymbol{H}) = -\mathbf{H}.\frac{\partial \mathbf{B}}{\partial \mathrm{t}} - \mathbf{E}.\mathbf{J} - \mathbf{E}.\frac{\partial \mathbf{D}}{\partial \mathrm{t}} \qquad \text{....(7)}$$

On rearranging we get

$$\boldsymbol{\nabla}.(\boldsymbol{E} \times \boldsymbol{H}) = -\mathbf{E}.\mathbf{J} - \mathbf{E}.\frac{\partial \mathbf{D}}{\partial \mathrm{t}} - \mathbf{H}.\frac{\partial \mathbf{B}}{\partial \mathrm{t}}$$

Or

$$-\boldsymbol{\nabla}.(\boldsymbol{E} \times \boldsymbol{H}) = \mathbf{E}.\mathbf{J} + \mathbf{E}.\frac{\partial \mathbf{D}}{\partial \mathrm{t}} + \mathbf{H}.\frac{\partial \mathbf{B}}{\partial \mathrm{t}} \qquad \text{...(8)}$$

Now

$$\boldsymbol{E}.\frac{\partial \boldsymbol{D}}{\partial t} = \varepsilon \boldsymbol{E}.\frac{\partial \boldsymbol{E}}{\partial t} = \frac{1}{2}\varepsilon\frac{\partial E^2}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}(\boldsymbol{E}.\boldsymbol{D}) = \frac{\partial}{\partial t}\left(\frac{1}{2}\boldsymbol{E}.\boldsymbol{D}\right)$$

and

$$\boldsymbol{H}.\frac{\partial \boldsymbol{B}}{\partial t} = \mu \boldsymbol{H}.\frac{\partial \boldsymbol{H}}{\partial t} = \frac{1}{2}\mu\frac{\partial H^2}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}(\boldsymbol{H}.\boldsymbol{B}) = \frac{\partial}{\partial t}\left(\frac{1}{2}\boldsymbol{H}.\boldsymbol{B}\right)$$

Making these substitutions in equation (8) we get

$$-\boldsymbol{\nabla}.(\boldsymbol{E}\times\boldsymbol{H}) = \boldsymbol{E}.\boldsymbol{J} + \frac{\partial}{\partial t}\left(\frac{1}{2}\boldsymbol{E}.\boldsymbol{D}\right) + \frac{\partial}{\partial t}\left(\frac{1}{2}\boldsymbol{H}.\boldsymbol{B}\right)$$

Taking the volume integral of the above equation we get

$$-\int \boldsymbol{\nabla}.(\boldsymbol{E}\times\boldsymbol{H})dV = \int(\boldsymbol{E}.\boldsymbol{J})dV + \int\frac{\partial}{\partial t}\left(\frac{1}{2}\boldsymbol{E}.\boldsymbol{D}\right)dV + \int\frac{\partial}{\partial t}\left(\frac{1}{2}\boldsymbol{H}.\boldsymbol{B}\right)dV$$

$$-\int \boldsymbol{\nabla}.(\boldsymbol{E}\times\boldsymbol{H})dV = \int(\boldsymbol{E}.\boldsymbol{J})dV + \frac{d}{dt}\int\left(\frac{1}{2}\boldsymbol{E}.\boldsymbol{D}\right)dV + \frac{d}{dt}\int\left(\frac{1}{2}\boldsymbol{H}.\boldsymbol{B}\right)dV$$

$$-\int \boldsymbol{\nabla}.(\boldsymbol{E}\times\boldsymbol{H})dV = \int(\boldsymbol{E}.\boldsymbol{J})dV + \frac{d}{dt}\int\frac{1}{2}(\boldsymbol{E}.\boldsymbol{D}+\boldsymbol{H}.\boldsymbol{B})\,dV$$

Applying Gauss divergence theorem on LHS we get

$$-\oint(\boldsymbol{E}\times\boldsymbol{H}).\boldsymbol{dS} = \int(\boldsymbol{E}.\boldsymbol{J})dV + \frac{d}{dt}\int\frac{1}{2}(\boldsymbol{E}.\boldsymbol{D}+\boldsymbol{H}.\boldsymbol{B})\,dV$$

On RHS

(i) First term represent the rate of transfer of EM energy due to the motion of the charge.
(ii) The second term represents the rate of transfer of EM energy in form of electromagnetic field

This equation is called Poynting theorem. It represents the **conservation of energy** in an electromagnetic field.

Therefore total EM energy flowing per unit time in entire volume is given by

$$-\oint(\boldsymbol{E}\times\boldsymbol{H}).\boldsymbol{dS}$$

So total outgoing EM energy from entire volume is given by

$$\oint(\boldsymbol{E}\times\boldsymbol{H}).\boldsymbol{dS}$$

If this volume in bounded by a surface area S then $\boldsymbol{E} \times \boldsymbol{H}$ represent outgoing energy from per unit area per unit time and it is called Poynting vector and denoted by **S**

So Poynting vector

$$\boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{H}$$

Poynting vector

The vector $\boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{H}$ is known as poynting vector. It is interpreted as the amount of field energy passing through unit area of the surface in a direction perpendicular to the plane containing **E** & **H** per unit time. As **E** & **H** are perpendicular to each other and also to the direction of propagation, therefore **S** having magnitude $EH\ sin90^0 = EH$ is also in the direction of wave propagation..

Unit of Poynting vector $is\ Joule/\ (m^2.sec)$ or $Watt/\ m^2$

Dimensions of Poynting vector are

$$\frac{Energy}{Area.Time} = \frac{ML^2T^{-2}}{L^2T}$$
$$= MT^{-3}$$

Electromagnetic waves in free space

The wave equation for an electrically free space or vacuum, containing neither free charge nor conduction current.

Maxwell's equations are

$$\boldsymbol{\nabla}.\boldsymbol{D} = \rho \qquad ...(1)$$

$$\boldsymbol{\nabla}.\boldsymbol{B} = 0 \qquad ...(2)$$

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \qquad ...(3)$$

$$\boldsymbol{\nabla} \times \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t} \qquad ...(4)$$

In free space $\rho = 0, \boldsymbol{J} = 0, \boldsymbol{D} = \varepsilon_0\boldsymbol{E}, \boldsymbol{B} = \mu_0\boldsymbol{H}$

Making these substitutions in above equations we get Maxwell`s equations in free space

$$\boldsymbol{\nabla}.\boldsymbol{E} = 0 \qquad ...(5)$$

$$\boldsymbol{\nabla}.\boldsymbol{H} = 0 \qquad ...(6)$$

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\mu_0 \frac{\partial \boldsymbol{H}}{\partial t} \qquad \dots (7)$$

$$\boldsymbol{\nabla} \times \boldsymbol{H} = \varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} \qquad \dots (8)$$

Taking curl of equation (7), we get

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{E}) = -\boldsymbol{\nabla} \times \left(\mu_0 \frac{\partial \boldsymbol{H}}{\partial t}\right)$$

$$\Rightarrow \qquad \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{E}) = -\mu_0 \frac{\partial}{\partial t} (\boldsymbol{\nabla} \times \boldsymbol{H}) \qquad \dots (9)$$

Putting the value of $\boldsymbol{\nabla} \times \boldsymbol{H}$ in equation (9) from equation (8) we get

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{E}) = -\mu_0 \frac{\partial}{\partial t} \left(\varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}\right)$$

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{E}) = -\mu_0 \varepsilon_0 \frac{\partial^2 \boldsymbol{E}}{\partial t^2} \qquad \dots (10)$$

Using vector identity we get

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{E}) = \boldsymbol{\nabla}(\boldsymbol{\nabla}.\boldsymbol{E}) - \nabla^2 \boldsymbol{E}$$

From equation (5) we have $\boldsymbol{\nabla}.\boldsymbol{E} = 0$,

Therefore $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{E}) = -\nabla^2 \boldsymbol{E}$

Putting this in equation (10) we get

$$-\nabla^2 \boldsymbol{E} = -\mu_0 \varepsilon_0 \frac{\partial^2 \boldsymbol{E}}{\partial t^2}$$

$$\nabla^2 \boldsymbol{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \boldsymbol{E}}{\partial t^2} \qquad \dots (11)$$

Similarly we can get

$$\nabla^2 \boldsymbol{H} = \mu_0 \varepsilon_0 \frac{\partial^2 \boldsymbol{H}}{\partial t^2} \qquad \dots (12)$$

By general wave equation we have

$$\nabla^2 \psi = \frac{1}{v^2} \frac{\partial^2 \psi}{\partial t^2} \qquad \dots (13)$$

Where ψ is the wave function which propagates with velocity v. Thus comparing equations (11) & (12) with the above equation we observe that field vectors E & H propagate as waves in free space and the velocity of propagation is

$$v = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$$

$$= \frac{1}{\sqrt{4\pi \times 10^{-7} \times \ 8.85 \times 10^{-12}}}$$

$$= \ 2.99 \times 10^8 m/s = c \text{ (Speed of light)}$$

Thus EM waves travel in free space with speed of light.

So equations (11) and (12) may be written as

$$\nabla^2 \boldsymbol{E} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{E}}{\partial t^2} = 0 \ \ \& \nabla^2 \boldsymbol{H} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{H}}{\partial t^2} = 0$$

<u>Solution of plane electromagnetic wave</u> (transverse nature of waves)

The equation for electric and magnetic field in free space are given by

$\nabla^2 \boldsymbol{E} \ = \mu_0 \varepsilon_0 \frac{\partial^2 \boldsymbol{E}}{\partial t^2}$ & $\nabla^2 \boldsymbol{H} = \mu_0 \varepsilon_0 \frac{\partial^2 \boldsymbol{H}}{\partial t^2}$

The solution of above equations may be written as:

$$\boldsymbol{E}(r,t) = \boldsymbol{E_0} e^{i(\boldsymbol{k}.\boldsymbol{r} - \omega t)}$$

$$\boldsymbol{H}(r,t) = \boldsymbol{H_0} e^{i(\boldsymbol{k}.\boldsymbol{r} - \omega t)}$$

Where E_0 and H_0 are amplitudes and k are is propagation vector defined as

$$\boldsymbol{k} = k\hat{n} = \frac{2\pi}{\lambda}\hat{n} = \frac{2\pi\nu}{c}\hat{n} = \frac{\omega}{c}\hat{n}$$

Here $\hat{n}$ is a unit vector in the direction of wave propagation.

Since in $\boldsymbol{E}$ and $\boldsymbol{H}$ there are no angular coordinates therefore

$$\nabla = \frac{\partial}{\partial r}$$

Now

$$\frac{\partial \boldsymbol{E}}{\partial r} = \frac{\partial}{\partial r} \boldsymbol{E_0} e^{i(\boldsymbol{k}.\boldsymbol{r}-\omega t)}$$

$$= ik\boldsymbol{E_0} e^{i(\boldsymbol{k}.\boldsymbol{r}-\omega t)}$$

$$= ik\boldsymbol{E}$$

$$\Rightarrow \quad \nabla = \frac{\partial}{\partial r} = ik$$

Now

$$\frac{\partial \boldsymbol{E}}{\partial t} = \frac{\partial}{\partial t} \boldsymbol{E_0} e^{i(\boldsymbol{k}.\boldsymbol{r}-\omega t)}$$

$$= (-i\omega)\boldsymbol{E_0} e^{i(\boldsymbol{k}.\boldsymbol{r}-\omega t)}$$

$$= -i\omega\boldsymbol{E}$$

$$\Rightarrow \quad \frac{\partial}{\partial t} = -i\omega$$

Making the substitutions for ∇ in equation (5) we get

$$i\boldsymbol{k}.\boldsymbol{E} = \boldsymbol{0}$$

$$\boldsymbol{k}.\boldsymbol{E} = 0$$

i.e. $\boldsymbol{k} \perp \boldsymbol{E}$

Similarly making the substitutions for ∇ in equation (6) we get

$$i\boldsymbol{k}.\boldsymbol{H} = \boldsymbol{0}$$

$$\boldsymbol{k}.\boldsymbol{H} = 0$$

i.e. $\boldsymbol{k} \perp \boldsymbol{H}$

Now making the substitutions for ∇ and $\frac{\partial}{\partial t}$ in equation $\boldsymbol{\nabla} \times \boldsymbol{E} = -\mu_0 \frac{\partial \boldsymbol{H}}{\partial t}$ we get

$$i(\boldsymbol{k} \times \boldsymbol{E}) = \mu_0 i\omega\boldsymbol{H}$$

$$\Rightarrow \quad (\boldsymbol{k} \times \boldsymbol{E}) = \mu_0 \omega\boldsymbol{H}$$

From above equation it is clear that field vector **H** is perpendicular to both **k** and **E**.

Thus we see that in EM wave electric field $\boldsymbol{E}$, magnetic field $\boldsymbol{H}$ and direction of propagation $\boldsymbol{k}$ are mutually perpendicular i.e. EM waves are transverse in nature.

This means electromagnetic field vectors E and H are perpendicular to the direction of propagation vector **k**. This implies that electromagnetic waves have transverse nature.

Relation between *E* and H

We know
$$(\boldsymbol{k} \times \boldsymbol{E}) = \mu_0 \omega \boldsymbol{H}$$

$$kE \sin 90^0 = \mu_0 \omega H$$

$$kE = \mu_0 \omega H$$

$$E = \mu_0 \frac{\omega}{k} H$$

$$E = \mu_0 c H \quad , \quad (v = \frac{\omega}{k} = c)$$

$$E = \mu_0 \frac{1}{\sqrt{\mu_0 \varepsilon_0}} H \quad , \quad (c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}})$$

$$E = \sqrt{\frac{\mu_0}{\varepsilon_0}} H$$

Characteristic Impedance

The ratio of magnitude of E to the magnitude of H is symbolized as Z_o and has the dimensions of electric resistance.

$$\boldsymbol{Z_o} = \left|\frac{\boldsymbol{E}}{\boldsymbol{H}}\right| = \frac{\boldsymbol{E_o}}{\boldsymbol{H_o}} = \mu_o c = \mu_o \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = \sqrt{\frac{\mu_o}{\varepsilon_o}} = 376.72 ohms$$

The ratio Z_o is a universal constant and is called characteristic impedance or wave impedance of free space.

Electromagnetic waves in non conducting media

A non charged, current free dielectrics are non conducting media,

Maxwell's equations are

$$\boldsymbol{\nabla}.\boldsymbol{D} = \rho \qquad \ldots(1)$$

$$\boldsymbol{\nabla}.\boldsymbol{B} = 0 \qquad \ldots(2)$$

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \qquad \dots (3)$$

$$\boldsymbol{\nabla} \times \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t} \qquad \dots (4)$$

In **non conducting media** $\rho = 0, \boldsymbol{J} = 0, \boldsymbol{D} = \varepsilon \boldsymbol{E}, \boldsymbol{B} = \mu \boldsymbol{H}$

Making these substitutions in above equations we get Maxwell`s equations in free space

$$\boldsymbol{\nabla} . \boldsymbol{E} = 0 \qquad \dots (5)$$

$$\boldsymbol{\nabla} . \boldsymbol{H} = 0 \qquad \dots (6)$$

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\mu \frac{\partial \boldsymbol{H}}{\partial t} \qquad \dots (7)$$

$$\boldsymbol{\nabla} \times \boldsymbol{H} = \varepsilon \frac{\partial \boldsymbol{E}}{\partial t} \qquad \dots (8)$$

Taking curl of equation (7), we get

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{E}) = -\boldsymbol{\nabla} \times \left(\mu \frac{\partial \boldsymbol{H}}{\partial t}\right)$$

$$\Rightarrow \qquad \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{E}) = -\mu \frac{\partial}{\partial t} (\boldsymbol{\nabla} \times \boldsymbol{H}) \qquad \dots (9)$$

Putting the value of $\boldsymbol{\nabla} \times \boldsymbol{H}$ in equation (9) from equation (8) we get

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{E}) = -\mu \frac{\partial}{\partial t} \left(\varepsilon \frac{\partial \boldsymbol{E}}{\partial t}\right)$$

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{E}) = -\mu\varepsilon \frac{\partial^2 \boldsymbol{E}}{\partial t^2} \qquad \dots (10)$$

Using vector identity we get

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{E}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} . \boldsymbol{E}) - \nabla^2 \boldsymbol{E}$$

From equation (5) we have $\qquad \boldsymbol{\nabla} . \boldsymbol{E} = 0$,

Therefore $\qquad \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{E}) = -\nabla^2 \boldsymbol{E}$

Putting this in equation (10) we get

$$-\nabla^2 \boldsymbol{E} = -\mu\varepsilon \frac{\partial^2 \boldsymbol{E}}{\partial t^2}$$

$$\nabla^2 \boldsymbol{E} = \mu\varepsilon \frac{\partial^2 \boldsymbol{E}}{\partial t^2} \qquad \dots (11)$$

Similarly we can get

$$\nabla^2 \boldsymbol{H} = \mu\varepsilon \frac{\partial^2 \boldsymbol{H}}{\partial t^2} \qquad \dots (12)$$

Depth of penetration (Skin depth)

When an EM wave propagate in a medium its amplitude decrease with the distance inside the medium from the surface. This phenomenon is known as attenuation.

The amplitude of an EM wave at a depth x is given by

$$E = E_0 e^{-\alpha x}$$

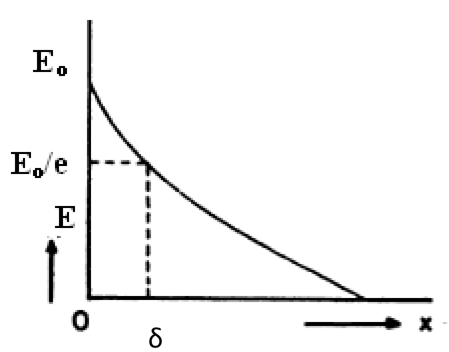
Where

E_0 is amplitude of the wave at the surface of the medium

α is attenuation constant it is given by

$$\alpha = \omega \left[\frac{\mu\varepsilon}{2} \left\{ \left(1 + \frac{\sigma^2}{\omega^2 \varepsilon^2}\right)^{\frac{1}{2}} - 1 \right\} \right]^{\frac{1}{2}}$$

The depth of penetration is defined as the depth in which the strength of electric field associated with the electromagnetic wave reduces to $\frac{1}{e}$ times to its initial value.



Now

$$E = E_0 e^{-\alpha x}$$

At skin depth $x = \delta,\ E = \frac{E_0}{e}$

$$\frac{E_0}{e} = E_0 e^{-\alpha\delta}$$

$$e^{-1} = e^{-\alpha\delta}$$

$$\alpha\delta = 1$$

$$\delta = \frac{1}{\alpha}$$

Thus,

$$skin\ depth = \frac{1}{attenuation\ constant}$$

For good conductors:

$$\frac{\sigma}{\omega\varepsilon} \gg 1$$

Therefore neglecting 1 with respect to $\frac{\sigma}{\omega\varepsilon}$ we get

$$\alpha = \omega\left\{\frac{\mu\varepsilon}{2}\left(\frac{\sigma}{\omega\varepsilon} - 1\right)\right\}^{\frac{1}{2}}$$

$$\alpha = \omega\left(\frac{\mu\varepsilon}{2}\frac{\sigma}{\omega\varepsilon}\right)^{\frac{1}{2}}$$

$$\alpha = \left(\frac{\mu\sigma\omega}{2}\right)^{\frac{1}{2}}$$

$$\alpha = \sqrt{\frac{\mu\sigma\omega}{2}}$$

Skin depth

$$\delta = \frac{1}{\alpha}$$

$$\delta = \sqrt{\frac{2}{\mu\sigma\omega}}$$

For poor conductors or insulators :

$$\frac{\sigma}{\omega\varepsilon} \ll 1$$

Therefore

$$\left(1+\frac{\sigma^2}{\omega^2\varepsilon^2}\right)^{\frac{1}{2}} \approx 1+\frac{\sigma^2}{2\omega^2\varepsilon^2}$$

$$\alpha = \omega\left\{\frac{\mu\varepsilon}{2}\left(1+\frac{\sigma^2}{2\omega^2\varepsilon^2}-1\right)\right\}^{\frac{1}{2}}$$

$$\alpha = \omega\left(\frac{\mu\varepsilon}{2}\frac{\sigma^2}{2\omega^2\varepsilon^2}\right)^{\frac{1}{2}}$$

$$\alpha = \frac{\sigma}{2}\left(\frac{\mu}{\varepsilon}\right)^{\frac{1}{2}}$$

$$\alpha = \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}}$$

$$\delta = \frac{2}{\sigma}\sqrt{\frac{\varepsilon}{\mu}}$$

From this expression it is clear that skin depth for insulator does not depend upon frequency of EM waves

Energy Density in Plane Electromagnetic Wave in Free Space:

In small volume dV, the energy of electric field $\boldsymbol{E}$ is $U_E = \frac{1}{2}\varepsilon_0\boldsymbol{E}^2$ dV,

In small volume dV, the energy of magnetic field $\boldsymbol{B}$ is $U_B = \frac{1}{2\mu_0}\boldsymbol{B}^2$ dV,

The electric energy per unit volume or electric field energy density U_E is given by

$$U_E = \frac{1}{2}\varepsilon_0\boldsymbol{E}^2 \qquad \text{..........(1)}$$

Similarly, the magnetic field energy density U_B is given by

$$U_B = \frac{1}{2\mu_0}\boldsymbol{B}^2 \quad \text{............(2)}$$

In electromagnetic field, sum of energy densities due to both **E** and **B is**

$$U = U_E + U_B = \frac{1}{2}\varepsilon_0\boldsymbol{E}^2 + \frac{1}{2\mu_0}\boldsymbol{B}^2, \ (\boldsymbol{B} = \mu\boldsymbol{H})$$

$$\frac{\boldsymbol{E}}{\boldsymbol{H}} = \sqrt{\frac{\boldsymbol{\mu_o}}{\boldsymbol{\varepsilon_o}}} \ \text{ or } H=E\sqrt{\frac{\boldsymbol{\varepsilon_o}}{\boldsymbol{\mu_o}}} \quad \text{.........(3)}$$

$$U = \frac{1}{2}\varepsilon_0\boldsymbol{E}^2 + \frac{1}{2}\mu_0\frac{\boldsymbol{\varepsilon_o}}{\boldsymbol{\mu_o}}\boldsymbol{E}^2$$

$$U = \varepsilon_0\boldsymbol{E}^2 \quad \text{......................(4)}$$

Hence total electromagnetic energy density $U = \varepsilon_0\boldsymbol{E}^2$

Therefore time average of energy density, u is obtained as

$$\langle u\rangle = \langle \varepsilon_0\boldsymbol{E}^2\rangle = \varepsilon_0 \langle (\boldsymbol{E_o}e^{i\boldsymbol{k}.\boldsymbol{r}-i\omega t})^2{}_{\text{real}}\rangle \text{ ....(5)}$$

We know that real part of $e^{i\boldsymbol{k}.\boldsymbol{r}-i\omega t}$ =cos $(\omega t - \boldsymbol{k}.\boldsymbol{r})$

$$\langle u\rangle = \varepsilon_0 E_0^2 \langle cos^2(\omega t - \boldsymbol{k}.\boldsymbol{r})\rangle$$

$$\text{Again, } \langle cos^2(\omega t - \boldsymbol{k}.\boldsymbol{r})\rangle = \frac{1}{2}$$

Therefore, $\langle u\rangle = \frac{1}{2}\varepsilon_0 E_0^2 = \varepsilon_0 E_{rms}^2 \quad (E_{rms} = E_0/\sqrt{2})$ .....(6)

We, know that the Poynting vector

$$\langle s\rangle = \frac{1}{Z_0}\frac{E_0^2}{2}\hat{n} = \frac{E_{rms}^2}{Z_0}\hat{n} \quad \text{.......(7)}$$

From Equation 6 and 7

$$\frac{\langle S\rangle}{\langle u\rangle} = \frac{\hat{n}}{Z_0\ \varepsilon_0} = \frac{\hat{n}}{\varepsilon_0\sqrt{\mu_0/\varepsilon_0}} = \frac{\hat{n}}{\sqrt{\mu_0\ \varepsilon_0}} = c\,\hat{n}$$

$$\langle s\rangle = c\hat{n}\langle u\rangle \quad \text{...8}$$

Or **Energy flux= velocity of light × energy density**

Thus the energy density associated with an electromagnetic wave in free space travels with a speed equal to velocity of light with which the field vectors propagate.

The ratio of electric and magnetic densities:

$$\frac{U_E}{U_B} = \frac{\frac{1}{2}\varepsilon_0 \boldsymbol{E}^2}{\frac{1}{2}\mu_0 \boldsymbol{H}^2} = \frac{\varepsilon_0 \boldsymbol{E}^2}{\mu_0 \boldsymbol{H}^2}$$

Again from equation $\frac{\boldsymbol{E}}{\boldsymbol{H}} = \sqrt{\frac{\boldsymbol{\mu_o}}{\boldsymbol{\varepsilon_o}}}$

$$\frac{E^2}{H^2} = \frac{\mu_o}{\varepsilon_o}, \text{ or } \mu_0 \boldsymbol{H}^2 = \varepsilon_0 \boldsymbol{E}^2$$

$$\frac{U_E}{U_B} = \frac{\varepsilon_0 \boldsymbol{E}^2}{\varepsilon_0 \boldsymbol{E}^2} = 1$$

Thus the electric field energy density is equal to magnetic field energy density.

Radiation Pressure When electromagnetic radiation strikes to a surface area, its momentum get changed. It means some momentum get transferred to the surface. If electromagnetic radiation is striking continuously to a surface then the momentum of the surface will be continuously changed. According to Newton`s second law the rate of change of momentum of the surface will be equal to the force exerted on the surface and the force per unit area on the surface is equal to the pressure. Therefore if electromagnetic radiation strikes to a surface then it exerts a pressure on that surface.

Relation between radiation Pressure and energy density – Let a plane EM wave incident normally on a perfectly absorbing surface having area A.

Also let this surface area absorbs energy U in time t.

Then momentum transferred to the surface is

$$P = \frac{U}{c} \qquad (E = Pc)$$

We know

$$U = SAt$$

Where S in magnitude of Poynting vector

Therefore

$$P = \frac{SAt}{c}$$

$$P = UAt \qquad\qquad (S = Uc)$$

Where U in energy density

By Newton`s second law

$$F = \frac{dP}{dt} = UA$$

Now Pressure

$$P = \frac{F}{A} = U$$

i.e.

$$P = U$$

i.e. Radiation pressure is equal to energy density.

UNIT Second Numerical: Electromagnetic Theory

Question 1. Derive Coulomb`s law from Maxwell`s first equation.

Solution. Maxwell`s first equation

$$\nabla.\boldsymbol{D} = \rho$$

Integrating over an entire volume we get

$$\int (\nabla.\boldsymbol{D})dV = \int \rho\, dV$$

Applying Gauss divergence theorem on LHS we get

$$\oint \boldsymbol{D}.\boldsymbol{dS} = \int \rho\, dV$$

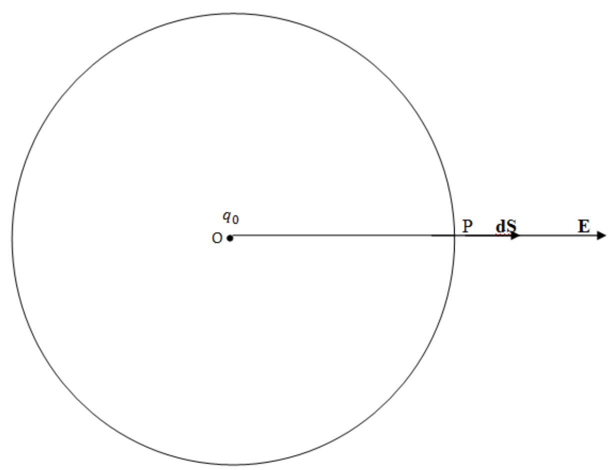
$$\oint (\varepsilon_0 \mathbf{E}).\boldsymbol{dS} = \int \rho\, dV$$

$$\varepsilon_0 \oint \mathbf{E}.\boldsymbol{dS} = q$$

$$\oint \mathbf{E}.\boldsymbol{dS} = \frac{q}{\varepsilon_0}$$

Consider a point charge q_0 at a point O

Now applying Gauss law for a spherical surface of radius r centered at O



We get

$$\oint \mathbf{E}.\boldsymbol{dS} = \frac{q_0}{\varepsilon_0} \ldots\ldots\ldots\ldots\ldots..(1)$$

For a positive point charge electric field will be radialy outward and so it will along $\boldsymbol{dS}$

$$\oint \mathrm{E}\; dS = \frac{q_0}{\varepsilon_0}$$

$$\mathrm{E}\oint dS = \frac{q_0}{\varepsilon_0}$$

$$E.4\pi r^2 = \frac{q_0}{\varepsilon_0}$$

$$E = \frac{q_0}{4\pi\varepsilon_0 r^2}$$

If an another point charge q is placed at point P then electrostatic force on that is

$$F = qE$$

$$F = \frac{q_0 q}{4\pi\varepsilon_0 r^2}$$

This is Coulomb`s law

Question 2. Derive equation of continuity from Maxwell`s fourth equation

Solution. Maxwell`s fourth equation is

$$\nabla \times \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t}$$

Taking divergence on both sides we get

$$\nabla.(\nabla \times \boldsymbol{H}) = \nabla.\left(\boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t}\right)$$

But $$\nabla.(\nabla \times \boldsymbol{H}) = 0$$

Therefore $$\nabla.\left(\boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t}\right) = 0$$

$$\nabla.\boldsymbol{J} + \nabla.\left(\frac{\partial \boldsymbol{D}}{\partial t}\right) = 0$$

$$\nabla.\boldsymbol{J} + \frac{\partial(\nabla.\boldsymbol{D})}{\partial t} = 0$$

But by Maxwell`s first equation we know

$$\nabla.\boldsymbol{D} = \rho$$

Putting the value of $\nabla.\boldsymbol{D}$ we get

$$\nabla.\boldsymbol{J} + \frac{\partial \rho}{\partial t} = 0$$

This is the equation of continuity.

Question 3. Earth receives solar energy from the Sun which is $\frac{2cal}{cm^2.Min}$. what are the amplitude of electric and magnetic fields of radiation?

or

The sunlight strikes upper atmosphere of earth with energy $1400\ \text{Watt/m}^2$. Calculate peak values of electric and magnetic fields at points.

Solution:

$$S = 2cal/(cm^2.Min)$$

$$= \frac{2 \times 4.2\ Joule}{(10^{-2}m)^2 \times 60\ s}$$

$$= 1400\ Joule/(m^2.s)$$

We know Poynting vector is given by

$$S = EXH = \text{EH}\sin 90 = \text{EH}$$

Therefore $$EH == 1400\frac{Joule}{m^2.s} \ldots \ldots \ldots \ldots \ldots \ldots \ldots (1)$$

We also know by characteristic impedance of vacuum is given by

$$\frac{E}{H} = 376.77\ \Omega \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (2)$$

Multiplying equations (1) and equation (2) we get

$$E^2 = 527478\ (N/C)^2$$

$$\Rightarrow \quad E = 726.27\ N/C$$

Putting the value of E in equation (1) we get

$$H = \frac{1400}{726.27}\ N/(A.m)$$

$$= 1.93\ N/(A.m)$$

$$E_0 = E\sqrt{2}$$

$$\Rightarrow \quad E_0 = 726.27\sqrt{2}\ N/C$$

$$\Rightarrow \quad E_0 = 1027.1\ N/C$$

$$H_0 = H\sqrt{2}$$

$\Rightarrow \quad H_0 = 1.93\sqrt{2}\, N/(A.m)$

$\Rightarrow \quad H_0 = 2.73\, N/(A.m)$

\ Question 4. Earth receives solar energy from the Sun which is 10 Joules per minute per cm^2 What are the amplitude of electric and magnetic fields of radiation?

Solution: $S = 10\, Joules/(cm^2.Min)$

$$= \frac{10\, Joules}{(10^{-2}m)^2 \times 60\, s}$$

$$= 1666.67\, Joule/(m^2.s)$$

We know Poynting vector is given by

$S = EXH$= EH sin90 = EH

Therefore $EH == 1666.67\, \frac{Joule}{m^2.s} \dots\dots\dots\dots\dots\dots\dots.(1)$

We also know by characteristic impedance of vacuum is given by

$$\frac{E}{H} = 376.77\, \Omega \dots\dots\dots\dots\dots\dots\dots\dots..(2)$$

Multiplying equations (1) and equation (2) we get

$$E^2 = 627951.25\, (N/C)^2$$

$\Rightarrow \quad E = 792.43\, N/C$

Putting the value of E in equation (1) we get

$$H = \frac{1666.67}{792.43}\, N/(A.m)$$

$$= 2.1\, N/(A.m)$$

$$E_0 = E\sqrt{2}$$

$\Rightarrow \quad E_0 = 792.43\,\sqrt{2}\, N/C$

$\Rightarrow \quad E_0 = 1120.67\, N/C$

$$H_0 = H\sqrt{2}$$

$\Rightarrow \quad H_0 = 2.1\sqrt{2}\, N/(A.m)$

$\Rightarrow \quad H_0 = 2.97\, N/(A.m)$

Question 5. If the upper atmosphere of earth receives the energy $1360\ Watt/m^2$ from the sun what will be the peak values of electric and magnetic fields at the layer.

Solution: $$S = 1360\ Watts/m^2$$

$$S = 1360 \frac{Joules}{(m^2.s)}$$

We know Poynting vector is given by

$$S = EXH = \text{EH}\ \sin 90 = \text{EH}$$

Therefore $$EH == 1360\ \frac{Joule}{m^2.s} \ldots\ldots\ldots\ldots\ldots\ldots\ldots.(1)$$

We also know by characteristic impedance of vacuum is given by

$$\frac{E}{H} = 376.77\ \Omega \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(2)$$

Multiplying equations (1) and equation (2) we get

$$E^2 = 512407.2\ (N/C)^2$$

$$\Rightarrow \qquad E = 715.83\ N/C$$

Putting the value of E in equation (1) we get

$$H = \frac{1360}{715.83}\ N/(A.m)$$

$$= 1.89\ N/(A.m)$$

$$E_0 = E\sqrt{2}$$

$$\Rightarrow \qquad E_0 = 715.83\sqrt{2}\ N/C$$

$$\Rightarrow \qquad E_0 = 1012.34\ N/C$$

$$H_0 = H\sqrt{2}$$

$$\Rightarrow \qquad H_0 = 1.89\sqrt{2}\ N/(A.m)$$

$$\Rightarrow \qquad H_0 = 2.67\ N/(A.m)$$

Question 6. Assuming that all the energy from a 100 watt lamp is radiated uniformly. Calculate the average values of the intensities of electric and magnetic fields of radiation at a distance of 2 m from the lamp.

Solution: We know Poynting vector is given by

$$S = \frac{P}{4\pi r^2}$$

$$\Rightarrow \qquad S = \frac{100\ W}{4 \times 3.14 \times (2\ m)^2}$$

$$\Rightarrow \qquad S = \frac{100\ Joule}{4 \times 3.14 \times 4\ m^2.s}$$

$$\Rightarrow \qquad S = 1.99\ Joule/(m^2.s)$$

We know $S = EXH$= EH sin90 = EH

$$S = EH$$

Therefore $EH = 1.99\ \frac{Joule}{m^2.s} \dots \dots \dots \dots \dots \dots \dots . (1)$

We also know by characteristic impedance of vacuum is given by

$$\frac{E}{H} = 376.77\ \Omega \dots \dots \dots \dots \dots \dots \dots \dots .. (2)$$

Multiplying equations (1) and equation (2) we get

$$E^2 = 1492\ (N/C)^2$$

$$\Rightarrow \qquad E = 6.22\ N/C$$

Putting the value of E in equation (1) we get

$$H = \frac{1.99}{6.22}\ N/(A.m)$$

$$= 0.05\ N/(A.m)$$

Question 7. A 500 watt lamp radiates power uniformly in all directions. Calculate the electric and magnetic field intensities at 1 m distance from the lamp.

Solution: We know Poynting vector is given by, $S = \frac{P}{4\pi r^2}$

$$\Rightarrow \qquad S = \frac{500\ W}{4 \times 3.14 \times (1\ m)^2}$$

$$\Rightarrow \qquad S = \frac{500\ Joule}{4 \times 3.14 \times 1\ m^2.s}$$

$$\Rightarrow \qquad S = 38.53\ Joule/(m^2.s)$$

We know $\qquad S = EXH$= EH sin90 = EH

$$S = EH$$

Therefore $\qquad EH = 38.53\ \frac{Joule}{m^2.s} \ldots\ldots\ldots\ldots\ldots\ldots\ldots.(1)$

We also know by characteristic impedance of vacuum is given by

$$\frac{E}{H} = 376.77\ \Omega \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(2)$$

Multiplying equations (1) and equation (2) we get

$$E^2 = 14516.94\ (N/C)^2$$

$$\Rightarrow \qquad E = 120.49\ N/C$$

Putting the value of E in equation (1) we get

$$H = \frac{38.53}{120.49}\ N/(A.m)$$

$$= 0.32\ N/(A.m)$$

Question 8. Calculate the magnitude of Poynting vector at the surface of the sun. Given the power radiated by sun is 3.8×10^{26} Watts and radius of the sun is 7×10^8 m.

Solution: We know Poynting vector is given by $\quad S = \frac{P}{4\pi r^2}$

$$\Rightarrow \qquad S = \frac{3.8 \times 10^{26}\ W}{4 \times 3.14 \times (7 \times 10^8\ m)^2}$$

$$\Rightarrow \qquad S = \frac{3.8 \times 10^{26}\ Joule}{4 \times 3.14 \times 49 \times 10^{16}\ m^2.s}$$

$$\Rightarrow \qquad S = 6.17 \times 10^7\ Joule/(m^2.s)$$

Question 9. In a material for which $\sigma = 5 \text{ S/m}$ and $\varepsilon_r = 1$the electric field intensity is $E = 250\sin(10^{10}t)\text{V/m}$. Find conduction and displacement current densities and the frequency at which both have equal amplitude.

Solution: Conduction current density is given by $J_c = \sigma E$

$$\Rightarrow \quad J_c = 5 \times 250 sin(10^{10}t)\ A/m^2$$

$$J_d = \varepsilon \frac{dE}{dt}$$

$$\Longrightarrow \quad J_d = \varepsilon_r \varepsilon_0 \frac{d}{dt}\{250 sin(10^{10}t)\}$$

$$\Longrightarrow \quad J_d = 1 \times 8.854 \times 10^{-12} \times 250\ \times 10^{10}\ \times cos(10^{10}t)$$

Question 10. The permeability, permittivity and conductivity of aluminum are $\mu_r = 1$, $\varepsilon_r = 1$ and $\sigma = 3.54 \times 10^7\ mho/m$. Find the skin depth if the wave enter in aluminum with frequency of 71.56 MHz.

Solution: Skin depth is given by , $\lambda = \sqrt{\frac{2}{\mu\sigma\omega}}$

$$\delta = \sqrt{\frac{2}{\mu_r \mu_0 \sigma 2\pi f}}$$

$$\delta = \sqrt{\frac{1}{\mu_r \mu_0 \sigma \pi f}}$$

$$\delta = \sqrt{\frac{1}{1 \times 4\pi \times 10^{-7}\ N/A^2 \times 3.54 \times 10^7\ mho/m \times \pi \times 71.56 \times 10^6 \text{ Hz}}}$$

$$\delta = 9.99 \times 10^{-6}\ m$$

Question 11. For silver, $\mu = \mu_0$ and $\sigma = 3 \times 10^7\ mhos/m$. Calculate the skin depth at $10^8\ Hz$ frequency. Given, $\mu_0 = 4\pi \times 10^{-7}\ N/A^2$

Solution: Skin depth is given by , $\lambda = \sqrt{\frac{2}{\mu\sigma\omega}}$

$$\delta = \sqrt{\frac{2}{\mu\sigma 2\pi f}}$$

$$\delta = \sqrt{\frac{1}{\mu\sigma\pi f}}$$

$$\delta = \sqrt{\frac{1}{4\pi \times 10^{-7}\ N/A^2 \times 3 \times 10^7\ mho/m \times \pi \times 10^8 \text{Hz}}}$$

$$\delta = 9.18 \times 10^{-6}\ m$$

Question 12. Find the skin depth at frequency 71.6 MHz in aluminum. The related parameters for aluminum are $\mu = \mu_0 = 4\pi \times 10^{-7}$ N/A^2 and $\sigma = 3.58 \times 10^7$ siemen/m

Solution: Skin depth is given by , $\lambda = \sqrt{\frac{2}{\mu\sigma\omega}}$

$$\delta = \sqrt{\frac{2}{\mu\sigma\omega}}$$

$$\delta = \sqrt{\frac{2}{\mu\sigma 2\pi f}}$$

$$\delta = \sqrt{\frac{1}{\mu\sigma\pi f}}$$

$$\delta = \sqrt{\frac{1}{4\pi \times 10^{-7}\ N/A^2 \times 3.58 \times 10^7\ mho/m \times \pi \times 71.610^6 \text{Hz}}}$$

$$\delta = 9.99 \times 10^{-6}\ m$$

Q. **What do you mean by black body? Explain black body spectrum with proper diagram.**

OR

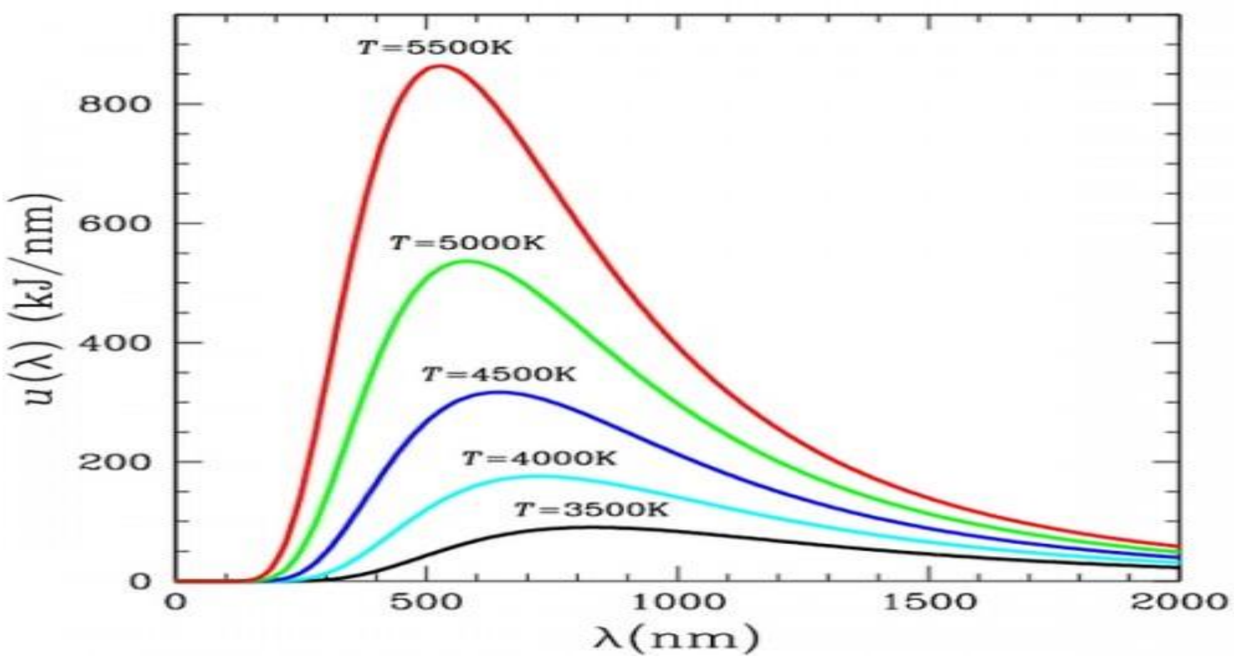
How the wavelength changes with increase of temperature?

Black body radiation:

Solids, when heated, emit radiation varying over a wide range of wavelengths. For example: when we heat solids, color changes which continue to change with a further increase in temperature. This change in color happens from a lower frequency region to a higher frequency region as the temperature increases. For example, in many cases, it changes from red to blue. *An ideal body which can emit and absorb radiation of all frequencies is called a black body. A black body is an object that absorbs 100% of the radiation incident on it.* Therefore it reflects no radiation and appears perfectly black. It is also a perfect emitter of radiation. At a particular temperature the black body would emit the maximum amount of energy possible for that temperature. The radiation emitted by such bodies is called black body radiation.

Black body spectrum

A graph was obtained between the energy density and wavelength at different temperature as shown in figure.



The experimental results are

1. The emission from black body at any temperature consist radiation of all wavelengths.

2. At a given temperature the energy is not uniformly distributed. As the temperature of black body increases the intensity of radiation for each wavelength increases.

3. The amount of radiant energy emitted is small at very short and very long wavelengths.

4. The wavelength corresponding to the maximum energy represented by the peak of the curve shifts towards shorter wavelengths as the temperature increases. This is called as Wein's displacement law. According to this law,

$$\boxed{\lambda_m T = \text{constant}}$$

This shows that, as the temperature increases, the black body emits the radiation of shorter wavelength such that product of temperature T and maximum wavelength λ_m is a constant.

Q. What is Stefan's law?

Stefan's law:

When a black body is heated, it emits the radiation of all possible wavelengths. The scientist Stefan gave the law for the energy distribution among the different wavelengths. Stefan's law gives the total energy radiated by a black body. According to Stefan's law, the total amount of the radiant energy by a black body per unit area per unit time due to all wavelengths is directly proportional to the fourth power of absolute temperature.

$$E \propto T^4$$

$$E = \sigma T^4$$

Where, the constant of proportionality σ is called the Stefan Boltzmann constant and has a value of $\sigma = 5.6704\times10^{-8}$ watt / (meter ^{2} Kelvin^4)

Q. What is Wien's displacement law?(2017-18)

Wien's displacement law-

Wien investigated the energy distribution over different wavelength. This law determines at what wavelength the intensity of radiation emitted from a blackbody reaches its maximum point. After this point, the intensity decreases as temperature increases. This creates the characteristic shape of blackbody radiation curves which shows that the maximum energy point shifts towards the shorter wavelengths side when the temperature of body is raised.

$$\boxed{\lambda_m T = b(\text{constant})}$$

Where λ_m is the wavelength corresponding to maximum energy emission from a black body at absolute temperature T.

b = Wien's displacement constant = 0.0029 m / T

Q. What is Wein's law?(2016-17)

Wien's Radiation Formula (Wien's Law)

Wien showed that the amount of radiation $E_\lambda d\lambda$ emitted by unit area of a black-body per second at a temperature of T Kelvin in the wavelength range λ & $\lambda+d\lambda$ is given by the formula,

$$E_\lambda d\lambda = \frac{A}{\lambda^5} f(\lambda, T) d\lambda$$

Where A is a constant & f(λ,T) is the function of the product of λ and T. This is the Wien's formula.

Therefore he also proved that the maximum energy emitted by a blackbody is proportional to the fifth power of its absolute temperature. Hence

$$(E_\lambda)_{max} \alpha \; T^5$$

$$\frac{(E_\lambda)_{max}}{T^5} = constant$$

It approaches the data at shorter wavelengths, but it deviates at longer wavelengths.

Q. What is Rayleigh-Jeans?

Rayleigh-Jeans law: Lord Rayleigh used the classical theories of electromagnetism and thermodynamics to show that the blackbody spectral energy distribution is given by:

$$E\lambda = \frac{8\pi kT}{\lambda^4}$$

Where k is Boltzmann's constant

It approaches the data at longer wavelengths, but it deviates at short wavelengths.

Q. What are the basic assumptions of quantum theory?

Basic assumptions of quantum theory of radiation and Planck's law:

In 1900 Planck developed a theory of blackbody radiation that leads to an equation for the energy density of the radiation. This equation is in complete agreement with experimental observations. He assumed that the atoms of the wall of a black body behave as oscillators and each has characteristic frequency of oscillation. The assumptions about atomic oscillator are,

1. The energy of an oscillator can have only certain discrete values E_n

$$E_n = nh\nu$$

Where, **n** is a positive integer called the quantum number, **ν** is the frequency of oscillation, **h** is Planck's constant

This says the energy is quantized. Each discrete energy value corresponds to a different quantum state.

2. The oscillators emit or absorb energy only in the form of packets of energy (hν) not continuously, when making a transition from one quantum state to another. The entire energy difference between the initial and final states in the transition is emitted or absorbed as a quantum of radiation,

$$\Delta E = \Delta n h\nu$$

$$E_2 - E_1 = (n_2 - n_1)h\nu$$

Q. Establish Planck radiation formula and show that the Wein's formula and Rayleigh-Jeans formula are special cases of Planck's formula.

Planck's law: To derive the Planks law, we shall first calculate the average energy of the a Plank oscillator of frequency *v.* if N be the total no of oscillators and E is their total energy,

Then average energy of an oscillator is given by,

$$E_{avg} = \frac{E}{N} \qquad \ldots(1)$$

The number of oscillator having energy nhv is given by the formula

$$N_n = N_0 e^{-\frac{nhv}{kt}} \qquad \ldots(2)$$

The total no. of oscillator is given by

$$N = N_0 + N_1 + N_2 + N_3 + N_4 + \ldots..$$

$$= N_0 + N_0 e^{-\frac{hv}{kt}} + N_0 e^{-\frac{2hv}{kt}} + N_0 e^{-\frac{3hv}{kt}} + \ldots..$$

$$N = N_0 \left[1 + e^{-\frac{hv}{kt}} + e^{-\frac{2hv}{kt}} + e^{-\frac{3hv}{kt}} + \ldots\ldots\right]$$

$$N = N_0 \frac{1}{1-e^{-\frac{h\nu}{kt}}}$$ ...(3) $\left[1+x+x2+x3+\ldots\ldots = \frac{1}{1-x}\right]$

Total energy of all oscillators is given by

$$E = N_0\times 0 + N_1\times h\nu + N_2\times 2h\nu + N_3\times 3\,h\nu + N_4\times 4\,h\nu + \ldots$$

On simplifying

$$E = N_0 e^{-\frac{h\nu}{kt}} \frac{h\nu}{\left(1-e^{-\frac{h\nu}{kt}}\right)^2}$$ ...(4)

Substituting the value of N and E from Eq. 3 & Eq. 4 in Eq. 1

$$\mathbf{E_{avg}} = \frac{\boldsymbol{h\nu}}{\boldsymbol{e^{h\nu/kt}-1}}$$ ...(5)

This is the expression for average energy of Planck's oscillator.

Now the energy density of radiation is in frequency range v and $v+dv$ is given by

$u_\nu dv$ = no. of oscillator per unit volume ×average energy

$$= \frac{8\pi\nu^2 d\nu}{c^3}\left(\frac{h\nu}{e^{h\nu/kt}-1}\right)$$

$$\boldsymbol{u_\nu dv = \frac{8\pi h\nu^3 d\nu}{c^3} \frac{1}{(e^{\frac{h\nu}{kt}}-1)}}$$ ...(6)

This is the Planck's radiation formula in terms of frequency.

To express it in terms of wavelength, we put

$$v = \frac{c}{\lambda} \quad \text{and} \quad dv = -\frac{c}{\lambda^2}d\lambda$$

Therefore $u_\lambda d_\lambda = u_\nu dv$

$$\boldsymbol{u_\lambda d_\lambda = \frac{8\pi hc\,d\lambda}{\lambda^5} \frac{1}{(e^{\frac{hc}{\lambda kt}}-1)}}$$

This is Planck's radiation formula in terms of wavelength.

Deduction of various laws using Planck's radiation formula

1. **Wien's law from Planck's formula**.

Planck's radiation formula in terms of wavelength for black body spectrum is given by

$$u_\lambda d_\lambda = \frac{8\pi hc\,d\lambda}{\lambda^5} \frac{1}{(e^{\frac{hc}{\lambda kt}}-1)}$$ ...(1)

For short wavelengths or when λ is small, $e^{\frac{hc}{\lambda kt}} \gg 1$, so 1 may be neglected in the denominator of above equation. Thus,

$$u_\lambda d_\lambda = \frac{8\pi hc}{\lambda^5} e^{-\frac{hc}{\lambda kt}} d\lambda$$ ...(2)

Substituting $8\pi hc = c_1$ and $hc/k=c_2$, equation 2 becomes

$$u_\lambda d_\lambda = \frac{c_1}{\lambda^5} e^{-\frac{c_2}{\lambda t}} d\lambda$$

This is Wien's law which agrees with experimental curve at short wavelengths region.

2. **Rayleigh-Jeans law from Planck's formula**:

   For long wavelengths or λ is large, $e^{\frac{hc}{\lambda kt}} \approx 1 + \frac{hc}{\lambda kt}$(neglecting terms with higher power of λ in denominator), therefore equation 1 becomes,

$$u_\lambda d_\lambda = \frac{8\pi hc d\lambda}{\lambda^5 (1+\frac{hc}{\lambda kt}-1)} = \frac{\mathbf{8\pi kT\lambda}}{\boldsymbol{\lambda^4}}$$

This is Rayleigh-Jeans law which agrees with experimental curves at long wavelengths region.

Q. What do you mean by wave particle duality?

Wave particle duality

A particle means an object with a definite position in space which cannot be simultaneously occupied by another particle & specified by their properties such as mass, momentum, kinetic energy, velocity etc. On the other hand, a wave means a periodically repeated pattern in space which is specified by its wavelength, amplitude, frequency, energy, momentum etc. Two or more waves can coexist in the same region and superimpose to form a resultant wave. The particle & wave properties of radiation can never be observed simultaneously. Radiation, sometimes behave as a wave (Interference, diffraction, etc) & at some other time as a particle (Photoelectric effect, Compton Effect etc), *i.e.*, it has a wave particle dualism.

Q. What is de-Broglie hypothesis of Matter Waves? Explain.

de-Broglie hypothesis of Matter waves

The wave associated with a moving particle is called matter waves or de-Broglie waves. According to de-Broglie's concept (1924), a moving particle always has a wave associated with it & motion of the particle is guided by that wave in similar manner as photon is controlled by wave. If a particle of mass m has momentum p and λ the wavelength of the wave associated with it, then according to de-Broglie hypothesis

$$\boldsymbol{\lambda = \frac{h}{p} = \frac{h}{mv}}$$

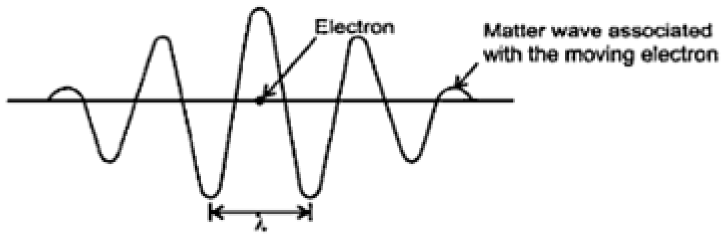


Fig.1.1

Expression for de-Broglie

According to Planck's theory of radiation, the energy of a photon is given by

$$\boldsymbol{E = h\nu} \qquad \text{.......(1)}$$

According to Einstein energy-mass relation

$$\boldsymbol{E = mc^2} \qquad \text{........(2)}$$

From eqs. (1) & (2), we get

$$h\nu = mc^2$$

$$h\frac{c}{\lambda} = mc^2 \quad (since\ c{=}\nu\lambda)$$

$$\frac{h}{\lambda} = mc$$

Or

$$\lambda = \frac{h}{mc}$$

For materialistic Particle

If we consider the case of a material particle of mass m and moving with a valocity v then the wavelength associated with this particle is given by

$$\lambda = \frac{h}{p} = \frac{h}{mv}$$

Diiferent expressions: -

1. If E is then kinetic energy of the material particle, then,

$$E_k = \frac{1}{2}mv^2 = \frac{m^2v^2}{2m} = \frac{p^2}{2m}$$

$$p = \sqrt{2mE_k}$$

Therfore $\boldsymbol{\lambda = \frac{h}{\sqrt{2mE_k}}}$

2. When a charged particle carrying a charge e is accelerated by a potential difference V volts, then E is given by

$$E = eV$$

Hence,the de-Broglie wavelength is given by

$$\lambda = \frac{h}{p} = \frac{h}{mv}$$

$$\lambda = \frac{h}{\sqrt{2mE_k}} = \lambda = \frac{h}{\sqrt{2meV}}$$

3. When a material particle is in thermal equilibrium at a temperature T, then

$$E = \frac{3}{2}kT$$

Where k = Boltzmann's constant = 1.38×10^{-23} J/K

So, de-Broglie wavelength at temperature T is given by

$$\lambda = \frac{h}{\sqrt{2mE_k}} = \lambda = \frac{h}{\sqrt{2m\frac{3}{2}kT}} \quad \text{or} \quad \lambda = \frac{h}{\sqrt{3mkT}}$$

4. **De-Broglie wavelength associated with electrons (Non-Relativistic case):-** let us consider the case of an electron of rest mass m_0 & charge e which is accelerated by a potential V volt from rest to velocity v, then

$\frac{1}{2}mv^2 = eV$ *or* $v = \sqrt{2eV/m_0}$

$$\lambda = \frac{h}{p} = \frac{h}{m_0 v} = \frac{h}{\sqrt{2eVm_0}}$$

$$\lambda = \frac{6.63*10^{-34}}{\sqrt{2\times 1.6\times 10^{-19}\times 2\times V\times 19.1\times 10^{-31}}}$$

$$\lambda = \frac{h}{\sqrt{2mqv}} = \frac{12.28}{\sqrt{V}}$$

Q. What are the Properties of Matter Waves?

Properties of Matter Waves

1. Lighter is the particle, greater is the wavelength associated with it.
2. Smaller is the velocity of the particle, greater is the wavelength associated with it.
3. When v = 0 then $\lambda = \infty$, i.e., wave becomes indeterminate & if v = ∞ then λ = o. This shows that matter waves are generated by the motion of particles.
4. The matter waves are not electromagnetic in nature.
5. The de-Broglie wavelength is independent of charge of the particle, therefore, the matter waves are generated by moving charged particles as well as by moving neutral particles.

Q. What do you mean by group velocity? (2017-18)

OR What is the difference between phase velocity and group velocity? (2013-14)

OR What do you mean by phase velocity and group velocity? (2014-15)

Wave velocity or Phase velocity:- Whan a monochromatic wave travels through a medium, its velocity of advancement of wave in the medium is called as wave velocity or the velocity of propogation of the planes of constant phase through medium is known as wave velocity (phase velocity).

Consider a wave whose displacement y is expressed as

$$y = a \sin (\omega t - kx)$$

where a is the amplitude, ω is angular frequency (= 2πυ) and k (= 2π/λ) is the propogation constant of the wave.

For a plane of constant phase,

$\omega t - kx = \text{constant}$

Differentiating, $\dfrac{d}{dt}(\omega t) - \dfrac{d}{dt}(kx) = 0$

$$\omega - k\frac{dx}{dt} = 0 \quad \text{or} \quad -k\frac{dx}{dt} = -\omega$$

$$\frac{dx}{dt} = \frac{\omega}{k} \quad \text{or} \quad \boxed{\text{Phase velocity, } v_p = \frac{\omega}{k}}$$

Group velocity:- The pulse consists of a number of waves slightly differing in frequency from one another. The superposition of such waves is called wave group. When such a group travels in the medium, the phase velocities are different for different components. Thus, the group velocity is the velocity with which the group (wave packet) is transmitted. Let the two waves can be represented by

$$y_1 = a \sin (\omega_1 t - k_1 x)$$

and

$$y_2 = a \sin (\omega_2 t - k_2 x)$$

Using principle of superposition of waves, we get

$$y = y_1 + y_2$$

or

$$y = a \sin (\omega_1 t - k_1 x) + a \sin (\omega_2 t - k_2 x)$$

or

$$y = 2a \sin \frac{(\omega_1 t - k_1 x) + (\omega_2 t - k_2 x)}{2} \cos \frac{(\omega_1 t - k_1 x) - (\omega_2 t - k_2 x)}{2}$$

or

$$y = 2a \cos \left[\frac{(\omega_1 - \omega_2)t}{2} - \frac{(k_1 - k_2)x}{2}\right] \sin \left[\frac{(\omega_1 + \omega_2)t}{2} - \frac{(k_1 + k_2)x}{2}\right]$$

or

$$y = A \sin \left[\frac{(\omega_1 + \omega_2)t}{2} - \frac{(k_1 + x_2)x}{2}\right]$$

Here,

$$A = 2a \cos \left[\frac{(\omega_1 - \omega_2)t}{2} - \frac{(k_1 - k_2)x}{2}\right]$$

Above equation can also be written as

$$y = 2a \cos \left(\frac{\Delta\omega}{2} t - \frac{\Delta k}{2} x\right) \sin (\omega t - kx)$$

The velocity with which this wave packet moves is given by

$$\therefore \qquad \frac{\Delta\omega}{2}t - \frac{\Delta k}{2}x = \text{constant}$$

or $$\frac{\Delta k}{2}x = \frac{\Delta\omega}{2}t \qquad \text{or} \qquad x = \frac{\Delta\omega}{\Delta k}t$$

or $$\frac{dx}{dt} = \frac{\Delta\omega}{\Delta k} \qquad \text{or} \quad \text{Group velocity, } v_g = \frac{\Delta\omega}{\Delta k}$$

In the limiting case, $\boxed{v_g = \frac{d\omega}{dk}}$

<u>Phase velocity of de-Broglie waves</u>:- According to de-Broglie's wave concept the wavelength of matter wave associated with moving particle is given by

$\boldsymbol{\lambda=\frac{h}{mv}}$, Where h is Planck's constant

And wave propagation constant $\mathbf{k=\frac{2\pi}{\lambda}=\frac{2\pi mv}{h}}$ ......(1)

If E is the energy of the particle then **E=hν or ν =**$\boldsymbol{\frac{E}{h}}$

And $\boldsymbol{\omega = 2\pi\nu=2\pi\frac{E}{h}}$

According to Einstein's mass energy relation

$\mathbf{E=mc^2}$, therefore $\boldsymbol{\omega =\frac{2\pi m}{h}c^2}$ ......(2)

Therefore phase velocity of matter wave is given by- $\mathbf{v_p =\frac{\omega}{k}}$

Substitutuing the values of ω and k from equation 1 and 2

$\mathbf{v_p =\frac{c^2}{v}}$

$\mathbf{v_p . v=c^2}$ **or** $\mathbf{v_p . v_g=c^2}$

Q. Obtain relation between group velocity and phase velocity.

<u>Relation between Group velocity & Phase velocity</u>:-

Phase velocity, $$v_p = \frac{\omega}{k} \quad \text{or} \quad \omega = v_p k$$

Group velocity, $$v_g = \frac{d\omega}{dk}$$

or $$v_g = \frac{d\omega}{d\left(\frac{2\pi}{\lambda}\right)} \qquad \left[\because k = \frac{2\pi}{\lambda}\right]$$

or $$v_g = \frac{d\omega}{-\frac{2\pi}{\lambda^2}\, d\lambda} \quad \text{or} \quad v_g = -\frac{\lambda^2}{2\pi}\frac{d\omega}{d\lambda}$$

or $$v_g = -\frac{\lambda^2}{2\pi}\frac{d}{d\lambda}(v_p k)$$

or $$v_g = -\frac{\lambda^2}{2\pi}\frac{d}{d\lambda}\left(\frac{2\pi v_p}{\lambda}\right) \quad \text{or} \quad v_g = -\lambda^2 \frac{d}{d\lambda}\left(\frac{v_p}{\lambda}\right)$$

or $$v_g = -\lambda^2\left[v_p\left(-\frac{1}{\lambda^2}\right) + \frac{1}{\lambda}\frac{dv_p}{d\lambda}\right] \quad \text{or} \quad \boxed{v_g = v_p - \lambda\frac{dv_p}{d\lambda}}$$

Equation shows that in a dispersive medium group velocity is less than the phase velocity. In a non-dispersive medium v_p is constant.

$$\therefore \qquad \frac{dv_p}{d\lambda} = 0$$

$$\boxed{v_g = v_p}$$

It means in a non-dispersive medium or in free sapce, the group velocityof a wave packet is equal to phase velocity.

Q1. Write the characteristics of wave function.(2015-16)

Q2. Give the physical significance of wave function ψ in quantum mechanics? (2011-12)

<u>Born interpretation of wave function & its Physical Significance</u>

The quantity whose variation builts up matter waves is called wave function (ψ). The value of wave function associated with a moving particle at a particulat point (x, y, z) in space at the time t is related to the possibility of finding the particle there at that time. The physical significance of the wave function is that the square of its absolute value at a point is proportional to the probability of experimantally finding the particle described by the wave function in a small element of volume dτ (dxdydz) at that point. **Acc. to Max Born** $|\psi|^2$ gives the probability of finding the particle in the state ψ, i.e., $|\psi|^2$ is a measure of probability density. For the total probability of finding the particle somewhere is unity, i.e., particle is certainly to be found somewhere in space.

$$\iiint |\psi|^2 dxdydz = 1$$

Ψ satisfying above requirement is said to be normalised.

<u>Charactistic of wave function</u>:-

1. It must be finite everywhere.
2. It must be single valued.
3. It must be continuous.Its first derivative should also be continous.

Q. Derive Schrödinger's Time dependent equations.

Equation of motion of matter waves

(a) **Schrodinger time dependent wave equation-** The differential equation of a wave motion of a particle in one-dimension can be written as

$$\frac{\partial^2\psi}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2} \qquad \text{........(1)}$$

The general solution of eq.(1) is given by

$$\psi(x,t) = Ae^{-i\omega[t-(x/v)]} \qquad \text{........(2)}$$

We know that ω = 2πν and v = νλ. So, eq.(2) can be written as

$$\psi(x,t) = Ae^{-2\pi i[\nu t-(x/\lambda)]} \qquad \text{........(3)}$$

Put ν = E/h and λ = h/p in eq.(3), we get

$$\psi(x,t) = Ae^{-\left(\frac{2\pi i}{h}\right)(Et-px)} \qquad \text{........(4)}$$

Now, differentiating eq.(4) twice with respect to x, we get

$$\frac{\partial^2\psi}{\partial x^2} = Ae^{-\left(\frac{2\pi i}{h}\right)(Et-px)}(2\pi ip/h)^2 = -\frac{4\pi^2 p^2}{h^2}\psi \quad \text{or } p^2 = \frac{1}{\psi}\frac{h^2}{4\pi^2}\frac{\partial^2\psi}{\partial x^2} \qquad \text{........(5)}$$

Now, differentiate eq.(4) with respect to t, we get

$$\frac{\partial\psi}{\partial t} = Ae^{-\left(\frac{2\pi i}{h}\right)(Et-px)}\left(-\frac{2\pi iE}{h}\right) = -\frac{2\pi iE}{h}\psi$$

$$or\ E = -\frac{1}{\psi}\frac{h}{2\pi i}\frac{\partial\psi}{\partial t} = \frac{1}{\psi}\frac{ih}{2\pi}\frac{\partial\psi}{\partial t} \qquad \text{........(6)}$$

If E & V be the total & potential energies of the particle respectively, then its kinetic energy $\frac{1}{2}mv^2$ is given by

$$\frac{1}{2}mv^2 = E - V \qquad or\ E = \frac{p^2}{2m} + V \qquad \text{.......(7)}$$

Putting the values from eqs.(5) & (6) in eq.(7), we get

$$\frac{1}{\psi}\frac{ih}{2\pi}\frac{\partial\psi}{\partial t} = \frac{h^2}{8\pi^2 m}\frac{1}{\psi}\frac{\partial^2\psi}{\partial x^2} + V \quad or \; -\frac{h^2}{8\pi^2 m}\frac{\partial^2\psi}{\partial x^2} + V\psi = \frac{ih}{2\pi}\frac{\partial\psi}{\partial t}$$

Substituting $\hbar = \frac{h}{2\pi}$, we get

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V\psi = \frac{ih}{2\pi}\frac{\partial\psi}{\partial t} \qquad \text{.........(8)}$$

This is the required shrodinger time dependent equation in one dimension. In three dimension the above equation can be written as-

$$or \; \left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)\psi = \frac{ih}{2\pi}\frac{\partial\psi}{\partial t}$$

Q. Derive Schrödinger's Time independent equations.

(b). Schrodinger time independent wave equation-

The shrodingers time independent equation can be obtained with help of time dependent equation. The differential equation of a wave motion of a particle in one-dimension can be written as

$$\frac{\partial^2\psi}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2} \qquad \text{........(1)}$$

The general solution of eq.(1) is given by

$$\psi(x,t) = Ae^{-i\omega[t-(x/v)]} \qquad \text{........(2)}$$

We know that ω = 2πν and v = νλ. So, eq.(2) can be written as

$$\psi(x,t) = Ae^{-2\pi i[\nu t-(x/\lambda)]} \qquad \text{........(3)}$$

Put ν = E/h and λ = h/p in eq.(3), we get

$$\psi(x,t) = Ae^{-\left(\frac{2\pi i}{h}\right)(Et-px)} \qquad \text{........(4)}$$

The wave function can be seperated into time dependent and space dependent parts.

$$\psi(x,t) = Ae^{\left(\frac{2\pi ipx}{h}\right)}\,e^{-\left(\frac{2\pi iEt}{h}\right)}$$

If $\psi(x) = \psi_\circ = Ae^{\left(\frac{2\pi ipx}{h}\right)}$ then,

$$\psi(x,t) = \psi_\circ\, e^{-\left(\frac{2\pi iEt}{h}\right)} \qquad \text{......(5)}$$

Now differentiationg equation 5 twicw with respect to x, we get

$$\frac{\partial^2\psi}{\partial x^2} = \frac{\partial^2\psi_\circ}{\partial x^2}\, e^{-\left(\frac{2\pi iEt}{h}\right)} \qquad \text{......(6)}$$

differentiationg equation 5 with respect to t, we get

$$\frac{\partial\psi}{\partial t} = \psi_\circ\, e^{-\left(\frac{2\pi iEt}{h}\right)}\left(-\,\frac{2\pi iE}{h}\right) \qquad \text{......(7)}$$

The well known shrodinger time dependent equation in one dimension is given by-

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V\psi = \frac{ih}{2\pi}\frac{\partial\psi}{\partial t}$$

Subtituting the values of ψ, $\frac{\partial^2\psi}{\partial x^2}$ and $\frac{\partial\psi}{\partial t}$ from equations 5,6,7 in above equation,

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi_\circ}{\partial x^2}\, e^{-\left(\frac{2\pi iEt}{h}\right)} + V\psi_\circ\, e^{-\left(\frac{2\pi iEt}{h}\right)} = \frac{ih}{2\pi}\,\psi_\circ\, e^{-\left(\frac{2\pi iEt}{h}\right)}\left(-\,\frac{2\pi iE}{h}\right)$$

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi_\circ}{\partial x^2} + V\psi_\circ = E\psi_\circ$$

$$\frac{\partial^2\psi_\circ}{\partial x^2} + \frac{2m}{\hbar^2}(E - V)\psi_\circ = 0$$

OR in general we can write above equation

$$\frac{\partial^2\psi}{\partial x^2} + \frac{2m}{\hbar^2}(E - V)\,\psi = 0$$

This is the required time dependent equation in one dimension . In three dimension the above equation can be written as-

$$\nabla^2\psi + \frac{2m}{\hbar^2}(E - V)\,\psi = 0 \qquad \text{........(8)}$$

For a free particle V = 0, hence the Schrodinger wave equation for a free particle can be expressed as

$$\nabla^2\psi + \frac{2m}{\hbar^2}E\,\psi = 0$$

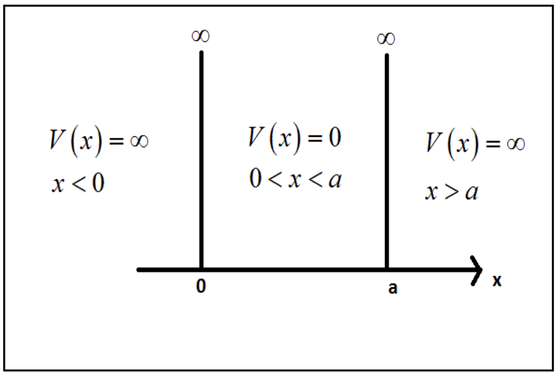
Q. Find the expression for Energy Eigen values and Eigen function for particle in one dimensional box.

<u>OR</u> Obtaion normalized Wave function for particle in one dimensional box.

<u>Particle in one-dimensional Box</u>

Let us consider the case of a particle of mass m moving along x-axis between two rigid walls A & B at x = 0 & x = a. The particle is free to move between the walls. The potential function is defined in the following way:

	V(x)	=	0	for	$0 < x < a$
and	V(x)	=	∞	for	$0 \geq x$ and $x \geq a$



Under this condition, particle is said to move in an infinitely deep potential well or infinite square wall.

The Schroedinger equation for the particle within the box (V= 0) is,

$$\nabla^2 \psi + \frac{2m}{\hbar^2} E\,\psi = 0 \qquad \text{.........(1)}$$

$$\frac{d^2\psi}{dx^2} + K^2\psi = 0 \qquad \text{..........(2)}$$

where

$$k = \sqrt{\frac{2mE}{\hbar^2}} \qquad \text{..........(3)}$$

The general solution of eq.(2) is of the for

$$\psi = A \sin kx + B \cos kx \qquad \text{..........(4)}$$

Apply the boundary condition, ψ = 0 at x = 0 & x = a to eq.(4)

$$0 = A \sin k \cdot 0 + B \cos k \cdot 0 \quad \Rightarrow \quad B = 0$$

$$0 = A \sin ka$$

or $$ka = \pm n\pi, \quad \text{where } n = 1, 2, 3, 4, \ldots \qquad \text{..........(5)}$$

From eq.(3) & eq.(5), we get

$$E_n = \frac{n^2 h^2}{8ma^2} \qquad \text{..........(6)}$$

From eq.(6), it is cleat that the particle can not have an arbitrary energy, but can have certain discrete energy corresponding to n = 1, 2, 3,…….. Each permitted energy is called eigen value of the particle & constitute the energy level of the system. The corresponding eigenfunction is given by

$$\psi_n = A \sin kx$$

To find the value of constant A we apply normalisation condition,

$$\int_0^a |\psi_n|^2\, dx = 1$$

$$A^2 \int_0^a \sin^2 \frac{n\pi}{a}\, dx = 1$$

or $$A^2 \int_0^a \frac{1}{2}\left\{1 - \cos\left(\frac{2n\pi x}{a}\right)\right\} dx = 1$$

or $$A^2\left\{\frac{a}{2}-\int_0^a \cos\frac{2n\pi x}{a}\,dx\right\}=1$$

$$A^2\,\frac{a}{2}=1 \qquad \text{or} \qquad A=\sqrt{\frac{2}{a}}$$

Therefore, the normalised wavefunction for n^{th} state is given by

$$\psi_n=\sqrt{\frac{2}{a}}\sin\frac{n\pi}{a}x \qquad \text{.........(7)}$$

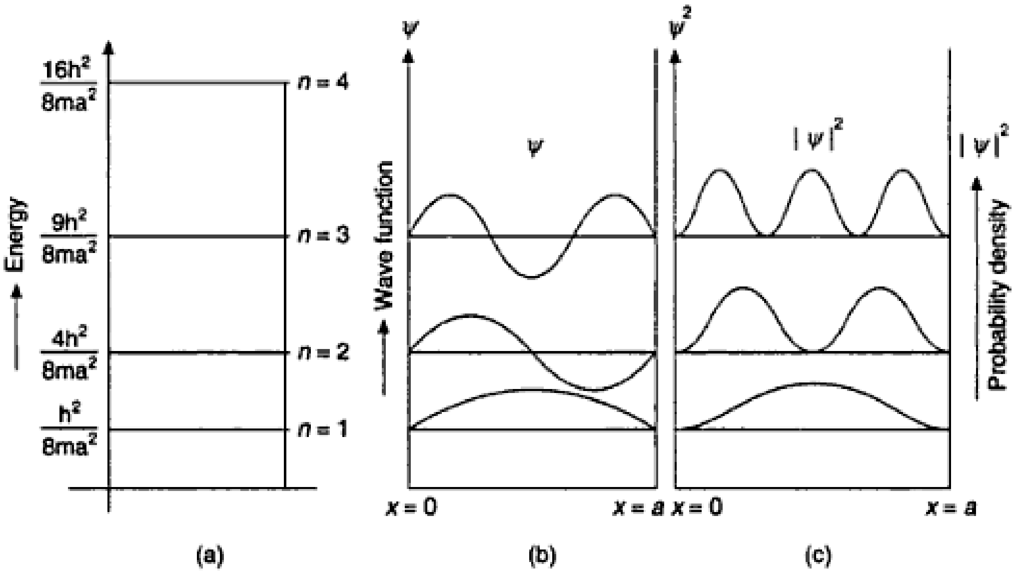


Fig. (a) Energy levels (b) wave function (c) Probability density

Lecture-23

Q. What is Compton effect? Derive expression for Compton shift.

Compton Effect

In 1921, Professor A.H. Compton discovered that when a monochromatic beam of high

frequency radiation is scattered by electrons, the scattered radiation contain the radiations of lower frequency or greater wavelength along with the radiations of unchanged frequency or wavelength, the radiations of unchanged wavelength in the scattered light are called unmodified radiations while the radiations of greater wavelength are called modified radiations. This phenomenon is called the Compton Effect. It provided evidence for the particle nature of light and Planck's postulates.

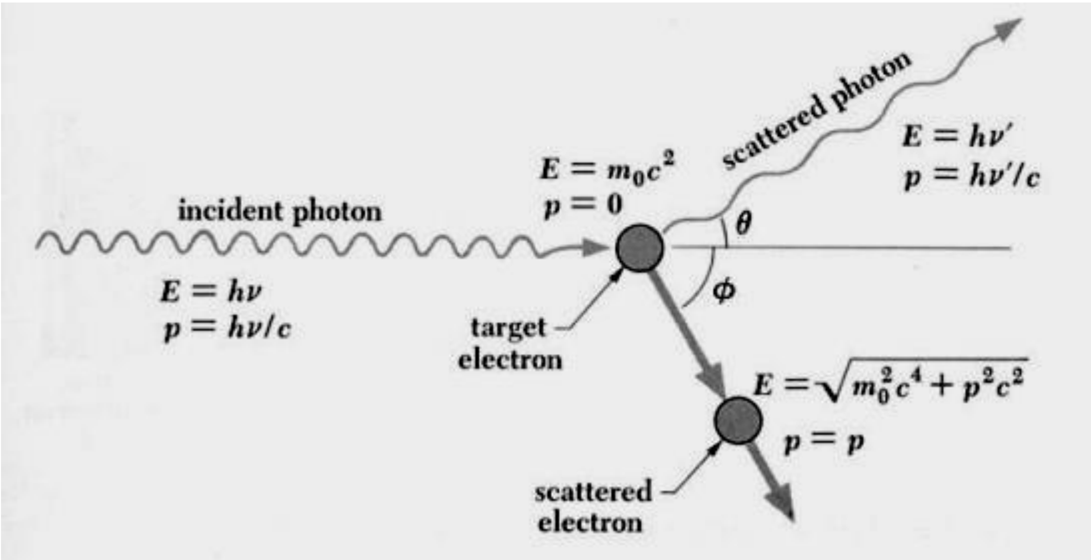
Quantum Explanation: The explanation was given by Compton which was based on

Quantum theory of light. According to quantum theory when photon of energy $h\nu$ strikes with the substance some of the energy of photon is transferred to the electrons, therefore the energy (or frequency) of photon reduces and wavelength increases.

Various assumptions were made for explaining the effect these were:

(i) Compton Effect is the result of interaction of an individual particle and free electron of target.

(ii) The collision is relativistic and elastic.
(iii) The laws of conservation of energy and momentum hold good.



The energy of the system before collision= $hv + m_oc^2$
The energy of the system after collision = $hv' + mc^2$

According to the principle of conservation of energy

$$hv + m_oc^2 = hv' + mc^2$$

or $mc^2 = (hv - hv') + m_oc^2$ ....(1)

According to the principle of conservation of linear momentum along and perpendicular
To the direction of incident photon (i.e., along x and y axis), we have

$$\frac{hv}{c} + 0 = \frac{hv\prime}{c}\cos\theta + mv\cos\Phi$$

Or

$$mvc\cos\Phi = hv - hv'\cos\theta \quad ....(2)$$

And (for y-axis)

$$0 = -\frac{hv\prime}{c}\sin\theta + mv\sin\Phi$$

Or

$mvc \sin \Phi = hv' \sin \theta$ ....(3)

Squaring (2) and (3) and then adding, we get

$m^2\, v^2 c^2 = (\, hv - hv' \cos \theta)^2 + (hv' \sin \theta)^2$

$m^2\, v^2 c^2 = (\, hv)^2 + (hv' \cos \theta)^2 - 2(hv)(hv') \cos \theta + (hv' \sin \theta)^2$

$m^2\, v^2 c^2 = (\, hv)^2 + (hv')^2 (\, [(\cos \theta)^2 + (\sin \theta)^2] - \, 2(hv)(hv') \cos \theta$

$m^2\, v^2 c^2 = (\, hv)^2 + (hv')^2 - \, 2(hv)(hv') \cos \theta$ ....(4)

Squaring equation (1), we get

$m^2\, c^4 = m_o{}^2 c^4 + (\, hv)^2 + (hv')^2 - \, 2(hv)(hv') + 2m_o c^2\, (hv \, - \, hv')$ ......(5)

Subtracting (4) from (5), we get

$m^2\, c^4 - m^2\, v^2 c^2 = m_o{}^2 c^4 + 2(hv)(hv')(co\, s\, \theta - 1) + 2m_o c^2\, (hv \, - \, hv')$......(6)

According to the theory of relativity

$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}$ or $m^2 = \frac{m_o{}^2}{(1 - \frac{v^2}{c^2})}$ or $m^2 \left(1 - \frac{v^2}{c^2}\right) = m_o{}^2$

Thus $m^2 c^2 - m^2\, v^2 = m_o{}^2 c^2$

Multiplying both sides by c^2, we get

$m^2 c^4 - m^2\, v^2 c^2 = m_o{}^2 c^4$ ......(7)

Using equation (7), equation (6) becomes

$m_o{}^2 c^4 = m_o{}^2 c^4 + 2(hv)(hv')(co\, s\, \theta - 1) + 2m_o c^2\, (hv \, - \, hv')$

$0 = 2(hv)(hv')(co\, s\, \theta - 1) + 2m_o c^2\, (hv \, - \, hv')$

$2(hv)(hv')(co\, s\, \theta - 1) = 2m_o c^2\, (hv \, - \, hv')$

$\frac{(v \, - \, v')}{v' v} = \frac{h}{m_o c^2}(1 - co\, s\, \theta)$

$\frac{1}{v'} - \frac{1}{v} = \frac{h}{m_o c^2}((1 - co\, s\, \theta)$ ......(8)

To find the relation in term of wavelength, let us substitute $v' = \frac{c}{\lambda'}$ and, $v = \frac{c}{\lambda}$

We thus have

$$\Delta\lambda = \lambda' - \lambda = \frac{h}{m_o c}((1 - co\,s\,\theta)$$

Compton shift

$$\Delta\lambda = \lambda' - \lambda = \frac{h}{m_o c}((1 - co\,s\,\theta) \quad ....(9)$$

Lecture-24

From above equations (8) and (9) following conclusions can be drawn

1. The wavelength of the scattered photon λ' is greater than the wavelength of incident photon λ.
2. Δ λ is independent of the incident wavelength.
3. Δ λ have the same value for all substance containing free electron
4. Δ λ only depend on the scattering angle ϕ.

1. when θ = 0; cos $\theta = 1$

$\Delta\lambda = \lambda' - \lambda = 0$

$\lambda' = \lambda$, the scattered wavelength is same as the incident wavelength in the direction of incidence.

2. when $\theta = 90°$; cos $\theta = 0$

$$\Delta\lambda = \lambda' - \lambda = \frac{h}{m_o c}$$

$$\Delta\lambda = \frac{h}{m_o c} = 0.02426 \mathring{A}$$

$= \lambda_c$; ....(10)

Where λ_c is called the Compton wavelength of the electron.

3. when $\theta = 180°$;

$$\Delta\lambda = \frac{2h}{m_o c} = 0.04652 \mathring{A} \quad ....(11)$$

Why Compton Effect is not observed in visible spectrum

The maximum change in wavelength $\Delta \lambda_{max}$ is 0.04652 Å or roughly 0.05 Å. This is very small, therefore cannot be observed for wavelength longer than few angstrom units.

For example- For X-ray, the incident radiation is about 1 Å , $\Delta \lambda_{max}$ is 0.05 Å therefore the percentage of incident radiation is about 5%(detectable)

For Visible radiation, the incident radiation is about 5000 Å, $\Delta \lambda_{max}$ is 0.05 Å therefore the percentage of incident radiation is about 0.001% (undetectable)

Direction of Recoil electron

Dividing equation (2) by (3) direction of recoil electron is given by

$$\tan\Phi = \frac{h\nu' \sin\theta}{h\nu - h\nu' \cos\theta}$$

$$\tan\Phi = \frac{\nu' \sin\theta}{\nu - \nu' \cos\theta}$$

$$\tan\Phi = \frac{\frac{c}{\lambda'}\sin\theta}{\frac{c}{\lambda} - \frac{c}{\lambda'}\cos\theta}$$

$$\tan\Phi = \frac{\lambda \sin\theta}{\lambda' - \lambda \cos\theta}$$

Kinetic energy of recoiled electron:

$$E = h\nu - h\nu'$$

$$E = \frac{hc}{\lambda} - \frac{hc}{\lambda'}$$

$$E = hc\left(\frac{1}{\lambda} - \frac{1}{\lambda'}\right)$$

$$\boldsymbol{E = hc\left(\frac{\lambda' - \lambda}{\lambda\lambda'}\right)}$$

<u>Question</u>

1. What is Compton Effect? (2017-18)
2. What do you mean by Compton shift? (2013-14)
3. What information do you derive about the nature of light from the Compton Effect experiment? (2011-12)
4. Why Compton Effect is not observable for visible light? (2012-13)

Example 1: (i) In the year 1911, Abbot found that the wavelength of maximum energy in the solar system was $\lambda_m = 4753$ Å. If the value of Wien's constant be 0.293 cm-deg, what is the effective solar temperature?

K

(ii) In an atomic explosion, the maximum temperature reached was of the order of 10^7 K. Calculate the wavelength of the maximum energy.

Solution: **(i)** According to Wien's law

$$\lambda_m T = \text{constant} = 0.293$$

Here $$\lambda_m = 4753 \text{ Å} = 4753 \times 10^{-8} \text{ cm}$$

$\therefore$ $$4753 \times 10^{-8}\, T = 0.293 \qquad \text{or} \qquad T = 6164.53 \text{ K}$$

Thus, **the effective temperature of the sun = 6164.53 K**

(ii) Again $$\lambda_m T = 0.293$$

Here, maximum temperature T is of the order of 10^7 K, therefore

$$\lambda_m = \frac{0.293}{T} = \frac{0.293}{10^7} = 2.93 \times 10^{-8} \text{ cm} = \mathbf{2.93\ \text{Å}}$$

Example 2: Using Wien displacement law, estimate the temperature of sun. Given $\lambda_m = 4900$ Å and Wien's constant = 0 .292 cm-K.

Solution: According to Wien's displacement law,

$$\lambda_m T = \text{constant} = b$$

where T K is the temperature of a perfect black body and λ_m is the maximum wavelength at which the spectral radiancy is maximum

For sun, $\lambda_m = 4900 \text{ Å} = 4900 \times 10^{-8}$ cm and $b = 0.292$ cm-K

Therefore, the temperature of the sun, $T = \dfrac{b}{\lambda_m} = \dfrac{0.292}{4900 \times 10^{-8}} = 5.959 \times 10^3$

or $T = \mathbf{5959\ K}$

Example 3: A body at 1500 K emits maximum energy at a wavelength 20,000 Å. If the sun emits maximum energy at wavelength 5500 Å, what would be the temperature of the sun?

Solution: According to Wien's displacement law,

$$\lambda_m T = \text{const.}$$

If sun emits maximum energy at wavelength λ'_m and the temperature of the sun is T', then

$$\lambda'_m T' = \text{const.}$$

or $\lambda_m T = \lambda'_m T'$ or $T' = \dfrac{\lambda_m T}{\lambda'_m}$

Here, $\lambda_m = 20{,}000$ Å, $T = 1500$ K and $\lambda'_m = 5500$ Å

∴ the temperature of sun, $T' = \dfrac{20{,}000 \times 1500}{5500} = \mathbf{5454\ K}$

Example 4: Deduce the frequency corresponding to the maximum energy density in the radiation emitted from a black body at temperature 1000 K. Given that, $k = 1.38 \times 10^{-23}$ JK^{-1} and $h = 6.6 \times 10^{-34}$ J-sec.

Solution: According to Wien's law

$$\lambda_m T = \text{const.} = \frac{hc}{5k} \quad ...(1)$$

The frequency corresponding to maximum wavelength λ_m is

$$\upsilon_m = \frac{c}{\lambda_m}$$

Thus from eqn. (1), we have

$$\upsilon_m = \frac{5kTc}{hc} = \frac{5kT}{h}$$

Here $k = 1.38 \times 10^{-23}$ J- K^{-1}, $T = 1000$ K and $h = 6.6 \times 10^{-34}$ J-sec

∴ $\upsilon_m = \dfrac{5 \times 1.38 \times 10^{-23} \times 1000}{6.6 \times 10^{-34}} = \mathbf{1.045 \times 10^{14}}$ ...

Example 5: The black body spectral energy distribution of radiation from moon shows two wavelength maximum at 14 μ and 5000 Å. What is the significance of such an observation? Calculate the corresponding temperature. Given b = 0.3cm-deg.

Solution: The wavelength maximum at 14 μ (14×10^{-4} cm) is due to moon's own radiation, whereas that at 5000 Å (5000×10^{-8} cm) coincides with the maximum in solar radiation. It is because of the fact that solar radiation is reflected from moon's disc.

According to Wien's displacement law

$$\lambda_m T = \text{const.} = b$$

For moon,
$$T = \frac{b}{\lambda_m} = \frac{0.3}{14 \times 10^{-4}} = \mathbf{214\ K}$$

For sun,
$$T = \frac{b}{\lambda_m} = \frac{0.3}{5000 \times 10^{-8}} = \mathbf{6000\ K}$$

Example 6: What is the wavelength of maximum intensity radiation, radiated from a source having temperature 3000 K? The Wien's constant is 0.3×10^{-2} m-K.

Solution: According to Wien's law

$$\lambda_m T = b \qquad \text{or} \qquad \lambda_m = \frac{b}{T}$$

Here
$$T = 3000 \text{ K} \qquad \text{and} \qquad b = 0.3 \times 10^{-2} \text{ m-K}$$

$\therefore$
$$\lambda_m = \frac{0.3 \times 10^{-2}}{3000} = \mathbf{0.1 \times 10^{-5}\ m = 10000\ Å}$$

Example 7: From Wien's law, we have

$$\lambda_m T = \frac{hc}{4.965k} = b$$

The letters have their usual meaning. Calculated the constant b. Abbot's measurements show that λ_m for the solar radiation is 4753 Å. Calculate the temperature of the surface of the photosphere of the sun. Given $h = 6.63 \times 10^{-34}$ J - s, $k = 1.38 \times 10^{-23}$ J - K^{-1} and $c = 3.0 \times 10^{8}$ m sec^{-1}.

Solution: According to the given problem,

$$b = \frac{hc}{4.965\,k}$$

$\therefore$
$$b = \frac{6.63 \times 10^{-34} \times 3.0 \times 10^{8}}{4.965 \times 1.38 \times 10^{-23}} = \mathbf{2.90 \times 10^{-3}\ m\text{-}K}$$

For solar radiation, $\lambda_m = 4753$ Å. Therefore,

$$T = \frac{b}{\lambda_m} = \frac{2.90 \times 10^{-3}}{4753 \times 10^{-10}} = 6.10 \times 10^{3} \text{ K} = \mathbf{6100\ K}$$

This is the colour temperature of the sun. It is higher than the brightness temperature.

Matter Waves		Electromagnetic Waves	
1.	Matter waves are associated with moving particles irrespective of whether the particles are charged or not.	1.	Electromagnetic waves are produced only by accelerated charged particles.
2.	Matter waves obtained by charged particles are associated with electric and magnetic fields.	2.	Electromagnetic waves are associated with electric and magnetic fields perpendicular to each other as well as to the direction of propagation of wave.
3.	Matter waves are neither emitted by the particles nor radiated into space. These are simply associated with the particles.	3.	Electromagnetic waves can be radiated into space.
4.	The velocity of matter waves depends upon the velocity of the material particles.	4.	Velocity of E.M. waves is constant in a given medium.
5.	The velocity of matter waves is generally less than the velocity of light.	5.	The velocity of electromagnetic waves is equal to the velocity of light.
6.	The matter waves have shorter wavelengths given by de-Broglie equation : $\lambda = \frac{h}{mv}$ where v is the velocity.	6.	The wavelength of electromagnetic waves are given by relation : $\lambda = \frac{c}{\upsilon}$ where υ is the frequency.

$\lambda = 7.16 \times 10$ Å or 0.00716 Å

5

Example 28: *Calculate the kinetic energy of an electron if its de-Broglie wavelength equals the wavelength of sodium light (5893Å).* [MTU., B.Tech. II Sem 2011]

Solution: The de-Broglie's wavelength of a wave associated with a particle is given by.

$$\lambda = \frac{h}{mv} \quad \text{or} \quad v = \frac{h}{m\lambda}$$

where m is the mass of a particle moving with a velocity v. The kinetic energy k of the particle is

$$K = \frac{1}{2} mv^2 = \frac{1}{2} m\left(\frac{h}{m\lambda}\right)^2$$

or

$$K = \frac{1}{2} \frac{h^2}{m\lambda^2}$$

In the absence of relativity mass m of electron is equal to its rest mass, that is, $m = 9.1 \times 10^{-31}$ kg.

Here $h = 6.63 \times 10^{-34}$ joule-sec and $\lambda = 5893 \times 10^{-10}$ m.

$\therefore$ Kinetic energy, $K = \dfrac{(6.63 \times 10^{-34})^2}{2 \times 9.1 \times 10^{-31} \times (5893 \times 10^{-10})^2}$

or

$$K = 6.955 \times 10^{-25} \text{ joule}$$

Example 42: Show that the de–Broglie wavelength for a material particle of rest mass m_0 and charge q accelerated from rest through a potential difference of V volts relativistically is given by

$l = h / \{2m_0qV[1 + qV / 2m_0c^2]\}^{1/2}$. **[UPTU, B. Tech. I Sem. Q. Bank, 2001]**

Solution: The de–Broglie wavelength of a material particle of momentum p is given by

$$\lambda = \frac{h}{p} \qquad ...(1)$$

The particle has charge q and accelerate to potential difference V, therefore its kinetic energy, $E_K = qV$. Since velocity is relativistic, we find momentum p by using relativistic formula as E

$$E^2 = p^2c^2 + m_0^2c^4,\ E = E_K + m_0c^2 = qV + m_0c^2$$

$$\therefore \quad p^2c^2 = E^2 - m_0^2c^4 = (qV + m_oc^2)^2 - m_o^2c^4$$

$$= q^2V^2 + m_0^2c^4 + 2m_0c^2qV - m_0^2c^4 = q^2V^2 + 2m_0c^2qV$$

or
$$p^2 = 2m_0qV + \frac{q^2V^2}{c^2} = 2m_0qV\left(1 + \frac{qV}{2m_0c^2}\right)$$

or
$$p = \sqrt{2m_0qV\left(1 + \frac{qV}{2m_0c^2}\right)} \qquad ...(2)$$

Substituting the value of p from equation(2) in equation (1), we get

$$\lambda = \frac{h}{\sqrt{2m_0qV\left(1+\frac{qV}{2m_0c^2}\right)}}$$

Example 68: At what rate will energy be emitted from a bulb whose filament has a temperature of 3600 K, if at 1800 K the energy is emitted at the rate of 16 watt.

Solution: From Stefan's law, we know that

$$E = \sigma T^4 \qquad \text{or} \qquad E \propto T^4$$

$$\therefore \quad \frac{E_1}{E_2} = \frac{T_1^4}{T_2^4}$$

Here, $E_1 = 16$ watt, $T_1 = 1800$ K, $T_2 = 3600$ K and $E_2 = ?$

$$\therefore \quad E_2 = E_1 \times \frac{T_2^4}{T_1^4} = 16 \times \left(\frac{3600}{1800}\right)^4 = \mathbf{256\ watt}$$

Example 69: Luminosity of Riget star in orion constellation is 17000 times that of our sun. If the surface temperature of the sun is 6000 K. Calculate the temperature of star.

Solution: According to Stefan's law, the total energy radiated per second per unit surface area from a black body at temperature TK is given by,

$$E = \sigma (T^4 - T_0^4)$$

where T_0 is the temperature of the surroundings and σ the Stefan's constant.

For $T >> T_0$, we can approximated the above expression as

$$E = \sigma T^4$$

If E_1 and E_2 be the luminosities of the star and the sun respectively, and T_1 and T_2 their respective temperatures, then

$$\frac{E_1}{E_2} = \frac{T_1^4}{T_2^4} \qquad \text{or} \qquad T_1^4 = \frac{E_1}{E_2} \times T_2^4$$

Here

$$\frac{E_1}{E_2} = 17000 \qquad \text{and} \qquad T_2 = 6000 \text{ K}$$

Thus,

$$T_1^4 = 17000 \times (6000)^4$$

or

$$T_1 = 6000 \times (17000)^{1/4} = 6000 \times 11.42 = \mathbf{68520\ K}$$

Example 70: ... enclosure whose walls are blackened and kept at ... or lost by the body when its the particle being anywhere in the box is same.

(10)

Example 75: Find the energy of an electron moving in one dimension in an infinitely high potential box of width 1Å (mass of the electron is 9.11 $\times 10^{-31}$ kg and h = 6.63×10^{-34} J–s).

[UPTU, B.Tech. I Sem. 2001, I Sem. 2005, II Sem. 2006, II Sem. 2007]

Solution : The energy of a particle of mass m moving in one dimension in an infinitely high potential box of width L is given by

$$E = \frac{n^2 h^2}{8mL^2}, \text{ where } n = 1, 2, 3, \ldots$$

The minimum energy of the particle is obtained by substituting $n = 1$ in the above formula, that is,

$$E = \frac{h^2}{8mL^2}$$

Here $m = 9.11 \times 10^{-31}$ kg, $h = 6.63 \times 10^{-34}$ joule–sec and $L = 1\text{ Å} = 10^{-10}$ m

$$\therefore \quad E = \frac{(6.63 \times 10^{-34})^2}{8 \times 9.11 \times 10^{-31} \times (10^{-10})^2} = 6.03 \times 10^{-18} \text{ joule}$$

$$= \frac{6.63 \times 10^{-18}}{1.6 \times 10^{-19}} \text{ eV} = 37.69 \text{ eV}$$

$\sqrt{2a}$

Example 80: ***An electron is bound in one dimensional potential box which has width 2.5×10^{-10}m. Assuming the height of the box to be infinite, calculate the lowest two permitted energy values of the electron.***

[UPTU, B.Tech. II Sem. 2001]

Solution: The energy of practical of mass m moving in one dimensional potential box of infinite height and of width L is given by

$$E_n = \frac{n^2 h^2}{8mL^2}, \text{ where } n = 1, 2, 3, \ldots$$

Here $h = 6.63 \times 10^{-34}$ J.s, $m = 9.1 \times 10^{-31}$kg and $L = 2.5 \times 10^{-10}$ m

$$\therefore \quad E_n = \frac{n^2 (6.63 \times 10^{-34})^2}{8 \times 9.1 \times 10^{-31} \times (2.5 \times 10^{-10})^2} = 9.66 \times 10^{-19} n^2 \text{joule}$$

or

$$E_n = \frac{9.66 \times 10^{-19}}{1.6 \times 10^{-19}} n^2 \text{eV} = 6.04\, n^2 \text{eV}$$

The lowest two permitted energy values of the electron is obtained by putting $n = 1$ and $n = 2$.

First lowest permitted energy value ($n = 1$) = 6.04 eV

Second lowest permitted energy value ($n = 2$) = 6.04×2^2 eV = **24.16 eV**

Unit IV Interference & Diffraction

Interference of Light.

When two waves of the same frequency travel approximately in the same direction and have a constant phase difference that remains constant with time, the resultant intensity of light is not uniformly distributed in space. The non-uniform distribution if the light intensity due to the superposition of two waves is called *interference.*

At some points the intensity is maximum and the interference at these points is called *constructive interference*. At some other points the intensity is minimum and the interference is called *destructive* i*nterference*. When two light waves are made to interfere, we get alternate dark and bright fringes of a regular or irregular in shape. These are called *interference fringes.*

Coherent Sources

Two Sources are said to be coherent if they emit continuous light waves of the same frequency or wavelength, nearly the same amplitude and having sharply defined phase difference that remains constant with time. Any two independent beams of light are always incoherent therefore there is no interference by two different light sources.

Essential Conditions of Sustained Interference

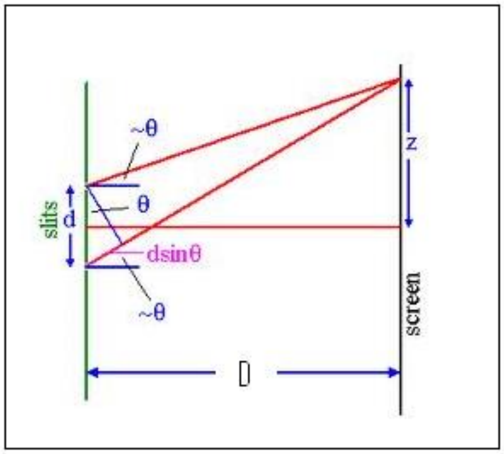
For good interference the following conditions must be satisfied:

1. The two interfering waves should be coherent.
2. The two interfering waves should emit light of the same frequency or wavelength.
3. The two coherent sources must be narrow.
4. The separation between the coherent sources should be as small as possible.
5. The distance of the screen from the two sources should be quite large.

Formation of Coherent Sources:

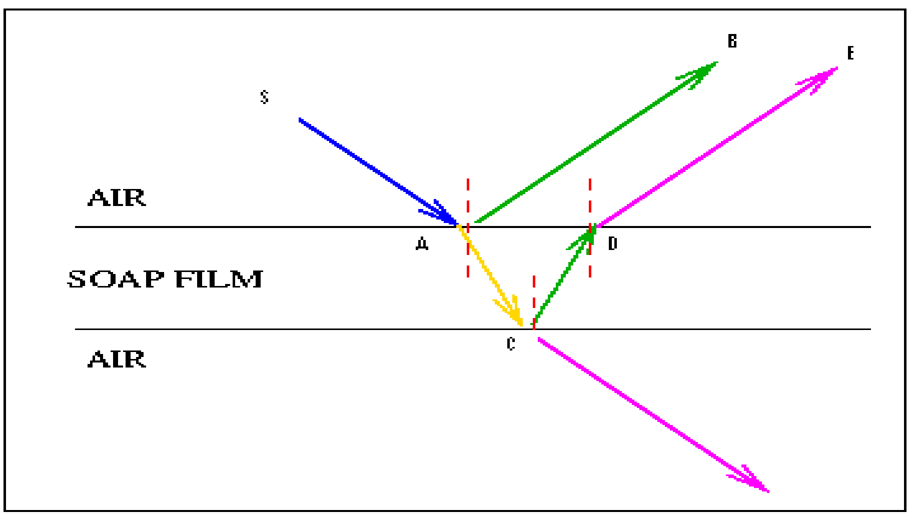
In some famous experimental set up the coherent sources are obtained in the following manner:

1. Young's Double-Slit Experiment:-In this method monochromatic light is passed from a small source through a slit and the light emerging from this slit is used to illuminate two other narrow slits which are very closed together and parallel to slit. Two narrow slit act as coherent sources. Alternate bright and dark equally spaced fringes are observed on a screen placed at some distance from the coherent sources.



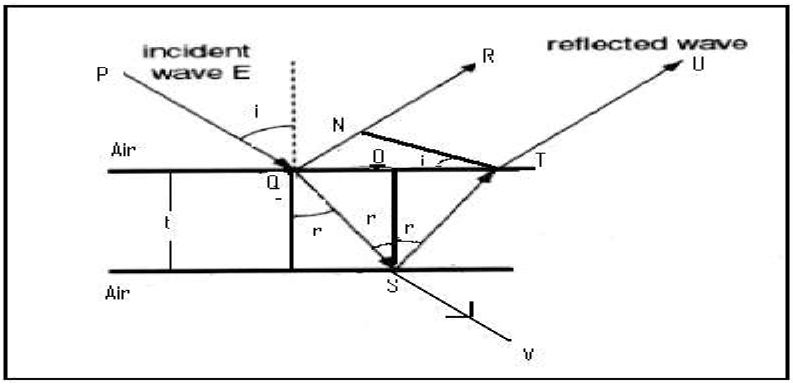
2. Thin Film:-

In this case two coherent beams are obtained by division of the amplitude of an incoming beam by partial reflection and refraction. A single wave train of monochromatic light be incident on the upper surface if a thin film of refractive index μ (>1) and thickness t at an acute angle. This ray is partly reflected along AB and partly refracted along AC. At C, the ray is again partly reflected along CD and then part of it is transmitted along DE. The rays AB and DE are act as coherent sources.



Newton & Hooke observed and developed the interference phenomenon due to the multiple reflections from the surface of thin transparent materials. When a thin film of oil spread on the surface of water or a thin glass plate, is illuminated by source of light, interference occurs between the light waves reflected from the sun & in between the light waves transmitted through film. As a result some brilliant colors are seen in that film.

(A) Interference due to reflected light in thin film: Consider a transparent film of thickness t & refractive index μ (>1). Let a ray of monochromatic light PQ be incident on the upper surface of the film.



PQ ray is partly reflected along QR and partly refracted along QS. At S, it is again partly reflected ST and partly refracted along SV and this process continues throughout the film.

The optical path difference,

$$\Delta = \text{path }(QS+ST) - \text{path }QN$$

$$\Delta = \mu\,(QS+ST) - QN \qquad \text{…………(1)}$$

In right angled Δ QSO,

$$\cos r = \frac{SO}{QS} \qquad or \qquad QS = \frac{t}{\cos r} \qquad \text{…………(2)}$$

Similarly in right angled Δ SOT,

$$\cos r = \frac{SO}{ST} \qquad or \qquad ST = \frac{t}{\cos r} \qquad \text{…………(3)}$$

In right angled Δ QTN,

$$\sin i = \frac{QN}{QT} \quad or \quad QN = QT\sin i = (QO+OT)\sin i \qquad \text{…………(4)}$$

Now again in Δ QSO and Δ SOT,

$$\tan r = \frac{QO}{OS} \qquad and \qquad \tan r = \frac{OT}{OS}$$

Putting these values in eq^n (4), we get

QN = (OS tan r + OS tan r) sin i

QN = 2t tan r(μ sin r) [From Snell's law, $\frac{\sin i}{\sin r} = \mu$]

QN = $2\mu t \frac{\sin^2 r}{\cos r}$ …………..(5)

Putting the values from eq^n (2),(3) & (5) in eq^n (1)

$$\Delta = \mu(\frac{t}{\cos r} + \frac{t}{\cos r}) - 2\mu t \frac{\sin^2 r}{\cos r}$$
$$\Delta = \frac{2\mu t}{\cos r}(1 - \sin^2 r) \qquad = \frac{2\mu t \cos^2 r}{\cos r}$$
$$\Delta = 2\mu t \cos r$$

But this is not actual, it is apparent because when the light is reflected from the surface of an optically denser medium, a phase change of π equivalent to a path difference of λ/2 occurs.

$$\therefore \quad \Delta = 2\mu t \cos r - \frac{\lambda}{2}$$

Condition for maxima & minima:

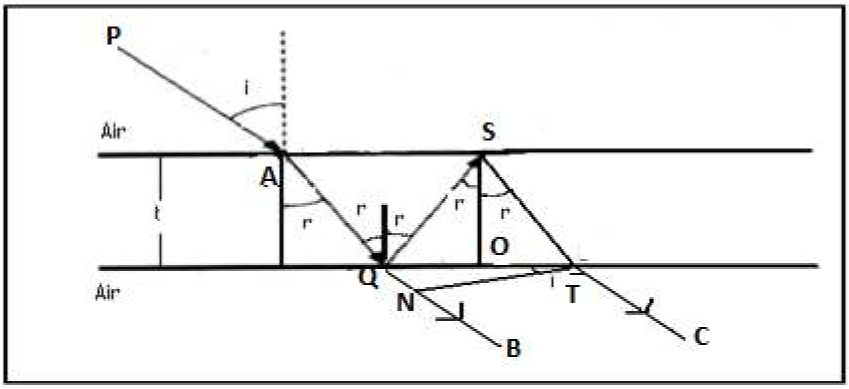
1. If Δ = nλ, where n = 0,1,2,3,…….constructive interference takes place and the film will appear bright

$$\therefore \quad 2\mu t \cos r - \frac{\lambda}{2} = n\lambda$$
$$or \quad 2\mu t \cos r = (2n+1)\lambda/2$$

2. 2. If Δ = (2n-1) λ/2, where n=1, 2, 3 … destructive interference takes place and the film will appear dark.

$$\therefore \quad 2\mu t \cos r = (2n-1)\lambda/2$$
$$or \quad 2\mu t \cos r = n\lambda$$

(B)Interference due to transmitted light in thin film: Consider a transparent film of thickness t & refractive index μ. Let a ray of monochromatic light be incident on the upper surface of the film.



The ray PA ray is refracted along AQ at an angle r.The refracted part AQ is partly reflected along QS and partly refracted along QB.The reflected part QS is again reflected from point S on the upper surface of the film along ST and finally emerges out through TC.

The optical path difference,

Δ = path (QS+ST) – path QN

$\Delta = \mu\,(QS+ST) - QN$ ....(1)

In right angled Δ QSO,

$$\cos r = \frac{SO}{QS} \qquad or \qquad QS = \frac{t}{\cos r} \qquad \text{............(2)}$$

Similarly in right angled Δ SOT,

$$\cos r = \frac{SO}{ST} \qquad or \qquad ST = \frac{t}{\cos r} \qquad \text{............(3)}$$

In right angled Δ QTN,

$$\sin i = \frac{QN}{QT} \quad or \quad QN = QT \sin i = (QO + OT) \sin i \quad \text{...........(4)}$$

Now again in Δ QSO and Δ SOT,

$$\tan r = \frac{QO}{OS} \qquad and \qquad \tan r = \frac{OT}{OS}$$

Putting these values in eq^n (4), we get

QN = (OS tan r + OS tan r) sin i

QN = 2t tan r(μ sin r) [From Snell's law, $\frac{\sin i}{\sin r} = \mu$]

QN = $2\mu t \frac{\sin^2 r}{\cos r}$ …………..(5)

Putting the values from eq^n (2),(3) & (5) in eq^n (1)

$$\Delta = \mu(\frac{t}{\cos r} + \frac{t}{\cos r}) - 2\mu t \frac{\sin^2 r}{\cos r}$$

$$\Delta = \frac{2\mu t}{\cos r}(1 - \sin^2 r) \qquad = \frac{2\mu t \cos^2 r}{\cos r}$$

$$\Delta = 2\mu t \cos r$$

Condition for maxima & minima:

1. If $\Delta = n\lambda$, where n = 0,1,2,3,…….constructive interference takes place and the film will appear bright

$$\therefore \quad 2\mu t \cos r = n\lambda$$

2. If $\Delta = (2n-1)\lambda/2$, where n=1, 2, 3 … destructive interference takes place and the film will appear dark.

$$\therefore \quad 2\mu t \cos r = (2n-1)\lambda/2$$

Intereference pattern in wedge-shaped film:-

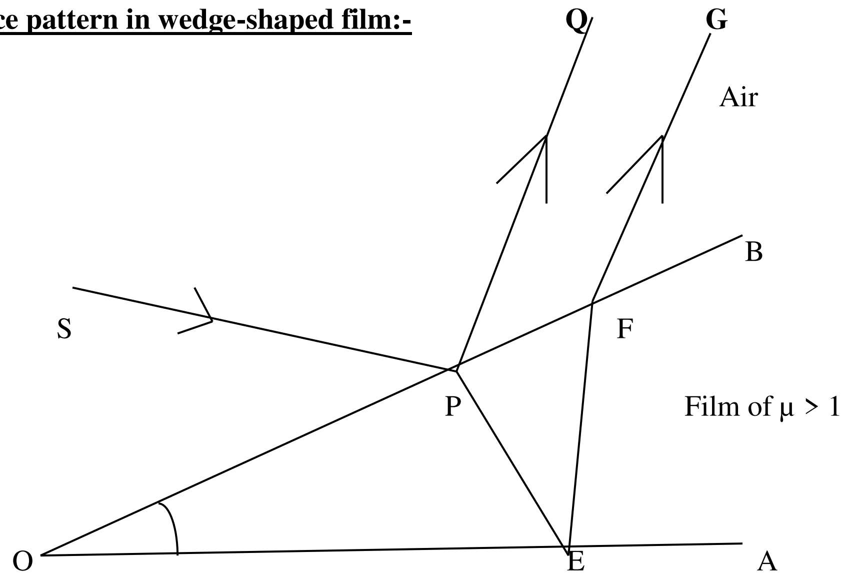


FIG (1)

A Wedge shaped thin film is one whose plane surfaces (OA and OB) are slightly inclined to each other at small angle θ and encloses a film of transparent material of refractive index μ as shown in fig (1).The thickness of the film increases from O to A.At the point of contact thickness is zero.When the upper surface (OB) of the film is illuminated by parallel beam of monochromatic light and the surface is viewed by reflected light,then the interference between two rays, one (PQ) reflected from the upper surface of the film and other (FG) obtained by internal reflection at the back surface and consequent transmission at the film surface (AB).Since both the rays PQ and FG are derived from the same incident ray SP they are coherent and on overlapping produce a system of equidistant bright and dark fringes.

When a beam of monochromatic light is incident normally at point P on the upper surface of the film, the path difference between the rays reflected from the upper surface and lower surfaces of the film is 2μt where t is the thickness of the film at P.As the rays reflected from the surface of denser medium therefore there occurs an additional path difference of $\frac{\lambda}{2}$ or phase change of π. Thus an additional path difference of $\frac{\lambda}{2}$ is introduced in the ray reflected from the upper surface.

Hence the effective path difference between the two rays = $2\mu t \cos(r+\theta) + \frac{\lambda}{2}$

For normal incidence r = 0 and for small angle of wedge cosθ =1

Hence effective path difference = $2\mu t + \frac{\lambda}{2}$

The condition for bright fringe or maximum intensity.

$$2\mu t + \frac{\lambda}{2} = \frac{2n\lambda}{2}$$

$$2\mu t = \frac{(2n-1)\lambda}{2}$$

Similarly the condition for dark fringe or maximum intensity is

$$2\mu t + \frac{\lambda}{2} = \frac{(2n+1)\lambda}{2}$$

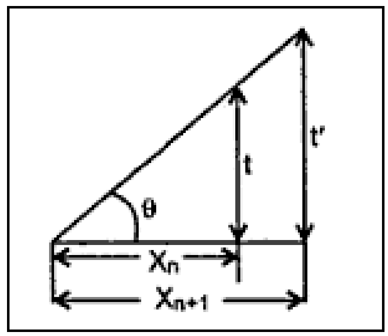
$$2\mu t = n\lambda$$

Fringe Width:-

For bright fringe,

Let x_n be the distance of nth bright fringe from the edge of the film, then

$$\tan\theta = \frac{t}{x_n} \qquad or \qquad t = x_n \tan\theta$$



Putting this value of t in above eq^n

$$\therefore 2\mu t\cos(r+\theta) = (2n+1)\lambda/2, \qquad where\, n = 0,1,2,3.........$$

$$2\mu\ x_n\tan(\theta)\mathrm{Cos}(r+\theta) = (2n+1)\lambda/2 \quad \text{--------------------(1)}$$

Similarly, if x_{n+1} is the distance of (n+1)th bright fringe, then

$$2\mu x_{n+1}\tan\theta\cos(r+\theta) = [2(n+1)+1]\frac{\lambda}{2} = (2n+3)\frac{\lambda}{2} \quad \text{-------------------(2)}$$

Subtracting eq^n (1) from eq^n (2),

$$x_{n+1} - x_n = \frac{\lambda}{2\mu\tan\theta\cos(r+\theta)} = \omega$$

For normal incidence, i = r = 0 and cos (r + θ) = cos θ

So, $\omega = \dfrac{\lambda}{2\mu \tan\theta \cos\theta} = \dfrac{\lambda}{2\mu \sin\theta}$

For every small value of θ, $\sin\theta \cong \theta$

$$\therefore \ \omega = \frac{\lambda}{2\mu\theta}$$

For air film, $\mu = 1$ $\qquad \therefore \quad \omega = \dfrac{\lambda}{2\theta}$

Necessity of an Extended Source

Let the light from a narrow source S is incident on **a thin film** Fig

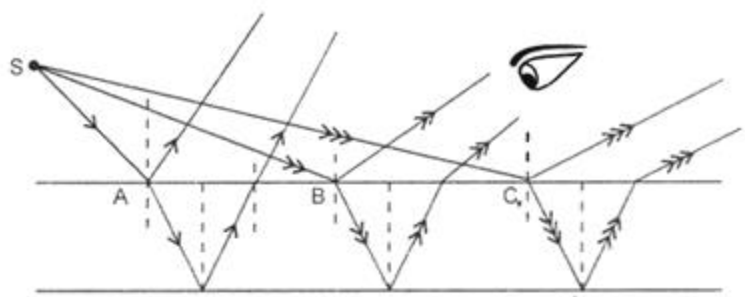


Fig.

For each incident ray, we get **a set of interferring rays. As obvious from** **the pair** of parallel interferring rays SA, **SB and SC are diversed in a wide field** of **view and due to** limited size of the pupil of the eye, the rays **from only a small portion of the film will enter the** eye. Thus, only a small area of the film will be **visible.**

Now, let us consider the case when **light from** an extended source **is incident on the thin** film Fig.

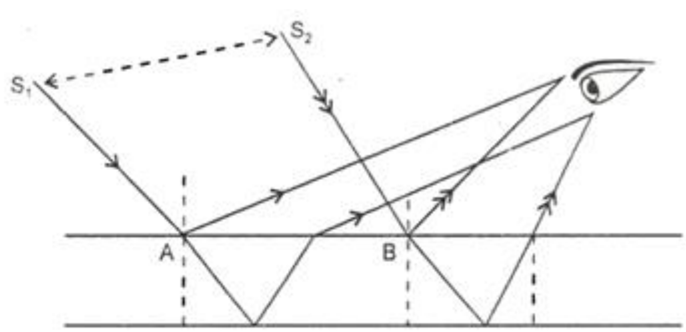
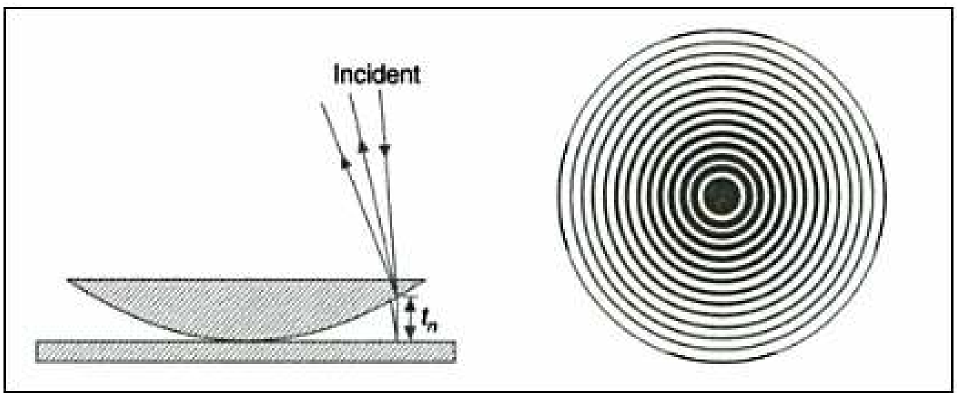


Fig.

The portion A of the film is **seen by the reflected light** coming from S_1. Similarly **the portion** B of the film is seen by the reflected **light** coming **from** other point S_2 of the **same** extended source. Thus, in the case of extended source of light, the incident rays from different **parts of the** same extended source are reflected from different parts **of the film** and enter the eye placed at a suitable position. Therefore, it is clear that an extended source of light is necessary for observing the whole film simultaneously by the eye.

Newton's Rings:

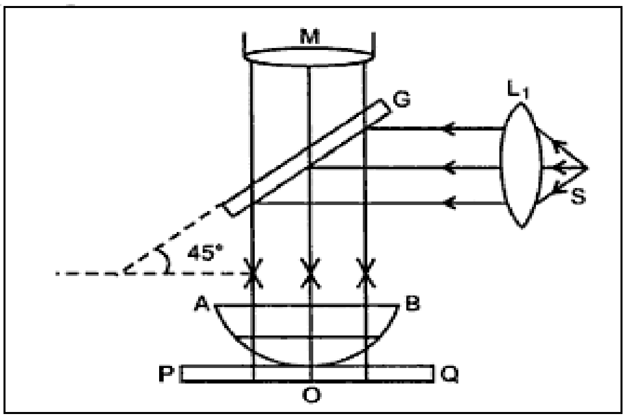
When a Plano-convex lens of large radius of curvature is placed with its convex surface in contact with a plane glass plate, an air film of gradually increasing thickness from the point of contact is formed between the upper surface of the plate and the lower surface of the lens. If monochromatic light is allowed to fall normally on this film, then alternate bright and dark concentric rings with their centre dark are formed. These rings are known as Newton's rings. Newton's rings are formed because of the interference between the waves reflected from the top & bottom surfaces of an air film formed between the lens and plate.



Experimental arrangement:-

A plano-convex lens L of large radius of curvature is placed on a plane glass plate P such that both of them are having a point of contact. Light from a monochromatic source is allowed to fall on a glass plate G inclined at an angle 45º to the incident beam. The light reflected from the glass plate falls normally on the air film enclosed between plano convex lens & plane glass plate. Light rays reflected upward from the air film, superimpose each other and interference takes place. Due to interference of these rays, alternate bright & dark concentric rings are seen, with the help of microscope.

The fringes are circular because the air film is symmetrical about the point of contact of the lens with the plane glass plate.



Theory: As the rings are observed in reflected light, the effective path difference is given by $2\mu t \cos r + \lambda/2$, where μ is the refractive index of the film, t is the thickness of the film at point of incidence. For normal incidence r = 0 and for air film $\mu = 1$, therefore path difference is $2\mu t + \lambda/2$

At t = 0, the effective path difference is $\lambda/2$.

This is the condition of minimum intensity. Hence the centre of Newton's ring is dark.

For constructive interference, $\Delta = n\lambda$

$$2\mu t + \lambda/2 = n\lambda$$

or $2\mu t = (2n - 1)\lambda/2$, where n = 1,2,3,…… ………(1)

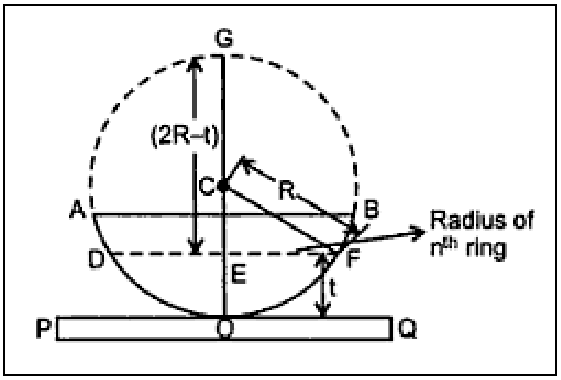
For destructive interference, $\Delta = (2n+1)\lambda/2$

$$2\mu t + \lambda/2 = (2n+1)\lambda/2$$

or $2\mu t = n\lambda$, where n = 0,1,2,3……… ……….(2)

Diameters of Bright & Dark Rings:-

Let R be the radius of curvature of the lens and r be the radius of a Newton's ring where film thickness is t.



From the property of a circle, DE×EF= OE ×EG

$$r \text{ x } r = t\,(2R\text{-}t)$$

$$r^2 = 2Rt - t^2 \qquad \text{..........(3)}$$

since t is very small as compared to R, hence

$$r^2 = 2Rt \quad \text{or} \quad t = \frac{r^2}{2R} \qquad \text{.........(4)}$$

For Bright rings, Substituting the value of t in eq^n(1),

$$2\mu\frac{r^2}{2R} = (2n-1)\lambda/2 \qquad or \qquad r^2 = \frac{(2n-1)\lambda R}{2\mu}$$

For nth ring. $r_n{}^2 = \dfrac{(2n-1)\lambda R}{2\mu}$

$$\frac{D_n{}^2}{4} = \frac{(2n-1)\lambda R}{2\mu} \qquad or \quad D_n{}^2 = \frac{2(2n-1)\lambda R}{\mu}$$

$$D_n = \sqrt{\frac{2(2n-1)\lambda R}{\mu}} \qquad \text{.................(6)}$$

For air film, $\mu = 1$

$$\therefore \quad D_n = \sqrt{2(2n-1)\lambda R}$$

Let $\sqrt{2\lambda R} = K$

Hence $D_n = K\sqrt{2n-1}$

Hence $D_n \propto \sqrt{2n-1}$

Thus, *the diameters of bright rings are proportional to the square root of the odd natural number*.

For Dark rings, Substituting the value of t in eq^n (2),

$$2\mu\frac{r^2}{2R} = n\lambda \qquad or \qquad r^2 = \frac{n\lambda R}{\mu}$$

For nth ring. $r_n^{\ 2} = \dfrac{n\lambda R}{\mu}$

$$\frac{D_n^{\ 2}}{4} = \frac{n\lambda R}{\mu} \qquad or \quad D_n^{\ 2} = \frac{4n\lambda R}{\mu}$$

For air film, μ = 1,

$$D_n = \sqrt{\frac{4n\lambda R}{\mu}} \qquad \text{.................(7)}$$

$$\therefore \quad D_n = \sqrt{4n\lambda R} \qquad or \qquad D_n = \sqrt{n}$$

Thus, *the diameters of dark rings are proportional to the square root of the natural number*.

<u>Determination of Wavelength of Sodium light using Newton's Ring:-</u>

Let D_n and $D_{(n+p)}$ be the diameters of the nth & (n+p)th dark rings respectively, then

$$D_n^{\ 2} = 4n\lambda R \qquad and \qquad D_{(n+p)}^{\ \ 2} = 4(n+p)\lambda R$$

where p is any number & R be the radius of curvature of the lens.

$$D^2{}_{(n+p)} - D^2{}_n = 4(n+p)\lambda R - 4n\lambda R = 4p\lambda R$$

Thus, $\quad or \qquad \lambda = \dfrac{D^2{}_{(n+p)} - D^2{}_n}{4pR}$

The same result shall be obtained for bright ring.

<u>Determination Of refractive index of a liquid:-</u>

$$D_n^{\,2} = 4n\lambda R \qquad and \qquad D_{(n+p)}^{\,2} = 4(n+p)\lambda R$$

$$D^2{}_{(n+p)} - D^2{}_n = 4(n+p)\lambda R - 4n\lambda R = 4p\lambda R$$

Let D_n' and $D_{(n+p)}'$ be the diameters of the n^{th}& $(n+p)^{th}$ dark rings with liquid as medium respectively, then

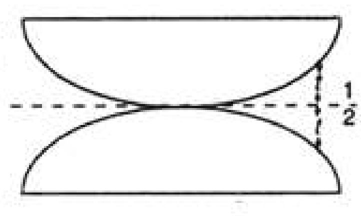
$$D_n'^{\,2} = 4n\lambda R \qquad and \qquad D_{(n+p)}'^{\,2} = 4(n+p)\lambda R$$

$$D'^2{}_{(n+p)} - D'^2{}_n = \frac{4p\lambda R}{\mu}$$

$$\therefore \ \mu = \frac{[D^2{}_{(n+p)} - D^2{}_n]air}{[D'^2{}_{(n+p)} - D'^2{}_n]liquid}$$

Newton's Rings with Both Surfaces Curved

In Newton's ring arrangement, if the plane glass plate is replaced by a curved surface, the rings are formed due to reflection from the upper and lower surfaces of the air film. The following two experimental arrangements are possible:



Case 1: When the lower surface is convex as shown in Fig.

The total thickness of the air film at any point as shown in the figure is given by

$$t = t_1 + t_2 \qquad \ldots(1)$$

If R_1 and R_2 be the radii of curvatures of the upper and lower curved surfaces respectively, then

$$t = \frac{r_n^2}{2R_1} + \frac{r_n^2}{2R_2}$$

or

$$2t = \frac{r_n^2}{R_1} + \frac{r_n^2}{R_2} \qquad \ldots(2)$$

Where r_n is the radius of curvature of nth ring correspondign to thickness t.
The condition for dark rings at normal incidence and air film is

$$2t = n\lambda \quad \text{...(3)}$$

From equations (2) and (3),

$$r_n^2\left(\frac{1}{R_1}+\frac{1}{R_2}\right) = n\lambda \quad \text{...(4)}$$

If D_n be the diameter of nth dark ring, then

$$r_n = \frac{D_n}{2} \quad \text{...(5)}$$

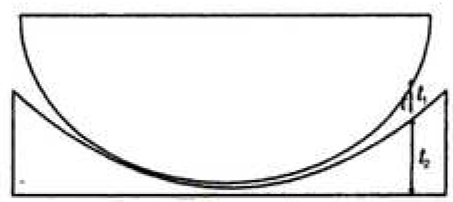
Equation (4) may be written in terms of D_n as

$$\frac{D_n^2}{4}\left(\frac{1}{R_1}+\frac{1}{R_2}\right) = n\lambda$$

or $$D_n^2 = \frac{4n\lambda}{\left(\frac{1}{R_1}+\frac{1}{R_2}\right)} \quad \text{...(6)}$$

Case 2: When the lower surface is concave as shown in Fig.
It is obvious from Fig.

$$t = t_1 - t_2 \quad \text{...(7)}$$



Substituting $t_1 = \frac{r_n^2}{2R_1}$ and $t_2 = \frac{r_n^2}{2R_2}$ in equation (7), we get

$$t = \frac{r_n^2}{2R_1} - \frac{r_n^2}{2R_2}$$

or $$2t = r_n^2\left(\frac{1}{R_1}-\frac{1}{R_2}\right) \quad \text{...(8)}$$

Substituting this value of $2t$ in equation (3), we get

$$r_n^2\left(\frac{1}{R_1}-\frac{1}{R_2}\right) = n\lambda$$

or $$\frac{D_n^2}{4}\left(\frac{1}{R_1}-\frac{1}{R_2}\right) = n\lambda$$

or $$D_n^2 = \frac{4n\lambda}{\left(\frac{1}{R_1}-\frac{1}{R_2}\right)} \quad \text{...(9)}$$

The equations (6) and (9) may be combined to give a general expression as

$$D_n^2 = \frac{4n\lambda}{\left(\frac{1}{R_1}\pm\frac{1}{R_2}\right)} \quad \text{...(10)}$$

FOR BRIGHT RINGS→

$$2\mu t = (2n-1)\frac{\lambda}{2} \qquad \text{--------(11)}$$

For air film μ=1, $2t = (2n-1)\frac{\lambda}{2}$ ---------------(12)

Now using equation (2) and (12) we get

$$r_n^2\left(\frac{1}{R_1}+\frac{1}{R_2}\right) = (2n-1)\frac{\lambda}{2}$$

$$r_n = \frac{D_n}{2}$$

Hence $\frac{D_n^2}{4}\left(\frac{1}{R_1}+\frac{1}{R_2}\right) = (2n-1)\frac{\lambda}{2}$

$$D_n^2 = \frac{2(2n-1)\lambda}{\left(\frac{1}{R_1}+\frac{1}{R_2}\right)} \qquad \text{---------------------------(13)}$$

Using equation (8) and (12)

$$r_n^2\left(\frac{1}{R_1}-\frac{1}{R_2}\right) = (2n-1)\frac{\lambda}{2}$$

$$r_n = \frac{D_n}{2}$$

Hence $\frac{D_n^2}{4}\left(\frac{1}{R_1}-\frac{1}{R_2}\right) = (2n-1)\frac{\lambda}{2}$

$$D_n^2 = \frac{2(2n-1)\lambda}{\left(\frac{1}{R_1}-\frac{1}{R_2}\right)} \qquad \text{------------------------(14)}$$

Using equation (13) and(14) we get

$$D_n^2 = \frac{2(2n-1)\lambda}{\left(\frac{1}{R_1}\pm\frac{1}{R_2}\right)}$$

Interference numericals

Q1. Light of wavelength 5893 Å is reflected at nearly normal incidence from a soap film of refractive index 1.42. What is the least thickness of the film that will appear (i) dark (ii) bright.

Solution (i)The condition for the darkness of the film in reflected system is given by

$$2\mu t cos r = n\lambda$$

For normal incidence, r=0 and cos r=1

Therefore

$$2\mu t = n\lambda$$

$$t = \frac{n\lambda}{2\mu}$$

For the least thickness of the film, n=1

$$t = \frac{\lambda}{2\mu}$$

$$t = \frac{5893 \times 10^{-8}}{2 \times 1.42}$$

=2075 Å

(ii) The condition for the brightness of the film in reflected system is given by

$$2\mu t = (2n-1)\frac{\lambda}{2}$$

$$t = (2n-1)\frac{\lambda}{4\mu}$$

For the least thickness of the film, n=1

$$t = \frac{\lambda}{4\mu}$$

$$t = \frac{5893 \times 10^{-8}}{4 \times 1.42}$$

=1037.5 Å

Q 2. A parallel beam of sodium light of 5880 Å is incident on a thin glass plate of refractive index 1.5 such that the angle of refraction in the plate is 60^0.Calculate the smallest thickness of the plate which will make it appear dark by reflection.

Solution: We know that the condition for dark band or fringe in the reflected light is

$$2\mu t cosr = n\lambda$$

$$t = \frac{n\lambda}{2\mu cosr}$$

$$t = \frac{5880 \times 10^{-8}}{2 \times 1.5 \times .5}$$

$$t = 3920 \times 10^{-8}\text{cm}$$

Q3. Calculate the thickness of the thinnest film (μ=1.4) in which interference of violet component (λ=4000 Å) of incident light can take place by reflection.

Solution:We know that the condition of condition of constructive interference in the reflected system is

$$2\mu t cosr = \frac{(2n+1)\lambda}{2}$$

For normal incidence r=0 or cosr=1

For for thinnest film n=0

Hence

$$t = \frac{\lambda}{4\mu cosr}$$

$$t= \frac{4000\times10^{-8}}{4\times1.4\times1}$$

t = 714.3 Å

Q4. A soap film of refractive index 1.43 is illuminated by white light incident at an angle of 30^0.The refracted light is examined by a spectroscope in which dark band corresponding to the wavelengths 6×10^{-7}m is observed .Calculate the thickness of the film.

Solution: The condition for the darkness of the film in reflected system is given by

$$2\mu t cos r = n\lambda$$

$$t = \frac{n\lambda}{2\mu cos r}$$

$$\mu = \frac{sin i}{sin r}$$

$$sin r = \frac{sin i}{\mu}$$

$$sin r = \frac{sin 30^0}{1.43}$$

$$sin r = \frac{1/2}{1.43} = .381$$

$$Cos r = \sqrt{1 - sin^2 r} = .92$$

$$t = \frac{1 \times 6 \times 10^{-7}}{2 \times 1.43 \times .92} = 2.28 \times 10^{-8} cm$$

Q5. A thin film of soap solution is illuminated by white light at an angle of incidence $\sin^{-1}(4/5)$. In reflected light two dark consecutive overlapping fringes are observed corresponding to wavelength 6.1×10^{-7} and 6×10^{-7}m. The refractive index for soap solution is 4/3. Calculate the thickness of the film

Solution:We know that the condition for dark band or fringe in the reflected light is

$$2\mu t cos r = n\lambda$$

If n and $(n+1)$ are the orders of consecutive dark bands for wavelengths λ_1 and λ_2 respectively, then

$$2\mu t cos r = n\lambda_1 \quad \text{and} \quad 2\mu t cos r = (n+1)\ \lambda_2$$

$$2\mu t cos r = n\lambda_1 \ = (n+1)\ \lambda_2 \text{ --------------------------------------(1)}$$

$$n\lambda_1 = (n+1)\ \lambda_2$$

$$n(\lambda_1 - \lambda_2) = \lambda_2$$

$$n = \frac{\lambda_2}{(\lambda_1 - \lambda_2)}$$

Put the value of n in equation (1),we get

$$2\mu t cosr = \frac{\lambda_1 \lambda_2}{(\lambda_1 - \lambda_2)}$$

$$t = \frac{\lambda_1 \lambda_2}{(\lambda_1 - \lambda_2)} \frac{1}{2\mu cosr} \text{ ------------------------------------(2)}$$

$$cosr = \sqrt{1 - sin^2\ r}$$

But from Snell's law $\mu = \frac{sini}{sinr}$

According to question $\mu = \frac{4}{3}$ and $sini = \frac{4}{5}$

$$\text{Hence } cosr = \sqrt{1 - \frac{(4/5)^2}{(4/3^2}}$$

$$= \sqrt{1 - \left(\frac{3}{5}\right)^2}$$

$$= \sqrt{\frac{16}{25}} = \frac{4}{5}$$

Put this value of $cosr$ in equation (2) and also $\lambda_1 = 6.1 \times 10^{-5}$cm, $\lambda_2 = 6 \times 10^{-5}$ cm, μ=$\frac{4}{3}$

$$t = \frac{6.1\times10^{-5}}{(6.1\times10^{-5} -} \frac{\times}{6\times10^{-5})} \frac{6\times10^{-5}}{\times 2 \times \frac{4}{3} \times \frac{4}{5}} \frac{\times 3 \times 5}{1}$$

$$t = .0017\ cm$$

Q6. Light of wavelength 5893 Å is reflected at nearly normal incidence from a soap film of refractive index 1.42. What is the least thickness of the film that will appear (i) dark (ii) bright.

Solution (i)The condition for the darkness of the film in reflected system is given by

$$2\mu t cosr = n\lambda$$

For normal incidence, r=0 and cos r=1

Therefore

$$2\mu t = n\lambda$$

$$t = \frac{n\lambda}{2\mu}$$

For the least thickness of the film , n=1

$$t = \frac{\lambda}{2\mu}$$

$$t = \frac{5893 \times 10^{-8}}{2 \times 1.42}$$

=2075 Å

(ii) The condition for the brightness of the film in reflected system is given by

$$2\mu t = (2n-1)\frac{\lambda}{2}$$

$$t = (2n-1)\frac{\lambda}{4\mu}$$

For the least thickness of the film , n=1

$$t = \frac{\lambda}{4\mu}$$

$$t = \frac{5893 \times 10^{-8}}{4 \times 1.42}$$

=1037.5 Å

Q7. Light of wavelength 6000 Å falls normally on a thin wedge shaped film of refractive index1.4 forming fringes that are 2 mm apart. Find the angle of the wedge.

Solution: If θ is the angle of wedge formed by a medium of refractive index μ,then for normal incidence the fringe width for wavelength λ is given by

$$W = \frac{\lambda}{2\mu}\frac{1}{\theta}$$

$$\lambda = 6000 \text{ Å} = 6000 \times 10^{-8}\text{cm},\ \mu = 1.4, \qquad W = 2mm = .2cm$$

$$\theta=\frac{6000\times10^{-8}}{2\times1.4\times.2}=10.71\times 10^{-5}\text{radian}=10.71\times 10^{-5}\times\frac{180\text{ degree}}{\pi}$$

$$\theta=.0061^0$$

Q8. Newton's rings are observed in reflected light of wavelength 6000 Å. The diameter of 10^{th} dark ring is 0.5 cm. Find the radius of curvature of the lens and the thickness of the air film.

Solution: The diameter of n^{th} dark ring is given by

$$D_n{}^2=4n\lambda R$$

$$R=\frac{D_n{}^2}{4n\lambda}$$

$$D_{n=.5\ cm,}\ \lambda=6000\ Å=6\times10^{-5}\text{cm and } n=10$$

$$R=\frac{.5\times.5}{4\times10\times6\times10^{-5}}=106\text{cm}$$

If t is the thickness of the film cprresponding to a ring of D diameter,then

$$2t=\frac{D^2}{4R}$$

$$t=\frac{D^2}{8R}$$

$$t=\frac{.5\times.5}{8\times106}=3\times10^{-4}\text{ cm}$$

Q9. Newton's rings are observed by keeping a spherical surface of 100 cm. radius on a plane glass plate. If the diameter of 15^{th} bright ring is 0.59 cm and the diameter of 5^{th} ring is 0.336 cm. What is the wavelength of light used?

Solution: If D_{n+p} and D_n be the diameters of $(n+p)^{th}$ and n^{th}bright ring then

$$\lambda=\frac{Dn+p^2-Dn^2}{4P}\frac{}{R}$$

D_{15}=.59 cm,D_5=.336 cm

P=10 and R=100 cm

$$\lambda=\frac{(.59)^2-(.336)^2}{4\times10\times100}$$

$$=5.88\times10^{-5}\text{cm}=5880\ \text{Å}$$

Q10. Newton's rings are formed in reflected light of wavelength 6000 Å with a liquid between the plane and curved surfaces.If the diameter of 6th bright ring is 3.1mm and the radius of curvature of the curved surface is 100 cm,calculate the refractive index of the liquid.

Solution:The diameter of nth bright ring is given by

$$Dn^2=\frac{2(2n-1)\lambda R}{\mu}$$

$$\mu=\frac{2(2n-1)\lambda R}{Dn^2}$$

In the given problem,n=6,λ=6000 Å=6000×10^{-8}cm, R=100cm and D_5=3.1mm=.31cm

$$\mu=\frac{2(2n-1)\lambda R}{Dn^2}$$

$$=\frac{2(12-1)\times6000\times10^{-8}\times100}{(.31)(.31)}$$

$$=1.373$$

Q11. In newton's ring experiment the diameter of 4th and 12th dark ring are .4 cm and .7 cm respectively.Deduce the diameter of 20th dark ring.

Solution:If D_{n+p} and D_n be the diameters of (n+p)th and nth dark ring respectively,then

$$D_{n+p}^2-D_n^2=4p\lambda R \text{ -------------------(1)}$$

In the given problem n=4,n+p=12,D_4=.4 cm and D_{12}=.7 cm

$$D_{12}^2-D_4^2=4\times8\lambda R\text{------------------(2)}$$

Suppose the diameter of20th dark ring is D_{20},

$$D_{20}^2-D_4^2=4\times16\ \lambda R \text{ ---------------------------------(3)}$$

Dividing equation (2) by (3), we get

$$\frac{D_{12}^2-D_4^2}{D_{20}^2-D_4^2}=\frac{4\times8\lambda R}{4\times16\ \lambda R}$$

$$=2(D_{12}^2-D_4^2)=(D_{20}^2-D_4^2)$$

$$\text{Or } D_{20}^2=2D_{12}^2-D_4^2$$

$$D_{20}^2=2\times(.7)\times(.7)-(.4)\times(.4)$$

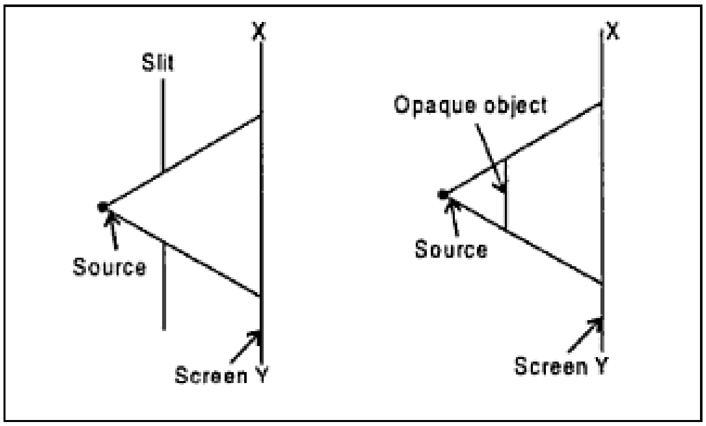
$$D_{20}^2=.98-.16=.82$$

So diameter of 20 th ring =.906 cm

DIFFRACTION

Q. What is meant by diffraction of light? Distinguish between Fresnal and Fraunhofer classes of diffraction.

Solution: Diffraction refers to various phenomena which occur when a wave encounters an obstacle. The diffraction phenomenon is described by the *bending of waves around small obstacles and the spreading out of waves past small openings*. Diffraction occurs with all waves including sound waves, water waves and electromagnetic waves such as visible light, X-rays and radio waves.



Diffraction of light can be divided into two classes:

1. Fraunhofer diffraction.
2. Fresnel diffraction.

In Fraunhofer diffraction,

a) Source and the screen are at infinite distance from the diffracting aperture.
b) Incident wave fronts on the diffracting obstacle are plane.
c) For this, single, double slits or gratings are used.
d) Plane diffracting wave fronts are converged by means of a convex lens to produce diffraction pattern.
e) The centre of the diffraction pattern is always bright.

Joseph von Fraunhofer
(1787–1826)
Fraunhofer, the famous German physicist, discovered 574 dark lines appearing in the solar spectrum. These lines are still sometimes called *Fraunhofer lines* in his honour. He also developed the theory of diffraction and invented the diffraction grating

In Fresnel diffraction,

a) Source and the screen are at finite distance from the diffracting aperture.
b) Incident wave fronts are spherical or cylindrical.
b) For obtaining Fresnal diffraction, zone plates are used.
c) Convex lens is not needed to converge the spherical wave fronts.
d) The centre of diffraction pattern may be bright or dark depending upon the number of Fresnel zones.


Augustin-Jean Fresnel

Q. What is the difference between interference and diffraction?

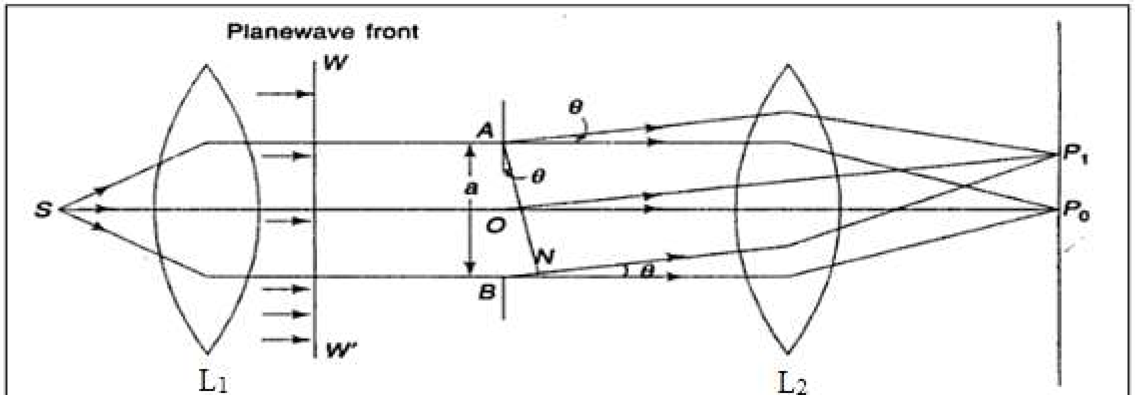
Solution:

	Interference	Diffraction
1.	Interference is due to interaction of light coming from two different wave fronts originating from the same source.	Diffraction is due to interaction of light coming from different parts of the same wave front.
2.	Interference fringes are of the same width.	Diffraction fringes are not of the same width.
3.	All bright fringes are of the same intensity.	All bright fringes are not of the same intensity.
4.	All points of minimum intensity are perfectly dark.	All points of minimum intensity are not perfectly dark.
5.	The spacing between fringes is uniform.	The spacing between fringes is not uniform.

Q. Give Fraunhofer diffraction at a single slit and show that the relative intensities of successive maxima are nearly

$1 : (4/9\pi^2) : (4/25\pi^2) : . \ . \ . \ . \ . \ . \ . \ . \ .$

<u>Solution:</u> Let a parallel beam of monochromatic light of wavelength λ, produced by a point source S be incident upon a converging lens (L_1) and emerging light from it, falls upon a slit AB of width '*a*' where it gets diffracted. If a converging lens (L_2) is placed in the path of the diffracted beam, a real image of the diffraction pattern is formed on the screen in the focal plane of the lens.



Fraunhoffer Diffraction Resulting from Single Slit

Path difference is given by,

$$BN = AB \sin\theta$$

$$= a \sin\theta$$

$$\text{Phase difference} = \frac{2\pi}{\lambda}(\text{a} \sin\theta)$$

The resultant amplitude at P_1 is given by,

$$R = \frac{\text{A}\sin\alpha}{\alpha} \qquad \dots (1)$$

where $2\,\alpha$ is the phase difference between initial and final oscillations which are producing due to the rays emerging from A and B.

$$2\alpha = \frac{2\pi}{\lambda}(\text{a} \sin\theta)$$

or $$\alpha = \frac{\pi}{\lambda}(\text{a} \sin\theta)$$

Resultant intensity, $$I = R^2 = \frac{A^2 sin^2\alpha}{\alpha^2} \quad \ldots (2)$$

Positions of Maxima and Minima:

Principal maximum or central maxima: The resultant amplitude given by eq^n (1) can be written as

$$R = \frac{A}{\propto}\left[\propto - \frac{\propto^3}{3!} + \frac{\propto^5}{5!} - \frac{\propto^7}{7!} + \cdots\right]$$

$$= A\left[1 - \frac{\propto^2}{3!} + \frac{\propto^4}{5!} - \frac{\propto^6}{7!} + \cdots\right]$$

If $\propto = 0$, the value of R will be maximum.

$$\therefore \quad \propto = \frac{\pi}{\lambda}\,(a \sin\theta) = 0$$

$$\therefore \quad \mathbf{\sin\boldsymbol{\theta} = 0 \; \boldsymbol{or} \; \boldsymbol{\theta} = 0}$$

Thus the maximum value of resultant intensity at P_0.This maxima is called Principal maxima.

Position of minima or secondary minima: For minimum intensity, sin α = 0 *i.e.*

$$\sin\alpha = 0$$

$$\propto = \pm n\pi \; or \pm \pi, \pm 2\pi, \pm 3\pi \ldots\ldots \pm n\pi$$

$$\frac{\pi}{\lambda}\,(a \sin\theta) = \pm n\pi$$

$$\boldsymbol{a} \sin\boldsymbol{\theta} = \pm \boldsymbol{n\lambda} \qquad \ldots\ldots(3)$$

where n = 1, 2, 3, 4 …….

Here n = 0 is not taken because for n = 0, we get the position of **Principal maxima**.

Position of maxima or secondary maxima:

Differentiate eq^n (2) with respect to α and equate to zero, that is,

$$\frac{dI}{d\propto} = \frac{d}{d\propto}\left[A^2\left(\frac{\sin\propto}{\propto}\right)^2\right] = 0$$

$$\therefore \quad A^2 . \frac{2\sin\propto}{\propto} . \frac{(\propto\cos\propto - \sin\propto)}{\propto^2} = 0$$

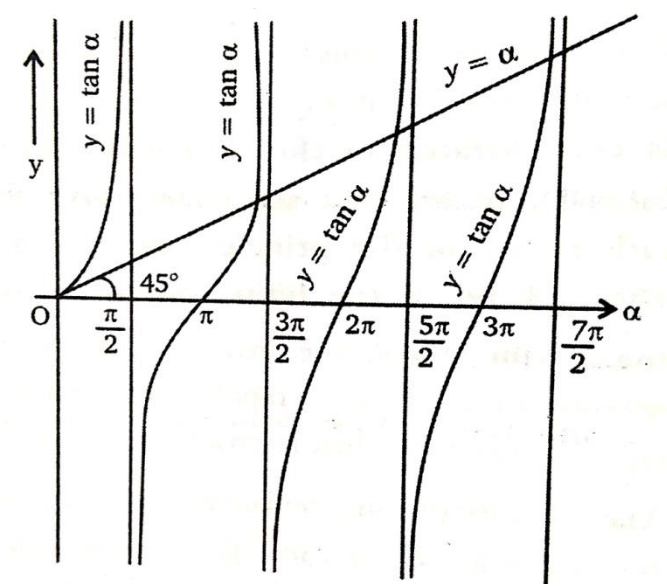
So that either $\sin\alpha = 0$ or $(\alpha\cos\alpha - \sin\alpha) = 0$

The condition $\sin\alpha = 0$ gives the position of minima. Hence position of secondary maxima are given by

$$(\alpha\cos\alpha - \sin\alpha) = 0$$

$$\boxed{\alpha = \tan\alpha} \quad ....(4)$$

On plotting graph for $y = \alpha$ and $y = \tan\alpha$



In graph, the points of intersection, are points of secondary maxima.

$$\propto = \mathbf{0}, \pm\frac{\mathbf{3\pi}}{\mathbf{2}}, \pm\frac{\mathbf{5\pi}}{\mathbf{2}}, \pm\frac{\mathbf{7\pi}}{\mathbf{2}} ....$$

Here, α = 0 is not possible because at this value, we get the central maxima. The direction of secondary maxima is given by

$$\boldsymbol{a sin\theta = \pm(2n+1)\frac{\lambda}{2}} \qquad ....(5)$$

1) For *n = 0*, $I = A^2 = I_0$ (principal maxima)

2) For n = 1,

$$I_1 = A^2\left[\frac{\sin(3\pi/2)}{3\pi/2}\right]^2 = \boldsymbol{A^2\frac{4}{9\pi^2}} = \frac{I_0}{22}\ (Ist\ secondary\ maxima)$$

Thus *intensity of first secondary maxima is about (1/22)th of the intensity of the central maxima.*

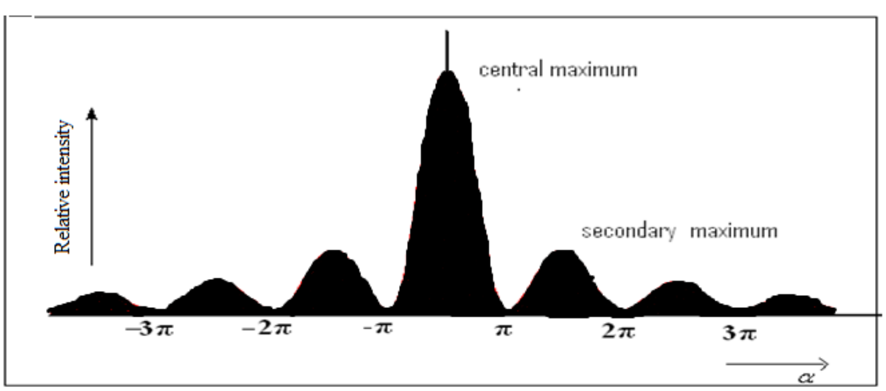
3) For *n* = 2,

$$I_2 = A^2\left[\frac{\sin(5\pi/2)}{5\pi/2}\right]^2 = \boldsymbol{A^2\frac{4}{25\pi^2}} = \frac{I_0}{61}\ (IInd\ secondary\ maxima)$$

Thus *intensity of second secondary maxima is about (1/61)th of the intensity of central maxima.*

The relative intensities of successive maxima are nearly

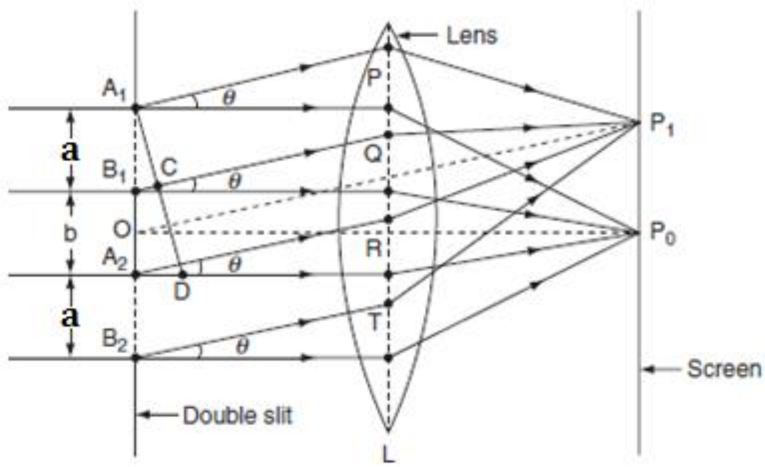
$$\boldsymbol{1 : \frac{4}{9\pi^2} : \frac{4}{25\pi^2} : \frac{4}{49\pi^2}} \dots$$



Q.	**Explain the Fraunhofer diffraction due to double slit. Explain missing order in double slit pattern.**

<u>Solution:</u> Let a monochromatic plane wave front of wave length 'λ' is incident normally on both the slits. The double slits have been represented as A_1B_1 and A_2B_2 in Fig. The slits are narrow and rectangular in shape. Let the width of both the slits be equal and it is 'a' and they are separated by length 'b'.

P_0 corresponds to the position of the central bright maximum. The intensity distribution on the screen is the combined effect of interference of diffracted secondary waves from the slits.



From the triangle A_1B_1C,

$$\sin\theta = \frac{B_1C}{A_1\,B_1} = \frac{B_1C}{a}$$

$$B_1C = a\sin\theta$$

phase difference,

$$2\propto = \frac{2\pi}{\lambda}\,(a\,sin\theta)$$

$$\propto = \frac{\pi}{\lambda}\,(a\,sin\theta)\quad . \quad . \quad . \;(1)$$

The diffracted wave amplitudes [A sinα / α] from the two slits, combine to produce interference. The path difference between the rays coming from corresponding points in the slits A_1B_1 and A_2B_2 can be found by drawing a normal from A_1 to A_2R. A_2D is the path difference between the waves from corresponding points of the slits.

In the Δ A_1A_2D,

$$\sin\theta = A_2D / A_1A_2$$

path difference, $A_2D = A_1A_2 \sin\theta = (a + b)\sin\theta$

The corresponding phase difference,

$$2\beta = \frac{2\pi}{\lambda}\,(a+b)\sin\theta. \quad . \quad . \;(2)$$

Applying the theory of interference on the wave amplitudes [(Asinα) / α] at the two slits gives the resultant wave amplitude (R).

The intensity at P_1 is

$$I = R^2 = 4A^2 \frac{sin^2 \propto}{\propto^2} cos^2\beta$$

$$= 4I_0 \ \frac{sin^2 \propto}{\propto^2} cos^2\beta \ \ldots\ldots (3)$$

[since $I_0 = A^2$]

Equation (3) represents the intensity distribution on the screen. In equation (3) the term $\cos^2\beta$ corresponds to interference and [$\sin^2\alpha$ / α^2] corresponds to diffraction. Now, we will look at the conditions for interference and diffraction maxima and minima.

Interference maxima and minima: If the path difference

$$A_2D = (a + b) \sin \theta = \pm n \lambda \ . \quad . \quad . \ (4)$$

where n = 1, 2, 3… then 'θ' gives the directions of the maxima due to interference of light waves coming from the two slits.

On the other hand if the path difference is odd multiples of $\lambda/2$ i.e. then θ gives the directions of minima due to interference of the secondary waves from the two slits.

$$A_2D = (a + b) \sin \theta = \pm(2n - 1)\frac{\lambda}{2} \quad . \quad . \quad . \ (5)$$

Diffraction maxima and minima:

For diffraction minima,

$$\boldsymbol{a \sin \theta = \pm n\lambda}$$

where n = 1, 2, 3… then θ gives the directions of diffraction minima. The ± sign indicates minima on both sides with respect to central maximum.

For diffraction maxima,

$$\boldsymbol{a \sin \theta = \pm(2n - 1)\frac{\lambda}{2}}$$

The ± sign indicates maxima on both sides with respect to central maximum.

Missing orders in double slit:

The direction of interference maxima are given as,

$$(a + b) \sin \theta = n\lambda \text{ .... } (6) \quad \text{where } n = 1, 2, 3, \text{ .....}$$

The directions of diffraction minima are given as,

$$a \sin \theta = m\lambda \text{ ....}(7) \qquad \text{where } m = 1, 2, 3, \ldots$$

Based on the relative values of a and *b,* certain orders of interference maxima are missing in the resultant pattern.

(i) Let $a = b$, then, $2\, a \sin\theta = n\lambda$ and $a \sin\theta = m\lambda$, on dividing (6) by (7), we get

$$\therefore \frac{n}{m} = 2 \quad \text{or} \quad n = 2m$$

If m = 1, 2, 3 … then n = 2, 4, 6… *i.e.*, the interference orders 2, 4, 6 … missed in the diffraction pattern

(ii) If $2a = b$, then $3a \sin \theta = n\lambda$ and $a \sin \theta = m\lambda$

$$\therefore \frac{n}{m} = 3 \quad \text{or} \quad n = 3m$$

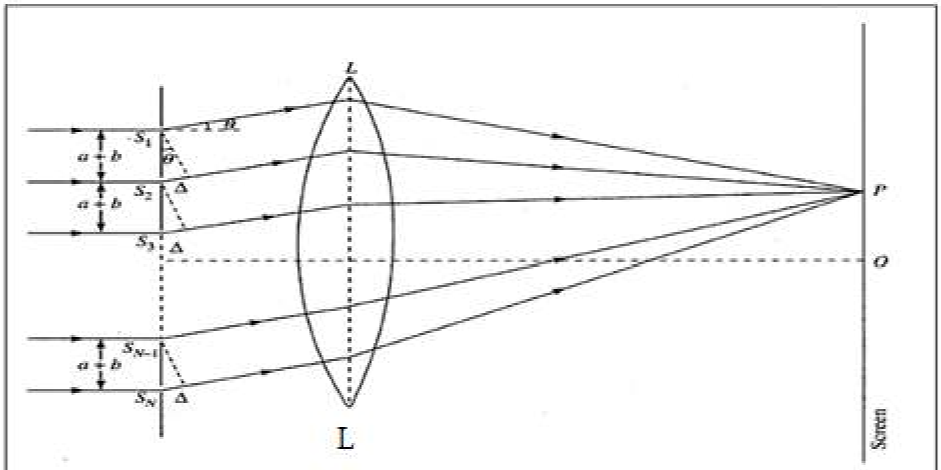
if m = 1, 2, 3… Then n = 3, 6, 9… *i.e.* the interference orders 3, 6, 9… are missed in the diffraction pattern

(iii) if $a + b = e$ *i.e.* $b = 0$, then two slits are joined. So, the diffraction pattern is due to a single slit of width $2a$.

Q. What do you mean by diffraction grating? Give construction and theory of plane transmission grating. Also give the conditions for central maxima, secondary minima and secondary maxima.

Solution: A plane diffraction grating is an arrangement consists of a large number of close, parallel, straight, transparent and equidistant slits of same width ***a***, with neighboring slits being separated by an opaque region of width ***b***. It may be constructed by ruling a large number of parallel and equidistant lines on a plane glass plate with the help of a diamond point. Ruled part is opaque and unruled part is transparent.

Let a monochromatic light incident on a plane diffraction grating consists of large number of N parallel slits, each of width a and separation b. Here **(a+b) is called grating element.**



Fraunhofer diffraction by a plane transmission grating.

The waves diffracted from each slit is equivalent to a single wave of amplitude

$$R = \frac{A \sin \alpha}{\alpha}$$

The path difference between the consecutive waves is same and equal to (a+b) sinθ.

$$\text{Phase difference } (2\beta) = \frac{2\pi}{\lambda}\,(a+b)\sin\theta$$

$$\text{where } \beta = \frac{\pi}{\lambda}\,(a+b)\sin\theta$$

Thus, the resultant amplitude at P is the resultant amplitude of N waves, each of amplitude R and common phase difference, 2β. Hence, the resultant amplitude at P is given by

$$R' = \frac{R \sin N\beta}{\sin\beta} = \frac{A\sin\alpha}{\alpha}\cdot\frac{\sin N\beta}{\sin\beta}$$

Resultant intensity at P is

$$I' = R'^2 = \frac{A^2 sin^2\alpha}{\alpha^2}\cdot\frac{sin^2 N\beta}{sin^2\beta} \quad \text{……….(1)}$$

The factor gives the intensity pattern due to a single slit while the factor *(sin^2 $N\beta$)/(sin^2 β)* gives the distribution of intensity due to interference from all the N points.

Principal maxima: From eq ^{n}(1), intensity will be maximum when

$$\sin\beta = 0 \text{ or } \beta = \pm n\pi, \qquad where\ n = 0,1,2,\ldots\ldots\ldots$$

But sin $N\beta$ is also equal to zero. Hence, sin $N\beta$ / sin β becomes indeterminate. Its limiting value can be evaluated by L'Hospital rule:

$$\lim_{\beta\to\pm n\pi}\frac{sinN\beta}{sin\beta} = \lim_{\beta\to\pm n\pi}\frac{\frac{d}{d\beta}(sinN\beta)}{\frac{d}{d\beta}(sin\beta)}$$

$$= \lim_{\beta\to\pm n\pi}\frac{NcosN\beta}{cos\beta} = N$$

Therefore, the intensity at $\beta = \pm n\pi$, is given by,

$$I_{max} = A^2\left(\frac{sin\propto}{\propto}\right)^2 N^2 \qquad \ldots\ldots\ldots\ldots(2)$$

$$\therefore\ \boldsymbol{I_{max} \propto N^2}$$

The direction of principal maxima are given by,

$$\beta = \pm n\pi$$

$$\frac{\pi}{\lambda}(a+b)\sin\theta = \pm n\pi$$

$$\boldsymbol{(a+b)\sin\theta = \pm n\lambda} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(3) \qquad n = 0, 1, 2,\ldots.$$

For ***n* = 0, *θ* = 0**, this gives the direction of zero order **principal maxima**. The values of n = 1, 2, 3,………gives the direction of first, second, third……order principal maxima.

Secondary minima: For minimum intensity, sin $N\beta$ = 0

$$N\beta = \pm m\pi$$

$$N\frac{\pi}{\lambda}(a+b)\sin\theta = \pm m\pi$$

$$(a+b)\sin\theta = \pm\frac{m\lambda}{N} \qquad \ldots\ldots\ldots\ldots(4)$$

where m can take all integral values except 0, N, $2N$, $3N$,……..

Secondary maxima: For secondary maxima,

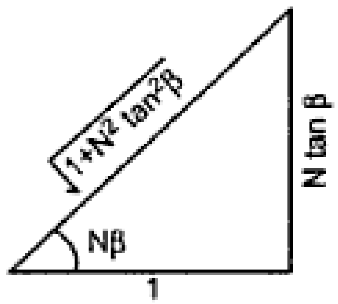
$$\frac{dI}{d\beta} = \frac{d}{d\beta}\left(\frac{A^2 sin^2\alpha}{\alpha^2} \cdot \frac{sin^2 N\beta}{sin^2\beta}\right) = 0$$

$$\frac{A^2 sin^2 \propto}{\alpha^2} \cdot 2\left[\frac{sinN\beta}{sin\beta}\right] \cdot \left[\frac{NcosN\beta sin\beta - sinN\beta cos\beta}{sin^2\beta}\right] = 0$$

$$NcosN\beta sin\beta - sinN\beta cos\beta = 0$$

$$\tan N\beta = N \tan\beta \quad \text{……………..(5)}$$

To find the intensity of secondary maxima, we make the use if the triangle shown below:



We have, $\sin N\beta = \frac{N\tan\beta}{\sqrt{(1+N^2 tan^2\beta)}}$

$$\frac{sin^2 N\beta}{sin^2\beta} = \frac{(N^2 tan^2\beta)/(1+N^2 tan^2\beta)}{sin^2\beta}$$

$$= \frac{(N^2 tan^2\beta)}{(1+N^2 tan^2\beta)sin^2\beta} = \frac{N^2}{1+(N^2-1)sin^2\beta}$$

Putting this value of $(sin^2 N\beta)/sin^2\beta$ in eq ^{n} (1)

$$I' = R'^2 = \frac{A^2 sin^2\alpha}{\alpha^2} \cdot \frac{N^2}{1+(N^2-1)sin^2\beta} \qquad ...(6)$$

Dividing eq ^{n} (6) by eq ^{n} (2), we get

$$\frac{Intensity\ of\ secondary\ maxima}{Intensity\ of\ primary\ minima} = \frac{I'}{I_{max}} = \frac{1}{1+(N^2-1)sin^2\beta}$$

Hence, the greater the value of N, the weaker are secondary maxima.

Q. What do you mean by missing order spectra in diffraction grating?

Solution: $(a+b)\sin\theta = n\lambda$, $n = 0,1,2,3\ldots\ldots$ . . .(8)

and $a\sin\theta = m\lambda$, $m = 1,2,3\ldots\ldots..$ . . . (9)

Dividing eq ^{n} (8 by eq ^{n} (9),

$$\frac{(a+b)}{a} = \frac{n}{m} \quad or \quad n = \frac{(a+b)}{a}m$$

This is the required condition of missing order spectra in the diffraction pattern.

i) When $b = a$, then $n = 2m$
Therefore, when $m = 1,2,3,\ldots\ldots.$missing orders are 2, 4, 6,……..

ii) When $b = 2a$, then $n = 3m$

Therefore, when $m = 1,2,3,\ldots\ldots.$missing orders are 3, 6, 9,…..

Q. Explain formation of spectra by diffraction grating.

Solution: The direction of the nth principal maxima is given by

$$(a+b)\sin\theta_n = n\lambda$$

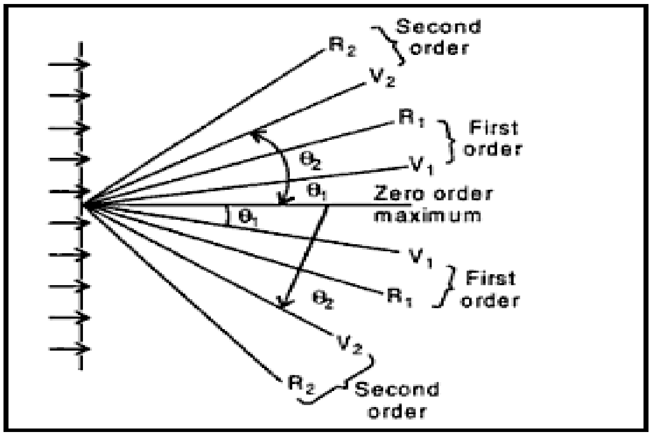
where θ_n is the angle of diffraction and (a+b) is the grating element. From this equation, it can be concluded that

1. The angle of diffraction θ_n is different for different order principal maxima for a given wavelength.
2. If we use white light then the light of different wavelength is diffracted in a different direction for a particular order. Longer the wavelength, greater the angle of diffraction. At central maxima ($n = 0$), all maxima of different wavelengths coincide to form a central image of color similar to incident light. For $n = 1$, all principal maxima of different wavelength form a spectrum of the first order. Similarly for $n = 2$, all principal maxima of different wavelength form a second order spectrum.
3. In grating spectra, violet colour is in the innermost position and red is the outermost position. As the order of the spectrum increase, the intensity decreases.

4. The maximum orders available in grating spectra can be obtained from the following condition. For n[th] principal maxima,

$$(a+b)\sin\theta = n\lambda$$

$$n = \frac{(a+b)\sin\theta}{\lambda}$$



Q. What do you mean by dispersive power of diffraction grating?

Solution: The **dispersive power** of a diffraction grating is defined as the **rate of change of the angle of diffraction with the change in the wavelength** of light used.

$$\boxed{\omega = \frac{d\theta}{d\lambda}}$$

For plane diffraction grating, $(a+b)\sin\theta = n\lambda$, ……….(10)

Differentiate this eq ^{n} with respect to λ, we get

$$(a+b)\cos\theta\frac{d\theta}{d\lambda} = n$$

$$or \qquad \frac{d\theta}{d\lambda} = \frac{n}{(a+b)\cos\theta} \qquad \text{……………(11)}$$

$$or \qquad \frac{d\theta}{d\lambda} = \frac{n}{(a+b)(1-\sin^2\theta)^{1/2}} \qquad \text{……………(12)}$$

From eq ^{n} (1), we have, $\sin\theta = \dfrac{n\lambda}{(a+b)}$

Substituting the above value of sin θ in eq ^{n} (3),

$$\boxed{\frac{d\theta}{d\lambda} = \frac{n}{(a+b)[1-(n\lambda/a+b)^2]^{1/2}} \quad or \quad \frac{d\theta}{d\lambda} = \frac{1}{[(\frac{a+b}{n})^2 - \lambda^2]^{1/2}}}$$

The above equation leads to the following conclusion:

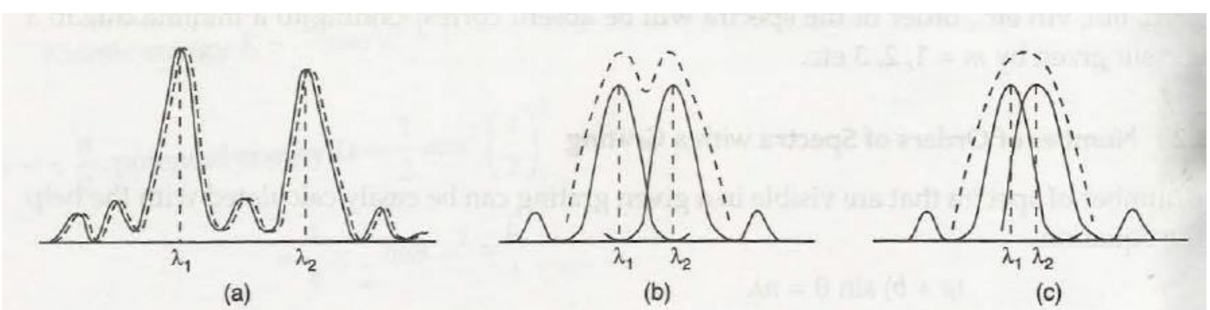
1) ω is directly proportional to the order of spectrum i.e. the higher is the order, greater is the dispersive power.
2) ω is inversely proportional to the (a+b) i.e. the dispersive power is greater for a grating having larger number of lines per cm.
3) ω is inversely proportional to the cos θ i.e. larger the value of θ, higher is the dispersive power.

Q. What do you mean by resolving power? Describe Rayleigh criterion of resolution.

Solution: The capacity of an optical instrument to show two close objects separately is called resolution and the ability of an optical instrument to just resolve the images of two close point objects is called its **resolving power**.

The Rayleigh criterion of resolution:

According to Rayleigh, the two point sources or two equally intense spectral lines are **just resolved** by an optical instrument when the central maximum of the diffraction pattern due to one source falls exactly on the first minimum of the diffraction pattern of the other and vice-versa.



(a) Fully resolved (b) Just resolved (c) Not resolved

The distribution of intensity in the grating is of the form

$$I = I_{max}\frac{sin^2 x}{x^2}$$

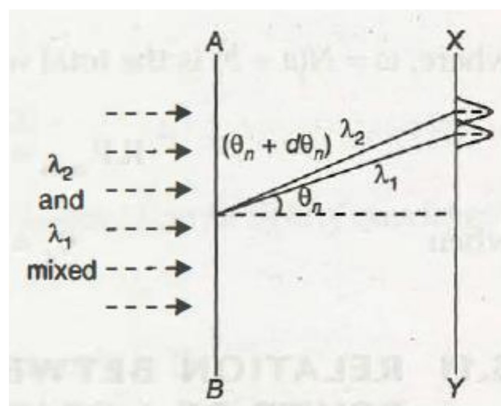
For first minimum, I is minimum if $x = \pi$. At mid point $x = \pi/2$. Thus, the total intensity at the midpoint of the two maxima is given by

$$I_{mid} = 2I_{max}\frac{sin^2(\pi/2)}{(\pi/2)^2} = \frac{8}{\pi^2}I_{max}$$

$$\boxed{\frac{I_{mid}}{I_{max}} = \frac{8}{\pi^2}}$$

Q. What do you mean by resolving power of diffraction grating? Write down the expression of resolution in case of grating.

Solution: The **resolving power of a grating** is defined as the ratio of the wavelength of any spectral line to the smallest wavelength difference between neighbouring lines for which the spectral can be just resolved at the wavelength λ. It can be expressed mathematically as $\boldsymbol{\lambda/d\,\lambda}$.



Expression for resolving power:

Let the direction of n^th principal maxima for wavelength λ_1 is given by

$$(a + b)\sin\theta_n = n\lambda_l$$

$$N(a + b)\sin\theta_n = Nn\lambda_1$$

and the first minima will be in the direction given by

$$N(a+b)\sin(\theta_n + d\theta_n) = m\lambda_1$$

where m is an integer except 0, N, $2N$ … because at these values condition of maxima will be satisfied.

The first minima adjacent to the n^th maxima will be in the direction $(\theta_n + d\theta_n)$ only when $m = (nN + 1)$. Thus

$$N(a + b)\sin(\theta_n + d\theta_n) = (nN + 1)\lambda_1 \qquad \text{.... (1)}$$

For just resolution, the principal maxima for the wavelength λ_2 must be formed in the direction $(\theta_n + d\theta_n)$, therefore

$$(a + b)\sin(\theta_n + d\theta_n) = n\lambda 2$$

$$N(a + b)\sin(\theta_n + d\theta_n) = Nn\lambda_2 \qquad \text{.... (2)}$$

Now equating (1) and (2),

$$(nN + 1)\,\lambda_1 = Nn\,\lambda_2$$

Let $\lambda_1 = \lambda$ and $\lambda_2 - \lambda_1 = d\lambda$ then $\lambda_2 = \lambda + d\lambda$

$$\Rightarrow (nN + 1)\,\lambda = Nn\,(\lambda + d\lambda)$$

$$\lambda = Nn\,d\lambda$$

Thus resolving power of grating is,

$$\boxed{\textbf{R.P.} = \lambda/d\lambda = nN}$$

Q. Give difference between dispersive power and resolving power of grating.

Solution:

	Dispersive power	**Resolving power**
1.	The dispersive power of a grating is defined as the rate of change of angle diffraction with the wavelength of light used ($d\theta/d\lambda$).	The resolving power of a grating is defined as the ratio of the wavelength of any spectral line to the smallest wavelength difference between neighboring lines for which the spectral lines can be just resolved.
2.	The dispersive power is independent of the number of lines on the grating element.	The resolving power increases as the number of lines on the grating surface increases.
3.	The dispersive power depends upon the grating element.	The resolving power is independent of the grating element.
4.	The dispersive power depends upon the number of lines per centimeter on the grating	The resolving power depends on the total number of lines on the ruled surface.

DIFFRACTION NUMERICALS

Q.1 Light of wavelength 5500 A^0 is incident normally on a slit of width $22\times\ 10^{-5}$cm. Calculate the angular position of the second minima.

Sol. For single slit diffraction, the angular position of minima is

$$sin\theta = 2\lambda / a \qquad\qquad (n = 2)$$

$$\theta = sin^{-1}\,(2\lambda/a)$$

$$= sin^{-1}\,(2\times 5500\times 10^{-8}) / (22\times\ 10^{-5})$$

$$= 30°$$

Q.2 A diffraction grating used at normal incidence gives a yellow line (λ = 6000 A^0) in a certain spectral order superimposed on a blue line (λ = 4800 A^0) of next higher order. If the angle of diffraction is $sin^{-1}(^3/_4)$, calculate the grating element.

Sol. The direction of principal maxima for normal incidence for wavelength λ_1 is

$$(a+b)\sin\theta = n\lambda_1 \qquad \ldots\ldots(1)$$

Let nth order maximum of λ_1 coincide with (n+1)th order maximum of λ_2, then

$$(a+b)\sin\theta = n\lambda_1 = (n+1)\,\lambda_2 \quad \text{or } n = \frac{\lambda_2}{\lambda_1 - \lambda_2}$$

From (1), $$(a+b)\sin\theta = \frac{\lambda_2}{\lambda_1 - \lambda_2}$$

$$(a+b) = \frac{\lambda_2}{(\lambda_1 - \lambda_2)\sin\theta}$$

$$= \frac{6000\times 10^{-8}\times 4800\times 10^{-8}}{(6000-4800)\times 10^{-8}\times(^3/_4)}$$

$$= 3.2\times 10^{-4}cm$$

Q.3 Light of wavelength 5000 A^0 is incident normally on a slit. The central maximum falls out 30^0 on both sides of the direction of incident light. Calculate the slit width.

Sol. For single slit, the direction of minima is given by,

$$a\sin\theta = n\lambda \quad \text{where } n = 1, 2, 3\ldots..$$

Therefore, the angular spread of the central maximum on either side of incident light is,

$$\sin\theta = \frac{\lambda}{a}$$

$$a = \frac{\lambda}{sin\theta} = \frac{5\times10^{-8}}{\sin 30} = 10^{-4}cm$$

Q.4 **Calculate the angle at which the first dark band and the next bright band are formed in the Fraunhofer diffraction pattern of a slit 0.3 mm wide (λ = 5890 A⁰).**

Sol. For single slit, the direction of minima is given by,

$$a\sin\theta = n\lambda \quad \text{where } n = 1, 2, 3\ldots..$$

For first dark band, n =1

$$\sin\theta = \frac{\lambda}{a} = \frac{5\times10^{-8}}{0.03} = 0.00196$$

$$\theta = \sin^{-1}(0.00196) = 0.112^{\circ}$$

The angle of diffraction $\theta^{'}$ corresponding to the first bright band on either side of the central maximum is given by

$$a\sin\theta^{'} = 3\lambda/2$$

$$\sin\theta^{'} = 1.5\frac{\lambda}{a}$$

$$\text{since } \frac{\lambda}{a} = 0.00196\text{, then}$$

$$\sin\theta^{'} = 0.00196\times1.5 = 0.00294$$

$$\theta^{'} = \sin^{-1}(0.00294) = 0.168^{\circ}$$

Q.5 **A single slit is illuminated by light composed of two wavelengths λ_1 and λ_2. One observes that due to Fraunhofer, the first minima obtained for λ_1 coincides with the second diffraction minima of λ_2. What is relation between λ_1 and λ_2?**

Sol. For single slit, the direction of minima is given by,

$$a\sin\theta = n\lambda \quad \text{where } n = 1, 2, 3\ldots..$$

For wavelength λ_1 , the direction of first minima is,

$$a\sin\theta_1 = \lambda_1$$

Similarly for wavelength λ_2 , the direction of second minima is,

$$a\sin\theta_2 = 2\lambda_2$$

For given problem,

$$\lambda_1 = 2\lambda_2$$

$$\lambda_2 = \frac{\lambda_1}{2}$$

Q.6 **In a plane transmission grating, the angle of diffraction for the second order principal maxima for $\lambda = 5\times 10^{-5}\ cm$ is 30^0. Calculate the number of lines in one cm of the grating surface.**

Sol. $n = 2, \quad \theta = 30^0, \quad \lambda = 5\times 10^{-5}\ cm$

For grating, $\quad (a+b)\sin\theta = n\lambda$

$$(a+b) = \frac{n\lambda}{\sin\theta}$$

$$Number\ of\ lines\ in\ one\ cm = \frac{1}{(a+b)} = \frac{\sin\theta}{n\lambda}$$

$$\frac{1}{(a+b)} = \frac{\sin 30^{\circ}}{2\times\mathbf{5\times 10^{-5}}} = 5000$$

Q.7 **Calculate the angle between the central image of a lamp filament and its first diffracted image produced by a fabric with 160 threads per cm ($\lambda = 6\times 10^{-5}\ cm$).**

Sol. $n = 1, \quad \lambda = 6\times 10^{-5}\ cm$,

$$(a+b) = \frac{1}{160}$$

For grating, $\quad (a+b)\sin\theta = n\lambda$

$$\sin\theta = \frac{n\lambda}{(a+b)} = \frac{1\times 6\times 10^{-5}}{(\frac{1}{160})} = 0.0096$$

$$\theta = \sin^{-1} 0.0096 = 33'$$

Q.8 **Light of wavelength, $\lambda = 5\times 10^{-5}\ cm$ is incident normally on a plane transmission grating of width 3 cm and 1500 lines. Find the angle of diffraction in first order.**

Sol. $n = 1, \quad \lambda = 5\times 10^{-5}\ cm$,

$$(a+b) = \frac{width\ of\ grating}{total\ number\ of\ lines\ on\ the\ grating} = \frac{3}{15000}\ cm$$

For grating, $\quad (a+b)\sin\theta = n\lambda$

$$\sin\theta = \frac{n\lambda}{(a+b)} = \frac{1\times5\times10^{-5}}{(\frac{3}{15000})} = \frac{1\times5\times10^{-5}\times15000}{3} = 0.25$$

$$\theta = \sin^{-1} 0.25 = 14°29'$$

Q.9 **A diffraction grating used at normal incidence gives a green line (5400 A^0) in a certain order *n* superimposed on the violet line (4050 A^0) of the next higher order. If the angle of diffraction is 30^0, calculate the value of *n*. Also find how many lines per cm are there in the grating?**

Sol. For grating, $(a+b)\sin\theta = n\lambda$

Let n^{th} order maximum of λ_1 coincide with $(n+1)^{th}$ order maximum of λ_2, then we have

$$(a+b)\sin\theta = n\,\lambda_1 = (n+1)\ \lambda_2$$

$$n\,\lambda_1 = (n+1)\ \lambda_2$$

$$n = \frac{\lambda_2}{\lambda_1 - \lambda_2}$$

We have, $(a+b)\sin\theta = \frac{\lambda_2\,\lambda_1}{\lambda_1 - \lambda_2}$

$$\sin\theta = \frac{\lambda_2\,\lambda_1}{(\lambda_1-\lambda_2)(a+b)} \quad \text{........(1)}$$

$$\lambda_1 = 5400\ A^{\circ} = 5400 \times 10^{-8} cm,\ \lambda_2 = 4050\ A^{\circ} = 4050 \times 10^{-8} cm, \qquad \lambda_1 - \lambda_2 = 1350\ \times 10^{-8} cm, \qquad \theta = 30^{\circ}$$

$$\therefore n = \frac{\lambda_2}{\lambda_1 - \lambda_2} = \frac{4050\times10^{-8}}{1350\ \times10^{-8}} = 3$$

Now from (1), $(a+b) = \frac{5400\times10^{-8}\times4050\times10^{-8}}{1350\ \times10^{-8}\times\sin 30°}$

Number of lines per cm = $\frac{1}{(a+b)} = 3086$

Q.10 **In a grating, which spectral line in 4^{th} order will overlap with 3^{rd} order line of 5461 A^0.**

Sol. For grating, $(a+b)\sin\theta = n\lambda$

Let n^{th} order maximum of λ_1 coincide with $(n+1)^{th}$ order maximum of λ_2, then we have

$$(a+b)\sin\theta = n\,\lambda_1 = (n+1)\ \lambda_2$$

$$n\,\lambda_1 = (n+1)\ \lambda_2$$

$$\lambda_2 = \frac{n\,\lambda_1}{(n+1)}$$

$$n = 3,\ \lambda_1 = 5400\ A^{\circ} = 5461 \times 10^{-8} cm\,, (n+1) = 4$$

Then,

$$\lambda_2 = \frac{3\times5461\times10^{-8}}{(4)} = 4096 \text{ A}^0$$

Q.11 A plane transmission grating has 40000 lines. Determine its resolving power in the second order for a wavelength of 5000 A^0.

Sol. $n = 2, N = 40000$

$$\text{Resolving power} = \frac{\lambda}{d\lambda} = nN = 2 \times 40000 = 80000$$

Q.12 Find the minimum number of lines in a plane diffraction grating required to just resolve the sodium doublet (5890 A^0 and 5896 A^0) in the (i) first order (ii) second order.

Sol. $n = 1, \lambda = \frac{\lambda_1+\lambda_2}{2} = \frac{5890+5896}{2} = 5893\ A^\circ,\ d\lambda = 5896 - 5890 = 6A^\circ$

$$\text{Resolving power} = \frac{\lambda}{d\lambda} = nN$$

$$N = \frac{1}{n} \cdot \frac{\lambda}{d\lambda} = \frac{5893}{1\times 6} = 982$$

For $n = 2$,

$$N = \frac{1}{n} \cdot \frac{\lambda}{d\lambda} = \frac{5893}{2\times 6} = 491$$

Q.13 Can D_1 and D_2 lines of sodium (Na) light be resolved for $\boldsymbol{\lambda_{D1}} = 5890\ \text{A}^0$, $\boldsymbol{\lambda_{D2}} = 5896\ \text{A}^0$ in second order. Number of lines in grating of 2.0 cm wide = 4500.

Sol. $n = 2, \lambda = \frac{\lambda_1+\lambda_2}{2} = \frac{5890+5896}{2} = 5893\ A^\circ,\ d\lambda = 5896 - 5890 = 6A^\circ$

$$\frac{\lambda}{d\lambda} = 2 = \frac{5893}{1\times 6}$$

$$N = \frac{5893}{12} = 491$$

Given $N = 4500 \gg$ required value. Hence lines can be resolved in second order.

UNIT – V OPTICAL FIBER

INTRODUCTION-fiber optics, the science of transmitting data, voice, and images by the passage of light through thin, transparent fibers. In telecommunications, fiber optic technology has virtually replaced copper wire in long-distance telephone lines, and it is used to link computers within local area networks. Fiber optics is also the basis of the fiberscope's used in examining internal parts of the body (endoscopy) or inspecting the interiors of manufactured structural products. The basic medium of fiber optics is a hair-thin fiber that is sometimes made of plastic but most often of glass. A typical glass optical fiber has a diameter of 125 micrometers (μm).

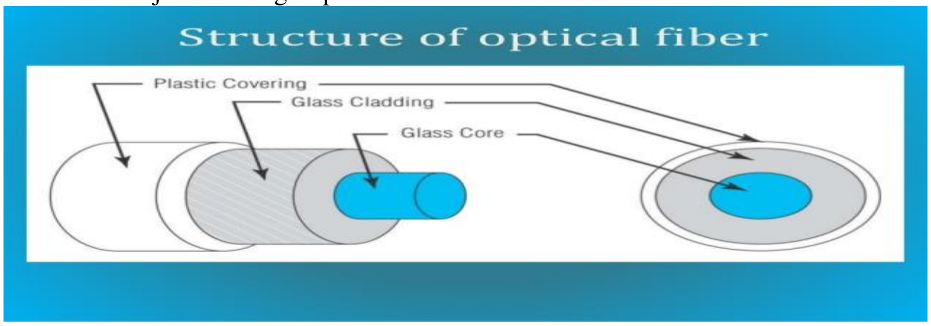
It consists of three regions—

(a) **CORE**- It is the innermost cylinder with diameter 10-100 μm and refractive index n_1.Light beam is kept within the core using the phenomenon of Total Internal reflection.

(b) **CLADDING**-The core is surrounded by a glass or plastic material with refractive index n_2 .It provides some strength to core and keep light waves within the core. The difference between n_1 and n_2 is 10^{-3} μm.

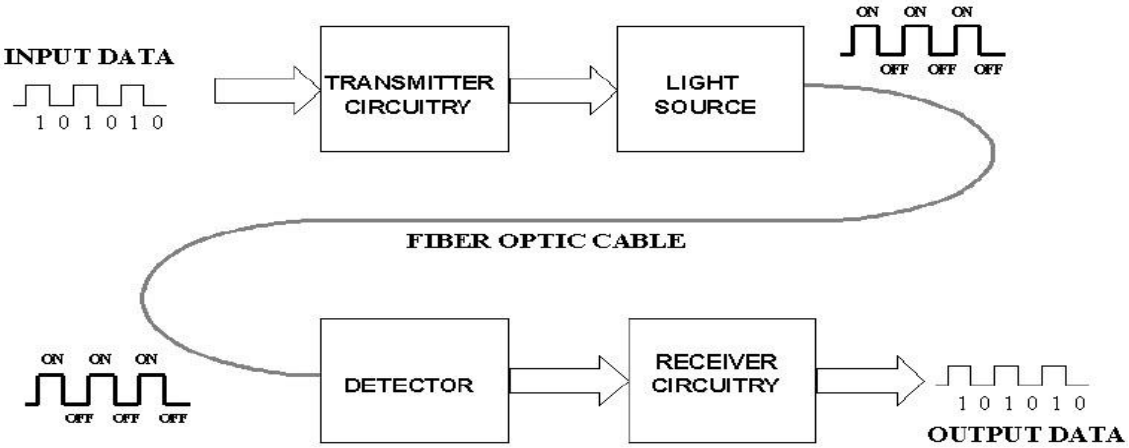
(c) **SHEATH**-The core cladding system is enclosed in an outer jacket to protect it from abrasions, contaminations and moisture.

Many such fibers with individual jackets are grouped as cables that contain hundreds of fibers.



OPTICAL FIBER COMMUNICATION-

A message to be conveyed is encoded in light wave and fed to fiber which is propagated through TIRs.



Transmitter (laser Diode) converts electrical signal to optical signal, the information is carried through optical fiber and receiver receives it and convert to electrical equivalent which than decoded to give original information.

Photo diodes are used as receivers and infrared light is used due to less attenuation and dispersion in optical fiber.

PROPAGATION MECHANISM IN OPTICAL FIBER ORTOTAL INTERNAL REFLECTION

All the information in optical fibers is carried out by the principle of **total internal reflection** and all the information carried out in the core of the optical fiber.

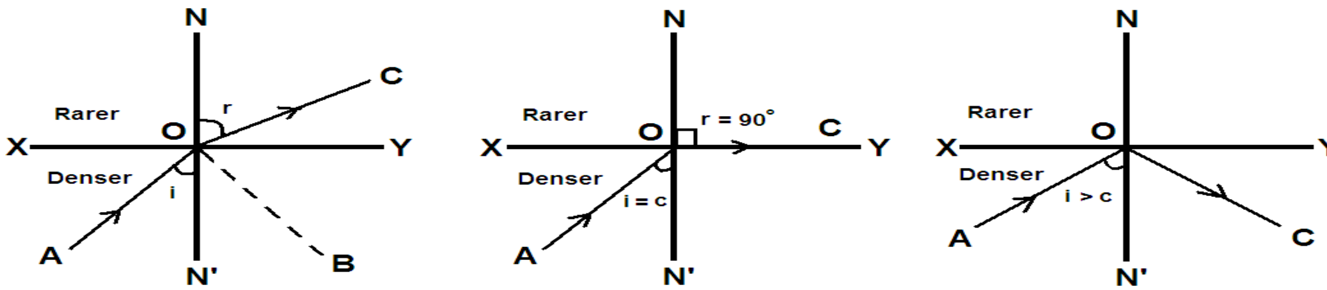
When light ray move from denser to rarer medium so as incidence angle increasesrefracted ray move away from the normal.

By Snell's Law

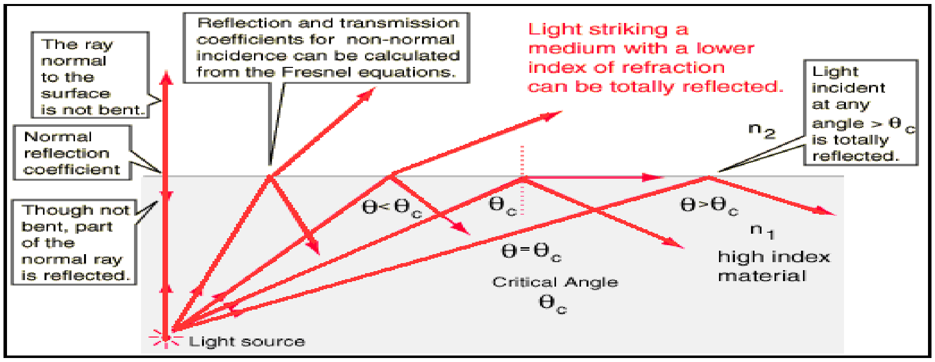
$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{v_1}{v_2} = \frac{n_2}{n_1}$$

i.e. We have angle of refraction is always greater than angle of incidence.

The incidence angle at which light ray grazes along the surface i.e. refracted ray makes an angle 90˚ with the normal is called critical angle.



Above critical angle refracted ray goes back to same medium and light is said to be total internally reflected.This effect is used in optical fiber to confine light in the core. The cladding surrounding the core, protects the core and provides an interface with a controlled index of reflection.

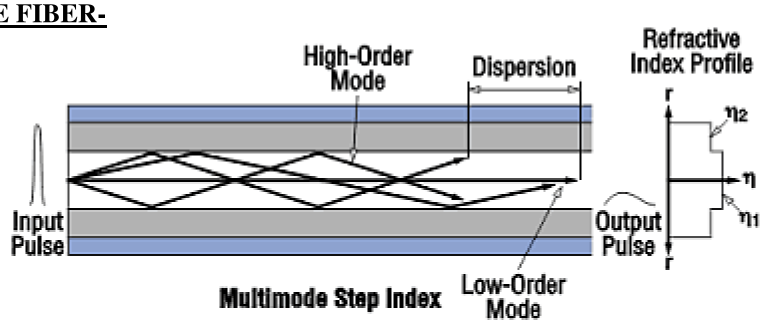


TYPES OF FIBER-

There are three types of optical fiber to meet different requirements-

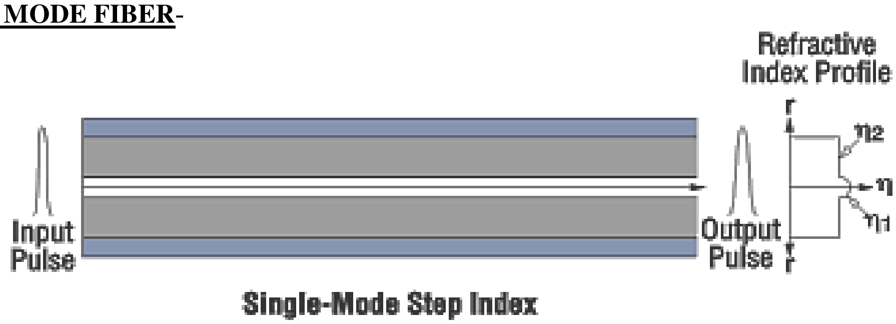
(a) Step Index Multi mode Fiber
(b) Step Index Single mode Fiber
(c) Graded Index Multi mode Fiber

STEP INDEX MULTI MODE FIBER-



It has a core diameter of about 100µm large enough to propagate many modes of light .Different mode travel different distance depending on their angle so time taken is also different hence Rays that enter with a shallower angle travel by a more direct path, and arrive sooner than those enter at steeper angles (which reflect many more times off the core/cladding boundaries as they travel the length of the fiber). The arrival of different modes of the light at different times is called **Modal Dispersion**. A narrow pulse gets broadened when passes through multimode fiber so it reduces information carrying capacity. These fibers are used for short distances 200m range.

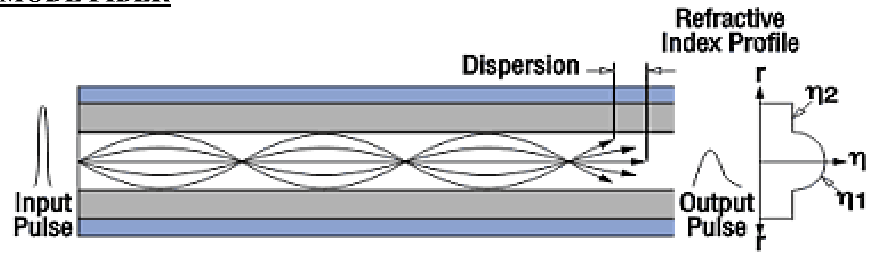
STEP INDEX SINGLE MODE FIBER-



The single mode step index fiber has the distinct advantage of low dispersion, as only one mode is transmitted. It has high information carrying capacity.

Core diameter is less about 10µm thus only one mode can propagate through the fiber so eliminates modal dispersion produced due to time difference of multimode.It requires highly directional source such as LASER diode because of small core diameter, hence the cost is higher than multimode step index fiber.SMF are used for communication greater than 200m and frequently used under sea water.

GRADED INDEX MULTI MODE FIBER



Multimode Graded Index

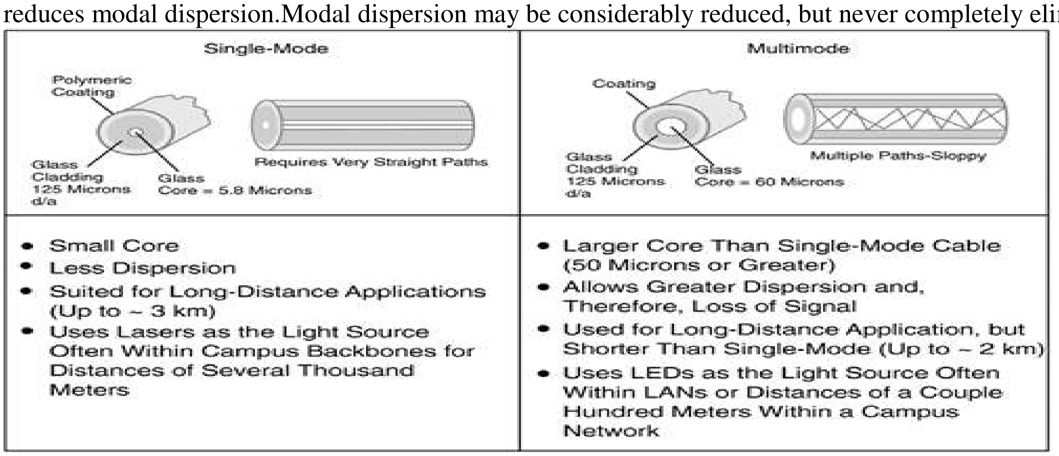
It is less expensive for overcoming modal dispersion. In step index, core has a constant value but in graded index refractive index of core has max value at axis and decreases to min. at core cladding boundary.

As ref. index

$$\eta \propto \frac{1}{v}.$$

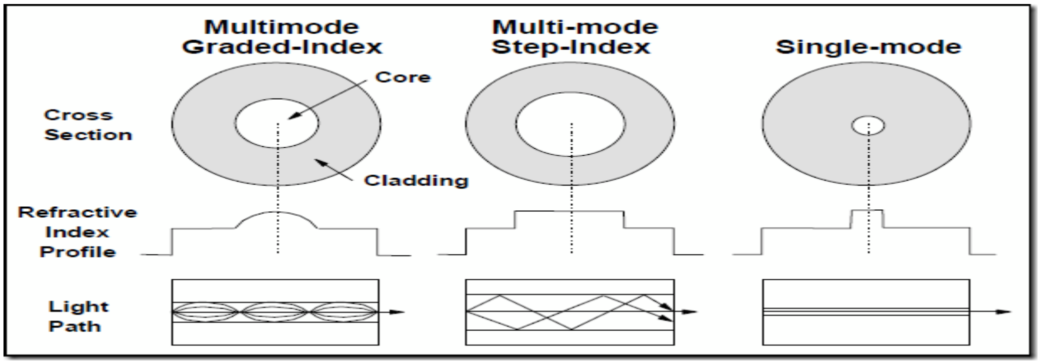
Different light modes in a graded-index multimode fiber still follow different lengths along the fiber, as in step-index multimode fiber. However their speeds differ because the speed of guided light changes with fiber core's refractive index.

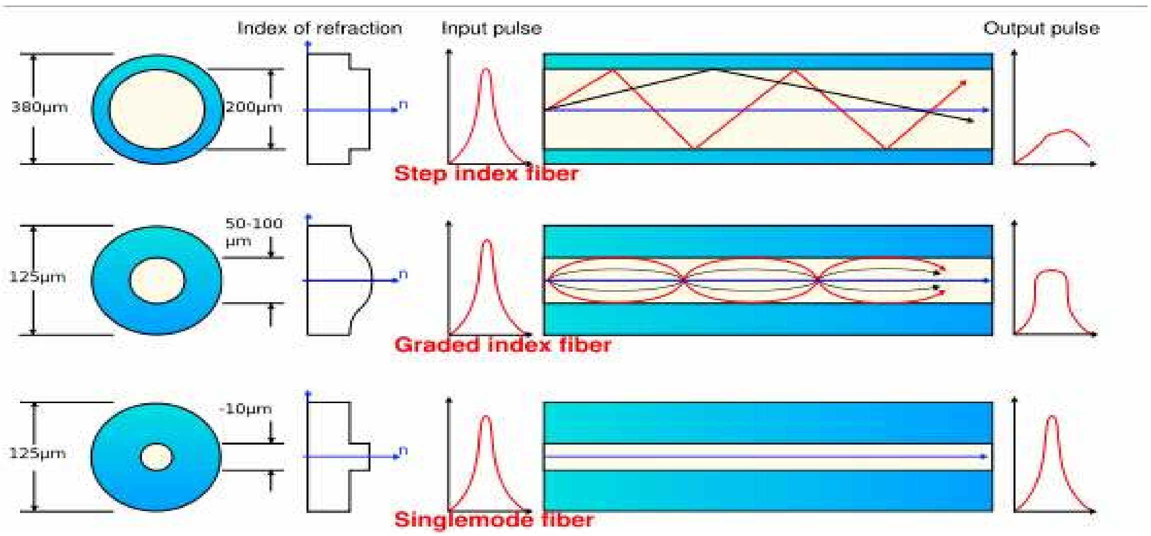
So farther the light goes from the center of the fiber, the faster its speed. So the speed difference compensate for the longer paths followed by the light rays that go farthest from the center of the fiber. This equalizing of transit times of different modes greatly reduces modal dispersion.Modal dispersion may be considerably reduced, but never completely eliminated.



COMPARISON BETWEEN STEP INDEX AND GRADED INDEX FIBER

Step index	**Graded index**
1. Refractive index of core is constant.	1. Refractive index of core is max at core axis and min. at core-cladding boundary.
2. Light ray reflect abruptly at core cladding boundary	2. Light raybend smoothly as reaches to cladding.
3. NA is large.	3. NA is small.
4. High attenuation.	4. Low attenuation.
5. pulse dispersion is large in multi mode fiber	5. pulse dispersion is small



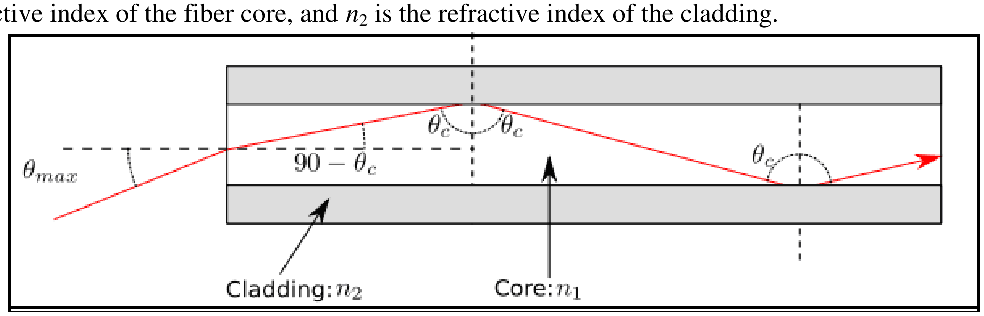


ACCEPTANCE ANGLE AND ACCEPTANCE CONE

As the ray makes angle greater than critical angle undergo TIR and those less than Critical angle are either absorbed or scattered.The half-angle of the cone (a) within which incident light is totally internally reflected is defined as Acceptance angle of the cone. It is equal to sin-1(NA).This angle is different for different fibers and depends on ref. index and diameter of core.The range of angles for which incident light is totally internally reflected is defined as Acceptance cone. A multimode fiber will only propagate light that enters the fiber within a certain cone, known as the acceptance cone of the fiber. The half-angle of this cone is called the acceptance angle θ_{max}. For step index multimode fiber, the acceptance angle is determined only by the indices of refraction:

$$n \sin \theta_{\max} = \sqrt{n_1^2 - n_2^2},$$

Where n_1 is the refractive index of the fiber core, and n_2 is the refractive index of the cladding.



When a light ray is incident from a medium of refractive index n to the core of index n_1 at the maximum acceptance angle, Snell's law at the medium–core interface gives

$$n \sin \theta_{\max} = n_1 \sin \theta_r.$$

as we have:

$$\sin \theta_r = \sin \left(90^\circ - \theta_c\right) = \cos \theta_c$$

$$\theta_c = \sin^{-1} \frac{n_2}{n_1}$$

Where θ_c is the critical angle for total internal reflection.

Substituting $\cos\theta_c$ for $\sin \theta_r$ in Snell's law we get:

$$\frac{n}{n_1}\sin\theta_{\max} = \cos\theta_c.$$

By squaring both sides

$$\frac{n^2}{n_1^2}\sin^2\theta_{\max} = \cos^2\theta_c = 1 - \sin^2\theta_c = 1 - \frac{n_2^2}{n_1^2}.$$

Solving, we find the formula stated above:

$$n\sin\theta_{\max} = \sqrt{n_1^2 - n_2^2},$$

$$\theta_{\max} = \sin^{-1}\sqrt{\frac{n_1^{\,2} - n_2^{\,2}}{n}},$$ For air n=1

Numerical aperture It defines the light acceptance capacity of fiber also known as figure of merit .sine of acceptance angle is NA.

i.e. $\sin\theta_{\max} = \mathrm{NA}$ and $$\mathrm{NA} = \sqrt{n_1^2 - n_2^2},$$

Fractional refractive index change- it is defined as the ratio of difference in ref. indices of core and cladding to ref. index of core, denoted by Δ

$$\Delta = \frac{n_1 - n_2}{n_1} = positive \text{ Or } \Delta = \frac{n_1^{\,2} - n_2^{\,2}}{2n_1^{\,2}} = \frac{NA}{2n_1^{\,2}}$$

$$\Rightarrow NA = n_1\sqrt{2\Delta}$$

<u>**V-NUMBER-**</u> The no. of modes supported by an optical fibers obtained by an important parameter called normalized frequency or cut-off parameter or V-parameter or V-number of fiber.

$$V = \frac{2\pi a}{\lambda}\sqrt{\left(n_1^{\,2} - n_2^{\,2}\right)} = \frac{2\pi a}{\lambda}NA = \frac{2\pi a}{\lambda}n_1\sqrt{2\Delta}$$

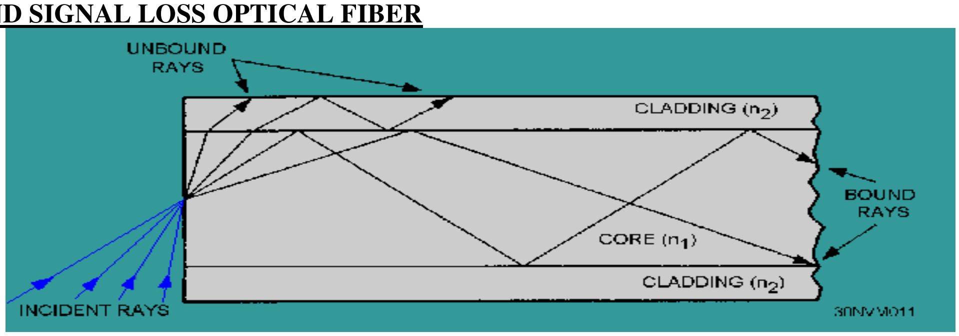
So no. of modes supported by step index fiber is $N_{\max} = \frac{V^2}{2}$

For single mode fiber V<2.405, For multi mode fiber V>2.405. Wavelength corresponding to V=2.405 is called cut-off wavelength of fiber.

$$\lambda_c = \frac{\lambda V}{2.405}$$

And no. of modes for graded index fiber $N_{\max} = \frac{V^2}{4}$

<u>**ATTENUATION AND SIGNAL LOSS OPTICAL FIBER**</u>



When guided through an optical fiber, the reduction in amplitude or intensity of light signal is called attenuation. Lower is its value, greater be the intensity of signal at receiving end.

Attenuation in an optical fiber is caused by absorption, scattering, and bending losses. **Attenuation** is the loss of optical power as light travels along the fiber. Signal attenuation is defined as the ratio of optical input power (P_i) to the optical output power (P_o). Optical input power is the power given to fiber from an optical source. Optical output power is the power received at the fiber end or optical detector. The following equation defines signal attenuation as a unit of length:

Attenuation loss α is measured in decibels i.e. in dB/Km.

$$\alpha = \frac{10}{L}\log_{10}\left(\frac{P_{out}}{P_{in}}\right)$$

To indicate loss we introduce negative sign as

$$\alpha = -\frac{10}{L}\log_{10}\left(\frac{P_{out}}{P_{in}}\right)$$

Here length (L) is expressed in kilometers. Therefore, the unit of attenuation is decibels/kilometer (dB/km).
Attenuation is caused by absorption, scattering, and bending losses. Each mechanism of loss is influenced by fiber-material properties and fiber structure.

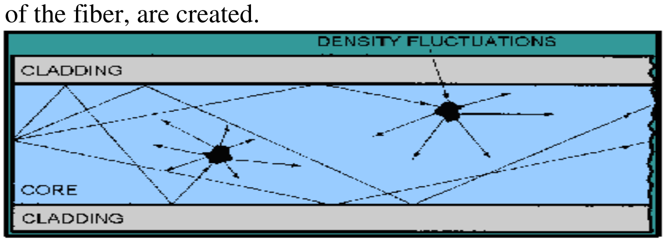
1. **ABSORPTION LOSS-** Absorption is a major cause of signal loss in an optical fiber. **Absorption** is defined as the portion of attenuation resulting from the conversion of optical power into another energy form, such as heat. Absorption in optical fibers is explained by three factors:
   a) Imperfections in the atomic structure of the fiber material
   b) The intrinsic or basic fiber-material properties
   c) The extrinsic (presence of impurities) fiber-material properties

(a)Imperfections in the atomic structure-Imperfections in the atomic structure induce absorption by the presence of missing molecules or oxygen defects. Absorption is also induced by the diffusion of hydrogen molecules into the glass fiber. Since intrinsic and extrinsic material properties are the main cause of absorption, they are discussed further.

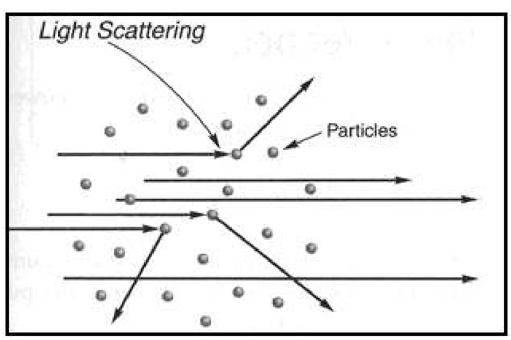
(b) INTRINSIC ABSORPTION-Intrinsic absorption is caused by basic fiber-material properties. If an optical fiber were absolutely pure, with no imperfections or impurities, then all absorption would be intrinsic. Intrinsic absorption sets the minimal level of absorption.

(c)EXTRINSIC ABSORPTION - Extrinsic absorption is caused by impurities introduced into the fiber material. Trace metal impurities, such as iron, nickel, and chromium, are introduced into the fiber during fabrication. **Extrinsic absorption** is caused by the electronic transition of these metal ions from one energy level to another.

2. SCATTERING - Basically, scattering losses are caused by the interaction of light with density fluctuations within a fiber. Density changes are produced when optical fibers are manufactured. During manufacturing, regions of higher and lower molecular density areas, relative to the average density of the fiber, are created.

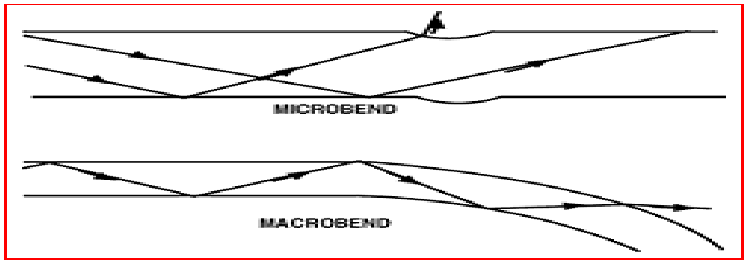


(a) RAYLEIGH SCATTERING LOSS- Rayleigh scattering is a loss mechanism arising from local microscopic fluctuation in density.Silica molecules move randomly in the molten state and freeze in place during fiber fabrication. Density fluctuation leads to random fluctuations of the refractive index on a scale smaller than the optical wavelength. Light scattering in such a medium is known as Rayleigh scattering.Scattering depends not on the specific type of material but on the size of the particles relative to the wavelength of light. The closer the wavelength is to the particle size, the more scattering. In fact, the amount of scattering increases rapidly as the wavelength decreases.



(b) WAVEGUIDE SCATTERING LOSS-An ideal single mode fiber with a perfect cylindrical geometry guides the optical mode without energy leakage into the cladding layer. But in reality, imperfections at the core-cladding interface, such as random core-radius variations, can lead to additional losses which contribute to the total fiber loss.

3. BENDING LOSS - Bending the fiber also causes attenuation. Bending loss is classified according to the bend radius of curvature: microbend loss or macrobend loss.
(a) Macrobends are small microscopic bends of the fiber axis that occur mainly when a fiber is cabled.**Macrobends** are bends having a large radius of curvature relative to the fiber diameter. Macrobend and Macrobend losses are very important loss mechanisms. Fiber loss caused by microbending can still occur even if the fiber is cabled correctly. During installation, if fibers are bent too sharply, Macrobend losses will occur.
(b) Microbend losses are caused by small discontinuities or imperfections in the fiber. Uneven coating applications and improper cabling procedures increase microbend loss. External forces are also a source of microbends. An external force deforms the cabled jacket surrounding the fiber but causes only a small bend in the fiber. Microbends change the path that propagating modes take, as shown in figure.**Microbend loss** increases attenuation.



SIGNAL DISPERSION
Dispersion is the phenomenon in which phase velocity of wave depends on its frequency.
While propagating through the fiber the phenomenon of spreading or broadening of pulses is pulse dispersion.
1.INTERMODAL OR MODAL DISPERSION-It is dominant source of dispersion in MMF and does not exist in SMF.
Due to different wavelength each ray takes different time so adjacent pulses overlap at output.
Pulse spreading determines min. separation between adjacent pulses hence determine max. Information carrying capacity of fiber. It also depends upon fiber distance and increases with increasing distance. Thus amplitude decreases and due to overlapping pulses become indistinguishable.

2. INTRAMODAL OR CHROMATIC DISPERSION-
It occurs in all types of fibers. Optical signal emits a band of frequency so due to propagation delay each transmitted wave gets broadened. This is called intramodal dispersion.
(a) Material Dispersion-Material dispersion occurs because the spreading of a light pulse is dependent on the wavelengths' interaction with the refractive index of the fiber core. Different wavelengths travel at different speeds in the fiber material. Different wavelengths of a light pulse that enter a fiber at one time exit the fiber at different times. Material dispersion is a function of the source spectral width. The spectral width specifies the range of wavelengths that can propagate in the fiber. Material dispersion is less at longer wavelengths.
(b) Waveguide dispersion-Waveguide dispersion occurs because the mode propagation constant is a function of the size of the fiber's core relative to the wavelength of operation. Waveguide dispersion also occurs because light propagates differently in the core than in the cladding.

APPLICATION OF OPTICAL FIBER-

1. These are used in broadcast T.V. and cable T.V.
2. Used for transmission of digital data generated by computer .In military it is used for secret communication, command and control link on ships and aircrafts.
3. In building so, optical fibers are used for to route sunlight from roof to other parts of buildings.
4. Are used to form sensors to measure physical and chemical parameters.
5. Optical fiber can couple two circuits without any electric link.
6. Used to provide signal amplification and carry light to display units.

ADVANTAGE OF OPTICAL FIBER OVER COAXIAL CABLE (OR COPPER WIRE)

1. Optical frequency range is very high i.e. light has a very high information carrying capacity hundreds times greater than that of copper wire.
2. Optical fibers are more economical in long run.
3. Optical fibers can accommodate much higher band width.
4. Optical fibers are light in weight so they are easier to transport and handle and no connection is required to connect transmitter and receiver.
5. Optical fibers can withstand environmental hazards better and have a long life.
6. Silica which is principle material for optical fiber is easily available, and much cheaper in cost as compare to copper wire.
7. Dielectric nature of optical fiber is an important advantageous feature. This provides optical immunity to electromagnetic interference.

LASER

LASER- The word "LASER" stands for Light Amplification by Stimulated Emission of Radiation. It is a device to produce strong intense, monochromatic, collimated, unidirectional and highly coherent beam of light; and depends on the phenomena of stimulated emission. The theoretical basis of laser was predicted by Einstein in 1917. The first laser device was developed by T. H. Maiman in 1960.

When stimulated emission occurs below infrared region of electromagnetic spectrum, the term is known as MASER (Microwave Amplification by Stimulated Emission of Radiation). When stimulated emission occurs in infrared region and above infrared region ie in visible and UV region, the term is known as LASER.

The phenomenon of stimulated emission was first used by Townes in 1954, in development of MASER. In 1958, Shawlow and Townes showed that MASER principle can be extended into visible region, and in 1960, Maiman built the first laser which is known as Ruby laser.

Difference between laser light and ordinary light-

	Ordinary Light	Laser light
1.	Light emitted is not monochromatic	Light emitted is highly monochromatic
2.	Light is not intense and bright	Light is intense and bright
3.	Light is emitted in all direction	Light is emitted only in one direction
4.	Light does not have high degree of coherence	Light have high degree of coherence
5.	Coherence time is not higher	Coherence time is much higher

Absorption Of Radiation- Let us consider two energy levels 1 and 2 of an atom with energies E_1 and E_2 as shown in figure 1.1.

An atom residing in energy state E_1 can absorb a photon and go to excited state with energy E_2, provided the photon energy $h\nu$ equals the energy difference $(E_2 - E_1)$. Therefore,

$$h\nu = E_2 - E_1 \quad \text{or} \quad \nu = (E_2 - E_1)/h$$

This process is called stimulated absorption or simply absorption.

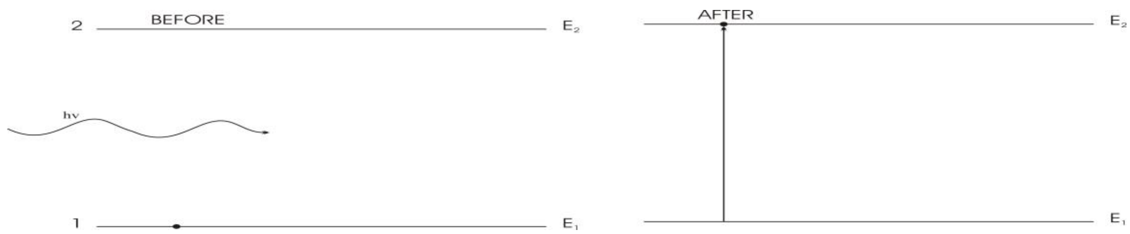


Fig. 1.1

Therefore, an atom in ground state with energy E1 absorbs an incident photon of energy h ν and goes to excited state with higher energy E_2. This process is known as stimulated absorption. The rate of absorption or rate of transition 1→2 is proportional to N_1 and also to energy density u(ν).

$$P_{12} = N_1 B_{12}\, u(\nu),\quad \text{where } B_{12} \text{ is known as Einstein's coefficient of absorption.}$$

The atom in excited state returns to ground state in following two form**.**

(a) Spontaneous emission **(b)** stimulated emission

<u>Spontaneous Emission</u>- Let us now consider an atom initially is in the higher state E_2 (figure 1.**2**) Higher energy state E_2 is not a stable state. After a short interval of time (10^{-8} Sec), the atom jumps to ground state $\mathbf{E_1}$ by emitting a photon of frequency ν .This is known as spontaneous emission of radiation.

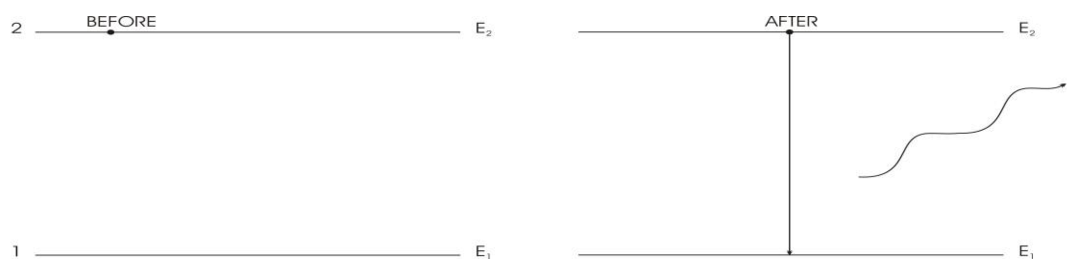


Fig.1.2

The emission of radiation from higher energy state to lower energy state without any external influence is called as spontaneous emission. The spontaneous emission is random in character. Therefore spontaneous emission is incoherent. The rate of fall of electrons from excited state E_2 to ground state E_1 is proportional to number of electrons in excited state E_2. The probability of spontaneous emission transition 2→1 is directly proportion to number of atoms in upper energy level

$P_{21} = N_2A_{21}$,   where A_{21} is known as Einstein's coefficient of spontaneous

emission absorption.

Stimulate Emission- According to Einstein, an atom in an excited energy state, under the influence of the of photon of frequency v incident upon it, jump to lower energy state. This transition produces a second photon which is identical to incident photon with respect to frequency, phase and propagation direction. This process is called stimulated or induced emission. The probability of stimulated emission transition $2\rightarrow1$ is directly proportion to number of atoms in upper energy level N_2 and energy density $u(\nu)$,

$$P_{21} = N_2B_{21}\, u(\nu),$$

So the process of forced emission of photons caused by the incident photon is called as stimulated emission.

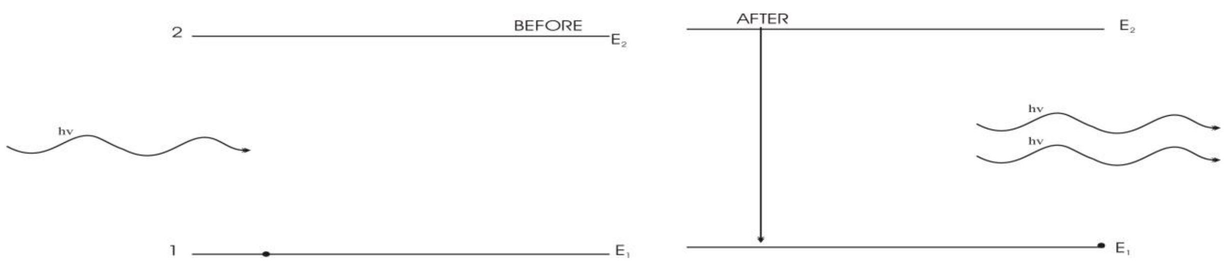


Fig. 1.3

Difference between spontaneous emission and stimulated emission

	Spontaneous emission	Stimulated emission
1	The emitted photons from various atoms have no phase relationship between them	The emitted photons have same frequency and are in phase with incident photons
2	Emitted radiation are non-coherent	Emitted radiation are coherent
3	Emitted photons can move in any direction	Foe every incident photon, there are two outgoing photons moving in same direction

4	The rate of emission is proportional to number of excited electrons in excited state E2	The rate of emission is proportional to number of excited electrons in excited state E2 and energy density of incident radiation
5	Spontaneous emission disfavours laser action	Stimulated emission favours laser action

Einstein relation between spontaneous And stimulated Emissions or transition probabilities

Let N_1 and N_2 be the number of atoms at any instant in the state 1 and 2 respectively. The probability of absorption that the number of atoms in state 1 absorbs a photon and rise to state 2 per unit time is given by

$$N_1 P_{12} = N_1 B_{12}\, u(\nu), \qquad \text{-------(1)}$$

The total probability of emission that the number of atoms in higher state 2 to drop to lower state 1, either spontaneously or under stimulation by emitting a photon, per unit time is the sum of two probabilities, that is

$$N_2 P_{21} = N_2 A_{21} + N_2 B_{21}\, u(\nu), \qquad \text{----(2)}$$

At thermal equilibrium, the absorption and emission probabilities are equal, we can write

$$N_1 P_{12} = N_2 P_{21}$$

$$N_1 B_{12}\, u(\nu) = N_2 A_{21} + N_2 B_{21}\, u(\nu)$$

$$u(\nu)\,(N_1 B_{12} - N_2 B_{21}) = N_2 A_{21}$$

$$u(\nu) = \frac{N_2 A_{21}}{(N_1 B_{12} - N_2 B_{21})}$$

$$u(\nu) = \frac{N_2 A_{21}}{N_2\left(\frac{N_1}{N_2} B_{12} - B_{21}\right)}$$

$$u(\nu) = \frac{A_{21}}{\left(\frac{N_1}{N_2} B_{12} - B_{21}\right)}$$

$$u(\nu) = \frac{A_{21}}{B_{21}} \left\{ \frac{1}{\left(\frac{N_1}{N_2}\right)\left(\frac{B_{12}}{B_{21}}\right) - 1} \right\} \quad \text{-------3}$$

According to Einstein, the probability of stimulated absorption is equal to the probability of stimulated emission, that is

$$B_{12} = B_{21}$$

Thus from equation 3

$$u(\nu) = \frac{A_{21}}{B_{21}} \left\{ \frac{1}{\left(\frac{N_1}{N_2}\right) - 1} \right\} \quad \text{-------(4)}$$

According to Boltzmann's distribution law, number of atoms N_1 and N_2 in energy states E_1 and E_2 at temperature T is given by

$$N_1 = N_0 \quad e^{-E_1/kT}, \; N_2 = N_0 \quad e^{-E_2/kT}$$

Where N_0 is the total number of atoms present and k is Boltzmann's constant

$$\frac{N_1}{N_2} = e^{(E_2 - E_1)/kT}, \text{ but } E_2 - E_1 = h\nu$$

$$\frac{N_1}{N_2} = e^{(h\nu)/kT} \quad \text{-----(5)}$$

Now using equation 5 in equation 4

$$u(\nu) = \frac{A_{21}}{B_{21}} \quad \frac{1}{(e^{(h\nu)/kT} - 1)} \quad \text{----(6)}$$

Now according to Planck's law the energy density of radiation $u(\nu)$ is given by,

$$u(\nu) = \frac{8\pi h \nu^3}{c^3} \left(\frac{1}{(e^{(h\nu)/kT} - 1)} \right) \quad \text{-----(7)}$$

On comparing equation (6) and (7), we get

$$\frac{A_{21}}{B_{21}} = \frac{8\pi h \nu^3}{c^3} \quad \text{---------(8)}$$

From above equation it is clear the ratio between spontaneous emission coefficient and stimulated emission coefficient is proportional to ν^3. It means probability of spontaneous emission increases rapidly with the energy difference between two states.

Necessary condition to achieve laser action----

1. **The rate of emission must be greater than the rate of absorption**- Under normal condition, the number of atoms in the upper energy level is always smaller than that in lower energy level. If by some means, the number of atoms in higher energy state is made greater than that in lower energy state, the emission rate will become greater than the absorption rate.

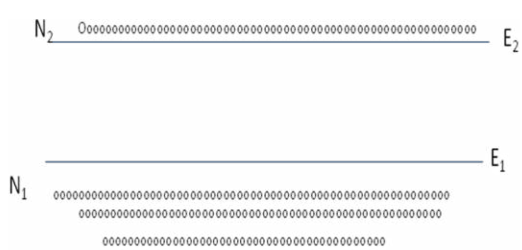
2. **The probability of spontaneous emission must be negligible in comparison to the probability of stimulated emission**- This condition can be achieved by taking working substance(active medium) such that its atom have metastable states which have a life time 10^{-3} sec. or more instead of usual 10^{-8}sec. If certain atoms are excited to metastable state the probability of spontaneous emission will be quite negligible.

3. **The coherent beam of light must be sufficiently amplified**-For this, active medium must be placed between two reflecting mirrors. The one of mirror is fully reflecting while the other is partially transmitting. The photons emitted by stimulating emission are reflected back and forth in the laser medium by these mirrors so that they are confined within the system long enough to allow them to stimulate further emission from other excited state. In this way reflected photons, traveling through the medium stimulate further emission as well as to produce amplified coherent and intense beam of light which is emitted by partially transmitting mirror.

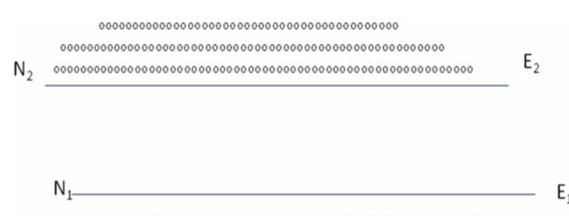
Population Inversion-

Let N_1 be the number of atoms in ground state and N_2 be the number of atoms in excited state.

For the laser action to take place, the higher energy levels should be more populated than the lower energy levels, i.e., $N_2>N_1$. The situation in which the number of atoms in the higher energy state exceeds that in the lower state ($N_2>N_1$) is known as "population inversion".



Normal condition $N_1>N_2$



After population inversion $N_2>N_1$

Various Pumping Methods

Pumping- The raising of atoms from lower energy level to upper energy level is called pumping. i. e. The process of achieving population inversion is known as " Pumping of atoms". There are several method for pumping.

1. **Optical Pumping --** The atoms are exposed to light radiation (photons) of energy h ν. The atoms in lower energy states absorb these radiations and go to excited state. In optical pumping the energy comes in the form of short flashes of light. This method was first used by Maiman in Ruby laser. This method is suitable for those medium which are transparent to light. This type of pumping is used in Ruby laser, Nd-YAG laser, etc.

2. **Electric discharge method-** In this method, the electrons are produced in an electric discharge tube. These electrons are accelerated to high velocities by strong electric field. These accelerated electrons collide with gas atoms. The accelerated electrons transfer energy to gas atoms. Some of the atoms gain energy and go to excited state. This method is known as electric discharge method.

   This method of pumping is used in gas laserlike argon laser, Helium **-** Neon laser, CO_2 laser, etc.

3. **Inelastic atom-atom collision-**In this method, accelerated electrons produced by electric discharge of gaseous medium (a mixture of two gases) collide with the atoms of one kind of gas (which are responsible for pumping). The excitation energy of these atoms are transferred to the atoms of other kind of gas (which are responsible for laser transition in their inelastic collision with them. In He-Ne laser, Helium is the pumping medium and Neon is the lasing medium.

4. **Chemical Pumping**- The energy necessary for pumping is generated by chemical reaction.As an example, the heat energy evolved, when hydrogen combines with fluorine to form hydrogen fluoride, is used for pumping the atoms in a CO_2 laser.

5. **Direct Conversion-** In this method, electric energy is applied to direct band semiconductor like GaAS. The combination of electrons and holes take place. During this process, electric energy is directly converted into light energy. This method of pumping is used in semiconductor laser.

Metastable state-We know that normally an atom in excited state has very short life time (~ 10^{-8} sec.). As atoms are continuously going to excited state by pumping process, they should remain in higher energy state until population in higher state (N_2) becomes greater than that in lower state (N_1).

An energy state which has a long lifetime is called metastable state. Metastable state can be obtained in a crystal containing impurity atoms. These levels lie in forbidden band gap of host crystal.

Optical Resonator- The optical resonator consists of two reflecting mirrors R_1 and R_2. The mirror R_1 is fully reflecting while the other mirror R_2 is partially reflecting. The active material is placed between them.

The photon generated due to transitions between the energy states of active material undergoes multiple reflections between the two mirrors. As the photons (light) bounces back and forth between the two mirrors, the intensity of light is increased enormously. Finally the intense and amplified beam (laser) is allowed to come out the partial mirror R_2. **Therefore the function of the optical resonator is to increase the intensity of laser beam.**

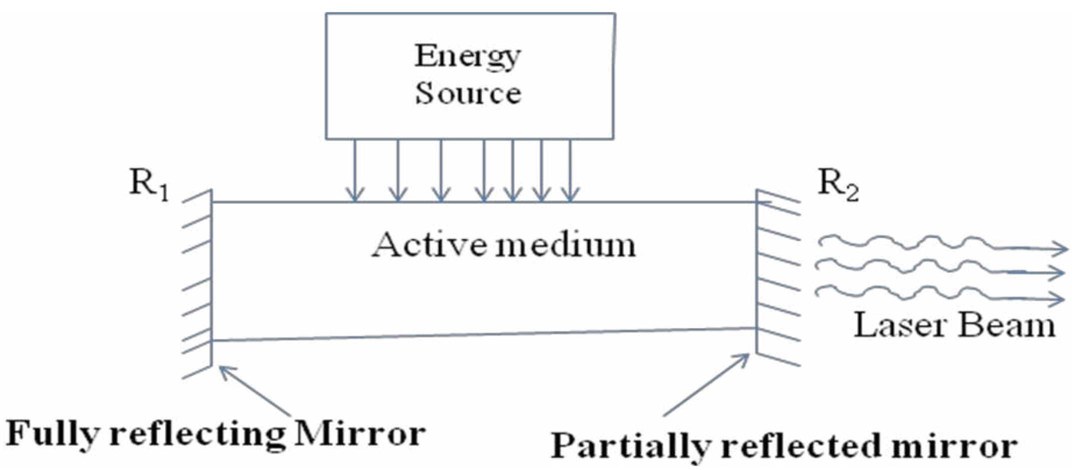


Fig.1.5 Optical Resonator

<u>RUBY LASER</u>-This is the first laser developed by Maiman in 1960 and is the solid state laser. A solid laser can be made by introducing impurity atoms into a crystal (by doping method).

It consists of a pink ruby cylindrical rod whose ends are optically flat and parallel (figure 1.6).one end is fully silvered and the other is only partially silvered. The partially reflecting end can be used as a window for laser output. This works as **Resonant Cavity**.

Ruby rod is surrounded by a helical xenon flash tube, which provides light to raise chromium ions to upper energy levels.

In the xenon flash tube, each flash lasts several mili seconds and in each flash a few thousand Joules of energy is consumed. Only a small part of this energy is used in pumping Cr^{+++} ions while rest energy is wasted by heating up the apparatus.

For cooling the apparatus water circulation (liquid nitrogen) in a glass tube is provided.

In Ruby laser, Cr^{+++} ions are the active material. The solid state Ruby laser (pulse laser) is a three level laser system. A systematic diagram of construction and energy level diagram is shown in Figure (1.6) & Figure (1.7) respectively.

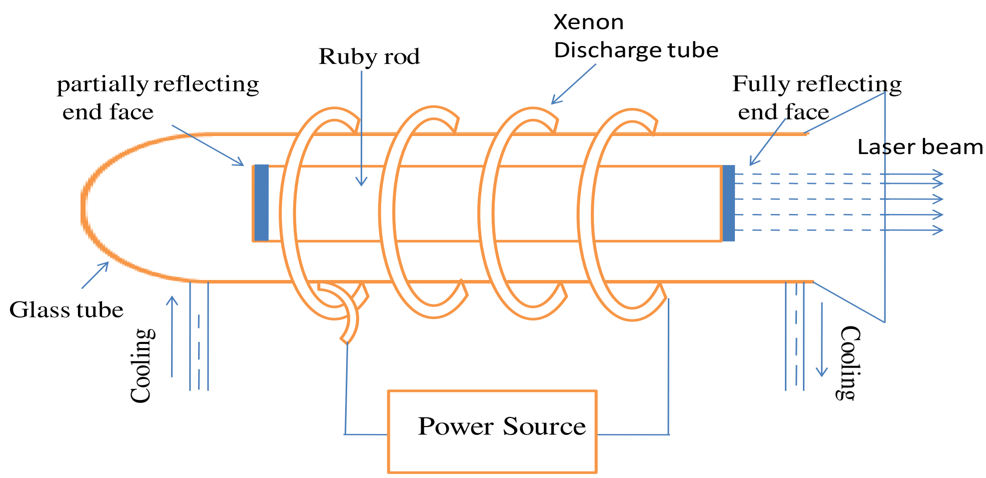


Fig. 1.6

Working-The ruby rod is a crystal of aluminium oxide (Al_2O_3) doped with 0.05% chromium oxide (Cr_2O_3), so that some of the Al^{+++} ions are replaced by Cr^{+++} ions. These "impurity" chromium ions give pink colors to the ruby and give rise to laser action. The crystal is in the form of cylindrical rod which is 2 to 20 cm in length and nearly 0.1 to 2.0 cm in diameter

A simplified energy- level diagram is shown in Figure 1.7. It consists of an upper short-lived energy level E_3 above its ground energy level E_1, the energy difference E_3-E_1 corresponds to a wavelength of about 5500 Å. There is an intermediate excited-state level E_2 which is metastable having a life time of 10^{-3} sec.

Normally, most of the chromium ions are in the ground state E_1.When a flash of light falls upon the ruby rod, the 5500 Å radiation photons are absorbed by the chromium ions which are pumped to the excited state E_3. The excited ions give up, by collision, part of their energy to the crystal lattice and decay to the metastable state E_2.The corresponding transition 2 is thus a radiation less transition. Since E_2 has a much longer life-time, the number of ions in this state goes on increasing while, due to pumping, the number in the ground state E_1 goes on decreasing. Thus, population inversion is established between the metastable state E_2 and the ground state E_1.

When an excited ion passes from the metastable state to the ground state, it emits a photon of wavelength 6943Å.This photon travels through the ruby rod and, if it is moving parallel to the axis of crystal, is reflected back and forth by the silvered ends until it stimulates an excited ion and causes it to emit a fresh photon, in phase with stimulating photon. This "stimulated" transition 4 is the laser transition. The process is repeated again and again because the photons repeatedly move along the crystal, being reflected from its ends. The photons thus multiply. When the photon beam becomes sufficiently intense, part of it emerges through the partially silvered end of the crystal.

Drawback in the three level lasers such as ruby-. The laser requires high pumping power because the laser transition terminates at the ground state and more than one half of the ground state atoms must be pumped up to the higher state to achieve population inversion.

The efficiency of ruby laser is very small. Only the green component of pumping light is utilized.

The laser output is not continuous. The output occurs in the form of pulses of microsecond duration.

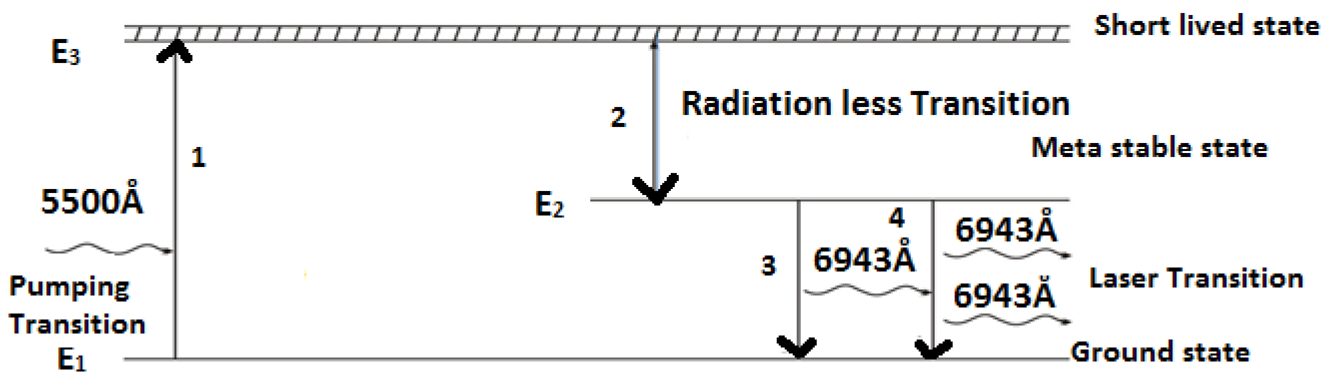


Figure 1.7

Characteristics of Ruby laser --

Type: Three level solid state laser

Active medium: Ruby rod is used as active medium

Pumping method: Optical pumping

Optical resonator: The two ends of Ruby rod which are polished with silver (one is fully silvered while the other is partially silvered) are used as optical resonatory.

Power output: 10^4- 10^6 watts.

Wavelength: The wavelength of output beam is 6943 Å.

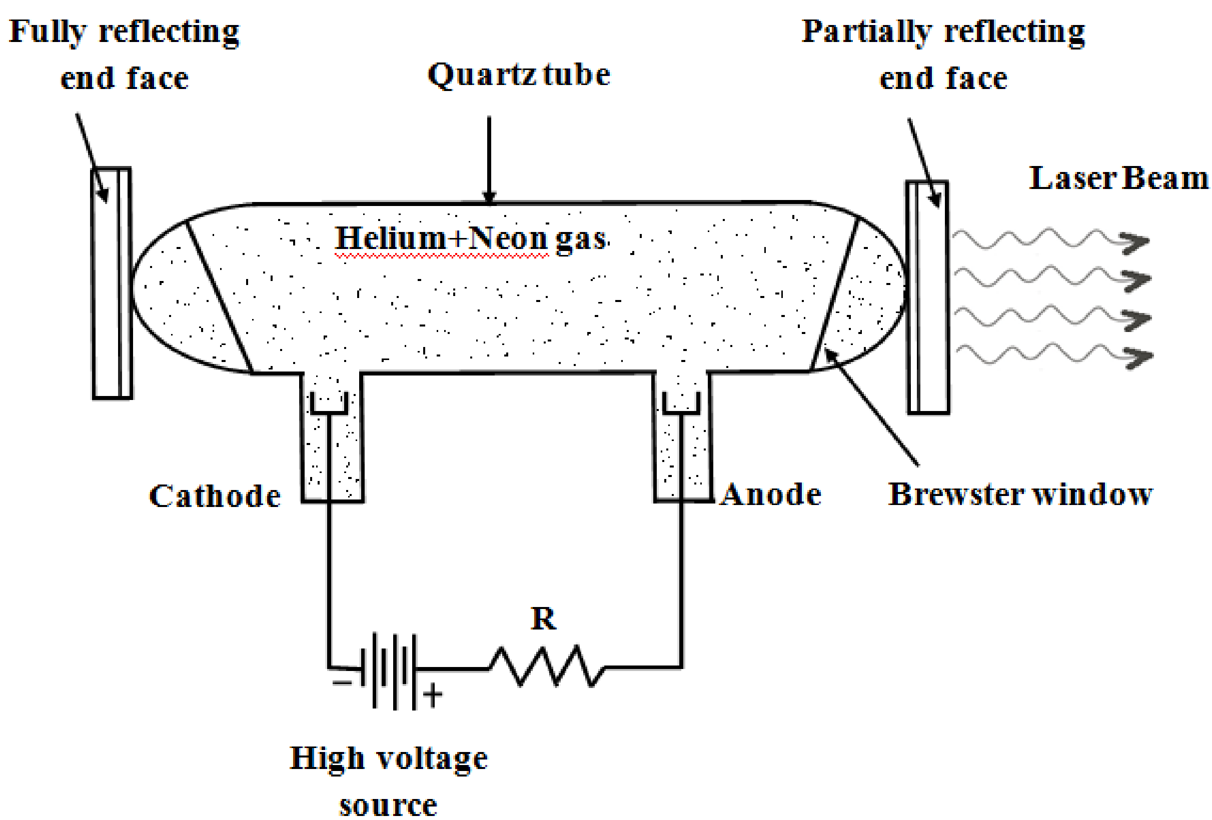
Nature of output: Pulsed beam of light.

Applications of ruby Laser

1. It is used in laboratory experiments.
2. It is used in soldering and wielding.
3. It is used to test the quality of the materials.
4. It is used in the treatment of detached ratina.
5. It is used in light detection and ranging (LIDAR).
6. It is used in the work of interferometry, holography.

Helium Neon Gas Laser—(Four Level Transition Laser)

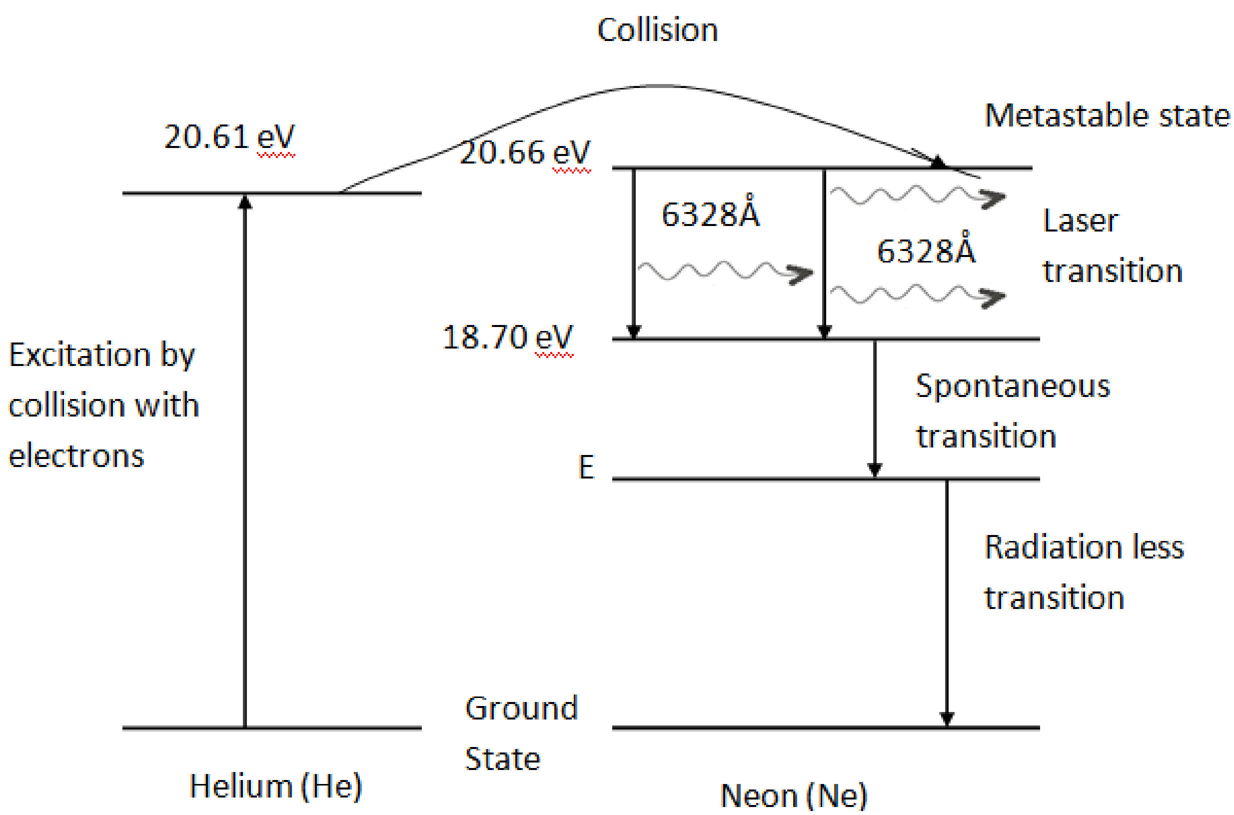
In 1961 A. Javan, W. Bennett and D. Herriot reported a continuous He-Ne gas laser. It is the first gas laser which was operated successfully. To get continuous and intense beam of laser, gas lasers are used. The spectral lines in a gas laser are narrow and well defined as compared to solid in which absorption bands are broad. A simplified diagram showing basic features of a gas laser is shown in figure ().



The resonant system is a long and thin optical cavity with mirrors at each ends. At one end, there is a perfect reflector while on another ends there is a partial reflector. Resonant system is also called a discharge tube.

He- Ne laser consists of a discharge (quartz) tube containing the helium and neon in the ration of 7 to 1 at a total pressure of about 1 torr (1 mm of Hg). The end faces of discharge tube are inclined at the polarizing angle so that the laser light is plane polarized. Such an arrangement is known as Brewster window. A powerful generator (R. F. Generator) is used to produce a discharge in the gas. Actual lasing atoms are neon atoms while helium is used just for selective pumping of the upper laser level of neon.

He-Ne gas laser is a four level laser. The first few energy levels of He and Ne atoms are shown in fig ().



When an electric discharge passes through gas, the electrons in discharge tube collide with He and Ne atoms and excite or pump them to metastable state 20.61eV and 20.66 eV respectively above the ground state. Some of excited He atoms transfer their energy to unexcited Ne atoms by collisions. Thus lighter He atoms help in achieving population inversion in heavier Ne atoms.

When an excited Ne atoms drop down spontaneously from metastable state at 20.66 eV to lower energy state at 18.70 eV, it emits a photon (6328Å) in visible region. This photon travels through mixture of gas and if it is moving parallel to axis of tube, is reflected back and forth by reflector ends until it stimulates an excited Ne atom and causes it to emit a fresh 6328Å photon in phase with stimulating photon.

The photon emitted spontaneously which do not move parallel to axis of tube escape through sides of tube. The stimulated transition from 20.66 eV level to 18.70 eV level is the laser transition. The two photons will knock out two more photons and the process is repeated again and again and photon thus multiplies. When this becomes sufficiently intense, a portion of it escapes through partially silvered end.

The Ne atoms drop down from 18.70 eV to lower metastable state E through spontaneous emission emitting incoherent light. From level E, Neon atoms are brought to ground state through collision with walls of tube. Hence final transition is radiation less.

Characteristics of He-Ne laser --

Type: Four level gas laser

Active medium: It uses a mixture of helium and neon gases as the active medium.

Pumping method: Electric discharge method.

Optical resonator: The two ends of Ruby rod which are polished with silver (one is fully silvered while the other is partially silvered) are used as optical resonatory.

Power output: 0.5 to 50 milliwatts.

Wavelength: The wavelength of output beam is 6328 Å.

Nature of output: continuous beam of light.

Applications of ruby Laser

1. It is used in laboratory experiments to produce interference and diffraction patterns.
2. It is used in optical communication without fibre for moderate distance.
3. It can be used to produce holograms i.e., 3 D photographs.
4. It is used in ophthalmology.

Significance of Pulse laser –As the flash lamp stop operating, the population of upper level decreases very rapidly and lasing action stops till the further operation of next flash. Hence Ruby laser is pulsed type of laser. During the period of operation of two flashes, laser output is oscillating. The output is called laser spiking.

Advantages of Gas laser (He-Ne) over solid state laser or Ruby laser-

In gas laser, light is produced as a continuous beam rather than ultra-short pulses as in Ruby laser. Gas laser beam is highly monochromatic and highly directional. This is because of fact that in gas lasers the crystalline imperfection, thermal distortion and scattering are almost absent like Ruby laser. Gas laser are capable of operating continuously without any need of cooling. Therefore **He-Ne laser is superior than Ruby laser.**

Difference between Ruby laser and He-Ne laser—

	Ruby Laser	**He-Ne Laser**
1	It produces a pulsed laser beam.	It produces a continuous laser beam.
2	It is a three level laser system.	It is a four level laser system.
3	Optical pumping method is used for pumping.	Electric discharge method is used for pumping.
4	It has active medium in solid state.	It has active medium in gaseous state.
5	Cooling arrangement is required.	Cooling arrangement is not required.
6	It emits light of wavelength 6943 Å.	It emits light of wavelength 6328 Å.

Characteristic Of Laser Beam-The laser beam has the following main characteristics

1. High monochromaticity, high intensity, high degree of coherence and high directionality of laser radiation is of special significance in advanced research.

2. **Directionality**- A laser beam is very narrow and can travel to long distances without any spread while ordinary light source radiates light in all direction.

3. Output in a laser beam is many millions times more concentrated then the best search light available.

4. The laser gives out light in to a narrow beam and its energy is concentrated in a extremely small region.

5. Because of extremely high intensity of laser beam, it can produce temperature of the order of 10^4 $^\circ$c at a focused point.

6. Because of its narrow bandwidth, laser beams can be focused on a very small area of the order of 10^{-6} cm^2

7. Laser beam is extremely bright.

8. The laser beam is completely coherent spatially as well as temporally.

Applications And Uses Of Laser Radiations-

1. In **industrial** and technical fields the laser beam is used for drilling fine holes in diamonds, teeth, paper clips, hard sheets and even in human hairs.

Laser cutting technology is widely used in the fabrication of space craft. It has been observed that finger prints can be detected under laser light where the normal method of obtaining finger prints through dusting powder is ineffective.

2. In **medicine**, micro-surgery has become possible due to narrow angular spread of the laser beam. The laser beam can be focused on harmful components to destroy them without seriously damaging the neighbouring regions. It can be used in the treatment of kidney stone, tumour, in cutting and sealing the small blood vessel in brain operations.

3. Laser has an important application in **optical communication and holography**. Optic communication is a method of transmitting information from one place to another by sending highly intense laser light through an optical fiber

4. **Atmospheric Studies**-Laser remote sensing is frequently used for precise measurement of ozone in the atmosphere. Atmospheric optics uses lasers for the measurement of traces of pollutant gases, temp., water vapour concentration. Pictures of clouds, wind movement can be obtained with laser beam. The data so obtained helps in weather forecasting.

5. Radio astronomers have been able to amplify very faint radio signals from space with the uses of lasers. With the help of lasers it is possibly to hear the bursts of light and radiation waves from stars which emitted them over a millions of years ago.

6. **Photography-** We can get 3-D photography using laser, known as Holography.

7**. Military Applications**- Due to very high energy density, a laser beam can be used to destroy very big object like aircraft missiles in a few second by directing laser beam into target. So it is called 'Death ray or ray weapon'. In laser gun, highly convergent beam is focused on enemy targetsat a short range.

Unit V.

Question. In a Ruby laser, total number of Cr^{+3} ions is 2.8×10^{19}. If the laser emits radiation of wavelength 7000 A^0. Calculate the energy of laser pulse.

Sol. Energy of laser pulse (E) = total no. of ions (n) × $energy\ of\ one\ photon$

$$E = nh\nu = \frac{nhc}{\lambda} = 2.8 \times 10^{19} \times \frac{(6.6 \times 10^{-34} \times 3 \times 10^{8})}{7 \times 10^{-7}} \text{ Joule}$$

$$= 7.94 \text{ Joule}$$

Question.

The CO_2 laser is one of the most powerful lasers. The energy difference between the two laser levels is 0.117 eV. Determine the frequency and wavelength of the radiation.

Solution :

$$\lambda\ (\text{Å}) = \frac{12400}{E(\text{eV})} = \frac{12400}{0.117}$$

or

$$\lambda = 105983\ \text{Å}$$

$$= \mathbf{10.5\ \mu m}$$

$$\therefore \quad \nu = \frac{c}{\lambda} = \frac{3 \times 10^{8}\ m/s}{10.5 \times 10^{-6}\ m} = \mathbf{2.9 \times 10^{13}\ Hz}$$

Question. Calculate the population ratio of two states in He-Ne laser that produce light of wavelength 6000 Å at 27 °C. (Dec 2018)

Solution.

$$\frac{N_2}{N_1} = e^{-\frac{(E_2 - E_1)}{kT}} = e^{\frac{-h\upsilon}{kT}} = e^{-\frac{hc}{\lambda kT}}$$

$$\frac{N_2}{N_1} = e^{-\frac{6.62 \times 10^{-34} \times 3 \times 10^{8}}{6000 \times 10^{-10} \times 1.38 \times 10^{-23} \times 300}}$$

$$\frac{N_2}{N_1} = e^{-80}$$

Question.

The Ruby laser has two states at 300 K and 500 K. If it emits light 7000 Å, then calculate relative population. (Given $K = 8.6 \times 10^{-5}$ eV/K)

Solution The population ratio,

$$\frac{N_2}{N_1} = e^{-(E_2 - E_1)/kT}$$

$$\therefore \quad E_2 - E_1 = h\nu = \frac{12400}{\lambda}$$

$$\therefore \quad E_2 - E_1 = \frac{12400}{7000} = 1.77 \text{ eV}$$

At 300 K,

$$\frac{N_2}{N_1} = \exp\left[\frac{-1.77}{8.6 \times 10^{-5} \times 300\,\text{K}}\right]$$

$$= e^{-65} = 5.9 \times 10^{-29}$$

At 500 K,

$$\frac{N_2}{N_1} = \exp\left[\frac{-1.77}{8.6 \times 10^{-5} \times 500\text{K}}\right]$$

$$= e^{-41.1} = 1.4 \times 10^{16}$$

Question. Compute the numerical aperture, acceptance angle and critical angle of an optical fibre from the following data: (core) μ_1=1.50 and μ_2(cladding) =1.45. (Dec 2018)

Solution: We know that, numerical aperture, $NA = \mu_1 \sqrt{(2\Delta)}$

where $$\Delta = \frac{\mu_1 - \mu_2}{\mu_1}$$

$$\therefore \quad \Delta = \frac{1.50 - 1.45}{1.50} = 0.033$$

So, $$NA = 1.50\sqrt{(2 \times 0.033)} = 1.50 \times 0.257 = \mathbf{0.385}$$

Acceptance angle, $$\alpha = \sin^{-1}(NA) = \sin^{-1}(0.385) = \mathbf{22.63^\circ}$$

According to Snell's law

$$\sin\theta_c = \frac{\mu_2}{\mu_1} \quad \text{or} \quad \theta_c = \sin^{-1}\left(\frac{\mu_2}{\mu_1}\right) = \sin^{-1}\left(\frac{1.45}{1.50}\right)$$

or $$\theta_c = \sin^{-1}(0.967) = \mathbf{75.3^\circ}$$

Question.

If the fractional difference between the core and cladding refractive indices of a fibre is* 0.0135 *and numerical aperture NA is* 0.2425, *calculate the refractive indices of the core and cladding materials.

Solution : We have that, $NA = \mu_1\sqrt{(2\Delta)}$

and $$\Delta = \frac{\mu_1 - \mu_2}{\mu_1}$$

where μ_1 and μ_2 are the refractive indices of core and cladding materials respectively.

Here $NA = 0.2425$ and $\Delta = 0.0135$

$$\therefore \qquad \mu_1 = \frac{NA}{\sqrt{2\Delta}} = \frac{0.2425}{\sqrt{2 \times 0.0135}} = \frac{0.2425}{0.1643} = \mathbf{1.476}$$

and $$\Delta = 0.0135 = \frac{\mu_1 - \mu_2}{\mu_1} = \frac{1.476 - \mu_2}{1.476}$$

or $$1.476 - \mu_2 = 0.0135 \times 1.476$$

or $$\mu_2 = 1.476 - 0.02 = \mathbf{1.456}$$

Question.

***A step index fibre has core refractive index* 1.466, *cladding refractive index* 1.46. *If the operating wavelength of the rays is* 0.85 μm, *calculate the cut-off parameter and the number of modes which the fibre will support. The diameter of core* = 50 μ*m*.**

Solution : We know that, cut-off parameter or cut-off number is,

$$V = \frac{2\pi a}{\lambda}\sqrt{\mu_1^2 - \mu_2^2}$$

where a is the radius of the core, μ_1 refractive index of the core, μ_2 the refractive index of cladding and λ operating wavelength.

Here $a = 50/2 = 25\,\mu\text{m}$, $\lambda = 0.85\,\mu\text{m}$, $\mu_1 = 1.466$ and $\mu_2 = 1.46$

$$\therefore \qquad V = \frac{2 \times 3.14 \times 25}{0.85}\sqrt{(1.466)^2 - (1.46)^2}$$

$$= 184.70\sqrt{(2.149 - 2.131)} = 184.70 \times 0.134$$

or $$V = \mathbf{24.75}$$

Number of modes, $$N = \frac{V^2}{2} = \frac{(24.75)^2}{2} \simeq \mathbf{306}$$

Question.

The core diameter of a multimode fibre is 70 μm and the relative refractive index difference is of 1.5%. It operates at a wavelength of 0.85 μm. If the refractive index of the core is 1.46, calculate (i) the refractive index of the cladding, (ii) the normalised frequency V-number of the fibre, and (iii) the total number of guided modes in the fibre.

Solution **:** In the given problem, radius, $a = 70/2 = 35$ μm, $\mu_1 = 1.46$ and relative index difference,

$\Delta = \dfrac{\mu_1 - \mu_2}{\mu_1} = 1.5\%$, where μ_1 is the refractive index of core and μ_2 that of cladding material

(i) $\Delta = \dfrac{1.5}{100} = \dfrac{\mu_1 - \mu_2}{\mu_1}$ or $\mu_1 - \mu_2 = 0.015\mu_1$

or $\mu_2 = (1 - 0.015)\mu_1$ or $\mu_2 = 0.985 \times 1.46 = \mathbf{1.438}$

(ii) The normalized frequency V-number $= \dfrac{2\pi a}{\lambda}\sqrt{\mu_1^2 - \mu_2^2}$

Here $\lambda = 0.85\,\mu\text{m}$

$$\therefore \quad V = \frac{2 \times 3.14 \times 35}{0.85}\sqrt{(1.46)^2 - (1.438)^2}$$

$$= 258.589\sqrt{(2.132 - 2.068)} \quad \text{or} \quad V = 258.589 \times 0.253 = \mathbf{65.42}$$

(iii) The total number of guided mode in the step-index fibre is,

$$N = \frac{V^2}{2} = \frac{(65.42)^2}{2} = \mathbf{2139\ modes}$$

Question.

Determine the core radius necessary for single mode operation at 0.82 μm of a step index fibre with $\mu_1 = 1.480$ and $\mu_2 = 1.478$. What is the numerical aperture, critical angle and maximum acceptance angle of the fibre.

Solution **:** For single mode operation, the cut-off parameter V should be less than 2.405. That is ,

$$V = \frac{2\pi a}{\lambda}\sqrt{\mu_1^2 - \mu_2^2} < 2.405$$

or $$a < \frac{2.405\,\lambda}{2\pi\sqrt{\mu_1^2 - \mu_2^2}}$$

Here $\mu_1 = 1.480, \mu_2 = 1.478$ and $\lambda = 0.82\,\mu\text{m}$

$$\therefore \quad a < \frac{2.405 \times 0.82}{2 \times 3.14 \times \sqrt{(1.480)^2 - (1.478)^2}}$$

or $$a < \frac{1.9721}{6.28 \times 0.768} \quad \text{or} \quad a < \frac{1.9721}{0.4823}$$

or $$a < \mathbf{4.089\,\mu m}$$

Numerical aperture, $NA = \sqrt{(\mu_1^2 - \mu_2^2)} = \sqrt{[(1.48)^2 - (1.478)^2]}$

or $NA = \mathbf{0.0768}$

Maximum acceptance angle, $\alpha = \sin^{-1}(\sqrt{(\mu_1^2 - \mu_2^2)} = \sin^{-1}\sqrt{[(1.48^2 - (1.478)^2]}$

or $\alpha = \sin^{-1}(0.0768) = \mathbf{4.4°}$

Critical angle, $$\theta_c = \sin^{-1}\left(\frac{\mu_2}{\mu_1}\right) = \sin^{-1}\left(\frac{1.478}{1.48}\right)$$

or $$\theta_c = \sin^{-1}(0.9986) = \mathbf{87^\circ}$$

Question.

A signal of 100 mW is injected into a fibre. The outcoming signal from the other end is 40 mW. What is the attenuation loss in dBs ?

Solution : Loss in decibel (dB) $= -10 \log_{10}\left(\frac{P_{out}}{P_{in}}\right)$

$$dB = -10 \log_{10}\left(\frac{40}{100}\right) = -10\,[\log_{10} 2 - \log_{10} 5]$$

$$= -10\,[0.3010 - 0.6990]$$

$$dB = \mathbf{3.98\ dB}$$