

Neutrosophic Extension of Weibull Distribution in Reliability Model



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award of the degree of**

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DECLARATION

I hereby declare that the project entitled,
“Neutrosophic Extension of Weibull Distribution in Reliability Model”,
is submitted by me for the partial fulfilment of the degree of M.Sc. Statistics & Computing, under
the guidance and supervision of **Assistant Professor Dr. Abhimanyu Singh Yadav, Department
of Statistics**. I also declare that it has not been submitted previously to this university, or any
institution for the fulfilment of the requirement of any degree or diploma.

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“Neutrosophic Extension of Weibull Distribution in Reliability Model”

Submitted by **Saunak Mitra (Examination Roll Number – 22419STC033)** for the partial fulfilment of the M.Sc. in Statistics & Computing for the session 2022 – 2024, has been successfully conducted and completed under my supervision. I recommend this project report for the evaluation.

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Abstract

The Neutrosophic Weibull Distribution (NWD) is a versatile probabilistic model that integrates the Neutrosophic Set Theory with the Weibull distribution. It provides a robust framework for handling uncertainty in real-world data, particularly in scenarios where traditional statistical methods may fall short. The (NWD) captures the imprecise, indeterminate, and contradictory aspects of data, offering a more comprehensive representation of uncertainty. This abstract explores the theoretical foundation, statistical properties, and potential applications of the proposed distribution. Weibull Distribution is widely used for Reliability/survival analysis. But in most of the real life cases it is impossible to identify and precisely account of all factors. To deal with this we move to neutrosophic approach. Though the neutrosophic weibull distribution model was proposed before by "Kawther Fawzi Hamza Alhasan and Florentin Smarandache" but here we derive all the theoretical derivations thoroughly and the simulate the data from the mentioned distribution and the approach of maximum likelihood estimation is applied to estimate the parameters and using those estimated values we validate our model based on the evaluated performance of the proposed model.

Introduction:-

The Neutrosophic Extension of the Weibull Distribution represents a significant advancement in reliability modeling, offering a nuanced approach to capturing uncertainties inherent in complex systems. Reliability analysis plays a pivotal role in various fields such as engineering, healthcare, and finance, where understanding and predicting the lifetimes of systems or components is critical for decision-making and risk management.

The traditional Weibull distribution has long been a cornerstone in reliability engineering due to its flexibility in modeling failure rates over time. However, conventional statistical methods often struggle to account for the inherent ambiguity and vagueness present in real-world data, especially in scenarios where information is incomplete or imprecise.

The Neutrosophic Extension of the Weibull Distribution addresses this challenge by integrating the principles of neutrosophic set theory into the reliability modeling framework. Neutrosophy, introduced by Florentin Smarandache, extends the concepts of classical set theory to accommodate indeterminacy, ambiguity, and contradictions through the notion of truth-membership, indeterminacy-membership, and falsehood-membership degrees.

In the context of reliability modeling, neutrosophic set theory enables us to represent and quantify uncertainties that arise from various sources such as measurement errors, subjective judgments, or incomplete information. By allowing for the characterization of vague or incomplete data, the neutrosophic extension enriches the descriptive power of the Weibull distribution, making it more adaptable to real-world scenarios where precise probabilistic modeling may be inadequate.

The incorporation of neutrosophic elements into the Weibull distribution empowers analysts to more accurately model and analyze complex systems' failure patterns, particularly in situations where traditional statistical approaches may fall short. Moreover, this extension opens avenues for exploring new dimensions of uncertainty and variability, leading to more robust and insightful reliability assessments.

In summary, the Neutrosophic Extension of the Weibull Distribution represents a promising frontier in reliability modeling, offering a principled framework for grappling with the inherent uncertainties and ambiguities present in real-world data. As researchers continue to explore and refine this approach, its potential applications across diverse domains are likely to expand, fostering deeper insights into the dynamics of complex systems and enhancing decision-making processes.

Weibull Distribution in Reliability Model:-

Weibull distribution is a very renowned distribution used in reliability. For example, we can consider the lifetime of a human follows this type of distribution. Where in the first part we can observe a decreasing hazard rate because when a new baby born they has a higher chance of failure and slowly the risk decreases. After a certain age the risk of failure becomes stable. so a constant hazard rate can be observed. and after age 60 again the chance of failure increases. So, now we can observe a increasing hazard rate. So, a bathtub type of hazard curve which follows Weibull distribution can be seen. The bathtub curve can be observed if we consider a mixture model of two weibull distribution one having shape parameter < 1 and the other one having shape parameter > 1 .

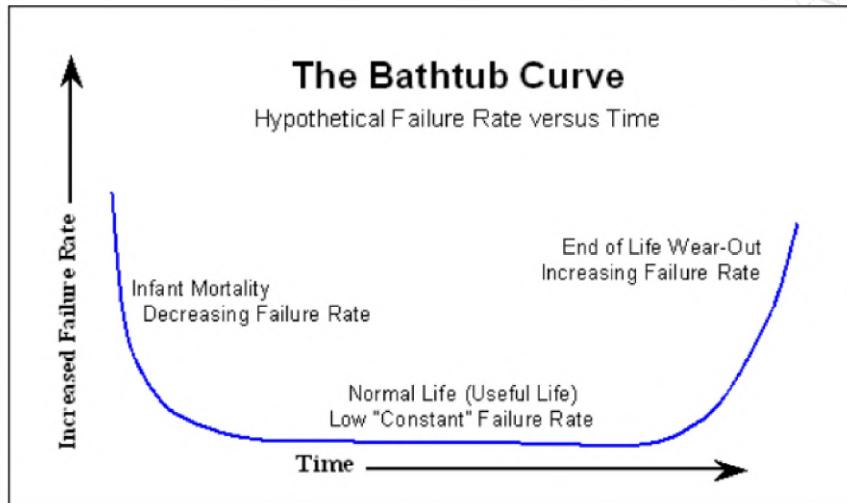


Figure 1: The Bathtub Curve

Weibull Distribution and it's characteristics:-

The pdf of weibull distribution is given by,

$$f_X(x;\lambda,k) = \frac{k}{\lambda} \left(\frac{x}{\lambda}\right)^{k-1} e^{-\left(\frac{x}{\lambda}\right)^k}, \quad x \geq 0$$
$$= 0 \quad , x < 0$$

So the cdf of weibull distribution will be,

$$F_X(x;\lambda,k) = \int_0^x f_T(t;\lambda,k) dt \quad , x \geq 0$$
$$= 0 \quad , x < 0$$

which implies,

$$F_X(x; \lambda, k) = \int_0^x \frac{k}{\lambda} \left(\frac{t}{\lambda}\right)^{k-1} e^{-\left(\frac{t}{\lambda}\right)^k} dt$$

Substituting $\left(\frac{t}{\lambda}\right)^k = p$ we get,

$$F_X(x; \lambda, k) = \int_0^{\left(\frac{x}{\lambda}\right)^k} e^{-p} dp$$

which simplifies to,

$$\begin{aligned} F_X(x; \lambda, k) &= 1 - e^{-\left(\frac{x}{\lambda}\right)^k}, \quad x \geq 0 \\ &= 0 \quad , x < 0 \end{aligned}$$

The graph of Weibull distribution depends on it's parameter, it looks like

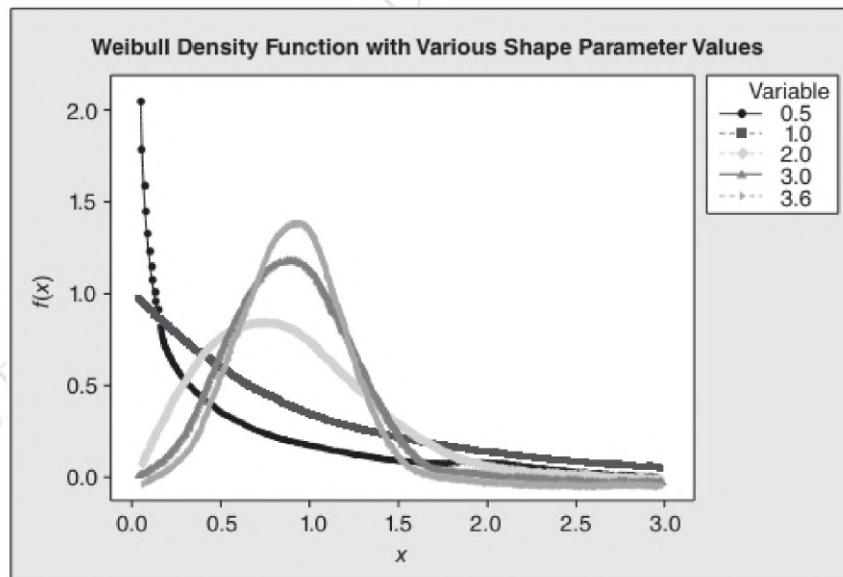


Figure 2: Weibull pdf for various values of the shape parameter k

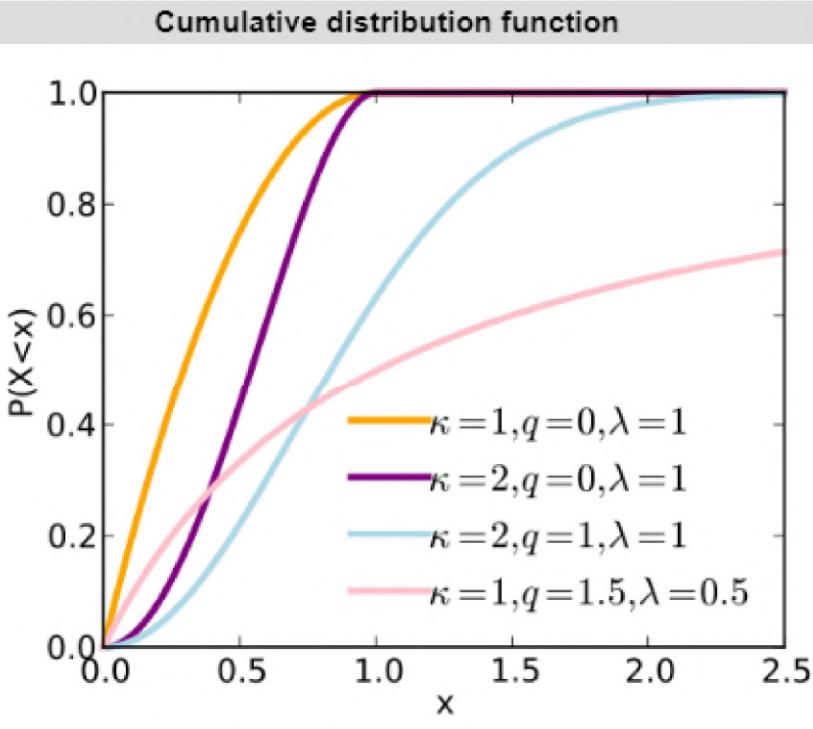


Figure 3: Weibull cdf for various values of the parameters

So, the Reliability function is given by,

$$\omega_X(x; \lambda, k) = 1 - F_X(x; \lambda, k) = e^{-\left(\frac{x}{\lambda}\right)^k}$$

The hazard function of Weibull distribution is given by, $h(x; \lambda, k) = \frac{f_X(x; \lambda, k)}{\omega_X(x; \lambda, k)} = \frac{k}{\lambda} \left(\frac{x}{\lambda}\right)^{k-1}$

The expectation of Weibull distribution is defined as,

$$\begin{aligned} E(x) &= \int_0^\infty x f_X(x; \lambda, k) dx \\ &= \int_0^\infty x \left(\frac{k}{\lambda}\right) \left(\frac{x}{\lambda}\right)^{k-1} e^{-\left(\frac{x}{\lambda}\right)^k} dx \end{aligned}$$

substituting $\left(\frac{x}{\lambda}\right)^k = t$ we get,

$$\left(\frac{k}{\lambda}\right) \left(\frac{x}{\lambda}\right)^{k-1} dx = dt$$

so it simplifies to,

$$E(x) = \lambda \int_0^\infty t^{\frac{1}{k}+1-1} e^{-t} dt$$

$$= \boxed{\lambda \Gamma\left(\frac{1}{k} + 1\right)}$$

the second order raw moment of order two of weibull distribution is,

$$E(x^2) = \int_0^\infty x^2 f_X(x; \lambda, k) dx$$

$$= \int_0^\infty x^2 \left(\frac{k}{\lambda}\right) \left(\frac{x}{\lambda}\right)^{k-1} e^{-\left(\frac{x}{\lambda}\right)^k} dx$$

substituting $\left(\frac{x}{\lambda}\right)^k = t$ we get,

$$\left(\frac{k}{\lambda}\right) \left(\frac{x}{\lambda}\right)^{k-1} dx = dt$$

so it simplifies to,

$$E(x^2) = \lambda^2 \int_0^\infty t^{\frac{2}{k}+1-1} e^{-t} dt$$

$$= \boxed{\lambda^2 \Gamma\left(\frac{2}{k} + 1\right)}$$

So the form of variance will be written as,

$$\text{var}(X) = E(x^2) - (E(x))^2$$

$$\text{i.e., } \text{var}(X) = \boxed{\lambda^2 \Gamma\left(\frac{2}{k} + 1\right) - \left(\lambda \Gamma\left(\frac{1}{k} + 1\right)\right)^2}$$

Similarly, third order raw moment $E(x^3)$ is equal to $\lambda^3 \Gamma\left(\frac{3}{k} + 1\right)$

So, the third order central moment can be denoted as, $\mu_3 = E(x^3) - 3E(x)E(x^2) + 2E(x)^3$

$$\text{So, } \mu_3 = \boxed{\lambda^3 \Gamma\left(\frac{3}{k} + 1\right) - 3\lambda^3 \Gamma\left(\frac{1}{k} + 1\right) \Gamma\left(\frac{2}{k} + 1\right) + 2\lambda^3 \Gamma\left(\frac{1}{k} + 1\right)^3}$$

So, the measure of skewness will be,

$$S_k = \frac{\mu_3}{\mu_2^{\frac{3}{2}}} = \left(\frac{\Gamma\left(\frac{3}{k} + 1\right) - 3\Gamma\left(\frac{1}{k} + 1\right)\Gamma\left(\frac{2}{k} + 1\right) + 2\Gamma\left(\frac{1}{k} + 1\right)^3}{\left(\Gamma\left(\frac{2}{k} + 1\right) - \left(\Gamma\left(\frac{1}{k} + 1\right)\right)^2\right)^{\frac{3}{2}}} \right)$$

Depending on the values of the parameters we can infer if the distribution is positively skewed, negatively skewed or almost symmetric. Kurtosis is defined as,

$$\frac{\mu_4}{\mu_2^2} = \left(\frac{\Gamma\left(\frac{4}{k} + 1\right) - 4\Gamma\left(\frac{3}{k} + 1\right)\Gamma\left(\frac{1}{k} + 1\right) + 6\Gamma\left(\frac{2}{k} + 1\right)\left(\Gamma\left(\frac{1}{k} + 1\right)\right)^2 - 3\left(\Gamma\left(\frac{1}{k} + 1\right)\right)^4}{\left(\Gamma\left(\frac{2}{k} + 1\right) - \left(\Gamma\left(\frac{1}{k} + 1\right)\right)^2\right)^2} \right)$$

Weibull distribution can be leptokurtic, mesokurtic, platykurtic based on it's shape parameter k. if $k > 2$ then the distribution can be leptokurtic.

What is meant by Neutrosophic ?-

Neutrosophy, introduced by Florentin Smarandache in the late 20th century, is a philosophical concept that deals with indeterminacy, contradiction, and incomplete information. Neutrosophy recognizes that in many real-world scenarios, elements can possess not only true and false values but also indeterminate values lying between true and false. This concept of indeterminacy is crucial in understanding the complexities of human perception and cognition. Neutrosophic logic, an extension of classical and fuzzy logic, provides a framework to handle indeterminacy by introducing the notion of neutrosophic set theory. In a neutrosophic set, elements can belong not only to the set (true), its complement (false), or both, but also to the indeterminate set (neither true nor false). This allows for a more nuanced representation of uncertainty in data and knowledge. Neutrosophic statistics, therefore, applies neutrosophic logic to statistical analysis, enabling the handling of uncertain, imprecise, or incomplete information in data. It extends traditional statistical methods to accommodate situations where classical approaches may fall short due to the presence of ambiguity or contradictory evidence. Neutrosophic statistics finds applications in various domains such as decision-making, pattern recognition, image processing, and artificial intelligence, where dealing with uncertainty is essential for making informed decisions and drawing meaningful conclusions from data.

Neutrosophic Sets :-

Neutrosophic sets, an extension of classical and fuzzy sets, incorporate the notion of indeterminacy. In a neutrosophic set, elements can belong not only to the set (true) or its complement (false), but also to an indeterminate subset (neither true nor false). This accommodates situations where the truth value of an element is uncertain, vague, or contradictory. Neutrosophic sets provide a framework for representing and reasoning with incomplete or imprecise information, allowing for a more flexible and nuanced approach to modeling uncertainty in various fields such as decision making, pattern recognition, and artificial intelligence.

Neutrosophic probability :-

Neutrosophic probability is a theory that extends classical probability theory to handle uncertainty in a more nuanced manner. It deals with situations where the probabilities of events are uncertain, imprecise, or indeterminate. Unlike classical probability, which assigns precise probabilities to events, neutrosophic probability allows for the representation of indeterminacy, ambiguity, and contradiction in probability assessments. It introduces the notion of neutrosophic events, where the likelihood of occurrence lies within a range of values rather than a single probability. Neutrosophic probability finds applications in decision making, risk analysis, and other domains where dealing with uncertain or incomplete information is essential.

The function that models the neutrosophic probability of a random variable x is called neutrosophic distribution: $NP(x) = (T(x), I(X), F(x))$, where $T(x)$ represents the probability that the value x occurs, $F(x)$ represents the probability that the value x does not occur, and $I(X)$ represents the indeterminate / unknown probability of value x to occur or not.

Neutrosophic statistics :- :-

Neutrosophic statistics is a statistical framework that extends traditional statistics to handle uncertainty, ambiguity, and indeterminacy in data analysis. It incorporates neutrosophic logic to accommodate situations where data may be vague or incomplete, allowing for a more comprehensive treatment of uncertainty. Neutrosophic statistics provides methods for analyzing and interpreting data sets that contain imprecise or contradictory information, enabling researchers and practitioners to make informed decisions in the presence of uncertainty. This approach finds applications in various fields such as decision making, pattern recognition, and machine learning, where dealing with uncertain or incomplete data is common.

Weibull Distribution in Neutrosophic approach:-

In neutrosophic distribution we proceed as same as classical way but the moments such as mean and the parameters are not proper or imprecise. Neutrosophic concept actually gives idea about indeterminacy value present in real world.

In neutrosophic context, the pdf of Weibull distribution is defined as,

$$f_X(x; \lambda, k_N) = \frac{k_N}{\lambda_N} \left(\frac{x}{\lambda_N} \right)^{k_N-1} e^{-\left(\frac{x}{\lambda_N}\right)^{k_N}}, \quad x \geq 0$$

where $k_N = [k_l, k_u]$ and $\lambda_N = [\lambda_l, \lambda_u]$

In classical model we use $k_l = k_u = k$ and $\lambda_l = \lambda_u = \lambda$ and if we take $k_N = [1, 1]$ then the distribution becomes neutrosophic exponential model.

The cumulative distribution function is jointly coupled form of df is given by,

$$F_N(x; \lambda_N, k_N) = 1 - e^{-\left(\frac{x}{\lambda_N}\right)^{k_N}}, \quad x \geq 0$$

The survival function or reliability function is defined as,

$$S_N(x; \lambda_N, k_N) = 1 - F_N(x; \lambda_N, k_N) = e^{-\left(\frac{x}{\lambda_N}\right)^{k_N}}$$

Hazard function of neutrosophic weibull distribution is defined as,

$$h_N(x; \lambda_N, k_N) = \frac{f_N(x; \lambda_N, k_N)}{S_N(x; \lambda_N, k_N)} = \frac{k_N}{\lambda_N} \left(\frac{x}{\lambda_N} \right)^{k_N-1}$$

various values of our neutrosophic scale and shape parameters gives us different graphs.

So, we consider 4 sets of values for which we observe the density curve and the cdf curve and the survival function curve as given below,

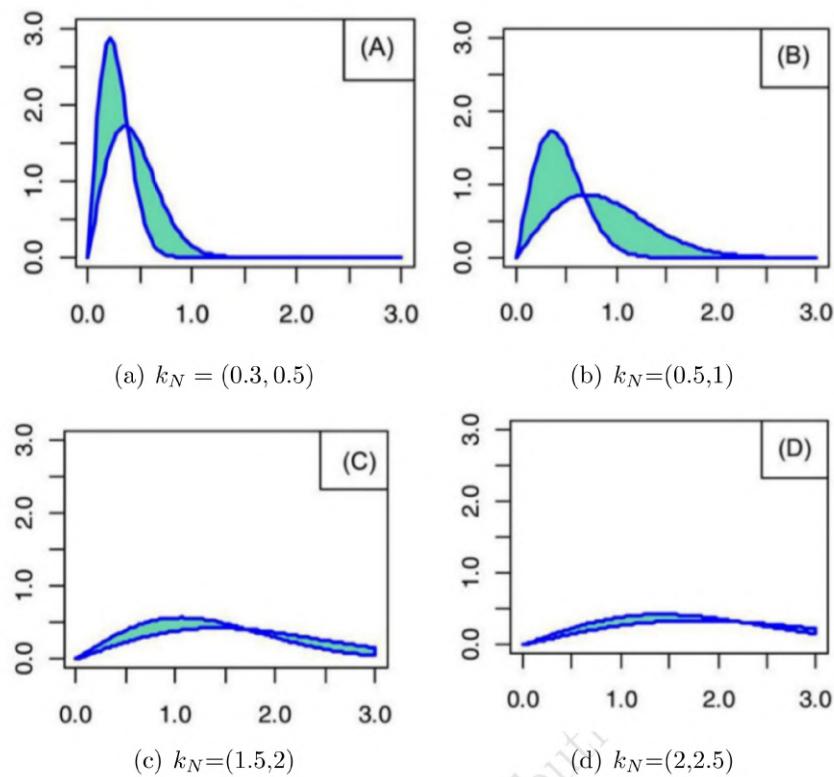


Figure 1: pdf of neutrosophic weibull distribution with different shape parameters

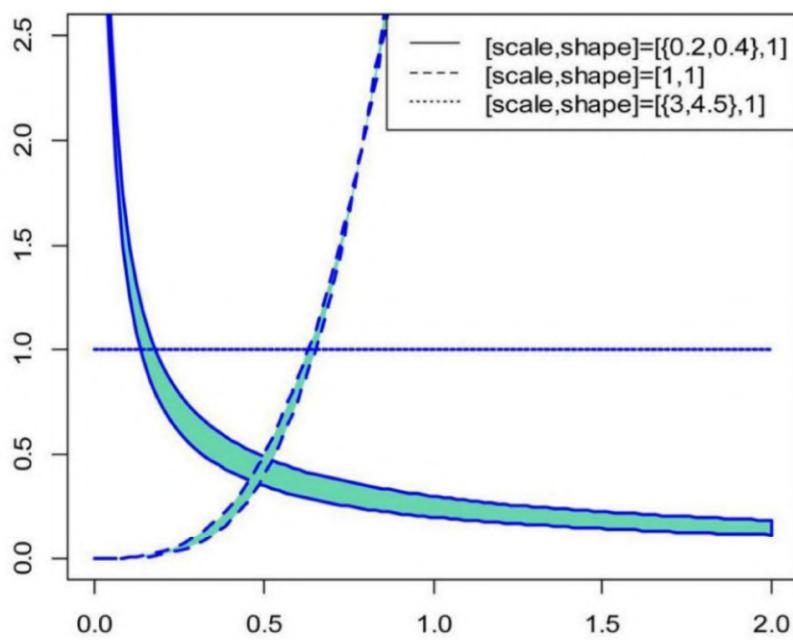


Figure 4: hazard function of weibull distribution at various shape and scale parameters

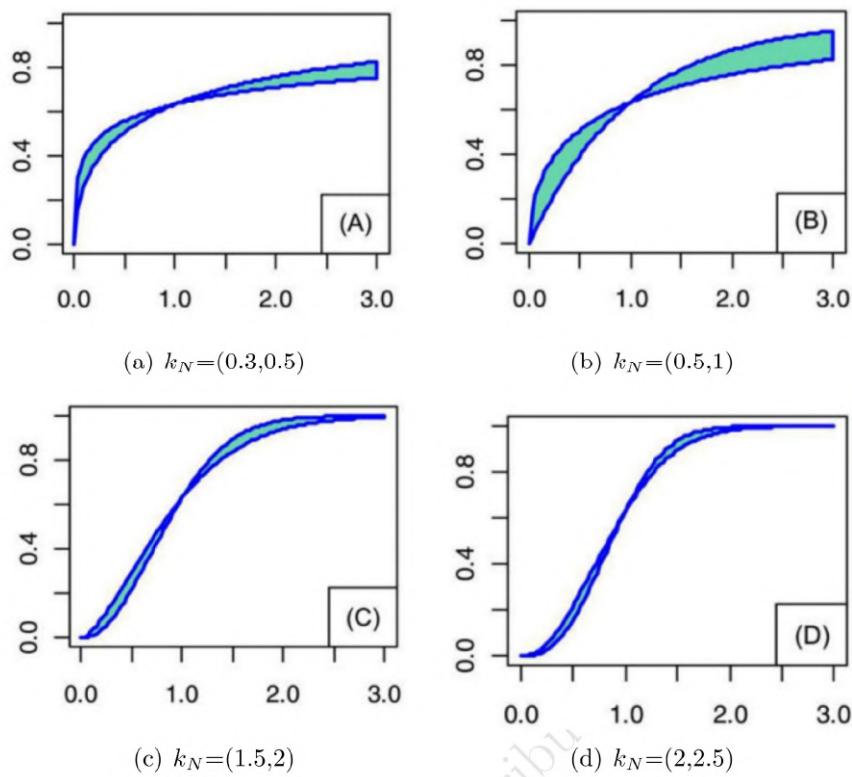


Figure 2: cdf of neutrosophic weibull distribution with different shape parameters

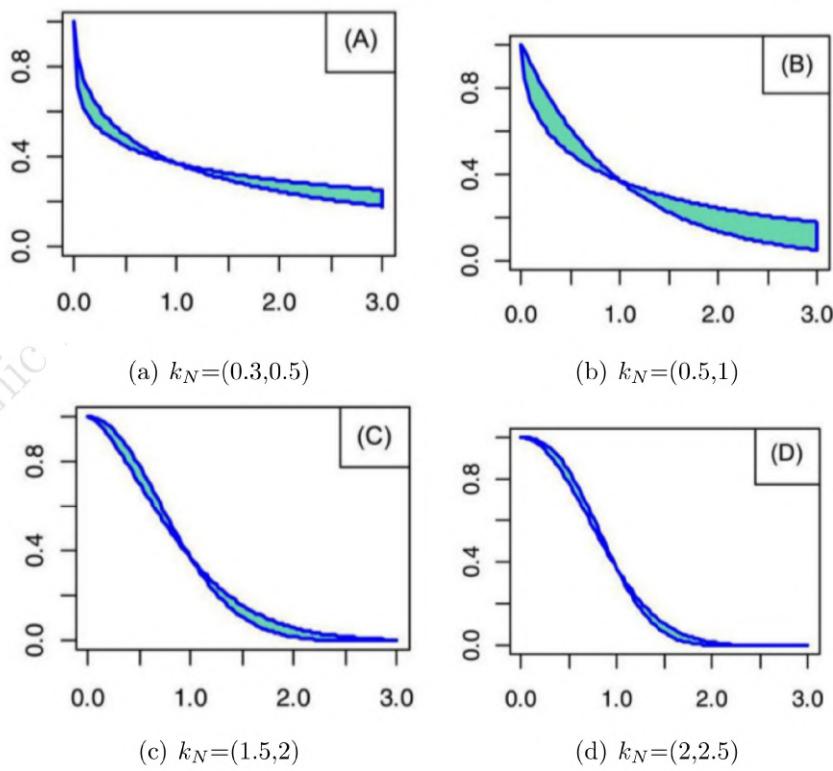


Figure 3: survival function curve of neutrosophic weibull distribution with different shape parameters

Theoritical derivations:-

The CDF of Neutrosophic Weibull Distribution is defined as,

$$F_N(x; \lambda_N, k_N) = \int_0^x f_N(t; \lambda_N, k_N) dt$$

which implies,

$$F_N(x; \lambda_N, k_N) = \left[\int_0^x \frac{k_l}{\lambda_l} \left(\frac{t}{\lambda_l} \right)^{k_l-1} e^{-\left(\frac{t}{\lambda_l}\right)^{k_l}} dt, \int_0^x \frac{k_u}{\lambda_u} \left(\frac{t}{\lambda_u} \right)^{k_u-1} e^{-\left(\frac{t}{\lambda_u}\right)^{k_u}} dt \right]$$

Substituting $\left(\frac{t}{\lambda_N}\right)^{k_N} = p$ we get,

$$F_N(x; \lambda_N, k_N) = \left[\int_0^{\left(\frac{x}{\lambda_l}\right)^{k_l}} e^{-p} dp, \int_0^{\left(\frac{x}{\lambda_u}\right)^{k_u}} e^{-p} dp \right]$$

which simplifies to,

$$F_N(x; \lambda_N, k_N) = \left[1 - e^{-\left(\frac{x}{\lambda_l}\right)^{k_l}}, 1 - e^{-\left(\frac{x}{\lambda_u}\right)^{k_u}} \right]$$

Which equals to,

$$\boxed{F_N(x; \lambda_N, k_N) = 1 - e^{-\left(\frac{x}{\lambda_N}\right)^{k_N}}}$$

By definition the survival function can be written in the form,

$$S_N(x) = 1 - F_N(x)$$

So, it becomes,

$$1 - \left[1 - e^{-\left(\frac{x}{\lambda_N}\right)^{k_N}} \right]$$

Which simplifies to,

$$\left[e^{-\left(\frac{x}{\lambda_l}\right)^{k_l}}, e^{-\left(\frac{x}{\lambda_u}\right)^{k_u}} \right]$$

$$\boxed{S_N(x) = e^{-\left(\frac{x}{\lambda_N}\right)^{k_N}}}$$

The hazard rate is given by,

$$h_N(x) = \frac{f_N(x)}{S_N(x)}$$

$$\begin{aligned} &= \left[\frac{\frac{k_l}{\lambda_l} \left(\frac{x}{\lambda_l} \right)^{k_l-1} e^{-\left(\frac{x}{\lambda_l}\right)^{k_l}}}{e^{-\left(\frac{x}{\lambda_l}\right)^{k_l}}}, \frac{\frac{k_u}{\lambda_u} \left(\frac{x}{\lambda_u} \right)^{k_u-1} e^{-\left(\frac{x}{\lambda_u}\right)^{k_u}}}{e^{-\left(\frac{x}{\lambda_u}\right)^{k_u}}} \right] \\ &= \left[\frac{k_l}{\lambda_l} \left(\frac{x}{\lambda_l} \right)^{k_l-1}, \frac{k_u}{\lambda_u} \left(\frac{x}{\lambda_u} \right)^{k_u-1} \right] \\ &= \boxed{\frac{k_N}{\lambda_N} \left(\frac{x}{\lambda_N} \right)^{k_N-1}} \end{aligned}$$

The Expectation of neutrosophic weibull distribution is given by,

$$\begin{aligned} E_N(X) &= \int_0^\infty x f_N(x; \lambda_N, k_N) dx \\ &= \int_0^\infty x \left(\frac{k_N}{\lambda_N} \right) \left(\frac{x}{\lambda_N} \right)^{k_N-1} e^{-\left(\frac{x}{\lambda_N}\right)^{k_N}} dx \\ &= \left[\int_0^\infty x \left(\frac{k_l}{\lambda_l} \right) \left(\frac{x}{\lambda_l} \right)^{k_l-1} e^{-\left(\frac{x}{\lambda_l}\right)^{k_l}} dx, \int_0^\infty x \left(\frac{k_u}{\lambda_u} \right) \left(\frac{x}{\lambda_u} \right)^{k_u-1} e^{-\left(\frac{x}{\lambda_u}\right)^{k_u}} dx \right] \end{aligned}$$

substituting $\left(\frac{x}{\lambda_N} \right)^{k_N} = p$ we get,

$$\left(\frac{k}{\lambda_N} \right) \left(\frac{x}{\lambda_N} \right)^{k_N-1} dx = dp$$

so it simplifies to,

$$\begin{aligned} E(X_N) &= \left[\frac{\lambda_l^{k_l+1}}{\lambda_l^{k_l}} \int_0^\infty p^{\frac{1}{k_l}+1-1} e^{-p} dp, \frac{\lambda_u^{k_u+1}}{\lambda_u^{k_u}} \int_0^\infty p^{\frac{1}{k_u}+1-1} e^{-p} dp \right] \\ &= \left[\lambda_l \Gamma \left(\frac{1}{k_l} + 1 \right), \lambda_u \Gamma \left(\frac{1}{k_u} + 1 \right) \right] \\ &= \boxed{\lambda_N \Gamma \left(\frac{1}{k_N} + 1 \right)} \end{aligned}$$

To get the variance in neutrosophic approach, we require the second order raw moment of neutrosophic Weibull distribution,

$$E(X_N^2) = \int_0^\infty x^2 f_N(x; \lambda_N, k_N) dx$$

$$= \int_0^\infty x^2 \left(\frac{k_N}{\lambda_N} \right) \left(\frac{x}{\lambda_N} \right)^{k_N-1} e^{-\left(\frac{x}{\lambda_N} \right)^{k_N}} dx$$

$$= \left[\int_0^\infty x^2 \left(\frac{k_l}{\lambda_l} \right) \left(\frac{x}{\lambda_l} \right)^{k_l-1} e^{-\left(\frac{x}{\lambda_l} \right)^{k_l}} dx , \quad \int_0^\infty x^2 \left(\frac{k_u}{\lambda_u} \right) \left(\frac{x}{\lambda_u} \right)^{k_u-1} e^{-\left(\frac{x}{\lambda_u} \right)^{k_u}} dx \right]$$

substituting $\left(\frac{x}{\lambda_N} \right)^{k_N} = p$ we get,

$$\left(\frac{k}{\lambda_N} \right) \left(\frac{x}{\lambda_N} \right)^{k_N-1} dx = dp$$

so it simplifies to,

$$\begin{aligned} E(X_N^2) &= \left[\frac{\lambda_l^{k_l+2}}{\lambda_l^{k_l}} \int_0^\infty p^{\frac{2}{k_l}+1-1} e^{-p} dp , \quad \frac{\lambda_u^{k_u+2}}{\lambda_u^{k_u}} \int_0^\infty p^{\frac{2}{k_u}+1-1} e^{-p} dp \right] \\ &= \left[\lambda_l^2 \Gamma \left(\frac{2}{k_l} + 1 \right) , \quad \lambda_u^2 \Gamma \left(\frac{2}{k_u} + 1 \right) \right] \\ &= \boxed{\lambda_N^2 \Gamma \left(\frac{2}{k_N} + 1 \right)} \end{aligned}$$

So, the variance becomes,

$$\begin{aligned} &= \left[\lambda_l^2 \Gamma \left(\frac{2}{k_l} + 1 \right) , \quad \lambda_u^2 \Gamma \left(\frac{2}{k_u} + 1 \right) \right] - \left[\lambda_l^2 \Gamma \left(\frac{1}{k_l} + 1 \right)^2 , \quad \lambda_u^2 \Gamma \left(\frac{1}{k_u} + 1 \right)^2 \right] \\ &= \left[\lambda_l^2 \Gamma \left(\frac{2}{k_l} + 1 \right) - \lambda_l^2 \Gamma \left(\frac{1}{k_l} + 1 \right)^2 \right] , \quad \left[\lambda_u^2 \Gamma \left(\frac{2}{k_u} + 1 \right) - \lambda_u^2 \Gamma \left(\frac{1}{k_u} + 1 \right)^2 \right] \end{aligned}$$

Therefore,
$$\boxed{Var_N(x) = \left[\lambda_N^2 \left(\Gamma \left(\frac{2}{k_N} + 1 \right) - \Gamma \left(\frac{1}{k_N} + 1 \right)^2 \right) \right]}$$

The rth order raw moment for Neutrosophic Weibull Distribution is defined as,

$$\begin{aligned}
 E(X_N^r) &= \int_0^\infty x^r f_N(x; \lambda_N, k_N) dx \\
 &= \int_0^\infty x^r \left(\frac{k_N}{\lambda_N}\right) \left(\frac{x}{\lambda_N}\right)^{k_N-1} e^{-\left(\frac{x}{\lambda_N}\right)^{k_N}} dx \\
 &= \left[\int_0^\infty x^r \left(\frac{k_l}{\lambda_l}\right) \left(\frac{x}{\lambda_l}\right)^{k_l-1} e^{-\left(\frac{x}{\lambda_l}\right)^{k_l}} dx , \int_0^\infty x^r \left(\frac{k_u}{\lambda_u}\right) \left(\frac{x}{\lambda_u}\right)^{k_u-1} e^{-\left(\frac{x}{\lambda_u}\right)^{k_u}} dx \right]
 \end{aligned}$$

substituting $\left(\frac{x}{\lambda_N}\right)^{k_N} = p$ we get,

$$\left(\frac{x}{\lambda_N}\right)^{k_N-1} dx = dp$$

so it simplifies to,

$$\begin{aligned}
 E(X_N^r) &= \left[\frac{\lambda_l^{k_l+r}}{\lambda_l^{k_l}} \int_0^\infty p^{\frac{r}{k_l}+1-1} e^{-p} dp , \frac{\lambda_u^{k_u+r}}{\lambda_u^{k_u}} \int_0^\infty p^{\frac{r}{k_u}+1-1} e^{-p} dp \right] \\
 &= \left[\lambda_l^r \Gamma\left(\frac{r}{k_l} + 1\right) , \lambda_u^r \Gamma\left(\frac{r}{k_u} + 1\right) \right] \\
 &= \boxed{\lambda_N^r \Gamma\left(\frac{r}{k_N} + 1\right)}
 \end{aligned}$$

So, the mean can be obtained by putting $r=1$ in the above expression,

$$\mu_{1N} = \mu'_{1N} = \lambda_N \Gamma\left(\frac{1}{k_N} + 1\right)$$

Similarly,

$$\mu_{2N} = \mu'_{2N} - \mu'_{1N}^2 = \lambda_N^2 \left[\Gamma\left(\frac{2}{k_N} + 1\right) - \left(\Gamma\left(\frac{1}{k_N} + 1\right) \right)^2 \right]$$

Now,

$$\mu_{3N} = \mu'_{3N} - 3\mu'_{2N}\mu'_{1N} + 2(\mu'_{1N})^3 = \boxed{\lambda_N^3 \left[\Gamma\left(\frac{3}{k_N} + 1\right) - 3\Gamma\left(\frac{1}{k_N} + 1\right) \Gamma\left(\frac{2}{k_N} + 1\right) + 2\Gamma\left(\frac{1}{k_N} + 1\right)^3 \right]}$$

Again,

$$\begin{aligned}
 \mu_{4N} &= \mu'_{4N} - 4\mu'_{3N}\mu'_{1N} + 6\mu'_{2N}\mu'_{1N}^2 - 3\mu'_{1N}^4 \\
 &= \boxed{\lambda_N^4 \left[\Gamma\left(\frac{4}{k_N} + 1\right) - 4\Gamma\left(\frac{3}{k_N} + 1\right) \Gamma\left(\frac{1}{k_N} + 1\right) + 6\Gamma\left(\frac{2}{k_N} + 1\right) \left(\Gamma\left(\frac{1}{k_N} + 1\right) \right)^2 - 3 \left(\Gamma\left(\frac{1}{k_N} + 1\right) \right)^4 \right]}
 \end{aligned}$$

The coefficient of skewness is defined as,

$$Sk_N = \frac{\mu_{3N}}{(\mu_{2N})^{\frac{3}{2}}}$$

$$So, Sk_l = \frac{\left[\Gamma\left(\frac{3}{k_l} + 1\right) - 3\Gamma\left(\frac{1}{k_l} + 1\right)\Gamma\left(\frac{2}{k_l} + 1\right) + 2\Gamma\left(\frac{1}{k_l} + 1\right)^3 \right]^{\frac{3}{2}}}{\left[\Gamma\left(\frac{2}{k_l} + 1\right) - \left(\Gamma\left(\frac{1}{k_l} + 1\right)\right)^2 \right]^{\frac{3}{2}}}$$

$$So, Sk_u = \frac{\left[\Gamma\left(\frac{3}{k_u} + 1\right) - 3\Gamma\left(\frac{1}{k_u} + 1\right)\Gamma\left(\frac{2}{k_u} + 1\right) + 2\Gamma\left(\frac{1}{k_u} + 1\right)^3 \right]^{\frac{3}{2}}}{\left[\Gamma\left(\frac{2}{k_u} + 1\right) - \left(\Gamma\left(\frac{1}{k_u} + 1\right)\right)^2 \right]^{\frac{3}{2}}}$$

where, $Sk_N = [Sk_l, Sk_u]$

The coefficient of kurtosis is given by,

$$\frac{\mu_{4N}}{\mu_{2N}^2}$$

Which equals to,

$$\frac{\left[\Gamma\left(\frac{4}{k_N} + 1\right) - 4\Gamma\left(\frac{3}{k_N} + 1\right)\Gamma\left(\frac{1}{k_N} + 1\right) + 6\Gamma\left(\frac{2}{k_N} + 1\right)\left(\Gamma\left(\frac{1}{k_N} + 1\right)\right)^2 - 3\left(\Gamma\left(\frac{1}{k_N} + 1\right)\right)^4 \right]}{\left[\Gamma\left(\frac{2}{k_N} + 1\right) - \left(\Gamma\left(\frac{1}{k_N} + 1\right)\right)^2 \right]^2}$$

The coefficient of excess kurtosis is defined as,

$$\frac{\mu_{4N}}{\mu_{2N}^2} - 3$$

After simplifying it becomes,

$$\frac{\left[-6\left(\Gamma\left(\frac{1}{k_N} + 1\right)\right)^4 + 12\Gamma\left(\frac{2}{k_N} + 1\right)\left(\Gamma\left(\frac{1}{k_N} + 1\right)\right)^2 - 4\Gamma\left(\frac{3}{k_N} + 1\right)\Gamma\left(\frac{1}{k_N} + 1\right) + \Gamma\left(\frac{4}{k_N} + 1\right) - 3\Gamma\left(\frac{2}{k_N} + 1\right)^2 \right]}{\left[\Gamma\left(\frac{2}{k_N} + 1\right) - \left(\Gamma\left(\frac{1}{k_N} + 1\right)\right)^2 \right]^2}$$

The median of NWD can be calculated by solving,

$$F_N(\xi_{med}) = \frac{1}{2}$$

$$\Rightarrow 1 - e^{-\left(\frac{\xi_{med}}{\lambda_N}\right)^{k_N}} = \frac{1}{2}$$

$$\Rightarrow e^{-\left(\frac{\xi_{med}}{\lambda_N}\right)^{k_N}} = \frac{1}{2}$$

which simplifies to,

$$\boxed{\xi_{med} = \lambda_N[\ln 2]^{k_N}}$$

Maximum Likelihood Estimation:-

To, estimate the neutrosophic parameters of NWD we use MLE approach. In MLE approach we take independent observation from the same distribution and based on that we estimate our parameters. As, we take sample from same distribution they became identical too. In this method we form the likelihood function based on those samples and then find the value of the parameters which maximizes the likelihood function. MLE is often considered because of it's minimum variance and asymptotic unbiasedness property. Let x_1, x_2, \dots, x_n be n i.i.d survival times which follows NWD. So, the joint density is given by,

$$\begin{aligned} \mathcal{L}(\vec{x}|\lambda_N, k_N) &= \prod_{i=1}^n f(x_i|\lambda_N, k_N) \\ &= \prod_{i=1}^n \left[\frac{k_N}{\lambda_N} \left(\frac{x_i}{\lambda_N} \right)^{k_N-1} e^{-\left(\frac{x_i}{\lambda_N}\right)^{k_N}} \right] \\ &= \left[\frac{k_N^n}{\lambda_N^n} \prod_{i=1}^n x_i^{k_N-1} e^{-\left(\frac{x_i}{\lambda_N}\right)^{k_N}} \right] \end{aligned}$$

Taking logarithm on both sides we get,

$$\ln(\mathcal{L}(\vec{x}|\lambda_N, k_N)) = n \ln(k_N) - n \ln(\lambda_N) + (k_N - 1) \sum_{i=1}^n \ln(x_i) - \sum_{i=1}^n \left(\frac{x_i}{\lambda_N} \right)^{k_N}$$

Now partially differentiating and equating them with 0 gives,

$$\left[\frac{\delta(\ln(\mathcal{L}(\vec{x}|\lambda_N, k_N)))}{\delta \lambda_N}, \frac{\delta(\ln(\mathcal{L}(\vec{x}|\lambda_N, k_N)))}{\delta k_N} \right] = [0, 0]$$

So, we can write them as,

$$\left\{ \begin{array}{l} \frac{\delta(\ln(\mathcal{L}(\vec{x}|\lambda_N, k_N)))}{\delta\lambda_l}, \frac{\delta(\ln(\mathcal{L}(\vec{x}|\lambda_N, k_N)))}{\delta\lambda_u} \\ \frac{\delta(\ln(\mathcal{L}(\vec{x}|\lambda_N, k_N)))}{\delta k_l}, \frac{\delta(\ln(\mathcal{L}(\vec{x}|\lambda_N, k_N)))}{\delta k_u} \end{array} \right\} = [0, 0] \quad \dots \dots \dots \quad (1)$$

So, these equations have no closed form we need to use numerical iterative techniques such as Newton-Raphson method or scoring methods to find the estimated parameters.

Simulation of Neutrosophic Weibull Distribution and Estimation of parameters:-

To, simulate data from neutrosophic weibull distribution we need the method of Monte Carlo Estimation. This approach enables us to generate computer based pseudo random number generator for generating random numbers. Here, we use the library package "ntsDists" in R to generate neutrosophic random numbers from weibull distribution and estimate the parameters using newton raphson iterative technique.

Here, we have generated 10000 samples from neutrosophic weibull distribution with scale parameter [9.5544,10.3370] and shape parameter the generated samples are,

[1.669639e+01 , 1.803153e+01] , [1.186682e+01 , 1.282987e+01] , [1.0519, 1.0553] [5.227939e+00 , 5.667158e+00] , [2.967674e+01 , 3.199053e+01] , [7.889716e+00 , 8.541229e+00] , [5.348771e+00 , 5.797715e+00] , [5.213533e+00 , 5.651591e+00] , [1.073665e+01 , 1.161173e+01] , [1.202369e+01 , 1.299892e + 01] , [2.685068e - 01, 2.938626e - 01].....

The density plot of the generated neutrosophic weibull distribution looks like,

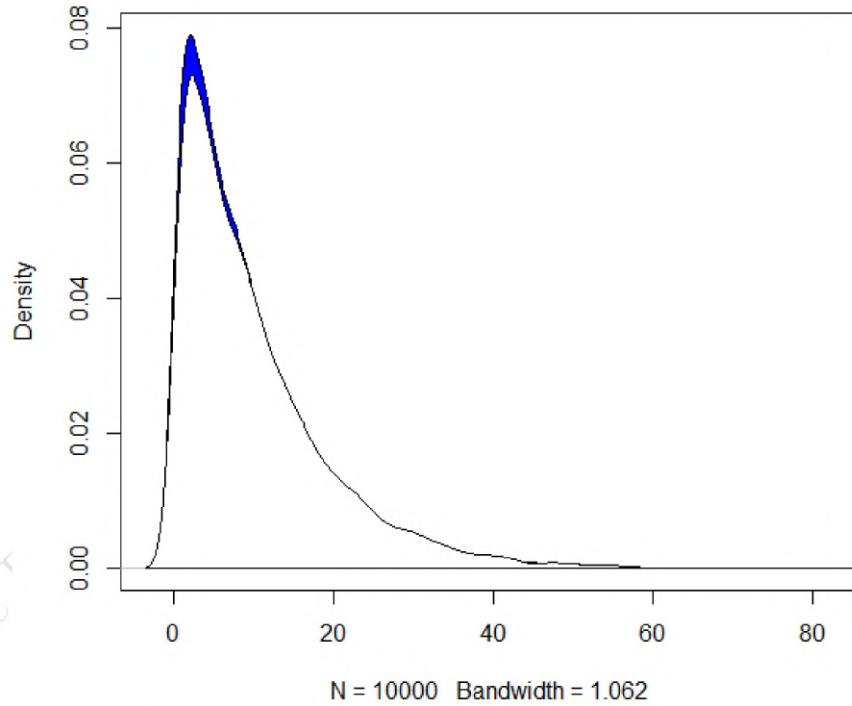


Figure 1: Neutrosophic Weibull pdf for various values of the shape and scale parameter

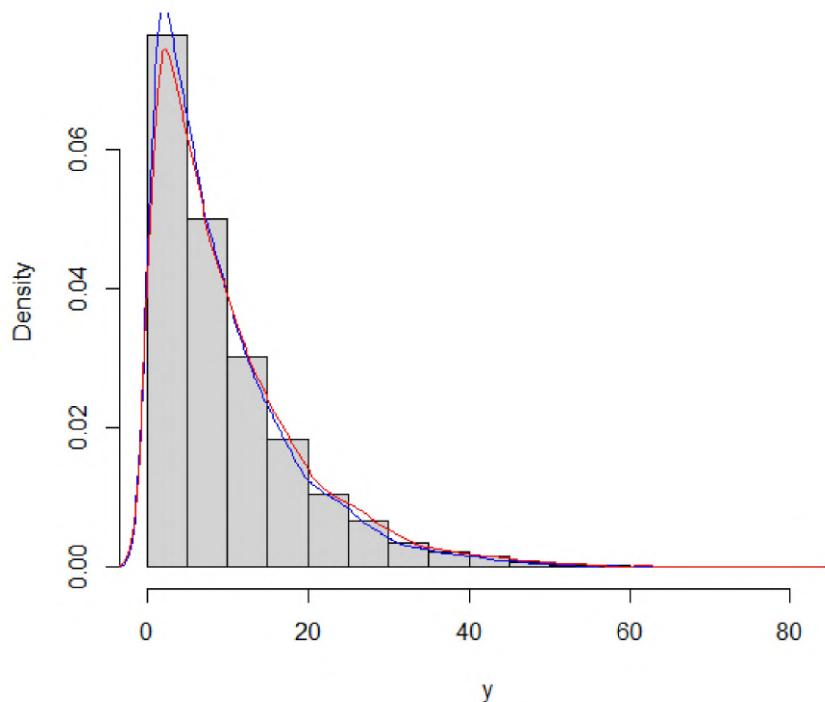


Figure 2: Histogram of Neutrosophic Weibull for various values of the shape and scale parameter

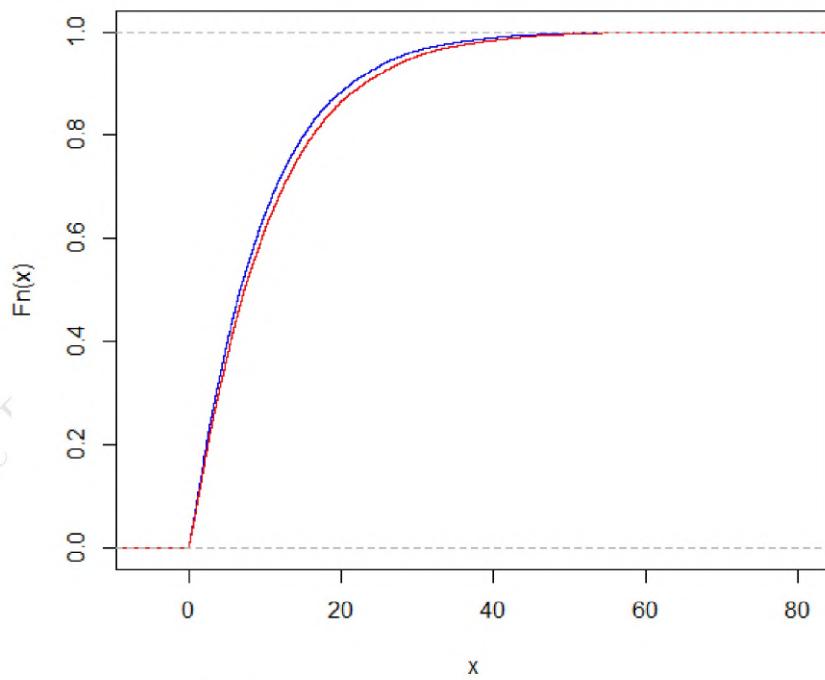


Figure 3: Neutrosophic Weibull cdf for various values of the shape and scale parameter

The maximum likelihood estimation for the parameters gives us the value as,

$$\text{MLE(shape parameter)}(\hat{k}) = [1.062414, 1.054294]$$

$$\text{MLE(scale parameter)}(\hat{\lambda}) = [9.687938, 10.352736]$$

So, the mean failure time based on estimated parameters is $= \hat{\lambda}_N \Gamma \left(\frac{1}{\hat{k}_N} + 1 \right)$

Which is equal to,

$$= \left[\hat{\lambda}_l \Gamma \left(\frac{1}{\hat{k}_l} + 1 \right), \hat{\lambda}_l \Gamma \left(\frac{1}{\hat{k}_l} + 1 \right) \right]$$

$$= \boxed{[9.460933, 10.13853]}$$

The sample mean is = [9.270696, 10.01847]

The variance based on MLE estimates is = $\left[\hat{\lambda}_N^2 \left(\Gamma \left(\frac{2}{\hat{k}_N} + 1 \right) - \Gamma \left(\frac{1}{\hat{k}_N} + 1 \right)^2 \right) \right]$
 $\left[\left[\hat{\lambda}_l^2 \Gamma \left(\frac{2}{\hat{k}_l} + 1 \right) - \hat{\lambda}_l^2 \Gamma \left(\frac{1}{\hat{k}_l} + 1 \right)^2 \right], \left[\hat{\lambda}_u^2 \Gamma \left(\frac{2}{\lambda k_u} + 1 \right) - \hat{\lambda}_u^2 \Gamma \left(\frac{1}{\hat{k}_u} + 1 \right)^2 \right] \right]$

Which is equal to,

$$= \boxed{[79.38225, 92.5476]}$$

The sample variance is equal to = [77.19573, 89.58036]

So, we represent the results of the analysis in the table below,

Estimated vs Observed Comparison		
Descriptive Measures	Observed Values	Estimated Values
Scale parameter	[9.5544, 10.3370]	[9.687938, 10.352736]
Shape parameter	[1.0519, 1.0553]	[1.062414, 1.054294]
mean	[9.270696, 10.01847]	[9.460933, 10.13853]
median	[6.743455, 7.304005]	[6.861323, 7.3127]
variance	[77.19573, 89.58036]	[79.38225, 92.5476]
skewness	[1.854021, 1.845088]	[1.860548, 1.851586]
kurtosis	[2.067178, 2.012699]	[1.90106, 2.028742]

Conclusions:-

Here, We discussed about the Neutrosophic Extension of Weibull distribution in Reliability analysis. At first, we develop the theoretical derivations such as mean, variance, skewness, kurtosis of the proposed model and then we perform simulation by generating data from Neutrosophic weibull distribution using monte carlo simulation with the help of R software and validate the proposed model by fitting the data by comparing the observed descriptive measures and estimated descriptive measures using the estimated parameters which we got from MLE method.

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Appendix:-

R code to generate neutrosophic weibull distribution,

```
> library(ntsDists)

###Generating 10000 samples from Neutrosophic Weibull Distribution

>y=rnsweibull(n=10000,scale=c(9.5544,10.3370),shape=c(1.0519,1.0553))

### The density curve of neutrosophic weibull distribution

> plot.new()
> plot(density(y[,1]), col="Blue")
> polygon(density(y[,1]), col="Blue")
> polygon(density(y[,2]), col="white")

### The histogram plot of neutrosophic weibull distribution

>plot.new()
>hist(y, freq = FALSE)
>lines(density(y[,1]), col="Blue")
>lines(density(y[,2]), col="red")

### The cdf plot of neutrosophic weibull distribution

>plot.new()
>plot(ecdf(y[,1]), col="Blue")
>lines(ecdf(y[,2]), col="red")

### measures based on sample

>mean(y[,1])
>mean(y[,2])
>var(y[,1])
>var(y[,2])

### MLE estimates

>install.packages("EnvStats")
>library(EnvStats)
>eweibull(y[,1], method = "mle")
>eweibull(y[,2], method = "mle")

###MLE estimates without using package

>data <- rweibull(10000, shape = shape parameter, scale = scale parameter)
>data
```

```

>likelihood <- function(params, data) {
  scale <- params[1]
  shape <- params[2]
  -sum(dweibull(data, shape = shape, scale = scale, log = TRUE))
}      # Define the likelihood function for Weibull distribution

>mle_result <- optim(c(2, 5), likelihood, data = data, method = "L-BFGS-B")
# Use optim function to obtain MLE of parameters

>mle_scale <- mle_result$par[1]
>mle_shape <- mle_result$par[2]      # Extract MLE estimates

>cat("MLE of shape parameter:", mle_shape, "\n")
>cat("MLE of scale parameter:", mle_scale, "\n")    # Print MLE estimates

####Comparing observed and esimated descriptive measures

# Mean of Weibull distribution
>mean_failure_time <- gamma(1+1/mle_shape)*mle_scale

# Calculate mean failure time and compare with sample mean
>sample_mean <- mean(data)

>cat("Mean failure time (MLE):", mean_failure_time, "\n")
>cat("Sample mean:", sample_mean, "\n")      # Print mean values

```

The python codes which can also be used,

####Generate samples for neutrosophic weibull distribution

```

import numpy as np
import matplotlib.pyplot as plt

def weibull_pdf(x, shape, scale):
    return (shape / scale) * (x / scale)**(shape - 1) *
    np.exp(-(x / scale)**shape)

def bounding_pdf(x, c):
    return c * np.exp(-c * x)

def acceptance_rejection(shape, scale, n_samples):
    max_pdf_ratio = weibull_pdf(scale, shape, scale) /
        bounding_pdf(scale, shape)
    samples = []
    while len(samples) < n_samples:
        x = np.random.exponential(scale=1/shape)

```

```

# Exponential distribution as a candidate
u = np.random.uniform(0, max_pdf_ratio * bounding_pdf(x, shape))
if u <= weibull_pdf(x, shape, scale):
    samples.append(x)
return samples

# Parameters for the Weibull distribution
shape = shape parameter
scale = scale parameter

# Number of samples to generate
n_samples = 1000

# Generate samples using acceptance-rejection method
samples = acceptance_rejection(shape, scale, n_samples)

# Plot histogram of generated samples
plt.hist(samples, bins=30, density=True, alpha=0.7, color='blue',
         label='Generated-samples')
# Plot the theoretical PDF
x_values = np.linspace(0, max(samples), 1000)
plt.plot(x_values, weibull_pdf(x_values, shape, scale), color='red',
          linewidth=2, label='Theoretical-PDF')
plt.xlabel('x')
plt.ylabel('Probability-Density')
plt.title('Samples-from-Weibull-distribution-(shape={}, scale={})'.format(shape, scale))
plt.legend()
plt.show()

```

####Plot of pdf of neutrosophic weibull distribution

```

import seaborn as sns
sns.kdeplot(samples, color='blue', label='Sample-1', linestyle='--',
             linewidth=2, shade=1)
sns.kdeplot(samples1, color='red', label='Sample-2', linestyle='—',
             linewidth=2, shade=1)

# Add labels and title
plt.xlabel('X')
plt.ylabel('Density')
plt.title('Kernel-Density-Estimate-(KDE)-Plot-of-Two-Different-Samples')
plt.legend()

# Show plot
plt.show()

```