A TEXTBOOK ON

ENGINEERING MATHEMATICS

Volume II

B. E./B. Arch. First Year/Second Part

(For Mechanical, Electronics, Computer, Civil, Electrical, Industrial, Agriculture and Architecture)

S. P. Shrestha
H. D. Chaudhary
P. R. Pokharel

Strictly based on new syllabus of T.U., IOE

A Textbook
on

Engineering Mathematics

Volume -II

(For B.E./B.Arch. First Year/Second Part)

Mechanical, Electronics, Computer, Civil, Electrical and Architecture

S. P. Shrestha
Professor
Institute of Engineering
Pulchowk Campus, Lalitpur.

H. D. Chaudhary
Associate Professor
Institute of Engineering
Pulchowak Campus, Lalitpur.

P. R. Pokharel
Lecturer
Tribhuvan University, Kathmandu.

VIDYARTHI PRAKASHAN (P.) Ltd.
PUBLISHER AND DISTRIBUTOR
Kamalpokhari, Kathmandu.

V
510
SHR
2016
(LS4US1)

Book : A Textbook on Engineering Mathematics II

Publisher : Vidyarthi Prakashan (P.) Ltd.

Kamalpokhari, Kathmandu, Nepal

Tel. No. 4227246, 4423333,4445202

First Edition : July, 2010

Second Edition : July, 2016

Price : Rs. 500/-

ISBN : 978- 99946-1-971-9

Layout : Vidyarthi Desk

Printed at : Alliance Printers

Contents

First User.
Siddhartha Shakya
073BME

Scanned by CamScanner

Preface

A suitable textbook of Engineering mathematics was a long felt need. Although there exists a lot of books on Engineering mathematics by foreign authors, but these are not suitable to the various courses for Nepalese Universities. In this context, authors have been inspired to bring this volume. This book " *A Text book on Engineering Mathematics-II* " is primarily written according to the new syllabus, 2010 of mathematics of **First Year Second Part** of Institute of Engineering (I.O.E.), Tribhuvan University. This book also covers the courses of other technical universities.

The subject matter is presented in a very systematic and logical manner. Every endeavor has been made to make the contents simple and lucid as possible. This book fairly contains large number of solved examples taken from various recently examination papers of different universities and Engineering colleges so that students may not face any difficulty while answering these problems in their final examinations.

In the preparation of this book, the authors have consulted various standard books of Engineering mathematics for which they are greatly indebted to the authors of those books.

Finally, authors take this opportunity to express their sincere thanks to the management of Vidyarthi Prakashan for the courtesy and the cooperation extended by them to the authors in the publishing of this book in short time and also thankful to Ms. Bhabani Lamsal for computer setting of this book on time.

Intimations of errors and suggestions for improvement will be highly appreciated and gratefully acknowledged

Authors

Chapter –1

Calculus of Several Variables

- Introduction
- Calculus of Two or More Variables
- Homogenous Functions
- Total Differential Coefficients
- Differentiation of Implicit Functions
- Increments
- Extrema of Functions of Two or Three Variables

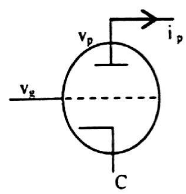
Chapter –1

Calculus of Several Variables

1.1 Introduction

In many Engineering problems, we come across a variable quantity, which depends for its value on two or more independent variables. To illustrate this, let us take an example of triode valve from Electronic Engineering. Here the electron flow from the cathode *c*, is controlled by the independent voltage v_g and v_p on the grid and the plate of the valve. Thus i_p the plate current, is a function of two independent variables v_p and v_g and to consider the variation of i_p we have to be familiar with the method of partial differentiation. There are large numbers of other engineering problems that necessitate a clear understanding of methods of partial differentiation.

A function of more than one variable (i.e. several variables) and the application of differential calculus to the function of several variables is known as *Calculus of several variables*. In this chapter we shall mainly concerned with the application of differential calculus to the functions of two or three variables.



1.1.1 Calculus of Two or More Variables

We have often known the quantities, which depend on two or more variables. For example, the area of a rectangle of length x and breadth y, is given by

$$A = x\,y.$$

So A has a definite value for a given pair of values of x and y. Similarly, the volume of the parallelepiped depends on three variables

x(= length) y (= breadth) and h(= height) is given by

$$v = x\,y\,h.$$

Thus the symbol z has a definite value for every pairs of value x and y is called function of two independent variables x and y and we write

$$z = f(x, y) \text{ or } \phi(x, y).$$

And the symbol u has definite value for every triad of value x, y and z is called function of three variables x, y and z and we write

$$u = f(x, y, z)$$

Here (x, y) be the coordinate of a point in the xy -plane and z as the height of the surface z = f(x, y).

A set of points lying within a circle having centre at (a, b) and radius δ is said to be neighborhood of (a, b) in the circular region

$$R : (x - a)^2 + (y - b)^2 < \delta^2.$$

i.e. $\{(x, y) : |x - a| < \delta, |y - b| < \delta\}$ is neighborhood of the point (a, b).

When z is a function of three or more variables x, y, t ……………, we represent the relation $z = f(x, y, t, \ldots\ldots)$.

1.1.2 Limit

Let $z = f(x, y)$ be a function of two independent variables x and y defined in the neighborhood of (a, b) then the function f(x, y) is said to have limit l as (x, y) tends to (a, b) if and only if corresponding to a positive number ε, there exists another positive number δ such that for every (x, y) in R

$(x - a)^2 + (y - b)^2 < \delta^2$ implies $|f(x, y) - l| < \varepsilon$.

We write $\lim\limits_{(x, y) \to (a, b)} f(x, y) = l$

1.1.3 Continuity

A function f(x, y) is said to be continuous at the point (a, b) if

$\lim\limits_{(x, y) \to (a, b)} f(x, y)$ exist and $\lim\limits_{(x, y) \to (a, b)} f(x, y) = f(a, b)$

If the function is continuous at all points of the neighborhood of (a, b), then it is said to be continuous in that region. A function that is not continuous at a point is said to be discontinuous at that point.

1.2 Partial Derivatives

The ordinary derivative of a function of several variables with respect to one of the independent variable, keeping all other independent variables constant is called *partial derivative* of the function with respect to the variable. Partial derivatives are used in vector calculus and differential geometry.

Thus if $u = f(x, y)$ be a function of two independent variables x and y ,then the derivative of u with respect x treating y as constant is called *partial derivative of u with respect to x* and is denoted by one of the symbols

$$\frac{\partial u}{\partial x},\ \frac{\partial f}{\partial x},\ f_x(x, y),\ f_x,\ u_x.$$

Thus

$$\frac{\partial u}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$ provided the limit exist.

Similarly, the partial derivative of u with respect to y, treating x as constant is called the *partial derivative of u with respect to y* and is denoted by

$$\frac{\partial u}{\partial y} \text{ or } \frac{\partial f}{\partial y},\ f_y(x, y),\quad u_y \quad f_y$$

Thus

$$\frac{\partial u}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$$ provided the limit exists.

The partial derivatives at a particular point (a, b) are often denoted by

$$\left(\frac{\partial u}{\partial x}\right)_{\text{at }(a, b)} \quad \text{or } f_x(a, b)$$

$$\left(\frac{\partial u}{\partial y}\right)_{\text{at }(a, b)} \quad \text{or } f_y(a, b)$$

$$\therefore \left(\frac{\partial u}{\partial x}\right)_{\text{at }(a, b)} = \lim_{h \to 0} \frac{f(a + h, b) - f(a, b)}{h}$$

$$\left(\frac{\partial u}{\partial y}\right)_{\text{at }(a, b)} = \lim_{k \to 0} \frac{f(a, b + k) - f(a, b)}{k}$$ provided these limit exist

In the same way, if u is a functions of three variables x, y, z, the partial derivative of u with respect to x, y and z are defined by

$$\frac{\partial u}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y, z) - f(x,y,z)}{\Delta x}$$

$$\frac{\partial u}{\partial y} = \lim_{\Delta x \to 0} \frac{f(x, y + \Delta y, z) - f(x, y, z)}{\Delta y}$$ and

$$\frac{\partial u}{\partial z} = \lim_{\Delta x \to 0} \frac{f(x, y, z + \Delta z) - f(x,y,z)}{\Delta z}$$ provided these limits exist.

In general, f_x and f_y are also functions of x and y and so these can be differentiated further partially with respect to x and y.

Thus $\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial x^2}$ or $\frac{\partial^2 f}{\partial x^2}$ or $\frac{\partial^2 u}{\partial x^2}$ or u_{xx} or f_{xx}

$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial^2 u}{\partial x^2}$ or $\frac{\partial^2 u}{\partial x \partial y}$ or $\frac{\partial^2 f}{\partial x \partial y}$ or u_{xy} or f_{xy}

$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial y \partial x}$ or $\frac{\partial^2 f}{\partial y \partial x}$ or u_{yx} or f_{yx}

and $\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial^2 u}{\partial y^2}$ or $\frac{d^2 f}{\partial y^2}$ or u_{yy} or f_{yy}

It can easily be verified that in all ordinary cases

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}.$$

Worked Out Examples

Ex. 1 Find the first and second partial derivatives of $x^3 + y^3 - 3axy$

Solution:

Let $u = x^3 + y^3 - 3axy$

Partially differentiating with respect to x and y respectively

$$\frac{\partial u}{\partial x} = 3x^2 - 3ay \text{ and } \frac{\partial u}{\partial y} = 3y^2 - 3ax$$

Also $\frac{\partial^2 u}{\partial x^2} = 6x,\quad \frac{\partial^2 u}{\partial y \partial x} = -3a$

$\frac{\partial^2 u}{\partial y^2} = 6y,\quad \frac{\partial^2 u}{\partial x \partial y} = -3a$

We observe that $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$.

Ex. 2 (i) If $u = x^2 + y^2 + z^2$, then show that $x\,u_x + y\,u_y + z\,u_z = 2u$

(ii) If $V = \sqrt{x^2 + y^2 + z^2}$, then show that $Vxx + Vyy + Vzz = \frac{2}{V}$

(iii) If $u = \log\sqrt{x^2 + y^2 + z^2}$, then show that

$$(x^2 + y^2 + z^2)\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = 1$$

(i) Solution:

Given $u = x^2 + y^2 + z^2$

Partially differentiating with respect to x, y and z respectively, we get

$u_x = 2x,\quad u_y = 2y,\quad u_z = 2z$

We have

$x\,u_x + y\,u_y + z\,u_z = 2x^2 + 2y^2 + 2z^2 = 2(x^2 + y^2 + z^2) = 2u$

$\therefore\ x\,u_x + y\,u_y + z\,u_z = 2u.$

(ii) Solution:

Given that $V = \sqrt{x^2 + y^2 + z^2}$

Partially differentiating with respect to x, y and z respectively, we get

$$V_x = \frac{1}{2}\frac{2x}{\sqrt{x^2+y^2+z^2}} = \frac{x}{\sqrt{x^2+y^2+z^2}}$$

$$V_{xx} = \frac{1}{\sqrt{x^2+y^2+z^2}} + x\left(\frac{-1}{2}\right)\frac{1.2x}{(x^2+y^2+z^2)^{3/2}}$$

$$= \frac{y^2 + z^2}{(x^2+y^2+z^2)^{3/2}}$$

Similarly $V_{yy} = \frac{z^2 + x^2}{(x^2+y^2+z^2)^{3/2}}$ and $V_{zz} = \frac{x^2 + y^2}{(x^2+y^2+z^2)^{3/2}}$

Now $V_{xx} + V_{yy} + V_{zz} = \frac{2(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{3/2}} = \frac{2}{\sqrt{x^2+y^2+z^2}} = \frac{2}{V}$

(iii) Solution:

Given $u = \log\sqrt{x^2+y^2+z^2}$

Partially differentiating with respect to x, y and z respectively, we get

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{x^2+y^2+z^2}}\,\frac{1.2x}{2\sqrt{x^2+y^2+z^2}} = \frac{x}{x^2+y^2+z^2}$$

$$\therefore \frac{\partial^2 u}{\partial x^2} = \frac{(x^2+y^2+z^2) - x.2x}{(x^2+y^2+z^2)^2} = \frac{y^2+z^2-x^2}{(x^2+y^2+z^2)^2}$$

Similarly

$$\frac{\partial^2 u}{\partial y^2} = \frac{z^2+x^2-y^2}{(x^2+y^2+z^2)^2} \qquad \frac{\partial^2 u}{\partial z^2} = \frac{x^2+y^2-z^2}{(x^2+y^2+z^2)^2}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{(x^2+y^2+z^2)}$$

$$(x^2+y^2+z^2)\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = 1$$

Ex. 3 (i) If $u = e^{xyz}$, then prove that $\frac{\partial^3 u}{\partial x \partial y \partial z} = (1 + 3xyz + x^2y^2z^2)\,e^{xyz}$

(ii) If $z(x + y) = x^2 + y^2$, then show that $\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)$

(iii) If $f(x, y, z) = e^{x/y} + e^{y/z} + e^{z/x}$, then show that

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = 0$$

(i) Solution:

Given $u = e^{xyz}$

Partially differentiating with respect to x, y and z respectively, we get

$$\frac{\partial u}{\partial z} = e^{xyz}\,.xy$$

$$\frac{\partial^2 u}{\partial y \partial z} = e^{xyz}x + xy\,e^{xyz}xz = (x + x^2yz)\,e^{xyz}$$

or $$\frac{\partial^3 u}{\partial x \partial y \partial z} = (x + x^2yz)\, e^{xyz}\, yz + e^{xyz}(1 + 2xyz)$$

$$\therefore \quad \frac{\partial^3 u}{\partial x \partial y \partial z} = e^{xyz}(1 + 3xyz + x^2y^2z^2)$$

(ii) Solution:

Given, $z(x + y) = x^2 + y^2$

Partially differentiating with respect to x and y respectively, we get

$$z + (x + y)\frac{\partial z}{\partial x} = 2x \quad \text{and} \quad z + (x + y)\frac{\partial z}{\partial y} = 2y$$

or $$\frac{\partial z}{\partial x} = \frac{2x - z}{x + y} \quad \text{and} \quad \frac{\partial z}{\partial y} = \frac{2y - z}{x + y}$$

Now $$\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = \left(\frac{2x - z}{x + y} - \frac{2y - z}{x + y}\right)^2 = 4\left(\frac{x - y}{x + y}\right)^2 \qquad \text{.............(1)}$$

Also $$4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right) = 4\left(1 - \frac{2x - z}{x + y} - \frac{2y - z}{x + y}\right)$$

$$= 4\,\frac{(x + y - 2x + z - 2y + z)}{x + y} = 4\left(\frac{2z - x - y}{x + y}\right)$$

$$= \frac{4}{(x + y)}\left[\frac{2(x^2 + y^2)}{x + y} - \frac{(x + y)}{1}\right]$$

$$= \frac{4}{(x + y)}\,\frac{(2x^2 + 2y^2 - x^2 - 2xy - y^2)}{(x + y)}$$

$$= 4\left(\frac{x - y}{x + y}\right)^2 \qquad \text{.........(2)}$$

From (1) and (2), we have

$$\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)$$

(iii) Solution: Given, $f(x, y, z) = e^{\frac{x}{y}} + e^{\frac{y}{z}} + e^{\frac{z}{x}}$

$$\therefore \quad \frac{\partial f}{\partial x} = \frac{1}{y}e^{\frac{x}{y}} + e^{\frac{z}{x}}\left(-\frac{z}{x^2}\right), \qquad \frac{\partial f}{\partial y} = e^{\frac{x}{y}}\left(\frac{-x}{y^2}\right) + e^{\frac{y}{z}}\,\frac{1}{z}$$

and $$\frac{\partial f}{\partial z} = e^{\frac{y}{z}}\left(\frac{-y}{z^2}\right) + e^{\frac{z}{x}}\,\frac{1}{x}$$

Now

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = \frac{x}{y}e^{\frac{x}{y}} - \frac{z}{x}e^{\frac{z}{x}} - \frac{x}{y}e^{\frac{x}{y}} + \frac{y}{z}e^{\frac{y}{z}} - \frac{y}{z}e^{\frac{y}{z}} + \frac{z}{x}e^{\frac{z}{x}}$$

$$= 0$$

Ex.4 (i) If $u = \frac{z}{x} + \frac{y}{z}$, then show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$

(ii) If $x = r\cos\theta$, $y = r\sin\theta$, prove that

$$\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} = \frac{1}{r}\left[\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2\right]$$

2060 Chaitra B. E.

(i) Solution:

Given that $$u = \frac{z}{x} + \frac{y}{z} \qquad \text{...(1)}$$

Partially differentiating (1) with respect to x, y and z respectively, we get

$$\frac{\partial u}{\partial x} = -\frac{z}{x^2}, \qquad \frac{\partial u}{\partial y} = \frac{1}{z} \quad \text{and} \quad \frac{\partial z}{\partial x} = \frac{1}{x} - \frac{y}{z^2}$$

Now

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = -\frac{-z}{x} + \frac{y}{z} + \frac{z}{x} - \frac{y}{z} = 0$$

(iii) Solution:

We have $x^2 + y^2 = r^2$ ..(1)

Partially differentiating with respect to x and y respectively, we get

$$2x = 2r\frac{\partial r}{\partial x}$$

or $$\frac{\partial r}{\partial x} = \frac{x}{r} = \frac{x}{\sqrt{x^2 + y^2}}$$

$$\therefore \quad \frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}} \qquad \text{....(2)}$$

and $$2y = 2r\frac{\partial r}{\partial y}$$

$$\frac{\partial r}{\partial y} = \frac{y}{r} = \frac{y}{\sqrt{x^2 + y^2}}$$

$$\therefore \quad \frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}} \qquad \text{.....(3)}$$

Again partially differentiating with respect to x to (2) and y to(3) respectively, we get

$$\frac{\partial^2 r}{\partial x^2} = \frac{r - x\frac{\partial r}{\partial x}}{r^2} = \frac{r - x\cdot\frac{x}{r}}{r^2} = \frac{r^2 - x^2}{r^3}$$

$$= \frac{x^2+y^2-x^2}{(x^2+y^2)^{3/2}} = \frac{y^2}{(x^2+y^2)^{3/2}}$$

and
$$\frac{\partial^2 r}{\partial y^2} = \frac{r - y\frac{\partial r}{\partial y}}{r^2} = \frac{r - y\cdot\frac{y}{r}}{r^2} = \frac{r^2 - y^2}{r^3}$$

$$= \frac{x^2+y^2-y^2}{(x^2+y^2)^{3/2}} = \frac{x^2}{(x^2+y^2)^{3/2}}$$

We have
$$\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} = \frac{y^2}{(x^2+y^2)^{3/2}} + \frac{x^2}{(x^2+y^2)^{3/2}}$$

$$= \frac{x^2+y^2}{(x^2+y^2)^{3/2}} = \frac{1}{\sqrt{x^2+y^2}} = \frac{1}{r}$$

and
$$\frac{1}{r}\left[\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2\right] = \frac{1}{r}\left(\frac{x^2}{r^2} + \frac{y^2}{r^2}\right) = \frac{1}{r}\ \frac{x^2+y^2}{r^2} = \frac{1}{r}$$

Exercise - 1

1. Evaluate f_x and f_y of the following function $f(x, y)$
   (i) $x^2y - x\sin(xy)$
   (ii) $\log(x^2+y^2)$
   (iii) $\tan^{-1}\frac{x^2+y^2}{x+y}$
   (iv) $\frac{x^2-y^2}{x^2+y^2}$
   (v) $\sin^{-1}(ax+by)$
2. Find the partial derivative of second orders of the following functions $f(x, y)$
   (i) $x^3+3x^2y+3xy^2+y^3$
   (ii) $e^{x^2+xy+y^2}$
   (iii) $\log(x^2y+xy^2)$
   (iv) $\tan^{-1}\frac{2xy}{x^2+y^2}$

3. (i) If $u = \sin^{-1}\frac{x}{y} + \tan^{-1}\frac{y}{x}$, then show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$
   (ii) If $u = x^2y + y^2z + z^2x$, then show $u_x + u_y + u_z = (x+y+z)^2$
   (iii) If $u = \log\frac{x^2+y^2}{x+y}$, then prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 1$
   (iv) If $u = \log(x^2+y^2+z^2)$, the show that $x\,u_{yz} = y\,u_{zx} = z\,u_{xy}$
4. If $u = \log(x^3+y^3+z^3-3xyz)$, then show that
   (i) $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3}{x+y+z}$
   (ii) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = -\frac{3}{(x+y+z)^2}$
   (iii) $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = -\frac{9}{(x+y+z)^2}$ [2072 Aswin, B. E.]
5. (i) If $u = f\left(\frac{y}{x}\right)$, then show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$
   (ii) If $V = z\tan^{-1}\frac{y}{x}$, then show that $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$
   (iii) If $u = \tan(y+ax) - (y-ax)^{3/2}$ then show that $\frac{\partial^2 u}{\partial x^2} = a^2\frac{\partial^2 u}{\partial y^2}$
   (iv) If $z = \phi(x+ay) + \psi(x-ay)$, then show that $\frac{\partial^2 z}{\partial x^2} = a^2\frac{\partial^2 z}{\partial y^2}$
6. (i) If $u = f(xyz)$ show that $x^2u_{xx} = y^2u_{yy} = z^2u_{zz}$
   (ii) If $u = \frac{1}{\sqrt{x^2+y^2+z^2}}$ prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$ [2072 Magh. B. E.]
7. If $x = r\cos\theta$, $y = r\sin\theta$, then show that
   (i) $\frac{\partial r}{\partial x} = \frac{\partial x}{\partial r}$ (ii) $\frac{1}{r}\frac{\partial x}{\partial \theta} = r\frac{\partial \theta}{\partial x}$ (iii) $\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = 0$
8. If $u = e^{r\cos\theta}\cos(r\sin\theta)$, $v = e^{r\cos\theta}\sin(r\sin\theta)$, then prove that
   $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$ and $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$
9. If $u = f(r)$, then prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r}f'(r)$

Answers

1. (i) $2xy - xy\cos(xy) - \sin(xy)$, $\quad x^2(1 - \cos xy)$

   (ii) $\dfrac{2x}{x^2+y^2}$, $\dfrac{2y}{x^2+y^2}$

   (iii) $\dfrac{x^2+2xy-y^2}{(x^2+y^2)^2+(x+y)^2}$, $\dfrac{y^2+2xy-x^2}{(x^2+y^2)^2+(x+y)^2}$

   (iv) $\dfrac{4xy^2}{(x^2+y^2)^2}$, $\quad -\dfrac{4x^2y}{(x^2+y^2)^2}$

   (v) $\dfrac{a}{\sqrt{1-(ax+by)^2}}$, $\dfrac{b}{\sqrt{1-(ax+by)^2}}$

2. (i) $6(x+y)$, $6(x+y)$, $\quad 6(x+y)$, $6(x+y)$

   (ii) $e^{x^2+xy+y^2}\{(2x+y)^2+2\}$, $\quad e^{x^2+xy+y^2}((2x+y)(x+2y)+1\}$,
   $e^{x^2+xy+y^2}\{(2x+y)(x+2y)+1\}$, $\quad e^{x^2+xy+y^2}\{(x+2y)^2+2\}$

   (iii) $-\dfrac{1}{x^2}-\dfrac{1}{(x+y)^2}$, $\quad -\dfrac{1}{(x+y)^2}$, $\quad -\dfrac{1}{(x+y)^2}$, $-\dfrac{1}{y^2}-\dfrac{1}{(x+y)^2}$

1.2.1 Homogenous Functions

An expressions in terms of two independent variables x and y of the form

$$a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + \dots\dots\dots + a_ny^n$$

in which every term is of the n^{th} degree, is called a *homogenous function of degree n.* It can be written as

$$x^n\left[a_0 + a_1\left(\frac{y}{x}\right) + a_2\left(\frac{y}{x}\right)^2 + \dots\dots\dots\dots + a_n\left(\frac{y}{x}\right)^n\right]$$

Thus if any function f(x, y) of two independent variables x and y can be expressed in the form $x^n\phi\left(\frac{y}{x}\right)$, then it is called a *homogeneous function of degree n in x and y.*

For instance, Let

$$f(x,y) = 2x^3 + 3x^2y + y^3 \text{ then}$$

$$f(x,y) = x^3\left[2 + 3\frac{y}{x} + \left(\frac{y}{x}\right)^3\right] = x^3\phi\left(\frac{y}{x}\right)$$

Also

$$f(tx,ty) = 2t^3x^3 + 3t^2x^2\,ty + t^3y^3$$
$$= t^3(2x^3 + 3x^2y + y^3) = t^3f(x,y)$$

Hence the function f of two variables is homogenous of degree three.

In general, a function f(x, y, z ………) is said to be homogenous function of degree n in x, y, z ……… if it can be expressed in the form $x^n\phi\left(\frac{y}{x}, \frac{z}{x}, \dots\right)$

or $\quad f(tx, ty, tz, \dots\dots) = t^n f(x, y, z, \dots\dots\dots)$ for every values of t.

Euler's theorem on Homogenous Functions

Theorem 1

If u be a homogenous function of degree n in x and y, then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$$

2065Kartik, B. E.

Proof

Given that u is a function of x and y of degree n, so that the function u can be written as

$$u = x^n f\left(\frac{y}{x}\right)$$

Partially differentiating with respect to x and y respectively, we get

$$\frac{\partial u}{\partial x} = nx^{n-1}f\left(\frac{y}{x}\right) + x^n f'\left(\frac{y}{x}\right)\left(\frac{-y}{x^2}\right) = nx^{n-1}f\left(\frac{y}{x}\right) - yx^{n-2}f'\left(\frac{y}{x}\right)$$

and $\frac{\partial u}{\partial y} = x^n f'\left(\frac{y}{x}\right)\frac{1}{x} = x^{n-1} f'\left(\frac{y}{x}\right)$

Hence $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\, x^n f\left(\frac{y}{x}\right) = nu.$

In general, if u is a function of degree in x, y, z ... ... then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} + \ldots\ldots\ldots = nu$$

1.2.2 Total Differential Coefficients

Let $u = f(x, y)$ where $x = \phi(t)$ and $y = \psi(t)$ then u can be expressed as a function of t alone and the ordinary derivative $\frac{du}{dt}$ is called *total differential coefficient* of u with respect to t

Find $\frac{du}{dt}$ without actually substituting the value of x and y in f(x, y).

For Let $u = f(x, y)$ be a function of x and y

Suppose Δt be increment of t whereas Δx, Δy and Δu be increments of x, y and u respectively.

Then $u + \Delta u = f(x + \Delta x, y + \Delta y)$

On subtraction

$$\Delta u = f(x + \Delta x, y + \Delta y) - f(x, y)$$

$$= \{f(x+\Delta x, y+\Delta y) - f(x, y+\Delta y)\} + \{f(x, y + \Delta y) - f(x, y)\}$$

$$\frac{\Delta u}{\Delta t} = \frac{f(x+\Delta x, y+\Delta y) - f(x, y+\Delta y)}{\Delta x}\frac{\Delta x}{\Delta t} + \frac{f(x, y+\Delta y) - f(x, y)}{\Delta y}\frac{\Delta y}{\Delta t}$$

Taking limits as $\Delta t \to 0$, Δx and Δy both tends to zero, we have

$$\frac{du}{dt} = \left[\lim_{\Delta x\to 0}\frac{f(x+\Delta x, y+\Delta y) - f(x, y+\Delta y)}{\Delta x}\right]\frac{dx}{dt}$$

$$+ \lim_{\Delta y\to 0}\frac{f(x, y+\Delta y) - f(x, y)}{\Delta y}\frac{dy}{dt}$$

$$= \left\{\frac{\partial f(x, y+\Delta y)}{\partial x}\right\}\frac{dx}{dt} + \frac{\partial f(x, y)}{\partial y}\frac{dy}{dt}$$

Suppose that $\partial f(x, y)$ to be continuous function of y.

$$= \frac{\partial f(x, y)}{\partial x}\frac{dx}{dt} + \frac{\partial f(x, y)}{\partial y}\frac{dy}{dt}$$

$$\therefore \quad \frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} \qquad \ldots\ldots\ldots(1)$$

Cor. 1:

In particular if $u = f(x, y)$ where y is a function x, then put $t = x$ to (1), we obtain

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial x}\frac{dy}{dx} \qquad \ldots\ldots\ldots\ldots(2)$$

Cor. 2:

If $u = f(x, y)$,then the total differential of u is defined as

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

Cor. 3:

If $u = f(x, y, z)$ where x, y, z are all functions of a single variable t, then we can similarly prove that

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}$$

1.2.3 Differentiation of Implicit Function

If $u = f(x, y) = c$ be implicit function which defines as a differentiable function of x, then (2) becomes

$$0 = \frac{du}{dx}$$

$$0 = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}$$

Which gives $\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$

$$\frac{dy}{dx} = \frac{-f_x}{f_y}, \quad f_y \neq 0$$

1.2.4 Increments

If $u = f(x, y)$ be function of two variables x and y and let Δx and Δy be increments of x and y respectively, then the increment Δu of $u = f(x, y)$ is

$$\Delta u = f(x + \Delta x, y + \Delta y) - f(x, y)$$

For instance,

Obtain the change in f(x, y) if (x, y) changes from (1, 2) to (1.01, 1.98) if

$f(x, y) = 3x^2 - xy$

Now $\Delta u = f(x + \Delta x, y + \Delta y) - f(x, y)$

or $\Delta u = [3(x + \Delta x)^2 - (x + \Delta x)(y + \Delta y)] - (3x^2 - xy)$

$= 6x\,\Delta x + 3(\Delta x)^2 - x\,\Delta y - y\,\Delta x - \Delta x\,\Delta y$

Since (x, y) changes from (1, 2) to (1.01, 1.98)

So that substituting $x = 1$, $y = 1$, $\Delta x = .01$, $\Delta y = -0.02$.

$\therefore\ \Delta u = 6(1)(0.01) + 3(0.01)^2 - 1(-0.02) - 2(0.01) - (0.01)(-0.02)$

$= 0.0605.$

It follows that the following definition.

If $u = f(x, y)$ and Δx and Δy be increments of x and y respectively, then the differentials dx and dy of the independent variables x and y are

$dx = \Delta x$ and $dy = \Delta y$.

The differential du of the independent variable u is defined by

$$du = f_x dx + f_y dy = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy.$$

1.2.5 Partial Derivatives of a Function of Two Functions

Theorem

If $z = f(x, y)$ where $x = \phi_1(u, v)$, $y = \phi_2(u, v)$ and u, v are independent variables, then

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \quad \text{and} \quad \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

Proof

We have $z = f(x, y)$

Differentiating it, we get

$$dz = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy \qquad \text{.........(1)}$$

Also $x = \phi_1(u, v)$, $\quad y = \phi_2(u, v)$

$$dx = \frac{\partial x}{\partial u}\,du + \frac{\partial x}{\partial v}\,dv, \qquad dy = \frac{\partial y}{\partial u}\,du + \frac{\partial y}{\partial v}\,dv$$

Substituting the values of dx, dy in (1), we get

$$dz = \frac{\partial z}{\partial x}\left(\frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv\right) + \frac{\partial z}{\partial y}\left(\frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv\right)$$

$$dz = \left(\frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}\right)du + \left(\frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}\right)dv \qquad \text{...........(2)}$$

Since z is a function of u, v .

$$dz = \frac{\partial z}{\partial u}\,du + \frac{\partial z}{\partial v}\,dv \qquad \text{...........(3)}$$

Comparing it with (2), we get

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

and

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

Cor.

It can also be generalized in the case of more than two variables.

If $X = f(x, y, z)$ where $x = \phi_1(u, v, w)$, $y = \phi_2(u, v, w)$, $z = \phi_3(u, v, w)$ and u, v, w are independent variables ,then we have

$$\frac{\partial X}{\partial u} = \frac{\partial X}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial X}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial X}{\partial z}\frac{\partial z}{\partial u}$$

$$\frac{\partial X}{\partial v} = \frac{\partial X}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial X}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial X}{\partial z}\frac{\partial z}{\partial v}$$

$$\frac{\partial X}{\partial w} = \frac{\partial X}{\partial x}\frac{\partial x}{\partial w} + \frac{\partial X}{\partial y}\frac{\partial y}{\partial w} + \frac{\partial X}{\partial z}\frac{\partial z}{\partial w}$$

1.2.6 Euler's Theorem on Homogenous Functions of Three Independent Variables

Theorem

If $f(x, y, z)$ be a homogenous function in x, y, z of degree n having continuous partial derivatives then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = n\,u$

Proof

Given that $u = f(x, y, z)$ be homogenous function with degree n, then it can be written as

$$u = x^n f\left(\frac{y}{x}, \frac{z}{x}\right) = x^n f(v, w) \qquad \text{.........(1)}$$

Where $v = \frac{y}{x}$, $\quad w = \frac{z}{x}$

Also $\frac{\partial v}{\partial x} = \frac{-y}{x^2}$, $\quad \frac{\partial v}{\partial y} = \frac{1}{x}$, $\quad \frac{\partial v}{\partial z} = 0$

and $\frac{\partial w}{\partial x} = \left(\frac{-z}{x^2}\right)$, $\quad \frac{\partial w}{\partial y} = 0$, $\quad \frac{\partial w}{\partial z} = \frac{1}{x}$

Differentiating (1) partially with respect to 'x', we have

$$\frac{\partial u}{\partial x} = nx^{n-1} f(v, w) + x^n\left(\frac{\partial f}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial x}\right)$$

$$\frac{\partial u}{\partial x} = nx^{n-1} f(v, w) - x^{n-2}\left(y\frac{\partial f}{\partial v} + z\frac{\partial f}{\partial w}\right)$$

Again, partially differentiating (1) with respect to y and z to respectively, we have

$$\frac{\partial u}{\partial y} = x^n \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} = x^{n-1}\frac{\partial f}{\partial v} \quad \text{and} \quad \frac{\partial u}{\partial z} = x^n \frac{\partial f}{\partial w}\frac{\partial w}{\partial z} = x^{n-1}\frac{\partial f}{\partial w}$$

Now $\quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nx^n f(v, w) - x^{n-1}\left(y\frac{\partial f}{\partial v} + z\frac{\partial f}{\partial w}\right)$

$$+ x^{n-1} y\frac{\partial f}{\partial v} + zx^{n-1}\frac{\partial f}{\partial w}$$

$$= nx^n f\left(\frac{y}{x}, \frac{z}{x}\right) = nu$$

$$\therefore \quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu$$

Worked Out Examples

Ex. 1. Verify Euler's theorem for the function

$u = f(x, y) = ax^2 + 2hxy + by^2$

Solution:

Here $\quad u = f(x, y) = ax^2 + 2hxy + by^2 \quad \ldots(1)$

$$f(tx, ty) = a(tx)^2 + 2h(tx)(ty) + b(ty)^2$$
$$= at^2x^2 + 2ht^2xy + bt^2y^2$$
$$= t^2(ax^2 + 2hxy + by^2) = t^2 f(x, y)$$

Hence the function u is homogenous of degree two. We have to verify the Euler's theorem for $\quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u$

For this, partially differentiating with respect to x and y to (1) respectively, we get

$$\frac{\partial u}{\partial x} = 2ax + 2hy, \quad \frac{\partial u}{\partial y} = 2hx + 2by$$

We have

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = x(2ax + 2hy) + y(2hx + 2by)$$
$$= 2ax^2 + 2hxy + 2hxy + 2b^2y^2$$
$$= 2(ax^2 + 2hxy + by^2) = 2u$$
$$\therefore \quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u$$

Hence Euler's theorem is verified.

Ex.2: Verify Euler's theorem of the function $u = \frac{x^{1/4} + y^{1/4}}{x^{1/5} + y^{1/5}}$ [2063Asadh, B. E.]

Solution:

Here $\quad u(x, y) = \frac{x^{1/4} + y^{1/4}}{x^{1/5} + y^{1/5}} \quad \ldots(1)$

or $\quad u(tx, ty) = \frac{t^{1/4}x^{1/4} + t^{1/4}y^{1/4}}{t^{1/5}x^{1/5} + t^{1/5}y^{1/5}} = \frac{t^{1/4}(x^{1/4} + y^{1/4})}{t^{1/5}(x^{1/5} + y^{1/5})}$

$$= t^{1/20}\frac{x^{1/4} + y^{1/4}}{x^{1/5} + y^{1/5}} = t^{1/20} u(x,y).$$

So the function is homogenous of degree $\frac{1}{20}$. We have to verify that Euler's theorem for

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{20}u.$$

Differentiating (1) partially with respect to x ,y and z respectively

$$\frac{\partial u}{\partial x} = \frac{(x^{1/5} + y^{1/5})(1/4)x^{-3/4} - (x^{1/4} + y^{1/4})(1/5)x^{-4/5}}{(x^{1/5} + y^{1/5})^2}$$

and, $\quad \frac{\partial u}{\partial y} = \frac{(x^{1/5} + y^{1/5})(1/4)y^{-3/4} - (x^{1/4} + y^{1/4})(1/5)y^{-4/5}}{(x^{1/5} + y^{1/5})^2}.$

We have,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = x\frac{(x^{1/5} + y^{1/5})(1/4)x^{-3/4} - (x^{1/4} + y^{1/4})(1/5)x^{-4/5}}{(x^{1/5} + y^{1/5})^2}$$
$$+ y\frac{(x^{1/5} + y^{1/5})(1/4)y^{-3/4} - (x^{1/4} + y^{1/4})(1/5)y^{-4/5}}{(x^{1/5} + y^{1/5})^2}$$
$$= \frac{(x^{1/5} + y^{1/5})(1/4)x^{1/4} - (x^{1/4} + y^{1/4})(1/5)x^{1/5}}{(x^{1/5} + y^{1/5})^2}$$
$$+ \frac{(x^{1/5} + y^{1/5})(1/4)y^{1/4} - (x^{1/4} + y^{1/4})(1/5)y^{1/5}}{(x^{1/5} + y^{1/5})^2}$$
$$= \frac{(1/4)(x^{1/5} + y^{1/5})(x^{1/4} + y^{1/4}) - (1/5)(x^{1/4} + y^{1/4})(x^{1/5} + y^{1/5})}{(x^{1/5} + y^{1/5})^2}$$

$$= \frac{1}{20}\frac{(x^{1/5}+y^{1/5})(x^{1/4}+y^{1/4})}{(x^{1/5}+y^{1/5})^2} = \frac{1}{20}\frac{x^{1/4}+y^{1/4}}{x^{1/5}+y^{1/5}} = \frac{1}{20}u.$$

$\therefore\ x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = \frac{1}{20}u$. Hence Euler's theorem is verified.

Ex. 3. If $u = \sin^{-1}\frac{x^2+y^2}{x+y}$, prove that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = \tan u$

Solution:

Here $\quad u = \sin^{-1}\frac{x^2+y^2}{x+y}$

or $\quad \sin u = \frac{x^2+y^2}{x+y}$

or $\quad Z = \frac{x^2+y^2}{x+y}$ where $Z = \sin u$

or $\quad Z(x,y) = \frac{x^2+y^2}{x+y}$ where $Z = \sin u$

or $\quad Z(tx,ty) = \frac{t^2x^2+t^2y^2}{tx+ty} = \frac{t^2(x^2+y^2)}{t(x+y)}$

$$= t^1\frac{x^2+y^2}{x+y} = t\,f(x,y)$$

Thus Z is a homogenous function of degree one. Hence by the Euler's theorem

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y} = 1.Z$$

or $\quad x\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+y\frac{\partial z}{\partial u}\frac{\partial u}{\partial y} = Z$

or $\quad x\cos u\frac{\partial u}{\partial x}+y\cos u\frac{\partial u}{\partial y} = \sin u$

$\therefore\quad x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = \tan u$

Ex.4: If $u = \log\frac{x^2+y^2}{x+y}$, show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = 1$ *2060/070 Bhadra B. E.*

Solution:

Here $\quad u = \log\frac{x^2+y^2}{x+y}$.

$\Rightarrow\quad e^u = \frac{x^2+y^2}{x+y}$

l
S10
SHR
2016
(159451)

$\Rightarrow\quad Z = \frac{x^2+y^2}{x+y}$ ...(1)

Where $\quad Z = e^u,\quad \frac{\partial Z}{\partial u} = e^u$

$Z(x,y) = \frac{x^2+y^2}{x+y}$

or $\quad Z(tx,ty) = \frac{t^2x^2+t^2y^2}{tx+ty} = t\frac{x^2+y^2}{x+y} = t^1 Z(x,y).$

So the function is homogenous of degree one. By using Euler's theorem

$$x\frac{\partial Z}{\partial x}+y\frac{\partial Z}{\partial y} = Z$$

or $\quad x\frac{\partial Z}{\partial u}.\frac{\partial u}{\partial x}+y\frac{\partial Z}{\partial u}.\frac{\partial u}{\partial y} = Z$

or $\quad xe^u\frac{\partial u}{\partial x}+ye^u\frac{\partial u}{\partial y} = e^u.$

$\therefore\quad x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = 1.$Proved.

Ex. 5: Find $\frac{du}{dt}$ if $u = \sin\left(\frac{x}{y}\right)$, $x = e^t$, $y = t^2$ *2070 Magh B. E.*

Solution:

Here $\quad u = \sin\left(\frac{x}{y}\right)$ ...(1)

and $\quad x = e^t,\quad y = t^2$ ....(2)

Differentiating (1) partially with respect to x and y respectively, we have

$$\frac{\partial u}{\partial x} = \frac{1}{y}\cos\left(\frac{x}{y}\right)\qquad \frac{\partial u}{\partial y} = -\frac{x}{y^2}\cos\left(\frac{x}{y}\right)$$

Differentiating (2) with respect to t, we get

$$\frac{dx}{dt} = e^t,\qquad \frac{dy}{dt} = 2t$$

We have

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} = \frac{1}{y}\cos\left(\frac{x}{y}\right)e^t + \left(\frac{-x}{y^2}\right)\cos\left(\frac{x}{y}\right)2t$$

$$= \frac{e^t}{t^2}\cos\left(\frac{e^t}{t^2}\right) - \frac{2e^t}{t^3}\cos\left(\frac{e^t}{t^2}\right) = \frac{(t-2)}{t^3}e^t\cos\left(\frac{e^t}{t^2}\right)$$

$$\therefore \quad \frac{du}{dt} = \frac{(t-2)}{t^3}e^t\cos\left(\frac{e^t}{t^2}\right)$$

Ex.6: Find the total derivative $\frac{du}{dx}$ when $u = x\log(xy)$ where $x^3 + y^3 + 3xy = 1$

Solution:

Here $\quad f(x, y) = x^3 + y^3 + 3xy - 1 = 0$

Differentiating it partially with respect to x and y respectively ,we get

$$f_x = 3x^2 + 3y, \quad f_y = 3y^2 + 3x$$

$$\frac{dy}{dx} = \frac{-f_x}{f_y} = -\frac{3(x^2+y)}{3(y^2+x)} = -\frac{x^2+y}{x+y^2}$$

Also $\quad u = x\log(xy)$

Differentiating it partially with respect to x and y respectively, we have

$$\frac{\partial u}{\partial x} = 1 + \log(xy) \text{ and } \frac{\partial u}{\partial y} = \frac{x}{y}$$

We have $\quad \frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx} = 1 + \log(xy) + \frac{x}{y}\cdot\left(-\frac{x^2+y}{x+y^2}\right)$

$$= 1 + \log xy - \frac{x(x^2+y)}{y(x+y^2)}$$

Ex. 7 : Find $\frac{dy}{dx}$ by using partial derivatives if $x^y = y^x$

Solution:

Let $\quad f(x, y) = x^y - y^x = 0$

Differentiating it partially with respect to x and y respectively, we have

$$f_x = y\,x^{y-1} - y^x\log y$$

$\frac{y}{x}\,\frac{y - x\log y}{x - y\log x}$

and $\quad f_y = x^y\log x - xy^{x-1}$

Now

$$\frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{yx^{y-1} - y^x\log y}{x^y\log x - xy^{x-1}} = \frac{y}{x}\frac{(y - x\log x)}{(x - y\log y)} \quad (\because x^y = y^x)$$

Ex. 8: If $z = f(x, y)$ and $x = e^u + e^{-v}$, $y = e^{-u} - e^v$, prove that

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$$

Solution:

Here $\quad x = e^u + e^{-v}, \quad y = e^{-u} - e^v$

Differentiating these partially with respect to u and v respectively

$$\frac{\partial x}{\partial u} = e^u, \qquad \frac{\partial y}{\partial u} = -e^{-u}$$

and $\quad \frac{\partial x}{\partial v} = -e^{-v}, \qquad \frac{\partial y}{\partial v} = -e^v$

Here. $\quad z = f(x, y)$

We have $\quad \frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} = e^u\frac{\partial z}{\partial x} - e^{-u}\frac{\partial z}{\partial y}$

and $\quad \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} = -e^{-v}\frac{\partial z}{\partial x} - e^v\frac{\partial z}{\partial y}$

Now

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = e^u\frac{\partial z}{\partial x} - e^{-u}\frac{\partial z}{\partial y} - \left(-e^{-v}\frac{\partial z}{\partial x} - e^v\frac{\partial z}{\partial y}\right)$$

$$= e^u\frac{\partial z}{\partial x} - e^{-u}\frac{\partial z}{\partial y} + e^{-v}\frac{\partial z}{\partial x} + e^v\frac{\partial z}{\partial y}$$

$$= (e^u + e^{-v})\frac{\partial z}{\partial x} - (e^{-u} - e^v)\frac{\partial z}{\partial y}$$

$$= u\frac{\partial z}{\partial x} - v\frac{\partial z}{\partial y}$$

$$\therefore \quad \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = u\frac{\partial z}{\partial x} - v\frac{\partial z}{\partial y}$$

Exercise - 2

1. Verify Euler's theorem for the following functions.

(i) $u = x f\left(\frac{y}{x}\right)$

(ii) $u = \sin^{-1}\frac{x}{y} + \tan^{-1}\frac{y}{x}$ **2063 Kartik, B. E.**

(iii) $u = x^n \tan^{-1}\left(\frac{y}{x}\right)$

(iv) $u = x^n \sin\left(\frac{y}{x}\right)$ **2071 Magh, B. E.**

(v) $u = \frac{x^2 y^2}{x^3 + y^3}$ **2061 Aswin B. E.**

(vi) $u = \frac{x^3 y}{x^2 + y^2}$

2. (i) If $u = \sin^{-1}\frac{x^2y^2}{x+y}$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3\tan u$

(ii) If $u = \log(x^2y)$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3$

(iii) If $\sin u = \frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}$, then show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$ **2064/069 Bhadra, B.E**

(iv) If $u = \tan^{-1}\frac{x^3+y^3}{x-y}$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$

(v) If $u = \operatorname{cosec}^{-1}\left(\frac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}\right)$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -\frac{1}{6}\tan u$

(vi) If $u = \cos^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, then show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -\frac{1}{2}\cot u.$$ **2065 Chaitra, B. E**

(vii). If $u = \log(x^2+y^2+z^2)$, then prove that

$$x\frac{\partial^2 u}{\partial y\,\partial z} = y\frac{\partial^2 u}{\partial z\,\partial x}$$ **2062 Jestha B. E.**

3. Find $\frac{dy}{dx}$ in the following functions

(i) $x^{2/3} + y^{2/3} = 4$

(iii) $2x^3 + x^2y + y^3 = 1$ (iv) $(\tan x)^y + (y)^{\tan x} = 0$

(v) $x^3 + y^3 = 3axy$ (vi) $x^py^q = (x+y)^{p+q}$ **2055 B. E.**

(vi) $x^y + y^x = a^b$ (vii) $e^x + e^y = 2xy$

4. Find $\frac{dz}{dt}$ in the following functions

(i) $z = x^3 - y^3$, $x = \frac{1}{t+1}$, $y = \frac{t}{t+1}$

(ii) $z = \log(x+y)$, $x = e^{-2t}$, $y = t^3 - t^2$

(iii) $z = \sin^{-1}(x-y)$, $x = 3t$, $y = 4t^3$

5. Find $\frac{dz}{dx}$ in the following functions

(i) $z = x\cos y + y\sin x$, $y = x^2 + 1$

(ii) $z = (y+x)e^{xy}$, $y = \frac{1}{x^2}$

6. If $u = f(x, y) = x^2 - 2xy + 3y$, find the differential du and the change Δu in u if f(x, y) changes from (1, 2) to (1.03, 1.99).

7. If $u = f(x, y) = x^3 + y^3$ find the differential du and change Δu in u if f(x, y) changes from (10, 10) to (10.1, 10.1).

8. If $u = f(x, y) = x^2 - 3x^3y^2 + 4x - 2y^3 + 6$. Find the differential du and change Δu in u if f(x, y) changes from (-2, 3) to (-2.02, 3.01)

9. The radius and altitude of a rigid circular cylinder are measured as 3 inches and 8 inches respectively with possible error in measurement of 0.05 inches. Find the maximum error in the calculated volume of the cylinder.

10. If the dimensions in inches of a rectangular box change from 9, 6, and 4 to 9.02, 5.97 and 4.01 respectively. Find the change in volume.

Answers

3. (i) $-\left(\frac{y}{x}\right)^{1/3}$

(ii) $-\left(\frac{2xy+6x^2}{x^2+3y^2}\right)$ (iii) $-\frac{\sec^2 x\,[y(\tan x)^{y-1}+\log y.\, y^{\tan x}]}{(\tan x)^y \log(\tan x)+\tan x\, y^{\tan x-1}}$

(iv) $\frac{x^2-ay}{ax-y^2}$ (v) $\frac{y}{x}$

(vi) $-\frac{yx^{y-1}+y^x\log x}{xy^{x-1}+x^y\log x}$ (vii) $\frac{e^x-2y}{2x-e^y}$

4. (i) $\frac{-3(t^2+1)}{(t+1)^4}$ (ii) $\frac{3t^2-2t-2e^{-2t}}{t^3-t^2+e^{-2t}}$

(iii) $\frac{3}{(1-t^2)^{1/2}}$

5. (i) $\cos(x^2+1)+(x^2+1)\cos x-2x^2\sin(x^2+1)+2x\sin x$

(ii) $e^{\frac{1}{x}}\left[1-\frac{1}{x}-\frac{2}{x^3}-\frac{1}{x^4}\right]$

6. $-0.070,\ -0.039$ 7. $600,\ 60.602.$

-0.069

8. $7.38,\ 7.490$ 9. 8.95 inch^3

10. -0.063906

1.3 Extrema of Functions of Two or Three Variables

We have already discussed about the theory of extreme values for functions of one variable. We now investigate the theory of functions of two or three variables. A maximum or minimum value of a function of two or three independent variables is known as extreme values of the function. So it is concerned with evaluation of maximum and minimum value of the function.

A function $f(x, y)$ is said to have a *maximum* or *minimum* at

$x = a,\ y = b$ according as

$$f(a+h,\ b+k) < f(a,b) \quad \text{or} \quad f(a+h,\ b+k) > f(a,b)$$

for all positive or negative small values of h and k.

1.3.1 Conditions for f(x, y) to be Maximum or Minimum

Theorem

The necessary conditions for $f(x, y)$ to have an extreme value at (a, b) is that $f_x(a, b) = 0$, $f_y(a, b) = 0$ provided these partial derivatives exist

If $f(a, b)$ is an extreme value of the function $f(x, y)$ of two variables, then it must also be an extreme value of both functions $f(x, b)$ and $f(a, y)$ of one variable and the necessary condition for that $f(x, b)$ and $f(a, y)$ have extreme values at $x = a$ and $y = b$ respectively is

$$f_x(a,b) = 0, \quad f_y(a,b) = 0$$

If these conditions are satisfied, then for small value of h and k, by Taylor 's theorem, we have

$f(a+h, b+k) - f(a, b)$

$$= \left(h\frac{\partial f}{\partial x}+k\frac{\partial f}{\partial y}\right)_{at(a,\,b)} + \frac{1}{2!}\left(h^2\frac{\partial^2 f}{\partial x^2}+2hk\frac{\partial^2 f}{\partial x\partial y}+k^2\frac{\partial^2 f}{\partial y^2}\right)+\ldots \text{gives}$$

$$= \frac{1}{2!}(h^2r+2hks+k^2t) \text{ where } r = f_{xx}(a,b),\ s = f_{xy}(a,b) \text{ and } t = f_{yy}(a,b)$$

$$= \frac{1}{2r}\left[(hr+ks)^2+k^2(rt-s^2)\right]$$

Since $(hr+ks)^2$ is always positive and $k^2(rt - s^2)$ will be positive if $rt - s^2 > 0$. Hence if $rt - s^2 > 0$, then $f(x, y)$ has a *maximum* or a *minimum* at (a, b) according as $r < 0$ or $r > 0$

If $rt - s^2 < 0$, then $f(x, y)$ has no maximum or minimum at (a, b) ,then this point is called *saddle point.*

If $rt - s^2 = 0$, then the further investigation is necessary.

Stationary Value

The value f(a, b) is called the *stationary value* of the function f(x, y)

if $f_x(a, b) = 0$ and $f_y(a, b) = 0$

i.e. the function is stationary at (a, b)

Thus, the stationary points are obtained by solving $f_x = 0$ and $f_y = 0$.

Thus, every extreme value is a stationary value but the converse may not be true. Stationary points that are not extreme points are called the *saddle points*.

1.3.2 Working Rule to Find the Maximum and Minimum Values of f(x, y)

(i) Find $\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}, \dfrac{\partial^2 f}{\partial x^2}, \dfrac{\partial^2 f}{\partial x \partial y}, \dfrac{\partial^2 f}{\partial y^2}$

(ii) Solve $\dfrac{\partial f}{\partial x} = 0$ and $\dfrac{\partial f}{\partial y} = 0$ simultaneously in x and y.

Let (a,b) ; (c ,d) ..... be the solutions of these equations

(iii) For each solution in step (ii) find $r = \dfrac{\partial^2 f}{\partial x^2}$, $s = \dfrac{\partial^2 f}{\partial x \partial y}$, $t = \dfrac{\partial^2 f}{\partial y^2}$

(iv) If $rt - s^2 > 0$ and

(a) $r < 0$, then f(x, y) has a maximum value

(b) $r > 0$, then f(x,y) has a minimum value

(v) If $rt - s^2 < 0$, f(x,y) has no extreme value at (a, b)

(vi) If $rt - s^2 = 0$, the case is doubtful and needs for further investigation.

The function f(x, y) of two variables x and y is said to be extreme value f(a, b) if $f_x(a, b) = 0$, $f_y(a, b) = 0$ and $f_{xx}(a, b) \cdot f_{yy}(a, b) - [f_{xy}(a, b)]^2 > 0$ and this extreme value is maximum or minimum according as

$f_{xx}(a, b) < 0$ or $\quad f_{xx}(a, b) > 0$.

Further investigation is necessary if

$f_{xx}(a, b)\, f_{yy}(a, b) - [f_{xy}(a, b)]^2 = 0$.

Definition

A function f(x, y, z) is said to have a *maximum* or *minimum* at x = a, y = b, z = c according as

$f(a + h, b + k, c + l) < f(a, b, c)$

or $f(a + h, b + k, c + l) > f(a, b, c)$ for all positive or negative small values of h, k and *l*.

In other words, to difference $f(a + h, b + k, c + l) - f(a, b, c)$ is of the same sign for all small values of h, k and *l*.

If the sign of difference is negative then f(a, b, c) is a maximum and if this sign of difference is positive f(a, b, c) is a minimum.

1.3.3 Necessary and Sufficient Conditions for a Function f(x, y, z) to have Extreme Value

Theorem

The necessary condition for f(a, b, c) to be minimum or maximum value of the function f are that all the partial derivatives f_x, f_y, f_z exist and vanish at (a, b, c)

A point (a, b, c) is called a *Stationary point* if all the first order partial derivatives of the function vanish at that point. Thus, the stationary points are determined by solving the following three equations simultaneously.

$f_x(x, y, z) = 0, \quad f_y(x, y, z) = 0, \quad f_z(x, y, z) = 0$

The necessary sufficient conditions for a function f(x, y, z) of three independent variables to have *Maximum* value at (a, b, c) are that

$f_x = 0, \quad f_y = 0, \quad f_z = 0$

$$f_{xx} < 0, \quad \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} > 0, \quad \begin{vmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{vmatrix} < 0$$

and *Minimum* value at (a, b, c) are that

$f_x = 0, \quad f_y = 0, \quad f_z = 0$

$$f_{xx} > 0, \quad \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} > 0, \quad \begin{vmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{vmatrix} > 0.$$

1.3.4 Stationary Values Under the Given Conditions

Sometimes it is required to find the stationary values of a function of two and three independent variables connected by some given conditions. With the help of given conditions, the function can reduce to a function of a single variable x or two variables x and y.

1. Stationary points of the function f(x, y) under the condition $\phi(x, y) = 0$

Given function is $\quad F = f(x, y) \quad$ ....(1)

under the condition $\quad \phi(x, y) = 0 \quad$ ....(2)

The function F of (1) reduce to a function of a single variable x with the help of (2)

To obtain stationary points, put $\dfrac{dF(x)}{dx} = 0$ which gives the value of x. After substituting the value of x to (2), we get y and thus will be obtain the stationary points.

2. Stationary points of the function f(x, y, z) under the condition $\phi(x, y, z) = 0$

Given function is $F = f(x, y, z)$ ....(1)

under the condition $\phi(x, y, z) = 0$ ....(2)

The function F of (i) reduce to a function of two variables x and y with the help of (2)

To obtain stationary point, put

$$F_x(x, y) = 0,\ F_y(x, y) = 0$$

Solving these we get the value of x, and y. After substituting the values of x and y to (2) we get z and thus will be obtained stationary points.

3. Stationary points of the function f(x, y, z) under the conditions $\phi_1(x, y, z) = 0$, and $\phi_2(x, y, z) = 0$.

Given function is $F = f(x, y, z)$ ....(1)

under the conditions $\phi_1(x, y, z) = 0$ ....(2)

and $\phi_2(x, y, z) = 0$ .....(3)

The function F of (i) reduces to a single variable x with the help of (2) and (3)

To obtain stationary point, put $\dfrac{dF(x)}{dx} = 0$ and solving this, we get the value of x. After substituting the value of x to (ii) and (iii), we get the value of x and y and thus will be obtained the stationary point.

1.3.5 Lagrange's Method of Undermined Multipliers

When the above method becomes impracticable, Lagrange's method proves very convenient. To find the stationary points, we use Lagrange's method of determined multipliers. We explain this method as follows.

1. To find the stationary point of a function f(x, y) under the condition $\phi(x, y) = 0$

A function f(x, y) of two variables x, y which are connected by the relation

$$\phi(x, y) = 0 \qquad ....(1)$$

For the function f(x, y) to have stationary values, it is necessary that $\dfrac{\partial f}{\partial x} = 0$

and $\dfrac{\partial f}{\partial y} = 0$

$$\therefore \quad df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0 \qquad ....(2)$$

Also differentiating (1), we get

$$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy = 0 \qquad ....(3)$$

Multiplying (3) by a parameter λ and adding to (2), we have

$$\left(\frac{\partial f}{\partial x} + \lambda\frac{\partial \phi}{\partial x}\right)dx + \left(\frac{\partial f}{\partial y} + \lambda\frac{\partial \phi}{\partial y}\right)dy = 0$$

This equation will be satisfied if

$$\frac{\partial f}{\partial x} + \lambda\frac{\partial \phi}{\partial x} = 0, \quad \frac{\partial f}{\partial y} + \lambda\frac{\partial \phi}{\partial y} = 0$$

These equations together with $\phi(x, y) = 0$ will determine the values of x, y and λ for which f is stationary.

The function f(x, y) has *Maximum* value at the stationary point if

$$\begin{vmatrix} 0 & \phi_x & \phi_y \\ \phi_x & f_{xx} & f_{xy} \\ \phi_y & f_{yx} & f_{yy} \end{vmatrix} > 0$$

and *Minimum* value if

$$\begin{vmatrix} 0 & \phi_x & \phi_y \\ \phi_x & f_{xx} & f_{xy} \\ \phi_y & f_{yx} & f_{yy} \end{vmatrix} < 0$$

2. To find the stationary point of a function f(x, y, z) under the condition $\phi(x, y, z) = 0$

A function f(x, y, z) of three variables x, y and z which are connected by the relation $\phi(x, y, z) = 0$ .....(1)

For the function f(x, y, z) to have stationary values, it is necessary that

$$\frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = 0, \quad \frac{\partial f}{\partial z} = 0$$

$$\therefore \quad \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = 0 \qquad ....(2)$$

Also differentiating (1), we get

$$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz = 0 \qquad ......(3)$$

Multiply (3) by a parameter λ and add to (2), then

$$\left(\frac{\partial f}{\partial x} + \lambda\frac{\partial \phi}{\partial x}\right)dx + \left(\frac{\partial f}{\partial y} + \lambda\frac{\partial \phi}{\partial y}\right)dy + \left(\frac{\partial f}{\partial z} + \lambda\frac{\partial \phi}{\partial z}\right)dz = 0$$

This equation will be satisfied if

$$\frac{\partial f}{\partial x}+\lambda\frac{\partial \phi}{\partial x}=0,\quad \frac{\partial f}{\partial y}+\lambda\frac{\partial \phi}{\partial y}=0,\quad \frac{\partial f}{\partial z}+\lambda\frac{\partial \phi}{\partial z}=0$$

These three equations together with (i) will determine the values of x, y, z and λ for which f is stationary.

Working rule:

(i) Write $F = f(x, y, z) + \lambda\, \phi(x, y, z)$

(ii) Obtain the equations $\frac{\partial F}{\partial x}=0,\ \frac{\partial F}{\partial y}=0,\ \frac{\partial F}{\partial z}=0$

(iii) Solve these equations together with $\phi(x, y, z) = 0$.

The value of x, y, z so obtained will give the stationary value of f(x, y, z).

The function f(x, y, z) has *maximum value* at the stationary point if

$$\begin{vmatrix} 0 & \phi_x & \phi_y \\ \phi_x & f_{xx} & f_{xy} \\ \phi_y & f_{yx} & f_{yy} \end{vmatrix} > 0,\qquad \begin{vmatrix} 0 & \phi_x & \phi_y & \phi_z \\ \phi_x & f_{xx} & f_{xy} & f_{xz} \\ \phi_y & f_{yx} & f_{yy} & f_{yz} \\ \phi_z & f_{zx} & f_{zy} & f_{zz} \end{vmatrix} < 0$$

and *minimum value* at the stationary points

$$\text{if}\quad \begin{vmatrix} 0 & \phi_x & \phi_y \\ \phi_x & f_{xx} & f_{xy} \\ \phi_y & f_{yx} & f_{yy} \end{vmatrix} < 0,\qquad \begin{vmatrix} 0 & \phi_x & \phi_y & \phi_z \\ \phi_x & f_{xx} & f_{xy} & f_{xz} \\ \phi_y & f_{yx} & f_{yy} & f_{yz} \\ \phi_z & f_{zx} & f_{zy} & f_{zz} \end{vmatrix} < 0$$

3. To find the stationary point of a function f(x, y, z) under the two conditions $\phi_1(x, y, z) = 0$, $\phi_2(x, y, z) = 0$

A function f(x, y, z) of three variables x, y, z which are connected by the relations

$$\phi_1(x, y, z) = 0 \qquad \text{...(1}$$

and

$$\phi_2(x, y, z) = 0 \qquad \text{..(2)}$$

For the function f(x, y, z) to have stationary values, it is necessary that

$$\frac{\partial f}{\partial x}=0,\quad \frac{\partial f}{\partial y}=0,\quad \frac{\partial f}{\partial z}=0$$

$$\therefore\quad df=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy+\frac{\partial f}{\partial z}dz=0 \qquad \text{....(3)}$$

Also differentiating (1) and (2) respectively, we get

$$d\phi_1=\frac{\partial \phi_1}{\partial x}dx+\frac{\partial \phi_1}{\partial y}dy+\frac{\partial \phi_1}{\partial z}dz=0 \qquad \text{.......(4)}$$

$$d\phi_2=\frac{\partial \phi_2}{\partial x}dx+\frac{\partial \phi_2}{\partial y}dy+\frac{\partial \phi_2}{\partial z}dz=0 \qquad \text{........(5)}$$

Multiply (4) by a parameter λ_1 and (5) by another parameter λ_2 and add to (3), then

$$\left(\frac{\partial f}{\partial x}+\lambda_1\frac{\partial \phi_1}{\partial x}+\lambda_2\frac{\partial \phi_2}{\partial x}\right)dx+\left(\frac{\partial f}{\partial y}+\lambda_1\frac{\partial \phi_1}{\partial y}+\lambda_2\frac{\partial \phi_2}{\partial y}\right)dy+\left(\frac{\partial f}{\partial z}+\lambda_1\frac{\partial \phi_1}{\partial z}+\lambda_2\frac{\partial \phi_2}{\partial z}\right)dz=0$$

This equation will be satisfied if $\frac{\partial f}{\partial x}+\lambda_1\frac{\partial \phi_1}{\partial x}+\lambda_2\frac{\partial \phi_2}{\partial x}=0$

$$\frac{\partial f}{\partial y}+\lambda_1\frac{\partial \phi_1}{\partial y}+\lambda_2\frac{\partial \phi_2}{\partial y}=0,\quad \frac{\partial f}{\partial z}+\lambda_1\frac{\partial \phi_1}{\partial z}+\lambda_2\frac{\partial \phi_2}{\partial z}=0$$

These three equations together with (1) and (2), $\phi(x, y, z) = 0$ will determine the values of x, y, z and λ for which f is stationary.

Working rule:

(i) Write $F = f(x, y, z) + \lambda_1\, \phi_1(x, y, z) + \lambda_2\, \phi_2(x, y, z)$

(ii) Obtain the equations $\frac{\partial F}{\partial x}=0,\ \frac{\partial F}{\partial y}=0,\ \frac{\partial F}{\partial z}=0$

(iii) Solve these equations together with

$$\phi_1(x, y, z) = 0,\ \phi_2(x, y, z) = 0.$$

The value of x, y, z so obtained will give the stationary value of f(x, y, z).

Worked Out Examples

Ex. 1: Examine the maxima and minima of the function

$$\mathbf{f(x, y) = x^3 + y^3 - 3x - 12y + 20}$$

Solution:

Here the function is

$$f(x, y) = x^3 + y^3 - 3x - 12y + 20.$$

Differentiating it partially with respect to x ,y and z respectively, we get

$$f_x = 3x^2 - 3,\quad f_y = 3y^2 - 12$$

For the stationary points $f_x = 0$, $f_y = 0$

$$x = \pm 1 \text{ and } y = \pm 2$$

Thus, the function has four stationary points

$(1, 2), (-1, 2), (1, -2), (-1, -2)$

Now $\quad f_{xx} = 6x, \quad f_{xy} = 0, \quad f_{yy} = 6y$

At (1, 2) $\quad f_{xx} = 6 >$ and $f_{xx} f_{yy} - (f_{xy})^2 = 72 > 0$.

Hence (1, 2) is a point of minima of the function .

At (- 1, 2),

$$f_{xx} = -6 < 0 \text{ and } f_{xx} f_{yy} - (f_{xy})^2 = -72 < 0$$

Hence the function has neither maximum nor minimum at (- 1, 2).

At (1, - 2),

$$f_{xx} = 6 > 0 \text{ and } f_{xx} f_{yy} - (f_{xy})^2 = -72 < 0$$

Hence the function has neither maximum nor minimum at (1, - 2).

At (- 1, -2),

$$f_{xx} = -6 < 0 \text{ and } f_{xx}f_{yy} - (f_{xy})^2 = 72 > 0$$

Hence the function has a maximum value at (- 1, -2).

The points (- 1, 2), (1, - 2) are saddle points.

Min. value = $f(1, 2) = 1 + 8 - 3 - 24 + 20 = 2$

Max. value = $f(-1, -2) = -1 - 8 + 3 + 24 + 20 = 38$.

Ex. 2:Find the maximum and minimum values of

(i) $x^3 - x^2 - y^2 + xy$ $\qquad$ **(ii)** $xy + \dfrac{a^3}{x} + \dfrac{a^3}{y}$

(iii) $x^3y^2(1 - x - y),\ x \neq 0,\ y \neq 0,\ x + y \neq 1$

(iv) $(x + y + z)^3 - 3(x + y + z) - 24\,xyz + 1$

(i) Solution:

Let $\quad f(x, y) = x^3 - x^2 - y^2 + xy$, then

$f_x = 3x^2 - 2x + y, \quad f_y = -2y + x$

$f_{xx} = 6x - 2, \quad f_{yx} = 1 = f_{xy}, \quad f_{yy} = -2$

For the extreme values,

$f_x = 0 \qquad f_y = 0$

$3x^2 - 2x + y = 0, \quad -2y + x = 0$

Solving these $\quad x = 0, \dfrac{1}{2}; \quad y = 0, \dfrac{1}{4}$.

Therefore the stationary points are (0, 0) and $\left(\dfrac{1}{2}, \dfrac{1}{4}\right)$.

At (0, 0),

$$f_{xx} = -2 < 0, \qquad f_{xx} f_{yy} - f_{xy}{}^2 = (-2)\times(-2) - 1 = 3 > 0$$

Hence f(x, y) has maximum value at (0, 0).

Max value = $f(0, 0) = 0 - 0 + 0 + 0 = 0$.

At $\left(\dfrac{1}{2}, \dfrac{1}{4}\right)$,

$$f_{xx} = 1 > 0, \qquad f_{xx} f_{yy} - f_{xy}{}^2 = -3 < 0.$$

Hence f has neither maximum nor minimum.

So the points $\left(\dfrac{1}{2}, \dfrac{1}{4}\right)$ is saddle point.

(ii)Solution:

Let $\quad f(x, y) = xy + \dfrac{a^3}{x} + \dfrac{a^3}{y}$

$f_x = y - \dfrac{a^3}{x^2}, \quad f_y = x - \dfrac{a^3}{y^2}, \quad f_{xx} = \dfrac{2a^3}{x^3}, f_{xy} = 1, \quad f_{yy} = \dfrac{2a^3}{y^3}$

For the extreme values, $\quad f_x = 0, \quad f_y = 0$

$$y - \frac{a^3}{x^2} = 0, \qquad x - \frac{a^3}{y^2} = 0$$

Solving these $\quad x^2y - a^3 = 0, \quad xy^2 - a^3 = 0$

$x = 0,\ y = 0$ and $x = a,\ y = a$.

Therefore, the stationary points are (0, 0) and (a, a).

At (a, a) $\quad f_{xx} = 2 > 0, \qquad f_{xx} f_{yy} - f_{xy}{}^2 = 3 > 0$.

Hence f(x, y) has minimum value at (a, a).

Min. value = $f(a, a) = a^2 + a^2 + a^2 = 3a^2$.

(iii)Solution:

Let $\quad f(x, y) = x^3y^2(1 - x - y), \quad x \neq 0, y \neq 0, x + y \neq 1$

then $\quad f_x = 3x^2y^2 - 4x^3y^2 - 3x^2y^3, \qquad f_y = 2x^3y - 2x^4y - 3x^3y^2$

$f_{yx} = 6x^2y - 8x^3y - 9x^2y^2 = f_{xy}, \quad f_{yy} = 2x^3 - 2x^4 - 6x^3y$

$f_{xx} = 6x^3y^2 - 12x^2y^2 - 6xy^3$

For the extreme values, $f_x = 0, f_y = 0$

$3x^2y^2 - 4x^3y^2 - 3x^2y^3 = 0$

and $\quad 2x^3y - 2x^4y - 3x^3y^2 = 0$

Since $x \neq 0, y \neq 0$,

$3 - 4x - 3y = 0$ and $2 - 2x - 3y = 0$

Solving these $x = \dfrac{1}{2}, y = \dfrac{1}{3}$.

Therefore the stationary point is $\left(\dfrac{1}{2}, \dfrac{1}{3}\right)$.

At $\left(\dfrac{1}{2}, \dfrac{1}{3}\right)$ $f_x = \dfrac{-17}{36} < 0, \ f_{xx} f_{yy} - f_{xy}{}^2 = \dfrac{15}{288} > 0$

Hence the function f(x, y) has maximum value at $\left(\dfrac{1}{2}, \dfrac{1}{3}\right)$.

Max. value = $f\left(\dfrac{1}{2}, \dfrac{1}{3}\right) = \dfrac{1}{72}\left(1 - \dfrac{1}{2} - \dfrac{1}{3}\right) = \dfrac{1}{432}$.

(iv) Solution:

Let $f(x,y) = (x + y + z)^3 - 3(x + y + z) - 24xyz + 1$

then $f_x = 3(x + y + z)^2 - 3 - 24yz$

$f_y = 3(x + y + z)^2 - 3 - 24xz$, $f_z = 3(x + y + z)^2 - 3 - 24xy$

$f_{xx} = 6(x + y + z)$, $f_{yx} = 6(x + y + z) - 24z$

$f_{yy} = 6(x + y + z)$, $f_{xz} = 6(x + y + z) - 24y$

$f_{zz} = 6(x + y + z)$, $f_{yz} = 6(x + y + z) - 24x$

For the extreme values, $f_x = 0$, $f_y = 0$, $f_z = 0$.

$(x + y + z)^2 - 1 - 8yz = 0$

$(x + y + z)^2 - 1 - 8zx = 0$

and $(x + y + z)^2 - 1 - 8xy = 0$

Solving these $x = y = z$, $x = 1, -1$; $y = 1, -1$; $z = 1, -1$

So the stationary points are (1, 1, 1) (- 1, - 1, - 1)

At (1, 1, 1), $f_{xx} = 18 > 0$

$$\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 18 & -6 \\ -6 & 18 \end{vmatrix} = 18 > 0$$

and
$$\begin{vmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yz} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{vmatrix} = \begin{vmatrix} 18 & -6 & -6 \\ -6 & 18 & -6 \\ -6 & -6 & 18 \end{vmatrix} = 3456 > 0.$$

Hence the function f(x, y) has minimum value at (1, 1, 1).

Min. value = $(1 +1+1)^3 -3(1 +1+1) -24 +1 = 27 -9 -24 +1 = -5$

At (- 1, - 1, - 1),

$$f_{xx} = -18 < 0, \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 18 & -6 \\ -6 & 18 \end{vmatrix} = 288 > 0$$

and
$$\begin{vmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{vmatrix} = \begin{vmatrix} -18 & 6 & 6 \\ 6 & -18 & 6 \\ 6 & 6 & -18 \end{vmatrix} = -3456 < 0$$

Hence, the function f(x, y) has maximum value at (-1,- 1,- 1)

Max. value = $(-1 -1-1)^3 -3(-1 -1-1) + 24 +1 = -27 +9 +24 +1 = 7$

Ex. 3: Show that $f(x, y) = y^2 + x^2y + x^4$ has a minimum at (0, 0)

Solution:

Given $f(x, y) = y^2 + x^2y + x^4$

$f_x = 2xy + 4x^3$, $f_y = 2y + x^2$

$f_{xx} = 2y + 12x^2$, $f_{yy} = 2$, $f_{xy} = 2x$

It can be easily verified that at (0, 0)

$f_x = 0$, $f_y = 0$, $f_{xx} = 0$, $f_{yy} = 2$, $f_{xy} = 0$

Thus at the origin $f_{xx} f_{yy} - (f_{xy})^2 = 0$

It is doubtful case and requires further investigation. The given function can be written as

$$f(x, y) = \left(y + \frac{x^2}{2}\right)^2 + \frac{3x^4}{4}$$

and $f(x, y) - f(0, 0) = \left(y + \frac{x^2}{2}\right)^2 + \frac{3x^4}{4}$ which is greater than zero for all values of x and y. Hence, f(x, y) is minimum at (0, 0).

Ex.4: Find the minimum value of $x^2 + y^2 + z^2$ connected by the relation $ax + by + cz = p$

2063/070 Bhadra, B. E.

Solution:

Let $f(x, y, z) = x^2 + y^2 + z^2$ ....(1)

which are connected by the relation

$ax + by + cz = p$

i.e. $z = \frac{(p - ax - by)}{c}$ ....(2)

With the help of (2), the function (1) reduces to

$$f = x^2 + y^2 + \frac{1}{c^2}(p - ax - by)^2$$

We have, $f_x = 2x - \frac{2a}{c^2}(p - ax - by)$, $f_{xx} = 2 + \frac{2a^2}{c^2}$

$f_y = 2y - \frac{2b}{c^2}(p - ax - by)$, $f_{xy} = \frac{2ab}{c^2}$, $f_{yy} = 2 + \frac{2b^2}{c^2}$

For the extreme values, $f_x = 0$, $f_y = 0$

$(a^2 + c^2)x + aby - ap = 0$

and $abx + (b^2 + c^2)y - bp = 0$

Solving these two equations, we obtain

$$x = \frac{ap}{a^2 + b^2 + c^2}, \quad y = \frac{bp}{a^2 + b^2 + c^2}$$

Here $f_{xx} = 2 + \frac{2a^2}{c^2} > 0$,

$$f_{xx} f_{yy} - f_{xy}^2 = 4\left(1 + \frac{a^2}{c^2}\right)\left(1 + \frac{b^2}{c^2}\right) - \frac{4a^2b^2}{c^4} = 4\left(1 + \frac{a^2}{c^2} + \frac{b^2}{c^2}\right) > 0$$

Hence the function is minimum at $x = \frac{ap}{a^2+b^2+c^2}$, $y = \frac{bp}{a^2+b^2+c^2}$

Putting these to (2), we get

$$z = \frac{cp}{a^2+b^2+c^2}$$

$$\text{Min. value} = \frac{a^2p^2+b^2p^2+c^2p^2}{(a^2+b^2+c^2)^2} = \frac{p^2}{a^2+b^2+c^2}.$$

Ex. 5: Obtain the maximum value of xyz such that $x + y + z = 24$

[2069 Bhadra, B. E]

Solution:

Let $f(x, y, z) = xyz$ ....(1)

with the condition

$$x + y + z = 24$$

i.e. $z = 24 - x - y$ ....(2)

So that the function f of (1) can be expressed as a functions of x and y

i.e. $f = xy(24 - x - y)$, then

$f_x = 24y - 2xy - y^2$

$f_y = 24x - x^2 - 2xy$,

$f_{xx} = -2y$, $f_{xy} = 24 - 2x - 2y$, $f_{yy} = -2x$

For the extreme values, $f_x = 0$, $f_y = 0$

$24y - 2xy - y^2 = 0$, $24x - x^2 - 2xy = 0$

or $24 - 2x - y = 0$ and $24 - x - 2y = 0$

Solving these, $x = 8$, $y = 8$, then from (2), $z = 8$

Therefore the stationary point is (8, 8, 8).

At (8, 8, 8),

$f_{xx} = -16 < 0$ and $f_{xx} f_{yy} - (f_{xy})^2 = 256 - 64 = 192 > 0$

Hence f has maximum value at (8, 8, 8)

Max. value = f(8, 8, 8) = 512.

Ex. 6: Find the minimum value of $x^2 + xy + y^2 + 3z^2$ under the conditio $x + 2y + 4z = 60$

[2072 Aswin, B. E]

Solution:

Let $f = x^2 + xy + y^2 + 3z^2$ ....(1

$\phi = x + 2y + 4z - 60 = 0$ ....(2

then we write

$$F = x^2 + xy + y^2 + 3z^2 + \lambda (x + 2y + 4z - 60)$$

where λ is Lagrange's multiplier

$F_x = 2x + y + \lambda$, $F_y = x + 2y + 2\lambda$, $F_z = 6z + 4\lambda$

For the stationary points

$F_x = 0$, $F_y = 0$, $F_z = 0$

$2x + y + \lambda = 0$, $x + 2y + 2\lambda = 0$, $6z + 4\lambda = 0$

Solving these three equations and (2), we have

$$\lambda = \frac{-90}{7},\quad x = 0,\quad y = \frac{90}{7},\quad z = \frac{60}{7}.$$

Therefore the stationary point is $\left(0, \frac{90}{7}, \frac{60}{7}\right)$.

From (1) and (2),

$f_x = 2x + y$, $f_y = x + 2y$, $f_z = 6z$

$f_{xx} = 2$, $f_{xy} = 1$, $f_{xz} = 0$, $f_{yz} = 1$, $f_{yz} = 0$,

$f_{yy} = 2$, $f_{zx} = 0$, $f_{zy} = 0$, $\phi_x = 1$, $\phi_y = 2$, $\phi_z = 4$

$$\text{Now} \begin{vmatrix} 0 & \phi_x & \phi_y \\ \phi_x & f_{xx} & f_{xy} \\ \phi_y & f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 0 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 2 \end{vmatrix} = -8 < 0$$

$$\text{and} \begin{vmatrix} 0 & \phi_x & \phi_y & \phi_z \\ \phi_x & f_{xx} & f_{xy} & f_{xz} \\ \phi_y & f_{yx} & f_{yy} & f_{yz} \\ \phi_z & f_{zx} & f_{zy} & f_{zz} \end{vmatrix} = \begin{vmatrix} 0 & 1 & 2 & 4 \\ 1 & 2 & 1 & 0 \\ 2 & 1 & 2 & 0 \\ 4 & 0 & 0 & 6 \end{vmatrix} = -12 < 0$$

$\therefore$ f has minimum value at (0, 90/7, 60/7)

$$\text{Min. value} = \frac{8100}{49} + \frac{10800}{49} = \frac{2700}{7}$$

Ex.7: Find the minimum value of $x^2 + y^2 + z^2$ subject to the condition $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$

[2069 oaush, B. E]

Solution:

Let $f = x^2 + y^2 + z^2$ ......(1)

Subject to the relation $\phi = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} - 1 = 0$ ......(2)

Then ,we write

$$F = x^2 + y^2 + z^2 + \lambda\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z} - 1\right) \quad ...(3)$$

where λ is a Lagrange's multiplier.

Differentiating (3) partially with respect to x ,y and z respectively, we get

$$F_x = 2x - \frac{\lambda}{x^2},\quad F_y = 2y - \frac{\lambda}{y^2},\quad F_z = 2z - \frac{\lambda}{z^2}$$

For the stationary points, put

$F_x = 0$, $F_y = 0$ and $F_z = 0$

$$2x - \frac{\lambda}{x^2} = 0, \quad 2y - \frac{\lambda}{y^2} = 0 \text{ and } 2z - \frac{\lambda}{z^2} = 0.$$

Solving these equations and (2), we find

$$x = \left(\frac{\lambda}{2}\right)^{\frac{1}{3}}, y = \left(\frac{\lambda}{2}\right)^{\frac{1}{3}} \text{ and } z = \left(\frac{\lambda}{2}\right)^{\frac{1}{3}}$$

Substituting the values of x ,y and z in (2) ,we get

$$\left(\frac{2}{\lambda}\right)^{\frac{1}{3}} + \left(\frac{2}{\lambda}\right)^{\frac{1}{3}} + \left(\frac{2}{\lambda}\right)^{\frac{1}{3}} - 1 = 0$$

or $\quad 3\left(\frac{2}{\lambda}\right)^{\frac{1}{3}} = 1$

$\therefore \quad \lambda = 54$

Then $\quad x = 3, y = 3$ and $z = 3$

Thus the stationary point is (3 ,3 , 3)

Differentiating (1) and (2) partially with respect to x ,y and z respectively,

$$f_x = 2x, \quad f_y = 2y, \quad f_z = 2z$$

$$\phi_x = -\frac{1}{x^2} = -\frac{1}{9}, \quad \phi_y = -\frac{1}{y^2} = -\frac{1}{9}, \quad \phi_z = -\frac{1}{z^2} = -\frac{1}{9}$$

$$f_{xx} = 2, \quad f_{xy} = 0, \quad f_{xz} = 0,$$
$$f_{yx} = 0, \quad f_{yy} = 2, \quad f_{yz} = 0$$
$$f_{zx} = 0, \quad f_{zy} = 0, \quad f_{zz} = 2$$

Now
$$\begin{vmatrix} 0 & \phi_x & \phi_y \\ \phi_x & f_{xx} & f_{xy} \\ \phi_y & f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 0 & -1/9 & -1/9 \\ -1/9 & 2 & 0 \\ -1/9 & 0 & 2 \end{vmatrix}$$

$$= \frac{1}{9}\begin{vmatrix} -1/9 & 0 \\ -1/9 & 2 \end{vmatrix} - \frac{1}{9}\begin{vmatrix} -1/9 & 2 \\ -1/9 & 0 \end{vmatrix}$$

$$= \frac{1}{9}\left(-\frac{2}{9}\right) - \frac{1}{9}\left(\frac{2}{9}\right) = -\frac{4}{81} < 0$$

$$\begin{vmatrix} 0 & \phi_x & \phi_y & \phi_z \\ \phi_x & f_{xx} & f_{xy} & f_{xz} \\ \phi_y & f_{yx} & f_{yy} & f_{yz} \\ \phi_z & f_{zx} & f_{zy} & f_{zz} \end{vmatrix} = \begin{vmatrix} 0 & -1/9 & -1/9 & -1/9 \\ -1/9 & 2 & 0 & 0 \\ -1/9 & 0 & 2 & 0 \\ -1/9 & 0 & 0 & 2 \end{vmatrix}$$

$$= \frac{1}{9}\begin{vmatrix} -1/9 & -1/9 & -1/9 \\ -1/9 & 0 & 2 \\ 0 & 0 & 2 \end{vmatrix} + 2\begin{vmatrix} 0 & -1/9 & -1/9 \\ -1/9 & 2 & 0 \\ -1/9 & 0 & 2 \end{vmatrix}$$

$$= \frac{1}{9}\left(\frac{1}{-9}\right)\left(\frac{2}{-9}\right) + 2\left[\left(\frac{1}{9}\right)\left(\frac{2}{-9}\right) - \frac{1}{9}\left(\frac{2}{9}\right)\right]$$

$$= \frac{2}{729} - \frac{8}{81} = \frac{2-72}{729} = -\frac{70}{729} < 0$$

$\therefore$ f has minimum value at (3, 3, 3)

Min. value = 9 + 9 + 9 = 27

Ex. 8: Find the extreme value of $x^2 + y^2 + z^2$ connected by the relations $x + z = 1$ and $2y + z = 2$ [2070 Magh B. E.]

Solution:

Let $\quad f = x^2 + y^2 + z^2$ ....(1)

connected by the relations

$x + z = 1$ .....(2)

$2y + z = 2$ ....(3)

From (2) and (3),

$$z = 1 - x, \quad y = \frac{1+x}{2}$$

The function f of (1) reduces to

$$F = x^2 + \left(\frac{1+x}{2}\right)^2 + (1-x)^2$$

Differentiating it with respect to x, we get

$$\frac{dF}{dx} = 2x + \frac{(1+x)}{2} - 2(1-x) = \frac{9x}{2} - \frac{3}{2}$$

$$\frac{d^2F}{dx^2} = \frac{9}{2} > 0$$

For the extreme values, put $\frac{dF}{dx} = 0$

or $\frac{9x}{2} - \frac{3}{2} = 0$

$\therefore$ $x = \frac{1}{3}$

Then from (2) and (3), we get

$z = \frac{2}{3}, \quad y = \frac{2}{3}$

Therefore the stationary point is (1/3, 2/3, 2/3)

$$\text{Min .Value} = \left(\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2 = \frac{1}{9} + \frac{4}{9} + \frac{4}{9} = 1.$$

Exercise - 3

1. Examine and find the maximum and minimum values of
   (i) $x^3 + y^3 - 3axy$ [2056 B.]
   (ii) $x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$
   (iii) $xy(a - x - y)$
   (iv) $2(x - y)^2 - x^4 - y^4$ [2064 Paush,066Kartik B.]
   (v) $4x^2 - xy + 4y^2 + x^3y + xy^3 - 4$
   (vi) $16 - (x + 2)^2 - (y - 2)^2$
   (vii) $8 - 4x + 4y - x^2 - y^2$
   (viii) $2x^2 - 3xy + 4y^2 + 5$
2. Examine and find the maximum and minimum values of
   (i) $y^2 + 2z^2 - 5x^4 + 4x^5$
   (ii) $-3x^2 + 6xz + 4y - 2y^2 - 6z^2$
   (iii) $35 - (2x + 3)^2 - (y - 4)^2 - (z + 1)^2$
   (iv) $8z + 2x^2 + 3y^2 + 4z^2 - 3xy$
   (v) $20 - x^2 - y^2 - z^2$ [2063Kartik, B.]
3. Show that following functions have a minimum at the points indicated
   (i) $x^2 + y^2 + z^2 + 2xyz$ at (0, 0, 0)
   (ii) $x^4 + y^4 + z^4 - 4xyz$ at (1, 1, 1)

4. Find the extreme values for the function $x^2 + y^2$ under the condition $x + 4y = 2$
5. Find the extreme values of $48 - (x - 5)^2 - 3(y - 4)^2$ subject to the condition $x + 3y = 9$
6. Find the extreme value of $x^2 + y^2 + z^2$ subject to the condition $x + y + z = 1$ [2060 Chaitra B. E.]
7. Find the maximum and minimum values of xy^2 under the condition $x + y = 1$
8. Find the maximum value of xyz under the condition $x + y + z = 8$
9. Find the minimum value of $x^2 + y^2 + z^2$ subject to the condition $x + y + z - 1 = 0$ and $xyz + 1 = 0$ [2061/071 Magh B. E.]
10. Find the minimum value $x^2 + y^2 + z^2$ when $x + y + z = 3a$ [2062/072 Magh, B. E.]

Answers

1. (i) Max. at (a, a); Max. value = $-a^3$
   (ii) Max. at (4, 0), Max. value = 112 ; Min. at (6, 0), Min. value = 108
   (iii) Extreme point $\left(\frac{a}{3}, \frac{a}{3}\right)$; Extreme value = $\frac{a^3}{27}$
   Max. value for a > 0 and Min. value for a < 0.
   (iv) Max. at ($\sqrt{2}$, - $\sqrt{2}$) and at (- $\sqrt{2}$, $\sqrt{2}$); Max. value = 8
   (v) Max. at $\left(\frac{3}{2}, \frac{-3}{2}\right)$ and at $\left(-\frac{3}{2}, \frac{3}{2}\right)$ Max.. value = $\frac{49}{8}$
   Min. at (0, 0); Min. value = - 4
   (vi) Max. at (- 2, 2); Max. value = 16
   (vii) Max. at (-2, 2); Max. value = 16
   (viii) Min. at (0, 0); Min. value = 5
2. (i) Min. at (1, 0, 0); Min. value = - 1 (ii) Max. at (0, 1, 0); Max. value = 2
   (iii) Max. at $\left(\frac{-3}{2}, 4, -1\right)$; Max. value = 35

(iv) Min. at (0, 0, - 1); Min. value = 4 (v) Max. at (0, 0, 0); Max value = 20

4. Min. value = $\frac{4}{27}$ 5. Max. value = 32 6. Min. value = $\frac{1}{3}$

7. Max. value = $\frac{4}{27}$ 8. Max. value = $\frac{512}{27}$, Min. value = 0

9. Min. value = 3 10. Min. value = $3a^2$

Corrections made by Siddharth

Chapter –2

Multiple Integral

- Introduction
- Double Integrals in Polar Co-ordinates
- Change of Variables
- Triple Integrals
- Cylindrical Coordinates
- Spherical Polar Coordinates
- Dritchlet's Integral
- Application of Double and Triple Integrals
- Volume

Chapter –2

Multiple Integral

2.1 Introduction

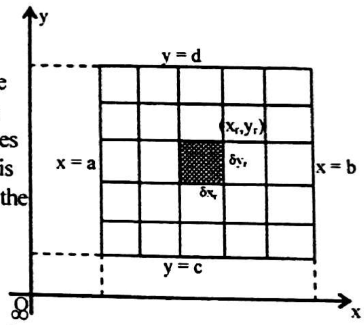
The integrable functions of two or three variables defined on the certain interval, called multiple integrals. In this chapter we shall define two or three variables of double integrals and triple integrals which are explained as follows.

2.2 Double Integrals

In a double integral we integrate a function f(x, y) of the independent variables x and y over a closed bounded region R in the xy - plane. On division of region R into rectangles by drawing parallel to x and y axes with n elementary areas δA_1, δA_2, …, δA_n. Choose a point (x_r, y_r) within r^{th} rectangles with area δA_r, then the sum

$$f(x_1, y_1)\,\delta A_1 + f(x_2, y_2)\,\delta A_2 + \ldots + f(x_r, y_r)\,\delta A_r + \ldots + f(x_n, y_n)\,\delta A_n$$

$$= \sum_{r=1}^{n} f(x_r, y_r)\,\delta A_r.$$



If this limit exists, as the length of the maximum diagonal of the rectangles approaches to zero when n approaches to infinity, then the limit of this sum is called double integral of f(x, y) over the region R and is denoted by

$$\iint_R f(x, y)\,dA$$

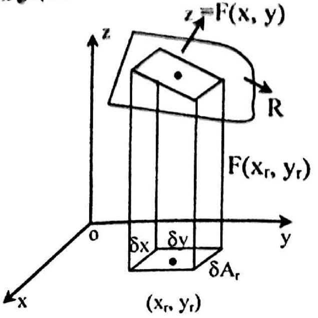
or $$\iint_R f(x, y)\,dA = \lim_{\substack{n\to\infty \\ dA\to 0}} \sum_{r=1}^{\infty} f(x_r, y_r)\,\delta A_r$$ provided the limit exists.

It can also be expressed as

$$\iint_R f(x, y)\,dx\,dy \quad \text{or} \quad \iint_R f(x, y)\,dy\,dx.$$

2.2.1 Evaluation of Double Integrals

The graph of the function of two variables F(x, y) is a surface, plotted in three dimension. The double integral of F(x, y) over the region R is the volume between the graph and the region R .

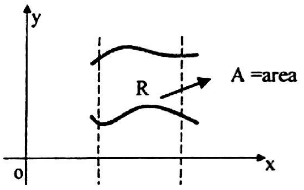


Thus, the double integral of F(x, y) is the volume defined as

$$\iint_R F(x, y)\,dx\,dy .$$

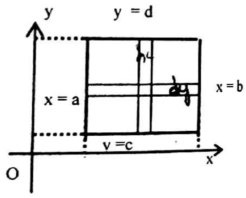
If F(x, y) =1, then the numerical value of the volume is equal to value of the area of the region R. Thus,

$$A = \iint_R dx\,dy.$$



(a) When the region R is rectangle and both pairs of limits are constants $a \le x \le b,\ c \le y \le d$

If the function of two variables F(x, y) is bounded by x = a, x = b ; y = c, y = d , then the double integral of F(x, y) in the region R is defined as



$$\iint_R F(x, y)\,dx\,dy = \int_a^b \int_c^d F(x, y)\,dy\,dx$$

$$= \int_c^d \int_a^b F(x,y)\,dx\,dy$$

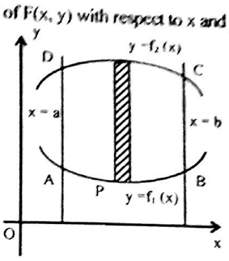
(b) When the region R : $a \le x \le b,\ f_1(x) \le y \le f_2(x)$

If the function of two variables F(x, y) is bounded by x = a, x = b ; y = f_1 (x),

y = f_2 (x), then the double integral of F(x, y) with respect to x and y in the region R is defined as

$$\iint_R F(x, y)\,dx\,dy$$

$$= \int_a^b \int_{f_1(x)}^{f_2(x)} F(x, y)\,dy\,dx$$

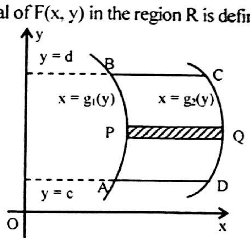


c) When the region R : $c \le y \le d,\ g_1(y) \le x \le g_2(y)$

If the function of two variables F(x, y) is bounded by y = c, y = d ; x = g_1(y), x = g_2(y), then the double integral of F(x, y) in the region R is defined as

$$\iint_R F(x, y)\,dx\,dy$$

$$= \int_c^d \int_{g_1(y)}^{g_2(y)} F(x, y)\,dx\,dy$$

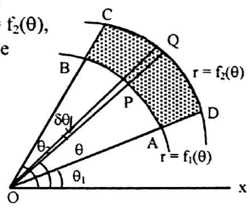


2.2.2 Double Integrals in Polar Co-ordinates

If a function of two variables F(r, θ) is bounded by θ = α, θ = β; r = f_1(θ), r = f_2(θ), then the double integral of F(r, θ) in the region R is defined as

$$\iint_R F(r, \theta)\,d\theta\,d\theta$$

$$= \int_\beta^\alpha \int_{f_1(\theta)}^{f_2(\theta)} F(r, \theta)\,dr\,d\theta.$$



To evaluate the integral, we first integrate with respect to r between limits r = f_1(θ) and r = f_2(θ), keeping θ fixed where r = f_1(θ), r = f_2 (θ) are variables of function θ and resulting function is integrated with respect to θ from θ = α to θ = β where α and β are constants.

In the figure, AB and CD are curves $r = f_1(\theta)$, $r = f_2(\theta)$ bounded by the lines $\theta = \alpha$ and $\theta = \beta$. Let PQ is small increment of θ with angular thickness $\delta\theta$, then that $\int_{\alpha}^{\beta}\int_{f_1(\theta)}^{f_2(\theta)} F(r, \theta)\, dr\, d\theta$ indicates that the integration is performed along PQ and the integration with respect to θ means rotation of the strip PQ from AC to BD.

Worked out Examples

Ex.1: Evaluate $\int_0^1 \int_1^2 (x^2 + y^2)\, dx\, dy$

Solution:

Here the integral is

$$\int_0^1 \int_1^2 (x^2+y^2)\, dx\, dy = \int_0^1 \left(\int_1^2 (x^2+y^2)\, dx \right) dy$$

$$= \int_0^1 \left[\frac{x^3}{3} + y^2 x \right]_1^2 dy = \int_0^1 \left[\frac{8}{3} + 2y^2 - \frac{1}{3} - y^2 \right] dy$$

$$= \int_0^1 \left[\frac{7}{3} + y^2 \right] dy = \left[\frac{7}{3} y + \frac{y^3}{3} \right]_0^1 = \frac{8}{3}$$

Ex .2 Show that $\int_0^1 dx \int_0^1 \frac{x-y}{(x+y)^3}\, dy \neq \int_0^1 dy \int_0^1 \frac{x-y}{(x+y)^3}\, dx$

Solution:

We have $\int_0^1 dx \int_0^1 \frac{x-y}{(x+y)^3}\, dy = \int_0^1 dx \int_0^1 \frac{-(x+y)+2x}{(x+y)^3}\, dy$

$$= \int_0^1 dx \int_0^1 \left[\frac{-(x+y)}{(x+y)^3} + \frac{2x}{(x+y)^3} \right] dy$$

$$= \int_0^1 dx \int_0^1 \left[-\frac{1}{(x+y)^2} + \frac{2x}{(x+y)^3} \right] dy$$

$$= \int_0^1 \left[\frac{1}{x+y} - \frac{x}{(x+y)^2} \right]_0^1 dx$$

$$= \int_0^1 \left[\frac{1}{x+1} - \frac{x}{(x+1)^2} - \frac{1}{x} + \frac{x}{(x+0)^2} \right] dx$$

$$= \int_0^1 \left[\frac{1}{x+1} - \frac{x+1-1}{(x+1)^2} - \frac{1}{x} + \frac{1}{x} \right] dx$$

$$= \int_0^1 \left[\frac{1}{x+1} - \frac{1}{x+1} + \frac{1}{(x+1)^2} \right] dx$$

$$= \int_0^1 \frac{1}{(x+1)^2} dx = - \left[\frac{1}{x+1} \right]_0^1 = - \left[\frac{1}{2} - 1 \right] = \frac{1}{2}.$$

And $\int_0^1 dy \int_0^1 \frac{x-y}{(x+y)^3}\, dx$

$$= \int_0^1 dy \int_0^1 \frac{(x+y)-2y}{(x+y)^3}\, dx$$

$$= \int_0^1 dy \int_0^1 \left[\frac{x+y}{(x+y)^3} - \frac{2y}{(x+y)^3} \right] dx$$

$$= \int_0^1 dy \int_0^1 \left[\frac{1}{(x+y)^2} - \frac{2y}{(x+y)^3} \right] dx$$

$$= \int_0^1 \left[-\frac{1}{x+y} + \frac{y}{(x+y)^2} \right]_0^1 dy$$

$$= \int_0^1 \left[-\frac{1}{y+1} + \frac{y}{(y+1)^2} + \frac{1}{y} - \frac{y}{(0+y)^2} \right] dy$$

$$= \int_0^1 \left[-\frac{1}{y+1} + \frac{y+1-1}{(y+1)^2} + \frac{1}{y} - \frac{1}{y} \right] dy$$

$$= \int_0^1 \left[-\frac{1}{y+1} + \frac{1}{y+1} - \frac{1}{(y+1)^2} \right] dy$$

$$= - \int_0^1 \frac{1}{(y+1)^2} dy = \left[\frac{1}{y+1} \right]_0^1 = \left[\frac{1}{2} - 1 \right] = -\frac{1}{2}$$

Hence $\int_0^1 dx \int_0^1 \frac{x-y}{(x+y)^3}\, dy \neq \int_0^1 dy \int_0^1 \frac{x-y}{(x+y)^3}\, dx$

Ex. 3: Evaluate $\int_1^2 \int_0^x \frac{dy\, dx}{y^2 + x^2}$

Solution:

Here the integral is $\int_1^2 \int_0^x \frac{dy\, dx}{y^2+x^2} = \int_1^2 \int_0^x \left[\frac{dy}{y^2+x^2} \right] dx$

$$= \int_1^2 \left[\frac{1}{x}\tan^{-1}\frac{y}{x}\right]_0^x dx = \int_1^2 \frac{1}{x}\left[\frac{\pi}{4} - 0\right] dx$$

$$= \frac{\pi}{4}\int_1^2 \frac{1}{x}\, dx = \frac{\pi}{4}\left[\log x\right]_1^2 = \frac{\pi}{4}\log 2.$$

Ex.4: Evaluate $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{1}{1+x^2+y^2}\, dx\, dy$

Solution:

Here the integral is

$$\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dx\, dy}{1+x^2+y^2} = \int_0^1 \int_0^{\sqrt{1+x^2}} \left(\frac{dy}{1+x^2+y^2}\right) dx$$

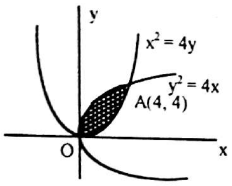
$$= \int_0^1 \left[\frac{1}{\sqrt{1+x^2}}\tan^{-1}\frac{y}{\sqrt{1+x^2}}\right]_0^{\sqrt{1+x^2}} dx$$

$$= \int_0^1 \frac{1}{\sqrt{1+x^2}}\frac{\pi}{4}\, dx = \frac{\pi}{4}\left[\log\left(x+\sqrt{1+x^2}\right)\right]_0^1$$

$$= \frac{\pi}{4}\log\left(1+\sqrt{2}\right).$$

Ex.5. Evaluate $\iint_R y\, dy\, dx$ **where R is the region bounded by the parabolas** $y^2 = 4x$ **and** $x^2 = 4y$

Solution:



Here the integral is

$\iint_R y\, dy\, dx$. The integration is first integrated with respect to y along vertical strip in which y extends from $y = x^2/4$ to $y = 2\sqrt{x}$ and those of x extends from $x = 0$ to 4.

$$\therefore \iint_R y\, dy\, dx = \int_0^4 \int_{x^2/4}^{2\sqrt{x}} y\, dy\, dx = \int_0^4 \left[\frac{y^2}{2}\right]_{x^2/4}^{2\sqrt{x}} dx$$

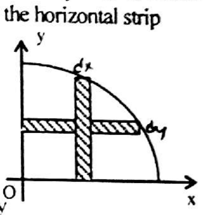
$$= \frac{1}{2}\int_0^4 \left(4x - \frac{x^4}{16}\right) dx = \frac{1}{2}\left[2x^2 - \frac{x^5}{80}\right]_0^4$$

$$= \frac{1}{2}\left[32 - \frac{16\times 64}{80}\right] = \frac{1}{2}\left[32 - \frac{64}{5}\right] = \frac{48}{5}.$$

Ex.6. Evaluate $\iint_R xy\, dx\, dy$ **where R is the quadrant of the circle** $x^2 + y^2 = a^2$ **and** $x \geq 0, y \geq 0$

Solution:

The region of integration is positive quadrant of the circle $x^2 + y^2 = a^2$ in which the integration is first integrated with respect to x along the horizontal strip which extends from $x = 0$ to $x = \sqrt{a^2 - y^2}$ and those of y varies from $y = 0$ to $y = a$.



$$\therefore \iint_R xy\, dx\, dy = \int_0^a \int_{x=0}^{\sqrt{a^2-y^2}} x\, dx\, y\, dy$$

$$= \int_0^a \left[\frac{x^2}{2}\right]_0^{\sqrt{(a^2-y^2)}} y\, dy = \int_0^a \frac{(a^2-y^2)y}{2}\, dy$$

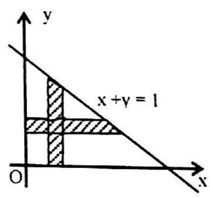
$$= \frac{1}{2}\left[\frac{a^2y^2}{2} - \frac{y^4}{4}\right]_0^a = \frac{1}{2}\left[\frac{a^4}{2} - \frac{a^4}{4}\right] = \frac{a^4}{8}.$$

Ex.7: Evaluate $\iint_R (x^2 + y^2)\, dx\, dy$ **over the region bounded by** $x = 0, y = 0, x + y = 1$

Solution:

We have $\iint_R (x^2 + y^2)\, dx\, dy$.

Here, the integration is first integrated with respect to x which extends from $x = 0$ to $x = 1 - y$ and those from y extends from $y = 0$ to $y = 1$. Thus



$$\iint_R (x^2 + y^2)\, dx\, dy = \int_0^1 \int_0^{1-y} (x^2 + y^2)\, dx\, dy$$

$$= \int_0^1 \left[\frac{x^3}{3} + xy^2\right]_0^{1-y} dy$$

$$= \int_0^1 \left[\frac{(1-y)^3}{3} + (1-y)y^2\right] dy$$

$$= \left[\frac{(1-y)^4}{-12} + \frac{y^3}{3} - \frac{y^4}{4}\right]_0^1 = \frac{1}{3} - \frac{1}{4} + \frac{1}{12} = \frac{1}{6}.$$

Ex.8. Evaluate $\int_0^\pi \int_0^x \sin y\, dy\, dx$

Solution:

We have

$$\int_0^\pi \int_0^x \sin y\, dy\, dx = \int_0^\pi \left(\int_0^x \sin y\, dy\right) dx = -\int_0^\pi \left[\cos y\right]_0^x dx$$

King in the

$$= -\int_0^{\pi} (\cos x - 1)\, dx$$

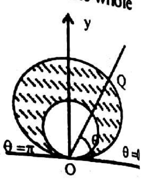
$$= \int_0^{\pi} (1 - \cos x)\, dx = [x - \sin x]_0^{\pi} = \pi.$$

Ex.9. Evaluate $\iint r^3\, dr\, d\theta$ over the area included between the circles $r = 2\sin\theta$ and $r = 4\sin\theta$ [2062/069 Poush, B. E.]

Solution:

Given circles are $r = 2\sin\theta$ and $r = 4\sin\theta$. We first integrate with respect to r in which the limits of r from $r = 2\sin\theta$ to $r = 4\sin\theta$ and to cover the whole region θ varies from $\theta = 0$ to π.

$$\text{Thus } \iint r^3\, dr\, d\theta = \int_0^{\pi}\left[\int_{2\sin\theta}^{4\sin\theta} r^3\, dr\right] d\theta$$



$$= \int_0^{\pi}\left[\frac{r^4}{4}\right]_{2\sin\theta}^{4\sin\theta} d\theta = 60\int_0^{\pi} \sin^4\theta\, d\theta$$

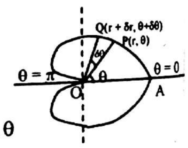
$$= 120\int_0^{\pi/2} \sin^4\theta\, d\theta = 120\,\frac{\sqrt{\pi}\,\Gamma\left(\frac{4+1}{2}\right)}{2\,\Gamma\left(\frac{4+2}{2}\right)}$$

$$= 120\,\frac{\sqrt{\pi}\,\frac{3}{2}\times\frac{1}{2}\sqrt{\pi}}{2\times 2\times 1} = \frac{45\pi}{2}.$$

Ex.10. Evaluate $\iint r\sin\theta\, dr\, d\theta$ over the area of the Cardioid $r = a(1 + \cos\theta)$ above the initial line [2066/072 Magh, B. E.]

Solution:

The integration is first integrated with respect to r in which the region covered by $r = 0$ to $r = a(1 + \cos\theta)$ which lies between $\theta = 0$ to $\theta = \pi$.



$$\text{Thus } \iint r\sin\theta\, dr\, d\theta = \int_0^{\pi}\int_0^{a(1+\cos\theta)} r\sin\theta\, dr\, d\theta$$

$$= \int_0^{\pi} \sin\theta\left[\frac{r^2}{2}\right]_0^{a(1+\cos\theta)} d\theta = \frac{a^2}{2}\int_0^{\pi} \sin\theta(1+\cos\theta)^2\, d\theta$$

Putting $1 + \cos\theta = t$ so that $-\sin\theta\, d\theta = dt$.

When $\theta = 0,\ t = 2;\ \theta = \pi,\ t = 0,$

$$= -\frac{a^2}{2}\int_2^0 t^2\, dt = -\frac{a^2}{2}\left[\frac{t^3}{3}\right]_2^0 = -\frac{a^2}{2}\left[0 - \frac{8}{3}\right] = \frac{4a^2}{3}.$$

Exercise - 4

1. Evaluate the following integrals

(i) $\int_0^3\int_1^2 xy(x + y)\, dx\, dy$ (ii) $\int_0^3\int_0^1 (x^2 + 3y^2)\, dy\, dx$

(iii) $\int_0^2\int_0^{\sqrt{(2x - x^2)}} x\, dx\, dy$ (iv) $\int_0^1\int_x^{\sqrt{x}} (x^2 + y^2)\, dx\, dy$

(v) $\int_0^1\int_0^{x^2} e^{y/x}\, dy\, dx$ (vi) $\int_0^a\int_0^{\sqrt{a^2 - y^2}} dx\, dy$

(vii) $\int_0^1\int_0^x e^{y/x}\, dy\, dx.$ (viii) $\int_1^{\log 8}\int_0^{\log y} e^{x+y}\, dx\, dy$

(ix) $\int_0^1\int_y^{\sqrt{y}} (x^2 + y^2)\, dy\, dx$ (x) $\int_0^a\int_0^{\sqrt{a^2 - y^2}} \sqrt{a^2 - x^2 - y^2}\, dy\, dx$

2. Evaluate $\iint_R (x + y)^2\, dx\, dy$ where R is the region bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ [2070 Magh, B. E.]

3. Evaluate $\iint_R xy(x + y)\, dx\, dy$, and R is region over the area between $y = x^2$ and $y = x$ [2069 071Bhadra, B. E.]

4. Evaluate $\iint_R xy\, dx\, dy$ where R is the region bounded by the x-axis, the ordinate $x = 2a$ and the curve $x^2 = 4ay$

5. Evaluate $\iint_R \sqrt{xy - y^2}\, dx\, dy$ where R is region bounded by the triangle with vertices (0, 0),(10,1) and (1, 1)

6. Evaluate $\iint_R xy\, dx\, dy$ where R is the region over the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in the first quadrant

2063 067Chaitra, B.E.

7. Evaluate $\iint_R x^2\, dx\, dy$ where R is the region in the first quadrant bounded by the hyperbola xy = 16 and the line y = x, y = 0 and x = 8

8. Evaluate $\iint_R y\, dy\, dx$ where R is the region bounded by $x^2 + 2y^2 = 4$ and the ordinate y = 0, $y = \sqrt{2}$

9. Evaluate $\iint_R r^3\, dr\, d\theta$ over the area bounded between the circles $r = 2\cos\theta$ and $r = 4\cos\theta$

10. Evaluate $\iint_R r^2 \sin\theta\, dr\, d\theta$ where R is the semi-circle $r = 2a\cos\theta$ above the initial line

11. Evaluate the following integrals

(i) $\int_0^{\pi/2}\int_0^{\pi/2} \sin(\theta + \phi)\, d\theta\, d\phi$ (ii) $\int_0^{\pi}\int_0^{a\sin\theta} r\, dr\, d\theta$

(iii) $\int_0^{\pi}\int_0^{a\theta} r^3\, dr\, d\theta$ (iv) $\iint_R \frac{r\, d\theta\, dr}{\sqrt{a^2 + r^2}}$ over one loop $r^2 = a^2\cos 2\theta$

Answers

1. (i) 24 (ii) 12 (iii) $\frac{\pi}{2}$ (iv) $\frac{3}{35}$ (v) $\frac{1}{2}$ (vi) $\frac{\pi a^2}{4}$

(vii) $\frac{1}{2}(e - 1)$ (viii) $8 \log 8 - 16 + e$ (ix) $\frac{3}{35}$ (x) $\frac{\pi a^3}{6}$

2. $\frac{\pi ab}{4}(a^2 + b^2)$ 3. $\frac{3}{56}$ 4. $\frac{a^4}{3}$ 5. 6 6. $\frac{a^2b^2}{8}$ 7. 448 8. $\frac{8}{3}$

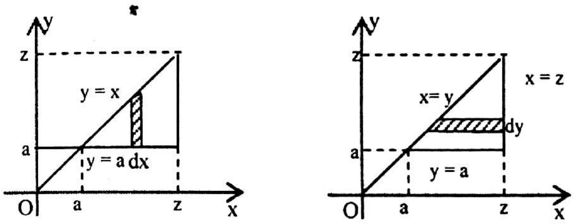
9. $\frac{45\pi}{2}$ 10. $\frac{2a^3}{3}$ 11.(i) 2 (ii) $\frac{\pi a^2}{4}$ (iii) $\frac{a^4\pi^5}{20}$ (iv) $\left(2 - \frac{\pi}{2}\right)a$

2.2.3 Change of Order of Integration

In Calculus, interchange of the order of integration is the methodology that transforms iterated integrals of the functions into other, hopefully simpler, integrals by changing the order in which the integrations are performed. In some cases, the order of integration can be validity interchanged, in others it cannot. Consider the iterated integral

$$\int_a^z dx \int_a^x h(y)\, dy$$

In this expression, the second integral is calculated first with respect to *y* and *x* is held constant - a strip of width *dx* is integrated first over the y direction - a strip of width *dx* is integrated first over the *y* direction- a strip of width *dx* in the *x* direction is integrated with respect to the y variable across the y direction, adding up an infinite amount of the rectangles of width *dy* along the y axis. This forms a three dimensional slice *dx* wide along the x-axis, from y = a, to y = x along the y axis, and in the z direction z = f(x, y). Notice that if the thickness *dx* is infinitesimal, *x* varies only infinitesimally on the slice. We can assume that *x* is constant.

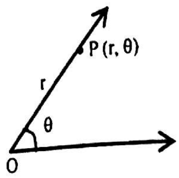


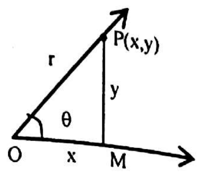
This integration is as shown in the left panel of the figure, but is inconvenient especially when the function h(y) is not easily integrated. The integral can be reduced to a single integration by reversing the order of integration as shown in the right panel of the figure. To accomplish this interchange of variables, the strip of width *dy* is first integrated from the line $x = y$ to the limit $x = z$, and then the result is integrated from $y = a$ to $y = z$ resulting in.

$$\int_a^z dx \int_a^x h(y)\, dy = \int_a^z h(y)\, dy \int_y^z dx = \int_a^z h(y)\,(z - y)\, dy$$

2.2.4 Change of Variables

To define the Polar Coordinates of a plane we need first to fix a point which will be called the *Pole* (or the origin) and a half-line starting from the pole. This half-line is *Polar Axis*. Positive values of the angle indicate angles measured in the counterclockwise direction from the Polar Axis.





We have

$$\cos\theta = \frac{OM}{OP} = \frac{x}{r} \qquad \therefore x = r\cos\theta,$$

$$\sin\theta = \frac{PM}{OP} = \frac{y}{r} \qquad \therefore y = r\sin\theta$$

$$x^2 + y^2 = r^2 \text{ and } \frac{y}{x} = \tan\theta.$$

So $$dx\,dy = J = \frac{\partial(x, y)}{\partial(r, \theta)}\,dr\,d\theta = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{vmatrix} dr\,d\theta$$

$$= \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix} dr\,d\theta$$

$$= (r\cos^2\theta + r\sin^2\theta)\,dr\,d\theta = r\,dr\,d\theta$$

$\therefore$ $dx\,dy = r\,dr\,d\theta$.

Hence $\iint_R f(x, y)\,dx\,dy = \iint_R f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$ where R is the region in the $r\theta$- plane corresponding to R in the xy-plane.

Worked out Examples

Ex. 1. Change the order of integration in the following integrals.

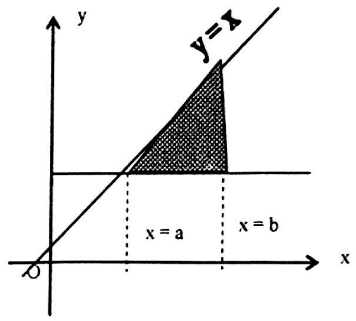
(i) $\int_a^b \int_a^x f(x, y)\,dx\,dy$ (ii) $\int_0^a \int_0^{\sqrt{(a^2 - x^2)}} f(x, y)\,dx\,dy$

(iii) $\int_0^{2a} \int_{x^2/4a}^{3a-x} f(x, y)\,dy\,dx$

(i) Solution:

We have $\int_a^b \int_a^x f(x, y)\,dx\,dy$.

Obviously the curve is bounded by y = a to y = x and x = a to x = b. Here, the integration is first integrated with respect to y along the vertical strip which extends from y = a to y = x. and then with respect to x along horizontal strip which extends x = a to x = b.



By changing order of integration we first integrate with respect to x along the horizontal strip which extends from x = y to x = b and those of y extends from y = a to y = b.

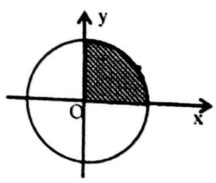
Hence $\int_a^b \int_a^x f(x, y)\,dx\,dy = \int_a^b \int_y^b f(x, y)\,dy\,dx$.

(ii) Solution:

We have $\int_0^a \int_0^{\sqrt{(a^2 - x^2)}} f(x, y)\,dx\,dy$.

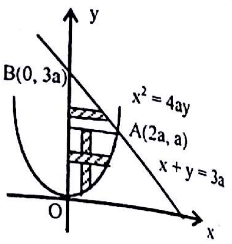
Here the integration is first integrated with respect to y along the vertical strip, which varies from y = 0 to $y = \sqrt{a^2 - x^2}$ i.e. $y^2 = a^2 - x^2$ and those of x varies from x = 0 to x = a.

Changing the order of integration, we first integrate with respect to x along horizontal strip which varies from $x = 0$ to $x = \sqrt{a^2 - y^2}$ and those from y varies from y = 0 to y = a.



$$\therefore \int_0^a \int_0^{\sqrt{(a^2 - x^2)}} f(x, y)\,dx\,dy = \int_0^a \int_0^{\sqrt{(a^2 - y^2)}} f(x, y)\,dy\,dx.$$

(iii) Solution:



We have $\int_0^{2a}\int_{x^2/4a}^{3a-x} f(x, y)\,dx\,dy$

Here the integration is first integrated with respect to y which extends from

$y = x^2/4a$ to $y = 3a - x$

i.e. $x^2 = 4ay$ to $x + y = 3a$ and

$x = 0$ to $x = 2a$.

By changing order of integration, we first integrate with respect to x along horizontal strip and that the region OAB splits up into two parts by the line AL the curve linear triangle OAL and the triangle ABL.

For the region OAL, x extends from $x = 0$ to $x = 2\sqrt{ay}$ and those of y from $y = 0$ to $y = a$, and for the region ALB, x extends from $x = 0$ to $x = 3a - y$ and those for y from $y = a$ to $y = 3a$.

$$\therefore \int_0^{2a}\int_{x^2/4a}^{3a-x} f(x, y)\,dx\,dy$$

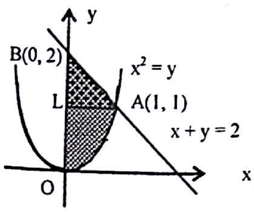
$$= \int_0^{a}\int_0^{2\sqrt{(ay)}} f(x, y)\,dy\,dx + \int_a^{3a}\int_0^{3a-y} f(x, y)\,dy\,dx.$$

Ex. 2. Change the order of integration $\int_0^1\int_{x^2}^{2-x} xy\,dx\,dy$ and hence evaluate the same

Solution:

The multiple integral is

$\int_0^1\int_{x^2}^{2-x} xy\,dx\,dy$. Here, we first integrate with respect to y along the vertical strip which extends from $y = x^2$ to $y = 2 - x$ and those for x extends from $x = 0$ to $x = 1$.



By changing the order of integration we first integrate with respect to x and then with respect to y.

Here, the region OAB splits up into two parts by the line AL the curve-linear OAL and triangle ABL.

For the region OAL, x extends from $x = 0$ to $x = \sqrt{y}$ and those for y extends from $y = 0$ to 1.

For the region ALB, x extends from $x = 0$ to $x = 2 - y$ and those for y extends from $y = 1$ to $y = 2$.

$$\int_0^1\int_{x^2}^{2-x} xy\,dx\,dy = \int_0^1 dy\int_0^{\sqrt{y}} xy\,dx + \int_1^2 dy\int_0^{2-y} xy\,dx$$

$$= \int_0^1 y\left[\frac{x^2}{2}\right]_0^{\sqrt{y}} dy + \int_1^2 y\left[\frac{x^2}{2}\right]_0^{2-y} dy.$$

$$= \frac{1}{2}\int_0^1 y^2\,dy + \frac{1}{2}\int_1^2 y(2-y)^2\,dy$$

$$= \frac{1}{2}\int_0^1 y^2\,dy + \frac{1}{2}\int_1^2 (4y - 4y^2 + y^3)\,dy$$

$$= \frac{1}{2}\left[\frac{y^3}{3}\right]_0^1 + \frac{1}{2}\left[2y^2 - \frac{4y^3}{3} + \frac{y^4}{4}\right]_1^2$$

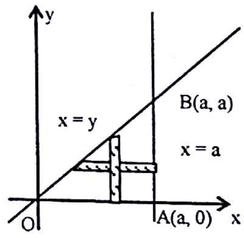
$$= \frac{1}{2}\cdot\frac{1}{3} + \frac{1}{2}\left[8 - \frac{32}{3} + 4 - 2 + \frac{4}{3} - \frac{1}{4}\right]$$

$$= \frac{1}{6} + \frac{1}{2}\left[\frac{39}{4} - \frac{28}{3}\right] = \frac{1}{6} + \frac{1}{2}\left(\frac{117 - 112}{12}\right) = \frac{9}{24} = \frac{3}{8}.$$

Ex.5. Evaluate $\int_0^a\int_y^a \frac{x\,dx\,dy}{x^2 + y^2}$ by changing order of integration

Solution:

Here, the integration is first integrated with respect to x along the horizontal strip which extends from $x = y$ to $x = a$ and those from $y = 0$ to $y = a$.



By changing order of integration, we first integrate with respect to y along vertical strip which extends $y = 0$ to $y = x$ and those for x extends from $x = 0$ to $x = a$.

Thus $\int_0^a\int_y^a \frac{x\,dx\,dy}{x^2+y^2} = \int_0^a\int_0^x \frac{dy}{x^2+y^2}\,x\,dx = \int_0^a\left[\frac{1}{x}\tan^{-1}\frac{y}{x}\right]_0^x x\,dx$

$$= \int_0^a\left(\frac{\pi}{4}-0\right)dx = \frac{a\pi}{4}.$$

Ex. 6. Evaluate $\int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} dy\,dx$ by changing order of integration

2058 Shravan B.E.

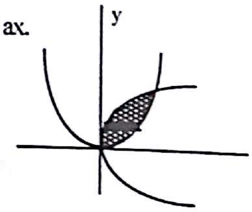
Solution:

Here the integral is

$$\int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} dy\,dx$$

The integration is first integrated with respect to y along the vertical strip which extends from $y = \frac{x^2}{4a}$ to $y = 2\sqrt{ax}$ and those for x from $x = 0$ to $x = 4a$.

i.e. $x^2 = 4ay$ to $y^2 = 4ax$.



By changing order of integration, we first integrate with respect to x along horizontal strip which extends from

$x = y^2/4a$ to $x = 2\sqrt{ay}$ and those for y extends from $y = 0$ to $y = 4a$.

Thus $\int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} dy\,dx = \int_0^{4a}\left[\int_{x=y^2/4a}^{2\sqrt{ay}} dx\right]dy = \int_0^{4a}\left(2\sqrt{ay}-\frac{y^2}{4a}\right)dy$

$$= \left[\frac{2\times 2}{3}\sqrt{a}\,y^{3/2}-\frac{y^3}{12a}\right]_0^{4a}$$

$$= \frac{32a^2}{3}-\frac{16a^2}{3} = \frac{16a^2}{3}.$$

Ex.7. Evaluate $\int_0^a\int_{\sqrt{ax}}^a \frac{y^2\,dy\,dx}{\sqrt{y^4-a^2x^2}}$ by changing order of integration

2070 Bhadra B.E.

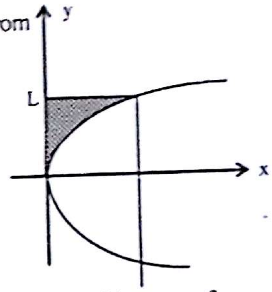
Solution:

Here the integration is first with respect to y along vertical strip which extends from

$y = \sqrt{ax}$ to $y = a$

i.e. $y^2 = ax$ to $y = a$ and those for x from

$x = 0$ to $x = a$



By changing order of integration, we first integrate with respect from x along horizontal strip which extends from $x = 0$ to $x = y^2/a$ and those for y from $y = 0$ to $y = a$.

Thus $\int_0^a\int_{\sqrt{ax}}^a \frac{y^2\,dy\,dx}{\sqrt{y^4-a^2x^2}} = \int_0^{4a}\int_{x=0}^{y^2/a} \frac{dx}{\sqrt{y^4-a^2x^2}}\,y^2\,dy$

$$= \int_0^a\int_{x=0}^{y^2/a}\frac{1}{a}\frac{dx}{\sqrt{\left(\frac{y^2}{a}\right)^2-x^2}}\,y^2\,dy$$

$$= \int_0^a\left[\frac{1}{a}\sin^{-1}\frac{ax}{y^2}\right]_0^{y^2/a} y^2dy = \frac{1}{a}\int_0^a\frac{\pi}{2}\,y^2\,dy$$

$$= \frac{1}{a}\left[\frac{\pi y^3}{6}\right]_0^a = \frac{\pi a^2}{6}.$$

Ex.8. Evaluate $\int_0^a\int_0^{\sqrt{(a^2-x^2)}} y^2\sqrt{x^2+y^2}\,dy\,dx$ by changing polar co-ordinates

Solution:

Here the integral is

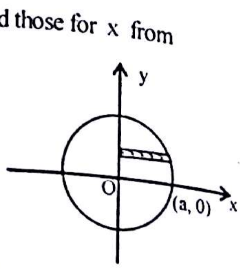
$$\int_0^a \int_0^{\sqrt{(a^2-x^2)}} y^2\sqrt{x^2+y^2}\,dy\,dx$$

The integration is first integrated with respect to y which extends from

$y = 0$ to $y = \sqrt{a^2 - x^2}$

i. e. $y = 0$ to $y^2 = a^2 - x^2$ and those for x from

$x = 0$ to $x = a$.



To change it into polar coordinates,

put $x = r\cos\theta$, $y = r\sin\theta$, $x^2 + y^2 = r^2$

and $dy\,dx = r\,dr\,d\theta$ in the given integral.

So $x^2 + y^2 = a^2$ becomes $r = a$.

Thus r varies from $r = 0$ to $r = a$ and those of θ varies from $\theta = 0$ to $\theta = \pi/2$.

Then $\displaystyle\int_0^a \int_0^{\sqrt{(a^2-x^2)}} y^2\sqrt{x^2+y^2}\,dy\,dx$

$$= \int_0^{\pi/2}\int_0^a r^2\sin^2\theta \; r.r\,dr\,d\theta$$

$$= \int_0^{\pi/2} \sin^2\theta\left[\int_0^a r^4\,dr\right]d\theta = \int_0^{\pi/2}\sin^2\theta\left[\frac{r^5}{5}\right]_0^a d\theta$$

$$= \frac{a^5}{5}\int_0^{\pi/2}\frac{(1-\cos 2\theta)}{2}\,d\theta$$

$$= \frac{a^5}{10}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi/2} = \frac{a^5}{10}\left[\frac{\pi}{2} - 0\right] = \frac{\pi a^5}{20}.$$

Ex. 9 .Evaluate $\displaystyle\int_0^2\int_0^{\sqrt{(2x-x^2)}} \frac{x\,dy\,dx}{\sqrt{x^2+y^2}}$ **by changing polar co-ordinates**

2062 Jestha, B.E

Solution:

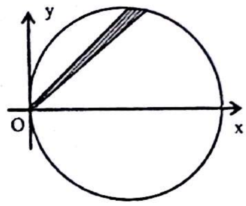
Here the integral is

$$\int_0^2\int_0^{\sqrt{(2x-x^2)}}\frac{x\,dy\,dx}{\sqrt{x^2+y^2}}.$$

The integral is first integrated with respect to y along vertical strip which extends from

$y = 0$ to $y = \sqrt{2x - x^2}$

i.e. $y = 0$ to $y^2 = 2x - x^2$ and then with respect to x as x varies from $x = 0$ to $x = 2$



To change it into polar coordinates, put $x = r\cos\theta$, $y = r\sin\theta$, $x^2 + y^2 = r^2$ and $dy\,dx = r\,dr\,d\theta$ in the given integral.

So $y^2 = 2x - x^2$ becomes $r^2\sin^2\theta = 2r\cos\theta - r^2\cos^2\theta$

$r^2 = 2r\cos\theta$ $\quad r = 0, 2\cos\theta$.

Clearly r varies from $r = 0$ to $r = 2\cos\theta$ and θ varies from $\theta = 0$ to $\theta = \pi/2$.

Then

$$\int_0^2\int_0^{\sqrt{(2x-x^2)}}\frac{x\,dy\,dx}{\sqrt{x^2+y^2}} = \int_{\theta=0}^{\pi/2}\int_{r=0}^{2\cos\theta}\frac{r\cos\theta\, r\,dr\,d\theta}{r}$$

$$= \int_0^{\pi/2}\cos\theta\int_0^{2\cos\theta} r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\cos\theta\left[\frac{r^2}{2}\right]_0^{2\cos\theta}d\theta$$

$$= \int_0^{\pi/2} 2\cos^2\theta\cos\theta\,d\theta = 2\int_0^{\pi/2}\cos^3\theta\,d\theta$$

$$= \frac{\sqrt{\pi}\,\Gamma\frac{(3+1)}{2}}{2\Gamma\left(\frac{3+1}{2}\right)} = \frac{\sqrt{\pi}}{\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}} = \frac{4}{3}.$$

Ex. 10.Evaluate $\displaystyle\int_0^1\int_0^{\sqrt{(1-y^2)}}\sqrt{\frac{1-x^2-y^2}{1+x^2+y^2}}\,dx\,dy$ **by changing to the polar co-ordinates**

Solution:

Here the integral is

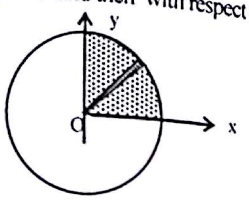
$$\int_0^1\int_0^{\sqrt{(1-y^2)}} \sqrt{\frac{1-x^2-y^2}{1+x^2+y^2}}\,dx\,dy$$

This integral is first integrated with respect to x as x varies from

$x = 0$ to $x = \sqrt{1-y^2}$

i.e. $x = 0$ to $x^2+y^2 = 1$ and then with respect to y as y varies from

$y = 0$ to $y = 1$.



To change it into polar coordinates, put $x = r\cos\theta$, $y = r\sin\theta$, $x^2+y^2=r^2$ and $dy\,dx = r\,dr\,d\theta$ in the given integral.

So $x^2+y^2 = 1$ becomes $r = a$. Clearly r varies from $r = 0$ to $r = 1$ and those for θ from $\theta = 0$ to $\theta = \pi/2$.

Then

$$\int_0^1\int_0^{\sqrt{(1-y^2)}} \sqrt{\frac{1-x^2-y^2}{1+x^2+y^2}}\,dx\,dy = \int_0^{\pi/2} d\theta \int_0^1 \sqrt{\frac{1-r^2}{1+r^2}}\, r\,dr$$

Putting $1+r^2=t^2$ so that $r\,dr = t\,dt$.

So

$$\int_0^1\int_0^{\sqrt{(1-y^2)}} \sqrt{\frac{1-x^2-y^2}{1+x^2+y^2}}\,dx\,dy = \int_0^{\pi/2}\int_1^{\sqrt{2}} \sqrt{2-t^2}\,dt\,d\theta$$

$$= \int_0^{\pi/2}\left[\frac{t\sqrt{2-t^2}}{2} + \frac{2}{2}\sin^{-1}\frac{t}{\sqrt{2}}\right]_1^{\sqrt{2}} d\theta$$

$$= \int_0^{\pi/2}\left(\frac{\pi}{2}-\frac{1}{2}-\frac{\pi}{4}\right) d\theta = \left(\frac{\pi}{4}-\frac{1}{2}\right)\frac{\pi}{2}$$

$$= \frac{\pi^2}{8} - \frac{\pi}{4}.$$

Exercise - 5

1. Change the order of integration and hence evaluate the same.

(i) $\int_0^3\int_1^{\sqrt{4-y}} (x+y)\,dx\,dy$ (ii) $\int_{-a}^{a}\int_0^{\sqrt{a^2-y^2}} dx\,dy$

(iii) $\int_0^a\int_{x^2/a}^{2a-x} xy\,dy\,dx$ (iv) $\int_0^\infty\int_x^\infty \frac{e^{-y}}{y}\,dy\,dx$

(v) $\int_0^1\int_x^{\sqrt{2-x^2}} \frac{x\,dy\,dx}{\sqrt{x^2+y^2}}$ [2061 Aswin, B.E.] (vi) $\int_0^1\int_0^{\sqrt{1-x^2}} y^2\,dx\,dy$

(vii) $\int_a^b\int_a^x xy\,dx\,dy$ (viii) $\int_0^a\int_0^x \frac{\cos y\,dy\,dx}{\sqrt{(a-x)(a-y)}}$ [2068 Bhadra, B. E.]

(ix) $\int_0^\infty\int_0^x x e^{-\frac{x^2}{y}}\,dy\,dx$ (x) $\int_0^{a/\sqrt{2}}\int_x^{\sqrt{a^2-x^2}} y^2\,dx\,dy$

2. Evaluate the following integrals by changing to polar co-ordinates

(i) $\int_0^a\int_x^a \frac{x\,dx\,dy}{x^2+y^2}$ (ii) $\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dy\,dx$ [2072 Aswin B. E]

(iii) $\int_0^a\int_0^{\sqrt{a^2-x^2}} \sqrt{x^2+y^2}\,dy\,dx$ (iv) $\int_0^1\int_x^{\sqrt{2x-x^2}} (x^2+y^2)\,dy\,dx$ [2067 071 Bhadra B. E.]

(v) $\int_0^{2a}\int_0^{\sqrt{2ax-x^2}} (x^2+y^2)\,dy\,dx$ [2067 Mangsir B. E.] (vi) $\int_0^a\int_0^{\sqrt{a^2-y^2}} (x^2+y^2)\,dy\,dx$

(vii) $\int_0^a\int_0^{\sqrt{a^2-x^2}} y^2\sqrt{(x^2+y^2)}\,dy\,dx$ [2070 072 Magh, B. E.] (viii) $\int_0^a\int_y^a \frac{x^2\,dx\,dy}{\sqrt{x^2+y^2}}$

(ix) $\int_0^2\int_0^{\sqrt{2x-x^2}} (x^2+y^2)\,dy\,dx$ (x) $\int_0^1\int_0^{\sqrt{1-x^2}} y^2\,dy\,dx$

Answers

1. (i) $\frac{241}{60}$ (ii) $\frac{\pi a^2}{2}$ (iii) $\frac{3a^4}{8}$

(iv) 1 (v) $1-\frac{1}{\sqrt{2}}$ (vi) $\frac{\pi}{16}$

(vii) $\frac{1}{8}(a^4+b^4-2a^2b^2)$ (viii) 2 sin a (ix) $\frac{1}{2}$

(x) $\left(\frac{2+\pi}{32}\right)a^4$

2. (i) $\frac{\pi a}{4}$ (ii) $\frac{\pi}{4}$ (iii) $\frac{\pi a^3}{6}$

(iv) $\frac{3\pi}{8}-1$ (v) $\frac{3\pi a^4}{4}$ (vi) $\frac{\pi a^4}{8}$

(vii) $\frac{\pi a^5}{20}$ (viii) $\frac{a^3}{3}\log\left(\sqrt{2}+1\right)$

(ix) $\frac{3\pi}{4}$ (x) $\frac{\pi}{16}$

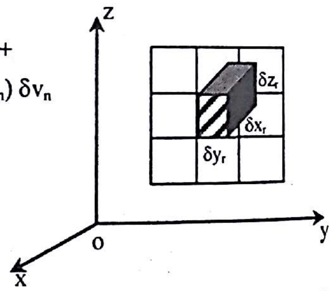
2.3 Triple Integrals

Consider of a function F(x, y, z) of three independent variables x, y and z defined at every point of three dimensional finite region v. Divide the region v into n elementary volumes $\delta v_1, \delta v_2, \ldots, \delta v_n$ and let $(x_1, y_1, z_1), (x_2, y_2, z_2), \ldots, (x_n, y_n, z_n)$ be any point to each elementary volumes.

Consider the sum

$$F(x_1, y_1, z_1)\,\delta v_1 + F(x_2, y_2, z_2)\,\delta v_2 + \ldots + F(x_n, y_n, z_n)\,\delta v_n$$

$$= \sum_{r=1}^{n} f(x_r, y_r, z_r)\,\delta v_r.$$



If this limit of the sum exists, as n tends to ∞ and δv_r tends 0, then it is called triple integral of F(x, y, z) over the region v, and is denoted by

$$\iiint_V F(x, y, z)\,dv = \lim_{\substack{n\to\infty \\ \delta v_r\to 0}} \sum_{r=1}^{n} F(x_r, y_r, z_r)\,\delta v_r.$$

For the purpose of evaluation, the triple integral can be written as the repeated integral as

$$\int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} F(x,y,z)\,dx\,dy\,dz.$$

For the repeated integral, we first integrate with respect to z between the limits $z = z_1(x, y)$ to $z = z_2(x, y)$ keeping x and y fixed. The resulting function is integrated with respect to y which extends from $y = y_1(x)$ to $y = y_2(x)$ keeping x constant. Finally, the result just obtained is integrated with respect to x which extends from $x = x_1$ to $x = x_2$, where x_1 and x_2 are constants. Thus

$$\iiint_V F(x, y, z)\,dv = \int_{x_1}^{x_2}\int_{y_1(x)}^{y_2(x)}\int_{z_1(x,y)}^{z_2(x,y)} F(x, y, z)\,dz\,dy\,dx.$$

2.3.1. Cylindrical Coordinates

Let OX, OY and OZ be three perpendicular axes and P (x, y, z) and P(r, θ, z) be the cartesian and cylindrical polar coordinates respectively, then

$$OM = x,\ OK = y \text{ and } PM = z,\ \ OL = r, \text{ and } \angle LOM = \theta.$$

We have $\cos\theta = \frac{OM}{OL} = \frac{x}{r}$

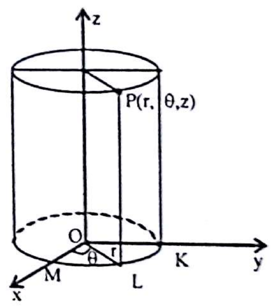
or $x = r\cos\theta$ and

or $\sin\theta = \frac{OK}{OL} = \frac{y}{r}$

or $y = r\sin\theta,\ \ z = z.$

So

$$dx\ dy\ dz = \frac{dx}{dr}\ \frac{dy}{d\theta}\ \frac{dz}{dz}\ dr\,d\theta\,dz$$



$$= J = \frac{\partial(x, y, z)}{\partial(r, \theta, z)}\, dr\, d\theta\, dz = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix} dr\, d\theta\, dz$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} dr\, d\theta\, dz$$

$$= (r\cos^2\theta + r\sin^2\theta)\, dr\, d\theta\, dz \quad = r\, dr\, d\theta\, dz.$$

$\therefore \quad dx\, dy\, dz = r\, dr\, d\theta\, dz.$

Hence, $\iiint_V F(x, y, z)\, dx\, dy\, dz = \iiint_V F(r\cos\theta, r\sin\theta, z)\, r\, dr\, d\theta\, dz$

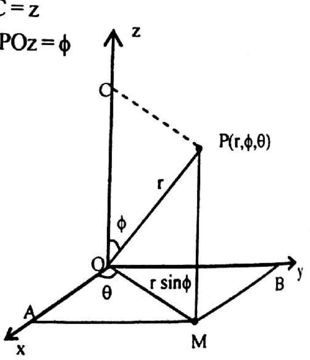
where v is the bounded region. The use of cylindrical coordinates is particularly appropriated for the problems that are an axis of symmetry of the solid. Instead of the volume element $dx\, dy\, dz$, we use in cylindrical coordinates, the volume of the element $r\, d\theta\ dr\ dz$ i. e. an element with a cross sectional area $r\, dr\, d\theta$ and altitude dz.

2.3.2. Spherical Polar Coordinates

Let OX,OY and OZ be three perpendicular axes and P(x, y,z)and P(r,ϕ ,θ) be the cartesian and cylindrical polar coordinates respectively, then

$OA = x,\ OB = y$ and $PM = OC = z$

$OP = r,\ \angle AOM = \theta$ and $\angle POz = \phi$



In right angle triangle OPM ,we have

$$\sin\phi = \frac{OM}{OP} = \frac{OM}{r}$$

$\therefore \quad OM = r\sin\phi$

From the right angle triangle AOM

$$\cos\theta = \frac{OA}{OM} = \frac{x}{r\sin\phi}$$

$\therefore \quad x = r\cos\theta\ \sin\phi$

From the right angle triangle BOM

$$\sin\theta = \frac{OB}{OM} = \frac{y}{r\sin\phi}$$

$\therefore \quad y = r\sin\theta\ \sin\phi$

From the right angle triangle POC

$$\cos\phi = \frac{OC}{OP} = \frac{z}{r}$$

$\therefore \quad z = r\cos\phi$

Now, $dx\, dy\, dz = \frac{dx}{dr}\frac{dy}{d\phi}\frac{dz}{d\theta}\, dr\, d\theta\, d\phi$

$$= J = \frac{\partial(x, y, z)}{\partial(r, \phi, \theta)}\, dr\, d\theta\, d\phi = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta} \end{vmatrix} dr\, d\theta\, d\phi$$

$$= \begin{vmatrix} \cos\theta\sin\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\sin\theta\cos\phi & r\cos\theta\sin\phi \\ \cos\phi & -r\sin\phi & 0 \end{vmatrix} dr\, d\theta\, d\phi$$

$$= (-r\sin\theta\sin\phi)\begin{vmatrix} \sin\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\phi & -r\sin\phi \end{vmatrix} dr\, d\theta\, d\phi$$

$$-r\cos\theta\sin\phi \begin{vmatrix} \cos\theta\sin\phi & r\cos\theta\cos\phi \\ \cos\phi & -r\sin\phi \end{vmatrix} dr\, d\theta\, d\phi$$

$$= [(-r\sin\theta\sin\phi)(-r\sin\theta\sin^2\phi - r\sin\theta\cos^2\phi)$$
$$-r\cos\theta\sin\phi\,(-r\cos\theta\sin^2\phi - r\cos\theta\cos^2\phi)]\, dr\, d\theta\, d\phi$$
$$= [r^2\sin^2\theta\sin\phi\,(\sin^2\phi + \cos^2\phi)$$
$$+ r^2\cos^2\theta\sin\phi\,(\sin^2\phi + \cos^2\phi)]\, dr\, d\theta\, d\phi$$
$$= (r^2\sin^2\theta\sin\phi + r^2\cos^2\theta\sin\phi)\, dr\, d\theta\, d\phi$$
$$= r^2\sin\phi\,(\sin^2\theta + \cos^2\theta)\, dr\, d\theta\, d\phi = r^2\sin\phi\ dr\, d\theta\, d\phi$$

$\therefore \quad dx\ dy\ dz = r^2\sin\phi\, dr\, d\theta\ d\phi.$

Hence, $\iiint_V F(x, y, z)\, dx\, dy\, dz$

$$= \iiint_V F(r\cos\theta\sin\phi, r\sin\theta\sin\phi, r\cos\phi)\, r^2\sin\phi\, dr\, d\theta\, d\phi$$

where v is the bounded region.

2.3.3 Diritchlet's Integral

Diritchlet's integral is an important integral for the evaluation of multiple integrals, which is stated as follows:

Theorem

If v is a region bounded by $x \geq 0$, $y \geq 0$ and $z \geq 0$ and $x + y + z \leq 1$, then

$$\iiint_V x^{l-1} y^{m-1} z^{n-1}\, dx\, dy\, dz = \frac{\Gamma l\, \Gamma m\, \Gamma n}{\Gamma(l + m + n + 1)}$$ [2059 Magh, B.E.]

Proof

The triple integral is first integrated with respect to z which extends from z = 0 to z = 1 - x - y and those for y extends from y = 0 to y = 1 - x, and x extends from x = 0 to x = 1.

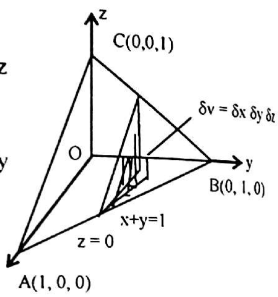
So $\iiint_V x^{l-1} y^{m-1} z^{n-1}\, dx\, dy\, dz$

$$= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} x^{l-1} y^{m-1} z^{n-1}\, dx\, dy\, dz$$

$$= \int_0^1 \int_0^{1-x} x^{l-1} y^{m-1} \int_0^{1-x-y} z^{n-1}\, dz\; dx\, dy$$

$$= \int_0^1 \int_0^{1-x} x^{l-1} y^{m-1} \left[\frac{z^n}{n}\right]_0^{1-x-y} dxdy$$

$$= \frac{1}{n} \int_0^1 \int_0^{1-x} x^{l-1} y^{m-1} (1-x-y)^n\, dy$$



Put $y = (1-x)t$ so that $dy = (1-x)\, dt$

$$= \frac{1}{n} \int_0^1 x^{l-1} \int_0^1 (1-x)^{m-1} t^{m-1} (1-x)^{n+1} (1-t)^n\, dt)$$

$$= \frac{1}{n} \int_0^1 \int_0^1 x^{l-1} (1-x)^{m+n} t^{m-1} (1-t)^n\, dt$$

$$= \frac{1}{n} \int_0^1 x^{l-1} (1-x)^{m+n+1-1}\, dx \int_0^1 t^{m-1} (1-t)^{n+1-1}\, dt$$

$$= \frac{1}{n}\, \beta(l, m+n+1)\; \beta(m, n+1) = \frac{1}{n} \frac{\Gamma l\, \Gamma(m+n+1) \Gamma m\;\, \Gamma(n+1)}{\Gamma(l+m+n+1)\; \Gamma(m+n+1)}$$

$$= \frac{1}{n} \frac{\Gamma l\, \Gamma m\;\, \Gamma(n+1)}{\Gamma(l+m+n+1)} = \frac{\Gamma l\, \Gamma m\, \Gamma n}{\Gamma(l+m+n+1)}$$

$$\therefore \iiint_V x^{l-1} y^{m-1} z^{n-1}\, dx\, dy\, dz = \frac{\Gamma l\, \Gamma m\, \Gamma n}{\Gamma(l+m+n+1)}.$$

Theorem

If R is region bounded by by $x \geq 0$, $y \geq 0$ and $x + y \leq h$, then prove that

$$\iint_R x^{l-1} y^{m-1}\, dx\, dy = h^{l+m} \frac{\Gamma(l)\, \Gamma(m)}{\Gamma(l + m + 1)}$$

Proof

Here the given integral is

$\iint_R x^{l-1} y^{m-1}\, dx\, dy$

Where R is the region bounded by $x \geq 0$, $y \geq 0$ and $x + y \leq h$.

So that y varies from

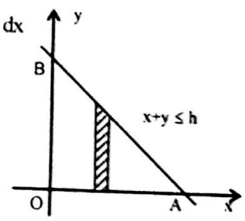
y = 0 to y = h - x and x varies from x = 0 to x = h.

$$\therefore \iint_R x^{l-1} y^{m-1}\, dx\, dy = \int_0^h \left[\int_0^{h-x} y^{m-1}\, dy\right] x^{l-1}\, dx$$

$$= \int_0^h \left[\frac{y^m}{m}\right]_0^{h-x} x^{l-1}\, dx$$

$$= \int_0^h \frac{(h-x)^m}{m}\, x^{l-1}\, dx$$

$$= \frac{1}{m} \int_0^h x^{l-1} (h-x)^m\, dx$$



Put x = ht so that dx = h dt

$$= \frac{1}{m} \int_0^1 h^{l-1} t^{l-1}\, h^m (1-t)^m\, h\, dt$$

$$= \frac{h^{l+m}}{m} \int_0^1 t^{l-1} (1-t)^{m+1-1}\, dt = \frac{h^{l+m}}{m}\, \beta(l, m+1)$$

$$= \frac{h^{l+1}}{m} \frac{\Gamma(m+1)\, \Gamma l}{\Gamma(m+1+l)} = \frac{h^{l+m}}{m} \frac{\Gamma(m+1)\, \Gamma l}{\Gamma(m+1+l)}$$

$$= \frac{h^{l+m}\, m\, \Gamma m\, \Gamma l}{m \Gamma(m+l+1)} = \frac{h^{l+m}\, \Gamma l\, \Gamma m}{\Gamma(l+m+1)}$$

$$\therefore \quad \iint_R x^{l-1} y^{m-1}\, dx\, dy = h^{l+m} \frac{\Gamma(l)\,\Gamma(m)}{\Gamma(l+m+1)}.$$

Workout Examples

Ex. 1 Evaluate $\int_0^1 \int_0^2 \int_1^2 x^2 yz\, dx\, dy\, dz$

Solution:

We have

$$\int_0^1 \int_0^2 \int_1^2 x^2yz\, dx\, dy\, dz = \int_0^1 \int_0^2 \int_1^2 z dz\, x^2 y\, dx\, dy$$

$$= \int_0^1 \int_0^2 \frac{1}{2} x^2 y\, [z^2]_1^2\, dx\, dy$$

$$= \frac{3}{2} \int_0^1 x^2 \int_0^2 y\, dy\, dx$$

$$= \frac{3}{2} \int_0^1 x^2 \left[\frac{y^2}{2}\right]_0^2 dx = \int_0^1 3x^2\, dx = 1$$

Ex.2 Evaluate $\int_0^1 \int_{y^2}^1 \int_0^{1-x} x\, dz\, dx\, dy$

Solution:

We have

$$\int_0^1 \int_{y^2}^1 \int_0^{1-x} x\, dz\, dx\, dy = \int_0^1 \int_{y^2}^1 x \int_0^{1-x} dz\, dx\, dy$$

= [illegible]

= [illegible]

= [illegible]

[illegible]

[illegible]

[illegible]

[illegible]

$$= \int_0^1 \int_1^2 \left[zx + yz + \frac{z^2}{2}\right]_2^3 dx\, dy$$

$$= \int_0^1 \int_1^2 \left(3x + 3y + \frac{9}{2} - 2x - 2y - 2\right) dy\, dx$$

$$= \int_0^1 \int_1^2 \left(x + y + \frac{5}{2}\right) dy\, dx$$

$$= \int_0^1 \left[xy + \frac{y^2}{2} + \frac{5y}{2}\right]_1^2 dx$$

$$= \int_0^1 (x+4)\, dx = \frac{9}{2}$$

Ex.4. Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{dz\, dy\, dx}{\sqrt{1-x^2-y^2-z^2}}$ **[2059 Shravan, B.E]**

Solution:

Let $I = \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{dz}{\sqrt{1-x^2-y^2-z^2}}\, dy\, dx$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} \left[\sin^{-1} \frac{z}{\sqrt{1-x^2-y^2}}\right]_0^{\sqrt{(1-x^2-y^2)}} dy\, dx$$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} \frac{\pi}{2}\, dy\, dx = \frac{\pi}{2} \int_0^1 \sqrt{1-x^2}\, dx = \frac{\pi^2}{8}.$$

Ex.5. Evaluate $\int_0^{\pi/2} \int_0^{a\sin\theta} \int_0^{\frac{a^2-r^2}{a}} r\, dz\, dr\, d\theta$

Solution:

Let $I = \int_0^{\pi/2} \int_0^{a\sin\theta} \int_0^{\frac{a^2-r^2}{a}} (r dz)\, dr\, d\theta = \int_0^{\pi/2} \int_0^{a\sin\theta} \left(r \frac{(a^2-r^2)}{a} dr\right) d\theta$

$$= \frac{1}{a} \int_0^{\pi/2} \left[\frac{a^2r^2}{2} - \frac{r^4}{4}\right]_0^{a\sin\theta} d\theta = \frac{1}{a} \int_0^{\pi/2} \left(\frac{a^4 \sin^2\theta}{2} - \frac{a^4 \sin^4\theta}{4}\right) d\theta$$

$$= \frac{a^3}{4} \int_0^{\pi/2} (2\sin^2\theta - \sin^4\theta)\, d\theta$$

$$=\frac{a^3}{4}\left[\frac{2\sqrt{\pi}\,\Gamma\frac{(2+1)}{2}}{2\Gamma\left(\frac{2+2}{2}\right)}-\frac{\sqrt{\pi}}{2}\frac{\Gamma\left(\frac{4+1}{2}\right)}{2\Gamma\left(\frac{4+2}{2}\right)}\right]$$

$$=\frac{a^3}{4}\left[\sqrt{\pi}\frac{1}{2}\sqrt{\pi}-\frac{\sqrt{\pi}\,\frac{3}{2}\times\frac{1}{2}\sqrt{\pi}}{2\times2\times1}\right]=\frac{a^3}{4}\left(\frac{\pi}{2}-\frac{3\pi}{16}\right)=\frac{5\pi a^3}{64}$$

Ex.6.Evaluate $\int_0^{\log 2}\int_0^{x}\int_0^{x+\log y} e^{x+y+z}\,dz\,dy\,dx$ **2070, Bhadra B. E.**

Solution:

Let $I=\int_0^{\log 2}\int_0^{x}\int_0^{x+\log y} e^{z}dz\; e^{x+y}\,dy\,dx$

$=\int_0^{\log 2}\int_0^{x}[e^z]_0^{x+\log y}\; e^{x+y}\,dydx$

$=\int_0^{\log 2}\int_0^{x}(ye^x-1)\,e^{x+y}\,dy.dx$

$=\int_0^{\log 2}\int_0^{x}(ye^y.e^x-e^y\,dy)\,e^x\,dx$

$=\int_0^{\log 2}[ye^{x+y}-e^{y+x}-e^y]_0^x\; e^x\,dx$

$=\int_0^{\log 2}(xe^{3x}-e^{3x}+e^x)\,dx=\left[\frac{xe^{3x}}{3}-\frac{e^{3x}}{9}-\frac{e^{3x}}{3}+e^x\right]_0^{\log 2}$

$=\frac{8}{3}\log 2-\frac{8}{9}-\frac{8}{3}+2+\frac{1}{9}+\frac{1}{3}-1=\frac{8}{3}\log 2-\frac{19}{9}$

Ex.7.Evaluate $\int_1^{e}\int_1^{\log y}\int_1^{e^x}\log z\,dz\,dx\,dy$ **2069/070 Poush, B. E.**

Solution:

Let $I=\int_1^{e}\int_1^{\log y}\int_1^{e^x}\log z\,dz\;dx\,dy=\int_1^{e}\int_1^{\log y}[z\log z-z]_1^{e^x}\;dx\,dy$

$=\int_1^{e}\int_1^{\log y}(xe^x-e^x+1)\,dx\;dy=\int_1^{e}[xe^x-e^x-e^x+x]_1^{\log y}\;dy$

$=\int_1^{e}[(y\log y-2y+\log y)-e+2e-1]\,dy$

$=\int_1^{e}(y\log y-2y+\log y+e-1)\,dy$

$=\left[\frac{y^2}{2}\log y-\frac{y^2}{4}-y^2+y\log y-y+ey-y\right]_1^e$

$=\frac{e^2}{2}-\frac{e^2}{4}-e^2+e-e+e^2-e+\frac{1}{4}+1+1-e+1$

$=\frac{e^2}{4}-2e+\frac{13}{4}=\frac{1}{4}(e^2-8e+13)$

Ex.8. Evaluate $\int_0^{a}\int_0^{x}\int_0^{x+y} e^{x+y+z}$ **dx dydz**

Solution:

Let I $=\int_0^{a}\int_0^{x}\int_0^{x+y} e^z\,dz\; e^{x+y}\,dx\,dy$

$=\int_0^{a}\int_0^{x}(e^{x+y}-1)\,e^{x+y}\,dy\,dx=\int_0^{a}\left[\frac{e^x e^{2y}}{2}-e^y\right]_0^x e^x\,dx$

$=\int_0^{a}\left(\frac{e^{3x}}{2}-e^x-\frac{e^x}{2}+1\right)e^x\,dx=\int_0^{a}\left(\frac{e^{4x}}{2}-\frac{3e^{2x}}{2}+e^x\right)dx$

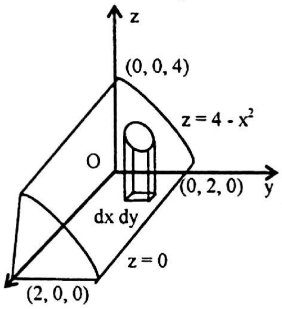
$=\left[\frac{e^{4x}}{8}-\frac{3e^{2x}}{4}+e^x\right]_0^a=\frac{e^{4a}}{8}-\frac{3e^{2a}}{4}+e^a-\frac{1}{8}+\frac{3}{4}-1$

$=\frac{1}{8}e^{4a}-\frac{3}{4}e^{2a}+e^a-\frac{3}{8}$

Ex.9.Evaluate $\iiint_R$ **(2x + y) dx dy dz where R is closed region bounded by the cylinder z =4 -x² and the planes x = 0, y = 0, y = 2, z = 0**

Solution:

Here the region bounded by the planes $z=0$ and the surface $z=4-x^2$ of the cylinder. The cylinder $z=4-x^2$ meets x-axis at (2, 0, 0) and y-axis at (0, 2, 0) in the given region, so that z extends from $z=0$ to $z=4-x^2$ and those for y extends from $y=0$ to $y=2$ and $x=0$ to 2.



$\iiint_R(2x+y)\,dx\,dy\,dz$

$=\int_0^2\int_{y=0}^{2}\int_{z=0}^{4-x^2}(2x+y)\,dx\,dy\,dz$

$=\int_0^2\int_0^2(4-x^2)(2x+y)\,dx\,dy$

$=\int_0^2\int_0^2[\{8x-2x^3+(4-x^2)\,y\}dy]dx$

$=\int_0^2\left[8xy-2x^3y+\frac{(4-x^2)\,y^2}{2}\right]_0^2dx=\int_0^2[16x-4x^3+2(4-x^2)]dx$

$=\left[8x^2-x^4+8x-\frac{2}{3}x^3\right]_0^2=32-16+16-\frac{16}{3}=\frac{80}{3}.$

Ex.10. Evaluate $\iiint x^{l-1} y^{m-1} z^{n-1}\, dx\, dy\, dz$ where x, y, z are all positive but $\left(\frac{x}{a}\right)^p + \left(\frac{y}{b}\right)^q + \left(\frac{z}{c}\right)^r \le 1$ [2067/70/072 Magh, B. E.]

Solution:

Let $I = \iiint x^{l-1} y^{m-1} z^{n-1}\, dx\, dy\, dz$

Putting $\left(\frac{x}{a}\right)^p = u$, $\left(\frac{y}{b}\right)^q = v$, $\left(\frac{z}{c}\right)^r = w$,

$x = a u^{1/p}$, $y = b v^{1/q}$, $z = c w^{1/r}$

$dx = \frac{a}{p} u^{\frac{1}{p}-1} du$, $dy = \frac{b}{q} v^{\frac{1}{p}-1} dv$, $dz = \frac{c}{r} w^{\frac{1}{r}-1} dw$

So $I = \iiint x^{l-1} y^{m-1} z^{n-1}\, dx\, dy\, dz$

$$= \iiint \left(au^{\frac{1}{p}}\right)^{l-1} \left(bv^{\frac{1}{q}}\right)^{m-1} \left(cw^{\frac{1}{r}}\right)^{n-1} \left(\frac{a}{p}\right) u^{\frac{1}{p}-1} \left(\frac{b}{q}\right) v^{\frac{1}{q}-1} \left(\frac{c}{r}\right) w^{\frac{1}{r}-1}\, du\, dv\, dw$$

$$= \frac{a^l b^m c^n}{pqr} \iiint u^{\frac{l}{p}-1} v^{\frac{m}{q}-1} w^{\frac{n}{r}-1}\, du\, dv\, dw \quad \text{where } u + v + w \le 1.$$

So by using Diritchlet's theorem

$$I = \frac{a^l b^m c^n}{prq} \frac{\Gamma \frac{l}{p} \Gamma \frac{m}{q} \Gamma \frac{n}{r}}{\Gamma\left(\frac{l}{p} + \frac{m}{q} + \frac{n}{r} + 1\right)}$$

Ex. 11. Prove that $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sqrt{x^2+y^2+z^2}\, e^{-(x^2+y^2+z^2)}\, dx\, dy\, dz = 2\pi$ by changing into spherical polar coordinates [2060 Chaitra, B.E.]

Solution:

Here $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sqrt{x^2+y^2+z^2}\, e^{-(x^2+y^2+z^2)}\, dx\, dy\, dz$

Put, $x = \cos\theta \sin\phi$, $y = \sin\theta \sin\phi$, $z = \cos\phi$, $x^2 + y^2 + z^2 = r^2$,

$dx\, dy\, dz = r^2 \sin\phi\, dr\, d\theta\, d\phi$

So r varies from $r = 0$ to $r = \infty$.

θ varies from $\theta = 0$ to $\theta = \pi$

ϕ varies from $\phi = -\pi$ to $\phi = \pi$.

We have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sqrt{x^2+y^2+z^2}\, e^{-(x^2+y^2+z^2)}\, dx\, dy\, dz$$

$$= \int_0^{\infty}\int_{-\pi}^{\pi}\int_0^{\pi} r e^{-r^2} r^2 (\sin\phi\, d\phi)\, d\theta\, dr$$

$$= \int_0^{\infty}\int_{-\pi}^{\pi} r e^{-r^2} r^2 \left[\frac{\cos\phi}{-1}\right]_0^{\pi} d\theta\, dr$$

$$= \int_0^{\infty}\int_{-\pi}^{\pi} r e^{-r^2} r^2 (1 + 1)\, d\theta\, dr$$

$$= 2\int_0^{\infty} re^{-r^2} r^2 \left[\theta\right]_{-\pi}^{\pi} dr = 2\pi \int_0^{\infty} r^2 e^{-r^2} (2r\, dr)$$

Putting $r^2 = t$, $2r\, dr = dt$.

When $r = 0, t = 0$, when $r = \infty$, $t = \infty$.

$$= 2\pi \int_0^{\infty} t e^{-t} dt = 2\pi \int_0^{\infty} t e^{-t} dt$$

$$= 2\pi \left[t \frac{e^{-t}}{-1} - 1 \times \frac{e^{-t}}{(-1)^2} \right]_0^{\infty} = 2\pi [0 + 1] = 2\pi$$

$$\therefore \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sqrt{x^2+y^2+z^2}\, e^{-(x^2+y^2+z^2)}\, dx\, dy\, dz = 2\pi.$$

Exercise - 6

1. Evaluate $\int_0^1 \int_0^z \int_{x-z}^{x+z} (x + y + z)\, dx\, dy\, dz$
2. Evaluate $\int_0^1 \int_0^{\sqrt{(1-x^2)}} \int_0^{\sqrt{(1-x^2-y^2)}} xyz\, dz\, dy\, dx$
3. Evaluate $\int_0^1 \int_0^{1-x} \int_0^{1-x-y} \frac{dx\, dy\, dz}{(x+y+z+1)^3}$
4. Evaluate $\int_0^4 \int_0^{2\sqrt{z}} \int_0^{\sqrt{(4z-x^2)}} dz\, dx\, dy$

5. Evaluate $\iiint_V (x - 2y + z)\, dx\, dy\, dz$ where v is the region bounded by $0 \le x \le 1,\ 0 \le y \le x^2,\ 0 \le z \le x + y$

6. Evaluate $\iiint_V x^2\, dx\, dy\, dz$ over the region v bounded by the planes $x = 0, y = 0, z = 0$ and $x + y + z = a$ [2069, Bhadra B. E.]

7. Evaluate $\iiint_V dx\, dy\, dz$ over the region bounded by the planes $x = 0, y = 0, z = 0, y = 8$ and cylinder $x^2 + z^2 = 9$

8. Evaluate $\iiint_V x\, dv$ where v is the region bonded by co-ordinate planes and the plane $x + y + z = 1$ [2072 Aswin B. E]

9. Evaluate $\iiint_V x\, dx\, dy\, dz$ where v is region in the first octant bounded by surface $x^{2/3} + y^{2/3} + z^{2/3} = 1$ [2065 Kartik, B.E]

10. Evaluate $\iiint_V x^{\alpha-1} y^{\beta-1} z^{\gamma-1}\, dx\, dy\, dz$ where v denotes the region in the first octant bounded by the surface $x^2 + y^2 + z^2 = 1$

Answers

1. 1 2. $\frac{1}{48}$ 3. $\frac{1}{2}\left(\log 2 - \frac{5}{8}\right)$

4. 8π 5. $\frac{8}{35}$ 6. $\frac{1}{6}a^5$ 7. 18π

8. $\frac{1}{24}$ 9. $\frac{3\pi}{1280}$ 10. $\frac{1}{8}\dfrac{\Gamma\frac{\alpha}{2}\Gamma\frac{\beta}{2}\Gamma\frac{\gamma}{2}}{\Gamma\left(\frac{\alpha}{2}+\frac{\beta}{2}+\frac{\gamma}{2}+1\right)}$

2.4 Application of Double and Triple Integrals

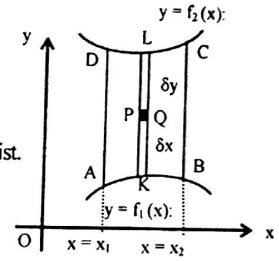
2.4.1 Areas Enclosed by Plane Curves

(i) Cartesian Coordinates

Let $y = f_1(x)$ and $y = f_2(x)$ be two curves, we have to find out the area enclosed by these two curves and ordinate $x = x_1, x = x_2$.

Divide this area into vertical strips. Take $P(x, y)$ and $Q(x + \delta x, y + \delta y)$ be two neighboring points so that δx be width of the strip then area of small rectangle $PQ = \delta x\, \delta y$.

Area of the strip $KL = \lim\limits_{\delta y \to 0} \Sigma\, \delta x\, \delta y$. Since width, δx is the same for all rectangles in these strips so that y varies from $y = f_1(x)$ to $y = f_2(x)$.



$\therefore$ Area of the strip KL

$$= \delta x \lim_{\delta y \to 0} \sum_{f_1(x)}^{f_2(x)} \delta y \text{ provided the limit exist.}$$

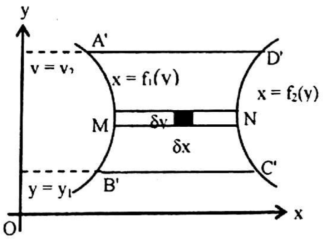
$$= \delta x \int_{f_1(x)}^{f_2(x)} dy$$

Adding all such strips from $x = x_1$ to $x = x_2$ we get the area ABCD

$$= \lim_{\delta x \to 0} \sum_{x_1}^{x_2} \delta x \int_{f_1(x)}^{f_2(x)} dy = \int_{x_1}^{x_2} dx \int_{f_1(x)}^{f_2(x)} dy$$

The required area $= \int_{x_1}^{x_2} \int_{f_1(x)}^{f_2(x)} dx\, dy$.

Similarly, if the region is bounded by the curves $x = f_1(y)$ and $x = f_2(y)$, then we can also find the area enclosed by these curves and $y = y_1$ to $y = y_2$ and get the area A'B'C'D'.



Thus,

The required area $= \int_{y_1}^{y_2} \int_{f_1(y)}^{f_2(y)} dx\, dy$

(ii) Polar Coordinates

Let $r = f(\theta)$ be equation of the enclosed curve and $P(r, \theta)$ $Q(r + \delta r, \theta + \delta\theta)$ be two neighboring points.

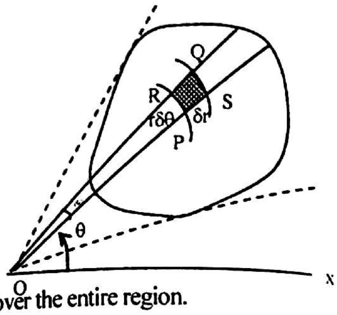
Making circular areas of radii r and $r + \delta r$ meeting OQ in R and OP in S so that arc $PR = r\,\delta\theta$ and $PS = \delta r$.

Therefore the area of curvilinear rectangle PRQS $= r\,\delta\theta . \delta r$.

The sum of such these curvilinear rectangles $\sum\sum r\,\delta\theta\,dr$ gives in the limit the area A.

$$\text{i.e. } A = \lim_{\substack{\delta r\to 0\\ \delta\theta\to 0}} \sum\sum r\,\delta\theta\,\delta r$$

$$= \iint_R r\,d\theta\,dr$$



provided the limit exist.

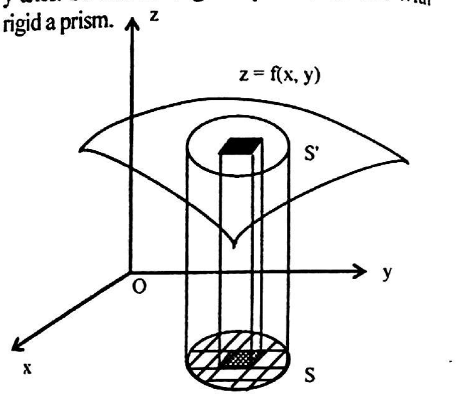
$$A = \int_{\alpha}^{\beta}\int_{r_1}^{r_2} r\,d\theta\,dr$$

Where the limits are to be chosen such that $\theta=\alpha$ to $\theta=\beta$ and $r=r_1$ to $r=r_2$ as to cover the entire region.

2.4.2 Volume

(i) Volume as double integrals

Let $z = f(x, y)$ be a surface the projection of the portion of the surface S on the xy - plane be area S. Divide S into elementary rectangles of area $\delta x\,\delta y$ by drawing lines parallel to x and y axes. So that its length is parallel to OZ with each of these rectangles as base rigid a prism.



$\therefore$ Volume of this prism between S and given surface $z = f(x, y)$ is $z\,\delta x\,\delta y$.

Therefore, the volume of the solid of cylinder bounded by the given surface S as base with generator parallel to z-axis.

$$= \lim_{\substack{\delta x\to 0\\ \delta y\to 0}} \sum\sum z\,\delta x\,\delta y = \iint z\,dx\,dy$$

provided the limit exist.

$v = \iint f(x, y)\,dx\,dy$ where the limits of integration is to be taken over the area S.

Note:

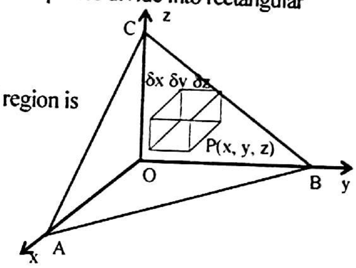
In polar coordinates replacing dx dy by $r\,dr\,d\theta$

$$\therefore \text{ volume} = \iint z\,r\,dr\,d\theta.$$

(ii) Volume as triple integral

If the solid planes parallel to the coordinate planes divide into rectangular parallelepiped of volume

$$\delta x\,\delta y\,\delta z.$$



Here, the total volume of the bounded region is

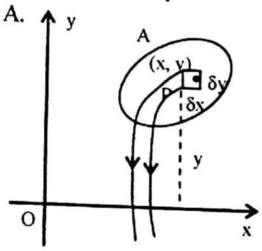
$$= \lim_{\substack{\delta x\to 0\\ \delta y\to 0\\ \delta z\to 0}} \sum\sum\sum \delta x\,\delta y\,\delta z$$

$$\text{Volume} = \iiint dx\,dy\,dz.$$

Where limits of integration is to be taken for bounded region.

(iii) Volume of solids of revolution

Let $y = f(x)$ be the equation of curve in the plane and P(x, y) be a point on the curve in which $\delta x\,\delta y$ be elementary plane area A.



If this elementary area revolves about x-axis, we get a ring of volume

$$\pi[(y+\delta y)^2 - y^2]\delta x \quad = 2\pi y\,\delta x\,\delta y$$

Hence, the total volume of solid obtained by revolving the area A about x-axis is

$$= \iint_A 2\pi y\,dx\,dy$$

In polar co-ordinates, putting $x = r\cos\theta$, $y = r\sin\theta$ and $dx\,dy = r\,dr\,d\theta$, we get

$$\text{Volume} = \iint_A 2\pi r\sin\theta\, r\,d\theta\,dr = \iint_A 2\pi r^2\sin\theta\,d\theta\,dr.$$

Similarly, the volume of solid obtained by revolving the area A about y-axis

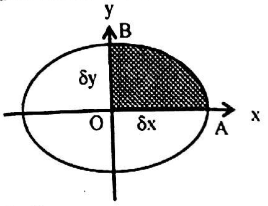
$$= \iint_A 2\pi x \, dx \, dy.$$

Worked out Examples

Ex.1. Find, by double integration, that the area of a plate in the form of a quadrant of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Solution:

Dividing the area of a quadrant of ellipse in the vertical strip of width dx, as y varies from $y = 0$ to $y = \frac{b}{a}\sqrt{a^2 - x^2}$ and then x varies from $x = 0$ to $x = a$.



$$\therefore \text{ the required area} = \int_0^a \int_0^{\frac{b}{a}\sqrt{a^2 - x^2}} dx\, dy$$

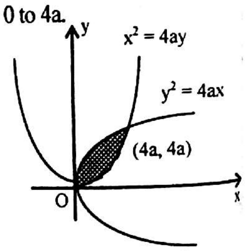
$$= \int_0^a \int_0^{\frac{b}{a}\sqrt{a^2 - x^2}} dy\, dx = \frac{b}{a}\int_0^a \sqrt{a^2 - x^2}\, dx$$

$$= \frac{b}{a}\left[\frac{x\sqrt{a^2 - x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a = \frac{b}{a}\frac{a^2}{2}\frac{\pi}{2} = \frac{\pi ab}{4}$$

Ex.2. Show, by double integration, that the area between the parabolas $y^2 = 4ax$ and $x^2 = 4ay$ is $\frac{16}{3}a^2$ $\boxed{\textit{2064 Paush, B.E}}$

Solution:

Dividing the area between two given parabolas into the vertical strip as y varies from $y = \frac{x^2}{4a}$ to $y = 2\sqrt{ax}$ and then x varies from $x = 0$ to $4a$.



$$\therefore \text{ Required area} = \int_0^{4a} \int_{x^2/4a}^{2\sqrt{ax}} dx\, dy$$

$$= \int_0^{4a} \left(2\sqrt{ax} - \frac{x^2}{4a}\right) dx$$

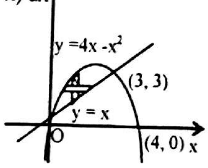
$$= \left[\frac{4}{3}\sqrt{a}\, x^{\frac{3}{2}} - \frac{x^3}{12a}\right]_0^{4a} = \frac{32a^2}{3} - \frac{16a^2}{3} = \frac{16a^2}{3}$$

Ex.3. Find, by double integration, the area lying between the parabola $y = 4x - x^2$ and the line $y = x$.

Solution:

In the figure, dividing the area bounded by $y = 4x - x^2$ and the line $y = x$ into vertical strip as y varies from $y = 4x - x^2$ to $y = x$ and then $x = 0$ to $x = 3$.

$$\therefore \text{ Required area} = \int_0^3 \int_x^{4x - x^2} dx\, dy = \int_0^3 (4x + x^2 - x)\, dx$$



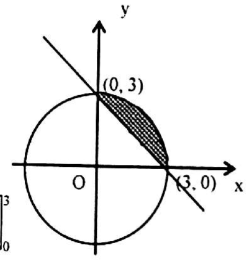
$$= \int_0^3 (3x - x^2)\, dx = \left[\frac{3x^2}{2} - \frac{x^3}{3}\right]_0^3$$

$$= \frac{27}{2} - 9 = \frac{9}{2}$$

Ex. 4. Find, by double integration, the smaller of the areas bounded by the curve $x^2 + y^2 = 9$ and the line $x + y = 3$

Solution:

Here the area enclosed between the circle $x^2 + y^2 = 9$ and the line $x + y = 3$ divide into vertical strip as y extends from $y = 3 - x$ to $y = \sqrt{9 - x^2}$ and then x varies from $x = 0$ to 3.



$$\text{Area} = \int_0^3 \int_{3-x}^{\sqrt{9-x^2}} dx\, dy$$

$$= \int_0^3 \left(\sqrt{9 - x^2} - 3 + x\right) dx.$$

$$= \left[\frac{x\sqrt{9 - x^2}}{2} + \frac{9}{2}\sin^{-1}\frac{x}{3} - 3x + \frac{x^2}{2}\right]_0^3$$

$$= \frac{9\pi}{4} - \frac{9}{2} = \frac{9}{4}(\pi - 2).$$

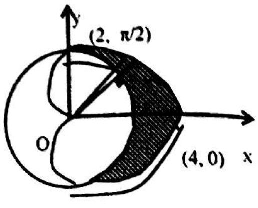
Ex. 5. Find, by double integration, the area outside the circle $r = 2$ and inside the cardioid $r = 2(1 + \cos\theta)$ $\boxed{\textit{2062 Bhadra, B.E.}}$

Solution:

Here the area is outside the circle $r = 2$ and in side the Cardioid $r = 2(1 + \cos\theta)$

The required area $= \iint r\, dr\, d\theta$ where the limit of integration are

$r = 2$ to $r = 2(1 + \cos\theta)$ and then $\theta = 0$ to $\pi/2$



$$\therefore \text{Required area} = 2\int_0^{\pi/2}\int_2^{2(1+\cos\theta)} r\,dr\,d\theta$$

$$= \int_0^{\pi/2} [r^2]_2^{2(1+\cos\theta)}\,d\theta$$

$$= 4\int_0^{\pi/2} (1 + 2\cos\theta + \cos^2\theta - 1)\,d\theta$$

$$= 2\int_0^{\pi/2} (4\cos\theta + 1 + \cos 2\theta)\,d\theta$$

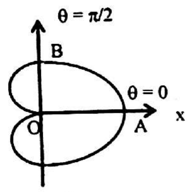
$$= 2\left[4\sin\theta + \theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/2} = 2\left[4 + \frac{\pi}{2} + 0\right] = 8 + \pi$$

Ex. 6. Find the area of the curve $r = a(1 + \cos\theta)$ by double integration

Solution:

Here the equation of a curve is

$$r = a(1 + \cos\theta)$$



The area of this curve is

$$= 2 \times \text{area OABO}$$

$$= 2\iint r\,dr\,d\theta$$

Where the limits of integration from

$r = 0$ to $r = a(1 + \cos\theta)$ and $\theta = 0$ to π.

$$\therefore \text{Area} = 2\int_0^{\pi}\int_0^{a(1+\cos\theta)} r\,dr\,d\theta = 2\int_0^{\pi}\left[\frac{r^2}{2}\right]_0^{a(1+\cos\theta)} d\theta$$

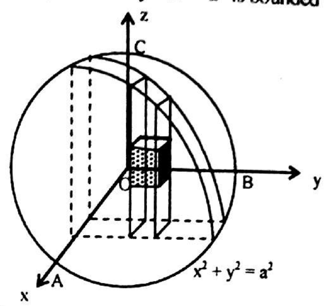
$$= 2\int_0^{\pi} a^2(1 + 2\cos\theta + \cos^2\theta)\,d\theta = a^2\int_0^{\pi}\left(1 + 2\cos\theta + \frac{1 + \cos 2\theta}{2}\right)d\theta$$

$$= a^2\left[\theta + 2\sin\theta + \frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_0^{\pi} = \frac{3\pi a^2}{2}$$

Ex.7. Find by double integration, the volume of the sphere $x^2 + y^2 + z^2 = a^2$

Solution:

In the figure OABC is the first octant of the sphere $x^2 + y^2 + z^2 = a^2$ is bounded by the planes $z = 0$, $x = 0$ and $y = 0$



Required volume $= 8\iint_R z\,dx\,dy$.

In the region R, y varies from $y = 0$ to $y = \sqrt{a^2 - x^2}$ and then x varies from $x = 0$ to $x = a$.

$$\text{Volume} = 8\int_0^a\int_{y=0}^{\sqrt{a^2-x^2}} \sqrt{a^2 - x^2 - y^2}\,dx\,dy$$

$$= 8\int_0^a\left[\frac{y\sqrt{a^2 - x^2 - y^2}}{2} + \frac{(a^2 - x^2)}{2}\sin^{-1}\frac{y}{\sqrt{a^2 - x^2}}\right]_0^{\sqrt{(a^2-x^2)}} dx$$

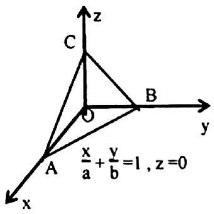
$$= 8\int_0^a \frac{(a^2 - x^2)}{2}\frac{\pi}{2}\,dx = \frac{4\pi a^3}{3}.$$

Ex.8. Find, by double integration, the volume of the tetrahedron bounded by the coordinate planes and the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

Solution:

Here the equation of the plane is

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$



This plane is bounded by the coordinate planes $x = 0$, $y = 0$ and $z = 0$.

$$\text{Volume} = \int_0^a\int_0^{\frac{b}{a}(a-x)} z\,dx\,dy$$

$$= \int_0^a\int_0^{\frac{b}{a}(a-x)} c\left(1 - \frac{x}{a} - \frac{y}{b}\right)dx\,dy = \int_0^a\left[c\left(y - \frac{xy}{a} - \frac{y^2}{2b}\right)\right]_0^{\frac{b}{a}(a-x)}$$

$$= c\int_0^a\left(\frac{b}{2} - \frac{bx}{a} + \frac{bx^2}{2a^2}\right)dx = c\left[\frac{bx}{2} - \frac{bx^2}{2a} + \frac{bx^3}{6a^2}\right]_0^a = \frac{abc}{6}.$$

Ex.9. Find, by double integration, the volume bounded by the cylinder $x^2+y^2=4$ and the planes $y+z=4$ and $z=0$ [2065 Chaitra B...]

Solution:

The plane $y+z=4$ cuts the cylinder $x^2+y^2=4$ so that $z=4-y$ is to be integrated over the circle $x^2+y^2=4$ in the xy - plane in which x varies from $=0$ to $x=\sqrt{4-y^2}$ and then y varies from - 2 to 2.

Required volume

$$=2\int_{-2}^{2}\int_{0}^{\sqrt{4-y^2}} z\,dx\,dy$$

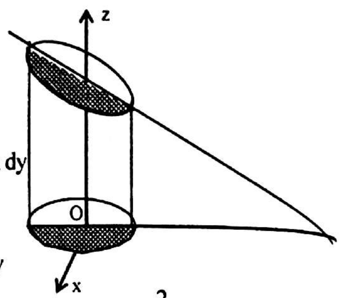
$$=2\int_{-2}^{2}\int_{0}^{\sqrt{4-y^2}}(4-y)\,dx\,dy$$

$$=2\int_{-2}^{2}(4-y)\sqrt{4-y^2}\,dy$$

$$=8\int_{-2}^{2}\sqrt{4-y^2}-2\int_{-2}^{2}y\sqrt{4-y^2}\,dy=16\int_{0}^{2}\sqrt{4-y^2}\,dy-0$$

$(\because y\sqrt{4-y^2}$ is odd function$)$

$$=16\left[\frac{y\sqrt{4-y^2}}{2}+\frac{4}{2}\sin^{-1}\frac{y}{2}\right]_0^2=16\pi$$



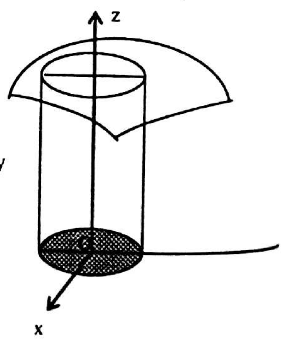
Ex. 10: Find, by double integration, the volume bounded by the plane $z=0$, surface $z=x^2+y^2+2$ and the cylinder $x^2+y^2=4$ [2068 Bhadra, B.E...]

Solution:

The surface $z=x^2+y^2+2$ is to be integrated over the circle $x^2+y^2=4$ in the xy- plane. So that x varies from $x=-\sqrt{4-y^2}$ to $x=\sqrt{4-y^2}$ and those for y varies from $y=-2$ to $y=2$.

$\therefore$ the required volume

$$=\int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} z\,dx\,dy$$



$$=\int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}}(x^2+y^2+2)\,dx\,dy$$

$$=2\int_{-2}^{2}\int_{0}^{\sqrt{4-y^2}}(x^2+y^2+2)\,dx\,dy$$

$$=2\int_{-2}^{2}\left[\frac{x^3}{3}+(y^2+2)x\right]_0^{\sqrt{4-y^2}}dy$$

$$=2\int_{-2}^{2}\left[\frac{1}{3}(4-y^2)^{3/2}+(y^2+2)\sqrt{4-y^2}\right]dy$$

$$=\frac{4}{3}\int_0^2(4-y^2)^{3/2}dy+4\int_0^2 y^2\sqrt{4-y^2}\,dy+8\int_0^2\sqrt{4-y^2}\,dy$$

Put $y=2\sin\theta$, $dy=2\cos\theta\,d\theta$

$$=\frac{64}{3}\int_0^{\pi/2}\cos^4\theta\,d\theta+64\int_0^{\pi/2}\sin^2\theta\cos^2\theta\,d\theta+32\int_0^{\pi/2}\cos^2\theta\,d\theta.$$

Using Gamma function

$$=\frac{64}{3}\frac{\sqrt{\pi}\,\Gamma\left(\frac{4+1}{2}\right)}{2\Gamma\left(\frac{4+2}{2}\right)}+64\frac{\Gamma\left(\frac{2+1}{2}\right)\Gamma\left(\frac{2+1}{2}\right)}{2\Gamma\left(\frac{2+2+2}{2}\right)}+32\frac{\sqrt{\pi}\,\Gamma\left(\frac{2+1}{2}\right)}{2\Gamma\left(\frac{2+2}{2}\right)}$$

$$=\frac{64}{3}\frac{\sqrt{\pi}\frac{3}{2}\times\frac{1}{2}\sqrt{\pi}}{2\times2\times1}+64\frac{\frac{1}{2}\sqrt{\pi}\ \frac{1}{2}\sqrt{\pi}}{2\times2\times1}+32\frac{\sqrt{\pi}\frac{1}{2}\sqrt{\pi}}{2\times1}$$

$$=4\pi+4\pi+8\pi=16\pi.$$

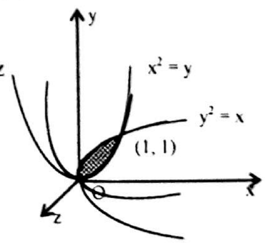
Ex. 11 .Find, by triple Integration, the volume of the region bounded by the surface $y=x^2$ and $x=y^2$ and the planes $z=0$, $z=3$

Solution:

Here the region bounded by the parabolas $y=x^2$ and $x=y^2$ and the plane $z=0$, $z=3$ so that x varies from $x=y^2$ to $x=\sqrt{y}$, $y=0$ to 1 & $z=0$ to 3.

$$\text{Volume}=\int_{z=0}^{3}\int_{y=0}^{1}\int_{x=y^2}^{\sqrt{y}}dx\,dy\,dz$$

$$=\int_0^3\int_0^1(\sqrt{y}-y^2)\,dy\,dz$$



$$= \int_0^3 \left[\frac{2y^{x}}{3} - \frac{y^3}{3}\right]_0^1 dz$$

$$= \int_0^3 \left(\frac{2}{3} - \frac{1}{3}\right) dz = \frac{1}{3}[z]_0^3 = 1.$$

Ex. 12. Find, by triple integral, the volume common to the cylinders $x^2 + y^2 = a^2$ and $x^2 + z^2 = a^2$ *2062/067 Chaitra, B. E*

Solution:

Here the volume common to the cylinders $x^2 + y^2 = a^2$ and $x^2 + z^2 = a^2$ region is bounded by

$$z = -\sqrt{a^2 - x^2} \text{ to } z = \sqrt{a^2 - x^2}$$

and y varies from

$$y = -\sqrt{a^2 - x^2} \text{ to } y = \sqrt{a^2 - x^2}$$

and those for x extends from

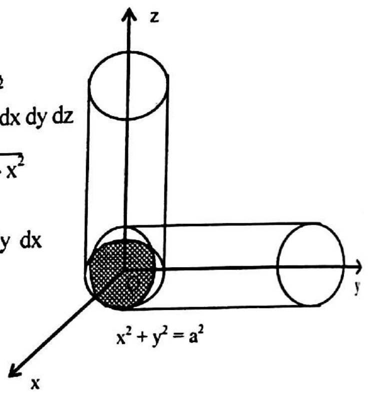
$$x = -a \text{ to } x = a.$$

$\therefore$ the required volume

$$= \int_{-a}^{a} \int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_{z=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dx\, dy\, dz$$

$$= 8\int_0^a \int_{y=0}^{\sqrt{a^2-x^2}} \left(\int_{z=0}^{\sqrt{a^2-x^2}} dz\right) dy\, dx$$

$$= 8\int_0^a \int_{y=0}^{\sqrt{a^2-x^2}} \left(\sqrt{a^2-x^2}\right) dy\, dx$$



$$= 8\int_0^a (a^2 - x^2)\, dx = 8\left[a^2 x - \frac{x^3}{3}\right]_0^a = 8\left[a^3 - \frac{a^3}{3}\right] = \frac{16a^3}{3}.$$

Ex. 16. The plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ meets the planes in A, B, C. Apply Dirichlet's integral to find the volume of tetrahedron OABC *2061Aswin B.E.*

Solution:

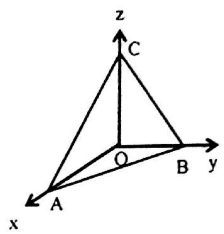
Here, the volume of the tetrahedron OABC is

$$= \iiint_V dx\, dy\, dz \text{ where v is the region bounded by the planes}$$

$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} \le 1$ and the coordinate planes $x = 0$, $y = 0$ and $z = 0$.

Putting $\frac{x}{a} = u$, $\frac{y}{b} = v$, $\frac{z}{c} = w$

$dx = a\,du$, $dy = b\,dv$, $dz = c\,dw$.



So volume $= \iiint_V a\,du\; b\,dv\; c\,dw$

$$= abc \iiint_V du\, dv\, dw.$$

Where $u + v + w \le 1$

$$= abc \iiint_V u^{1-1} v^{1-1} w^{1-1} du\, dv\, dw$$

By using Dirichlet's integral

$$= abc \frac{\Gamma 1\, \Gamma 1\, \Gamma 1}{\Gamma(1+1+1+1)} = \frac{abc}{6}.$$

Ex. 17. A right cone with semi-verical angle α, has its vertex at the center of a sphere and its axis coincident with the diameter of the sphere $x^2 + y^2 + z^2 = a^2$ passing through that point. Find the volume common to the cone and sphere

Solution:

Here the volume of the common to the cone and sphere is

$$= \iiint_V r^2 \sin\phi\; dr\; d\theta\; d\phi$$

where v is the region common to the sphere $r = a$ and cone with semi-vertical angle α.

$$\text{volume} = \int_0^a \int_0^{2\pi} \int_0^{\alpha} r^2 \left(\left(\sin\phi\; d\phi \right) d\theta \right) dr$$

$$= \int_0^a \int_0^{2\pi} r^2 \left[\frac{\cos\phi}{-1}\right]_0^{\alpha} d\theta\; dr$$

$$= (1 - \cos\alpha) \int_0^a \int_0^{2\pi} r^2\; d\theta\; dr$$

$$= 2\pi(1 - \cos\alpha) \int_0^a r^2\; dr = 2\pi(1 - \cos\alpha)\left[\frac{r^3}{3}\right]_0^a$$

$$= \frac{2\pi a^3}{3}(1 - \cos\alpha).$$

Ex. 18. Find the volume of the solid bounded by the surface $(x^2+y^2+z^2)^3 = 27a^3xyz$

Solution:

Here the volume of the common to the cone and sphere is

$$= \iiint_V r^2 \sin\phi \; dr \; d\theta \; d\phi$$

where v is the region bounded by the surface

$$(x^2+y^2+z^2)^3 = 27a^3xyz.$$

Putting, $x = r\cos\theta\sin\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\phi$, $x^2+y^2+z^2 = r^2$.

So $(x^2+y^2+z^2)^3 = 27a^3xyz$ becomes

$r^6 - 27a^3r^3\sin^2\phi\cos\theta\sin\theta\cos\phi$,

or $r^3(r^3 - 27a^3\sin^2\phi\cos\theta\sin\theta\cos\phi) = 0$,

So r varies from $r = 0$ to $r = 3a(\sin^2\phi\cos\theta\sin\theta\cos\phi)^{1/3}$, θ varies from $\theta = 0$ to $\theta = \pi/2$ and ϕ varies from $\phi = 0$ to $\phi = \pi/2$.

Volume

$$= 4\int_0^{\pi/2}\int_0^{\pi/2}\int_0^{3a(\sin^2\phi\cos\theta\sin\theta\cos\phi)^{1/3}} r^2[(\sin\phi\, d\phi)\, d\theta]\, dr$$

$$= 4\int_0^{\pi/2}\int_0^{\pi/2}\left[\frac{r^3}{3}\right]_0^{3a(\sin^2\phi\cos\theta\sin\theta\cos\phi)^{1/3}} (\sin\phi\, d\phi)\, d\theta$$

$$= 4 \times \frac{27a^3}{3}\int_0^{\pi/2}\int_0^{\pi/2}[\sin^2\phi\cos\theta\sin\theta\cos\phi]\sin\phi\, d\phi\, d\theta$$

$$= 36a^3\int_0^{\pi/2}\int_0^{\pi/2}[\sin^3\phi\cos\phi\, d\phi]\cos\theta\sin\theta\, d\theta$$

$$= 36a^3\int_0^{\pi/2}\left[\frac{\sin^4\phi}{4}\right]_0^{\pi/2}\cos\theta\sin\theta\, d\theta$$

$$= 9a^3\int_0^{\pi/2}\sin\theta\cos\theta\, d\theta$$

$$= 9a^3\left[\frac{\sin^2\theta}{2}\right]_0^{\pi/2} = \frac{9a^3}{2}.$$

Exercise - 7

1. Find, by double integration, the area bounded by the parabola $y = x^2$ and the line $y = 2x + 3$
2. Find, by double integration, the area enclosed by the pair of curves $y = 2 - x$ and $y^2 = 2(2 - x)$
3. Find, by double integration, the smaller of the areas bounded by the ellipse $4x^2 + 9y^2 = 36$ and the straight line $2x + 3y = 6$
4. Find, by double integration, the area of the region bounded by $y^2 = x^3$ and $y = x$
5. Find, by double integration, the area of the region enclosed by curves $x^2 + y^2 = a^2$, $x + y = a$ in first quadrant
6. Find, by double integration, the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
7. Show that by double integration that area lying inside the circle $r = a\sin\theta$ and outside the cardioid $r = a(1 - \cos\theta)$ is $\frac{a^2}{4}(4 - \pi)$
8. Find, by double integration, the area common the cardioids $r = a(1 - \cos\theta)$ and $r = a(1 + \cos\theta)$ *2061 Aswin, B.E.*
9. Find the area of the curve $r^2 = a^2\cos 2\theta$ by double integration
10. Find by double integration, volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$
11. Calculate the volume of the solid bounded by the surface $x = 0$, $y = 0$, $x + y + z = 1$ and $z = 0$
12. Find the volume of the solid bounded by the surface $z = 0$, $x^2 + y^2 = 1$, $x + y + z = 3$
13. Find, by triple integration, volume of ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$
14. Find, by triple integration, volume of the sphere $x^2 + y^2 + z^2 = a^2$

15. Show that the area in the first quadrant enclosed by the curve $\left(\frac{x}{a}\right)^{\alpha} + \left(\frac{y}{b}\right)^{\beta} = 1, \alpha > 0, \beta > 0$, is given by $\frac{ab}{\alpha+\beta} \frac{\Gamma\left(\frac{1}{\alpha}\right)\Gamma\left(\frac{1}{\beta}\right)}{\Gamma\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)}$

Answers

1. $10\frac{2}{3}$
2. $\frac{2}{3}$
3. $\frac{3}{2}(\pi - 2)$
4. $\frac{1}{10}$
5. $\frac{a^2}{4}$ (π-
6. πab
8. $a^2\left(\frac{3\pi}{2} - 4\right)$
9. a^2
10. $\frac{4\pi abc}{3}$
11. $\frac{1}{6}$
12. 3π
13. $\frac{4}{3}\pi abc$
14. $\frac{4\pi a^3}{3}$

Chapter -3

Three Dimensional Solid Geometry

- Introduction
- Plane
- Straight Line: Symmetrical and General Form
- Coplanar Lines
- Shortest Distance
- Sphere
- Plane Section of a Sphere
- Tangent Planes and Lines to the Sphere
- Cone: Right Circular Cone
- Cylinder: Right Circular Cylinder

Chapter –3

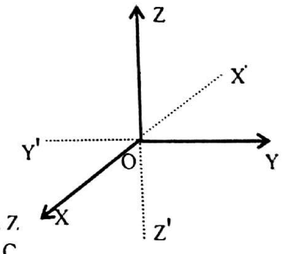
Three Dimensional Solid Geometry

3.0 Introduction

In practical life, we cannot expect a point to always lie in a plane, rather it lies on the three dimensional space in which we are living. It is essential to introduce three dimensional geometry in which the position of a point in space is obtained by three real numbers. In three-dimensional analytic geometry, it is necessary to describe co-ordinate system, ratio formula, locus, projection, direction ratios and direction cosines with reference to three mutually perpendicular axes.

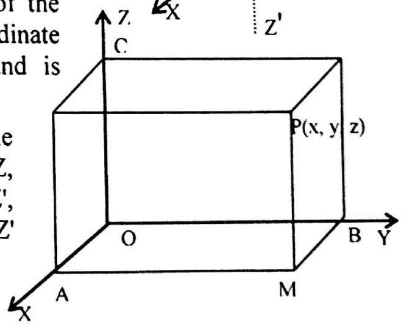
3.01 Cartesian Co-ordinates

If three mutually perpendicular straight lines X'OX, Y'OY and Z'OZ meet at point O, then the lines OX, OY and OZ are called positive x-, y - and z-axis (or simply axes) whereas OX', OY', OZ' are negative x - , y- , and z-axis respectively. The planes XOY, YOZ and ZOX are known as xy -, yz -, zx - planes respectively. It is also known as co-ordinate planes. The point O as origin whose coordinate is (0, 0, 0)



Let P be any point on the space. Draw PM perpendicular on xy-plane and draw lines MB and MA parallel to x-axis and y-axis.

Then OA = x, OB = y and OC = z. The triad (x, y, z) is referred to cartesian co-ordinates or simply coordinates of the point P with respect to the coordinate axes X'OX, Y'OY and Z'OZ and is usually denoted by P(x, y, z).

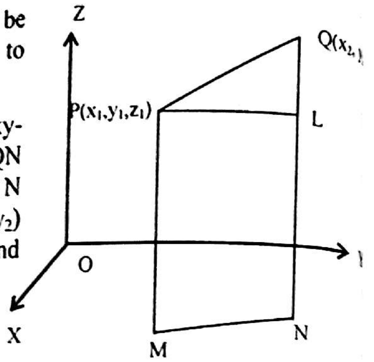


The co-ordinate planes divide the space into eight parts OXYZ, OX'YZ, OXY'Z, OXYZ', OX'YZ', OX'Y'Z, OXY'Z' and OX'Y'Z' which are called *octants*.

3.02 Distance Between Two Points

To find the distance between two the given points in space.

Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be two points in space with respect to coordinate axes OX, OY and OZ.



Draw PM, QN perpendiculars on xy-plane and PL perpendicular on QN so that the co-ordinates of M and N in xy-plane are (x_1, y_1) and (x_2, y_2) respectively with respect to OX and OY. Then

$$MN = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

From Right angle triangle PLQ, we have

$$PQ^2 = PL^2 + LQ^2 = (MN)^2 + (QN - LN)^2$$

$$PQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$$

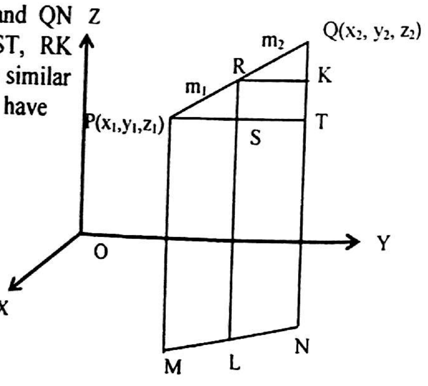
$$\therefore \quad PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

3.03 The Ratio Formula

To find the coordinates of the point which divides the line joining tw points in a given ratio.

Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be two points in space and $R(x, y, z)$ b a point which divides the PQ internally in the ratio $m_1 : m_2$.

Draw the lines PM, RL and QN parallel to Z-axis and PST, RK parallel to MLN. Then from similar triangles PSR and RKQ, we have



$$\frac{RS}{QK} = \frac{PR}{RQ}$$

But $RS = RL - LS = z - z_1$

and $QK = QN - KN = z_2 - z$

$$\frac{z - z_1}{z_2 - z} = \frac{m_1}{m_2}$$

or $$z = \frac{m_1 z_2 + m_2 z_1}{m_1 + m_2}$$

Similarly

$$x = \frac{m_1 x_2 + m_2 x_1}{m_1 + m_2} \text{ and } y = \frac{m_1 y_2 + m_2 y_1}{m_1 + m_2}.$$

Hence the coordinates of R are

$$\left(\frac{m_1 x_2 + m_2 x_1}{m_1 + m_2}, \frac{m_1 y_2 + m_2 y_1}{m_1 + m_2}, \frac{m_1 z_2 + m_2 z_1}{m_1 + m_2}\right)$$

Cor 1:
If R divides the line joining two given points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ externally in the ratio $m_1 : m_2$. Then the coordinates of R are

$$\left(\frac{m_1 x_2 - m_2 x_1}{m_1 - m_2}, \frac{m_1 y_2 - m_2 y_1}{m_1 - m_2}, \frac{m_1 z_2 - m_2 z_1}{m_1 - m_2}\right), \; m_1 \neq m_2$$

Cor 2:
If R divides the line joining two given points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ internally in the ratio $k : 1$, then the coordinates of R are

$$\left(\frac{kx_2 + x_1}{k + 1}, \frac{ky_2 + y_1}{k + 1}, \frac{kz_2 + z_1}{k + 1}\right)$$

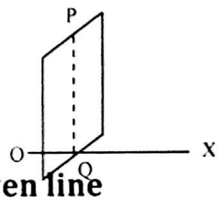
Cor 3.
The coordinates of the middle point of the line joining the points (x_1, y_1, z_1) and (x_2, y_2, z_2) are

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$$

3.04 Projection

Projection of a point on a line

The projection of a point on the given line is the foot of the perpendicular drawn from the point P to the line. Here Q is the projection of the point P on the given line OX.



Projection of a line segment on a given line

Let AB be the line segment and OX be given line. The projection of the points A and B on the given line is A' and B' respectively. Where A' and B' are feet of the perpendiculars drawn from the point A and B respectively.

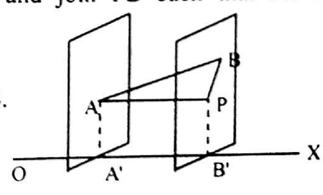
Then the projection of the line segment AB is the line segment A'B'.

Draw AP parallel and equal to A'B' and join PB such that PB is perpendicular to AP.

$$A'B' = AP = AB \cos\theta.$$

Where θ is the angle between AP and AB.

Hence, the projection of the line AB on



the given line OX is AB cos θ, where θ is the angle between AB and OX.

Cor.

If P_1, P_2, ... P_n be n arbitrary points in the space, then sum of the projection of the line segments P_1P_2, P_2P_3 ...., $P_{n-1}P_n$ on any line is equal to the projection of line segment P_1P_n on the same line.

3.05 Direciton Cosines and Direction Ratios

Direction cosines of any straight line in space is the cosines angles made by the line with positive direction of the axes. If α, β, γ be angles made by the line with positive direction of axes, then cos α, cos β, cos γ are direction cosines of the line.

In general, the direction cosines of the line are denoted by l, m, n. Thus

$$l = \cos\alpha,\quad m = \cos\beta,\quad n = \cos\gamma.$$

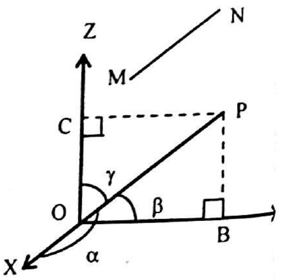
Cor.

Direction cosines of the line OX is cos0° cos90°, cos90° i.e. 1, 0, 0. Similarly, the direction cosines of the line OY and OZ are 0, 1, 0 and 0, 0, 1, respectively.

A relation in l, m, n of a line:

If l, m, n be the direction cosines of a line, then $l^2 + m^2 + n^2 = 1$

Let MN be a line in space and its directions cosines be l, m, n. Draw a line OP which is parallel to MN, then l, m, n are also the direction cosines of the line OP. If it makes angles α, β, γ with axes, then



$$l = \cos\alpha,\quad m = \cos\beta \text{ and } n = \cos\gamma.$$

Draw PA, PB and PC perpendiculars on the axes and suppose (x, y, z) be coordinates of P such that

$$OP = \sqrt{x^2 + y^2 + z^2}.$$

Then $OA = x,\ OB = y,\ OC = z$

$$\therefore\quad l = \cos\alpha = \frac{OA}{OP} = \frac{x}{r} \qquad \text{i.e. } x = lr$$

$$m = \cos\beta = \frac{OB}{OP} = \frac{y}{r} \qquad \text{i.e. } y = mr$$

$$n = \cos\gamma = \frac{OC}{OP} = \frac{z}{r} \qquad \text{i.e. } z = nr$$

Squaring and adding, we get

$$x^2 + y^2 + z^2 = (l^2 + m^2 + n^2)\,r^2 = (l^2 + m^2 + n^2)\,r^2$$

or $\quad r^2 = (l^2 + m^2 + n^2)\,r^2$

$\therefore\quad l^2 + m^2 + n^2 = 1.$

Direction cosines of a Line

To find the direction cosines of the lines whose direction ratios are given

Let a, b, c be the direction ratios of a line and let l, m, n be the directions cosines of the line, then

$$\frac{l}{a} = \frac{m}{b} = \frac{n}{c} = \pm\frac{\sqrt{l^2 + m^2 + n^2}}{\sqrt{a^2 + b^2 + c^2}} = \pm\frac{1}{\sqrt{a^2 + b^2 + c^2}}$$

$$\therefore\ l = \pm\frac{a}{\sqrt{a^2 + b^2 + c^2}},\quad m = \pm\frac{b}{\sqrt{a^2 + b^2 + c^2}},\quad n = \pm\frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

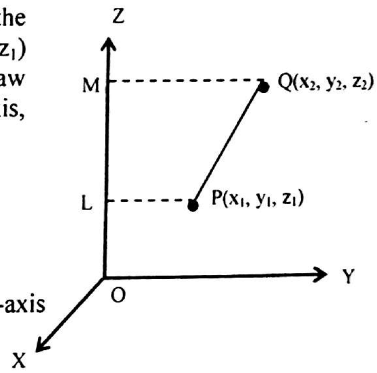
Note:

The direction cosines of the line AB is to be taken in positive sign and BA is to be taken negative sign in the above values of l, m, n.

Direction Ratios of the line joining two points

To find the direction ratios of a line joining two points (x_1, y_1, z_1), (x_2, y_2, z_2)

Let l, m, n be direction cosines of the line joining two given points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$. Suppose PQ = r. Draw Perpendiculars PL and QM on z-axis, then



$OM = z_2$ and $OL = z_1$, so that

LM = Projection of PQ on z-axis

= PQ cos γ

where γ is the angle between PQ and z-axis

or $\quad OM - OL = PQ\cos\gamma,$

or $\quad z_2 - z_1 = r\,n$

$$\therefore\quad n = \frac{z_2 - z_1}{r}$$

Similarly, we can prove that

$$m = \frac{y_2 - y_1}{r} \text{ and } l = \frac{x_2 - x_1}{r}$$

Hence the direction cosines of the line PQ are

$$\frac{x_2 - x_1}{r}, \frac{y_2 - y_1}{r}, \frac{z_2 - z_1}{r}$$

where $PQ = r = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$.

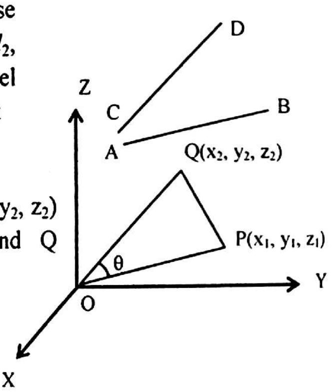
Thus the direction ratios of the line joining the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are $x_2 - x_1,\ y_2 - y_1,\ z_2 - z_1$.

3.06 Angle Between Two Lines

To find the angle between two lines whose direction cosines are l_1, m_1, n_1, and l_2, m_2, n_2

Let AB and CD be two lines whose direction cosines are l_1, m_1, n_1 and l_2, m_2, n_2 respectively. Draw OP parallel to AB and OQ parallel to CD. So that

$\angle POQ = \theta$.



Join P and Q let (x_1, y_1, z_1) and (x_2, y_2, z_2) be the two coordinates of P and Q respectively, then

$$OP = r_1 = \sqrt{x_1^2 + y_1^2 + z_1^2}$$

$$OQ = r_2 = \sqrt{x_2^2 + y_2^2 + z_2^2}.$$

We have

$x_1 = l_1r_1, \quad y_1 = m_1r_1, \quad z_1 = n_1r_1$

$x_2 = l_2r_2, \quad y_2 = m_2r_2, \quad z_2 = n_2r_2$

$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

From triangle OPQ, we have

$$\cos\theta = \frac{OP^2 + OQ^2 - PQ^2}{2OP.OQ}$$

$$= \frac{r_1^2 + r_2^2 - \left(x_2^2 + y_2^2 + z_2^2\right) - \left(x_1^2 + y_1^2 + z_1^2\right) + 2(x_1x_2 + y_1y_2 + z_1z_2)}{2r_1r_2}$$

$$= \frac{r_1^2 + r_2^2 - r_2^2 - r_1^2 + 2r_1r_2(l_1l_2 + m_1m_2 + n_1n_2)}{2r_1r_2}$$

$$\therefore \quad \cos\theta = l_1l_2 + m_1m_2 + n_1n_2$$

Cor 1:

The cosine angle between two lines whose direction ratios are a_1, b_1, c_1 and a_2, b_2, c_2 is given by

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

Cor. 2:

The two lines are *perpendicular* to each other if

$l_1l_2 + m_1m_2 + n_1n_2 = 0$ or $a_1a_2 + b_1b_2 + c_1c_2 = 0$

and *parallel* if $\frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}$ or $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

3.07 Plane

A plane is a surface such that if any two points be taken on it, then the line joining them lies wholly in the surface.

To prove that general equation of first degree in x, y, z represents plane.

Let the first degree equation in x, y, z be

$$ax + by + cz + d = 0 \quad \text{......(1)}$$

where a, b, c and d are constants

By definition of the plane, taking $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ any two points on the locus of equation (1), we have

$$ax_1 + by_1 + cz_1 + d = 0 \quad \text{....(2)}$$

and

$$ax_2 + by_2 + cz_2 + d = 0 \quad \text{...(3)}$$

Multiplying equation (2) by k and adding to (3), we get

$$a(kx_1 + x_2) + b(ky_1 + y_2) + c(kz_1 + z_2) + d(k + 1) = 0$$

or $\dfrac{a(kx_1 + x_2)}{k+1} + \dfrac{b(ky_1 + y_2)}{k+1} + \dfrac{c(kz_1 + z_2)}{k+1} + d = 0$

It shows that the point $\left(\dfrac{kx_1 + x_2}{k+1}, \dfrac{ky_1 + y_2}{k+1}, \dfrac{kz_1 + z_2}{k+1}\right)$ divides the line in the ratio k:1. Hence for all the points, the line joining any two points on the locus lie wholly on the locus. Thus, a general equation of first degree equation in x, y, z always represents a plane.

3.08 One-Point Form of the Equation of the Plane

To find the equation of the plane passes through a given point

Let $P(x_1, y_1, z_1)$ be any given point and let the equation of plane be

$$ax + by + cz + d = 0$$

Since it passes through the point (x_1, y_1, z_1)

$$ax_1 + by_1 + cz_1 + d = 0$$

$$\therefore \quad d = -(ax_1 + by_1 + cz_1)$$

Substituting the value of d in equation (1), we get

$$ax + by + cz - (ax_1 + by_1 + cz_1) = 0$$

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0.$$

This is the equation of a plane through a given point (x_1, y_1, z_1) where a, b, c are constant which will be determined by any given conditions

Cor.

The equation of a *plane* passing through the origin (0, 0, 0) is

$$ax + by + cz = 0$$

3.09 Three Points Form of the Equation of a Plane

To find the equation of a plane which passes through three given points (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3)

We know that the equation of a plane through (x_1, y_1, z_1) is

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$

Since it passes through (x_2, y_2, z_2) and (x_3, y_3, z_3), then (1) gives

$$a(x_2 - x_1) + b(y_2 - y_1) + c(z_2 - z_1) = 0$$

and $\quad a(x_3 - x_1) + b(y_3 - y_1) + c(z_3 - z_1) = 0$

Eliminating a, b, c from (1), (2) and (3), we get

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

is the required equation of the plane.

Cor.

The four points (x_1, y_1, z_1), (x_2, y_2, z_2) (x_3, y_3, z_3) and (x_4, y_4, z_4) lies on a plane if and only if

$$\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0$$

3.10 Intercept Form of the Equation of a Plane

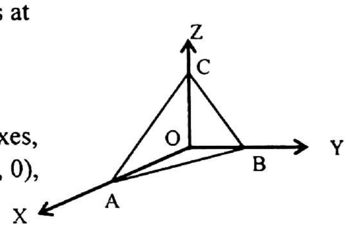
To find the equation of a plane in intercepts form

Let the general equation of the plane be

$$Ax + By + Cz + D = 0 \qquad \text{......(1)}$$

Let ABC be a plane which cuts the axes at A, B, C such that

$$OA = a, \ OB = b \text{ and } OC = c.$$



If the plane meets the coordinate axes, then it passes through the points A(a, 0, 0), B(0, b, 0), C(0, 0, c)

So that

$$A\,a + D = 0, \quad Bb + D = 0 \text{ and } Cc + D = 0$$

$$\therefore \quad A = \frac{-D}{a}, \quad B = \frac{-D}{b} \text{ and } C = -\frac{D}{c}$$

Substituting these values of A, B, C in (1), we get

$$-\frac{D}{a}x - \frac{D}{b}y - \frac{D}{c}z + D = 0$$

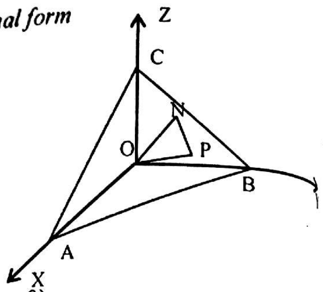
But $D \neq 0$,

$$\therefore \quad \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$$

This is the required equation of plane in intercept form.

3.11 Normal Form of the Equation of the Plane

To find the equation of a plane in the normal form



Let ABC be a plane and ON = p, the normal to the plane. Take any variable point P(x, y, z) on the plane. Since l, m, n be the direction cosines of ON.

The projection of a line joining two points O(0, 0, 0) and P(x, y, z) on the given line ON with direction cosines l, m, n is

$$ON = (x - 0)\,l + (y - 0)\,m + (z - 0)\,n$$

or $\quad p = l x + m y + n z$

$\therefore \quad lx + my + nz = p$ is the required equation of a plane in normal form.

3.12 Reduction of General Equation of the Plane to the Normal Form

Let the general equation of the plane be

$$ax + by + cz + d = 0 \qquad \ldots\ldots(1)$$

And the equation of plane as the normal form

$$lx + my + nz = p \qquad \ldots\ldots(2)$$

Comparing (1) and (2), we get

$$\frac{l}{a} = \frac{m}{b} = \frac{n}{c} = \frac{-p}{d}$$

or $$\frac{l}{a} = \frac{m}{b} = \frac{n}{c} = \pm\frac{\sqrt{l^2 + m^2 + n^2}}{\sqrt{a^2 + b^2 + c^2}} = \frac{-p}{d}$$

or $$\frac{l}{a} = \frac{m}{b} = \frac{n}{c} = \pm\frac{1}{\sqrt{a^2 + b^2 + c^2}} = \frac{-p}{d}.$$

This gives

$$l = \pm\frac{a}{\sqrt{a^2 + b^2 + c^2}}, \qquad m = \pm\frac{b}{\sqrt{a^2 + b^2 + c^2}},$$

$$n = \pm\frac{c}{\sqrt{a^2 + b^2 + c^2}} \quad \text{and} \quad p = \pm\frac{-d}{\sqrt{a^2 + b^2 + c^2}}$$

Now p is always positive. Thus

when d is positive

$$p = \frac{d}{\sqrt{a^2 + b^2 + c^2}}, \qquad l = \frac{-a}{\sqrt{a^2 + b^2 + c^2}}$$

$$m = -\frac{b}{\sqrt{a^2 + b^2 + c^2}}, \qquad n = \frac{-c}{\sqrt{a^2 + b^2 + c^2}}$$

when d is negative

$$p = \frac{-d}{\sqrt{a^2 + b^2 + c^2}}, \qquad l = \frac{a}{\sqrt{a^2 + b^2 + c^2}}$$

$$m = \frac{b}{\sqrt{a^2 + b^2 + c^2}} \quad \text{and} \quad n = \frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

Thus, the equations of the plane in the normal form are

$$-\frac{ax}{\sqrt{a^2 + b^2 + c^2}} - \frac{by}{\sqrt{a^2 + b^2 + c^2}} - \frac{cz}{\sqrt{a^2 + b^2 + c^2}} = \frac{d}{\sqrt{a^2 + b^2 + c^2}}$$

and $$\frac{ax}{\sqrt{a^2 + b^2 + c^2}} + \frac{by}{\sqrt{a^2 + b^2 + c^2}} + \frac{cz}{\sqrt{a^2 + b^2 + c^2}} = -\frac{d}{\sqrt{a^2 + b^2 + c^2}}$$

according as d is positive and negative respectively.

Note:

This shows that the direction cosines of the normal to the plane $ax + by + cz + d = 0$ are proportional to a, b, c.

3.13 Angle Between Two Planes

The angle between two planes is equal to the angle between their normals.

To find the angle between two planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$

Let the equations of two planes be

$$a_1x + b_1y + c_1z + d_1 = 0$$

and $$a_2x + b_2y + c_2z + d_2 = 0.$$

Then the direction ratios of the normals to the two planes are a_1, b_1, c_1 and a_2, b_2, c_2 respectively. So

$$l_1 = \frac{a_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}}, \quad m_1 = \frac{b_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}}, \quad n_1 = \frac{c_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}}$$

and $l_2 = \dfrac{a_2}{\sqrt{a_2^2+b_2^2+c_2^2}}$, $m_2 = \dfrac{b_2}{\sqrt{a_2^2+b_2^2+c_2^2}}$, $n_2 = \dfrac{c_2}{\sqrt{a_2^2+b_2^2+c_2^2}}$

If θ be the angle between them, then

$$\cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2$$

$$\therefore \quad \cos\theta = \frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$$

Cor. 1:

If the two planes will be perpendicular (i.e. their normals are perpendicular),then $\theta = 90°$ and we have

$$\cos 90° = \frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$$

$$0 = \frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$$

$$\therefore \quad a_1a_2 + b_1b_2 + c_1c_2 = 0$$

Cor. 2:

If the two planes will be parallel (i.e. their normals are parallel) ,then $\theta = 0°$ and we have

$$\cos 0° = \frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$$

or $$1 = \frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$$

Squaring both sides, we get

$$(a_1^2+b_1^2+c_1^2)(a_2^2+b_2^2+c_2^2) = (a_1a_2+b_1b_2+c_1c_2)^2$$

or $$(a_1b_2 - a_2b_1)^2 + (b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2 = 0$$

The sum of the square equal to zero only when

$$(a_1b_2 - a_2b_1) = 0,\ (b_1c_2 - b_2c_1) = 0 \text{ and } (c_1a_2 - c_2a_1) = 0$$

or $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2}$, $\quad \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$ and $\dfrac{c_1}{c_2} = \dfrac{a_1}{a_2}$

$\therefore \quad \dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$. This is condition of parallelism.

Cor. 3:

Any plane parallel to the plane $ax + by + cz + d = 0$ is

$ax + by + cz + d_1$ where d_1 be different from d which will be determined by the given condition in which the direction ratios of their normals are same.

Cor. 4:

The equation of a plane through the two planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ is

$$a_1x + b_1y + c_1z + d_1 + \lambda(a_2x + b_2y + c_2z + d_2).$$

Where λ is constant which will be determined by the given conditions.

3.14 Perpendicular Distance of the Given Point From a Plane

Let the general equation of a plane be

$$ax + by + cz + d = 0 \qquad \text{.......(1)}$$

and let $P(x_1, y_1, z_1)$ be a given point of the plane (1) and PL is perpendicular distance of $P(x_1, y_1, z_1)$.

So, the direction cosines of the normal PL to the plane (1) are

$$l = \frac{a}{\sqrt{a^2+b^2+c^2}},\ m = \frac{b}{\sqrt{a^2+b^2+c^2}},\ n = \frac{c}{\sqrt{a^2+b^2+c^2}}$$

Take Q(f, g, h) be any point on the plane (1), we have

$$af + bg + ch + d = 0$$

or $$d = -(af + bg + ch) \qquad \text{.....(2)}$$

We have

PL = The projection of the line joining two points $P(x_1, y_1, z_1)$ and Q(f, g, h) on the given line PL with given direction cosines.

$$PL = (x - f)l + (y - g)m + (z - h)n$$

$$= (x - f)\frac{a}{\sqrt{a^2+b^2+c^2}} + (y - g)\frac{b}{\sqrt{a^2+b^2+c^2}} + (z - h)\frac{c}{\sqrt{a^2+b^2+c^2}}$$

$$= \frac{(x_1 - f)a + (y_1 - g)b + (z_1 - h)c}{\sqrt{a^2+b^2+c^2}}$$

$$= \frac{ax_1 + by_1 + cz_1 - (af + bg + ch)}{\sqrt{a^2 + b^2 + c^2}}$$

$$= \frac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2 + b^2 + c^2}} \quad \text{by virtue of (2).}$$

The sign of the radical in this is taken positive or negative according as positive or negative.

Cor. 1:

The perpendiculars drawn from two points to the plane are taken to same sign if the points lie on the *same side* and *different signs* if they lie on the *opposite side of the plane.*

Therefore the two points (x_1, y_1, z_1) and (x_2, y_2, z_2) lie on the same side or an opposite sides of the plane $ax + by + cz + d = 0$ according as the expressions $ax_1 + by_1 + cz_1 + d$ and $ax_2 + by_2 + cz_2 + d$ are of the same sign or of opposite signs.

Cor. 2: Planes bisecting the angles between the two planes

Let the general equation of two planes be

$$a_1x + b_1y + c_1z + d_1 = 0 \quad \text{......(1)}$$

and $$a_2x + b_2y + c_2z + d_2 = 0 \quad \text{......(2)}$$

If P(x, y, z) be any point on either of the bisectors, then the perpendicular distances of the point P(x, y, z) from the planes (1) and (2) must be equal in magnitude

i.e. $$\frac{a_1x + b_1y + c_1z + d_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}} = \pm \frac{a_2x + b_2y + c_2z + d_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}.$$

These are the required equations of the bisecting planes.

If the angle between one of the bisector and one of the given planes is less than $\frac{\pi}{4}$, then the bisector is called acute- angled bisector and another bisector is called obtuse- angled bisector.

Cor. 3:

The equation of bisector which contains the origin is

$$\frac{a_1x + b_1y + c_1z + d_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}} = \frac{a_2x + b_2y + c_2z + d_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}$$ provided d_1 and d_2 are positive.

Worked Out Examples

Ex. 1. Find the equation of the plane which passes through the point (3, - 3, 1) and

(i) parallel to the plane $2x + 3y + 5z + 6 = 0$

(ii) normal to the line joining the points (3, 2,-1) and (2, -1, 5)

(iii) perpendicular to the planes $7x + y + 2z = 6$ and $3x + 5y - 6z - 8 = 0$

(i) Solution:

The equation of any plane parallel to the given plane $2x + 3y + 5z + 6$ is

$$2x + 3y + 5z + k = 0 \quad \text{....(1)}$$

Since it passes through the point (3, - 3, 1), the point must satisfy in (1),so

$$2\times 3 + 3(-3) + 5\times 1 + k = 0$$

or $6 - 9 + 5 + k = 0$

or $2 + k = 0$

$\therefore$ $k = -2$

Putting the value of k in (1), we get

$2x + 3y + 5z - 2 = 0$ is the required equation of the plane.

(ii) Solution:

The equation of plane passing through the point (3, - 3, 1) is

$$a(x - 3) + b(y + 3) + c(z - 1) = 0 \quad \text{.......(1)}$$

The direction cosines of the line joining the points (3, 2, - 1) and (2, - 1, 5) are proportional to

$2-3, -1-2, 5+1$

i. e. $-1, -3, 6.$

Since this line is normal to the plane (1), we have

$$\frac{a}{1} = \frac{b}{3} = \frac{c}{-6} = k(\text{ say})$$

$$a = k,\ b = 3k \text{ and } c = -6k$$

Substituting these values of a, b, c in (1), we get

$$k(x - 3) + 3k(y + 3) - 6k(z - 1) = 0$$

But $k \neq 0$, $x - 3 + 3y + 9 - 6z + 6 = 0$

$\therefore$ $x + 3y - 6z + 12 = 0$

(iii)Solution:

The equation plane passing through the point (3, - 3, 1) is

$$a(x - 3) + b(y + 3) + c(z - 1) = 0 \quad \text{.......(1)}$$

Since it is perpendicular to the planes

$$7x + y + 2z = 6 \text{ and } 3x + 5y - 6z = 8.$$

Thus, by using condition of perpendicularity, we get

$$7a + b + 2c = 0$$

and $\quad 3a + 5b - 6c = 0.$

Solving these, we obtain

$$\frac{a}{1} = \frac{b}{-3} = \frac{c}{-2} = k(\text{ say})$$

$$\therefore\ a = k,\ b = -3k \text{ and } c = -2k.$$

Substituting these values of a, band c in (1), we get

$$a(x-3) + b(y+3) + c(z-1) = 0$$

$$k(x-3) - 3k(y+3) - 2k(z-1) = 0$$

But $\quad k \neq 0, \quad x - 3 - 3y - 9 - 2z + 2 = 0$

or $\quad x - 3y - 2z - 10 = 0$

$\therefore \quad x - 3y - 2z = 10.$

Ex.2. The plane $lx + my = 0$ is rotated about its line of intersecti with the plane $z = 0$ through an angle α. Prove that t equation to the plane in its new position is $lx + m$ $\sqrt{l^2 + m^2}\ \tan \alpha z = 0$

Solution:

Any equation of plane through $lx + my = 0$ and $z = 0$ is

$$lx + my + \lambda z = 0 \qquad \text{......(}$$

where λ is constant.

Since the line $lx + my = 0$ makes an angle α with the plane (1). Then

$$\cos\alpha = \frac{l^2 + m^2}{\sqrt{l^2 + m^2}\sqrt{l^2 + m^2 + \lambda^2}}$$

or

$$\cos\alpha = \frac{\sqrt{l^2 + m^2}}{\sqrt{l^2 + m^2 + \lambda^2}}$$

Squaring both sides, we get

$$(l^2 + m^2 + \lambda^2)\cos^2\alpha = (l^2 + m^2)$$

or $\quad \lambda^2\cos^2\alpha = (l^2 + m^2)(1 - \cos^2\alpha)$

or $\quad \lambda^2\cos^2\alpha = (l^2 + m^2)\sin^2\alpha$

or $\quad \lambda^2 = (l^2 + m^2)\,.\tan^2\alpha$

$\therefore \quad \lambda = \pm\sqrt{l^2 + m^2}\ \tan\alpha.$

Substituting the value of λ in (1), we get

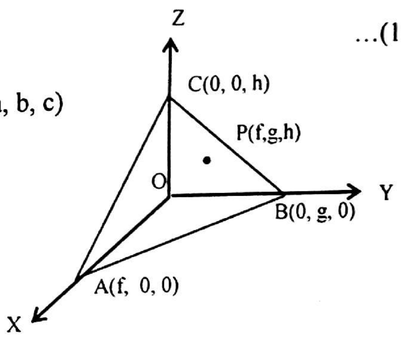
$$lx + my \pm \sqrt{l^2 + m^2}\ \tan\alpha\, z = 0$$

Ex.3. A variable plane passes through the fixed point (a, b, c) and meets the coordinate axes in A, B, C. Show that the locus of the point common to the planes through A, B, C parallel to the co-ordinate planes is $\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 1$

Solution:

Let P(f, g, h) be any point common to the planes through A, B and C such that OA = f, OB = g, OC = h. Then its equation is

$$\frac{x}{f} + \frac{y}{g} + \frac{z}{h} = 1 \qquad \text{...(1)}$$



Since it passes through fixed point (a, b, c)

$$\frac{a}{f} + \frac{b}{g} + \frac{c}{h} = 1.$$

The locus of the point (f, g, h) is

$$\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 1.$$

Ex. 4. Find the locus of the point which moves such that the sum of the squares of the distances of a point from the three planes $x + y + z = 0$, $x - z = 0$ and $x - 2y + z = 0$ is 9

Solution:

Let (f, g, h) be any point such that its distances from the three planes

$x + y + z = 0$, $x - z = 0$ and $x - 2y + z = 0$ are

$$\frac{f + g + h}{\sqrt{3}},\quad \frac{f - h}{\sqrt{2}} \text{ and } \frac{f - 2g + h}{\sqrt{6}}$$

Since $\quad \left(\frac{f + g + h}{\sqrt{3}}\right)^2 + \left(\frac{f - h}{\sqrt{2}}\right)^2 + \left(\frac{f - 2g + h}{\sqrt{6}}\right)^2 = 9$

or $\dfrac{f^2+g^2+h^2+2fg+2gh+2fh}{3} + \dfrac{f^2+h^2-2fh}{2} + \dfrac{f^2+4g^2+h^2-4fg-4gh+2fh}{6} = 9$

or $2f^2+2g^2+2h^2+4fg+4gh+4fh+3f^2+3g^2-6fh+f^2+4g^2+h^2-4fg-4gh+2fh=54$

or $6f^2+6g^2+6h^2=54$

or $f^2+g^2+h^2=9$

Hence the locus of the point (f, g, h) is

$$x^2+y^2+z^2=9.$$

Ex. 5. Find the equation of the bisectors of that angle between t planes $2x-y+2z+3=0$ and $3x-2y+6z+8=0$. Find a acute angle bisector and obtuse angle bisector

Solution:

Here, the given two planes are

$$2x-y+2z+3=0 \quad \text{....(1)}$$

and $$3x-2y+6z+8=0 \quad \text{....(2)}$$

The equation of the planes bisecting the angles between the planes (1) a (2) are

$$\frac{2x-y+2z+3}{\sqrt{4+1+4}} = \pm\frac{3x-2y+6z+8}{\sqrt{9+4+36}}$$

or $5x-y-4z-3=0$ ....(3)

and $23x-13y+32z+45=0$ ....(4)

If θ be angle between normals to (1) and (3), then

$$\cos\theta = \frac{2.5+(-1)(-1)+2(-4)}{\sqrt{4+1+4}\sqrt{25+1+16}} = \frac{1}{\sqrt{32}}$$

and $$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\sqrt{1-\cos^2\theta}}{\cos\theta} = \frac{\sqrt{1-\cos^2\theta}}{\cos\theta}$$

$$= \frac{\sqrt{1-\frac{1}{32}}}{\frac{1}{\sqrt{32}}} = \frac{\sqrt{31}}{1} = \sqrt{31} > 1.$$

So, $\theta > \frac{\pi}{4}$. Hence $5x-y-4z-3=0$ is obtuse angle bisector and other plane $23x-13y+32z+45=0$ is acute angle bisector

Exercise - 8

1. Find the equation of a plane passing through the points (1,1, 0), (3,-1, 1) and (-1, 0, 2)
2. Show that four points (0, - 1, 0), (2, 1, - 1) (1, 1, 1) and (3, 3, 0) are coplanar. Find the equation of plane through them
3. Find the equation of the plane through origin with direction cosines proportional to 2, 1, - 2 and 5, 2, - 3
4. Find the length and direction cosines of the normal from the origin to the plane $6x-3y+2z=14$
5. Find the equation of the plane through the point (2, 1, 0) and perpendicular to the planes $2x-y-z=5$ and $x+2y-3z=3$
6. Find the equation of the plane through the points (2, 2, 1) and (9, 3, 6) and perpendicular to the plane $2x+6y+6z=9$
7. Find the equation of the plane through the points (2, 2, 1), (1, - 2, 3) and parallel to the line joining the points (2, 1,-3) and (-1,5, - 8)
8. Find the equation of a plane passing through the line of intersection of the planes $2x+3y+10z=8$ and $2x-3y+7z=2$ and perpendicular to the plane $3x-2y+4z=5$
9. Find the angle between the following planes.
   (i) $x-2y-z=5$ and $2x-y+z=2$
   (ii) $x+2y+3z=4$ and $3x+4y+5z=10$
10. Find the equation of a plane through the points (1, 2, 3) and (7, 5, 6) parallel to x-axis
11. A plane meets the coordinate axes at A, B, C such that the centroid of the triangle ABC is the point (a, b, c). Show that the equation of the plane is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=3$
12. Find the equation of the planes bisecting the angle between the planes $x+2y+2z-9=0$, $4x-3y+12z+13=0$ and specify the one which bisects the acute angle

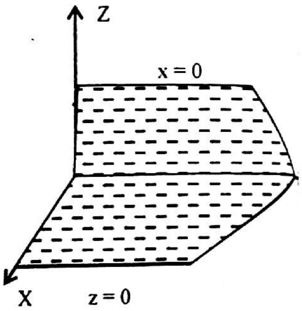
Answers

1. $x + 2y + 2z - 3 = 0$
2. $4x - 3y + 2z = 3$
3. $x - 4y - z = 0$
4. $2; \frac{6}{7}, \frac{-3}{7}, \frac{2}{7}$
5. $x - 5y - 2z + 7 = 0$
6. $3x + 4y - 5z = 9$
7. $12x - 11y - 16z + 14 = 0$
8. $28x - 17y + 19z = 0$
9. (i) $\frac{\pi}{3}$ (ii) $\cos^{-1}\frac{13}{5\sqrt{7}}$
10. $y - z + 1 = 0$
12. (i) $25x + 17y + 62z - 78 = 0$ (ii) $x + 35y - 10z - 156 = 0$

Equation of plane of (i) bisects the acute angle

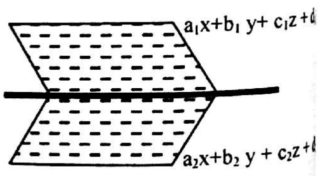
3.1 Straight Line

A straight line can be regarded as the locus of the common points of two intersecting planes. In a rectangular coordinate axes OX, OY and OZ, the planes $z = 0$ and $x = 0$ intersect y-axis, $x = 0$ and $y = 0$ intersect z-axis and $y = 0$ and $z = 0$ intersect x-axis. Thus we see that the point P(x, y, z) lies on the y-axis if and only if $z = 0$ and $x = 0$.



Hence, OY is a straight line whose equation is $z = 0$ and $x = 0$

Similarly, OZ and OX are straight lines whose equations are $x = 0$, $y = 0$ and $y = 0$, $z = 0$ respectively.



In general, consider equation of two planes

$$a_1x + b_1y + c_1z + d_1 = 0$$

and

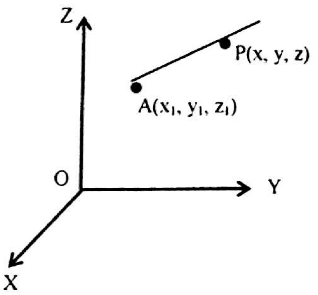
$$a_2x + b_2y + c_2z + d_2 = 0.$$

The locus of the intersection of these two planes is a straight line so that equation of straight line in general form is $a_1x + b_1y + c_1z + d_1 = 0$
$a_2x + b_2y + c_2z + d_2 = 0$.

It is also known as equation of straight line in *unsymmetrical* form.

3.1.1 Equation of Straight Line in Symmetrical Form

To find the equation of the line passing through the point $A(x_1, y_1, z_1)$ with direction cosines l, m, n



Let P(x, y, z) be any variable point on the given line through $A(x_1, y_1, z_1)$ such that AP = r. Then the direction cosines of the line joining two points are

$$l = \frac{x - x_1}{r},\quad m = \frac{y - y_1}{r},\quad n = \frac{z - z_1}{r}$$

So that $\frac{x - x_1}{l} = r,\quad \frac{y - y_1}{m} = r,\quad \frac{z - z_1}{n} = r$

or $\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}\ (= r),$

This is required equation of straight line in *Symmetrical* form.

Note:

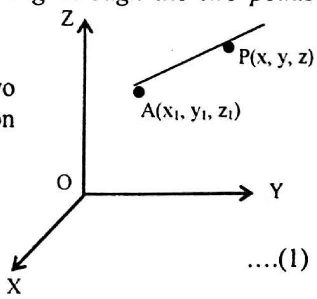
Any point of the straight line is $(lr + x_1, mr + y_1, nr + z_1)$

Cor. 1.

The equation of a straight line through (x_1, y_1, z_1) and having direction ratios a, b, c is

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}.$$

To find the equation of a straight line passing through the two points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$



Let $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ be two points in a straight line, then its direction ratio are

$x_2 - x_1, y_2 - y_1, z_2 - z_1$.

Its direction cosines are

$$l = \frac{x_2 - x_1}{AB},\ m = \frac{y_2 - y_1}{AB},\quad n = \frac{z_2 - z_1}{AB}. \qquad ....(1)$$

Let P (x, y, z) be any point on the line, then the equation of a line through the point A (x_1, y_1, z_1) with dcs l, m, n is

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$$

Eliminating l, m, n from (1) and (2), we get

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}.$$

3.1.2. Reduction of the Genral Equation of a Line to the Symmetrical Form

The equation of a line in the general form is

$$a_1x + b_1y + c_1z + d_1 = 0 \quad \ldots(1)$$
$$a_2x + b_2y + c_2z + d_2 = 0 \quad \ldots(2)$$

Let l, m, n be direction cosines of the line of intersection of two planes. Since the line lies on both planes (1) and (2), it is perpendicular to the normals to these planes.

The direction cosines of the normal to the planes (1) and (2) are proportional to a_1, b_1, c_1 and a_2, b_2, c_2 respectively.

Therefore $\quad la_1 + mb_1 + nc_1 = 0$

$\quad la_2 + mb_2 + nc_2 = 0$

Solving these, we get

$$\frac{l}{b_1c_2 - b_2c_1} = \frac{m}{c_1a_2 - c_2a_1} = \frac{n}{a_1b_2 - a_2b_1}.$$

Thus the direction cosines of the line are proportional to

$$b_1c_2 - b_2c_1,\ c_1a_2 - c_2a_1,\ a_1b_2 - a_2b_1.$$

To find a point on the line

Putting z = 0 in the given equations, we have

$$a_1x + b_1y + d_1 = 0$$
$$a_2x + b_2y + d_2 = 0$$

Solving these by cross multiplication, we get

$$x = \frac{b_1d_2 - b_2d_1}{a_1b_2 - a_2b_1},\quad y = \frac{a_2d_1 - a_1d_2}{a_1b_2 - a_2b_1}$$

Therefore any point of the line is

$$\left(\frac{b_1d_2 - b_2d_1}{a_1b_2 - a_2b_1}, \frac{a_2d_1 - a_1d_2}{a_1b_2 - a_2b_1}, 0\right)$$

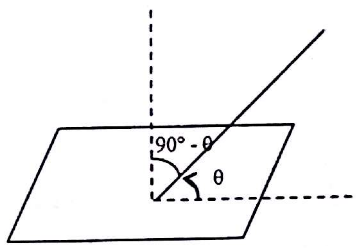
Thus the equation of the line in the symmetrical form is

$$\frac{x - \dfrac{b_1d_2 - b_2d_1}{a_1b_2 - a_2b_1}}{b_1c_2 - b_2c_1} = \frac{y - \dfrac{a_2d_1 - a_1d_2}{a_1b_2 - a_2b_1}}{c_1a_2 - c_2a_1} = \frac{z - 0}{a_1b_2 - a_2b_1}$$

3.1.3. Angle Between a Line and a Plane

To find the angle between the line $\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$ *and the plane* $ax + by + cz + d = 0$

Let θ be the angle between the plane and the line, then 90° - θ be the angle between the line and normal to the plane. Now the direction ratios of the line are



l, m, n

and the direction ratios of the normal to the plane are

a, b, c.

We have

$$\cos(90° - \theta) = \frac{la + mb + nc}{\sqrt{l^2 + m^2 + n^2}\sqrt{a^2 + b^2 + c^2}}$$

or

$$\sin\theta = \frac{al + bm + cn}{\sqrt{l^2 + m^2 + n^2}\sqrt{a^2 + b^2 + c^2}}$$

Cor.

If the line is *parallel* to the plane (perpendicular to normal to the plane), then θ = 0°, we have

$$\sin 0° = \frac{al + bm + cn}{\sqrt{l^2 + m^2 + n^2}\sqrt{a^2 + b^2 + c^2}}$$

$$0 = al + bm + cn$$

$$\therefore \quad al + bm + cn = 0$$

If the line is *perpendicular* to the plane (*parallel* to normal to the plane), then θ = 90°, we have

$$\sin 90° = \frac{al + bm + cn}{\sqrt{l^2 + m^2 + n^2}\sqrt{a^2 + b^2 + c^2}}$$

$$1 = \frac{al + bm + cn}{\sqrt{l^2 + m^2 + n^2}\sqrt{a^2 + b^2 + c^2}}$$

Squaring both sides, we get

$$(l^2 + m^2 + n^2)(a^2 + b^2 + c^2) = (al + bm + cn)^2$$

or $$(am - bl)^2 + (bn - cm)^2 + (an - lc)^2 = 0$$

The sum of the square equal to zero only when

$$am - bl = 0, \quad bn - cm = 0 \quad \text{and} \quad an - lc = 0$$

or $$\frac{l}{a} = \frac{m}{b}, \quad \frac{m}{b} = \frac{n}{c} \quad \text{and} \quad \frac{n}{c} = \frac{l}{a}$$

$$\therefore \quad \frac{l}{a} = \frac{m}{b} = \frac{n}{c}.$$

3.1.4. Condition for a Line to lie in a Plane

To find the condition that the line $\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$ *may lie in the plane* $ax + by + cz + d = 0$

Here, the equation of the line is

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n} = r \text{ (say)}$$

Any point of this line is

$$(lr + x_1, mr + y_1, nr + z_1).$$

If this point will lie on the plane $ax + by + cz + d = 0$, then the point must satisfy in this plane, so

$$a(lr + x_1) + b(mr + y_1) + c(nr + z_1) + d = 0$$

or $$(al + bm + cn)r + (ax_1 + by_1 + cz_1 + d) = 0$$

It is true for all values of r, if

$$al + bm + cn = 0 \text{ and}$$

$$ax_1 + by_1 + cz_1 + d = 0$$

These two equations are the required conditions for the line may lie in the plane.

3.1.5. Plane Containing the line

To find the equation of a plane containing the line $\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$

Here, the equation of the line is

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n} = r \text{ (say)} \quad \ldots(1)$$

Let the equation of a plane containing the line (1) be

$$ax + by + cz + d = 0 \quad \ldots\ldots(2)$$

If the plane (2) is contained by the line (1), then the condition is

$$ax_1 + by_1 + cz_1 + d = 0 \quad \ldots.(3)$$

where $$al + bm + cn = 0 \quad \ldots(4)$$

Subtracting (3) from (2), we get

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0 \quad \ldots.(3)$$

Therefore the equation of the plane containing the line (1) is

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$

where $$al + bm + cn = 0.$$

To find the equation of a plane containing the line $a_1x + b_1y + c_1z + d_1 = 0$, $a_2x + b_2y + c_2z + d_2 = 0$

Here, the equation of the line is

$$a_1x + b_1y + c_1z + d_1 = 0 = a_2x + b_2y + c_2z + d_2 \quad \ldots(1)$$

The equation of a plane containing the line (1) is

$$a_1x + b_1y + c_1z + d_1 + \lambda(a_2x + b_2y + c_2z + d_2) = 0.$$

Where λ is arbitrary constant which will be determined by the given conditions.

Worked Out Examples

Ex.1 Find the coordinates of the point of intersection of the line $\frac{x+1}{1} = \frac{y+3}{3} = \frac{z-2}{-2}$ **with the plane** $3x + 4y + 5z = 5$

Solution:

Here the equation of the line is

$$\frac{x+1}{1} = \frac{y+3}{3} = \frac{z-2}{-2} = r \text{ (say)}$$

Any point of this line is

$$(r-1,\ 3r-3,\ -2r+2)$$

If this point will lie on the plane $3x + 4y + 5z = 5$, then the point m satisfy in the plane, so

$$3(r-1) + 4(3r-3) + 5(-2r+2) = 5$$

or $\quad 3r - 3 + 12r - 12 - 10r + 10 = 5$

or $\quad 5r = 10$

$\therefore \quad r = 2$

So $\quad (r-1,\ 3r-3,\ -2r+2) = (2-1,\ 6-3,\ -4+2) = (1,\ 3,\ -2)$

Hence, the co-ordinate of the point of intersection of the line and gi plane is $(1, 3, -2)$.

Ex. 2 Find the point where the line joining (2, - 3, 1), (3, - 4, - cuts the plane 2x + y + z = 7

Solution:

The equation of a line joining two points $(2, -3, 1)$ and $(3, -4, -5)$ is

$$\frac{x-2}{3-2} = \frac{y+3}{-4+3} = \frac{z-1}{-5-1}$$

or $\quad \dfrac{x-2}{1} = \dfrac{y+3}{-1} = \dfrac{z-1}{-6} = r \text{ (say)}$

Any point of this line is

$$(r+2,\ -r-3,\ -6r+1)$$

If this point will lie on the plane $2x + y + z = 7$, then the point m satisfy in the plane, so

$$2(r+2) + (-r-3) - 6r + 1 = 7$$

or $\quad 2r + 4 - r - 3 - 6r + 1 = 7$

or $\quad -5r = 5$

$\therefore \quad r = -1.$

So $\quad (r+2,\ -r-3,\ -6r+1) = (-1+2,\ 1-3,\ 6+1) = (1,\ -2,\ 7)$

Hence the coordinate of the point where the line cuts the plane is

$(1, -2, 7)$.

Ex. 3. Find k so that the lines $\dfrac{x-1}{-3} = \dfrac{y-2}{2k} = \dfrac{z-3}{2}$ and $\dfrac{x-1}{3k} = \dfrac{y-5}{1} = \dfrac{z-6}{-5}$ may be perpendicular to each other

Solution:

Here, the equations of the two lines are

$$\frac{x-1}{-3} = \frac{y-2}{2k} = \frac{z-3}{2} \qquad \text{....(1)}$$

and

$$\frac{x-1}{3k} = \frac{y-5}{1} = \frac{z-6}{-5} \qquad \text{...(2)}$$

The direction ratios of the line (1) are

$a_1 = -3, \quad b_2 = 2k, \quad c_1 = 2$

The direction ratios of the line (2) are

$a_2 = 3k, \quad b_2 = 1, \quad c_2 = -5$

If the two lines (1) and (2) are perpendicular, then

$$a_1 a_2 + b_1 b_2 + c_1 c_2 = 0$$

or $\quad -3(3k) + 2k.1 + 2.(-5) = 0$

or $\quad -9k + 2k - 10 = 0$

$\therefore \quad k = -\dfrac{10}{7}.$

Ex.4. Find the distance of the point (3, - 4, 5) from the plane 2x + 5y - 6z = 16 measured along a line with direction cosines proportional to 2, 1, - 2

Solution:

The equation of a line passing through the point $P(3, -4, 5)$ with direction ratios $2, 1, -2$ is

$$\frac{x-3}{2} = \frac{y+4}{1} = \frac{z-5}{-2} = r \text{ (say)}$$

Any point of this line is $Q(2r+3,\ r-4,\ -2r+5)$.

If this point will lie on the plane $2x + 5y - 6z = 16$, then the point must satisfy in the plane, so

$$2(2r+3) + 5(r-4) - 6(-2r+5) = 16$$

or $\quad 4r + 6 + 5r - 20 + 12r - 30 = 16$

or $\quad 21r = 60$

$\therefore \quad r = \dfrac{20}{7}$

So $\quad (2r+3, r-4, -2r+5) = \left(\frac{40}{7}+3, \frac{20}{7}-4, -\frac{40}{7}+5\right) = \left(\frac{61}{7}, -\frac{8}{7}, -\frac{5}{7}\right)$

Hence the point of intersection is $Q\left(\frac{61}{7}, -\frac{8}{7}, -\frac{5}{7}\right)$

Now $\quad PQ = \sqrt{\left(\frac{61}{7}-3\right)^2 + \left(\frac{-8}{7}+4\right)^2 + \left(\frac{-5}{7}-5\right)^2}$

$$= \sqrt{\frac{1600}{49}+\frac{400}{49}+\frac{1600}{49}} = \frac{60}{7}.$$

Ex.5. Find the distance of the point (1, -2, 3) from the pl[ane] $x - y + z = 5$ measured parallel to the line $\frac{x}{2} = \frac{y}{3} = \frac{z}{-6}$

2072 Aswin, ...

Solution:

The equation of a line passing through the point P(1, -2, 3) with direction rati[os] 2, 3, - 6 is

$$\frac{x-1}{2} = \frac{y+2}{3} = \frac{z-3}{-6} = r \text{ (say)}$$

Any point of this line is Q(2r + 1, 3 r -2, - 6r + 3). If this point will lie on the plane $x - y + z = 5$, then the point must satisfy in the plane, so

$$2r+1-3r+2-6r+3 = 5,$$

or $\quad -7r = -1,$

$\therefore \quad r = \frac{1}{7}.$

So $\quad (2r+1, 3r-2, -6r+3) = \left(\frac{2}{7}+1, \frac{3}{7}-2, -\frac{6}{7}+3\right) = \left(\frac{9}{7}, -\frac{11}{7}, \frac{15}{7}\right)$

Hence the point of intersection is $Q\left(\frac{9}{7}, -\frac{11}{7}, \frac{15}{7}\right)$

Now $\quad PQ = \sqrt{\left(\frac{9}{7}-1\right)^2 + \left(-\frac{11}{7}+2\right)^2 + \left(\frac{15}{7}-3\right)^2}$

$$= \sqrt{\frac{4}{49}+\frac{9}{49}+\frac{36}{49}} = 1.$$

Ex. 6. Find the image of the point P(1, 3, 4) in the plane $2x - y + z + 3 = 0$

Solution:

The equation of a line through the point P(1, 3, 4) and normal to the plane

$2x - y + z + 3 = 0$ is

$$\frac{x-1}{2} = \frac{y-3}{-1} = \frac{z-4}{1} = r\text{(say)}$$

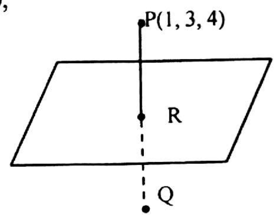
Any point of this line is $(2r+1, -r+3, r+4)$

If this point Q(2r + 1, -r + 3, r + 4) is the image of P, then the middle point of PQ is

$$R\left(\frac{1+2r+1}{2}, \frac{3-r+3}{2}, \frac{4+r+4}{2}\right) = R\left(r+1, \frac{6-r}{2}, \frac{r+8}{2}\right)$$

This point must lie on the plane $2x - y + z + 3 = 0$,

So, we have



$$2(r+1) - \frac{6-r}{2} + \frac{r+8}{2} + 3 = 0$$

or $\quad 2r+2-3+\frac{r}{2}+\frac{r}{2}+4+3 = 0$

or $\quad 3r+6 = 0$

$\therefore \quad r = -2$

When r = -2, the coordinate of Q is

$$(2r+1, -r+3, r+4) = (-4+1, 2+3, -2+4) = (-3, 5, 2)$$

Hence the image of P is Q(- 3, 5, 2).

Ex. 7. Find the points in which the line $\frac{x+1}{-1} = \frac{y-12}{5} = \frac{z-7}{2}$ cuts the surface $11x^2 - 5y^2 + z^2 = 0$

Solution:

Here, the equation of the line is

$$\frac{x+1}{-1} = \frac{y-12}{5} = \frac{z-7}{2} = r\text{(say)}$$

Any point of this line is $(-r-1, 5r+12, 2r+7)$

If this point cuts the surface $11x^2 - 5y^2 + z^2 = 0$, then the point must satisfy the surface. So, we have

$$11(-r-1)^2 - 5(5r+12)^2 + (2r+7)^2 = 0$$

or $\quad 11(r^2+2r+1) - 5(25r^2+120r+144) + (4r^2+28r+49) = 0$

or $\quad 11r^2+22r+11-125r^2-600r-720+4r^2+28r+49 = 0$

or $\quad -110r^2-550r-660 = 0$

or $\quad r^2+5r+6 = 0$

or $(r+3)(r+2)=0$ $\therefore$ $r = -2, -3$

When $r = -2$, so the point is

$$(-r-1, 5r+12, 2r+7) = (2-1, -10+12, -4+7) = (1, 2, 3)$$

When $r = -3$, so the point is

$$(-r-1, 5r+12, 2r+7) = (3-1, -15+12, -6+7) = (2, -3, 1)$$

Therefore, the points are $(1, 2, 3)$ and $(2, -3, 1)$.

Ex. 8. If the axes are rectangular so that the equation to the through (α, β, γ) at right angle to the lines $\frac{x}{l_1} = \frac{y}{m_1}$; $\frac{x}{l_2} = \frac{y}{m_2} = \frac{z}{n_2}$ is $\frac{x-\alpha}{m_1n_2 - m_2n_1} = \frac{y-\beta}{n_1l_2 - n_2l_1} = \frac{z-\gamma}{l_1m_2 - l_2m_1}$

Solution:

The equation of a line through (α, β, γ) having direction cosines l, m, n is

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$$

Since it is perpendicular to both lines $\frac{x}{l_1} = \frac{y}{m_1} = \frac{z}{n_1}$ and $\frac{x}{l_1} = \frac{y}{m_2} = \frac{z}{n_2}$

We have

$$ll_1 + mm_1 + nn_1 = 0$$
$$ll_2 + mm_2 + nn_2 = 0$$

Solving these, we get

$$\frac{l}{m_1n_2 - m_2n_1} = \frac{m}{n_1l_2 - n_2l_1} = \frac{n}{l_1m_2 - l_2m_1}$$

Eliminating l, m, n from (1) and (2), we have

$$\frac{x-\alpha}{m_1n_2 - m_2n_1} = \frac{y-\beta}{n_1l_2 - n_2l_1} = \frac{z-\gamma}{l_1m_2 - l_2m_1}$$

Ex. 9. Find the length of the perpendicular from the p (3, - 1, 11) to the line $\frac{x}{2} = \frac{y-2}{3} = \frac{z-3}{4}$. Also obtai equation of perpendicular 2067/072 Magh,

Solution:

Here, the equation of the line is

$$\frac{x}{2} = \frac{y-2}{3} = \frac{z-3}{4} = r \text{ (say)}$$

Any point of this line is

$$N(2r, 3r+2, 4r+3)$$

If PN is perpendicular to the given line, the direction ratios of the line joining the point (3, - 1, 11) to N(2r, 3r + 2, 4r + 3) are

$$2r-3, \quad 3r+2+1, \quad 4r+3-11.$$

i.e.

$$2r-3, \quad 3r+3, \quad 4r-8.$$

Then, using the condition of perpendicularity

$$2(2r-3) + 3(3r+3) + 4(4r-8) = 0$$

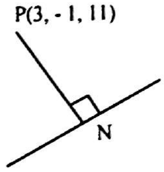
or $4r - 6 + 9r + 9 + 16r - 32 = 0$

or $29r - 29 = 0$

or $r = 1$.

$\therefore$ $N(2r, 3r+2, 4r+3) = N(2, 3+2, 4+3)$

So, $= N(2, 5, 7)$.



Hence, the coordinates of N is (2, 5, 7).

Thus, the length of the perpendicular

$$PN = \sqrt{(3-2)^2 + (-1-5)^2 + (11-7)^2}$$
$$= \sqrt{1+36+16} = \sqrt{53}.$$

The equation of the perpendicular joining two points P(3, - 1, 11) and N(2, 5, 7) is

$$\frac{x-3}{2-3} = \frac{y+1}{5+1} = \frac{z-11}{7-11}$$

$$\therefore \quad \frac{x-3}{1} = \frac{y+1}{-6} = \frac{z-11}{4}$$

Ex. 10. Express the equation of the line
$x + 2y + 3z - 6 = 0$, $3x + 4y + 5z - 2 = 0$ in symmetrical form

Solution:

Let l, m, n be the direction cosines of the line of intersection of two planes

$$x + 2y + 3z - 6 = 0, \quad 3x + 4y + 5z - 2 = 0.$$

The direction ratios of the normal to these planes are

$$1, 2, 3 \text{ and } 3, 4, 5.$$

Since the line is perpendicular to both normal to the planes, using the condition of perpendicularity, we have

$$l + 2m + 3n = 0$$
$$3l + 4m + 5n = 0$$

Solving these, we get

$$\frac{l}{10-12} = \frac{m}{9-5} = \frac{n}{4-6}$$

$$\therefore \quad \frac{l}{1} = \frac{m}{-2} = \frac{n}{1}$$

Thus the direction cosines of the line of intersection of the given planes are proportional to 1, - 2, 1.
To find a point on the line, putting z = 0 in the given equations of planes, we have

$$x + 2y - 6 = 0 \text{ and}$$
$$3x + 4y - 2 = 0$$

Solving these, we get

$$\frac{x}{20} = \frac{y}{-16} = \frac{1}{-2}$$

$\therefore$ $x = -10,\ y = 8$

The point on the line is (- 10, 8, 0)
Thus the equation of the line in symmetrical form is

$$\frac{x+10}{1} = \frac{y-8}{-2} = \frac{z}{1}.$$

Ex. 11. Find the angle between the lines x-2y+z = 0, x + y - z = 0 and x + 2y + z = 0, 8x + 12y + 5z = 0

Solution:

Here, the equations of the lines are

$$x - 2y + z = 0,\quad x + y - z = 0 \qquad \text{.........(1)}$$
and $$x + 2y + z = 0,\quad 8x + 12y + 5z = 0 \qquad \text{.........(2)}$$

Let l_1, m_1, n_1 and l_2, m_2, n_2 be direction cosines of the lines of intersection of (1) and (2) respectively. The direction cosines of the normal to the planes of(1) are proportional to

1, - 2, 1, and 1, 1, - 1.

Since the line of intersection of (1) is perpendicular to both normals to the planes, using the condition of perpendicularity ,we have

$$l_1 - 2m_1 + n_1 = 0 \text{ and}$$
$$l_1 + m_1 - n_1 = 0$$

Solving these, we get

$$\frac{l_1}{1} = \frac{m_1}{2} = \frac{n_1}{3} = \frac{\sqrt{l_1^2 + m_1^2 + n_1^2}}{\sqrt{1+4+9}} = \frac{1}{\sqrt{14}}$$

$$\therefore \quad l_1 = \frac{1}{\sqrt{14}},\quad m_1 = \frac{2}{\sqrt{14}} \text{ and } n_1 = \frac{3}{\sqrt{14}}.$$

Also, the direction cosines of the normals to the planes of(2) are proportional to

1, 2, 1, and 8, 12, 5.

Since the line of intersection of (2) is perpendicular to both normals to the planes, using the condition of perpendicularity ,we have

$$l_2 + 2m_2 + n_2 = 0 \text{ and}$$
$$8l_2 + 12m_2 + 5n_2 = 0$$

Solving these, we get

$$\frac{l_2}{-2} = \frac{m_2}{3} = \frac{n_2}{-4}$$

or $$\frac{l_2}{2} = \frac{m_2}{-3} = \frac{n_2}{4} = \frac{\sqrt{l_2^2 + m_2^2 + n_2^2}}{\sqrt{4+9+16}} = \frac{1}{\sqrt{29}}$$

$$\therefore \quad l_2 = \frac{2}{\sqrt{29}},\quad m_2 = \frac{-3}{\sqrt{29}} \text{ and } n_2 = \frac{4}{\sqrt{29}}$$

If θ be the angle between them, then

$$\cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2$$
$$= \frac{1.2 + 2.-3 + 3.4}{\sqrt{14}\sqrt{29}} = \frac{8}{\sqrt{406}}$$

$$\therefore \quad \theta = \cos^{-1}\left(\frac{8}{\sqrt{406}}\right).$$

Ex. 12. Find the angle between the line $\frac{x+1}{2} = \frac{y}{3} = \frac{z-3}{6}$ and the plane 3x + y + z = 7

Solution:

Here, the equation of the line is

$$\frac{x+1}{2} = \frac{y}{3} = \frac{z-3}{6}$$

Its direction ratios are 2, 3 ,6 .
The equation of the plane is

$$3x + y + z = 7$$

The direction ratios of the normal to the plane are 3 ,1 ,1.
If θ be the angle between the line and the plane, then

$$\cos\theta = \frac{3\times 2 + 1\times 3 + 1.6}{\sqrt{9+1+1}\sqrt{4+9+36}} = \frac{15}{7\sqrt{11}}$$

or $$\cos\theta = \frac{15}{\sqrt{539}}$$

$$\therefore \quad \theta = \cos^{-1}\left(\frac{15}{\sqrt{539}}\right).$$

Ex. 13. Prove that the line $\frac{x-3}{2} = \frac{y-4}{3} = \frac{z-5}{4}$ is parallel to the plane $4x + 4y - 5z = 0$

Solution:

Here, the equation of the line is

$$\frac{x-3}{2} = \frac{y-4}{3} = \frac{z-5}{4}$$

The direction ratio of this line are 2, 3, 4.

The equation of the plane is

$$4x + 4y - 5z = 0$$

The direction ratios of normal to the plane are 4, 4, - 5.

Using the condition of parallelism,

$$2 \times 4 + 3 \times 4 + 4 \times (-5) = 0$$

or $$8 + 12 - 20 = 0$$

$\therefore$ $$0 = 0.$$

Hence the line is parallel to the plane.

Ex. 14. Find the equation of plane containing the line $\frac{x-1}{2} = \frac{y-3}{1} = \frac{z-13}{2}$ and is perpendicular to the plane $x + y + z = 3$

Solution:

Any equation of plane containing the line $\frac{x-1}{2} = \frac{y-3}{1} = \frac{z-13}{2}$ is

$$a(x-1) + b(y-3) + c(z-13) = 0 \qquad .....(1)$$

where $$2a + b + 2c = 0 \qquad .....(2)$$

Since (1) is perpendicular to the plane $x + y + z = 3$, using the condition of perpendicularity, we have

$$a + b + c = 0 \qquad .....(3)$$

Solving (2) and (3), we obtain

$$\frac{a}{-1} = \frac{b}{0} = \frac{c}{1} \qquad .....(4)$$

Eliminating a, b, c between (1) and (4), we obtain

$$x - z + 12 = 0$$

Ex. 15. Find the equation of the plane containing the line $\frac{x-2}{2} = \frac{y-3}{4} = \frac{z-4}{5}$ and parallel to the co-ordinate axes.

Solution:

Any equation of plane through the line $\frac{x-2}{2} = \frac{y-3}{4} = \frac{z-4}{5}$ is

$$a(x-2) + b(y-3) + c(z-4) = 0 \qquad .....(1)$$

where $$2a + 4b + 5c = 0 \qquad ....(2)$$

Since equation (1) is parallel to co-ordinate plane $x = 0$, using the condition of parallelism, we have

$$a \times 1 + b \times 0 + c \times 0 = 0$$

$\therefore$ $$a = 0$$

Then from (2)

$$4b + 5c = 0$$

$$\frac{a}{0} = \frac{b}{5} = \frac{c}{-4} \qquad ....(3)$$

Eliminating a, b, c from (1) and (3), we obtain

$$5y - 4z + 1 = 0.$$

If the equation plane (1) is parallel to the plane $y = 0$, then $b = 0$ and from (2),

$$2a + 5c = 0$$

$$\frac{a}{5} = \frac{b}{0} = \frac{c}{-2} \qquad ....(4)$$

Eliminating a, b, c from (1) and (4), we obtain

$$5x - 2z - 2 = 0$$

If the plane (1) is parallel to the plane $z = 0$, then $c = 0$ and from (2),

$$2a + 4b = 0$$

or $$\frac{a}{-2} = \frac{b}{1} = \frac{c}{0} \qquad ....(5)$$

Eliminating a, b, c from (1) and (5), we get

$$2x - y - 1 = 0$$

Ex. 16. Find the equation of the plane containing the line $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$ and passes through (x_1, y_1, z_1)

Solution:

Any equation of plane containing the line $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$ is

$$a(x-\alpha) + b(y-\beta) + c(z-\gamma) = 0 \quad \ldots(1)$$

where $\quad al + bm + cn = 0 \quad \ldots(2)$

Since the equation of plane (1) passes through the point (x_1, y_1, z_1)

$$a(x_1-\alpha) + b(y_1-\beta) + c(z_1-\gamma) = 0 \quad \ldots(3)$$

Eliminating a, b, c from (1), (2) and (3), we get

$$\begin{vmatrix} x-\alpha & y-\beta & z-\gamma \\ l & m & n \\ x_1-\alpha & y_1-\beta & z_1-r \end{vmatrix} = 0$$ is the required equation of plane.

Exercise -9

1. Find the point where the line joining the points (2,1,3), (4,-2,5) cuts the plane $2x + y - z - 3 = 0$
2. Find where the line $\frac{x-1}{2} = \frac{y-2}{-3} = \frac{z+3}{4}$ meets the plane $2x + 4y - z + 1 = 0$
3. Show that the equation of the straight line through (a, b, c) parallel to the x-axis are $y = b$, $z = c$
4. Find the equation of the line through the point (- 2, 3, 4) and parallel to the planes $2x + 3y + 4z = 5$ and $4x + 3y + 5z = 6$
5. Find the equation of the line through (1, 2, - 1) and perpendicular to each of the lines $\frac{x}{1} = \frac{y}{0} = \frac{z}{-1}$ and $\frac{x}{3} = \frac{y}{4} = \frac{z}{5}$
6. Prove that the lines $x = ay + b$, $z = cy + d$ and $x = a'y + b'$, $z = c'y + d'$ are perpendicular if $aa' + cc' + 1 = 0$ [2067 Chaitra B.E.]
7. Find the distance of the point (3, - 4, 5) from the plane $2x + 5y - 6z = \ldots$ measured parallel to the line $\frac{x}{2} = \frac{y}{1} = \frac{z}{-2}$
8. Find the image of the point (2, -1, 3) in the plane $3x - 2y - z - 9 = 0$ [2069 Bhadra B.E.]
9. Find the distance of the point (1, -2, 3) from the plane $x - y + z = \ldots$ measured parallel to the line $\frac{x}{2} = \frac{y}{3} = \frac{z}{-6}$

10. Express the equation of a line $x + y + z + 1 = 0$, $4x + y - 2z + 2 = 0$ in symmetrical form.
11. Show that the equation of the perpendicular from the point (α, β, γ) to the plane $ax + by + cz + d = 0$ is $\frac{x-\alpha}{a} = \frac{y-\beta}{b} = \frac{z-\gamma}{c}$ and also find the perpendicular distance of the point (α, β, γ) from the plane
12. Find the distance from the point (3, 4, 5) to the point where the line $\frac{x-3}{1} = \frac{y-4}{2} = \frac{z-5}{2}$ meets the plane $x + y + z = 2$
13. Find the angle between the lines $x + 2y + z - 5 = 0 = x + y - 2z - 3$ and $2x - y - z = 0 = 7x + 10y - 8z$
14. Find the equation of the plane through the point (1, 1, 1) and perpendicular to the line $x - 2y + z = 2$, $4x + 3y - z + 1 = 0$.
15. Find the equation of a plane through the origin containing the line $\frac{x-1}{5} = \frac{y-2}{4} = \frac{z-3}{5}$
16. Find the equation of the plane containing the line $a_1x+b_1y+c_1z+d_1=0$, $a_2x + b_2y + c_2z + d_2 = 0$ and parallel to the line $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$
17. Prove that the plane through the point (α, β, γ) and the line $x = py + q = rz + s$ is $\begin{vmatrix} x & py+q & rz+s \\ \alpha & p\beta+q & r\gamma+s \\ 1 & 1 & 1 \end{vmatrix} = 0$ [2068 Bhadra B. E.]
18. Obtain equation of the plane passing through the line of intersection of two planes $7x - 4y + 7z + 16 = 0$ and $4x + 3y - 2z + 13 = 0$ and perpendicular to the plane $x - y - 2z + 5 = 0$
19. Find the equation of the plane through the line $2x + 3y - 5z - 4 = 0$, $3x - 4y + 5z - 6 = 0$ and parallel to the coordinate axes [2070 Magh. B. E.]
20. Find the equation of the plane through the points (1, 0, -1) and (3, 2, 2) and parallel to the line $\frac{x-1}{1} = \frac{1-y}{2} = \frac{z-2}{3}$

Answers

1. (0,4,1)

2. $\left(\frac{10}{3}, -\frac{3}{2}, -\frac{2}{3}\right)$

4. $\frac{x+2}{1} = \frac{y-3}{2} = \frac{z-4}{-2}$

5. $\frac{x-1}{1} = \frac{y-2}{-2} = \frac{z+1}{1}$

7. $\frac{60}{7}$

8. $\left(\frac{26}{7}, -\frac{15}{7}, \frac{17}{7}\right)$

9. 1

10. $\frac{x+1/3}{-1} = \frac{y+2/3}{2} = \frac{z}{-1}$

11. $\frac{a\alpha + b\beta + c\gamma + d}{\sqrt{a^2+b^2+c^2}}$

12. 6

13. $\cos^{-1}\frac{5}{\sqrt{119}}$

14. $x - 5y - 11z + 15 = 0$

15. $x - 5y - 3z = 0$

16. $\frac{a_1x+b_1y+c_1z+d_1}{a_1 l + b_1 m + c_1 n} = \frac{a_2x+b_2y+c_2z+d_2}{a_2 l + b_2 m + c_2 n}$

18. $47x - 11y + 29z + 119 = 0$

19. $17x - 25z = 0,\ 5z - 17x + 34 = 0,\ 5x - y - 10 = 0$

20. $3y - 2z - 2 = 0$

3. 2. Coplannar Lines

In three-dimensional geometry, the lines and planes are correlated .The two straight lines are said to be *Coplanar* if they lie on the same plane.

3. 2.1 Conditions for the Lines are to be Coplanar

The three cases for the lines are to be coplanar are as follows:

To find the condition that the straight lines $\frac{x-x_1}{l_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1}$ *and* $\frac{x-x_2}{l_2} = \frac{y-y_2}{m_2} = \frac{z-z_2}{n_2}$ *are coplanar*

Here, the equations of straight lines are

$$\frac{x-x_1}{l_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1} \qquad ...(1)$$

and

$$\frac{x-x_2}{l_2} = \frac{y-y_2}{m_2} = \frac{z-z_2}{n_2} \qquad ...(2)$$

The equation of any plane containing the line (1) is

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0 \qquad ......(3)$$

where

$$al_1 + bm_1 + cn_1 = 0 \qquad ......(4)$$

The line (2) will lie on the plane (3) if it is parallel to the plane (3) and its point (x_2, y_2, z_2) lies on the plane, we have

$$al_2 + bm_2 + cn_2 = 0 \qquad ....(5)$$

and

$$a(x_2 - x_1) + b(y_2 - y_1) + c(z_2 - z_1) = 0 \qquad ....(6)$$

Eliminating a, b, c from (4), (5) and (6), we get

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0 \text{ is the required condition.}$$

Also eliminating a, b, c from (3), (4) and (5), we get

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0 \text{ is the equation of the plane}$$

containing the lines (1) and (2).

Find the condition that the equation of lines $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$ $a_1x + b_1y + c_1z + d_1 = 0,\ a_2x + b_2y + c_2z + d_2 = 0$ *are coplanar*

Here, the equations of the lines are

$$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n} \qquad ....(1)$$

$$a_1x + b_1y + c_1z + d_1 = 0,\ a_1x + b_2y + c_2z + d_2 = 0 \qquad ...(2$$

The equation of any plane through the lines

$a_1x + b_1y + c_1z + d_1 = 0,\ a_2x + b_2y + c_2z + d_2 = 0$ is

$$a_1x + b_1y + c_1z + d_1 + \lambda\,(a_2x + b_2y + c_2z + d_2) = 0$$

or $$(a_1 + \lambda a_2)\,x + (b_1 + \lambda b_2)y + (c_1 + \lambda c_2)\,z + d_1 + \lambda d_1 = 0 \qquad .....(3)$$

The lines (1) and (2) are coplanar if the plane (3) is contained in the line (1) so that

$$(a_1 + \lambda a_2)\,x_1 + (b_1 + \lambda b_2)\,y_1 + (c_1 + \lambda c_2)\,z_1 + d_1 + \lambda d_1 = 0$$

or $$\lambda = -\frac{a_1x_1 + b_1y_1 + az_1 + d_1}{a_2x_1 + b_{21}y_1 + c_2z_1 + d_2} \qquad ....(4)$$

where

$$(a_1 + \lambda a_2)\,l + (b_1 + \lambda b_2)\,m + (c_1 + \lambda c_2)\,n = 0$$

or $\lambda = -\dfrac{a_1 l + b_1 m + c_1 n}{a_2 l + b_2 m + c_2 n}$ ......(5)

From (4) and (5), we have $\dfrac{a_1x_1 + b_1y_1 + c_1z_1 + d_1}{a_2x_1 + b_2y_1 + c_2z_1 + d_2} = \dfrac{a_1 l + b_1 m + c_1 n}{a_2 l + b_2 m + c_2 n}$ is the required condition.

To find the condition that the lines $a_1x + b_1y + c_1z + d_1 = 0$, $a_2x + b_2y + c_2z + d_2 = 0$; $a_3x + b_3y + c_3z + d_3 = 0$, $a_4x + b_4y + c_4z + d_4 = 0$ may intersect or coplanar

Here, the equations of two lines are

$$a_1x + b_1y + c_1z + d_1 = 0,\ a_2x + b_2y + c_2z + d_2 = 0 \quad ...(1)$$
$$a_3x + b_3y + c_3z + d_3 = 0,\ a_4x + b_4y + c_4z + d_4 = 0 \quad ...(2)$$

If the two lines intersect, then their point of intersection lies on each of the four planes representating the two lines. Hence the required condition is obtained by eliminating x, y, z between these four equations and is

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = 0$$

If this condition is satisfied, then the equations of the lines are coplanar and the intersection point is obtained by solving any three of four equations of planes.

Worked Out Examples

Ex.1. Show that the lines $\dfrac{x+1}{-3} = \dfrac{y-3}{2} = \dfrac{z+2}{1}$ and $\dfrac{x}{1} = \dfrac{y-7}{-3} = \dfrac{z+7}{2}$ are coplanar. Also, find the equation of the plane containing them [2069 Poush, B.E.]

Solution:

The equations of the lines are

$$\frac{x+1}{-3} = \frac{y-3}{2} = \frac{z+2}{1} \quad ...(1)$$

and $$\frac{x}{1} = \frac{y-7}{-3} = \frac{z+7}{2} \quad ...(2)$$

Here
$l_1 = -3,\ m_1 = 2,\ n_1 = 1$
$x_1 = -1,\ y_1 = 3,\ z_1 = -2$
$l_2 = 1,\ m_2 = -3,\ n_2 = 2$ and
$x_2 = 0,\ y_2 = 7,\ z_2 = -7$

If the two lines (1) and (2) are coplanar, then we have

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}$$

or $$\begin{vmatrix} -1-0 & 3-7 & -2+7 \\ -3 & 2 & 1 \\ 1 & -3 & 2 \end{vmatrix} = 0$$

or $$\begin{vmatrix} -1 & -4 & 5 \\ -3 & 2 & 1 \\ 1 & -3 & 2 \end{vmatrix} = 0$$

or $-1(4+3) + 4(-6-1) + 5(9-2) = 0$

or $-35 + 35 = 0$

$\therefore$ $0 = 0$.

Hence the lines (1) and (2) are coplanar.

Also, the equation of the plane containing the given lines (1) and (2) is

$$\begin{vmatrix} x+1 & y-3 & z+2 \\ -3 & 2 & 1 \\ 1 & -3 & 2 \end{vmatrix} = 0$$

Scanned by CamScanner

or $(x+1)(4+3) - (y-3)(-6-1) + (z+2)(9-2) = 0$

or $7(x+1) + 7(y-3) + 7(z+2) = 0$

$\therefore$ $x + y + z = 0.$

Ex.2. Show that the lines $\frac{x-1}{2} = \frac{y+1}{-3} = \frac{z+10}{8}$, $\frac{x-4}{1} = \frac{y+3}{-4} = \frac{z+1}{7}$ are coplanar. Find their common point and the equation of the plane in which they lie

2067 Chaitra B.E.

Solution:

Here, the equations of the lines are

$$\frac{x-1}{2} = \frac{y+1}{-3} = \frac{z+10}{8} = r\text{ (say)} \qquad ...(1)$$

and

$$\frac{x-4}{1} = \frac{y+3}{-4} = \frac{z+1}{7} \qquad ...(2)$$

Any point of the line (1) is

$$(2r+1, -3r-1, 8r-10)$$

If this point lies on the line (2), then we have

$$\frac{2r+1-4}{1} = \frac{-3r-1+3}{-4} = \frac{8r-10+1}{7}$$

or $$\frac{2r-3}{1} = \frac{-3r+2}{-4} = \frac{8r-9}{7}$$

Taking first and second ratios,

$$\frac{2r-3}{1} = \frac{-3r+2}{-4} \qquad \therefore\ r = 2$$

This value of r clearly satisfies the equation $\frac{-3r+2}{-4} = \frac{8r-9}{7}$

Hence the lines (1) and (2) intersect (i.e. coplanar) and their point of intersection is $(5, -7, 6)$.

The equation of plane in which they lie is

$$\begin{vmatrix} x-1 & y+1 & z+10 \\ 2 & -3 & 8 \\ 1 & -4 & 7 \end{vmatrix} = 0$$

or $(x-1)(-21+32) - (y+1)(14-8) + (z+10)(-8+3) = 0$

or $11(x-1) - 6(y+1) - 5(z+10) = 0$

$\therefore$ $11x - 6y - 5z = 67.$

Ex.3. Show that the lines $\frac{x+5}{3} = \frac{y+4}{1} = \frac{z-7}{-2}$ and $3x + 2y + z - 2 = 0$, $x - 3y + 2z - 13 = 0$ are coplanar and find the equation to the plane in which they lie

Solution:

Here, the equations of the lines are

$$\frac{x+5}{3} = \frac{y+4}{1} = \frac{z-7}{-2} = r\text{(say)} \qquad ...(1)$$

and $$3x + 2y + z - 2 = 0,\ x - 3y + 2z - 13 = 0 \qquad ....(2)$$

Any point of the line (1) is $P(3r-5, r-4, -2r+7)$.

This point which lies on one of the plane $3x + 2y + z - 2 = 0$ of (2) if

$3(3r-5) + 2(r-4) + (-2r+7) - 2 = 0$

or $9r - 15 + 2r - 8 - 2r + 7 - 2 = 0$

or $9r = 18$

$\therefore$ $r = 2$

The point P will also lie in the plane $x - 3y + 2z - 13 = 0$ of (2) if

$3r - 5 - 3(r-4) + 2(-2r+7) - 13 = 0$

or $3r - 5 - 3r + 12 - 4r + 14 - 13 = 0$

or $-4r = -8$

$\therefore$ $r = 2.$

Since these two values of r are equal, the given lines intersect and hence coplanar. Putting $r = 2$ in the coordinates of P, we get

$P(3r-5, r-4, -2r+7) = (6-5, 2-4, -4+7) = (1, -2, 3).$

Hence the coordinate of the point of intersection is $P(1, -2, 3)$.

The equation of a plane containing the line (2) is

$$3x + 2y + z - 2 + \lambda(x - 3y + 2z - 13) = 0 \qquad ....(3)$$

If this plane contains the line (1), then the point (- 5, - 4, 7) must lie in this plane (3), so

$$- 15 - 8 + 7 - 2 + \lambda(- 5 + 12 + 14 - 13) = 0$$

or $$- 15 - 8 + 7 - 2 + \lambda(- 5 + 12 + 14 - 13) = 0$$

or $$- 18 + 8\lambda = 0$$

$\therefore$ $$\lambda = \frac{9}{4}.$$

Substituting the values of λ in (1), we get

$$21x - 19y + 22z - 125 = 0$$

Ex. 4. Show that the lines x - 3y + 2z + 4 = 0 = 2x + y + 4z + 1 and 3x + 2y + 5z - 1 = 0 = 2y + z intersect and find the coordinates of their point of intersection

Solution:

Here, the equations of the lines in general form are

$$x - 3y + 2z + 4 = 0 = 2x + y + 4z + 1 \quad \text{....(1)}$$

and $$3x + 2y + 5z - 1 = 0 = 2y + z \quad \text{....(2)}$$

Putting z = 0 in (2), we get y = 0 and $x = \frac{1}{3}$

Hence the coordinate of the line of intersection of two planes of (2) is $\left(\frac{1}{3}, 0, 0\right)$.

Let l, m, n be directions cosines of the line of intersection of these two planes of (2), so that it is perpendicular to the normals to their planes of (2), we use the condition of perpendicularity

$$3l + 2m + 5n = 0$$

$$2l + n = 0$$

Solving these, we get

$$\frac{l}{8} = \frac{m}{3} = \frac{n}{-6}$$

So, the equation of a line through the point $\left(\frac{1}{3}, 0, 0\right)$ with direction ratios 8, 3, - 6 is

$$\frac{x - 1/3}{8} = \frac{y - 0}{3} = \frac{z - 0}{-6} = r(\text{say})$$

Any point of this line is P(8r + 1/3, 3r, - 6r).

If this point lies on the plane x - 3y + 2z + 4 = 0 of (1), then the point must satisfy this plane, so

$$8r + \frac{1}{3} - 9r - 12r + 4 = 0$$

or $$-13r + \frac{13}{3} = 0$$

$\therefore$ $$r = \frac{1}{3}.$$

The point P will also lie on the plane 2x + y + 4z + 1 = 0 of (1) if

$$2\left(8r + \frac{1}{3}\right) + 3r - 24r + 1 = 0$$

or $$16r + \frac{2}{3} - 21r + 1 = 0$$

or $$-5r + \frac{5}{3} = 0$$

$\therefore$ $$r = \frac{1}{3}.$$

Since the values of r are equal, the lines (1) and (2) intersect, and hence coplanar.

Putting the value of r in the coordinate of P, we get

$$\left(8r + \frac{1}{3}, 3r, - 6r\right) = \left(\frac{8}{3} + \frac{1}{3}, 1, - 2\right) = (3, 1, - 2).$$

Hence the coordinate of point of intersection is (3, 1, - 2).

Exercise - 10

1. Show that the lines $\frac{x - 5}{4} = \frac{y - 7}{4} = \frac{z + 3}{-5}$; $\frac{x - 8}{7} = \frac{y - 4}{1} = \frac{z - 5}{3}$ are coplanar. Find their point of intersection and equation of plane in which they lie [2070 Magh. B. E.]

2. Prove that the lines $x = \frac{y - 2}{2} = \frac{z + 3}{3}$ and $\frac{x - 2}{2} = \frac{y - 6}{3} = \frac{z - 3}{4}$ are coplanar and find their plane and point of intersection.

3. Prove that the lines $\frac{x - a}{a'} = \frac{y - b}{b'} = \frac{z - c}{c'}$ and $\frac{x - a'}{a} = \frac{y - b'}{b} = \frac{z - c'}{c}$ intersect and find the coordinates of the point of intersection and the equation of the plane in which they lie.

4. Show that the lines $\frac{x-a+d}{\alpha-\delta}=\frac{y-a}{\alpha}=\frac{z-a-d}{\alpha+\delta}$ and $\frac{x-b+c}{\beta-\gamma}=\frac{y-b}{\beta}=\frac{z-b-c}{\beta+\gamma}$ are coplanar and find the equation of the plane in which they lie.

5. Show that $\frac{x+4}{3}=\frac{y+6}{5}=\frac{z-1}{-2}$ and $3x-2y+z+5=0=2x+3y+4z-4$ are coplanar. Find their point of intersection.

6. Prove that the lines $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ and $4x-3y+1=0=5x-3z+2$ are coplanar. Also, find their point of intersection [2071/072 Ashwin, B.E.]

7. Prove that the line $x+y+z-3=0,\ 2x+3y+4z-5=0$ is coplanar with the line $4x-y+5z-7=0,\ 2x-5y-z-3=0$. Obtain the equation of plane in which they lie

Answers

1. $(1,3,2)$; $17x-47y-24z+172=0$
2. $(2,6,3)$; $x-2y+z+7=0$
3. $(a+c', b+b', c+c')$; $\sum(a'b-ab')=0$
4. $x-2y+z=0$
5. $(2,4,-3)$
6. $(-1,-1,-1)$
7. $x+2y+3z=2$

3.3. Shortest Distance

Two straight lines that do not lie on one plane i.e. they are neither parallel nor intersecting are called *skew lines*. Such lines posses a common perpendicular that is the line of shortest distance between them.

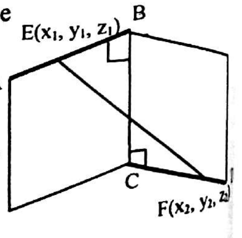
Find the magnitude and the equation of the line of shortest distance between two straight lines.

Let the equations of two skew lines AB and CD be

$$\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1}$$

and

$$\frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_1}{n_2}\ \text{respectively.}$$



Clearly, $E(x_1, y_1, z_1)$ and $F(x_2, y_2, z_2)$ are two points on the line AB and CD respectively.

Let l, m, n be direction cosines of the line of shortest distance BC. Since BC is perpendiculars to both AB and CD.

$$l\,l_1 + m\,m_1 + n\,n_1 = 0$$

and

$$l\,l_2 + m\,m_2 + n\,n_2 = 0$$

Solving these, we get

$$\frac{l}{m_1n_2-m_2n_1}=\frac{m}{n_1l_2-n_2l_1}=\frac{n}{l_1m_2-l_2m_2}$$

$$=\frac{\sqrt{l^2+m^2+n^2}}{\sqrt{(m_1n_2-m_2n_1)^2+(n_1l_2-n_2l_1)^2+(l_1m_2-l_2m_1)^2}}$$

$$=\frac{1}{\sqrt{\Sigma(m_1n_2-m_2n_1)^2}}$$

$$\therefore l=\frac{m_1n_2-m_2n_1}{\sqrt{\Sigma(m_1n_2-m_2n_1)^2}},\ m=\frac{n_1l_2-n_2l_1}{\sqrt{\Sigma(m_1n_2-m_2n_1)^2}},\ n=\frac{l_1m_2-l_2m_2}{\Sigma(m_1n_2-m_2n_1)^2}\quad ..(1)$$

Therefore the length of S.D. (BC) = projection of the line joining the points $E(x_1, y_1, z_1)$ and $F(x_2, y_2, z_2)$ on the line BC with direction cosines l, m, n

$$=(x_2-x_1)\,l+(y_2-y_1)\,m+(z_2-z_1)\,n$$

where l, m, n have the values as given by (1).

The equation of plane containing the lines AB and BC is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l & m & n \end{vmatrix}=0 \qquad .....(2)$$

The equation of plane containing the line CD and BC is

$$\begin{vmatrix} x-x_2 & y-y_2 & z-z_2 \\ l_2 & m_2 & n_2 \\ l & m & n \end{vmatrix}=0 \qquad ......(3)$$

Hence, the equation of the planes (2) and (3) represents the line of shortest distance.

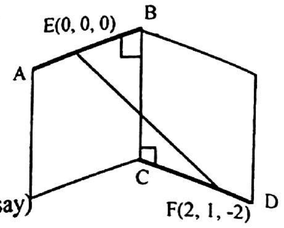
Worked Out Examples

Ex. 1. Find the magnitude and the equation of shortest distance between the lines $\frac{x}{2}=\frac{y}{-3}=\frac{z}{1}$ and $\frac{x-2}{3}=\frac{y-1}{-5}=\frac{z+2}{2}$. [2062 B.E.]

Solution:

- **First Method:**

Here, the equations of the lines are



$$\frac{x}{2}=\frac{y}{-3}=\frac{z}{1}=r\text{(say)} \qquad ...(1)$$

and
$$\frac{x-2}{3}=\frac{y-1}{-5}=\frac{z+2}{2}=r'\text{(say)} \qquad ...(2)$$

Let l, m, n be the direction cosines of the line of shortest distance BC. Since the line, BC is perpendicular to both AB and CD. Using condition of perpendicularity

$$2l-3m+n=0$$
$$3l-5m+2n=0$$

Solving these, we get

$$\frac{l}{1}=\frac{m}{1}=\frac{n}{1}=\frac{\sqrt{l^2+m^2+n^2}}{\sqrt{1+1+1}}=\frac{1}{\sqrt{3}}$$

Length of shortest distance BC = projection of EF on BC.

$$=(x_2-x_1).l+(y_2-y_1).m+(z_2-z_1).n$$
$$=(2-0)\frac{1}{\sqrt{3}}+(1-0)\frac{1}{\sqrt{3}}+(-2-0)\frac{1}{\sqrt{3}}=\frac{1}{\sqrt{3}}$$

The equations of the line of shortest distance (BC) are

$$\begin{vmatrix} x & y & z \\ 2 & -3 & 1 \\ 1 & 1 & 1 \end{vmatrix}=0 \text{ and } \begin{vmatrix} x-2 & y-1 & z+2 \\ 3 & -5 & 2 \\ 1 & 1 & 1 \end{vmatrix}=0$$

i.e. $4x+y-5z=0$ and $7x+y-8z=31$

Second Method:

Any point on the line (1) is $B(2r,-3r,r)$ and any point on the line(2) is $C(3r'+2,-5r'+1,2r'-2)$.

Then the direction cosines of the line of shortest distance BC are proportional to $3r'+2-2r, -5r'+1+3r, 2r'-2-r$.

Since the line of BC is perpendicular to both lines (1) and (2), using the condition of perpendicularity

$$2(3r'+2-2r)-3(-5r'+1+3r)+(2r'-2-r)=0$$

or $$6r'+4-4r+15r'-3-9r+2r'-2-r=0$$

$\therefore$ $$23r'-14r-1=0 \qquad ....(3)$$

and $$3(3r'+2-2r)-5(-5r'+1+3r)+2(2r'-2-r)=0$$

or $$9r'+6-6r+25r'-5-15r+4r'-4-2r)=0$$

$\therefore$ $$38r'-23r-13=0 \qquad ....(4)$$

Solving the equations (3) and (4), we get

$$r=\frac{31}{3};\quad r'=\frac{19}{3}.$$

So that the coordinates $B\left(\frac{62}{3},\frac{-93}{3},\frac{31}{3}\right)$ and $C\left(\frac{63}{3},-\frac{92}{3},\frac{32}{3}\right)$ are the points on the lines (1) and (2) respectively.

Length of S. D. $$=\sqrt{\left(\frac{62}{3}-\frac{63}{3}\right)^2+\left(\frac{92}{3}-\frac{93}{3}\right)^2+\left(\frac{31}{3}-\frac{32}{3}\right)^2}$$
$$=\sqrt{\frac{1}{9}+\frac{1}{9}+\frac{1}{9}}=\frac{\sqrt{3}}{3}=\frac{1}{\sqrt{3}}.$$

In addition, the equations of the line of shortest distance (BC) is

$$\frac{x-\frac{62}{3}}{1}=\frac{y+\frac{93}{3}}{1}=\frac{z-\frac{31}{3}}{1}$$

Third Method:

The equation of the plane containing the line (1) and parallel to theline (2)is

$$\begin{vmatrix} x & y & z \\ 2 & -3 & 1 \\ 3 & -5 & 2 \end{vmatrix}=0$$

or $$x+y+z=0 \qquad ....(5)$$

Now the shortest distance between the two lines

= Length of the perpendicular drawn from the point of $(2,1,-2)$ of line (2) on the plane (5)

$$= \frac{2+1-2}{\sqrt{4+1+1}} = \frac{1}{\sqrt{3}}$$

The equation of the plane through the line(1) and perpendicular to the plane(5) is

$$\begin{vmatrix} x & y & z \\ 2 & -3 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

$\therefore \quad 4x + y - 5z = 0$

The equation of the plane through the line (2) and perpendicular to the plane (5) is

$$\begin{vmatrix} x-2 & y-1 & z+2 \\ 3 & -5 & 2 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

$\therefore \quad 7x + y - 8z - 31 = 0.$

Hence the equation of the line of S.D. are

$4x + y - 5z = 0, 7x + y - 8z - 31 = 0.$

Ex.2. Find the length and equation of the S.D. between the lines $\frac{x-3}{3} = \frac{y-8}{-1} = \frac{z-3}{1}$; $2x - 3y + 27 = 0, 2y - z + 20 = 0$ [2067/071 Bhadra, B.E]

Solution:

Here, the equations of lines are

$$\frac{x-3}{3} = \frac{y-8}{-1} = \frac{z-3}{1} \quad \text{......(1)}$$

and $\quad 2x - 3y + 27 = 0, \; 2y - z + 20 = 0$ ......(2)

The equation of any plane through the line (2) is

$2x - 3y + 27 + \lambda(2y - z + 20) = 0$

or $\quad 2x + (2\lambda - 3)y - \lambda z + 20\lambda + 27 = 0$ ......(3)

It will be parallel to the line (1) if

$2.3 + (2\lambda - 3)(-1) + (-\lambda).1 = 0$

$\therefore \quad \lambda = 3.$

Substituting the value of λ in (3), we get

$2x + 3y - 3z + 87 = 0$ ......(4)

S.D. = length of perpendicular drawn form the point (3, 8, 3) of the line (1) on the plane (4)

$$= \frac{6 + 24 - 9 + 87}{\sqrt{4+9+9}} = \frac{108}{\sqrt{22}}$$

The equation of the plane through the line (1) and perpendicular to (4) is

$$\begin{vmatrix} x-3 & y-8 & z-3 \\ 3 & -1 & 1 \\ 2 & 3 & -3 \end{vmatrix} = 0$$

or $\quad y + z - 11 = 0$

The equation of a plane through the line (2) and perpendicular to (4) is

$2x - 3y + 27 + k(2y - z + 20) = 0$

or $\quad 2x + (2k - 3)y - kz + 20k + 27 = 0$ ....(5)

It is perpendicular to (4) if

$4 + 6k - 9 + 3k = 0$

$\therefore \quad k = \frac{5}{9}$

Substituting the value of k in (5), we get

$18x - 17y - 5z + 343 = 0$

The equations of the line of S.D. are

$y + z - 11 = 0, \quad 18x - 17y - 5z + 343 = 0.$

Ex. 3. Find the length and equation of the line of shortest distance between the lines $3x - 9y + 5z = 0$, $x + y - z = 0$; and $6x + 8y + 3z - 13 = 0$, $x + 2y + z - 3 = 0$

Solution:

The equations of lines are

$3x - 9y + 5z = 0, \quad x + y - z = 0$ ....(1)

and $\quad 6x + 8y + 3z - 13 = 0, \; x + 2y + z - 3 = 0.$ ....(2)

On putting z = 0 in (1), we get

$3x - 9y = 0$

and $\quad x + y = 0.$

Solving these two equations, we get

$x = 0, y = 0.$

Any point of the line of intersection of two planes of (1) is (0, 0, 0).

Let l, m, n be direction cosines of the line of intersection of two planes of (1), then using condition of perpendicularity

$3l - 9m + 5n = 0$

$l + m - n = 0$

Solving these two relations, we get

$$\frac{l}{1} = \frac{m}{2} = \frac{n}{3}$$

The equation of a line through (0, 0, 0) having direction cosines proportional to 1, 2, 3 is

$$\frac{x}{1} = \frac{y}{2} = \frac{z}{3} \qquad \text{.........(3)}$$

The equation of a plane through the line (2) is

$$6x + 8y + 3z - 13 + \lambda(x + 2y + z - 3) = 0$$

$$(6 + \lambda)x + (8 + 2\lambda)y + (3 + \lambda)z - 13 - 3\lambda = 0 \qquad \text{......(4)}$$

It is parallel to the line (3) if

$$(6 + \lambda).1 + (8 + 2\lambda).2 + (3 + \lambda)3 = 0$$

$$\therefore \quad \lambda = \frac{-31}{8}.$$

Substituting the value of λ in (4), we get

$$17x + 2y - 7z - 11 = 0 \qquad \text{..........(5)}$$

S.D. = Length of perpendicular drawn from any point (0, 0, 0) of the line (3) on the plane (5)

$$= \frac{11}{\sqrt{289 + 4 + 49}} = \frac{11}{\sqrt{342}}$$

The equation of a plane through the line (3) and perpendicular to the plane (5) is

$$\begin{vmatrix} x & y & z \\ 1 & 2 & 3 \\ 17 & 2 & -7 \end{vmatrix} = 0$$

$$\therefore \quad 10x - 29y - 16z = 0$$

The equation of a plane through the line (2) is

$$6x + 8y + 3z - 13 + k(x + 2y + z - 3) = 0$$

or $$(6 + k)x + (8 + 2k)y + (3 + k)z - 13 - 3k = 0 \qquad \text{.......(6)}$$

It is perpendicular to (5) if

$$(6 + k).17 + (8 + 2k).2 + (3 + k).-7 = 0$$

$$\therefore \quad k = -\frac{97}{14}$$

Substituting this to (6), we get

$$13x + 82y + 55z - 109 = 0$$

Hence the equations of the line of S.D. are

$$10x - 29y - 16z = 0, \quad 13x + 82y + 55z - 109 = 0$$

Ex. 4. Show that the S. D. between z - axis and the line $ax + by + cz + d = 0, a'x + b'y + c'z + d' = 0$ is $\dfrac{dc' - d'c}{\sqrt{(ac' - a'c)^2 + (bc' - b'c)^2}}$

Solution:

The equation of a plane through the given two planes is

$$ax + by + cz + d = 0, \quad a'x + b'y + c'z + d' = 0 \text{ is}$$

$$ax + by + cz + d + \lambda(a'x + b'y + c'z + d') = 0$$

or $$(a + \lambda a')x + (b + \lambda b') + (c + \lambda c')z + d + \lambda d' = 0. \qquad \text{.......(1)}$$

It will be parallel to z-axis whose direction cosines are 0, 0, 1. The normal to the plane is perpendicular to the z-axis if

$$0(a + \lambda a') + 0.(b + \lambda b') + 1(c + \lambda c') = 0$$

$$\therefore \quad \lambda = -\frac{c}{c'}.$$

Substituting the value of λ in (1), we get

$$(ac' - a'c)x + (bc' - b'c)y + (dc' - d'c) = 0 \qquad \text{.......(2)}$$

S.D = Length of perpendicular drawn from the point (0, 0, 0) of z-axis on the plane (2)

$$= \frac{dc' - d'c}{\sqrt{(ac' - a'c)^2 + (bc' - b'c)^2}}$$

Exercise - 11

1. Find the magnitude and the equation of the line of S. D. between the lines $\frac{x-3}{1} = \frac{y-5}{2} = \frac{z-7}{-3}$ and $\frac{x+1}{3} = \frac{y+2}{-4} = \frac{z+3}{1}$

2. Find the S. D. between the lines $\frac{x-6}{3} = \frac{y-7}{-1} = \frac{z-4}{1}$ and

$\frac{x}{-3} = \frac{y+9}{2} = \frac{z-2}{4}$. Find also the equation of shortest distance and the points

2069 Bhadra, B.E.

3. Find the magnitude and the equation of the S.D. between the lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $\frac{x-2}{3} = \frac{y-4}{4} = \frac{z-5}{5}$ 2068/069, Poush B.E.

4. Show the shortest distance between a diagonal of a rectangular parallelopiped whose edges are a, b, c and the edges not meeting it are $\frac{bc}{\sqrt{b^2+c^2}}, \frac{ca}{\sqrt{c^2+a^2}}, \frac{ab}{\sqrt{a^2+b^2}}$ 2065, B.E.

5. Show that shortest distance between the lines $x + a = 2y = -12z$ and $x = y + 2a = 6z - 6a$ is 2a

6. Find the magnitude of the line of the shortest distance between the lines $\frac{x}{4} = \frac{y+1}{3} = \frac{z-2}{2}$; $5x - 2y - 3z + 6 = 0,\ x - 3y + 2z - 3 = 0$

2072 Magh, B.E.

Answers

1. $7\sqrt{3}$; $5x - 4y - z + 12 = 0, 5x + 2y - 7z = 0$
2. $3\sqrt{30}$; $\frac{x-3}{2} = \frac{y-8}{5} = \frac{z-3}{-1}$; (3, 8, 3) and (- 3, - 7, 6)
3. $\frac{1}{\sqrt{6}}$; $11x + 2y - 7z + 6 = 0,\ 7x + y - 5z + 7 = 0$ 6. $\frac{34}{13\sqrt{6}}$;

3.4. Sphere

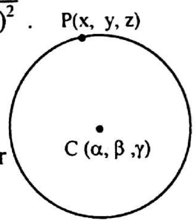
Definition

The locus of a point, which moves always at a constant distance from a fixed point, is called a *Sphere*. The constant distance is called radius and the fixed point is centre of the sphere.

3.4.1. Equation of a Sphere with Centre (α, β, γ) and Radius r

Let P(x, y, z) be any variable point on the sphere with centre C(α, β, γ) and r be radius ,then

$$CP = \sqrt{(x-\alpha)^2 + (y-\beta)^2 + (z-r)^2}\ .$$



By the definition of a sphere

$$CP = r$$

or $$\sqrt{(x-\alpha)^2 + (y-\beta)^2 + (z-r)^2} = r$$

Squaring both sides, we get

$$(x-\alpha)^2 + (y-\beta)^2 + (z-r)^2 = r^2$$

is the equation of a sphere.

In particular, the equation of the sphere with centre origin C(0,0,0) and radius 'a' is

$$x^2 + y^2 + z^2 = a^2$$

3.4.2. General Equation of a Sphere

To prove that the equation $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ represents a sphere with centre (- u, - v, - w) and radius $\sqrt{u^2+v^2+w^2-d}$

Here, the equation in x, y, z is

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$$

This can be written as

$$(x^2 + 2ux) + (y^2 + 2vy) + (z^2 + 2wz) = -d$$

or $$(x^2 + 2ux + u^2) + (y^2 + 2vy + v^2) + (z^2 + 2wz + w^2) = u^2 + v^2 + w^2 - d$$

or $\quad (x + u)^2 + (y + v)^2 + (z + w)^2 = u^2 + v^2 + w^2 - d$ which represents a sphere.

Comparing it with $(x - \alpha)^2 + (y - \beta)^2 + (z - \gamma)^2 = r^2$, we find

$$\text{Centre} = (\alpha, \beta, \gamma) = (-u, -v, -w), \quad \text{Radius} = \sqrt{u^2 + v^2 + w^2 - d}$$

Thus the characteristics of the general equation of a sphere are as follows:

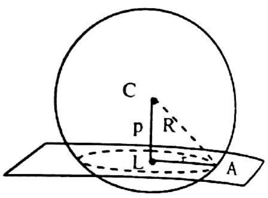
(i) It is of the second degree equation in x, y, z.

(ii) The coefficients of x^2, y^2, z^2 are equal

and (iii) There are no terms containing yz, zx and xy.

3.5 Plane Section of a Sphere

If a plane intersects a sphere, then a circle is formed . This circle made of section of a sphere by a plane, so there are different sizes and shapes of the circle. If the plane intersects a sphere at its centre, then a circle so formed is called a *great circle* otherwise it becomes a small circle.



Thus the equation of sphere

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \text{ and the plane}$$

$$Ax + By + Cz + D = 0 \text{ taken together represent a } \textit{Circle.}$$

Let R be radius of the sphere, p be the length of perpendicular drawn from the centre (- u, - v, - w) of the sphere on the plane Ax + By + Cz + D = 0 and L be the centre of the circle, then the radius of the circle LA is $\quad r = \sqrt{R^2 - p^2}$

3.5.1 Equation of a Sphere Through the Circle

Let the sphere and the plane be given by the equations

$$S \equiv x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \text{ and}$$

$$U \equiv Ax + By + Cz + D = 0 \,.$$

The equation of a sphere through the circle of intersection of the sphere S = 0 and the plane U = 0 is

$S + \lambda U = 0$ where λ is constant which will be determined by the given condition in which the point of intersection of the sphere S = 0 and the plane U = 0 satisfy it.

Note:

The equation of intersection of two spheres $S_1 = 0$ and $S_2 = 0$ is

$$S_1 - S_2 = 0 \text{ which represents a plane.}$$

So, either the equation of sphere

$$S_1 = 0, \quad S_1 - S_2 = 0$$

or $\quad S_2 = 0, \quad S_1 - S_2 = 0$ represent a circle.

3.5.2 Equation of a Sphere Through Four Given Points

To find the equation of a sphere passing through four points

Let equation of sphere in general form be

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \qquad .....(1)$$

Since it passes through four given points (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) and (x_4, y_4, z_4),

$$x_1^2 + y_1^2 + z_1^2 + 2ux_1 + 2vy_1 + 2wz_1 + d = 0$$

$$x_2^2 + y_2^2 + z_2^2 + 2ux_2 + 2vy_2 + 2wz_2 + d = 0$$

$$x_3^2 + y_3^2 + 2ux_3 + 2vy_3 + 2wz_3 + d = 0 \text{ and}$$

$$x_4^2 + y_4^2 + z_4^2 + 2ux_4 + 2vy_4 + 2wz_4 + d = 0$$

Eliminating the four constants u, v, w and from the above equations (1), (2), (3), (4) and (5) with the help of determinants, we get

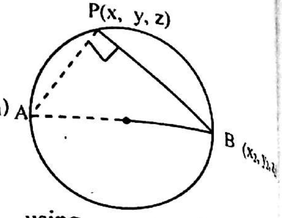
$$\begin{vmatrix} x^2 + y^2 + z^2 & x & y & z & 1 \\ x_1^2 + y_1^2 + z_1^2 & x_1 & y_1 & z_1 & 1 \\ x_2^2 + y_2^2 + z_2^2 & x_2 & y_2 & z_2 & 1 \\ x_3^2 + y_3^2 + z_3^2 & x_3 & y_3 & z_3 & 1 \\ x_4^2 + y_4^2 + z_4^2 & x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0$$

is the required equation.

3.5.3 Sphere With a Given Diameter

To find the equation of a sphere which has (x_1, y_1, z_1) and (x_2, y_2, z_2) as the extremities of a diameter

Let P(x, y, z) be any point on the sphere having $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ as ends of diameter, then AP and BP are at right angles.



Now the direction ratios of AP are

$$x - x_1, \quad y - y_1, \quad z - z_1$$

and those of BP are

$$x - x_2, \quad y - y_2, \quad z - z_2.$$

Since AP and BP are perpendicular, using condition of perpendicularity

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + (z - z_1)(z - z_2) = 0$$

This is the required equation.

Worked Out Examples

Ex.1. Find the equation of a sphere with centre (3, - 2, 1) and radius 3

Solution:

Here, the equation of sphere with centre (3, - 2, 1) and radius 3 is

$$(x - 3)^2 + [y - (-2)]^2 + (z - 1)^2 = (3)^2$$

or $$x^2 - 6x + 9 + y^2 + 4y + 4 + z^2 - 2z + 1 = 9$$

$\therefore$ $$x^2 + y^2 + z^2 - 6x + 4y - 2z + 5 = 0$$

Ex.2. Show that the equation of sphere passes through the points (- a, b, c) (a, - b, c) (a, b, - c) and origin is

$$(x^2 + y^2 + z^2) - (a^2 + b^2 + c^2)\left(\frac{x}{a} + \frac{y}{b} + \frac{z}{c}\right) = 0$$

Solution:

Here, the equation of sphere in general form is

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \quad .....(1)$$

Since it passes through the points (-a, b, c), (a, -b, c), (a, b, -c) and (0, 0, 0), then

$$a^2 + b^2 + c^2 - 2ua + 2vb + 2wc + d = 0 \quad ....(2)$$

$$a^2 + b^2 + c^2 + 2ua - 2vb + 2wc + d = 0 \quad ....(3)$$

$$a^2 + b^2 + c^2 + 2ua + 2vb - 2wc + d = 0 \quad ....(4)$$

$$d = 0 \quad ....(5)$$

Adding (2) & (3), (3) & (4), (4) & (1), we get

$$2(a^2 + b^2 + c^2) + 4wc = 0,$$

$$2(a^2 + b^2 + c^2) + 4ua = 0,$$

$$2(a^2 + b^2 + c^2) + 4vb = 0,$$

$$u = -\frac{(a^2 + b^2 + c^2)}{2a}, \; v = \frac{-(a^2 + b^2 + c^2)}{2b} \; w = -\frac{(a^2 + b^2 - c^2)}{2c}, \; d = 0$$

After substituting the values of u, v, w & d = 0 to equation (1), we get

$$x^2 + y^2 + z^2 - (a^2 + b^2 + c^2)\left(\frac{x}{a} + \frac{y}{b} + \frac{z}{c}\right) = 0.$$

Ex.3. Find the equation of a sphere which passes through the points (1, 0, 0) (0, 1, 0) (0, 0, 1) and has its radius as small as possible.

Solution:

Here, the general equation of a sphere is

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \quad ......(1)$$

Since it passes through the points (1, 0, 0), (0, 1, 0) & (0, 0, 1).

$$(1)^2 + (0)^2 + (0)^2 + 2u.1 + 2v.0 + 2w.0 + d = 0$$

or $$2u + d + 1 = 0$$

$\therefore$ $$u = \frac{-d - 1}{2} \quad ......(2)$$

Similarly $$(0)^2 + (1)^2 + (0)^2 + 2u.0 + 2v.1 + 2w.0 + d = 0$$

or $$2v + d + 1 = 0$$

$\therefore$ $$v = \frac{-d - 1}{2} \quad ......(3)$$

and $$(0)^2 + (0)^2 + (1)^2 + 2u.0 + 2v.0 + 2w.1 + d = 0$$

or $$2w + d + 1 = 0$$

$$\therefore \quad w = \frac{-d-1}{2} \qquad \text{.....(4)}$$

We have, the radius of sphere

$$R = \sqrt{u^2 + v^2 + w^2 - d}$$

Substituting the values of u ,v and w in this relation, we get

$$R = \sqrt{\left(\frac{-d-1}{2}\right)^2 + \left(\frac{-d-1}{2}\right)^2 + \left(\frac{-d-1}{2}\right)^2 - d}$$

or $\qquad R = \frac{1}{2}\sqrt{3d^2 + 2d + 3}$

Squaring, we get $\qquad R^2 = \frac{1}{4}(3d^2 + 2d + 3)$

Since R^2 is as small as possible, it is small if first derivative of R^2 be zero.

i. e. $\qquad \frac{1}{4}\frac{d}{dd}(3d^2 + 2d + 3) = 0$

or $\qquad 6d + 2 = 0$

$\therefore \qquad d = -\frac{1}{3}.$

Now $\qquad 2u = -d - 1 = \frac{1}{3} - 1 = -\frac{2}{3} \qquad \therefore \quad u = -\frac{1}{3},$

$$2v = -d - 1 = \frac{1}{3} - 1 = -\frac{2}{3} \qquad \therefore \quad v = -\frac{1}{3},$$

$$2w = -d - 1 = \frac{1}{3} - 1 = -\frac{2}{3} \qquad \therefore \quad w = -\frac{1}{3}.$$

Substituting the values of u, v, w and d in the equation (1), we get

$$3(x^2 + y^2 + z^2) - 2(x + y + z) - 1 = 0.$$

Ex. 4. A variable plane passes through a fixed point (a, b, c) cuts the coordinate axes in the points A, B, C. Show that the locus of the centre of the sphere OABC is $\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 2$

Solution:

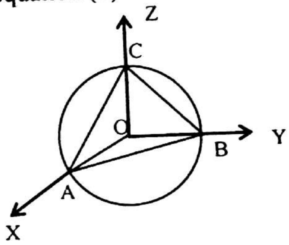
Here, the general equation of the sphere through the origin is

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz = 0 \qquad \text{.....(1)}$$

Since the plane cuts the axes OX, OY an OZ at the points A(a', 0, 0), B (0, b' 0) and C (0, 0, c') respectively. So the equation of a plane in intercept form is

$$\frac{x}{a'} + \frac{y}{b'} + \frac{z}{c'} = 1 \qquad \text{....(2)}$$

Also the equation of sphere (2) passes through the points A (a', 0, 0), (0, b' 0) and (0, 0, c'), these points must satisfy the equation (1).



We have

$a'^2 + 2ua' = 0 \qquad \therefore \quad a' = -2u$

$b'^2 + 2ub' = 0 \qquad \therefore \quad b' = -2v$ and

$c'^2 + 2wc' = 0 \qquad \therefore \quad c' = -2w$

Substituting the values of a' ,b'and c' in (2), we get

$$\frac{x}{-2u} + \frac{y}{-2v} + \frac{z}{-2w} = 1$$

Since it passes through the point (a, b, c)

$$\frac{a}{-2u} + \frac{b}{-2v} + \frac{c}{-2w} = 1$$

or $\qquad \frac{a}{-u} + \frac{b}{-v} + \frac{c}{-w} = 2.$

So the locus of centre (- u, - u, - w) of the sphere is

$$\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 2.$$

Ex. 5. A sphere of radius k passes through the origin and meets the axes in A, B, C. Prove that the centroid of the triangle ABC lies on the sphere $9(x^2 + y^2 + z^2) = 4k^2$ *2064 Paush B. E.*

Solution:

Here, the general equation of sphere through origin is

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz = 0 \qquad \text{........(1)}$$

Since the equation (1) meets the axes at A, B, C. So the coordinates of A, B and C are $(-2u, 0, 0)$, $(0, -2v, 0)$ and $(0, 0, -2w)$ respectively.

Let O'(f, g, h) be the centroid of the triangle ABC. Then

$$f = \frac{-2u+0+0}{3}, \quad g = \frac{0-2v+0}{3}, \quad h = \frac{0+0+-2w}{3}$$

$$\therefore \quad u = -\frac{3f}{2}, \qquad v = -\frac{3g}{2}, \qquad w = \frac{3h}{2}$$

The radius of the sphere $= u^2 + v^2 + w^2$ and given that

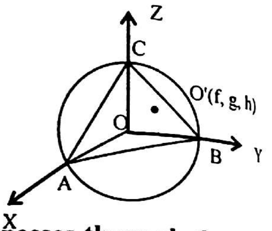
$$u^2 + v^2 + w^2 = k^2$$

or $$\frac{9f^2}{4} + \frac{9g^2}{4} + \frac{9h^2}{4} = k^2$$

$$9(f^2 + g^2 + h^2) = 4k^2.$$

So the locus of (f, g, h) is

$$9(x^2 + y^2 + z^2) = 4k^2.$$



Ex.6. Find the equation of sphere which passes through the points (1, - 3, 4), (1, - 5, 2) and (1, - 3, 0) and whose centre lies on the plane x + y + z = 0 [2062 Jestha B.E.]

Solution:

Here, the general equation of sphere is

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \qquad \text{......(1)}$$

Since it passes through the points (1,-3, 4), (1, - 5, 2) and (1,-3,0). Then

$$1 + 9 + 16 + 2u.1 + 2v(-3) + 2w.4 + d = 0$$

or $$26 + 2u - 6v + 8w + d = 0 \qquad \text{......(2)}$$

$$1 + 25 + 4 + 2u.1 + 2v(-5) + 2w.2 + d = 0$$

or $$30 + 2u - 10v + 4w + d = 0 \qquad \text{......(3)}$$

$$1 + 9 + 0 + 2u.1 + 2v(-3) + 2w.0 + d = 0$$

or $$10 + 2u - 6v + d = 0 \qquad \text{......(4)}$$

Since the centre of (1) lies on the plane $x + y + z = 0$. Therefore

$$u + v + w = 0 \qquad \text{......(5)}$$

On subtraction (2) and (3), we get

$$v + w = 1 \qquad \text{......(6)}$$

On subtraction (3) and (4), we get

$$v - w = 5 \qquad \text{.......(7)}$$

Solving (6) and (7), we get $v = 3$ and $w = -2$

So from (5), $u = -1$ and from (3), $d = 10$.

Substituting the values of u, v, w and d in (1), we get

$$x^2 + y^2 + z^2 - 2x + 6y - 4z + 10 = 0$$

Ex.7. Find the centre and the radius of the sphere $x^2 + y^2 + z^2 - 6x - 4y + 10z + 12 = 0$

Solution:

Here, the general equation of the sphere is

$$x^2 + y^2 + z^2 - 6x - 4y + 10z + 12 = 0 \qquad \text{......(1)}$$

Comparing it with general equation of the sphere

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \qquad \text{......(2)}$$

We find $u = -3$, $v = -2$, $w = 5$, $d = 12$

Hence centre is $(-u, -v, w) = (3, 2, -5)$ and

$$\text{Radius} = \sqrt{u^2 + v^2 + w^2 - d} = \sqrt{9 + 4 + 25 - 12} = \sqrt{26}.$$

Ex. 8. A sphere of constant radius k passes through the origin O and meets the axes in A, B, C. Prove that the locus of the foot of the perpendicular from O to the plane ABC is given by $$(x^2 + y^2 + z^2)^2 \left(\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2}\right) = 4k^2$$

Solution:

Here, the equation of the plane ABC is

$$\frac{x}{\alpha} + \frac{y}{\beta} + \frac{z}{\gamma} = 1 \qquad \text{.....(1)}$$

Then coordinates of A, B C are $(\alpha, 0, 0)$, $(0, \beta, 0)$ and $(0, 0, \gamma)$ respectively.

The equation of sphere passing through OABC is

$$x^2 + y^2 + z^2 - \alpha x - \beta y - \gamma z = 0 \quad \text{....(2)}$$

Radius $k = \sqrt{\frac{\alpha^2}{4} + \frac{\beta^2}{4} + \frac{\gamma^2}{4}}$ and centre $= \left(\frac{\alpha}{2}, \frac{\beta}{2}, \frac{\gamma}{2}\right)$

or $\alpha^2 + \beta^2 + \gamma^2 = 4k^2$ .....(3)

The direction ratios of the normal to the plane (1) are

$$\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}.$$

Therefore the equation of the line through 0(0, 0, 0) and perpendicular to the plane (1) is

$$\frac{x}{1/\alpha} = \frac{y}{1/\beta} = \frac{z}{1/\gamma} = r(\text{say}) \quad \text{.....(4)}$$

Any point of the line (4) is $\left(\frac{r}{\alpha}, \frac{r}{\beta}, \frac{r}{\gamma}\right)$. This point lies on the plane (1) if

$$\frac{r}{\alpha^2} + \frac{r}{\beta^2} + \frac{r}{\gamma^2} = 1$$

or $$r = \frac{1}{\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2}} = \frac{1}{\alpha^{-2} + \beta^{-2} + \gamma^{-2}} \quad \text{.......(v)}$$

Let (x', y', z') be the foot of the perpendicular drawn from O to the plane (1).Then

$$x' = \frac{r}{\alpha}, \; y' = \frac{r}{\beta}, \; z' = \frac{r}{\gamma}$$

Squaring and adding, we get

$$x'^2 + y'^2 + z'^2 = \frac{r^2}{\alpha^2} + \frac{r^2}{\beta^2} + \frac{r^2}{\gamma^2} = r^2(\alpha^{-2} + \beta^{-2} + \gamma^{-2})$$

and $$\frac{1}{x'^2} + \frac{1}{y'^2} + \frac{1}{z'^2} = \frac{\alpha^2}{r^2} + \frac{\beta^2}{r^2} + \frac{\gamma^2}{r^2} = \frac{1}{r^2}(\alpha^2 + \beta^2 + \gamma^2)$$

Now $(x'^2 + y'^2 + z'^2)^2 \left(\frac{1}{x'^2} + \frac{1}{y'^2} + \frac{1}{z'^2}\right)$

$$= r^4 (\alpha^{-2} + \beta^{-2} + \gamma^{-2})^2 \frac{1}{r^2} (\alpha^2 + \beta^2 + \gamma^2)$$

$$= r^2 (\alpha^{-2} + \beta^{-2} + \gamma^{-2})^2 (\alpha^2 + \beta^2 + \gamma^2)$$

$$= \frac{1}{(\alpha^{-2} + \beta^{-2} + \gamma^{-2})^2} (\alpha^{-2} + \beta^{-2} + \gamma^{-2})^2 \, 4k^2 = 4k^2$$

So that the locus of the point (x', y', z') is

$$(x^2 + y^2 + z^2)^2 \left(\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2}\right) = 4k^2$$

Ex. 9. Show that the equation of sphere having its centre on the planes $5y + 2z = 0$ and $2x - 3y = 0$ and passing through the points (2, - 1, - 1), (0, - 2, - 4) is $x^2 + y^2 + z^2 - 6x - 4y + 10z + 12 = 0$

Solution:

Here, the general equation of a sphere is

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \quad \text{....(1)}$$

Its centre is (- u, - v - w).Also given that the centre (- u, - v, - w) of the sphere lies on the plane $5y + 2z = 0$ and $2x - 3y = 0$.

$\therefore$ $5v + 2w = 0$

or $2w = - 5v$ ....(2)

and $2u - 3v = 0$

or $2u = 3v$ ...(3)

The sphere (1) passes through the points (0,-2,-4) and(2,-1,-1). So

$$0^2 + (-2)^2 + (-4)^2 + 0.u - 4v - 8w + d = 0$$

or $-4v - 8w + d = -20$ ....(4)

and $(2)^2 + (-1)^2 + (-1)^2 + 4u - 2v - 2w + d = 0$

or $4u - 2v - 2w + d = -6$ ....(5)

Solving (2), (3), (4) and (5), we get

$$u = -3, \; v = -2, \; w = 5, \; d = 12$$

Putting these values in (1), we get

$$x^2 + y^2 + z^2 + 2(-3)x + 2.(-2)y + 2(5)z + 12 = 0$$

or $x^2 + y^2 + z^2 - 6x - 4y + 10z + 12 = 0.$

Ex.10. Find the equation of sphere described on the join of (3, - 5, 2) (0, 1, 0) as diameter

Solution:

Here, the equation of a sphere described on the join of (3, - 5, 2) and (0, 1, 0) as diameter is

$$(x - 3)(x - 0) + (y + 5)(y - 1) + (z - 2)(z - 0) = 0$$

or $\quad x^2 - 3x + y^2 + 5y - y - 5 + z^2 - 2z = 0$

$\therefore \quad x^2 + y^2 + z^2 - 3x + 4y - 2z - 5 = 0$

Ex.11. Find the equation of the sphere through the circle $x^2 + y^2 + z^2 - 3x + 4y - 2z - 5 = 0, 5x - 2y + 4z + 7 = 0$ as a great circle [2068 Bhadra B. E.]

Solution:

Here, the equation of a sphere through the given circle is

$$x^2 + y^2 + z^2 - 3x + 4y - 2z - 5 + \lambda (5x - 2y + 4z + 7) = 0$$

or $\quad x^2 + y^2 + z^2 + (5\lambda - 3)x + (4 - 2\lambda)y + (4\lambda - 2)z + 7\lambda - 5 = 0 \quad ..(1)$

Comparing it with $\quad x^2 + y^2 + z^2 + 2u x + 2vy + 2w z + d = 0.$

We find $\quad u = \frac{5\lambda - 3}{2}, \; v = \frac{4 - 2\lambda}{2}, \; w = \frac{4\lambda - 2}{2}$ and $d = 7\lambda - 5$

$$\text{Centre} = \left(-\frac{5\lambda - 3}{2}, -\frac{4 - 2\lambda}{2}, -\frac{4\lambda - 2}{2}\right).$$

Since the circle is great circle, so its centre must lie on the plane

$5x - 2y + 4z + 7 = 0$ if

$$5\left(-\frac{5\lambda - 3}{2}\right) - 2\left(-\frac{4 - 2\lambda}{2}\right) + 4\left(-\frac{4\lambda - 2}{2}\right) + 7 = 0$$

or $\quad -25\lambda + 15 + 8 - 4\lambda - 16\lambda + 8 + 14 = 0$

or $\quad -45\lambda = -45$

$\therefore \quad \lambda = 1$

Substituting the value of λ in the equation (1), we get

$$x^2 + y^2 + z^2 + 2x + 2y + 2z + 2 = 0.$$

Ex. 12. Obtain the equation of the sphere through the circle $x^2 + y^2 + z^2 = 9, x - 2y + 2z = 5$ as a great circle also determine its centre and radius [2069 Bhadra B. E.]

Solution:

Here, the equation of a sphere through the given circle is

$$x^2 + y^2 + z^2 - 9 + \lambda(x - 2y + 2z - 5) = 0$$

or $\quad x^2 + y^2 + z^2 + \lambda x - 2\lambda y + 2\lambda z - 5\lambda - 9 = 0 \quad ...(1)$

Comparing it with $\quad x^2 + y^2 + z^2 + 2u x + 2vy + 2w z + d = 0.$

We find $\quad u = \frac{\lambda}{2}, \; v = -\lambda, \; w = 2\lambda$ and $d = -5\lambda - 9$

Center $= \left(-\frac{\lambda}{2}, \lambda, -\lambda\right)$ Since the circle is great circle of this sphere. So its centre lie on the plane $\quad x - 2y + 2z = 5$ if

$$-\frac{\lambda}{2} - 2\lambda + 2.(-\lambda) = 5$$

or $\quad \lambda = -\frac{10}{9}$

Substituting the value of λ in (1), we get

$$9(x^2 + y^2 + z^2) - 10x + 20y - 20z - 31 = 0.$$

Therefore the centre of sphere is $\left(\frac{5}{9}, -\frac{10}{9}, \frac{10}{9}\right)$ and

Radius $= \sqrt{\left(\frac{5}{9}\right)^2 + \left(-\frac{10}{9}\right)^2 + \left(\frac{10}{9}\right)^2 + \frac{31}{9}} = \frac{2}{3}\sqrt{14}.$

Ex. 13. Find the equation of the sphere through the circle $x^2 + y^2 = 4, z = 0$ and is intersected by the plane $x + 2y + 2z = 0$ in a circle of radius 3 [2070 Bhadra B. E.]

Solution:

Here, the equation of a sphere through the given circle is

$$x^2 + y^2 + z^2 - 4 + \lambda z = 0 \quad ......(1)$$

and its center is $(-u, -v, -w) = \left(0, 0, \frac{-\lambda}{2}\right)$, Radius $= \sqrt{\frac{\lambda^2}{4} + 4}$

Also the length of perpendicular from the centre $\left(0, 0, -\frac{\lambda}{2}\right)$ of the sphere to the plane $x + 2y + 2z = 0$ is

$$p = \left|\frac{1.0 + 2.0 + 2.(-\lambda/2)}{\sqrt{1 + 4 + 4}}\right| = \frac{\lambda}{3}, \text{ radius of circle } (r) = 3.$$

We have $\quad r^2 = R^2 - p^2$

or $\quad 9 = \frac{\lambda^2}{4} + 4 - \frac{\lambda^2}{9}$

or $\quad 5 = \frac{5\lambda^2}{36}$

$\therefore \quad \lambda = \pm 6$

Substituting the value of λ in (1), we get

$x^2 + y^2 + z^2 - 4 \pm 6z = 0.$

Ex. 14. Prove that the circles

$x^2 + y^2 + z^2 - 2x + 3y + 4z - 5 = 0,\ 5y + 6z + 1 = 0$

and $x^2 + y^2 + z^2 - 3x - 4y + 5z - 6 = 0,\ x + 2y - 7z = 0$ lie on the same sphere and find its equation $\boxed{\text{2060 Bhadra B.E.}}$

Solution:

Here, the equation of a sphere through the circle

$x^2 + y^2 + z^2 - 2x + 3y + 4z - 5 = 0,\ 5y + 6z + 1 = 0$ is

$x^2 + y^2 + z^2 - 2x + 3y + 4z - 5 + \lambda\,(5y + 6z + 1) = 0$

or $\quad x^2 + y^2 + z^2 - 2x + (5\lambda + 3)y + (4 + 6\lambda)z + \lambda - 5 = 0$ ....(1)

and the equation of a sphere through the circle

$x^2 + y^2 + z^2 - 3x - 4y + 5z - 6 = 0,\ x + 2y - 7z = 0$ is

$x^2 + y^2 + z^2 - 3x - 4y + 5z - 6 + \lambda'\,(x + 2y - 7z) = 0$

or $\quad x^2 + y^2 + z^2 + (\lambda' - 3)x + (2\lambda' - 4)y + (5 - 7\lambda')z - 6 = 0$ ..(2)

If the equations of two given circles lie on same sphere, then for some values of λ and λ' the equations (1) and (ii) are identical. Thus

$$\left.\begin{array}{ll} -2 = \lambda' - 3, & 5\lambda + 3 = 2\lambda' - 4 \\ 4 + 6\lambda = 5 - 7\lambda' \quad \text{and} & \lambda - 5 = -6 \end{array}\right\} \quad \text{....(3)}$$

Solving two of these equations, we get $\lambda' = 1,\ \lambda = -1$.

Clearly these value of λ and λ' satisfy the remaining two equations of (3). So they are consistent. It shows that two circles lie on same sphere.

Substituting the value of either λ in (1) or λ' in (2), we get

$x^2 + y^2 + z^2 - 2x - 2y - 2z - 6 = 0$ is the required equation of the sphere in which the given circles lie.

Ex. 15. A variable plane is parallel to the given plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 0$ and meets the axes in A, B, C. Prove that the circle ABC lies on the cone $yz\left(\frac{b}{c} + \frac{c}{b}\right) + zx\left(\frac{c}{a} + \frac{a}{c}\right) + xy\left(\frac{a}{b} + \frac{b}{a}\right) = 0$ $\boxed{\text{2069 Poush, B. E.}}$

Solution:

Here, the equation of any plane parallel to the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 0$ is

$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = \lambda$

or $\quad \frac{x}{a\lambda} + \frac{y}{b\lambda} + \frac{z}{c\lambda} = 1$ ....(1)

This plane meets the axes at $A(a\lambda, 0, 0)$, $B(0; b\lambda, 0)$ and $C(0, 0, c\lambda)$ respectively.

The equation of any sphere through origin is

$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz = 0$ ....(2)

Since it meets the axes at A, B and C.

$\therefore \quad a^2\lambda^2 + 2ua\lambda = 0$

or $\quad u = \frac{-\lambda}{2}$

$b^2\lambda^2 + 2vb\lambda = 0$

or $\quad v = \frac{-b\lambda}{2}$

and $\quad c^2\lambda^2 + 2wc\lambda = 0$

or $\quad w = \frac{-c\lambda}{2}$

So the equation of sphere (2) gives

$x^2 + y^2 + z^2 - a\lambda x - b\lambda y - c\lambda z = 0$

or $\quad x^2 + y^2 + z^2 - \lambda\,(ax + by + cz) = 0$ ....(3)

Eliminating λ between (1) and (3), we get

$x^2 + y^2 + z^2 - (ax + by + cz)\left(\frac{x}{a} + \frac{y}{b} + \frac{z}{c}\right) = 0.$

or $\quad yz\left(\frac{b}{c} + \frac{c}{b}\right) + zx\left(\frac{c}{a} + \frac{a}{c}\right) + xy\left(\frac{a}{b} + \frac{b}{a}\right) = 0$

Ex. 16. Obtain the centre and radius of the circle

$x^2 + y^2 + z^2 + x + y + z = 4, \quad x + y + z = 0$ $\boxed{\text{2068 Chaitra B. E.}}$

Solution:

Here, the equation of the sphere is

$x^2 + y^2 + z^2 + x + y + z = 4$ ....(1)

Comparing it with $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$

We find $u = \frac{1}{2},\ v = \frac{1}{2},\ w = \frac{1}{2}$ and $d = -4$

Center of the sphere $(-u, -v, -w) = \left(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right)$

and radius of the sphere $R = \sqrt{\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + 4} = \sqrt{\frac{19}{4}}$

The equation of the plane is

$$x + y + z = 0 \qquad \ldots(2)$$

The equation of a line through centre $\left(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right)$ of the sphere and perpendicular to the plane (2) is

$$\frac{x + 1/2}{1} = \frac{y + 1/2}{1} = \frac{z + 1/2}{1} = r \text{ (say)}$$

Any point of this line is $\left(r - \frac{1}{2}, r - \frac{1}{2}, r - \frac{1}{2}\right)$. If this point lies on the plane (2), then it is the centre of the circle. Thus

$$r - \frac{1}{2} + r - \frac{1}{2} + r - \frac{1}{2} = 0$$

or $\quad 3r - \frac{3}{2} = 0$

$\therefore \quad r = \frac{1}{2}$

So the centre of the circle $\left(r - \frac{1}{2}, r - \frac{1}{2}, r - \frac{1}{2}\right) = (0, 0, 0)$

Length of the perpendicular from the centre $\left(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right)$ of the sphere to the plane $x + y + z = 0$ is

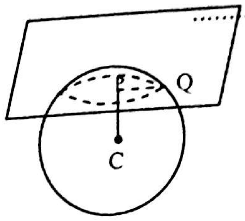
$$p = \left|\frac{1.\left(-\frac{1}{2}\right) + 1.\left(-\frac{1}{2}\right) + 1.\left(-\frac{1}{2}\right)}{\sqrt{1+1+1}}\right| = \left|\frac{3}{2\sqrt{3}}\right| = \frac{\sqrt{3}}{2}$$

$\therefore \quad p = \frac{\sqrt{3}}{2}$

Radius of the circle $r = \sqrt{R^2 - p^2} = \sqrt{\frac{19}{4} - \frac{3}{4}} = \sqrt{\frac{16}{4}} = 2.$

Ex. 17. Show that the equation of circle lying on the sphere $x^2 + y^2 + z^2 - 49 = 0$ having its centre at (2, - 1, 3) is $x^2 + y^2 + z^2 - 49 = 0, 2x - y + 3z - 14 = 0$

Solution:



Here the equation of the sphere is

$$x^2 + y^2 + z^2 - 49 = 0$$

Its centre is $(-u, -v, -w) = C(0, 0, 0)$.

The centre of the circle is $P(2, -1, 3)$.

The direction ratios of the line CP are

$$2 - 0, -1 - 0, 3 - 0$$

or $\quad 2, -1, 3.$

The equation of any plane through the point $P(2, -1, 3)$ is

$$a(x - 2) + b(y + 1) + c(z - 3) = 0 \qquad \ldots(2)$$

It will become plane of the circle if it is perpendicular to the line CP.

So $\quad a = 2, \quad b = -1, \quad c = 3.$

$\therefore$ the equation of the plane (2) becomes

$$2(x - 2) - (y + 1) + 3(z - 3) = 0$$

or $\quad 2x - 4 - y - 1 + 3z - 9 = 0$

$\therefore \quad 2x - y + 3z - 14 = 0$

Hence the equation of required circle is

$$x^2 + y^2 + z^2 - 49 = 0, \quad 2x - y + 3z - 14 = 0$$

Exercise - 12

1. Find the equation of sphere through the points (0, 0, 0), (0, 1, - 1), (- 1, 2, 0) and (1, 2, 3). Also, find its centre and the radius
2. Find the equation of sphere passing through the points (3, 0, 2), (-1, 1, 1), (2, -5, 4) and having the centre on the plane $2x+3y+4z = 6$
3. A sphere of constant radius 2k passes through the origin and meets the axes in A, B, C. Show that the locus of the centroid of the tetrahedron OABC is the sphere $x^2 + y^2 + z^2 = k^2$
4. Find the equation of the sphere described on the join of (0, 1, 0), (3, - 5, 2) as diameter
5. Find the equation of sphere through the circle $x^2 + y^2 + z^2 - 9 = 0$, $x + 3y + 4z - 2 = 0$ and the point (0, 0, 0)
6. Find the centre and radius of the circles

(i) $x^2 + y^2 + z^2 - 8x + 4y + 8z - 45 = 0, \quad x - 2y + 2z - 3 = 0$ **2067/070/072 Magh, B. E.**

(ii) $x^2 + y^2 + z^2 + 12x - 12y - 16z + 111 = 0, \quad 2x + 2y + z - 17 = 0$

7. Find the equation of the sphere having the circle
   (i) $x^2 + y^2 + z^2 + 10y - 4z - 8 = 0, \quad x + y + z = 3$ as a great circle
   (ii) $x^2 + y^2 + z^2 + 7y - 2z + 2 = 0, \quad 2x + 3y + 4z - 8 = 0$ as a great circle
8. Find the equation of sphere, which passes through the point $(0, -2, 3)$ and the circle $x^2 + y^2 = 4, \quad z = 0$
9. Find the equation of the sphere through the circle $x^2 + y^2 + z^2 - 2x + 3y + 4z - 5 = 0$, $x^2 + y^2 + z^2 - 3x - 4y + 5z - 6 = 0$ and passing through the point $(1, 1, 2)$
10. Find the equation of the sphere through the circle $x^2 + y^2 + z^2 - 2x - 3y + 4z + 8 = 0$, $x^2 + y^2 + z^2 + 4x + 5y - 6z + 2 = 0$ and having its centre on the plane $4x - 5y - z = 3$

Answers

1. $7(x^2+y^2+z^2) - 15x - 25y - 11z = 0; \left(\frac{15}{14}, \frac{25}{14}, \frac{11}{14}\right); \frac{\sqrt{971}}{14}$
2. $x^2+y^2+z^2+4y-6z=1$
4. $x^2+y^2+z^2-3x+4y-2z=0$
5. $2(x^2+y^2+z^2)-9x-27y-36z=0$
6. (i) $\left(\frac{13}{3}, -\frac{8}{3}, -\frac{10}{3}\right); 4\sqrt{5}$ (ii) $(-4, 8, 9); 4$
7. (i) $x^2+y^2+z^2-4x+6y-8z+4=0$
   (ii) $x^2+y^2+z^2-2x+4y-6z+10=0$
8. $x^2+y^2+z^2-3z-4=0$
9. $x^2+y^2+z^2-\frac{24x}{7}-7y+\frac{38z}{7}-\frac{45}{7}=0$
10. $x^2+y^2+z^2+7x+9y-11z-1=0$

3.6 .Tangent Planes and Lines

The locus of all the tangent lines at point P of the sphere is called the *tangent plane* to the sphere at P.

To find the equation of tangent plane at $P(x_1, y_1, z_1)$ of the sphere $x^2 + y^2 + z^2 = a^2$

The equation of the sphere is

$$x^2+y^2+z^2=a^2 \qquad \ldots(1)$$

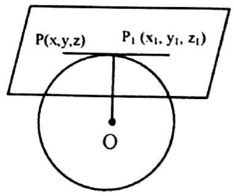
Centre of the sphere is O (0, 0, 0). Let P(x, y, z) be any point on the plane in which the plane is tangent at $P_1(x_1, y_1, z_1)$ to the sphere (1), then the direction ratios of P_1P are

$$x - x_1, \; y - y_1, \; z - z_1$$

The direction ratios of radius OP_1 are

$$x_1 - 0, \; y_1 - 0, \; z_1 - 0.$$

Since OP_1 is normal to the tangent plane at P_1 i.e. $OP_1 \perp P_1P$.



$\therefore \quad x_1(x - x_1) + y_1(y - y_1) + z_1(z - z_1) = 0$

or $\quad xx_1 + yy_1 + zz_1 = x_1^2 + y_1^2 + z_1^2.$

Since the point $P_1(x_1, y_1, z_1)$ lies on the sphere(1)

$\therefore \quad x_1^2 + y_1^2 + z_1^2 = a^2$

Hence $\quad xx_1 + yy_1 + zz_1 = a^2.$

This is the required equation of tangent plane.

Cor.:

Similarly, from the above method the equation of tangent plane at (x_1, y_1, z_1) to the sphere

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \text{ is}$$
$$xx_1+yy_1+zz_1+u(x+x_1)+v(y+y_1)+w(z+z_1)+d=0.$$

To find the condition that the line $\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$ touches the sphere $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ at the point $P(x_1, y_1, z_1)$.

Here the equation of the sphere is

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \qquad \ldots(1)$$

and the equation of the line is

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}=r\ \text{(say)} \qquad \ldots(2)$$

Any point of this line is

$$(x_1+lr,\ y_1+mr,\ z_1+nr)$$

If this point lies on the sphere (1), then we have

$$(x_1+lr)^2+(y_1+mr)^2+(z_1+nr)^2+2u(x_1+lr)$$
$$+2v(y_1+mr)+2w(z_1+nr)+d=0$$

or
$$r^2(l^2+m^2+n^2)+2r[l(u+x_1)+m(v+y_1)+n(w+z_1)]$$
$$+x_1^2+y_1^2+z_1^2+2ux_1+2vy_1+2wz_1+d=0 \qquad \ldots(3)$$

This is quadratic equation in r.

Since (x_1, y_1, z_1) lies on the sphere (1).

$$x_1^2+y_1^2+z_1^2+2ux_1+2vy_1+2wz_1+d=0 \qquad \ldots(4)$$

$\therefore$ (3) reduces to

$$r^2(l^2+m^2+n^2)+2r[l(u+x_1)+m(v+y_1)+n(w+z_1)]=0$$

Clearly the one values of r is zero. The line (2) to be tangent, the two values of r should be zero. The other value of r is zero if

$$l(u+x_1)+m(v+y_1)+n(w+z_1)=0 \qquad \ldots(5)$$

This is the required condition for the line (2) to be tangent to the sphere (1).

Eliminating l, m ,n from (2) and(5) ,we get

$$xx_1+yy_1+zz_1+u(x+x_1)+v(y+y_1)+w(z+z_1)+d=0$$

is the equation of the tangent plane.

Cor.2:

The condition for a plane (or a line) touches a sphere is that the perpendicular distance of centre from the plane (or the line) = the radius.

Worked Out Examples

Ex.1. Find the equation of the tangent plane to the sphere $3(x^2+y^2+z^2)-2x-3y-4z-22=0$ at the point (1, 2, 3)

Solution:

Here, the equation of the sphere is

$$x^2+y^2+z^2-\frac{2x}{3}-y-\frac{4z}{3}-\frac{22}{3}=0$$

Comparing it with $x^2+y^2+z^2+2ux+2vy+2wz+d=0$

We find $u=-\frac{1}{3},\ v=-\frac{1}{2},\ w=-\frac{2}{3}$ and $d=-\frac{22}{3}$

The equation of tangent plane at (1, 2, 3) to the sphere is

$$xx_1+yy_1+zz_1+u(x+x_1)+v(y+y_1)+w(z+z_1)+d=0$$
$$x.1+y.2+z.3+\left(-\frac{1}{3}\right)(x+1)+\left(-\frac{1}{2}\right)(y+2)+\left(-\frac{2}{3}\right)(z+3)-\frac{22}{3}=0$$

or $\quad 4x+9y+14z-64=0.$

Ex.2. Find the equation of the tangent planes to the sphere $x^2+y^2+z^2-4x+2y-6z+5=0$ which are parallel to the plane $2x+y-z=0$

Solution:

Here the equation of the sphere is

$$x^2+y^2+z^2-4x+2y-6z+5=0 \qquad \ldots(1)$$

Comparing it with $x^2+y^2+z^2+2ux+2vy+2wz+d=0$

We find $\quad u=-2,\ v=1,\ w=-3$ and $d=5$.

The equation of a plane parallel to the given plane $2x+y-z=0$ is

$$2x+y-z+\lambda=0 \qquad \ldots\ldots\ldots(2)$$

where λ is any constant.

The plane (2) will touch the sphere (1) if the length of perpendicular from the centre (2, - 1, 3) of the sphere (1) to the plane (2) is equal to the radius of the sphere.

Radius of the sphere $R=\sqrt{u^2+v^2+w^2-d}=\sqrt{4+1+9-5}=3$

Length of perpendicular from the centre (2, - 1, 3) to the plane (2) is

$$\pm\frac{2.2+1.(-1)-1.3+\lambda}{\sqrt{4+1+1}}$$

Thus $\quad \pm\frac{\lambda}{\sqrt{6}}=3$ or $\lambda=\pm 3\sqrt{6}$.

$\therefore$ the required tangent planes are $2x+y-z\pm 3\sqrt{6}$

Ex. 3. Prove that the plane $2x-2y+z+12=0$ touches the sphere $x^2+y^2+z^2-2x-4y+2z=3$ and find the point of contact

Solution:

Here the equation of a plane is

$$2x-2y+z+12=0 \qquad \ldots(1)$$

and the equation of the sphere is

$$x^2+y^2+z^2-2x-4y+2z-3=0$$

Comparing it with $x^2+y^2+z^2+2ux+2vy+2wz+d=0$ ....(2)

We find $u=-1,\ v=-2,\ w=1$ and $d=-3$

Center of the sphere is $(-u,-v,-w)=(1,2,-1)$

Radius of the sphere is $R=\sqrt{u^2+v^2+w^2-d}=\sqrt{1+4+1+3}=3.$

Also length of perpendicular from the centre $(1,2,-1)$ of the sphere to the plane is

$$p=\frac{2.1-2.2+1.(-1)+12}{\sqrt{4+4+1}}=\frac{9}{3}=3$$ is equal to the radius of the sphere.

Hence the plane touches the sphere.

The equation of a line through the centre $(1,2,-1)$ of the sphere (2)and perpendicular to the plane (1) is

$$\frac{x-1}{2}=\frac{y-2}{-2}=\frac{z+1}{1}=r(\text{say})$$

Any point of this line is $(2r+1,-2r+2,r-1)$. This point will be tangential to the sphere if this point satisfies to the given plane (1). So

$$2(2r+1)-2(-2r+2)+r-1+12=0$$

or $\quad 4r+2+4r-4+r-1+12=0$

or $\quad 9r=-9$

$\therefore \quad r=-1$

Thus the point of contact is $(-1,4,-2)$.

Ex. 4. Find the equations of the tangent planes to the sphere $x^2+y^2+z^2+6x-2z+1=0$ which passes through the line $x+z-16=0,\ 2y-3z+30=0$ [2070/072 Ashwin, B.E.]

Solution:

Here the equation of the sphere is

$$x^2+y^2+z^2+6x-2z+1=0 \quad ...(1)$$

Comparing it with $x^2+y^2+z^2+2ux+2vy+2wz+d=0$

We find $u=3,\ v=0,\ w=-1$ and $d=1$

Centre of the sphere is $(-u,-v,-w)=(-3,0,1)$

Radius of the sphere is $R=\sqrt{u^2+v^2+w^2}=\sqrt{9+0+1-1}=3.$

The equation of the tangent planes passes through the line

$x+z-16=0,\ 2y-3z+30=0$ is

$$x+z-16+\lambda(2y-3z+30)=0$$

or $\quad x+2\lambda y+(1-3\lambda)z+30\lambda-16=0 \quad ....(2)$

Length of perpendicular from the centre $(-3,0,1)$ of the sphere (1)to the plane (2) is

$$=\pm\frac{1.(-3)+2\lambda.0+(1-3\lambda).1+30\lambda-16}{\sqrt{1+4\lambda^2+1-6\lambda+9\lambda^2}}$$

$$=\pm\frac{27\lambda-18}{\sqrt{13\lambda^2-6\lambda+2}}$$

The plane (2) will be tangent plane to the sphere (1) if length of perpendicular from the centre of the sphere to the plane (2) is equal to the radius of the sphere.

Thus we have $\quad \pm\dfrac{27\lambda-18}{\sqrt{13\lambda^2-6\lambda+2}}=3$

or $\quad (9\lambda-6)^2=13\lambda^2-6\lambda+2$

or $\quad 81\lambda^2-108\lambda+36=13\lambda^2-6\lambda+2$

or $\quad 68\lambda^2-102\lambda+34=0$

or $\quad 2\lambda^2-3\lambda+1=0$

or $\quad (2\lambda-1)(\lambda-1)=0$

$\therefore \quad \lambda=1,\ \frac{1}{2}$

Therefore the required planes are $2x+2y-z-2=0,\ x+2y-2z+14=0$

Ex. 5. Find the equation of the sphere which passes through the circle $x^2+y^2+z^2-5=0,\ x+2y+3z-3=0$ and touch the plane $4x+3y-15=0$

Solution:

The equation of the sphere through the circle

$x^2+y^2+z^2-5=0,\ x+2y+3z-3=0$ is

$$x^2+y^2+z^2-5+\lambda(x+2y+3z-3)=0 \quad ......(1)$$

Comparing it with $x^2+y^2+z^2+2ux+2vy+2wz+d=0$

We find $\quad u=\frac{\lambda}{2},\ v=\lambda,\ w=\frac{3\lambda}{2}$ and $d=-5-3\lambda$

Center of the sphere is $(-u,-v,-w)=\left(-\frac{\lambda}{2},-\lambda,-\frac{3\lambda}{2}\right)$

Radius of the sphere is $R = \sqrt{u^2+v^2+w^2-d} = \sqrt{\frac{\lambda^2}{4}+\lambda^2+\frac{9\lambda^2}{4}+3\lambda+5}$

The equation of the plane is $4x+3y-15=0$ ....(2)

Length of the perpendicular form the centre $\left(-\frac{\lambda}{2}, -\lambda, -\frac{3\lambda}{2}\right)$ of the sphere (1) to the tangent plane (2) is

$$= \pm \frac{4(-\lambda/2)+3(-\lambda)-15}{\sqrt{16+9}} = \pm \frac{-2\lambda-3\lambda-15}{\sqrt{25}}$$

$$= \pm \frac{-5\lambda-15}{5} = \pm(-\lambda-3)$$

The plane (2) will touch the sphere (1) if the length of the perpendicular from the centre of the sphere to the plane is equal to the radius of the sphere.

Thus we have, $\pm(-\lambda-3) = \sqrt{\frac{7}{2}\lambda^2+3\lambda+5}$

or $\lambda^2+6\lambda+9 = \frac{7}{2}\lambda^2+3\lambda+5$

or $5\lambda^2-6\lambda-8=0$

$\therefore$ $\lambda = 2, -\frac{4}{5}$

Therefore the two required spheres are

$x^2+y^2+z^2+2x+4y+6z-11=0$; $5(x^2+y^2+z^2)-4x-8y-12z-13=0$

Ex. 6. Show that the centre of the sphere which touch the line $y = mx$, $z = c$; $y = -mx$, $z = -c$ lie upon the conicoid $mxy + cz(1+m^2) = 0$

Solution:

Let the equation of sphere be $x^2+y^2+z^2+2ux+2vy+2wz+d=0$ .....(1)

This sphere touch the line $y = mx$, $z = c$ .....(2)

Solving (1) and (2), we get

$$x^2+m^2x^2+c^2+2ux+2vmx+2wc+d=0$$

or $x^2(1+m^2)+2(u+vm)x+(c^2+2wc+d)=0$ .....(3)

The line (2) will touch the sphere(1) if both values of x given by (3) are coincident.

$$(u+vm)^2 = (1+m^2)(c^2+2wc+d) \quad .....(4)$$

Also the sphere (1) touches the line $y=-mx$, $z=-c$ .....(5)

Solving (1) and (5), we get

$$x^2+m^2x^2+c^2+2ux-2vmx-2wc+d=0$$

$$(1+m^2)x^2+2(u-vm)x+c^2-2wc+d=0 \quad ...(6)$$

The line (5) will touch the sphere(1) if both values of x given by (6) are coincident.

$$(u-vm)^2 = (1+m^2)(c^2-2wc+d) \quad ....(7)$$

Subtracting (7) from (4), we get

$$4uvm = (1+m^2)\,4wc$$

or $(-u)(-v)\,m = -(1+m^2)(-w)\,c$ ....(8)

So the locus of the centre $(-u, -v, -w)$ of the sphere is

$$mxy = -(1+m^2)zc$$

$\therefore$ $mxy+(1+m^2)cz=0$

Ex. 7 Find condition for the spheres $x^2+y^2+z^2+2ux+2vy+2wz+d=0$ and $x^2+y^2+z^2+2u_1x+2v_1y+2w_1z+d_1=0$ to intersect orthogonal.

Solution:

Let A and B be the centers of the two spheres

$x^2+y^2+z^2+2ux+2vy+2wz+d=0$. ....(1)

and $x^2+y^2+z^2+2u_1x+2v_1y+2w_1z+d_1=0$ ....(2)

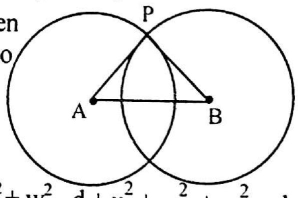
The coordinates of A and B are $A(-u, -v, -w)$ and $B(-u_1, -v_1, -w_1)$ respectively.

$AB^2 = (u-u_1)^2+(v-v_1)^2+(w-w_1)^2$

The radii of the spheres (1) and (2) are

$PA^2 = u^2+v^2+w^2-d$ and $PB^2 = u_1^2+v_1^2+w_1^2-d_1$ respectively.

Let P be any point on the sphere of intersection ,then the two spheres (1) and (2) intersect orthogonally so that the radii AP and PB are at right angles.



Therefore

$$AB^2 = PA^2+PB^2$$

or $(u-u_1)^2+(u-u_1)^2+(w-w_1)^2 = u^2+v^2+w^2-d+u_1^2+v_1^2+w_1^2-d$

or $2uu_1+2vu_1+2ww_1 = d+d_1$.

This is the required condition for the two spheres to intersect orthogonally.

Ex. 8 Show that the equation of sphere which touches the sphere $x^2+y^2+z^2+2x-6y+1=0$ at the point (1, 2, - 2) and passes through the point (1, - 1, 0) is $x^2+y^2+z^2+24x-17y-22z-43=0$ 2068, Bhadra B. E.

Solution:

Here the equation of sphere is

$$x^2+y^2+z^2+2x-6y+1=0 \quad \ldots(1)$$

Comparing it with $x^2+y^2+z^2+2ux+2vy+2wz+d=0$

We find $u=1,\ v=-3,\ w=0$ and $d=1$.

The equation of tangent plane at the point $(1, 2, -2)$ to the sphere (1) is

$$xx_1+yy_1+zz_1+u(x+x_1)+v(y+y_1)+w(z+z_1)+d=0$$

$$x+2y-2z+1(x+1)-3(y+2)+0.(z-2)+1=0$$

$$x+2y-2z+x+1-3y-6+1=0$$

or $\quad x+2y-2z+x+1-3y-6+1=0$

or $\quad 2x-y-2z-4=0 \quad \ldots\ldots(2)$

Therefore the equation of sphere through (1) and (2) is

$$x^2+y^2+z^2+2x-6y+1+\lambda(2x-y-2z-4)=0 \quad \ldots(3)$$

Since it passes through $(1, -1, 0)$.

So $\quad 1+1+0+2+6+1+\lambda(2+1-0-4)$

or $\quad 11-\lambda=0$

$\therefore \quad \lambda=11$

So the equation of the sphere (3) gives

$$x^2+y^2+z^2+24x-17y-22z-43=0$$

Ex. 9. Find the equation of the tangent planes to the sphere $x^2+y^2+z^2+2x-4y+6z-7=0$ which intersect in the line $6x-3y-23=0=3z+2$ [2063 Kartik, B.E.]

Solution:

Here the equation of the sphere is

$$x^2+y^2+z^2+2x-4y+6z-7=0 \quad \ldots(1)$$

Comparing it with $x^2+y^2+z^2+2ux+2vy+2wz+d=0$

We find $u=1,\ v=-2,\ w=3$ and $d=-7$

Centre of the sphere is $(-u,-v,-w)=(-1, 2, -3)$

Radius of the sphere is $R=\sqrt{u^2+v^2+w^2-d}=\sqrt{1+4+9+7}=\sqrt{21}$.

The equation of plane through the given line $6x-3y-23=0,\quad 3z+2=0$ is

$$6x-3y-23+\lambda(3z+2)=0 \quad \ldots\ldots(2)$$

If the plane (2) touches the sphere (1), then, the radius of the sphere equals to the perpendicular distance from centre $(-1, 2, -3)$ of the sphere (1) to the plane (2)

Thus $\quad \sqrt{21}=\dfrac{-35-7\lambda}{\sqrt{45+9\lambda^2}}$

or $\quad 2\lambda^2-7\lambda-4=0$

or $\quad (2\lambda+1)(\lambda-4)=0$

$\therefore \quad \lambda=-\dfrac{1}{2}, 4.$

Thus the two tangent planes are obtained from (2) by putting $\lambda=-\dfrac{1}{2}, 4$ which are $4x-2y-z-16=0$ and $2x-y+4z-5=0$.

Ex. 10. Find the equation of sphere which has its centre at the origin and which touches the line $2x+y=0,\ y+z+1=0$

Solution:

Here the equation of a sphere with centre origin and radius r is

$$x^2+y^2+z^2=r^2 \quad \ldots\ldots(1)$$

Since the sphere (1) touches the line $\quad 2x+y=0,\quad y+z+1=0 \quad \ldots(2)$

i. e. $x=-\dfrac{y}{2},\quad z=-1-y.$

Then (1) gives

$$\frac{y^2}{4}+y^2+(1+y)^2=r^2$$

or $\quad 9y^2+8y+4(1-r^2)=0$

This is quadratic equation in y so it has two roots. The line (2) is tangent to the sphere (1) if

$$(8)^2=4.(9).4(1-r^2)$$

or $\quad 64=144(1-r^2)=0$

$\therefore \quad r^2=\dfrac{5}{9}.$

So equation of sphere (1) gives

$$x^2+y^2+z^2=\frac{5}{9},$$

$\therefore \quad 9(x^2+y^2+z^2)=5.$

Exercise - 13

1. Find the equation of tangent plane to the spheres

(i) $x^2+y^2+z^2-x+3y+2z-3=0$ at the point $(1, 1, -1)$

(ii) $x^2+y^2+z^2-2x-4y+2z-3=0$ at the point $(-1,4,-2)$

2. Find the equation of tangent planes to the sphere $x^2 + y^2 + z^2 = ...$ which are parallel to the plane $3x + 4y + 5z = 7$

3. Find the two tangent planes to the sphere $x^2 + y^2 + z^2 - 4x + 2y - 6z + 5 = 0$ which are parallel to the plane $2x + 2y - z = 0$

4. Prove that the plane $2x - y + 2z - 14 = 0$ touches the sphere $x^2 + y^2 + z^2 - 4x + 2y - 4 = 0$. Find the point of contact.

5. Obtain the equation of tangent planes to the sphere $x^2 + y^2 + z^2 = ...$ which passes through the line $x + y - 6 = 0$, $x - 2z - 3 = 0$

6. For what value of λ the plane $x + y + z = \lambda\sqrt{3}$ touches the sphere $x^2 + y^2 + z^2 - 2x - 2y - 2z - 6 = 0$.

7. If any tangent plane to the sphere $x^2 + y^2 + z^2 = r^2$ makes intercepts a, b, c on the coordinate axes. Prove that $a^{-2} + b^{-2} + c^{-2} = r^{-2}$

8. Obtain the equation of the tangent planes to the sphere $x^2 + y^2 + z^2 ... 6x - 2z + 1 = 0$ which passes through the line $3(16 - x) = 3z = 2y + 30$

9. Find the equations of the spheres passing through the circle $x^2 + y^2 + z^2 - 6x - 2z + 5 = 0$, $y = 0$ and touching the plane $3y + 4z + 5 = 0$.

10. Obtain the equation of spheres through the circle $x^2 + y^2 + z^2 = 1$,

$2x + 4y + 5z = 6$ and touching the plane $z = 0$

Answers

1. (i) $x + 5y - 6 = 0$ (ii) $2x - 2y + z + 12 = 0$
2. $3x + 4y + 5z \pm 15\sqrt{2}$
3. $2x + 2y - z + 10 = 0$; $2x + 2y - z - 8 = 0$
4. $(4, -2, 2)$
5. $2x + y - 2z - 9 = 0$, $x + 2y + 2z - 9 = 0$
6. $\sqrt{3} \pm 3)$
8. $x + 2y - 2z + 14 = 0$; $2x + 2y - z - 2 = 0$
9. $x^2 + y^2 + z^2 - 6x - \frac{11}{4}y - 2z + 5 = 0$, $x^2 + y^2 + z^2 - 6x - 4y - 2z + 5 = 0$
10. $x^2 + y^2 + z^2 - 2x - 4y - 5z + 5 = 0$, $5(x^2 + y^2 + z^2) - 2x - 4y - 5z + 1 = 0$

3.7. Cone

A cone is a surface generated by a *straight line* which passes through a *fixed point* and intersects a *given curve* (or touches a given surface)

The straight line is called *generator* the fixed point is called *vertex* and the given curve or the surface is *guiding curve*.

1. To find the equation of a cone whose vertex and the guiding curve are given

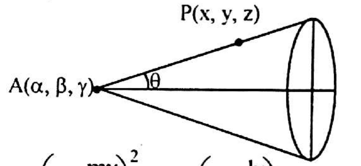
Let the equation of the conic as the guiding curve of the cone be

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0, \quad z = 0 \qquad \text{.........(1)}$$

Let the point $A(\alpha, \beta, \gamma)$ be the vertex and the conic (1) and the equation of the generator AP through $A(\alpha, \beta, \gamma)$ be

$$\frac{x - \alpha}{l} = \frac{y - \beta}{m} = \frac{z - \gamma}{n} \qquad \text{..........(2)}$$

α β γ
x y z


P(x, y, z)
A(α, β, γ)
θ

This line (2) meets the plane $z = 0$

at the point $P\left(\alpha - \frac{l\gamma}{n}, \beta - \frac{m\gamma}{n}, 0\right)$.

If this point lies on the conic (1), then

$$a\left(\alpha - \frac{l\gamma}{n}\right)^2 + 2h\left(\alpha - \frac{l\gamma}{n}\right)\left(\beta - \frac{m\gamma}{n}\right) + b\left(\beta - \frac{m\gamma}{n}\right)^2 + 2g\left(\alpha - \frac{l\gamma}{n}\right) + 2f\left(\beta - \frac{m\gamma}{n}\right) + c = 0 \qquad \text{.............(3)}$$

Eliminating l, m, n from (2) and (3), we get

$$a\left(\alpha - \frac{x - \alpha}{z - \gamma}\cdot\gamma\right)^2 + 2h\left(\alpha - \frac{x - \alpha}{z - \gamma}\gamma\right)\left(\beta - \frac{y - \beta}{z - \gamma}\gamma\right) + b\left(\beta - \frac{y - \beta}{z - \gamma}\cdot\gamma\right)^2 + 2g\left(\alpha - \frac{x - \alpha}{z - \gamma}\gamma\right) + 2f\left(\beta - \frac{y - \beta}{z - \gamma}\cdot\gamma\right) + c = 0$$

or $a(\alpha z - \gamma x)^2 + 2h(\alpha z - \gamma x)(\beta z - \gamma y) + b(\beta z - \gamma y)^2 + 2g(\alpha z - \gamma x)(z - \gamma) + 2f(\beta z - \gamma y)(z - \gamma) + c(z - \gamma)^2 = 0$ .. ...(4).

This is the required equation of the cone.

2. Equation of cone with vertex at origin

To show that general equation of second degree represents a cone whose vertex is at origin is homogeneous and conversely

Here the general equation of second degree in x, y, z is

$$ax^2 + by^2 + cz^2 + 2hxy + 2fyz + 2gzx + 2ux + 2vy + 2wz + d = 0 \qquad \text{......(1)}$$

Let the equation (1) represents a cone with vertex is at the origin O(0, 0, 0) and

$P(x', y', z')$ be a point on the cone represented by the equation (1).

The equation of a line passing through O (0,0,0) and $P(x', y', z')$ is

$$\frac{x-0}{x'-0} = \frac{y-0}{y'-0} = \frac{z-0}{z'-0}$$

or $$\frac{x}{x'} = \frac{y}{y'} = \frac{z}{z'} = r \text{ (say)} \quad \ldots(2)$$

where r is the distance of the point P from the vertex O.

Any point on the generator (2) is (rx', ry', rz') which will lie on the cone (1) if this point satisfies the equation of cone (1).

i.e. $r^2(ax'^2 + by'^2 + cz'^2 + 2hx'y' + 2fy'z' + 2gz'x') + 2r(ux' + uy' + wz') + d = 0$

This implies for all values of r

$$ax'^2 + by'^2 + cz'^2 + 2hx'y' + 2fy'z' + 2gz'x' = 0$$

$$ux' + vy' + wz' = 0 \quad \ldots(3)$$

$$d = 0 \quad \ldots(5) \quad \ldots(4)$$

From equation (4), it shows that $u = u = w = 0$, otherwise the point (x', y', z') will lie on the plane $ux + vy + wz = 0$ which is a contradiction. Hence the required *equation of cone* becomes

$ax^2 + by^2 + cz^2 + 2hxy + 2fyz + 2gzx = 0$ which is necessarily homogeneous equation.

Conversely

We have to show that every homogenous equation of second degree in x, y, z represents a cone with its vertex at the origin.

Let the homogeneous equation of second degree in x, y, z be

$$ax^2 + by^2 + cz^2 + 2hxy + 2fyz + 2gzx = 0 \quad \ldots(6)$$

Let $P(x', y', z')$ be any point on the locus represented by (6), then

$$ax'^2 + by'^2 + cz'^2 + 2hx'y' + 2fy'z' + 2gz'x' = 0 \quad \ldots(7)$$

Multiplying it by r^2, we get

$$A r^2x'^2 + b r^2y'^2 + cr^2z'^2 + 2hrx'\, ry' + 2f ry'\, rz' + 2g\, rz'\, rx' = 0$$

It shows that (rx', ry', rz') lies on the locus of (6) and this is any point on the line joining the origin O to the point P. Therefore OP lies completely on the locus of (6). Thus the equation (6) represents a cone with vertex is at the origin.

3.7.1 Condition that the General Equation of Second Degree Represent a Cone

To find the condition that the general equation of second degree may represent a cone

Here the general equation of second degree is

$$ax^2 + by^2 + cz^2 + 2hxy + 2yz + 2gzx + 2ux + 2vy + 2wz + d = 0 \ldots(1)$$

Let the equation (1) represent sa cone with vertex at (α, β, γ). Shifting the origin to the point (α, β, γ) by transformations

$$x = x + \alpha,\ y = y + \beta,\ z = z + \gamma.$$

The equation (1) reduces to

$$a(x+\alpha)^2 + b(y+\beta)^2 + c(z+\gamma)^2 + 2h(x+\alpha)(y+\beta) + 2f(y+\beta)(z+\gamma)$$
$$+ 2g(z+\gamma)(x+\alpha) + 2u(x+\alpha) + 2v(y+\beta) + 2w(z+\gamma) + d = 0$$

or $$ax^2 + by^2 + cz^2 + 2hxy + 2fyz + 2gzx + 2\{(a\alpha + h\beta + g\gamma + u)x$$
$$+ (h\alpha + b\beta + f\gamma + v)y + (g\alpha + f\beta + c\gamma + w)z\} + a\alpha^2 + b\beta^2$$
$$+ c\gamma^2 + 2h\alpha\beta + 2f\beta\gamma + 2g\gamma\alpha + 2u\alpha + 2v\beta + 2w\gamma + d = 0 \quad \ldots(2)$$

Since equation (2) represents a cone with vertex at the new origin. It must be second degree homogenous equation in x, y, and z.. The equation (2) be comes homogenous of second degree if the following conditions are satisfied.

$$a\alpha + h\beta + g\gamma + u = 0 \quad \ldots(3)$$

$$h\alpha + b\beta + f\gamma + v = 0 \quad \ldots(4)$$

$$g\alpha + f\beta + c\gamma + w = 0 \quad \ldots(5)$$

and $$a\alpha^2 + b\beta^2 + c\gamma^2 + 2h\alpha\beta + 2f\beta\gamma + 2g\gamma\alpha + 2u\alpha + 2v\beta + 2w\gamma + d = 0$$

or $$\alpha(a\alpha + h\beta + g\gamma + u) + \beta(h\alpha + b\beta + f\gamma + v) + \gamma(g\alpha + f\beta + c\gamma + w)$$
$$+ u\alpha + v\beta + w\gamma + d = 0.$$

With the help of (3), (4) and (5), it reduces to

$$u\alpha + v\beta + w\gamma + d = 0 \quad \ldots(6)$$

Eliminating α, β, γ from equation (3), (4), (5) and (6), we get

$$\begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & d \end{vmatrix} = 0$$

This is the required condition for the equation (1) of the second degree to represents a cone.

Note:

Let the equation

$$F(x, y, z) \equiv ax^2 + by^2 + cz^2 + 2hxy + 2fyz + 2gzx + 2ux + 2vy + 2wz + d = 0$$

represent a cone.

Making the equation homogeneous by introducing a new variable t, we get

$$F(x, y, z, t) = ax^2 + by^2 + cz^2 + 2hxy + 2fyz + 2gzx$$
$$+ 2uxt + 2vyt + 2wzt + dt^2 = 0$$

Differentiating partially, we get

$$\frac{\partial F}{\partial x} = 2(ax + hy + gz + ut)$$

$$\frac{\partial F}{\partial y} = 2(hx + by + fz + vt)$$

$$\frac{\partial F}{\partial z} = 2(gx + fy + cz + wt)$$

$$\frac{\partial F}{\partial t} = 2(ux + vy + wz + dt).$$

In the above equations

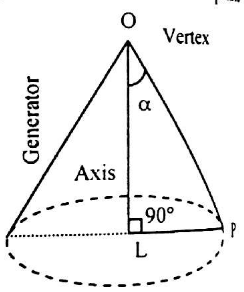
$\frac{\partial F}{\partial x} = 0, \frac{\partial F}{\partial y} = 0, \frac{\partial F}{\partial z} = 0, \frac{\partial F}{\partial t} = 0$, taking t = 1 are in fact the equations (3), (4), (5) and (6) respectively. Thus the equation $F(x, y, z) = 0$ represents a cone iff $\frac{\partial F}{\partial x} = 0, \frac{\partial F}{\partial y} = 0, \frac{\partial F}{\partial z} = 0, \frac{\partial F}{\partial t} = 0$ are consistent.

3.7.2 Right Circular Cone

A right circular cone is a surface generated by a line which passes through a fixed point and makes a constant angle with fixed line through the fixed point. The fixed point is called the *vertex* the fixed line, the *axis* and the fixed angle the *semi-vertical angle* of the cone.

To prove that every section of a right circular cone by a plane perpendicular to its axis is a circle.

Let O L be axis, α be semi-vertical angle of the right circular cone and a plane perpendicular to the axis OL meet it at L.



Let P be any point on the section. Since OL is perpendicular to the plane where the plane contains the line LP i.e. OL ⊥ LP

$$\therefore \quad \tan \angle LOP = \frac{PL}{OL}$$

or
$$\tan \alpha = \frac{PL}{OL}$$

$$PL = OL \tan \alpha$$

Hence PL is constant for every position of the point P of the section. Hence the section is a circle with centre L.

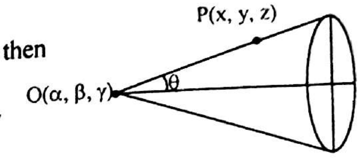
To find the equation of a right circular cone with vertex $O(\alpha, \beta, \gamma)$, the semi-vertical angle θ and the axis is $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$

Here, the axis of the right circular cone is

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$$

The direction ratios of the axis (1) of the cone are

l, m, n



Let P(x, y, z) be any point on the cone, then the direction ratios of OP are

$$x - \alpha, \quad y - \beta, \quad z - \gamma$$

If θ is semi-vertical angle, then

$$\cos\theta = \frac{l(x-\alpha) + m(y-\beta) + n(z-\gamma)}{\sqrt{(l^2 + m^2 + n^2)\{(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2\}}}$$

or
$$\{l(x-\alpha) + m(y-\beta) + n(z-\gamma)\}^2 = (l^2 + m^2 + n^2)\{(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2\}\cos^2\theta$$

Cor. 1:

If we put $\alpha = 0, \beta = 0, \gamma = 0$, the equation of a right circular cone with vertex at the origin is

$$(lx + my + nz)^2 = (l^2 + m^2 + n^2)(x^2 + y^2 + z^2)\cos^2\theta$$

Cor. 2:

The equation of a cone with vertex at the origin and axis as z-axis, then $l = 0, m = 0, n = 1$ and the equation of right circular cone is $x^2 + y^2 = z^2 \tan^2\theta$

Worked Out Examples

Ex. 1. Find the equation to the cone with vertex at the origin and which passes through the curve of intersection of the plane $lx + my + nz = p$ and the surface $ax^2 + by^2 + cz^2 = 1$

2061 Aswin, B. E.

Solution:

Here the equation of the surface is

$$ax^2 + by^2 + cz^2 = 1 \quad \text{.......(1)}$$

The equation of the plane is

$$lx + my + nz = p$$

or
$$\frac{lx + my + nz}{p} = 1 \quad \text{....(2)}$$

We know that equation of cone with vertex at the origin is homogenous. So making homogenous to the equation (1) with the help of (2), we get

$$ax^2 + by^2 + cz^2 = \left(\frac{lx + my + nz}{p}\right)^2$$

$\therefore \quad p^2(ax^2+by^2+cz^2) = (lx + my + nz)^2$ is the required equation of cone.

Ex.2. Prove that the general equation of the cone of second degree passing through the axes is hxy + fyz + gzx = 0 [2063 Bhadra B.E.]

Solution:

The cone passing through the axes will have the origin at vertex. Let its general equation be

$$ax^2 + by^2 + cz^2 + 2hxy + 2fyz + 2gzx = 0 \qquad \text{.......(1)}$$

Since it passes through the co-ordinate axes. Therefore the direction cosines of the axes are

$$1, 0, 0; 0, 1, 0; 0, 0, 1 \text{ respectively.}$$

It will satisfy the equation (1). Thus, we get

$$a = 0,\ b = 0,\ c = 0.$$

Hence the equation of the cone becomes

$$hxy + fyz + gzx = 0$$

Ex.3. Prove that a cone of second degree can be found to pass through two sets of rectangular axes through the same origin

Solution:

Let the general equation of the cone with vertex at the origin is

$$ax^2 + by^2 + cz^2 + 2hxy + 2fyz + 2gzx = 0 \qquad \text{.......(1)}$$

Let the co-ordinate axes be one of rectangular axes through which the cone passes. The equation of the cone passing through co-ordinate axes is

$$h\,xy + fyz + g\,zx = 0 \qquad \text{.......(2)}$$

Let the second set of rectangular axes through the same origin be OP, OQ, OR having direction cosines $l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3$ respectively, then

$$m_1n_1 + m_2n_2 + m_3n_3 = 0 \qquad \text{.......(3)}$$

$$n_1l_1 + n_2l_2 + n_3l_3 = 0 \qquad \text{.......(4)}$$

$$l_1m_1 + l_2m_2 + l_3m_3 = 0 \qquad \text{.......(5)}$$

If the cone (2) passes through OP and OQ, then

$$hl_1m_1 + fm_1n_1 + gn_1l_1 = 0 \qquad \text{.......(6)}$$

and

$$hl_2m_2 + fm_2n_2 + gn_2l_2 = 0 \qquad \text{.......(7)}$$

Adding (6) and (7), we get

$$h(l_1m_1 + l_2m_2) + (m_1n_1 + m_2n_2) + g(n_1l_1 + n_2l_2) = 0$$

Using (3), (4) and (5), it becomes

$$hl_3m_3 + fm_3n_3 + gn_3l_3 = 0$$

It shows that (2) also passes through OR.

Ex.4. The plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ meets the co-ordinate axes in A, B, C. Prove that the equation of the cone generated by the lines drawn from O to meet the circle ABC is $xy\left(\frac{a}{b} + \frac{b}{a}\right) + yz\left(\frac{b}{c} + \frac{c}{b}\right) + zx\left(\frac{c}{a} + \frac{a}{c}\right) = 0$

Solution:

Here the equation of the plane is

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \qquad \text{..........(1)}$$

The plane (1) meets the axes at A(a, 0, 0), B(0, b, 0), C(0, 0, c) respectively. So the equation of a sphere through OABC is

$$x^2 + y^2 + z^2 - ax - by - cz = 0 \qquad \text{.........(2)}$$

The circle ABC is represented by the equations (1) and (2). Thus the cone generated by the lines drawn from O to meet the circle ABC is obtained by making equation (2) homogeneous with the help of (1). So

$$x^2 + y^2 + z^2 - (ax + by + cz)\left(\frac{x}{a} + \frac{y}{b} + \frac{z}{c}\right) = 0$$

$$\therefore \quad xy\left(\frac{a}{b} + \frac{b}{a}\right) + yz\left(\frac{b}{c} + \frac{c}{b}\right) + zx\left(\frac{c}{a} + \frac{a}{c}\right) = 0$$

Ex.5. The section of a cone whose guiding curve is the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,\ z = 0$ by the plane x = 0 is a rectangular hyperbola. Prove that the locus of the vertex is the surface $\frac{x^2}{a^2} + \frac{y^2 + z^2}{b^2} = 1$

Solution:

Let (α, β, γ) be the vertex of the cone, then the equation of a generator of the cone is

$$\frac{x - \alpha}{l} = \frac{y - \beta}{m} = \frac{z - \gamma}{n} \qquad \text{......(1)}$$

The line (1) will meet the plane z = 0 in the point given by $\left(\alpha - \frac{l\gamma}{n}, \beta - \frac{m\gamma}{n}, 0\right)$ which also lie on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ in the plane z = 0 if

$$\frac{1}{a^2}\left(\alpha-\frac{l\gamma}{n}\right)^2+\frac{1}{b^2}\left(\beta-\frac{m\gamma}{n}\right)^2=1 \quad \text{....(2)}$$

Eliminating l, m, n from (1) and (2), we get

$$\frac{1}{a^2}\left(\alpha - \frac{x-\alpha}{z-\gamma}\cdot\gamma\right)^2 + \frac{1}{b^2}\left(\beta-\frac{y-\beta}{z-\gamma}\cdot\gamma\right)^2=1$$

$$\frac{1}{a^2}(\alpha z - \gamma x)^2 + \frac{1}{b^2}(\beta z - \gamma y)^2 = (z-\gamma)^2 \quad \text{....(3)}$$

This is the equation of the cone with vertex (α, β, γ). Given that the cone (3) is intersected by the plane x = 0. Then it becomes

$$\frac{\alpha^2 z^2}{a^2}+\frac{1}{b^2}(\beta z-\gamma y)^2=(z-\gamma)^2$$

or
$$\frac{\gamma^2 y^2}{b^2}+\left(\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}-1\right)z^2-\frac{2\beta\gamma yz}{b^2}+2\gamma z-\gamma^2=0$$

This represents a rectangular hyperbola in yz plane if

Coeff. of y^2 + Coeff. of $z^2 = 0$

or
$$\frac{\gamma^2}{b^2}+\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}-1=0$$

Hence the locus of (α, β, γ) is

$$\frac{x^2}{a^2}+\frac{y^2+z^2}{b^2}=1.$$

Ex. 6. Find the equation of the cone with vertex (α, β, γ) and guiding curve is parabola $y^2 = 4ax$, z = 0 [2068 Bhadra, B.E]

Solution:

Let (α, β, γ) be the vertex of the cone, then the equation of generator is

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n} \quad \text{....(1)}$$

This line meets the plane z = 0 in the point given by $\left(\alpha - \frac{l\gamma}{n}, \beta - \frac{m\gamma}{n}, 0\right)$ which also lie on the parabola $y^2 = 4ax$ in the plane z = 0 if

$$\left(\beta-\frac{m\gamma}{n}\right)^2=4a\left(\alpha-\frac{l\gamma}{n}\right) \quad \text{..........(2)}$$

Eliminating l, m, n from (1) and (2), we get

$$\left(\beta-\frac{y-\beta}{z-\gamma}\cdot\gamma\right)^2=4a\left(\alpha-\frac{x-\alpha}{z-\gamma}\cdot\gamma\right)$$

or $(\beta z-\gamma y)^2 = 4a(\alpha z-\gamma x)(z-\gamma)$

This is the required equation of cone.

Ex.7. Prove that the lines drawn through the point (α, β, γ) whose direction cosines satisfy the relation $al^2 + bm^2 + cn^2 = 0$, generate the cone $a(x - \alpha)^2 + b(y - \beta)^2 + c(z - \gamma)^2 = 0$

Solution:

Here the equation of a line through the point (α, β, γ) with direction cosines l, m, n is

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n} \quad \text{....(1)}$$

The direction cosines of the line (1) l, m, n satisfy the relation

$$al^2+bm^2+cn^2=0 \quad \text{....(3)}$$

Eliminating l, m, n from (1) and (2), we get

$$a(x-\alpha)^2+b(y-\beta)^2+c(z-\gamma)^2=0.$$

This is the required equation of the cone.

Ex.8. Find the equation of the cone whose vertex is the origin and base the circle x = a, $y^2 + z^2 = b^2$ and prove that the section of the cone by a plane parallel to xy-plane is a hyperbola

Solution:

Here the equation of a line with vertex at origin is

$$\frac{x-0}{l}=\frac{y-0}{m}=\frac{z-0}{n}$$

or
$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n} \quad \text{..........(1)}$$

The line (1) will meet the plane x = a in the point given by $\left(a, \frac{am}{l}, \frac{an}{l}\right)$ which also lie on the circle $y^2 + z^2 = b^2$ in the plane x = a if

$$\left(\frac{am}{l}\right)^2+\left(\frac{an}{l}\right)^2=b^2 \quad \text{.....(2)}$$

Eliminating l, m, n from (1) and (2), we get

$$a^2y^2+a^2z^2=b^2x^2$$

or $a^2(y^2+z^2) = b^2x^2$ is required equation of cone.

If this cone is intersected by a plane parallel to the XOY plane i.e. z = k, then it gives

$$a^2y^2 + a^2k^2 = b^2x^2$$

or $$\frac{x^2}{k^2a^2/b^2} - \frac{y^2}{k^2} = 1$$

This is a hyperbola.

Ex. 9. The plane through OX and OY include an angle α, prove that their line of intersection lies on the cone

$$z^2(x^2+y^2+z^2) = x^2y^2\tan^2\alpha$$

[2067/072 Magh, B.E.]

Solution:

The equation of a plane through OX is

$$by + cz = 0$$

or $$y + \frac{c}{b}z = 0$$

or $$y + n_1z = 0 \quad \text{...(1)}$$

and the equation of a plane through OY is

$$ax + cz = 0$$

or $$x + \frac{c}{a}z = 0$$

or $$x + n_2z = 0 \quad \text{...(2)}$$

The direction ratios of the normals to the planes (1) and (2) are

0, 1, n_1, and 1, 0, n_2 respectively.

If α be the angle between them, then

$$\cos\alpha = \frac{0.1 + 1.0 + n_1n_2}{\sqrt{(0+1+n_1^2)}\sqrt{1+0+n_2^2}}$$

or $$\cos\alpha = \frac{n_1n_2}{\sqrt{1+n_1^2}\sqrt{1+n_2^2}} \quad \text{...(3)}$$

Now we have to find the locus of the line of intersection of the planes $y + n_1z = 0$

and $x + n_2z = 0$ subject to the condition (3). So eliminating n_1, n_2, from (1), (2) and (3), we get

$$\cos\alpha = \frac{\left(\frac{-y}{z}\right).\left(\frac{-x}{z}\right)}{\sqrt{\left(1+\frac{y^2}{z^2}\right)}\sqrt{\left(1+\frac{x^2}{z^2}\right)}} = \frac{xy}{\sqrt{y^2+z^2}\sqrt{x^2+z^2}}$$

$$\cos^2\alpha\,(y^2+z^2)(x^2+z^2) = x^2y^2$$

or $$\cos^2\alpha\,(y^2x^2+y^2z^2+z^2x^2+z^4) = x^2y^2$$

or $$x^2y^2\cos^2\alpha + z^2\cos^2\alpha\,(x^2+y^2+z^2) = x^2y^2$$

or $$z^2\cos^2\alpha\,(x^2+y^2+z^2) = x^2y^2(1-\cos^2\alpha)$$

or ∴ $$z^2(x^2+y^2+z^2) = x^2y^2\tan^2\alpha$$

Ex.10. Find the equation of the cone with vertex at the origin and passes through the curves of intersection given by the equations $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ and $\frac{1}{2z}\left(\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2}\right) = 1$

Solution:

Here, the equations of the curves are

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad \text{.............(1)}$$

and $$\frac{1}{2z}\left(\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2}\right) = 1 \quad \text{.............(2)}$$

We know that the equation of the cone with vertex at the origin is homogenous. So making (1) homogeneous with the help of (2), we get

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = \frac{1}{4z^2}\left(\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2}\right)^2.$$

This is the required equation of cone.

Ex.11. Show that $4x^2 - y^2 + 2z^2 + 2xy - 3yz + 12x - 11y + 6z + 4 = 0$ represents a cone with vertex (- 1, - 2, - 3).

Solution:

Here the general equation of second degree in x,y,z is

$$4x^2 - y^2 + 2z^2 + 2xy - 3yz + 12x - 11y + 6z + 4 = 0$$

Making this equation homogenous by introducing a new variable t, we get

$$F(x, y, z, t) = 4x^2 - y^2 + 2z^2 + 2xy - 3yz + 12xt - 11yt + 6zt + 4t^2 = 0$$

Now equating to zero the partial derivatives of F, we get

$$\frac{\partial F}{\partial x} = 8x + 2y + 12t = 0 \quad \text{.....(1)}$$

$$\frac{\partial F}{\partial y} = 2x - 2y - 3z - 11t = 0 \quad \text{.....(2)}$$

$$\frac{\partial F}{\partial z} = 4z - 3y + 6t = 0 \quad \text{.....(3)}$$

$$\frac{\partial F}{\partial t} = 12x - 11y + 6z + 8t = 0 \quad \text{.....(4)}$$

Solving equations (1), (2) and (3) by taking $t = 1$, we get

$x = -1, y = -2, z = -3.$

These values of x, y, z also satisfy the equation (4). Hence the equations are consistent .So the given equation represents a cone with vertex $(-1, -2, -3)$.

Ex. 12. Prove that the equation $ax^2+by^2+cz^2+2ux+2vy+2wz+d=0$ represents a cone if $\frac{u^2}{a} + \frac{u^2}{b} + \frac{w^2}{c} = d$ *2066/072 Ashwin, B.E.*

Solution:

Here the general equation of second degree in x,y,z is

$$ax^2 + by^2 + cz^2 + 2ux + 2vy + 2wz + d = 0$$

Making this equation homogenous by introducing a new variable t, we get

$$F(x, y, z, t) = ax^2 + by^2 + cz^2 + 2uxt + 2vyt + 2wzt + dt^2 = 0$$

Now equating to zero the partial derivatives of F, we get

$$\frac{\partial F}{\partial x} = 2ax + 2ut = 0 \quad \text{...(1)}$$

$$\frac{\partial F}{\partial y} = 2by + 2vt = 0 \quad \text{...(2)}$$

$$\frac{\partial F}{\partial z} = 2cz + 2wt = 0 \quad \text{.......(3)}$$

$$\frac{\partial F}{\partial t} = 2ux + 2vy + 2wz + 2dt = 0 \quad \text{......(4)}$$

Solving the equations (1), (2) and (3) taking $t = 1$, we get

$$x = -\frac{u}{a},\ y = -\frac{v}{b},\ z = -\frac{w}{c}.$$

These values must satisfy equation (4),then

$$-2u.\frac{u}{a} - 2v.\frac{v}{b} - 2w.\frac{w}{c} + 2d = 0$$

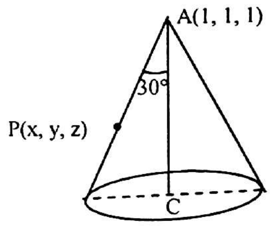
or $$\frac{u^2}{a} + \frac{v^2}{b} + \frac{w^2}{c} = d$$

Ex.13. Find the equation of a right circular cone with vertex (1, 1, 1) and axis is the line $\frac{x-1}{-1} = \frac{y-1}{2} = \frac{z-1}{3}$ and semi vertical angle 30° *2070 Magh, B. E.*

Solution:

Here the axis of the cone is

$$\frac{x-1}{-1} = \frac{y-1}{2} = \frac{z-1}{3} \quad \text{...(1)}$$



The direction ratios of the axis of the cone are

$-1, 2, 3.$

Let P(x, y, z) be any point on the cone with vertex A(1, 1, 1),the direction ratios of AP are

$x - 1, y - 1, z - 1.$

Semi vertical angle is $\angle PAC = 30°$

$$\cos 30° = \frac{(x-1)(-1) + (y-1).2 + (z-1).3}{\sqrt{(x-1)^2 + (y-1)^2 + (z-1)^2}\sqrt{1+4+9}}$$

or $$\frac{\sqrt{3}}{2} = \frac{-x + 2y + 3z - 4}{\sqrt{14(x^2 + y^2 + z^2 - 2x - 2y - 2z + 3)}}$$

or $$3 \times 14(x^2 + y^2 + z^2 - 2x - 2y - 2z + 3) = 4(x - 2y - 3z + 4)^2$$

or $$19x^2 + 13y^2 + 3z^2 + 8xy + 12xz - 24yz - 58x - 10y + 6z + 31 = 0$$

Ex. 14. Prove the plane ax + by + cz = 0 cuts cone yz + zx + xy = 0 in perpendicular lines if $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=0$

Solution:

Here the equation of the cone is

$$yz + zx + xy = 0 \quad \cdots(1)$$

and the equation of the plane is

$$ax + by + cz = 0 \quad \cdots(2)$$

Let $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ be the equation of any one of the two lines in which the plane (2) cuts the cone(1), so that

$$al + bm + cn = 0 \quad \cdots\cdots(3)$$

and

$$mn + nl + lm = 0 \quad \cdots\cdots(4)$$

Eliminating l from the equations (3) and (4), we get

$$b\left(\frac{m}{n}\right)^2 + (b+c-a)\frac{m}{n} + c = 0.$$

This equation is quadratic in $\frac{m}{n}$, so it has two roots say $\frac{m_1}{n_1}$ and $\frac{m_2}{n_2}$.

$$\therefore \quad \frac{m_1m_2}{n_1n_2} = \frac{c}{b}$$

or

$$\frac{m_1m_2}{1/b} = \frac{n_1n_2}{1/c} \quad \cdots\cdots(5)$$

Similarly, eliminating m from (3) and (4), we get

$$\frac{l_1l_2}{1/a} = \frac{n_1n_2}{1/c} \quad \cdots\cdots(6)$$

From (5) and (6) $\quad \frac{l_1l_2}{1/a} = \frac{m_1m_2}{1/b} = \frac{n_1n_2}{1/c}$

Two lines are perpendicular if

$$l_1l_2 + m_1m_2 + n_1n_2 = 0$$

or $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=0$

Exercise - 14

1. (i) Find the equation of the cone with vertex (1, 1, 0) and guiding curve is $y = 0, x^2 + z^2 = 4$

   (ii) Find the equation of the cone with vertex (α, β, γ) and base $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, z = 0$ [2063 Kartik .B.E.]

   (iii) Find the equation of cone with vertex (3, 1, 2) and base $2x^2 + 3y^2 = 1, z = 1$ [2069 Bhadra, B. E.]

2. Find the equation of cone whose vertex is origin and guiding curve is $\frac{x^2}{4}+\frac{y^2}{9}+\frac{z^2}{1}=1, x+y+z=1$

3. Find the equation of a cone whose vertex at origin and base is the circle $y^2 + z^2 = b^2, x = a$

4. Find the equation of cone whose vertex at the origin and direction cosines of the generators satisfy the relation $3l^2 - 4m^2 + 5n^2 = 0$

5. Show that the equation of cone whose vertex is origin whose guiding curve is circle represented by $x^2 + y^2 + z^2 - ax - by - cz = 0$ $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ is $c(a^2+b^2)xy + a(b^2+c^2)yz + b(c^2+a^2)zx = 0$

6. Find the equation of right circular cone whose vertex at origin and axis is the line $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$ with vertical angle 30° [2070/071 Bhadra, B. E.]

7. Find the equation of right circular cone whose vertex is at the origin and semi-vertical angle α and having axis of z as its axis

8. Find the equation of cone whose vertical angle is $\frac{\pi}{2}$ with vertex at the origin and its axis along the line $x = -2y = z$

Answers

1. (i) $x^2 - 3y^2 + z^2 - 2xy + 8y - 4 = 0$
   (ii) $b^2(\alpha z - \gamma x)^2 + a^2(\beta z - \gamma y) = a^2b^2(z - \gamma)^2$
   (iii) $2x^2 + 3y^2 + 20z^2 - 6yz - 12xz + 12x + 6y - 26z - 7 = 0$
2. $27x^2 + 32y^2 + 72(xy + yz + zx) = 0$
3. $a^2(y^2 + z^2) = b^2x^2$
4. $3x^2 - 4y^2 + 5z^2 = 0$
6. $19x^2 + 13y^2 + 3z^2 - 8xy - 24yz - 12zx = 0$
7. $x^2 + y^2 = z^2\tan^2\alpha$
8. $x^2 + 7y^2 + z^2 + 8xy + 8yz - 16zx = 0$

3.8. Cylinder

A cylinder is a surface generated by a straight line which is parallel to a *fixed line* and satisfies one more condition i.e. it may intersect or touch a *given curve*. The given curve is called *guiding curve* and the straight line in any position is called the *generator* and the fixed line, the *axis* of the cylinder.

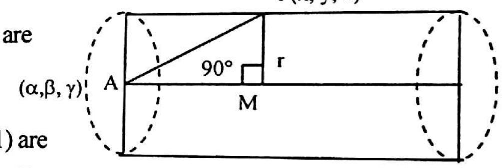
3.8.1 Right Circular Cylinder

A right circular cylinder is a surface generated by a straight line which is parallel to a *fixed line* and is at a *constant distance* from it. The constant distance is called *radius* of cylinder.

To find the equation of right circular cylinder whose axis is the line $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$ *and whose radius is r* 2069 Bhadra, B. E.

Here, the axis of the cylinder is

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} \qquad \ldots(1)$$



The direction ratios of the line (1) are

l, m ,n

The direction cosines of the line (1) are

$$\frac{l}{\sqrt{l^2+m^2+n^2}}, \frac{m}{\sqrt{l^2+m^2+n^2}}, \frac{n}{\sqrt{l^2+m^2+n^2}}.$$

A point on the line (1) is A (α, β, γ). Let P(x, y, z) be any point on the cylinder.

So $\quad AP = \sqrt{(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2}$.

Draw PM perpendicular to the axis AM.

Then PM = r and

AM = projection of AP on AM(axis).

$$= \frac{(x-\alpha)l + (y-\beta)m + (z-\gamma)n}{\sqrt{l^2+m^2+n^2}}$$

From the right angle triangle PAM

$$(AP)^2 = (AM)^2 + (PM)^2$$

$$[(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2] = \frac{[l(x-\alpha) + m(y-\beta) + n(z-\gamma)]^2}{l^2+m^2+n^2} + r^2$$

This is required equation of right circular cylinder.

Worked Out Examples

Ex. 1. Find the equation of cylinder whose generating lines have direction cosines l, m, n and which passes through the circumference of the fixed circle $x^2 + z^2 = a^2$, $y = 0$

Solution:

Let $P(x_1, y_1, z_1)$ be any point of the cylinder so that the equation of the generator which is parallel to the axis $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$ is

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n} \quad \dots(1)$$

and guiding curve is the circle $x^2 + z^2 = a^2, y = 0 \quad \dots(2)$

The generator (1) cuts the plane y = 0, where

$$\frac{x - x_1}{l} = \frac{-y_1}{m} = \frac{z - z_1}{n}$$

or $\quad x = x_1 - \frac{l y_1}{m}$ and $z = z_1 - \frac{n y_1}{m}$

But these values of x and z satisfy the equation (2) $x^2 + z^2 = a^2$

$$\therefore \quad \left(x_1 - \frac{l y_1}{m}\right)^2 + \left(z_1 - \frac{n y_1}{m}\right)^2 = a^2$$

Hence the locus of (x_1, y_1, z_1) is

$$(m x - l y)^2 + (m z - n y)^2 = a^2 m^2$$

This is the required equation of the cylinder.

Ex. 2. Find the equation of the cylinder whose generators are parallel to the line $x = \frac{-y}{2} = \frac{z}{3}$ and whose guiding curve is $x^2 + 2y^2 = 1, z = 3$

[2061 Aswin/062 Jestha]

Solution:

Here, the equation of axis of the cylinder is

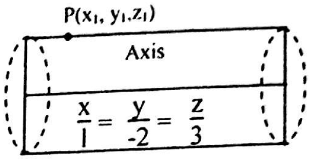
$$x = \frac{y}{-2} = \frac{z}{3} \quad \dots(1)$$

Let $P(x_1, y_1, z_1)$ be any point of the cylinder so that the equation of generator passing through $P(x_1, y_1, z_1)$ and parallel to axis (1) is

$$\frac{x - x_1}{1} = \frac{y - y_1}{-2} = \frac{z - z_1}{3} \quad \dots(2)$$

and the guiding curve is circle $x^2 + 2y^2 = 1$, $z = 3$. ....(3)

The generator (2) cuts the plane z = 3 where



$$\frac{x - x_1}{1} = \frac{y - y_1}{-2} = \frac{3 - z_1}{3}$$

or $\quad x = x_1 + 1 - \frac{z_1}{3}, \quad y = y_1 - 2 + \frac{2z_1}{3}$

But these values of x and y satisfy the equation $x^2 + 2y^2 = 1$.

$$\therefore \quad \left(x_1 + 1 - \frac{z_1}{3}\right)^2 + 2\left(y_1 - 2 + \frac{2z_1}{3}\right)^2 = 1$$

or $\quad 3x_1^2 + 6y_1^2 + 3z_1^2 + 8y_1z_1 - 2z_1x_1 - 18z_1 - 24y_1 + 24 = 0$

Hence the locus of (x_1, y_1, z_1) is

$$3x^2 + 6y^2 + 3z^2 + 8yz - 2zx - 18z - 24y + 24 = 0$$

Ex. 3. Find the equation of the right circular cylinder of radius 2 whose axis is the line $\frac{x-1}{2} = \frac{y}{3} = \frac{z-3}{1}$

Solution:

Here, the equation of the axis of the right circular cylinder is

$$\frac{x-1}{2} = \frac{y}{3} = \frac{z-3}{1}$$

The direction cosines of the axis are proportional to 2,3,1. Its actual direction cosines are $\frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}, \frac{1}{\sqrt{14}}$.

We know that the equation of right circular cylinder with radius 2 is

$$(x-1)^2 + (y-0)^2 + (z-3)^2 - \left(\frac{2}{\sqrt{14}}(x-1) + \frac{3}{\sqrt{14}}(y-0) + \frac{1}{\sqrt{14}}(z-3)\right)^2 = (2)^2$$

or $\quad (x-1)^2 + y^2 + (z-3)^2 - \frac{1}{14}[4(x-1)^2 + 9y^2 + (z-3)^2 + 12(x-1)y + 6y(z-3) + 4(x-1)(z-3)] = 4$

or $\quad 14(x-1)^2 + 14y^2 + 14(z-3)^2 - 4(x-1)^2 - 9y^2 - (z-3)^2 - 12(x-1)y - 6y(z-3) - 4(x-1)(z-3)] = 56$

or $\quad 10(x-1)^2 + 5y^2 + 13(z-3)^2 - 12xy + 12y - 6yz + 18y - 4xz + 12x + 4z - 12 = 56$

or $\quad 10x^2 - 20x + 10 + 5y^2 + 13z^2 - 78z + 117 - 12xy + 12y - 6yz + 18y - 4xz + 12x + 4z - 12 - 56 = 0$

$\therefore \quad 10x^2 + 5y^2 + 13z^2 - 12xy - 6yz - 4xz - 8x + 30y - 74z + 59 = 0$

is the equation of the cylinder.

Ex. 4. Obtain the equation of right circular cylinder of radius 4 and axis the line $x = 2y = -z$ [2069 Poush, B.E.]

Solution:

Here the equation of the axis of the cylinder is

$$\frac{x}{2} = \frac{y}{1} = \frac{z}{-2} \quad \ldots(1)$$

The direction ratios of the axis are

$2, 1, -2.$

The direction cosines of the axis are

$\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}.$

We know that the equation of right circular cylinder with radius 4 and given axis is

$$x^2 + y^2 + z^2 - \left(\frac{2}{3}x + \frac{1}{3}y - \frac{2}{3}z\right)^2 = (4)^2,$$

or $\quad x^2 + y^2 + z^2 - \frac{1}{9}(2x + y - 2z)^2 = 16,$

or $\quad 5x^2 + 8y^2 + 5z^2 + 4yz + 8zx - 4xy - 144 = 0.$

Ex. 5. Find the equation of the right circular cylinder whose guiding curve is the circle $x^2 + y^2 + z^2 - x - y - z = 0, x + y + z = 1$ [2067 Mangsir, B.E.]

Solution:

Here the equation of the circle is

$$x^2 + y^2 + z^2 - x - y - z = 0, x + y + z = 1 \quad \ldots(1)$$

Comparing the sphere of (1) with $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$

We find $\quad u = -\frac{1}{2}, v = -\frac{1}{2}, w = -\frac{1}{2} \quad$ and $d = 0$

Centre of the sphere is $(-u, -v, -w) = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$

Radius of the sphere $R = \sqrt{u^2 + v^2 + w^2 - d} = \sqrt{\frac{1}{4} + \frac{1}{4} + \frac{1}{4}} = \sqrt{\frac{3}{4}}.$

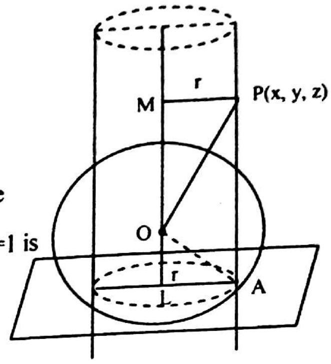
The equation of a line through the centre $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ of the sphere and

perpendicular to the plane of the circle (1) is

$$\frac{x - 1/2}{1} = \frac{y - 1/2}{1} = \frac{z - 1/2}{1}$$

So the axis of the cylinder is

$$\frac{x - 1/2}{1} = \frac{y - 1/2}{1} = \frac{z - 1/2}{1}$$



The length of the perpendicular from the centre $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ of the sphere to the plane $x + y + z = 1$ is

$$p = \frac{\frac{1}{2} + \frac{1}{2} + \frac{1}{2} - 1}{\sqrt{1^2 + 1^2 + 1^2}} = \frac{1}{2\sqrt{3}}$$

We have radius of the circle $= \sqrt{R^2 - p^2} = \sqrt{\frac{3}{4} - \frac{1}{12}} = \sqrt{\frac{2}{3}}$

Thus radius of the cylinder $(r) = \sqrt{\frac{2}{3}}$

If $P(x, y, z)$ be any point on the cylinder, then

$$OP^2 = OM^2 + MP^2$$

$$\left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 + \left(z - \frac{1}{2}\right)^2 = \left[\frac{1}{\sqrt{3}}\left(x - \frac{1}{2}\right) + \frac{1}{\sqrt{3}}\left(y - \frac{1}{2}\right) + \frac{1}{\sqrt{3}}\left(z - \frac{1}{2}\right)\right]^2 + \frac{2}{3}$$

$x^2 + y^2 + z^2 - yz - zx - xy - 1 = 0$ is the required equation of the cylinder.

Exercise - 15

1. Find the equation of cylinder whose guiding curve is conic $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0, z = 0$ axis is the line $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$
2. Find the equation of the cylinder whose generator are parallel to the line $\frac{x}{1} = \frac{y}{-2} = \frac{z}{3}$ and whose guiding curve is $x^2 + 2y^2 = 1, z = 0$
3. Find the equation of the cylinder which passes through $y^2 = 4ax, z = 0$ and whose generators are parallel to the line $\frac{x}{1} = \frac{y}{1} = \frac{z}{1}$ [2063 Kartik, B.E.]

4. Find the equation of the right circular cylinder of radius 2 whose axis passes through (1, 2, 3) and has direction ratios 2, - 3, 6

5. Find the equation of the right circular cylinder of radius 2 whose axis is the line $\frac{x-1}{2} = \frac{y-2}{1} = \frac{z-3}{2}$ [2070 Bhadra, B.E]

6. Find the equation of the right circular cylinder having for its base the circle $x^2 + y^2 + z^2 = 9$, $x - y + z = 3$ [2071 Bhadra, B.E]

Answers

1. $a(nx - lz)^2 + 2h(nx - lz)(ny - mz) + b(ny - mz)^2 + 2gn(nx - lz) + 2fm(ny - mz) + cn^2 = 0$
2. $3x^2 + 6y^2 + 3z^2 + 8yz - 2zx - 3 = 0$
3. $(y - z)^2 - 4a(x - z) = 0$
4. $45x^2 + 40y^2 + 13z^2 + 36yz - 24zx + 12xy - 42x - 280y - 126z + 294 = 0$
5. $5x^2 + 8y^2 + 5z^2 - 4xy - 4yz - 8zx + 22x - 16y - 14z - 10 = 0$
6. $x^2 + y^2 + z^2 + xy + yz - zx = 9$

Chapter -4

Solution of Differential Equations in Series and Special Functions

- Initial Value Problems
- Linear Dependence and Independence
- Ordinary and Singular Points
- Solution of Differential Equation by Power Series Method
- Legendre's Equations
- Legendre's Functions
- Recurrence Relations for Legendre's Functions
- Bessel's Equation
- Bessel's Function
- Recurrence Relations for Bessel's Functions

Chapter-4

Solution of Differential Equations in Series and Special Functions

The solution of differential equation can also be found in the form of an infinite series and every solution in infinite series have its own region of convergence. The series solution of certain differential equation gives rise to special functions such as Bessel' functions, Lagurre's Polynomial, Legendre functions, Hermite's Polynomial. These special functions have many applications in engineering. In this chapter we discuss the procedures for obtaining the solution in series of a linear differential equations of first and second order.

To obtain the solution in series of a differential equation, we consider the power series is an infinite series of the form.

$$y = \sum_{m=0}^{\infty} c_m x^m = c_o + c_1 x + c_2 x^2 + c_3 x^3 + \dots\dots\dots\dots$$

Also, we know that the solution of first order differential equation contains one arbitrary constant whereas two arbitrary constants is contained in the solution of second order linear differential equation.

In the solution of first order differential equation, the solution in series must be in terms of only one constant.

In the second order linear differential equation the solution in series must be in terms of two arbitrary constants.

Initial Value Problems

Any differential equation together with an initial condition is called initial value problems. As we have known that solution of n^{th} order differential equation contains n arbitrary numbers of constants. The n arbitrary number of constants will be determined by given n initial conditions.

The general solution of second order linear differential equation

$$\frac{d^2y}{dx} + P_1\frac{dy}{dx} + P_2 y = Q \text{ is} \qquad y = c_1 f_1(x) + c_2 f_2(x)$$

Where c_1 and c_2 be two arbitrary constants which will be determined by given two initial conditions $y(x_0) = k_0$, $y'(x_0) = k_1$ in which c_1 and c_2 have definite value. After substituting the value of c_1 and c_2 to the general solution we get particular solution of given differential equation.

Linear Dependence and Independence

The functions $y_1(x), y_2(x)$ ....,$y_n(x)$ on the interval I are said to be *Linearly Dependent* if there exist scalars $k_1, k_2, \ldots, k_n$ such that

$$k_1 y_1(x) + k_2 y_2(x) + \ldots + k_n y_n(x) = 0$$

with at least one $k_i \neq 0$ for $i = 1, 2, \ldots, n$.

The functions $y_1(x), y_2(x), \ldots y_n(x)$ on an interval I are said to be Linearly *Independent* if there exist scalars $k_1, k_2 \ldots k_n$ such that

$$k_1 y_1(x) + k_2 y_2(x) + \ldots + k_n y_n(x) = 0$$

with $k_1 = 0, k_2 = 0, \ldots, k_n = 0$.

Furthermore, the following criterion of linear independence and dependence of solution of second order differential equation will be helpful. Suppose $y = y_1(x)$ and $y = y_2(x)$ be two solutions of second order differential equation, then the determinant $\begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix}$ is called *Wronskin determinant* or briefly *Wronskian* and denoted by

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix}.$$

Wronskian extends to the n^{th} order. It uses the Wronskian W of n solutions defined as the n^{th} order determinant and denoted by

$$W(y_1, y_2, \ldots y_n) = \begin{vmatrix} y_1 & y_2 & \ldots y_n \\ y'_1 & y'_2 & \ldots y'_n \\ \vdots & \vdots & \vdots \\ y_1^{n-1} & y_2^{n-1} & \ldots y_n^{n-1} \end{vmatrix}$$

Theorem

If $P_1, P_2, \ldots P_n$ are continuous on an open interval I, then the solutions y_1, $y_2, \ldots y_n$ of the equation $\frac{d^2y}{dx^2} + P_1\frac{d^{n-1}y}{dx^{n-1}} + P_2\frac{d^{n-2}y}{dx^{n-2}} + \ldots P_n y = 0$ are linearly dependent in I if and only if the Wronskian $W(y_1, y_2, \ldots y_n) = 0$ for some x_0 on I. If there is an $x = x_1$ in I at which $W(y_1, y_2, \ldots y_n) \neq 0$, then y_1, y_2 ....y_n are linearly independent on I

Proof

If $y_1, y_2, \ldots y_n$ are linearly dependent solutions of the given equation, then there exist constants $k_1, k_2, \ldots k_n$ not all zero such that

$$k_1y_1 + k_2y_2 + \ldots + k_ny_n = 0$$

Differentiating with respect to x at n - 1 times, we get

$$k_1 y'_1 + k_2y'_2 + \ldots + k_ny'_n = 0,$$

$$k_1 y''_1 + k_2 y''_2 + \ldots + k_ny''_n = 0,$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$k_n y_1^{n-1} + k_2y_2^{n-1} + \ldots + k_ny_n^{n-1} = 0$$

Eliminating $k_1, k_2, \ldots, k_n$ from these, we get

$$\begin{vmatrix} y_1 & y_2 & \ldots y_n \\ y'_1 & y'_2 & \ldots y'_n \\ \vdots & \vdots & \vdots \\ y_1^{n-1} & y_2^{n-1} & \ldots y_n^{n-1} \end{vmatrix} = 0$$

$\therefore$ $W(y_1, y_2, \ldots y_n) = 0$

Conversely, suppose $W(y_1, y_2, \ldots, y_n) = 0$ for some $x = x_0$ in I, we have to show that $y_1, y_2 \ldots y_n$ are linearly dependent.

i. e. $k_1 y_1 + k_2y_2 + \ldots + k_ny_n = 0$ with $k_1, k_2 \ldots k_n$ are not all zeros.

Differentiating it at n - 1 times, we get

$$k_1y'_1 + k_2y'_2 + \ldots + k_ny'_n = 0$$
$$k_1y''_1 + k_2y''_2 + \ldots + k_ny''_n = 0$$
$$\vdots \qquad \vdots \qquad \vdots$$
$$k_1y_1^{n-1} + k_2y_2^{n-1} + \ldots + k_ny_n^{n-1} = 0.$$

Since $\begin{vmatrix} y_1 & y_2 & \ldots y_n \\ y'_1 & y'_2 & \ldots y'_n \\ \vdots & \vdots & \vdots \\ y_1^{n-1} & y_2^{n-1} & \ldots y_n^{n-1} \end{vmatrix} = 0$ and the system has a solution in which $k_1, k_2 \ldots k_n$ are not all zeros.

Therefore, $y_1, y_2 \ldots y_n$ are linearly dependent.

To show linearly independence, given that $W(y_1, y_2, \ldots y_n) \neq 0$

$$\text{or}\quad \begin{vmatrix} y_1 & y_2 & \cdots\cdots & y_n \\ y'_1 & y'_2 & \cdots\cdots & y'_n \\ \vdots & \vdots & & \vdots \\ y_1^{n-1} & y_2^{n-1} & \cdots\cdots & y_n^{n-1} \end{vmatrix} \neq 0$$

Now $k_1y_1 + k_2y_2 + \dots\dots + k_ny_n = 0$. Differentiating it n - 1 times, we obtain the system of linear equations

$$k_1y'_1 + k_2y'_2 + \dots\dots + k_ny'_n = 0$$
$$k_1y_1'' + k_2y_2'' + \dots\dots + k_ny_n'' = 0$$
$$\vdots \qquad \vdots \qquad \vdots$$
$$k_1y_1^{n-1} + k_2y_2^{n-1} + \dots\dots + k_ny_n^{n-1} = 0.$$

Since $W(y_1, y_2, \dots y_n) \neq 0$,

$$\Rightarrow \quad k_1 = 0, k_2 = 0, \dots\dots, k_n = 0$$

This shows that $y_1, y_2, \dots\dots y_n$ are linearly independent on I.

Worked Out Examples

Ex:1. Solve $y'' - 4y' + 4y = 0$ given that $y(0) = 0$, $y'(0) = -3$.

Solution:

Here the differential equation is

$$y'' - 4y' + 4y = 0$$
$$\text{or} \quad (D^2 - 4D + 4)y = 0$$

This is linear differential equation of second order with constant coefficient so its auxiliary equation is

$$m^2 - 4m + 4 = 0$$
$$\therefore \quad m = 2, 2$$

So its general solution is

$$y = (c_1x + c_2)e^{2x} \quad \dots\dots(1)$$

Using the initial condition $y(0) = 0$ in (1), we get

$$0 = 0 + c_2 \qquad \therefore c_2 = 0$$

Differentiating (1) with respect to x, we get

$$y' = 2(c_1x + c_2)e^{2x} + c_1e^{2x} \quad \dots\dots(2)$$

Using the initial condition $y'(0) = -3$ in (2), we get

$$-3 = 2c_2 + c_1 \qquad \therefore c_1 = -3$$

Putting the value of c_1 and c_2 in (1), we get

$y = -3xe^{2x}$ is the required solution.

Ex: 2. Solve the initial value problem $y'' + 6y' + 9y = 0$ given that $y(0) = -4$, $y'(0) = 14$

Solution:

Here the differential eqaution is

$$y'' + 6y' + 9y = 0$$
$$\text{or} \quad (D^2 + 6D + 9)y = 0$$

Its auxiliary equation is

$$m^2 + 6m + 9 = 0$$
$$\therefore \quad m = -3, -3$$

So its general solution is

$$y = (c_1x + c_2)e^{-3x} \quad \dots(1)$$

Using the initial condition $y(0) = -4$ in (1), we get

$$y'(0) = 14$$

Differentiating (1), we get

$$y' = -3(c_1x + c_2)e^{-3x} + c_1e^{-3x}$$

Form (1), $\quad -4 = c_2$

Form (2), $\quad 14 = -3c_2 + c_1$

$$\therefore \quad c_1 = 2.$$

Hence the equation (1) becomes $y = (2x - 4)e^{-3x}$ is the solution.

Ex: 3. Solve $y'' - 16y = 0$, $y(0) = 1$, $y'(0) = 20$

Solution:

Here, the differential equation is

$$y'' - 16y = 0,$$
$$\text{or} \quad (D^2 - 16)y = 0.$$

Its auxiliary equation is

$$m^2 - 16 = 0$$
$$\therefore \quad m = \pm 4$$

So its general solution is

$$y = c_1e^{4x} + c_2e^{-4x}. \quad \dots\dots(1)$$

Using the initial condition $y(0) = 1$ in (1), we get

$$1 = c_1 + c_2 \quad \dots(2)$$

Differentiating (1) with respect to x, we get

$$y'(x) = 4c_1e^{4x} - 4c_2e^{-4x}$$

Using the initial condition $y'(0) = 20$ in (2), we get

$$20 = 4c_1 - 4c_2,$$

or $\quad 5 = c_1 - c_2.$ ....(3)

Solving the equations (2) and (3), we get

$$c_1 = 3, c_2 = -2$$

Substituting the value of c_1 and c_2 in (1), we get

$$y = 3e^{4x} - 2e^{-4x}.$$

Ex: 4. Solve $y'' + y' - 2y = -6\sin 2x - 18\cos 2x$, $y(0) = 0$, $y'(0) = 1$

2067 Chaitra B.E.

Solution:

Here, the differential equation is

$$y'' + y' - 2y = -6\sin 2x - 18\cos 2x,$$

or $\quad (D^2 + D - 2)y = -6\sin 2x - 18\cos 2x.$

Its auxiliary equation is

$$m^2 + m - 2 = 0,$$

or $\quad (m + 2)(m - 1) = 0$

$\therefore \quad m = -2, 1.$

So $\quad$ C. F. $= c_1e^{-2x} + c_2e^{x}.$

$$\text{P. I.} = \frac{1}{D^2 + D - 2}(-6\sin 2x - 18\cos 2x)$$

$$= -\frac{6}{D^2 + D - 2}(\sin 2x) - \frac{18}{D^2 + D - 2}(\cos 2x).$$

Putting $D^2 = -2^2 = -4$, we get

$$\text{P. I.} = -\frac{6}{-4 + D - 2}(\sin 2x) - \frac{18}{-4 + D - 2}(\cos 2x)$$

$$= -\frac{6}{D - 6}(\sin 2x) - \frac{18}{D - 6}(\cos 2x)$$

$$= -\frac{6(D + 6)}{D^2 - 36}(\sin 2x) - \frac{18(D + 6)}{D^2 - 36}(\cos 2x).$$

Putting $D^2 = -2^2 = -4$, we get

$$\text{P. I.} = -\frac{6(D + 6)}{-40}(\sin 2x) - \frac{18(D + 6)}{-40}(\cos 2x)$$

$$= \frac{3}{20}(2\cos 2x + 6\sin 2x) + \frac{9}{20}(6\cos 2x - 2\sin 2x)$$

$$= 3\cos 2x$$

So its general solution is

$$y = c_1e^{-2x} + c_2e^{x} + 3\cos 2x \quad ....(1)$$

Using the initial condition $y(0) = 0$ in (1), we get

$$0 = c_1 + c_2 + 3 \quad ....(2)$$

Differentiating (1) with respect to x, we get

$$y'(x) = -2c_1e^{-2x} + c_2e^{x} - 6\sin 2x \quad ....(3)$$

Using the initial condition $y'(0) = 0$ in (3), we get

$$0 = -2c_1 + c_2 \quad ....(4)$$

Solving the equations (2) and (4), we get

$$c_1 = -1, c_2 = 2.$$

Substituting the value of c_1 and c_2 in (1), we get

$$y = c_1e^{-2x} + c_2e^{x} + 3\cos 2x.$$

Ex.5. Test whether the solutions of $y''' - 2y'' - y' + 2y = 0$ are linearly independent or dependent

2071 Bhadra, B. E.

Solution:

Here the differential equation is

$$y''' - 2y'' - y' + 2y = 0$$

or $\quad (D^3 - 2D^2 - D + 2)y = 0$

Its auxiliary equation is

$$m^3 - 2m^2 - m + 2 = 0$$

or $\quad m^2(m - 2) - 1(m - 2) = 0$

or $\quad (m^2 - 1)(m - 2) = 0$

$\therefore \quad m = 2, -1, 1$

So the solutions of the equation are

$$y_1 = e^{2x},\ y_2 = e^{-x},\ y_3 = e^{x}$$

Now

$$W(y_1, y_2, y_3) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y'_1 & y'_2 & y'_3 \\ y''_1 & y''_2 & y''_3 \end{vmatrix} = \begin{vmatrix} e^{2x} & e^{-x} & e^{x} \\ 2e^{2x} & -e^{-x} & e^{x} \\ 4e^{2x} & e^{-x} & e^{x} \end{vmatrix}$$

$$= e^{2x}e^{-x}e^{x}\begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ 4 & 1 & 1 \end{vmatrix} = e^{2x}\,10 \neq 0$$

Hence e^{2x}, e^{-x}, e^{x} are linearly independent.

Ex.7. Show that the solutions of $x^2y''' - 3xy'' + 3y' = 0$, $x > 0$ are linearly independent.

2060 Chaitra, B.E.

Solution:

Here the differential equation is

$$x^2y''' - 3xy'' + 3y' = 0, x > 0$$

or $$(x^3D^3 - 3x^2D^2 + 3xD)y = 0$$

This is homogenous so put $x = e^z$, $z = \log x$, $xD = \delta$,

$x^2D^2 = \delta(\delta - 1)$ $x^3D^3 = \delta(\delta - 1)(\delta - 2)$ in (1), we get

$$[\delta(\delta - 1)(\delta - 2) - 3\delta(\delta - 1) + 3\delta]\, y = 0$$

or $$(\delta^3 - 6\delta^2 + 8\,\delta)\, y = 0$$

Its auxiliary equation is

$$m^3 - 6m^2 + 8m = 0$$

or $m(m^2 - 6m + 8) = 0$

$\therefore$ $m = 0, 2, 4$

Its general solution is

$$y = c_1 + c_2e^{2z} + c_3e^{4z} = c_1 + c_2x^2 + c_3x^4$$

So $y_1 = 1$, $y_2 = x^2$ and $y_3 = x^4$

Now

$$W(y_1, y_2, y_3) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} 1 & x^2 & x^4 \\ 0 & 2x & 4x^3 \\ 0 & 2 & 12x^2 \end{vmatrix} = 16x^3$$

Since $x > 0$,

$\therefore$ $W(y_1, y_2, y_3) \neq 0$

It shows that $1, x^2, x^4$ are linearly independent.

Exercise - 16

Solve the following initial value problems

1. $y'' + y' - 2y = 0, y(0) = 4, y'(0) = -5$ 2065, Chaitra B.E.
2. $y'' - y = 0, y(0) = 5, y'(0) = -3$
3. $y'' + 2y' + 5y = 0, y(0) = 1, y'(0) = 5$ 2069, Bhadra B.E.
4. $(D^2 + 4D + 5)y = 0, y(0) = 1, y'(0) = -3$
5. $y'' - 4y' + 3y = 10e^{-2x}, y(0) = 1, y'(0) = 3$ 2068, Bhadra B.E.
6. $y'' - 3y' + 2y = 4x + e^{3x}$, $y(0) = 1$, $y'(0) = -1$

7. $\frac{d^2x}{dt^2} + \mu x = 0, \mu > 0$, given that $x = 1$ and $\frac{dx}{dt} = 0$ when $t = \frac{\pi}{2\sqrt{\mu}}$
8. $(D^2 - 4D + 5)y = 0$ given that $y = 1$ and $\frac{dy}{dx} = 2$ when $x = 0$
9. Show that the solution of $y''' + 3y'' + 3y' + y = 0, x > 0$ are linearly independent
10. Show that the solution of $(D^4 + 4D^3 + 6D^2 + 4D + 1)y = 0$ $x > 0$, are linearly independent
11. Show that the solution of $(D^3 + D^2 + D + 1)y = 0, x > 0$, are linearly independent
12. Test whether the solution of $(D^2 - 3D + 2)\, y = 0$ are linearly dependent or independent
13. Verify that $D^4y = 0, x > 0$ have linearly independent solutions
14. Verifying that $D^3y = 0, x > 0$, have linearly independent solutions
15. Show that the solutions of $(D^4 + 5D^2 + 4)y = 0$, $x > 0$ are linearly independent
16. Show that $y_1 = \cos x$, $y_2 = \sin x$ are independent solutions of $y'' + y = 0$

Answers

1. $y = e^x + 3e^{-2x}$
2. $y = 4e^x + e^{-x}$
3. $y = e^{-x}(\cos 2x + 3\sin 2x)$
4. $y = -e^{-2x}\sin x + e^{-2x}\cos x$
5. $y = \frac{[illegible]}{3}e^x\, [illegible]\, e^{3x} + \frac{2}{3}e^{-2x}$
6. $y = 2x + 3 + \frac{1}{2}(e^{3x} - e^x) - 2e^{2x}$
7. $x = \sin\sqrt{\mu}\, t$
8. $y = e^{2x}\cos x$

12. Linearly independent

4.1. Solution of Differential Equation by Power Series Method

The idea of the power series method for solving differential equation

$$P_0\frac{d^2y}{dx^2} + P_1\frac{dy}{dx} + P_2y = 0, \qquad \text{......(1)}$$

where P_0, P_1, P_2 are functions of x and $P_0 \neq 0$, based on mathematical justification as follows:

(i) Assume its solution to be the form

$$y = c_0 + c_1x + c_2x^2 + c_3x^3 + \text{..........} + c_nx^n + \text{............} \qquad \text{...(2)}$$

(ii) Calculate $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ from (2) and substitute the values of

y, $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ in (1).

(iii) Equating the coefficients of various powers of x with zero gives the value of c_2, c_3, c_4........ in terms of c_0, c_1.

(iv) Substituting the values of c_2, c_3, c_4,..........., in the equation (2), we get required solution in series of the differential equation (1).

4.1.1 Ordinary and Singular Points

A point x = a is called an *ordinary point* if $P_0(x) \neq 0$ at x = a

If $P_0(x) = 0$ at x = a , then x = a is *singular point*

When x = 0 is a singular point of (1) then its solution can be expressed as in

the form $y = x^r \sum_{m=0}^{\infty} c_o x^m = \sum_{m=0}^{\infty} c_o x^{m+r}$

When x = a is an ordinary point of (1) ,then its solution can be expressed as in the form

$$y = c_0 + c_1(x - a) + c_2(x - a)^2 + c_3(x - a)^3 + \text{......}$$

Worked Out Examples

Ex:1. Solve y' = y in series

Solution:

Here, the equation is $\frac{dy}{dx} = y$ ....(1)

Assume that the general solution of (1) is

$$y = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + \ldots \quad \ldots(2)$$

where c_0, c_1, c_2 .... are constants.

Differentiating, (2), we get

$$\frac{dy}{dx} = c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + \ldots\ldots\ldots$$

Putting the value of y and $\frac{dy}{dx}$ to the equation (1), we get

$$(c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + \ldots\ldots\ldots) - (c_0 + c_1x + c_2x^2 + c_3x^3 + \ldots\ldots\ldots) = 0$$

or $c_1 - c_0 + (2c_2 - c_1)x + (3c_3 - c_2)x^2 + (4c_4 - c_3)x^3 + \ldots = 0.$

Equating the coefficient of various powers of x to zero, we get

$c_1 - c_0 = 0$ i. e. $c_1 = c_0$

$2c_2 - c_1 = 0$ i. e. $c_2 = \frac{c_0}{2}$

$3c_3 - c_2 = 0$ i. e $c_3 = \frac{c_0}{6}$

$4c_4 - c_3 = 0$ i. e. $c_4 = \frac{c_0}{24}$ and so on.

Putting these in equation (2), we get

$$y = c_0 + \frac{c_0x}{1!} + \frac{c_0}{2!}x^2 + \frac{c_0}{3!}x^3 + \frac{c_0}{4!}x^4 + \ldots\ldots\ldots$$

$$= c_0\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots\right)$$ is the required solution in series.

Ex: 2. Solve (1 - x) y'= y by the power series method

Solution:

Here the differential equation is

$$(1 - x)\, y' = y \quad \ldots(1)$$

Assume that the general solution in series be

$$y = \sum_{m=0}^{\infty} c_mx^m = c_0 + c_1 + c_2x^2 + \ldots\ldots\ldots + c_nx^n + \ldots\ldots \quad \ldots(2)$$

Differentiating (2) with respect to x, we get

$$y' = \sum_{m=1}^{\infty} mc_mx^{m-1} = c_1 + 2c_2x + 3c_2x^2 + \ldots\ldots\ldots + nc_nx^{n-1} + \ldots\ldots$$

Substituting these values of y and y' in the given differential equation (1), we get

$$(1 - x)(c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + \ldots\ldots\ldots + nc_nx^{n-1} + \ldots) - (c_0 + c_1x + c_2x^2 + c_3x^3 + \ldots\ldots\ldots + c_nx^n + \ldots) = 0$$

or $(c_1 - c_0) + (2c_2 - 2c_1)x + (3c_3 - 3c_2)x^2 + (4c_4 - 4c_3)x^3 + \ldots = 0$

Equating the coefficient of various powers of x to zero, we get

$c_1 - c_0 = 0$ i.e. $c_1 = c_0$

$2c_2 - 2c_1 = 0$ i.e. $c_2 = c_1 = c_0$

$3c_3 - 3c_2 = 0$ i.e. $c_3 = c_0$ and so on.

Substituting these values in (2), we get

$$y = c_0(1 + x + x^2 + x^3 + x^4 + \ldots\ldots) = c_0(1 - x)^{-1}$$

$= \frac{c_0}{1 - x}$ is the required solution.

Ex: 3. Solve $(x + 1)\, y' = 3y$ by the power series method

Solution:

Here the differential equation is

$$(x + 1)y' = 3y \quad \ldots\ldots(1)$$

Assume that the general solution of (1) be

$$y = c_0 + c_1x + c_2x^2 + c_3x^3 + \ldots\ldots + c_nx^n + \ldots\ldots\ldots \quad \ldots(2)$$

Differentialting (2), we get

$$y' = c_1 + 2c_2x + 3c_3x^2 + \ldots\ldots\ldots + nc_nx^{n-1} + \ldots\ldots\ldots$$

Substituting these values of y, $\frac{dy}{dx}$ in (1), we get

$$(x + 1)(c_1 + 2c_2x + 3c_3x^2 + \ldots\ldots + nc_nx^{n-1} + \ldots\ldots\ldots) - 3(c_0 + c_1x + c_2x^2 + c_3x^3 + \ldots\ldots\ldots + c_nx^n + \ldots\ldots) = 0$$

or $(c_1 - 3c_0) + (2c_2 - 2c_1)x + (3c_3 - c_2)x^2 + \ldots\ldots\ldots = 0$

Equating to zero the coefficient of various powers of x, we get

$c_1 - 3c_0 = 0$ i.e. $c_1 = 3c_0$

$2c_2 - 2c_1 = 0$ i.e. $c_2 = c_1 = 3c_0$

$3c_3 - c_2 = 0$ i.e. $c_3 = \frac{c_2}{3} = c_0$ and so on.

Putting these in equation (2), we get

$$y = c_0(1 + 3x + 3x^2 + x^3 + \ldots\ldots)$$

Ex: 4. Solve $y' + 2xy = 0$ by the power series method [2069 Bhadra]

Solution:

Here the differential equation is

$$y' + 2xy = 0$$

Suppose the general solution of (1) be

$$y = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + \ldots + c_nx^n + \ldots$$

Differentiating (2) with respect to x, we get

$$y' = c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 \ldots + nc_nx^{n-1} + \ldots$$

Substituting these in equation (1), we get

$$(c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + \ldots + nc_nx^n + \ldots)$$
$$+ 2x(c_0 + c_1x + c_2x^2 + \ldots + c_nx^n + \ldots) = 0$$

Equating the coefficient of the various powers of x to zero, we get

$c_1 = 0$

$2c_0 + 2c_2 = 0$ i.e. $c_2 = -c_0$

$3c_3 + 2c_1 = 0$ i.e. $c_3 = 0$

$4c_4 + 2c_2 = 0$ i.e. $c_4 = -\frac{c_2}{2} = \frac{c_0}{2}$

$5c_5 + 2c_3 = 0$ i.e. $c_5 = 0$

$6c_6 + 2c_4$ i.e. $c_6 = -\frac{1}{3}c_4 = -\frac{c_0}{6}$

$7c_7 + 2c_5$ i.e. $c_7 = 0$ and so on.

With these coefficients (2) becomes

$$y = c_0\left(1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \ldots\right) = c_0e^{-x^2}$$ is the required solution.

Ex: 5. Solve $y'' - y = 0$ by the power series method [2072 Magh]

Solution:

Here the differential equation is

$$y'' - y = 0.$$

Suppose its general solution be

$$y = c_0 + c_1x + c_2x^2 + c_3x^3 + \ldots nc_nx^n + \ldots$$

Differentiating (2) with respect to x, we get

$$y' = c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + 5c_5x^4 + \ldots + nc_nx^{n-1}$$

and $$y'' = 2c_2 + 3.2c_3x + 4.3c_4x^2 + 5.4c_5x^3 + \ldots + n(n-1)x^{n-2}$$

Putting these to (1) becomes

$$(2c_2 + 3.2c_3x + 4.3c_4x^2 + 5.4c_5x^3 + \ldots + n(n-1)x^{n-2} + \ldots)$$
$$-(c_0 + c_1x + c_2x^2 + c_4x^4 + \ldots + c_nx^n + \ldots) = 0$$

Equating the coefficient of various powers of x to zero, we get

$2c_2 - c_0 = 0$ i.e. $c_2 = \frac{c_0}{2!}$

$3.2c_3 - c_1 = 0$ i.e. $c_3 = \frac{c_1}{3!}$

$4.3c_4 - c_2 = 0$ i.e. $c_4 = \frac{c_0}{4!}$

$5.4c_5 - c_3 = 0$ i.e. $c_5 = \frac{c_1}{5!}$

$6.5c_6 - c_4 = 0$ i.e. $c_6 = \frac{c_0}{6!}$ and so on.

Putting these to (2), we get

$$y = c_0 + c_1x + \frac{c_0}{2!}x^2 + \frac{c_1}{3!}x^3 + \frac{c_1}{4!}x^4 + \frac{c_1}{5!}x^5 + \frac{c_0}{6!}x^6 + \ldots$$
$$= c_0\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots\right) + c_1\left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \ldots +\right)$$

is the required solution.

Ex: 6 Solve $y'' - 4xy' + (4x^2 - 2)y = 0$ by the power series method

[2068/070/071/072 Aswin, B. E.]

Solution:

Here the differential equation is

$$y'' - 4xy' + (4x^2 - 2)y = 0 \qquad \ldots(1)$$

Suppose the general solution of (1) be

$$y = c_0 + c_1x + c_2x^2 + c_3x^3 + \ldots + c_nx^n + \ldots \qquad \ldots(2)$$

Differentiating (2) with respect to x, we get

$$y' = c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + \ldots + nc_nx^{n-1} + \ldots$$

and $$y'' = 2c_2 + 3.2c_3x + 4.3c_4x^2 + \ldots + n(n-1)c_nx^{n-2} + \ldots$$

Substituting these to (1), we get

$$(2c_2 + 3.2c_3x + 4.3c_4x^2 + 5.4c_5x^3 + \ldots + n(n-1)c_nx^{n-2} + \ldots)$$
$$- 4x(c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + 5c_5x^4 + \ldots + nc_nx^{n-1} + \ldots)$$
$$+ (4x^2 - 2)(c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + \ldots + c_nx^n + \ldots) = 0$$

or $$(2c_2 - 2c_0) + (6c_3 - 6c_1)x + (12c_4 - 10c_2 + 4c_0)x^2$$
$$+ (20c_5 - 14c_3 + 4c_1)x^3 + (30c_6 - 18c_4 + 4c_2)x^4 + \ldots = 0$$

Equating the coefficient of various powers of x to zero, we get

$c_2 = c_0$

$6c_3 = 6c_1$

$12c_4 - 10c_2 + 4c_0 = 0$ i.e. $c_3 = c_1$

i.e $c_4 = \frac{c_0}{2}$

$20c_5 - 14c_3 + 4c_1 = 0$ i.e. $c_5 = \frac{c_1}{2}$

$30c_6 - 18c_4 + 4c_2 = 0$ i.e. $c_6 = \frac{5c_0}{30} = \frac{c_0}{6}$ and so on.

Putting these to (2), we get

$$y = c_0 + c_1 x + c_0 x^2 + c_1 x^3 + \frac{c_0}{2}x^4 + \frac{c_1}{2}x^5 + \frac{c_0}{6}x^6 + \ldots\ldots$$

$$= c_0\left(1 + x^2 + \frac{x^4}{2} + \frac{x^6}{6} + \ldots\ldots\ldots\right) + c_1\left(x + x^3 + \frac{x^5}{2} + \ldots\right)$$

is the general solution.

4.1.2. Solution of Differential Equation by Frobenius Method

We have assumed that by the power series method the series solution of differential equation started with a constant c_0. Frobenus method is suitable for solving linear differential equation with variable coefficient in which the series solution must be taken as its first term $c_0 x^r$.

Here the differential equation is

$$P_0\frac{d^2y}{dx^2} + P_1\frac{dy}{dx} + P_2 y = 0 \quad \ldots(1)$$

If x = 0 is a *regular singularity*, then it has following procedure to solve the differential equation in series

i. Let $y = x^r \sum_{m=0}^{\infty} c_m x^m = c_0 x^r + c_1 x^{r+1} + c_2 x^{r+2} + \ldots\ldots\ldots\ldots$ be the general solution of (1).

ii. Substitute $y, \frac{dy}{dx}, \frac{d^2y}{dx^2}$ in (1).

iii. Equating the coefficient of lowest degree term in x to zero, gives quadratic equation in r known as the *indicial equation*.

iv. Indicial equation gives two roots which may be (a) distinct and not differing by integer (b) equal (c) distinct and differing by an integer.

Case I:

When r_1 and r_2 be distinct and not differing by an integer, then the

complete solution is $y = a_1(y)_{at\ r_1} + a_2(y)_{at\ r_2}$

Case II:

When r_1 and r_2 are equal, the complete solution is

$$y = a_1(y)_{at\ r_1} + a_2\left(\frac{\partial y}{\partial r}\right)_{at\ r_1}$$

Case III:

When r_1 and r_2 are distinct and differ by an integer ($r_1 < r_2$) and if the some of the coefficients of y series become infinite at $r = r_1$, then we modify to the form of y by replacing c_0 by $b_0(r - r_1)$ and get the complete solution as

$$y = c_1(y)_{at\ r_1} + c_2\left(\frac{\partial y}{\partial r}\right)_{at\ r_1}.$$

Worked Out Examples

Ex. 1: Solve the differential equation $y'' + \frac{1}{2x}y' + \frac{1}{4x}y = 0$

Solution:

The differential equation is ...(1)

$$4xy'' + 2y' + y = 0$$

Here x = 0 is regular singularity, so let the general solution of (1) be

$$y = x^r \sum_{m=0}^{\infty} c_m x^m = \sum_{m=0}^{\infty} c_m x^{m+r} \quad \ldots(2)$$

Differentiating (2), we get

$$y' = \sum_{m=0}^{\infty} c_m (m+r)\, x^{m+r-1}$$

$$y'' = \sum_{m=0}^{\infty} c_m (m+r)(m+r-1)\, x^{m+r-2}$$

Substituting these in (1), we get

$$4\sum_{m=0}^{\infty} c_m (m+r)(m+r-1)\, x^{m+r-1} + 2\sum_{m=0}^{\infty} c_m (m+r)\, x^{m+r-1}$$

$$+\sum_{m=0}^{\infty} c_m x^{m+r+1} = 0$$

or $\quad 4r(r-1)c_0x^{r-1} + 4(r+1)r\,c_1x^r + 4(r+2)(r+1)c_2x^{r+1} + \ldots\ldots$ ...(3)

$$+ 2r\,c_0\,x^{r-1} + 2(r+1)\,c_1x^r + 2(r+2)\,c_2x^{r+1} + \ldots\ldots$$

$$+ c_0x^r + c_1x^{r+1} + \ldots\ldots\ldots = 0$$

The lowest power of x is x^{r-1}. Equating its coefficient to zero, we get an indicial equation

$$4r(r-1) + 2r = 0$$

Thus $\quad 2r^2 - r = 0$ which gives $r = 0, \frac{1}{2}$.

Equating the coefficient of x^{r+s} in (3) to zero by taking $m + r - 1 = r + s$ i.e. $m = s + 1$ in the first two series and $m = s$ in last series, we set

$$4(s+r+1)(r+s)c_{s+1} + 2(s+r+1)c_{s+1} + c_s = 0$$

$\therefore \quad c_{s+1} = \dfrac{-c_s}{(2s+2r+2)(2s+2r+1)}$ where $s = 0, 1, 2, 3\ldots\ldots$

$$c_1 = \frac{-c_0}{2(r+1)(2r+1)}$$

$$c_2 = \frac{-c_1}{(2r+4)(2r+3)} = \frac{c_0}{(2r+2)(2r+4)(2r+1)(2r+3)}$$

$$c_3 = \frac{-c_2}{(2r+6)(2r+5)}$$

$$= \frac{-c_0}{(2r+2)(2r+4)(2r+6)(2r+1)(2r+3)(2r+5)} \text{ and so on.}$$

When $r = 0$

$$c_1 = -\frac{c_0}{2}, \qquad c_2 = \frac{c_0}{2.4.1.3}, \qquad c_3 = -\frac{c_0}{2.4.6.1.3.5} \text{ etc.}$$

Therefore, the solution is given by

$$y_1 = c_0\left(1 - \frac{1}{2!}x + \frac{1}{4!}x^2 - \frac{1}{6!}x^3 + \ldots\ldots\ldots\right).$$

When $r = \frac{1}{2}$

$$c_1 = \frac{-c_0}{3.2}, \qquad c_2 = \frac{c_0}{3.5.2.4}, \qquad c_3 = \frac{-c_0}{3.5.7.2.4.6} \text{ etc.}$$

Therefore, the solution is given by

$$y_2 = c_0\left(1 - \frac{1}{3!}x + \frac{1}{5!}x^2 - \frac{1}{7!}x^3 + \ldots\ldots\ldots\ldots\ldots\ldots\right)$$

Hence the complete solution is

$$y = a_1y_1 + a_2y_2$$

$$y = c_1\left(1 - \frac{1}{2!}x + \frac{1}{4!}x^2 - \frac{1}{6!}x^3 + \ldots\ldots\right) + c_2\left(1 - \frac{1}{3!}x + \frac{1}{5!}x^2 - \frac{1}{7!}x^3 + \ldots\ldots\right)$$

Where $c_1 = a_1c_0$, $c_2 = b_2c_0$.

Ex. 2. Solve in series the equation $x\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + xy = 0$

Solution:

The differential equation is

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} + xy = 0$$

Here $x = 0$ is singular regularity. So substituting

$$y = x^r \sum_{m=0}^{\infty} c_mx^m = \sum_{m=0}^{\infty} c_mx^{m+r} \qquad \ldots\ldots(2)$$

$$y' = \sum_{m=0}^{\infty} (m+r)\,c_m\,x^{m+r-1}$$

$y'' = \sum_{m=0}^{\infty} (m+r)(m+r-1)\,c_m\,x^{m+r-2}$ in the equation (1), we get

$$\sum_{m=0}^{\infty} (m+r)(m+r-1)\,c_m\,x^{m+r-1} + \sum_{m=0}^{\infty} (m+r)\,c_m\,x^{m+r-1}$$

$$+ \sum_{m=0}^{\infty} c_m\,x^{m+r+1} = 0. \qquad \ldots\ldots\ldots(3)$$

or $\quad r(r-1)c_0\,x^{r-1} + (r+1)r\,c_1x^r + (r+2)(r+1)\,c_2x^{r+1} + \ldots\ldots\ldots$

$$+ rc_0\,x^{r-1} + (r+1)\,c_1x^r + (r+2)\,c_2x^{r+1} + \ldots\ldots\ldots\ldots\ldots$$

$$+ c_0x^{r+1} + c_1x^{r+2} + \ldots\ldots\ldots = 0$$

The lowest powers of x is x^{r-1}. Equating its coefficient to zero, we get indicial equation $r(r-1)\,c_0 + rc_0 = 0$, $c_0 \neq 0$ which gives $r = 0, 0$ and equating the coefficient of x^r, $c_1 = 0$. By equating the sum of the coefficient of x^{r+s} in (3) to zero by taking $m + r - 1 = r + s$ i.e. $m = s + 1$ in first two series and $m + r + 1 = r + s$ i.e. $m = s - 1$ in last series, we get

$$(s+r+1)(s+r)\,c_{s+1} + (s+r+1)\,c_{s+1} + c_{s-1} = 0$$

$$c_{s+1} = \frac{-c_{s-1}}{(s+r+1)^2} \qquad (s = 1, 2, 3, \ldots\ldots)$$

$$c_2 = -\frac{-c_0}{(r+2)^2}, \qquad c_3 = \frac{-c_1}{(r+3)^2} = 0 \quad (\because c_1 = 0)$$

$$c_4 = \frac{-c_2}{(r+4)^2} = \frac{c_0}{(r+4)^2(r+2)^2}$$

Clearly $c_3 = 0,\ c_5 = 0,\ c_7 = 0,\ c_9 = 0$ ....

Therefore, the solution is given by

$$y = c_0x^r\left[1 - \frac{x^2}{(r+2)^2} + \frac{x^4}{(r+4)^2(r+2)^2} - \frac{x^6}{(r+6)^2(r+4)^2(r+2)^2} + \ldots\right]$$

Now putting r = 0, we obtain one series solution as

$$y_1 = c_0\left(1 - \frac{x^2}{2^2} + \frac{x^4}{2^2.4^2} - \frac{x^6}{2^2.4^2.6^2} + \ldots\ldots\ldots\right)$$ and second is given by $\left(\frac{\partial y}{\partial r}\right)_{at\ r=0}$.

$$\therefore\ y_2 = \left(\frac{\partial y}{\partial r}\right)_{at\ r=0}$$

$$= c_o x^r \log x\left[1 - \frac{x^2}{(r+2)^2} + \frac{x^4}{(r+4)^2(r+2)^2} - \frac{x^6}{(r+6)^2(r+4)^2(r+2)^2} \ldots\right]$$

$$+ c_0x^r\left[\frac{2x^2}{(r+2)^3} + \left\{-\frac{2}{(r+2)^3(r+4)^2} - \frac{2}{(r+2)^2(r+4)^3}\right\}x^4 + \ldots\right]$$

Putting r = 0, we get

$$y_2 = c_0\log x\left(1 - \frac{x^2}{2^2} + \frac{x^4}{2^2.4^2} - \frac{x^6}{2^2.4^2.6^2} + \ldots\right) + c_0\left(\frac{x^2}{2^2} - \frac{3x^4}{2^3.4^2} + \ldots\right)$$

Hence the complete solution is

$$y = a_1y_1 + a_2y_2$$

$$\text{or}\quad y = c_1\left[1 - \frac{x^2}{2^2} + \frac{x^4}{2^2.4^2} - \frac{x^6}{2^2 4^2 6^2} + \ldots\right]$$

$$+ c_2\left[\log x\left(1 - \frac{x^2}{2^2} + \frac{x^4}{2^2.4^2} - \frac{x^6}{2^2.4^2.6^2} + \ldots\right) + \left(\frac{x^2}{2^2} - \frac{3x^4}{2^3.4^2} + \ldots\right)\right]$$

where $c_1 = a_1 c_0 + a_2 c_0$.

Ex.3. Solve the equation $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - 4)y = 0$ in series form

Solution:

The general solution is

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2-4)y = 0$$

Here x = 0 is a singular point. So substituting

$$y = x^r\sum_{m=0}^{\infty} c_mx^m = \sum_{m=0}^{\infty} c_mx^{m+r} \qquad \ldots\ldots (2)$$

$$= c_0x^r + c_0x^{r+1} + c_2x^{r+2} + c_3x^{r+3} + \ldots$$

$$y' = \sum_{m=0}^{\infty}(m+r)c_mx^{m+r-1}$$

and $$y'' = \sum_{m=0}^{\infty}(m+r)(m+r-1)c_mx^{m+r-2}$$ in (1), we get

$$\sum_{m=0}^{\infty}(m+r)(m+r-1)c_mx^{m+r} + \sum_{m=0}^{\infty}(m+r)c_mx^{m+r} + \sum_{m=0}^{\infty}c_mx^{m+r+2} - 4\sum_{m=0}^{\infty}c_mx^{m+r} = 0$$

or $$\sum_{m=0}^{\infty}(m+r)(m+r-1)c_mx^{m+r} + \sum_{m=0}^{\infty}(m+r)c_mx^{m+r} + \sum_{m=0}^{\infty}c_mx^{m+r+2} - 4\sum_{m=0}^{\infty}c_mx^{m+r} = 0$$

or $$\sum[(m+r)^2 - 4]\, c_mx^{m+r} + \sum c_m x^{m+r+2} = 0$$

or $$(r^2-4)c_o x^r + [(r+1)^2 - 4]c_1x^{r+1} + [(r+2)^2 - 4]c_2x^{r+2} + \ldots + c_o x^{r+2} + c_1x^{r+3} + c_2x^{r+4} + \ldots\ldots = 0.$$

The lowest powers of x is x^r. Equating its coefficient to zero, we get

$$(r^2-4)c_o = 0$$

But $c_0 \neq 0\ \therefore\ r = \pm 2$.

The roots are differing by an integer. Equating to zero the sum of the coefficient of x^{s+r+2} by taking m = s + 2 in first series and m = s in last series, we obtain the recurrence relation

$$[(s+r+2)^2 - 4]c_{s+2} + c_s = 0$$

$$c_{s+2} = \frac{-c_s}{(s+r+2)^2 - 4} \qquad (s = 0, 1, 2, 3 \ldots)$$

$$c_2 = \frac{-c_0}{(r+2)^2-4}, \qquad c_3 = \frac{-c_1}{(r+3)^2-4}$$

$$c_4 = \frac{-c_2}{(r+4)^2-4} = \frac{-c_0}{\{(r+2)^2-4\}\{(r+4)^2-4\}} \text{ etc.}$$

Equating the coefficient of x^{r+1} to zero, we get

$$[(r+1)^2 - 4]\, c_1 = 0$$
$$(r^2 + 2r - 3)\, c_1 = 0$$

Clearly $r = \pm 2$ is not satisfied in $r^2 + 2r - 3 = 0$ $\therefore c_1 = 0$

Thus $c_3 = 0,\ c_5 = 0,\ c_7 = 0$ ....

Putting these values of coefficients in (2), we get

$$y = c_0 x^r \left[1 - \frac{x^2}{\{(r+2)^2 - 4\}} + \frac{x^4}{\{(r+4)^2 - 4\}\{(r+2)^2 - 4\}} - \ldots\ldots\right] \quad \ldots(3)$$

If we put $r = -2$ in this series, the coefficient of x^4 and onwards becomes infinite.

Hence put $c_0 = b_0 (r+2)$. Then

$$y = b_0 (r+2) x^r \left[1 - \frac{x^2}{\{(r+2)^2 - 4\}} + \frac{x^4}{\{(r+4)^2 - 4\}\{(r+2)^2 - 4\}} + \ldots\right]$$

$$= b_0 x^r \left[(r+2) - \frac{(r+2)x^2}{r(r+4)} + \frac{x^4}{r(r+4)(r+6)} - \frac{x^6}{r(r+4)^2(r+6)(r+8)} + \ldots\right]$$

Now putting $r = -2$ in this, we get one series solution as

$$y_1 = b_0 x^{-2}\left[-\frac{x^4}{2^2.4} + \frac{x^6}{2^3.4.6} - \frac{x^8}{2^3.4^2.6.8} + \ldots\ldots\right]$$

$$= b_0\left[-\frac{x^2}{16} + \frac{x^4}{12.16} - \frac{x^6}{12.16.32} + \ldots\ldots\right]$$

If we put $r = 2$ in the series (3), we get

$$y_2 = c_0 x^2\left[1 - \frac{x^2}{12} + \frac{x^4}{32.12} - \ldots\ldots\right] = c_0\left[x^2 - \frac{x^4}{12} + \frac{x^6}{32.12} - \ldots\ldots\right]$$

$$= -16\frac{c_0}{b_0}\, b_0\left[\frac{-x^2}{16} + \frac{x^4}{12.16} - \frac{x^6}{16.32.12} + \ldots\ldots\right] = \left(\frac{-16c_0}{b_0}\right) y_1$$

It shows that y_2 and y_1 are linearly dependent. Other solution is found by

$\left(\frac{\partial y}{\partial r}\right)_{at\ r=-2}$ i.e. $y_2 = \left(\frac{\partial y}{\partial r}\right)$

$$= b_0 x^r \log x\left[(r+2) - \frac{(r+2)}{r(r+4)}x^2 + \frac{x^4}{r(r+4)(r+6)} - \frac{x^6}{r(r+4)^2(r+6)(r+8)} + \ldots\right]$$

$$+ b_0 x^r\left[1 - \left\{\frac{1}{r(r+4)} - \frac{(r+2)}{r^2(r+4)} - \frac{(r+2)}{r(r+4)^2}\right\}x^2 + \ldots\ldots\right]$$

$$y_2 = \left(\frac{\partial y}{\partial r}\right)_{at\ r=-2} = b_0 x^{-2}\log x\left[-\frac{x^4}{16} + \frac{x^6}{12.16} - \frac{x^8}{16.32.12} + \ldots\ldots\right]$$

$$+ b_0 x^{-2}\left[1 + \frac{x^2}{2^2} + \frac{x^4}{2^2.4^2} + \ldots\ldots\right]$$

$$y_2 = b_0\left[\log x\left(\frac{-x^2}{16} + \frac{x^4}{12.16} - \frac{x^6}{12.16.32} + \ldots\right) + \left(\frac{1}{x^2} + \frac{1}{2^2} + \frac{x^2}{2^2.4^2} + \ldots\right)\right]$$

Hence the general solution is $y = a_1 y_1 + a_2 y_2$

$$y = a_1 b_0\left[-\frac{x^2}{16} + \frac{x^4}{12.16} - \frac{x^6}{12.16.32} + \ldots\ldots\right]$$

$$+ a_2 b_0\left[\log x\left\{\frac{-x^2}{16} + \frac{x^4}{12.16} - \frac{x^6}{12.16.32} + \ldots\right\} + \left\{\frac{1}{x^2} + \frac{1}{2^2} + \frac{x^2}{2^2.4^2} \ldots\right\}\right]$$

$$y = c_1 + c_2 \log x\left(\frac{-x^2}{16} + \frac{x^4}{12.16} - \frac{x^6}{12.16.32} + \ldots\ldots\right)$$

$$+ c_2\left(\frac{1}{x^2} + \frac{1}{2^2} + \frac{x^2}{2^2.4^2} + \ldots\ldots\right)$$ where $c_1 = a_1 b_0,\ c_2 = a_2 b_0$.

Exercise - 17

Solve the following differential equations by the power series method

1. $y' - y = 0$
2. $y' + y = 0$
3. $(x+1)y' + y = 0$
4. $y' - 3x^2 y = 0$
5. $(1 - x^4)y' = 4x^3 y$
6. $(x - 3)y' = xy$
7. $y' = 2xy$
8. $(x+1)y' - (2x+3)y = 0$
9. $y'' + 4y = 0$
10. $y'' - 3y' + 2y = 0$
11. $(1 - x^2)y'' - 2xy' + 2y = 0$
12. $y'' - 9y = 0$
13. $y'' - y = x$
14. $y'' + xy' + y = 0$ *2070 Magh, B. E.*
15. $y'' + x^2 y = 0$ *2061 Aswin, B. E.*

Solve the following differential equations by the Frobenius method

16. $x^2 y'' + 2x^2 y - 2y = 0$
17. $2xy'' + 6y' + y = 0$
18. $xy'' (3 + 2x)y' + 8y = 0$

Answers

1. $y = c_0 e^x$
2. $c_0 e^{-x}$
3. $c_0 (1 + x)^{-1}$
4. $y = c_0\left(\left(1 + \frac{x^3}{3!} + \frac{x^6}{2!} + \frac{x^9}{3!} + \ldots\right)\right.$
5. $y = c_0 (1 + x^4 + x^8 + \ldots)$

6. $y = c_o\left(1 - \frac{x^2}{6} - \frac{x^3}{27} + \ldots\ldots\right)$

7. $y = c_o\left(1 + x^2 + \frac{x^4}{2} + \frac{x^6}{3} + \ldots\ldots\right)$

8. $y = c_o\left(1 + 3x + 4x^2 + \frac{10x^3}{3} + 2x^4 + \ldots\ldots\right)$

9. $y = c_o\left(1 - 2x^2 + \frac{2x^4}{3} - \frac{4x^6}{45} + \ldots\right) + c_1\left(x - \frac{2x^3}{3} + \frac{2x^5}{15} - \ldots\ldots\right)$

10. $y = c_o + c_1x + \left(\frac{3c_1}{2} - c_o\right)x^2 + \left(\frac{7}{6} - c_o\right)x^3 + \ldots\ldots$

11. $y = c_o\left(1 - x^2 - \frac{x^4}{3} - \frac{x^6}{5} - \ldots\right) + c_1x$

12. $y = c_o\left(\frac{9x^2}{2!} - \frac{81x^4}{4!} + \frac{729x^6}{6!} + \ldots\right) + c_1\left(x + \frac{27x^3}{3!} + \frac{243x^5}{5!} + \ldots\right)$

13. $y = c_o\left(1 + \frac{x^2}{2} + \frac{x^4}{4!} + \ldots\right) + c_1\left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots\right)$

14. $y = c_o\left(1 - \frac{x^2}{2} + \frac{x^4}{2.4} - \frac{x^6}{2.4.6} + \ldots\right) + c_1\left(x - \frac{x^3}{3} + \frac{x^5}{3.5} - \frac{x^7}{3.5.7} + \ldots\right)$

15. $y = c_0\left(1 - \frac{x^4}{12} + \ldots\ldots\right) + c_1\left(x - \frac{x^5}{20} + \ldots\ldots\right)$

16. $y = c_0x^{-1}(1 + x) + x^2c_3\left(1 - x + \frac{3}{5}x^2 - \frac{4}{5}x^3 + \ldots\right)$

17. $y = c_0\left(1 - \frac{x}{2.3} + \frac{x^2}{2^3.\ 3.\ 4} - \frac{x^3}{2^4.\ 3^2.4.5} + \ldots\right)$

18. $y = c_0\left(1 - \frac{8}{3}x + \frac{2^2.4.5}{24}x^2 - \frac{2^3.6}{1.2.3^2}x^3 + \ldots\right)$

4. 2. Legendre's Equation

The second order differential equation of the form

$(1 - x^2)y'' - 2xy' + n(n + 1)y = 0$ is called the *Legendre's equation.*

4. 2.1. Solution of Legendre's Equation in Series

To find the general solution of the Legendre's differential equation [2070 Magh. B.E.]

The differential equation is

$$(1 - x^2)y'' - 2x\ y' + n(n + 1)y = 0. \quad \ldots(1)$$

Here. $P_0(x) = 1 - x^2 \neq 0$ when $x = 0$ so that $x = 0$ is an ordinary point.

Let the general solution of (1) be

$$y = c_0 + c_1x + c_2x^2 + \ldots\ldots = \sum_{m=0}^{\infty} c_mx^m \quad \ldots\ldots(2)$$

Differentiating (2) with respect to x, we get

$$y' = \sum_{m=1}^{\infty} m\,c_m x^{m-1}$$

$$y'' = \sum_{m=2}^{\infty} m(m-1)\,c_mx^{m-1}$$

Substituting these to (1), we get

$$(1-x^2)\sum_{m=2}^{\infty} m(m-1)\,c_mx^{m-2} - 2x\sum_{m=1}^{\infty} m\,c_mx^{m-1} + n(n+1)\sum_{m=0}^{\infty} c_mx^m = 0$$

or $$\sum_{m=2}^{\infty} m(m-1)\,c_mx^{m-2} - \sum_{m=2}^{\infty} m(m-1)\,c_mx^m$$

$$-2\sum_{m=1}^{\infty} m\,c_mx^m + n(n+1)\sum_{m=0}^{\infty} c_mx^m = 0$$

or $2.1c_2 + 3.2c_3x + 4.3c_4x^2 + \ldots + (s+2)(s+1)\,c_{s+2}x^s + \ldots$

$- 2.1c_2x^2 - \ldots\ldots - s(s-1)\ c_sx^s - \ldots\ldots$

$- 2.1c_1x - 2.2\,c_2x^2 - \ldots\ldots - 2sc_sx^s + \ldots\ldots\ldots$

$+ n(n+1)\,c_0 + n(n+1)c_1x + n(n+1)c_2x^2 + \ldots\ldots + n(n+1)\,c^sx^s + \ldots\ldots = 0$

Equating the coefficient of various powers of x to zero, we get

$$2c_2 + n(n+1)\,c_0 = 0,$$

$$6c_3 - 2c_1 + n(n+1)\,c_1 = 0.$$

In general

$$(s+2)(s+1)\,c_{s+2} - s(s-1)\,c_s - 2sc_s + n(n+1)\,c_s = 0$$

or $$(s+2)(s+1)\,c_{s+2} + (n-s)(n+s+1)\,c_s = 0$$

or $$c_{s+2} = -\frac{(n-s)(n+s+1)}{(s+2)(s+1)}\,c_s$$

This is called *recurrence relation*. It gives each coefficient in terms of arbitrary constants c_0 and c_1.

When s = 0, 1, 2, ...

$$c_2 = -\frac{n(n+1)}{2!}c_0,\quad c_3 = -\frac{(n-1)(n+2)}{3!}c_1$$

$$c_4 = -\frac{(n-2)(n+3)}{4.3}c_2 = \frac{(n-2)(n+3)n(n+1)}{4!}c_0$$

$$c_5 = -\frac{(n-3)(n+4)}{5.4}c_3 = \frac{(n-3)(n-1)(n+2)(n+4)}{5!}c_1$$

and so on.

Putting the values of c_2, c_3, c_4, c_5 etc. in (2), we get

$$y = c_0\left(1 - \frac{(n+1)}{2!}x^2 + \frac{(n-2)n(n+1)(n+3)}{4!}x^4 + \ldots\right)$$
$$+ c_1\left(x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!}x^5 + \ldots\right)$$

$\therefore$ $y = c_0y_1(x) + c_1y_2(x)$ is the required general solution of *Legendre's equation* where

$$y_1(x) = 1 - \frac{n(n+1)}{2!}x^2 + \frac{(n-2)(n+1)(n+3)}{4!}x^4 - \ldots$$

$$y_2(x) = x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!}x^5 - \ldots$$

4. 3. Legendre's Functions

Here the Legendre's differential equation is

$$(1-x^2)y'' - 2x\ y' + n(n+1)y = 0$$

The two general solutions of this equation in series are

$$y_1(x) = 1 - \frac{n(n+1)}{2!}x^2 + \frac{(n-2)(n+1)(n+3)}{4!}x^4 - \ldots \quad \ldots(1)$$

and $y_2(x) = x - \dfrac{(n-1)(n+2)}{3!}x^3 + \dfrac{(n-3)(n-1)(n+2)(n+4)}{5!}x^5 - \ldots$ ...(2)

If n is a positive *even integer*, then the series (1) terminates at the term in x^n and y_1 becomes a polynomial of degree n.

If n is a positive *odd integer*, then the series (2) becomes a polynomial of degree n.

If n is a *positive integer* and with the constant c_1 so chosen that

$$c_1 = \frac{1.3.5\ldots\ldots(2n-1)}{n!}$$, then the series (2) is called the *Legendre's function of the first kind* and denoted by $P_n(x)$ for different values of n such that $P_n(1) = 1$.

Again if n is a positive integer and with the constant c_0 so chosen that

$c_0 = \dfrac{n!}{1.3.5\ldots\ldots(2n-1)}$, then the series (2) is called the *Legendre's function of the second kind* and denoted by $Q_n(x)$.

4.3.1. Rodrigue's Formula

Theorem

If $P_n(x)$ is a polynomial function of Legendre's equation of degree n, then $P_n(x) = \dfrac{1}{2^n.\ n!}\dfrac{d^n}{dx^n}(x^2-1)^n$ 2067 Mangsir B. E.

This formula is known as Rodrigue's formula.

Proof

Let $v = (x^2-1)^n$ then $v_1 = \dfrac{dv}{dx} = n(x^2-1)^{n-1}.\ 2x$

or $(x^2-1)v_1 = 2nx(x^2-1)^n$

or $(x^2-1)v_1 = 2n\ x\ v$

or $(1-x^2)v_1 + 2n\ x\ v = 0$

Differentiating it (n +1) times by Leibnitz's theorem, we get

$$(1-x^2)v_{n+2} + (n+1)(-2x)v_{n+1} + \frac{(n+1).\ n}{2}(-2)\ v_n + 2n\ [\ x\ v_{n+1} + (n+1)\ v_n] = 0$$

or $(1-x^2)\dfrac{d^2(v_n)}{dx^2} - 2x\dfrac{d(v_n)}{dx} + n(n+1)\ v_n = 0$

This is Legendtre's differential equation and its solution is $P_n(x)$.

So $P_n(x) = c\ v_n = \dfrac{d^n}{dx^n}(x^2-1)^n$ ...(1)

$$= c\frac{d^n}{dx^n}[(x-1)^n(x+1)^n]$$

$$= c[(x-1)^n\frac{d^n}{dx^n}(x+1)^n + {}^nC_1.n(x-1)^n\frac{d^{n-1}}{dx^{n-1}}(x+1)^n + \ldots + (x+1)^n\frac{d^n}{dx^n}(x-1)^n]$$

$$= c[(x-1)^n\ n! + {}^nC_1.n(x-1)^n\frac{d^{n-1}}{dx^{n-1}}(x+1)^n + \ldots. + (x+1)^n\ n!$$

To determine the constants c, putting x =1, we get

$$P_n(1) = c\ [\ 0 + 0 + 0 + \ldots\ldots + 2^n\ n!]$$

or $1 = c\ 2^n\ n!$

$\therefore$ $c = \dfrac{1}{2^n\ n!}$

Substituting the values of c in (1), we get

$$P_n(x) = \frac{1}{2^n\, n!}\frac{d^n}{dx^n}(x^2-1)^n.$$

4.3.2. Legendre's Polynomials

Theorem

Prove that $P_n(x) = \sum_{r=0}^{N} \frac{(-1)^r (2n-2r)!}{2^n r!\,(n-r)!\,(n-2r)!} x^{n-2r}$

The Rodrigue' formula

$$P_n(x) = \frac{1}{2^n\, n!}\frac{d^n}{dx^n}(x^2-1)^n \quad ...(1)$$

Putting n = 0, 1, 2, 3, 4,... in (1), we get

$$P_0(x) = 1, \quad P_1(x) = \frac{1}{2}\frac{d}{dx}(x^2-1) = x$$

$$P_2(x) = \frac{1}{2^2(2)!}\frac{d^2}{dx^2}(x^2-1)^2 = \frac{1}{8}\frac{d^2}{dx^2}(x^4-2x^2+1) = \frac{1}{2}(3x^2-1)$$

$$P_3(x) = \frac{1}{2^3(3)!}\frac{d^3}{dx^3}(x^2-1)^3 = \frac{1}{48}\frac{d^2}{dx^2}(x^6-3x^4+3x^2-1) = \frac{1}{2}(5x^3-3x)$$

Similarly, $P_4(x) = \frac{1}{8}(35x^4-30x^2+3)$, $P_5(x) = \frac{1}{8}(63x^5-70x^3+15x)$

$$P_6(x) = \frac{1}{6}(231x^6-351x^4+105x^2-5)$$

From (1), by using binomial theorem

$$P_n(x) = \frac{1}{2^n\, n!}\frac{d^n}{dx^n}\sum_{r=0}^{n} {}^nC_r (x^2)^{n-r}(-1)^r$$

$$= \frac{1}{2^n\, n!}\frac{d^n}{dx^n}\sum_{r=0}^{n} (-1)^r \frac{n!}{r!\,(n-r)!} x^{2n-2r}$$

$$= \frac{1}{2^n\, n!}\sum_{r=0}^{n} (-1)^r \frac{n!}{r!\,(n-r)!}\frac{d^n}{dx^n}(x^{2n-2r})$$

$$= \frac{1}{2^n\, n!}\sum_{r=0}^{N} (-1)^r \frac{(-1)^r\,(2n-2r)!}{r!\,(n-r)!\,(n-2r)!} x^{n-2r}$$

$$\therefore P_n(x) = \sum_{r=0}^{N} (-1)^r \frac{(-1)^r\,(2n-2r)!}{r!\,(n-r)!\,(n-2r)!} x^{n-2r}$$

where $N = \frac{n}{2}$ or $\frac{n-1}{2}$ according as n is even or odd.

Ex. 1. Express $f(x) = x^3 - 5x^2 + x + 2$ in terms of Legendre's polynomials **2070 Bhadra, B. E.**

...(1)

Solution:

Here, the function is

$$f(x) = x^3 - 5x^2 + x + 2$$

We know that Legendre's polynomial are

$$P_0(x) = 1, \quad x = P_1(x)$$

$$P_2(x) = \frac{1}{2}(3x^2-1)$$

or $3x^2 - 1 = 2P_2(x)$

$\therefore$ $x^2 = \frac{1}{3} + \frac{2}{3}P_2(x)$

and $P_3(x) = \frac{1}{2}(5x^3-3x)$

or $5x^3 - 3x = 2P_3(x)$

$\therefore$ $x^3 = \frac{3}{5}x + \frac{2}{5}P_3(x)$

So (1) can be expressed as

$$f(x) = \frac{3}{5}x + \frac{2}{5}P_3(x) - 5\left[\frac{1}{3} + \frac{2}{3}P_2(x)\right] + x + 2$$

$$f(x) = \frac{3}{5}x + \frac{2}{5}P_3(x) - \frac{5}{3} - \frac{10}{3}P_2(x) + x + 2$$

$$= \frac{2}{5}P_3(x) - \frac{10}{3}P_2(x) + \frac{8}{5}x + \frac{1}{3}$$

$$= \frac{2}{5}P_3(x) - \frac{10}{3}P_2(x) + \frac{8}{5}P_1(x) + \frac{1}{3}P_0(x)$$

Ex. 2. Show that $x^4 = \frac{1}{35}[8P_4(x) + 20P_2(x) + 7P_0(x)]$

Solution:

We know that Legendre's polynomial are

$P_0(x) = 1, \quad P_2(x) = \frac{1}{2}(3x^2 - 1)$

and $\quad P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$

Now

$\frac{1}{35}[8P_4(x) + 20P_2(x) + 7P_0(x)]$

$$= \frac{1}{35}[35x^4 - 30x^2 + 3 + 10(3x^2 - 1) + 7]$$

$$= \frac{1}{35}[35x^4 - 30x^2 + 3 + 30x^2 - 10 + 7] = x^4.$$

4.3.3 Generating Function for $P_n(x)$

The function $(1 - 2xt + t^2)^{-1/2}$ is called the and the *generating function of* Legendre's polynomial function $P_n(x)$.

Theorem

If $P_n(x)$ is the Legendre's polynomials of order n, then

$$(1 - 2xt + t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)\, t^n$$

Proof

Since $\quad (1 - z)^{-1/2} = 1 + \frac{1}{2}z + \frac{\frac{1}{2}\cdot\frac{3}{2}}{2!}z^2 + \frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}}{3!}z^3 + \ldots\ldots$

$$= 1 + \frac{2!}{(1!)^2 2^2}z + \frac{4!}{(2!)^2 2^4}z^2 + \frac{6!}{(3!)^2 2^6}z^3 + \ldots\ldots$$

$\therefore \quad [1 - t(2x - t)]^{-1/2} = 1 + \frac{2!}{(1!)^2 2^2}t(2x - t) + \frac{4!}{(2!)^2 2^4}t^2(2x-t)^2 + \ldots$

$$+ \frac{(2n-2r)!}{[(n-r)!]^2\, 2^{2n-2r}}t^{n-r}(2x - t)^{n-r} + \ldots + \frac{(2n)!}{(n!)^2 2^{2n}}t^n(2x-t)^n + \ldots(1)$$

The term in t^n from the term containing $t^{n-r}(2x - t)^{n-r}$

$$= \frac{(2n-2r)!}{[(n-r)!]^2\, 2^{2n-2r}}\, t^{n-r}\, {}^{(n-r)}C_r\, (-t)^r (2x)^{n-2r}$$

$$= \frac{(2n-2r)!}{[(n-r)!]^2\, 2^{2n-2r}} \times \frac{(n-r)!}{r!\,(n-2r)!}(-1)^r t^n (2x)^{n-2r}$$

$$= \frac{(-1)^r (2n-2r)!}{2^n\, r!\,(n-r)!\,(n-2r)!}\, x^{n-2r}\, t^n$$

Collecting all terms in t^n which will occur the term containging $t^n(2x - t)^n$ and the preceding terms, we see that terms in t^n is

$$= \sum_{r=0}^{N} \frac{(-1)^r (2n-2r)!}{2^n\, r!\,(n-r)!\,(n-2r)!}\, x^{n-2r}\, t^n = P_n(x)\, t^n$$

Where $N = \frac{n}{2}$ or $\frac{n-1}{2}$ according as n is even or odd.

Hence (1) may be written as

$$[1 - 2xt + t^2]^{-1/2} = \sum_{n=0}^{\infty} P_n(x)\, t^n$$

Ex.3. Show that **(i) $P_n(1) = 1$ (ii) $P_n(-x) = (-1)^n P_n(x)$**

Solution:

We know that the generating function is

$$[1 - 2xt + t^2]^{-1/2} = \sum_{n=0}^{\infty} P_n(x)\, t^n \qquad \ldots(1)$$

(i) Putting x = 1 in (1), we get

$$[1 - 2t + t^2)]^{-1/2} = \sum_{n=0}^{\infty} P_n(1)\, t^n$$

or $\quad (1 - t)^{-1} = \sum_{n=0}^{\infty} P_n(1)\, t^n$

or $\quad 1 + t + t^2 + t^3 + \ldots + t^n + \ldots = \sum_{n=0}^{\infty} P_n(1)\, t^n$

Equating the coefficient of t^n, we get

$$P_n(1) = 1$$

(ii) Replacing x by (-x) in (1), we get

$$[1 + 2xt + t^2]^{-1/2} = \sum_{n=0}^{\infty} P_n(-x)\, t^n \qquad \ldots(2)$$

Again, replacing t by (-t) in (1), we get

$$[1 + 2xt + t^2]^{-1/2} = \sum_{n=0}^{\infty} P_n(x)(-t)^n$$

or $[1+2xt+t^2]^{-1/2} = \sum_{n=0}^{\infty} P_n(x)(-1)^n t^n$

On division (2) and (3)

$$\sum_{n=0}^{\infty} P_n(x)(-1)^n t^n = \sum_{n=0}^{\infty} P_n(-x)\, t^n$$

Equating the coefficient of t^n, we get

$$P_n(-x) = (-1)^n P_n(x).$$

Ex. 4 . Show that (i) $P_{2n}(0) = (-1)^n \dfrac{2n!}{2^{2n}(n!)^2}$ (ii) $P_{2n+1}(0) = 0$

Solution:

The generating function of the Legendrre's Polynomial is

$$[1-2xt+t^2]^{-1/2} = \sum_{n=0}^{\infty} P_n(x)\, t^n$$

Putting x = 0, we get

$$[1+t^2]^{-1/2} = \sum_{n=0}^{\infty} P_n(0)\, t^n$$

or $$1-\frac{1}{2}t^2+\frac{1.3}{2.4}t^4 - \ldots + (-1)^n\frac{1.3.5\ldots..(2n-1)}{2.4.6\ldots\ldots..(2n)}t^{2n}+\ldots.. = \sum_{n=0}^{\infty} P_n(0)\, t^n$$

(i) Equating the coefficient of t^{2n} on both sides, we get

$$P_{2n}(0) = (-1)^n\frac{1.3.5\ldots..(2n-1)}{2.4.6\ldots\ldots..(2n)}t^{2n}$$

$$= (-1)^n\frac{1.2.3.4.5\ldots..(2n-1)(2n)}{[2.4.6\ldots\ldots..(2n)]^2}t^{2n}$$

$$= (-1)^n\frac{(2n)!}{[2^n\, 1.2.3.4\ldots\ldots..n]^2}t^{2n} = (-1)^n\frac{(2n)!}{[2^n(n!)]^2}t^{2n}$$

$$\therefore \quad P_{2n}(0) = (-1)^n\frac{(2n)!}{[2^n(n!)]^2}t^{2n}$$

(ii) Equating the coefficient of t^{2n+1} on both sides, we get

$$P_{2n+1}(0) = 0$$

4.3.4 Recurrence Properties for Legendre's Functions

(i) $(2n+1)\,x\,P_n(x) = (n+1)\,P_{n+1}(x) + nP_{n-1}(x)$

We know that

$$[1-2xt+t^2]^{-1/2} = \sum_{n=0}^{\infty} P_n(x)\, t^n \qquad \ldots(1)$$

Partially differentiating (1) with respect to t, we get

$$-\frac{1}{2}(1-2xt+t^2)^{-3/2}(-2x+2t) = \sum_{n=0}^{\infty} n\,P_n(x)\, t^{n-1}$$

or $$(x-t)(1-2xt+t^2)^{-1/2} = (1-2xt+t^2)\sum_{n=0}^{\infty} n\,P_n(x)\, t^{n-1}$$

or $$(x-t)\sum_{n=0}^{\infty} P_n(x)\, t^n = (1-2xt+t^2)\sum_{n=0}^{\infty} n\,P_n(x)\, t^{n-1}$$

Equating the coefficient of t^n on both sides, we get

$$x\,P_n(x) - P_{n-1}(x) = (n+1)\,P_{n+1}(x) - 2x\,n\,P_n(x) + (n-1)\,P_{n-1}(x)$$

$$\therefore \quad (2n+1)x\,P_n(x) = (n+1)\,P_{n+1}(x) + n\,P_{n-1}(x)$$

(ii) $nP_n(x) = xP_n'(x) - P_{n-1}'(x)$

We know that

$$[1-2xt+t^2]^{-1/2} = \sum_{n=0}^{\infty} P_n(x)\, t^n \qquad \ldots(1)$$

Partially differentiating (1) with respect to x, we get

$$-\frac{1}{2}(1-2xt+t^2)^{-3/2}(-2t) = \sum_{n=0}^{\infty} P_n'(x)\, t^n$$

or $$t(1-2xt+t^2)^{-3/2} = \sum_{n=0}^{\infty} P_n'(x)\, t^n \qquad \ldots(2)$$

Partially differentiating (1) with respect to t, we get

$$-\frac{1}{2}(1-2xt+t^2)^{-3/2}(-2x+2t) = \sum_{n=0}^{\infty} n\,P_n(x)\, t^{n-1}$$

or $\quad (x-t)(1-2xt+t^2)^{-3/2} = \sum_{n=0}^{\infty} n P_n(x)\, t^{n-1}$

or $\quad (x-t)(1-2xt+t^2)^{-3/2} = \sum_{n=0}^{\infty} n P_n(x)\, t^{n-1}$

On division (2) and (3), we get

$$\therefore \quad t \sum_{n=0}^{\infty} n P_n(x)\, t^{n-1} = (x-t) \sum_{n=0}^{\infty} P_n'(x)\, t^n$$

or $\quad \sum_{n=0}^{\infty} n P_n(x)\, t^n = (x-t) \sum_{n=0}^{\infty} P_n'(x)\, t^n$

Equating the coefficient of t^n on both sides, we get

$$n P_n(x) = x P_n'(x) - P_{n-1}'(x)$$

(iii) $\quad (2n+1)\, P_n(x) = P_{n+1}'(x) - P_{n-1}'(x)$

We have the recurrence relation (i) is

$$(2n+1)x\, P_n(x) = (n+1)\, P_{n+1}(x) + n\, P_{n-1}(x)$$

Differentiating (1) with respect to x, we get

$$(2n+1)x\, P_n'(x) + (2n+1)P_n(x) = (n+1)\, P_{n+1}'(x) + n P_{n-1}'(x)$$

Also the recurrence relation (ii) is

$$n P_n(x) = x P_n'(x) - P_{n-1}'(x)$$

or $\quad x P_n'(x) = n P_n(x) + P_{n-1}'(x)$

From (2) and (3)

$$(2n+1)[n P_n(x) + P_{n-1}'(x)] + (2n+1)P_n(x) = (n+1)\, P_{n+1}'(x) + n P_{n-1}'(x)$$

or $\quad (n+1)(2n+1)P_n(x) + (2n+1-n)P_{n-1}'(x) = (n+1)P_{n+1}'(x)$

or $\quad (n+1)(2n+1)P_n(x) + (n+1)P_{n-1}'(x) = (n+1)P_{n+1}'(x)$

$\therefore \quad (2n+1)P_n(x) = P_{n+1}'(x) - P_{n-1}'(x)$

(iv) $\quad P_n'(x) = x\, P_{n-1}'(x) + nP_{n-1}(x)$

We have the recurrence relation (i) is

$$(2n+1)x\, P_n(x) = (n+1)\, P_{n+1}(x) + n\, P_{n-1}(x)$$

Differentiating (1) with respect to x, we get

$$(2n+1)x\, P_n'(x) + (2n+1)P_n(x) = (n+1)\, P_{n+1}'(x) + n P_{n-1}'(x)$$

or $\quad (n+1)P_{n+1}'(x) = (2n+1)x\, P_n'(x) + (2n+1)P_n(x) - n P_{n-1}'(x)$

Also the recurrence relation (ii) is

$$n P_n(x) = x P_n'(x) - P_{n-1}'(x) \qquad \ldots(3)$$

$$P_{n-1}'(x) = x P_n'(x) - n P_n(x)$$

From (2) and (3)

$$(n+1)P_{n+1}'(x) = (2n+1)x P_n'(x) + (2n+1)P_n(x) - n[x P_n'(x) - n P_n(x)]$$

or $\quad (n+1)P_{n+1}'(x) = (2n+1-n)\, x P_n'(x) + (2n+1+n^2)P_n(x)$

or $\quad (n+1)P_{n+1}'(x) = (n+1)\, x P_n'(x) + (n+1)^2 P_n(x)$

or $\quad P_{n+1}'(x) = x P_n'(x) + (n+1)P_n(x)$

Replacing n by n - 1, we get

$$P_n'(x) = x P_{n-1}'(x) + n P_{n-1}(x).$$

(v) $\quad (1-x^2)P_n'(x) = n\,[P_{n-1}(x) - xP_n(x)]$

We have the recurrence relation (ii) is

$$n P_n(x) = x P_n'(x) - P_{n-1}'(x) \qquad \ldots(1)$$

or $\quad x P_n'(x) = P_{n-1}'(x) + n P_n(x)$

Also the recurrence relation (iv) is

$$P_n'(x) = x P_{n-1}'(x) + n P_{n-1}(x) \qquad \ldots(2)$$

Multiplying (1) by x and subtracting from (2), we get

$$(1-x^2)P_n'(x) = n[P_{n-1}(x) - x P_n(x)]$$

(vi) $\quad (1-x^2)P_n'(x) = (n+1)[xP_n(x) - P_{n+1}(x)]$

We have the recurrence relation (i) is

$$(2n+1)x\, P_n(x) = (n+1)P_{n+1}(x) + n\, P_{n-1}(x)$$

or $\quad [n+(n+1)]x\, P_n(x) = (n+1)\, P_{n+1}(x) + n\, P_{n-1}(x)$

or $\quad n\, x P_n(x) + (n+1)\, x P_n(x) = (n+1)\, P_{n+1}(x) + n\, P_{n-1}(x)$

or $\quad (n+1)[x P_n(x) - P_{n+1}(x)] = n[P_{n-1}(x) - x P_n(x)] \qquad \ldots(1)$

Also the recurrence relation (v) is

$$(1-x^2)P_n'(x) = n[P_{n-1}(x) - x P_n(x)]$$

or $\quad n[P_{n-1}(x) - x P_n(x)] = (1-x^2)P_n'(x) \qquad \ldots(2)$

From (1) and (2), we get

$$(n+1)[x P_n(x) - P_{n+1}(x)] = (1-x^2)P_n'(x)$$

$\therefore \quad (1-x^2)P_n'(x) = (n+1)[x P_n(x) - P_{n+1}(x)]$

4.4. Bessel's Equation

One of the most important differential equations in applied mathematics of the form $x^2y'' + xy' + (x^2 - n^2)y = 0$ is called *Bessel's equation* of order n.

Here the Bessel's equation is

N K

$$x^2y'' + xy' + (x^2 - n^2)y = 0$$

To solve it, we use Frobenius method. Let its general solution be ...(1)

$$y = \sum_{m=0}^{\infty} c_m x^{m+r}$$ ...(2)

Differentiating (2), we get

$$y' = \sum_{m=0}^{\infty} (m + r)c_m x^{m+r-1}$$

$$y'' = \sum_{m=0}^{\infty} (m + r)(m + r - 1)c_m x^{m+r-2}$$

Substituting these in (1), we get

$$x^2 \sum_{m=0}^{\infty} (m + r)(m+r - 1)c_m x^{m+r-2} + x \sum_{m=0}^{\infty} (m + r)c_m x^{m+r-1}$$

$$+ (x^2 - n^2) \sum_{m=0}^{\infty} c_m x^{m+r} = 0$$

or $$\sum_{m=0}^{\infty} (m + r)(m + r - 1)c_m x^{m+r} + \sum_{m=0}^{\infty} (m + r)c_m x^{m+r}$$

$$+ \sum_{m=0}^{\infty} c_m x^{m+r+2} - n^2 \sum_{m=0}^{\infty} c_m x^{m+r} = 0$$

or $$\sum_{m=0}^{\infty} \{(m + r)^2 - n^2\}c_m x^{m+r} + \sum_{m=0}^{\infty} c_m x^{m+r+2} = 0$$

or $$(r^2 - n^2)c_0 x^r + \{(r+1)^2 - n^2\}c_1 x^{r+1} + \{(r+2)^2 - n^2\}c_2 x^{r+2} + \ldots + c_0 x^{r+2} + c_1 x^{r+3} + c_2 x^{r+4} + \ldots = 0$$

Equating to zero the coefficient of x^r, we get the indicial equation

$$c_0(r^2 - n^2) = 0$$

But $c_0 \neq 0$ $\quad \therefore \quad r = \pm n$.

Also equating the coefficient of x^{r+1} to zero, we get

$$c_1\{(r + 1)^2 - n^2\} = 0.$$

This gives $\quad c_1 = 0$ as $(r + 1)^2 - n^2 \neq 0$.

Equating the coefficient of x^{m+r+2} to zero, the recurrence relation in terms of the coefficient c_m is given by

$$c_{m+2}\{(m + r + 2)^2 - n^2\} + c_m = 0$$ ....(3)

or $$c_{m+2} = \frac{-c_m}{(m + r + 2)^2 - n^2}$$

where $m = 0, 1, 2, 3$ ....

Therefore, $$c_1 = c_3 = c_5 = \ldots\ldots\ldots = 0$$

and $$c_2 = \frac{-c_0}{(r + 2)^2 - n^2}$$

$$c_4 = \frac{-c_2}{(r + 4)^2 - n^2} = \frac{c_0}{\{(r + 4)^2 - n^2\}\{(r + 2)^2 - n^2\}} \text{ etc.}$$

$$\therefore \; y = c_0 x^r\left[1 - \frac{1}{\{(r+2)^2 - n^2\}}x^2 + \frac{1}{\{(r+4)^2 - n^2\}\{(r+2)^2 - n^2\}}x^4 + \ldots\right]$$

For $r = n$, we get one of the independent solution in the form of the series in ascending powers of x as

$$y_1 = c_0 x^n\left[1 - \frac{x^2}{4(n+1)} + \frac{x^4}{4^2.2!(n+1)(n+2)} - \frac{x^6}{4^3.3!(n+1)(n+2)(n+3)} + \ldots\right]$$

where c_0 is an arbitrary constant.

For $r = -n$, we get other indepednednt solution in series of descending powers of x as

$$y_2 = c_0 x^{-n}\left[1 - \frac{x^2}{4(-n+1)} + \frac{x^4}{4^2.2!(-n+1)(-n+2)} - \frac{x^6}{4^3.3!(-n+1)(-n+2)(-n+3)} + \ldots\right]$$

When n is not integral or zero, thus the complete solution of given equation is

$$y = a_1y_1 + a_2y_2.$$

4.5. Bessel's Function

The Bessel's differential equation of order n is

$$x^2y'' + xy' + (x^2 - n^2)y = 0$$ ...(1)

The solution of this equation in series of ascending powers of x is

$$y = c_0 x^n\left[1 - \frac{x^2}{4(n+1)} + \frac{x^4}{4^2.2!(n+1)(n+2)} - \frac{x^6}{4^3.3!(n+1)(n+2)(n+3)} + \ldots + \frac{(-1)^m x^{2m}}{4^m m!(n+1)(n+2)\ldots\ldots(n+m)} + \ldots\ldots\right]$$

where c_0 is an arbitrary constant.

If we put $c_0 = \dfrac{1}{2^n\,\Gamma(n+1)}$, then the solution in series is called *Bessel's function* of the first kind of order n and is denoted by $J_n(x)$.

Thus

$$\text{or } J_n(x)=\frac{x^n}{2^n\,\Gamma(n+1)}\sum_{m=0}^{\infty}\frac{x^{2m}}{4^m\, m!(n+1)(n+2)(n+3)\ldots(n+m)}$$

$$\text{or } J_n(x)=\sum_{m=0}^{\infty}\frac{(-1)^m}{m!\,\Gamma(n+m+1)}\left(\frac{x}{2}\right)^{n+2m}$$

Putting -n for n, other solution of the Bessel's equation in series of descending power of x is

$$J_{-n}(x)=\sum_{m=0}^{\infty}\frac{(-1)^m}{m!\,\Gamma(m-n+1)}\left(\frac{x}{2}\right)^{-n+2m}$$

When n is not integral or zero, the complete solution of Bessel's equation may be expressed in the form

$$y = AJ_n(x) + BJ_{-n}(x)$$

Note

$$J_{n+1}(x)=\sum_{m=0}^{\infty}\frac{(-1)^m}{m!\,\Gamma(n+1+m+1)}\left(\frac{x}{2}\right)^{(n+1)+2m}$$

4.5.1. Linear Depedence of Bessel's function Jn(x) and J-n(x)

Show that $J_{-n}(x) = (-1)^n J_n(x)$ where (i) n is a positive integer (ii) n is any integer [2067, Chaitra B.E.]

Proof

(i) When n is positive integer

$$\text{We have } J_{-n}(x)=\sum_{m=0}^{\infty}\frac{(-1)^m}{m!\,\Gamma(m-n+1)}\left(\frac{x}{2}\right)^{-n+2m}$$

Since Γ (a negative integer or zero) tends to infinity, each term in the summation is zero so long as $m-n+1 \le 0$ i.e., $m \le n-1$.

$\Gamma(m-n+1)$ is finite when $m \ge n$.

Thus

$$J_{-n}(x)=\sum_{m=0}^{n-1}\frac{(-1)^m}{m!\,\Gamma(m-n+1)}\left(\frac{x}{2}\right)^{-n+2m}+\sum_{m=n}^{\infty}\frac{(-1)^m}{m!\,\Gamma(m-n+1)}\left(\frac{x}{2}\right)^{-n+2m}$$

$$=0+\sum_{m=n}^{\infty}\frac{(-1)^m}{m!\,\Gamma(m-n+1)}\left(\frac{x}{2}\right)^{-n+2m}$$

Taking $m = n + s$ so that $s = 0, 1, 2, \ldots$, we get

$$J_{-n}(x)=\sum_{s=0}^{\infty}\frac{(-1)^{n+s}}{(n+s)!\,\Gamma(s+1)}\left(\frac{x}{2}\right)^{n+2s}$$

Since $\Gamma(s+1) = s!$ and $\Gamma(n+s+1) = (n+s)!$

$$\therefore\quad J_{-n}(x) = (-1)^n\sum_{s=0}^{\infty}\frac{(-1)^s}{s!\,\Gamma(n+s+1)}\left(\frac{x}{2}\right)^{n+2s}$$

$$J_{-n}(x) = (-1)^n J_n(x).$$

(ii) When n = 0, the result is obvious.

When n is negative integer, then n = - p where p is positive integer.

From the first case,

$$J_{-p}(x) = (-1)^p J_p(x)$$

$$\text{or}\quad J_p(x)=\frac{J_{-p}(x)}{(-1)^p}$$

$$\text{or}\quad J_p(x) = (-1)^{-p} J_{-p}(x)$$

Since p = - n

$$\therefore\quad J_{-n}(x) = (-1)^n J_n(x).$$

This is true for all integral values of n. It shows that Bessel's functions are $J_n(x)$ and $J_{-n}(x)$ which are linearly dependent.

4.5.2. Recurrence Relations for Bessel's Functions

1. $xJ'_n(x) = nJ_n(x) - xJ_{n+1}(x)$ [2063 Kartik, B.E.]

We have

$$J_n(x)=\sum_{m=0}^{\infty}\frac{(-1)^m}{m!\,\Gamma(m+n+1)}\left(\frac{x}{2}\right)^{n+2m}\qquad\ldots(1)$$

Differentiating (1) with respect to x, we get

$$J_n'(x)=\sum_{m=0}^{\infty}\frac{(-1)^m(n+2m)}{m!\,\Gamma(m+n+1)}\,\frac{1}{2}\left(\frac{x}{2}\right)^{n+2m-1}$$

$$\therefore \quad xJ'_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m (n+2m)}{m!\,\Gamma(m+n+1)} \left(\frac{x}{2}\right)^{n+2m}$$

$$= \sum_{m=0}^{\infty} \frac{n(-1)^m}{m!\,\Gamma(m+n+1)} \left(\frac{x}{2}\right)^{n+2m} + \sum_{m=0}^{\infty} \frac{2m(-1)^m}{m!\,\Gamma(m+n+1)} \left(\frac{x}{2}\right)^{n+2m}$$

$$= nJ_n(x) + x \sum_{m=0}^{\infty} \frac{(-1)^m}{(m-1)!\,\Gamma(n+m+1)} \left(\frac{x}{2}\right)^{n+2m-1}$$

Putting $m - 1 = s$. Then

$$xJ'_n(x) = nJ_n(x) + x \sum_{s=0}^{\infty} \frac{(-1)^{s+1}}{s!\,\Gamma(n+s+2)} \left(\frac{x}{2}\right)^{n+2s+1}$$

$$= nJ_n(x) - x \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,\Gamma(n+1+s+1)} \left(\frac{x}{2}\right)^{(n+1)+2s}$$

$$= nJ_n(x) - x J_{n+1}(x)$$

Thus $\quad xJ_n'(x) = nJ_n(x) - xJ_{n+1}(x)$.

2. $\quad \mathbf{xJ_n'(x) = -nJ_n(x) + xJ_{n-1}(x)}$

We have

$$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(n+m+1)} \left(\frac{x}{2}\right)^{n+2m}$$

Differentiating it with respect to x, we get

$$J_n'(x) = \sum_{m=0}^{\infty} \frac{(-1)^m (n+2m)}{m!\,\Gamma(m+n+1)} \frac{1}{2} \left(\frac{x}{2}\right)^{n+2m-1}$$

or $$xJ_n'(x) = \sum_{m=0}^{\infty} \frac{(-1)^m \{(2n+2m)-n\}}{m!\,\Gamma(m+n+1)} \left(\frac{x}{2}\right)^{n+2m}$$

$$= -n \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+n+1)} \left(\frac{x}{2}\right)^{n+2m} + \sum_{m=0}^{\infty} \frac{(-1)^m\, 2(n+m)}{m!\,\Gamma(m+n+1)} \left(\frac{x}{2}\right)^{n+2m}$$

$$= -nJ_n(x) + x \sum_{m=0}^{\infty} \frac{(-1)^m (m+n)}{m!\,(m+n)\,\Gamma(m+n)} \left(\frac{x}{2}\right)^{n+2m-1}$$

$$= -nJ_n(x) + x \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(n+m)} \left(\frac{x}{2}\right)^{n+2m-1}$$

$$= -nJ_n(x) + x \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(n-1+m+1)} \left(\frac{x}{2}\right)^{n-1+2m}$$

$$= -nJ_n(x) + xJ_{n-1}(x)$$

$\therefore \quad xJ_n'(x) = -nJ_n(x) + xJ_{n-1}(x)$.

3. $\quad \mathbf{2J_n'(x) = J_{n-1}(x) - J_{n+1}(x)}$

We have

$$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+n+1)} \left(\frac{x}{2}\right)^{n+2m}$$

Differentiating it with respect to x, we get

$$J_n'(x) = \sum_{m=0}^{\infty} \frac{(-1)^m (n+2m)}{m!\,\Gamma(m+n+1)} \frac{1}{2} \left(\frac{x}{2}\right)^{n+2m-1}$$

$$2J_n'(x) = \sum_{m=0}^{\infty} \frac{(-1)^m (n+m+m)}{m!\,\Gamma(m+n+1)} \left(\frac{x}{2}\right)^{n+2m-1}$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m (n+m)}{m!\,\Gamma(m+n+1)} \left(\frac{x}{2}\right)^{n+2m-1}$$

$$+ \sum_{m=0}^{\infty} \frac{(-1)^m\, m}{m!\,\Gamma(m+n+1)} \left(\frac{x}{2}\right)^{n+2m-1}$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m (n+m)}{m!\,(m+n)\,\Gamma(m+n)} \left(\frac{x}{2}\right)^{n+2m-1}$$

$$+ \sum_{m=1}^{\infty} \frac{(-1)^m\, m}{m(m-1)!\,\Gamma(m+n+1)} \left(\frac{x}{2}\right)^{n+2m-1}$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma((n-1)+m+1)} \left(\frac{x}{2}\right)^{n-1+2m}$$

$$+ \sum_{m=1}^{\infty} \frac{(-1)^m}{(m-1)!\,\Gamma(m+n+1)} \left(\frac{x}{2}\right)^{n+2m-1}$$

Taking m - 1 = s in the second series as s = 0, 1, 2,...

$$= J_{n-1}(x) - \sum_{s=0}^{\infty} \frac{(-1)^{s+1}}{s!\,\Gamma((s+1)+n+1)} \left(\frac{x}{2}\right)^{n+2s+1}$$

$$= J_{n-1}(x) - \sum_{s=0}^{\infty} \frac{(-1)^{s}}{s!\,\Gamma(s+(1+n)+1)} \left(\frac{x}{2}\right)^{n+1+2s}$$

$$= J_{n-1}(x) - J_{n+1}(x)$$

$$\therefore \quad 2J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$$

4. $2nJ_n(x) = x[J_{n-1}(x) + J_{n+1}(x)]$

We have

$$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+n+1)} \left(\frac{x}{2}\right)^{n+2m}$$

$$2nJ_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m 2n}{m!\,\Gamma(m+n)+1} \left(\frac{x}{2}\right)^{n+2m}$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m (2n+2m-2m)}{m!\,\Gamma(m+n)+1} \frac{x}{2} \left(\frac{x}{2}\right)^{n+2m-1}$$

$$= x \sum_{m=0}^{\infty} \frac{(-1)^m (n+m)}{m!\,\Gamma(m+n+1)} \left(\frac{x}{2}\right)^{n+2m-1}$$

$$- x \sum_{m=1}^{\infty} \frac{(-1)^m\, m}{(m-1)!\,\Gamma(n+m+1)} \left(\frac{x}{2}\right)^{n+2m-1}$$

$$= x \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(n+m)} \left(\frac{x}{2}\right)^{n+2m-1} - x \sum_{m=0}^{\infty} \frac{(-1)^{s+1}}{s!\,\Gamma(n+s+2)} \left(\frac{x}{2}\right)^{n+2s+1}$$

Taking m - 1 = s in the second series, we get

$$= x \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(n-1+m+1)} \left(\frac{x}{2}\right)^{(n-1)+2m}$$

$$- x \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,\Gamma(n+1+s+1)} \left(\frac{x}{2}\right)^{(n+1)+2s}$$

$$= xJ_{n-1}(x) - xJ_{n+1}(x)$$

$$2nJ_n(x) = xJ_{n-1}(x) + xJ_{n+1}(x)$$

5. $\frac{d}{dx}[x^{-n}J_n(x)] = -x^{-n}J_{n+1}$ [2068/070 Magh, B. E.]

We have

$$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(n+m+1)} \left(\frac{x}{2}\right)^{n+2m}$$

$$\text{Thus } \frac{d}{dx}[x^{-n}J_n(x)] = \frac{d}{dx}\left[x^{-n} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(n+m+1)} \left(\frac{x}{2}\right)^{n+2m}\right]$$

$$= \frac{d}{dx}\left[\sum_{m=0}^{\infty} \frac{(-1)^m 2^{-n}}{m!\,\Gamma(n+m+1)} \left(\frac{x}{2}\right)^{2m}\right]$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m 2m\,.2^{-n}}{m!\,\Gamma(n+m+1)} \frac{1}{2} \left(\frac{x}{2}\right)^{2m-1}$$

$$= x^{-n} \sum_{m=1}^{\infty} \frac{(-1)^m}{(m-1)!\,\Gamma(n+m+1)} \left(\frac{x}{2}\right)^{n+2m-1}$$

Taking m - 1 = s, we get

$$= x^{-n} \sum_{s=0}^{\infty} \frac{(-1)^{s+1}}{s!\,\Gamma(n+1+s+1)} \left(\frac{x}{2}\right)^{n+1+2s}$$

$$= -x^{-n} \sum_{s=0}^{\infty} \frac{(-1)^{s}}{s!\,\Gamma(n+1+s+1)} \left(\frac{x}{2}\right)^{n+1+2s}$$

$$= -x^{-n}J_{n+1}(x)$$

$$\therefore \quad \frac{d}{dx}[x^{-n}J_n(x)] = -x^{-n}J_{n+1}(x)$$

6. $\frac{d}{dx}[x^{n}J_n(x)] = x^{n}J_{n-1}(x)$ [2069 Bhadra B. E.]

We have

$$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(n+m+1)} \left(\frac{x}{2}\right)^{n+2m}$$

Thus $\frac{d}{dx}[x^n J_n(x)] = \frac{d}{dx} \sum_{m=0}^{\infty} \frac{(-1)^m 2^n}{m!\,\Gamma(n+m+1)} \left(\frac{x}{2}\right)^{2n+2m}$

$$= 2^n \sum_{m=0}^{\infty} \frac{(-1)^m (2n+2m)}{m!\,\Gamma(n+m+1)} \frac{1}{2} \left(\frac{x}{2}\right)^{2n+2m-1}$$

$$= 2^n \sum_{m=0}^{\infty} \frac{(-1)^m (n+m)}{m!\,\Gamma(n+m+1)} \left(\frac{x}{2}\right)^{2n+2m-1}$$

$$= x^n \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(n+m)} \left(\frac{x}{2}\right)^{n+2m-1}$$

$$= x^n \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(n-1+m+1)} \left(\frac{x}{2}\right)^{n-1+2m}$$

$$= x^n J_{n-1}(x)$$

$$\therefore \quad \frac{d}{dx}[x^n J_n(x)] = x^n J_{n-1}(x)$$

7. $x J_n'(x) = nJ_n(x) - xJ_{n+1}(x)$

We know that

$$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+n+1)} \left(\frac{x}{2}\right)^{n+2m}$$

Differentiating it with respect to x ,we get

$$J_n'(x) = \sum_{m=0}^{\infty} \frac{(-1)^m (n+2m)}{m!\,\Gamma(m+n+1)} \frac{1}{2} \left(\frac{x}{2}\right)^{n-2m-1}$$

or $$J_n'(x) = \sum_{m=0}^{\infty} \frac{(-1)^m (n+2m)}{m!\,\Gamma(m+n+1)} \frac{1}{2} \left(\frac{x}{2}\right)^{-1} \left(\frac{x}{2}\right)^{n+2m}$$

or $$J_n'(x) = \sum_{m=0}^{\infty} \frac{(-1)^m (n+2m)}{m!\,\Gamma(m+n+1)} \frac{1}{2} \frac{2}{x} \left(\frac{x}{2}\right)^{n+2m}$$

or $$x J_n'(x) = \sum_{m=0}^{\infty} \frac{(-1)^m (n+2m)}{m!\,\Gamma(m+n+1)} \left(\frac{x}{2}\right)^{n+2m}$$

or $$x J_n'(x) = n \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+n+1)} \left(\frac{x}{2}\right)^{n+2m} + \sum_{m=0}^{\infty} \frac{2m(-1)^m}{m\,(m-1)!\,\Gamma(m+n+1)} \left(\frac{x}{2}\right)^{n+2m}$$

or $$x J_n'(x) = nJ_n(x) + \sum_{m=0}^{\infty} \frac{2(-1)^m}{(m-1)!\,\Gamma(m+n+1)} \left(\frac{x}{2}\right)^{n+2m}$$

Put $m - 1 = s$,

$$x J_n'(x) = nJ_n(x) + x \sum_{s=0}^{\infty} \frac{(-1)^{s+1}}{s!\,\Gamma(s+1+n+1)} \left(\frac{x}{2}\right)^{n+2s+2-1}$$

or $$x J_n'(x) = nJ_n(x) - x \sum_{s=0}^{\infty} \frac{(-1)^{s}}{s!\,\Gamma((n+1)+s+1)} \left(\frac{x}{2}\right)^{(n+1)+2s}$$

$$\therefore \quad x J_n'(x) = nJ_n(x) - xJ_{n+1}(x).$$

Worked Out Examples

Ex. 1: Prove that $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$

Solution:

The Bessel's function is

$$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+n+1)} \left(\frac{x}{2}\right)^{n+2m} \quad \ldots(1)$$

Putting n = 1/2 in (1), we get

$$J_{1/2}(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma\left(m+\frac{1}{2}+1\right)} \left(\frac{x}{2}\right)^{\frac{1}{2}+2m}$$

$$= \left(\frac{x}{2}\right)^{\frac{1}{2}} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma\left(m+\frac{3}{2}\right)} \left(\frac{x}{2}\right)^{2m}$$

$$= \left(\frac{x}{2}\right)^{\frac{1}{2}} \left[\frac{1}{\Gamma\left(\frac{3}{2}\right)} - \frac{1}{1!\,\Gamma\left(\frac{5}{2}\right)}\left(\frac{x}{2}\right)^2 + \frac{1}{2!\,\Gamma\left(\frac{7}{2}\right)}\left(\frac{x}{2}\right)^4 - \ldots\ldots\right]$$

$$= \sqrt{\frac{x}{2}}\,\frac{1}{\Gamma\left(\frac{1}{2}\right)}\left[\frac{1}{\frac{1}{2}} - \frac{1}{\frac{3}{2}\cdot\frac{1}{2}}\left(\frac{x}{2}\right)^2 + \frac{1}{2!\,\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}}\left(\frac{x}{2}\right)^4 - \ldots\ldots\right]$$

$$= \sqrt{\frac{x}{2}}\,\frac{1}{\sqrt{\pi}}\left[2 - \frac{4}{3}\,\frac{x^2}{4} + \frac{4}{5.3}\,\frac{x^4}{16} - \ldots\ldots\right]$$

$$= \sqrt{\frac{x}{2\pi}}\left[2 - \frac{x^2}{3} + \frac{x^4}{5.4.3} - \ldots\right]$$

$$= \sqrt{\frac{x}{2\pi}}\,2\left[1 - \frac{x^2}{3\times 2} + \frac{x^4}{5\times 4\times 3\times 2} - \ldots\right]$$

$$= \sqrt{\frac{x}{2\pi}}\,\frac{2}{x}\left[1 - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right] = \sqrt{\frac{2}{\pi x}}\,\sin x$$

$$\therefore\quad J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\,\sin x.$$

Ex. 2: Prove that $J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos x$

Solution:

The Bessel's function is

$$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+n+1)}\left(\frac{x}{2}\right)^{n+2m} \qquad \ldots(1)$$

Putting n = -1/2 in (1), we get

$$J_{-1/2}(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma\left(m-\frac{1}{2}+1\right)}\left(\frac{x}{2}\right)^{-\frac{1}{2}+2m}$$

$$= \left(\frac{x}{2}\right)^{-\frac{1}{2}} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma\left(m+\frac{1}{2}\right)}\left(\frac{x}{2}\right)^{2m}$$

$$= \left(\frac{x}{2}\right)^{-\frac{1}{2}}\left[\frac{1}{\Gamma\left(\frac{1}{2}\right)} - \frac{1}{1!\,\Gamma\left(\frac{3}{2}\right)}\left(\frac{x}{2}\right)^2 + \frac{1}{2!\,\Gamma\left(\frac{5}{2}\right)}\left(\frac{x}{2}\right)^4 - \ldots\ldots\right]$$

$$= \left(\frac{x}{2}\right)^{-\frac{1}{2}}\frac{1}{\Gamma\left(\frac{1}{2}\right)}\left[1 - \frac{1}{\frac{1}{2}}\left(\frac{x}{2}\right)^2 + \frac{1}{2!\,.\frac{3}{2}\cdot\frac{1}{2}}\left(\frac{x}{2}\right)^4 - \ldots\ldots\right]$$

$$= \sqrt{\frac{2}{x}}\,\frac{1}{\sqrt{\pi}}\left[1 - \frac{2x^2}{4} + \frac{x^4}{3\times 2\times 4} - \frac{x^6}{6!}\ldots\ldots\right]$$

$$= \sqrt{\frac{2}{\pi x}}\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}\ldots\ldots\right] = \sqrt{\frac{2}{\pi x}}\cos x$$

$$\therefore\quad J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos x.$$

Ex. 3: Prove that $J_{3/2}(x) = \sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x} - \cos x\right)$ **2072 Magh, B. E.**

Solution:

The recurrence relation for Bessel's function is

$$J_{n+1}(x) = \frac{2n}{x}J_n(x) - J_{n-1}(x).$$

Putting n = 1/2 in this relation, we get

$$J_{3/2}(x) = \frac{2}{x}\times\frac{1}{2}\,J_{1/2}(x) - J_{-1/2}(x)$$

$$= \frac{1}{x}\sqrt{\frac{2}{\pi x}}\sin x - \sqrt{\frac{2}{\pi x}}\cos x$$

$$= \sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x} - \cos x\right)$$

$$\therefore\quad J_{3/2}(x) = \sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x} - \cos x\right).$$

Ex. 4: Prove **$4J''_n(x) = J_{n-2}(x) - 2J_n(x) + J_{n+2}(x)$** **2066/070 Bhadra, B. E.**

Solution:

The recurrence relation for Bessel's function is

$$2J_n'(x) = J_{n-1}(x) - J_{n+1}(x). \qquad \ldots(1)$$

Differentiating both sides of (1), we get

$$2J_n''(x) = J_{n-1}'(x) - J_{n+1}'(x). \qquad \ldots(2)$$

Replacing n by n - 1 in (1), we get

$$2J_{n-1}'(x) = J_{n-2}(x) - J_n(x),$$

$$\therefore\quad J_{n-1}'(x) = \frac{1}{2}\left[J_{n-2}(x) - J_n(x)\right]. \qquad \ldots(3)$$

Replacing n by n + 1 in (1), we get

$$2J_{n+1}'(x) = J_n(x) - J_{n+2}(x).$$

$$\therefore\quad J_{n+1}'(x) = \frac{1}{2}\left[J_n(x) - J_{n+2}(x)\right] \qquad \ldots(4)$$

Putting the values of $J_{n-1}'(x)$ and $J_{n+1}'(x)$ from (3) and (4) in (2), we get

$$2J_n''(x) = \frac{1}{2}\left[J_{n-2}(x) - J_n(x)\right] - \frac{1}{2}\left[J_n(x) - J_{n+2}(x)\right]$$

or, $4 J_n''(x) = J_{n-2}(x) - J_n(x) - J_n(x) + J_{n+2}(x)$

$\therefore$ $4 J_n''(x) = J_{n-2}(x) - 2 J_n(x) + J_{n+2}(x).$

Ex. 5: Prove that $J_2(x) - J_0(x) = 2J_0''(x)$

Solution:

The recurrence relation for Bessel's function is

$$2J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$$

Differentiating with respect to x to (1), we get

$$2J_n''(x) = J_{n-1}'(x) - J'_{n+1}(x)$$

Replacing n by n-1 in (1), we get

$$2J_{n-1}'(x) = J_{n-2}(x) - J_n(x),$$

$$\therefore \quad J_{n-1}'(x) = \frac{1}{2}[J_{n-2}(x) - J_n(x)]$$

Replacing n by n+1 in (1), we get

$$2J_{n+1}'(x) = J_n(x) - J_{n+2}(x),$$

$$\therefore \quad J_{n+1}'(x) = \frac{1}{2}[J_n(x) - J_{n+2}(x)]$$

From (2), (3) and (4), we get

$$2J_n''(x) = \frac{1}{2}[J_{n-2}(x) - J_n(x)] - \frac{1}{2}[J_n(x) - J_{n+2}(x)],$$

or $$2J_n''(x) = \frac{1}{2}J_{n-2}(x) + \frac{1}{2}J_{n+2}(x) - J_n(x).$$

Putting n = 0 on both sides, we get

$$2J_0''(x) = \frac{1}{2}J_{-2}(x) + \frac{1}{2}J_2(x) - J_0(x),$$

or $$2J_0''(x) = \frac{1}{2}(-1)^2J_2(x) + \frac{1}{2}J_2(x) - J_0(x),$$

or $$2J_0''(x) = \frac{1}{2}J_2(x) + \frac{1}{2}J_2(x) - J_0(x),$$

or $$2J_0''(x) = J_2(x) - J_0(x),$$

$$\therefore \quad J_2(x) - J_0(x) = 2J_0''(x).$$

Ex. 6: Prove that $J_3(x) + 3J'_0(x) + 4J_0'''(x) = 0$

Solution:

The recurrence relation for Bessel's function is

$$2J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$$

Differentiating with respect to x to (1), we get

$$2J_n''(x) = J_{n-1}'(x) - J'_{n+1}(x)$$

Replacing n by n-1 in (1), we get

$$2J_{n-1}'(x) = J_{n-2}(x) - J_n(x),$$

$$\therefore \quad J_{n-1}'(x) = \frac{1}{2}[J_{n-2}(x) - J_n(x)]$$

Replacing n by n+1 in (1), we get

$$2J_{n+1}'(x) = J_n(x) - J_{n+2}(x),$$

$$\therefore \quad J_{n+1}'(x) = \frac{1}{2}[J_n(x) - J_{n+2}(x)] \qquad \ldots(4)$$

From (2), (3) and (4), we get

$$2J_n''(x) = \frac{1}{2}[J_{n-2}(x) - J_n(x)] - \frac{1}{2}[J_n(x) - J_{n+2}(x)], \qquad \ldots(5)$$

or $$4J_n''(x) = J_{n-2}(x) + J_{n+2}(x) - 2J_n(x) \qquad \ldots(6)$$

Differentiating with respect to x to (5), we get

$$4J_n'''(x) = J_{n-2}'(x) + J_{n+2}'(x) - 2J_n'(x)$$

Replacing n by n-2 in (1), we get

$$2J_{n-2}'(x) = J_{n-3}(x) - J_{n-1}(x),$$

$$\therefore \quad J_{n-2}'(x) = \frac{1}{2}[J_{n-3}(x) - J_{n-1}(x)] \qquad \ldots(7)$$

Replacing n by n+2 in (1), we get

$$2J_{n+2}'(x) = J_{n+1}(x) - J_{n+3}(x),$$

$$\therefore \quad J_{n+2}'(x) = \frac{1}{2}[J_{n+1}(x) - J_{n+3}(x)] \qquad \ldots(8)$$

From (1), (6), (7) and (8), we get

$$4J_n'''(x) = \frac{1}{2}[J_{n-3}(x) - J_{n-1}(x)] + \frac{1}{2}[J_{n+1}(x) - J_{n+3}(x)] - J_{n-1}(x) + J_{n+1}(x),$$

or $$4J_n'''(x) = \frac{1}{2}J_{n-3}(x) - \frac{3}{2}J_{n-1}(x) + \frac{3}{2}J_{n+1}(x) - \frac{1}{2}J_{n+3}(x),$$

Putting n = 0 on both sides of it, we get

$$4J_0'''(x) = \frac{1}{2}J_{-3}(x) - \frac{3}{2}J_{-1}(x) + \frac{3}{2}J_1(x) - \frac{1}{2}J_3(x),$$

or $$4J_0'''(x) = \frac{1}{2}(-1)^3J_3(x) - \frac{3}{2}(-1)^1J_1(x) + \frac{3}{2}J_1(x) - \frac{1}{2}J_3(x),$$

or $$4J_0'''(x) = -\frac{1}{2}J_3(x) + \frac{3}{2}J_1(x) + \frac{3}{2}J_1(x) - \frac{1}{2}J_3(x),$$

or $$4J_0'''(x) = -J_3(x) + 3J_1(x),$$

or $$4J_0'''(x) = -J_3(x) - 3J_0'(x), \qquad (\because J_1(x) = -J_0'(x))$$

$$\therefore \quad 4J_0'''(x) + 3J_0'(x) + J_3(x) = 0.$$

Exercise - 18

Prove that the following Legendre's Functions

1. $\frac{2}{5}P_3(x) + \frac{3}{5}P_1(x) = x^3$

2. $x^5 = \frac{8}{63}\left[P_5(x) + \frac{7}{2}P_3(x) + \frac{27}{8}P_1(x)\right]$

3. $1 + x + x^3 = P_0(x) + \frac{8}{5}P_1(x) + \frac{2}{5}P_3(x)$ 2072 Aswin, B.E.

Express the following in terms of Legendre's polynomials:

4. $f(x) = 1 + x - x^2$

5. $f(x) = 5x^3 + x$

6. $f(x) = x^3 - 5x^2 + 6x + 1$

Prove the following Bessel's Functions: 2068/072 Magh, B.E.

7. $J_{-3/2}(x) = \sqrt{\frac{2}{\pi x}}\left(-\frac{\cos x}{x} - \sin x\right)$

8. $J_{5/2}(x) = \sqrt{\frac{2}{\pi x}}\left\{\frac{3 - x^2}{x^2}\sin x - \frac{3}{x}\cos x\right\}$ 2065 Chaitra, B.E.

9. $J_{-5/2}(x) = \sqrt{\frac{2}{\pi x}}\left\{\frac{3}{x}\sin x + \frac{3 - x^2}{x^2}\cos x\right\}$ 2071/072 Aswin, B.E.

10. $J_4(x) = \left(\frac{48}{x^3} - \frac{8}{x}\right)J_1(x) + \left(1 - \frac{24}{x^2}\right)J_0(x)$

11. $J_{n+3}(x) + J_{n+5}(x) = \frac{2}{x}(n + 4)J_{n+4}(x)$

12. $J'_0(x) = -J_1(x)$

13. $J_{-1/2}(x) = J_{1/2}(x)\cot x$

14. $[J_{1/2}(x)]^2 + [J_{-1/2}(x)]^2 = \frac{2}{\pi x}$

Answers

3. $f(x) = \frac{2}{3}P_0(x) + P_1(x) - \frac{2}{3}P_2(x)$ 4. $f(x) = 2P_3(x) + 4P_1(x)$

5. $f(x) = \frac{2}{5}P_3(x) - \frac{10}{3}P_2(x) + \frac{33}{5}P_1(x) - \frac{2}{3}P_0(x)$

Chapter -5

Vector Algebra and Calculus

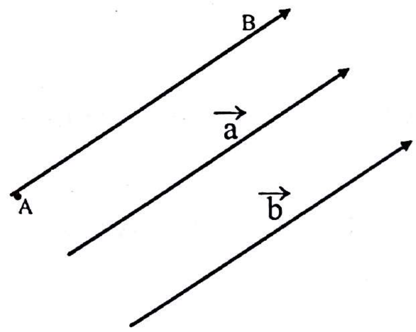
- Introduction
- Two and Three Dimensional Vectors
- Scalar and Vector Products
- Reciprocal System of Vectors
- Applications of Vectors: Lines and Planes
- Vector Calculus -Scalar and Vector Fields
- Derivatives–Velocity and Acceleration
- Gradient Divergence and Curl
- Directional Derivatives

Chapter -5

Vector Algebra and Calculus

5.1 Introduction

Vector analysis possesses a remarkable power and using in mathematical treatment of physical and engineering problems. This branch of mathematics provides a convenient shorthand methods by means of which the mathematical relations between physical quantities can be expressed compactly and as such a problem whose solution is susceptible to non vectorical methods, can be solved expeditiously by vector methods. As many modern books on engineering topics such as elasticity, hydrodynamics etc. frequently use vector methods. An engineering student must acquire the ability to interpret the various physical expressions such gradient, divergence, curl etc. The quantities such as area, volume, arc length temperature, and time have magnitude only and can be completely specified by a single real number. Such quantities are called *scalar quantities* and corresponding real number is *scalar*. The other types of quantities such as velocity, acceleration, forces have both magnitude and direction and are often represented by a directed line segment. Such quantities are called *vector quantities* or simply a *vector*.



If a vector extends from the initial point to the end point then the initial point is called *origin* of vector whereas its end point is called *terminal point* and is often denoted by two letters with an arrow over them. Thus the line segment AB denotes the vector by $\overrightarrow{AB}$ where A is the *origin* and B the *terminus* with length of AB is the magnitude of $\overrightarrow{AB}$, denoted by $|\overrightarrow{AB}|$.

5.2. Two and Three Dimensional Vectors

The two dimensional vector spaces $\vec{v}$ is the set all ordered (a, b) of real numbers called vector subject to the following axioms:

(i) *Addition of vectors*

If $\vec{a} = (a_1, a_2)$ and $\vec{b} = (b_1, b_2)$, then

$$\vec{a} + \vec{b} = (a_1 + b_1, a_2 + b_2)$$

(ii) *Multiplication of vectors by scalar*

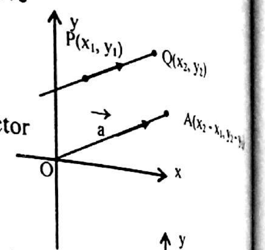
If $\vec{a} = (a_1, a_2)$ and x is scalar, then

$$x\vec{a} = x(a_1, a_2) = (xa_1, xa_2)$$

Some useful properties

1. The zero vector $\vec{O} = (0, 0)$ and negative of a vector $\vec{a} = (a_1, a_2)$ is

$$-\vec{a} = (-a_1, -a_2)$$

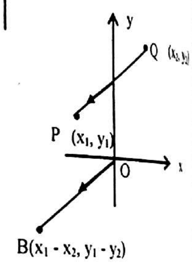


2. The magnitude of a vector $\vec{a} = (a_1, a_2)$ is

$$|\vec{a}| = \sqrt{a_1^2 + a_2^2}$$

3. If $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two points, then the vector $\vec{a}$ corresponds to $\overrightarrow{PQ}$ is

$$\vec{a} = (x_2 - x_1, y_2 - y_1)$$



and the vector $\vec{b}$ corresponds to $\overrightarrow{QP}$ is

$$\vec{b} = (x_1 - x_2, y_1 - y_2)$$

4. If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are three vectors then

(i) $\vec{a} + \vec{b} = \vec{b} + \vec{a}$

(ii) $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$

(iii) $\vec{a} + \vec{o} = \vec{a}$

(iv) $\vec{a} + (-\vec{a}) = \vec{o}$

5. If $\vec{a} = (a_1, a_2)$, then $\vec{a} = a_1\vec{i} + a_2\vec{j}$ where $\vec{i} = (1, 0)$, $\vec{j} = (0, 1)$

6. If $\vec{a}$ is a non zero vector then a unit vector of the vector $\vec{a}$ is $\frac{\vec{a}}{|\vec{a}|}$. The special unit vector $\vec{i} = (1, 0)$, $\vec{j} = (0, 1)$.

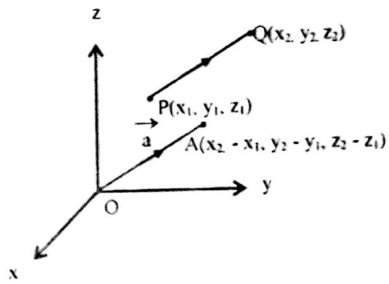
The three-dimensional vector space $\vec{v}$ is the set of all ordered triples of real numbers subject to the following axioms:

If $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$ and x is scalar, then

(i) $\vec{a} + \vec{b} = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$

(ii) $x\vec{a} = (xa_1, xa_2, xa_3)$

7. **Some useful Vectors:**



(i) Zero vector:

$$\vec{0} = (0, 0, 0)$$

(ii) Negative Vector:

$$-\vec{a} = -(a_1, a_2, a_3) = (-a_1, -a_2, -a_3)$$

(iii) Modulus of a vector:

If $\vec{a}$ is a non zero vector then the modulus of the vector $\vec{a}$ is

$$|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

(iv) Difference of two vectors:

If $\vec{a}$ and $\vec{b}$ are two vectors, then their difference is defined as

$$\vec{a} - \vec{b} = \vec{a} + (-\vec{b}) = (a_1 - b_1, a_2 - b_2, a_3 - b_3)$$

8. If $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are two points, then the vector $\vec{a}$ corresponds to $\overrightarrow{PQ}$ is

$$\vec{a} = (x_2 - x_1, y_2 - y_1, z_2 - z_1).$$

And the vector $\vec{b}$ corresponds to $\overrightarrow{QP}$ is

$$\vec{b} = (x_1 - x_2, y_1 - y_2, z_1 - z_2).$$

9. A vector $\vec{a}$ is unit vector if $|\vec{a}| = 1$. The special unit vectors are $\vec{i} = (1, 0, 0)$, $\vec{j} = (0, 1, 0)$, $\vec{k} = (0, 0, 1)$.

Any vector $\vec{a} = (a_1, a_2, a_3)$ can be expressed as a linear combination of $\vec{i}$, $\vec{j}$ and $\vec{k}$ i.e. $\vec{a} = (a_1, a_2, a_3) = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$.

5.3. Scalar and Vector Products

5.3.1 Scalar or Dot Product of Two Vectors

The scalar product or dot product of two vectors $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$ is defined as

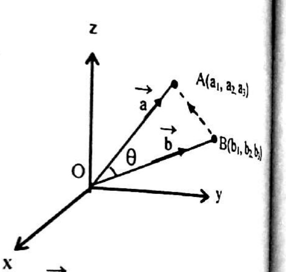
$$\vec{a}.\vec{b} = (a_1, a_2, a_3).(b_1, b_2, b_3)$$
$$= a_1b_1 + a_2b_2 + a_3b_3$$

If θ be the angle between non-zero vectors $\vec{a}$ and $\vec{b}$, then

$$\vec{a}.\vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta$$

Geometrical interpretation

The scalar product of two vectors $\vec{a}.\vec{b}$ is defined as the product of magnitude of $\vec{a}$ and the projection of $\vec{b}$ on $\vec{a}$ or the product of magnitude of $\vec{b}$ and the projection of $\vec{a}$ on $\vec{b}$ as in the figure.



$$\vec{a}.\vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta = |\vec{a}|\,(OB\cos\theta) = |\vec{a}|\,OM$$
$$= (\text{magnitude of } \vec{a})\ (\text{projection of } \vec{b} \text{ on } \vec{a})$$

Also

$$\vec{a}.\vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta = |\vec{b}|\,(OA\cos\theta) = |\vec{b}|\,OL$$
$$= (\text{magnitude of } \vec{b})\ (\text{projection of } \vec{a} \text{ on } \vec{b})$$

Properties of scalar product of two vectors

It follows from the definition that

(i) $\vec{a}.\vec{a} = |\vec{a}|^2$

(ii) $\vec{a}.\vec{b} = \vec{b}.\vec{a}$ *(Commutative)*

(ii) $\vec{a}.(\vec{b} + \vec{c}) = \vec{a}.\vec{b} + \vec{a}.\vec{c}$ *(Distributive)*

(iv) $m\vec{a}.n\vec{b} = mn(\vec{a}.\vec{b}) = mn\vec{a}.\vec{b} = \vec{a}.mn\vec{a}$ *(Associative)*

(v) Two vectors $\vec{a}$ and $\vec{b}$ are perpendicular if $\vec{a}.\vec{b} = 0$

(vi) If $\vec{i}, \vec{j}, \vec{k}$ are three unit vectors, then

$$\vec{i}.\vec{i} = 1, \quad \vec{j}.\vec{j} = 1, \quad \vec{k}.\vec{k} = 1.$$
$$\text{and } \vec{i}.\vec{j} = 0, \quad \vec{j}.\vec{k} = 0, \quad \vec{k}.\vec{i} = 0$$

Unit vectors $\vec{i}, \vec{j}, \vec{k}$, modulus and direction cosines of a vector $\vec{r}$

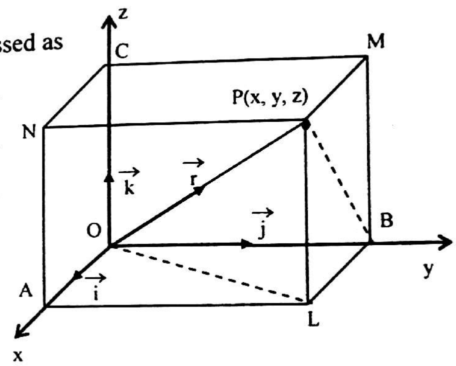
(i) Let $\vec{i}, \vec{j}, \vec{k}$ be the unit vectors along three rectangular co-ordinate axes defined by

$\vec{i} = (1,0,0)$, $\vec{j} = (0,1,0)$ and $\vec{k} = (0,0,1)$ respectively, then every vector can be expressed in terms of unit vectors $\vec{i}, \vec{j}, \vec{k}$.

For example

The vector (x, y, z) can be expressed as

$$x\vec{i} + y\vec{j} + z\vec{k}.$$



Let $\overrightarrow{OP}$ be the vector denoted by $\vec{r}$ and P(x, y, z) be any point in space, then

OA = x, OB = y and OC = z.

and $\overrightarrow{OA} = x\vec{i}$, $\overrightarrow{OB} = y\vec{j}$,

$\overrightarrow{OC} = z\vec{k}$

Again $\overrightarrow{OP} = \overrightarrow{OB} + \overrightarrow{BP} = \overrightarrow{OB} + \overrightarrow{BL} + \overrightarrow{LP}$

$= \overrightarrow{OB} + \overrightarrow{OA} + \overrightarrow{OC} = x\vec{i} + y\vec{j} + z\vec{k}$

$\therefore \quad \vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

Note:

If the vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$ are parallel to the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ respectively, then $\vec{r} = x\vec{a} + y\vec{b} + z\vec{c}$.

(ii) Modulus of $\overrightarrow{OP}$

From the above figure, we have

$$OP^2 = OL^2 + PL^2 = OB^2 + BL^2 + DL^2$$

$= OB^2 + OA^2 + OC^2$

$OP^2 = y^2 + x^2 + z^2$

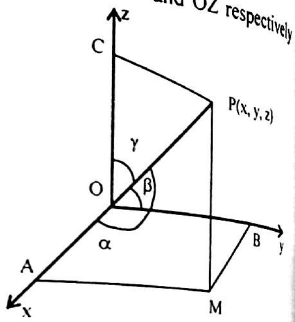
$OP = \sqrt{x^2 + y^2 + z^2}$

Thus if $\overrightarrow{OP} = \vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$, then

$$OP = |\overrightarrow{OP}| = \sqrt{x^2 + y^2 + z^2}$$

(iii) Direction cosines of $\overrightarrow{OP}$

The line OP makes the angles α, β, γ with OX, OY and OZ respectively then from the figure



We have

$OA = x = OP\cos\alpha$

$OB = y = OP\cos\beta$

$OC = z = OP\cos\gamma$

$\therefore \quad \cos\gamma = \dfrac{OC}{OP} = \dfrac{z}{\sqrt{x^2 + y^2 + z^2}}$

Similarly

$\cos\beta = \dfrac{OB}{OP} = \dfrac{y}{\sqrt{x^2 + y^2 + z^2}}$

and $\quad \cos\alpha = \dfrac{OA}{OP} = \dfrac{x}{\sqrt{x^2 + y^2 + z^2}}$

Hence cos α, cos β, cos γ are the direction cosines of the line OP.
Therefore the direction cosines of OP are

$$\frac{x}{\sqrt{x^2+y^2+z^2}}, \frac{y}{\sqrt{x^2+y^2+z^2}}, \frac{z}{\sqrt{x^2+y^2+z^2}}$$

Also if $\overrightarrow{OP} = x\vec{i} + y\vec{j} + z\vec{k}$, then

$$\widehat{OP} = \frac{\overrightarrow{OP}}{|\overrightarrow{OP}|} = \frac{x\vec{i} + y\vec{j} + z\vec{k}}{\sqrt{x^2+y^2+z^2}}$$

$$= \frac{x}{\sqrt{x^2+y^2+z^2}}\vec{i} + \frac{y}{\sqrt{x^2+y^2+z^2}}\vec{j} + \frac{z}{\sqrt{x^2+y^2+z^2}}\vec{k}$$

This shows that the direction cosines of the line OP is the coefficients of $\vec{i}, \vec{j}, \vec{k}$, in the unit vector $\widehat{OP}$.

5.3.2 Vector Product or Corss Product of Two Vectors

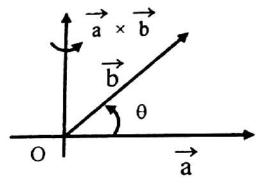
Let $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$ be two vector then the vector product or cross product of two vectors is defined by

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= (a_2b_3 - a_3b_2)\vec{i} - (a_1b_3 - a_3b_1)\vec{j} + (a_1b_2 - a_2b_1)\vec{k}$$

Again, if $\hat{n}$ be unit vector normal to the plane of $\vec{a}$ and $\vec{b}$, then

$\vec{a} \times \vec{b} = |\vec{a}|\,|\vec{b}| \sin\theta\, \hat{n}$ where θ be angle between two vectors $\vec{a}$ and $\vec{b}$.



In addition, we have

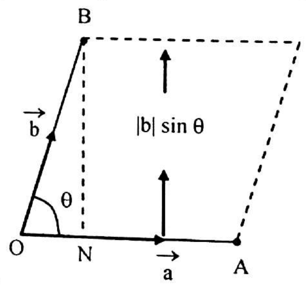
$|\vec{a} \times \vec{b}| = ab\sin\theta$

$\sin\theta = \dfrac{|\vec{a} \times \vec{b}|}{ab}$

Geometrical interpretation

The vector or cross product of two vectors $\vec{a} \times \vec{b}$ represents a vector which is perpendicular to the vectors $\vec{a}$ and $\vec{b}$ respectively. The magnitude of $\vec{a} \times \vec{b}$ i.e. $|\vec{a} \times \vec{b}|$ is equal to the area of parallelogram of which $\vec{a}$ and $\vec{b}$ are the adjacent sides.

If θ be the angle between two vectors $\vec{a}$ and $\vec{b}$ with initial point O. Let C be the point such that segments OA and OB are adjacent sides of a parallelogram with vertices O, A, B and C as illustrated in figure.



An altitude of parallelogram is $|\vec{b}|\sin\theta$.

and $|\vec{a} \times \vec{b}| = |\vec{a}|\,|\vec{b}|\sin\theta =$ OA (OB sin θ)

= (OA) (BN) = Area of the parallelogram OACB

Properties of cross product

Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be vectors and m and n are scalars then

(i) $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$

(ii) $(m\vec{a}) \times \vec{b} = m(\vec{a} \times \vec{b}) = \vec{a} \times (m\vec{b})$

(iii) $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$

(iv) $(\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$

(v) For unit vectors,

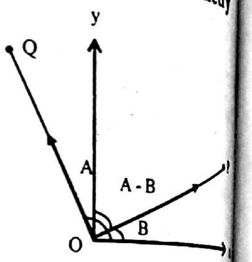
$$\vec{i} \times \vec{i} = 0, \quad \vec{j} \times \vec{j} = 0, \quad \vec{k} \times \vec{k} = 0$$

$$\vec{i} \times \vec{j} = \vec{k}, \quad \vec{j} \times \vec{k} = \vec{i}, \quad \vec{k} \times \vec{i} = \vec{j}$$

$$\vec{j} \times \vec{i} = -\vec{k}, \quad \vec{k} \times \vec{j} = -\vec{i}, \quad \vec{i} \times \vec{k} = -\vec{j}$$

Applications of scalar and vector product to plane Geometry

1. cos(A - B) = cosA cosB + sinA sinB



Let OX and OY be x-axis and y-axis such that

$\angle POX = B$, $\angle QOX = A$

$\angle POQ = A - B$ and $OP = r_1$, $OQ = r_2$.

Then the coordinates of P and Q are

$(r_1 \cos B,\ r_1 \sin B)$ and $(r_2 \cos A,\ r_2 \sin A)$ respectively.

$$\overrightarrow{OP} = (r_1 \cos B)\,\vec{i} + (r_1 \sin B)\,\vec{j}$$

$$\overrightarrow{OQ} = (r_2 \cos A)\,\vec{i} + (r_2 \sin A)\,\vec{j}$$ where $\vec{i}$ and $\vec{j}$ are unit vectors along the axes OX and OY respectively. Since angle between OP and OQ be A – B.

$$\therefore \quad \overrightarrow{OP}\cdot\overrightarrow{OQ} = |\overrightarrow{OP}|\,|\overrightarrow{OQ}| \cos(A - B)$$

$$\overrightarrow{OP}\cdot\overrightarrow{OQ} = r_1 r_2 \cos(A - B)$$

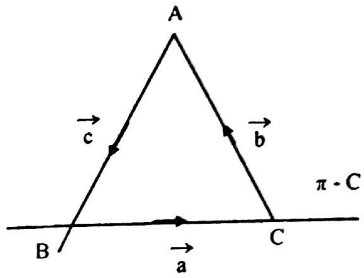
Also $$\overrightarrow{OP}\cdot\overrightarrow{OQ} = \{(r_1 \cos B)\,\vec{i} + (r_1 \sin B)\,\vec{j}\}$$
$$= \{(r_2 \cos A)\,\vec{i} + (r_2 \sin A)\,\vec{j}\}$$

$$\overrightarrow{OP}\cdot\overrightarrow{OQ} = r_1 r_2 \cos A \cos B + r_1 r_2 \sin A \sin B$$

$$r_1 r_2 \cos(A - B) = r_1 r_2 (\cos A \cos B + \sin A \sin B)$$

$$\therefore \cos(A - B) = \cos A \cos B + \sin A \sin B$$

2. $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$



Let ABC be a triangle in which

$$\overrightarrow{AB} = \vec{c},\ \overrightarrow{BC} = \vec{a} \text{ and } \overrightarrow{CA} = \vec{b}.$$

So that $|\overrightarrow{AB}| = c$, $|\overrightarrow{BC}| = a$ and $|\overrightarrow{CA}| = b$

$$\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = 0$$

$$\therefore \quad \vec{c} + \vec{a} + \vec{b} = 0$$

or $\quad \vec{c} = -(\vec{a} + \vec{b})$

or $\quad c^2 = (\vec{a} + \vec{b})^2 = a^2 + b^2 + 2\,\vec{a}\cdot\vec{b}$

The angle between the vectors $\vec{a}$ and $\vec{b}$ is $\pi - C$ so that

$$c^2 = a^2 + b^2 + 2ab\cos(\pi - C)$$

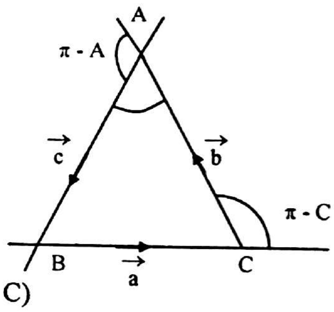
$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$\therefore \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

3. b = c cosA + a cosC

Let ABC be a triangle in which $\overrightarrow{AB} = \vec{c}$, $\overrightarrow{BC} = \vec{a}$ and $\overrightarrow{CA} = \vec{b}$ with

$|\overrightarrow{AB}| = c$, $|\overrightarrow{BC}| = a$ and $|\overrightarrow{CA}| = b$.



Then $\quad \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = 0$

$$\vec{c} + \vec{a} + \vec{b} = 0$$

or $\quad \vec{b} = -\vec{c} - \vec{a}$

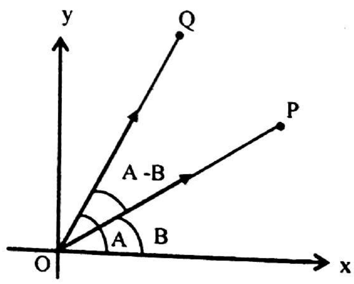
or $\quad \vec{b}\cdot\vec{b} = -\vec{c}\cdot\vec{b} - \vec{a}\cdot\vec{b}$

or $\quad b^2 = -cb\cos(\pi - A) - ab\cos(\pi - C)$

$$b^2 = bc\cos A + ab\cos C,$$

$$\therefore \quad b = c\cos A + a\cos C$$

4. sin(A - B) = sinA cosB - cosAsinB



Let OX and OY be rectangular axes with unit vectors $\vec{i}$ and $\vec{j}$ respectively and P and Q be two points in the plane so that $\angle POX = B$, $\angle QOX = A$, $\angle QOP = A - B$.

$OP = r_1, \quad OQ = r_2.$

Then the coordinates of P and Q are

$(r_1 \cos B, r_1 \sin B)$ and $(r_2 \cos A, r_2 \sin A)$ respectively.

So $\overrightarrow{OP} = (r_1 \cos B)\,\vec{i} + (r_1 \sin B)\,\vec{j}$

$\overrightarrow{OQ} = (r_2 \cos A)\,\vec{i} + (r_2 \sin A)\,\vec{j}$

The angle between $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ is A - B so that

$\overrightarrow{OP} \times \overrightarrow{OQ} = |\overrightarrow{OP}|\,|\overrightarrow{OQ}| \sin(A - B)\,\hat{n}$ where $\hat{n}$ is unit vector.

$= (OP)(OQ)\sin(A - B)\,\hat{n}$

$\{(r_1 \cos B)\,\vec{i} + (r_1 \sin B)\,\vec{j}\} \times \{(r_2 \cos A)\,\vec{i} + (r_2 \sin A)\,\vec{j}\}$

$= r_1 r_2 \sin(A - B)\,\hat{n}$

or $r_1 r_2 (\sin A \cos B - \cos A \sin B)\,\vec{k} = r_1 r_2 \sin(A - B)\,\hat{n}$

Taking modulus on both sides, we get

$r_1 r_2 (\sin A \,.\cos B - \cos A.\sin B) = r_1 r_2 \sin(A - B)$

$\therefore \quad \sin(A - B) = \sin A \cos B - \cos A \sin B$

Worked Out Examples

Ex.1 Prove that the points $2\vec{i}-\vec{j}+\vec{k}$, $\vec{i}-3\vec{j}-5\vec{k}$, $3\vec{i}-4\vec{j}-4\vec{k}$ are the vertices of a right angle triangle. Find also other angles.

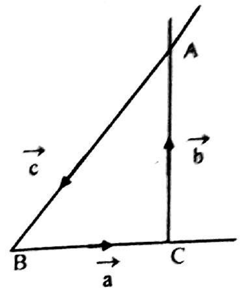
Solution:

Let the given points be A, B and C

Let O be the origin and A ,B and C be the position vectors, then

$\overrightarrow{OA} = 2\vec{i} - \vec{j} + \vec{k}$

$\overrightarrow{OB} = \vec{i} - 3\vec{j} - 5\vec{k},$



and $\overrightarrow{OC} = 3\vec{i} - 4\vec{j} - 4\vec{k}$

Now, $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= \vec{i} - 3\vec{j} - 5\vec{k} - 2\vec{i} + \vec{j} - \vec{k}$

$\therefore \quad \vec{c} = -\vec{i} - 2\vec{j} - 6\vec{k}$

Also $|\overrightarrow{AB}| = c = \sqrt{1 + 4 + 36} = \sqrt{41}$

Again, $\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$

$= 3\vec{i} - 4\vec{j} - 4\vec{k} - \vec{i} + 3\vec{j} + 5\vec{k} = 2\vec{i} - \vec{j} + \vec{k}$

$\therefore \quad \vec{a} = 2\vec{i} - \vec{j} + \vec{k}$

Also $|\overrightarrow{BC}| = a = \sqrt{4 + 1 + 1} = \sqrt{6}$

Again, $\overrightarrow{CA} = \vec{b} = \overrightarrow{OA} - \overrightarrow{OC}$

$= 2\vec{i} - \vec{j} + \vec{k} - 3\vec{i} + 4\vec{j} + 4\vec{k} = -\vec{i} + 3\vec{j} + 5\vec{k}$

$\therefore \quad \vec{b} = -\vec{i} + 3\vec{j} + 5\vec{k}$

Also $|\overrightarrow{CA}| = b = \sqrt{1 + 9 + 25} = \sqrt{35}$

Clearly, $c^2 = a^2 + b^2$. Hence, the triangle ABC is right angle.

Again, $\vec{a}.\vec{c} = (2\vec{i} - \vec{j} + \vec{k}).(-\vec{i} - 2\vec{j} - 6\vec{k})$

$= -2 + 2 - 6 = -6$

$\vec{a}.\vec{b} = (2\vec{i} - \vec{j} + \vec{k}).(-\vec{i} + 3\vec{j} + 5\vec{k})$

$= -2 - 3 + 5 = 0$

$\therefore \quad \vec{a}$ is perpendicular to $\vec{b}$

$\vec{b}.\vec{c} = (-\vec{i} + 3\vec{j} + 5\vec{k}).(-\vec{i} - 2\vec{j} - 6\vec{k})$

$= 1 - 6 - 30 = -35$

We have, $\vec{a}.\vec{c} = ac\cos(\pi - B)$

$-6 = -\sqrt{6}\sqrt{41}\cos B \quad$ or $\cos B = \sqrt{\dfrac{6}{41}}$

$\therefore \quad B = \cos^{-1}\sqrt{\dfrac{6}{41}}$

and $\vec{b}.\vec{c} = bc\cos(\pi - A)$

$-35 = -\sqrt{35}\sqrt{41}\cos A$

or $\quad \cos A = \sqrt{\dfrac{35}{41}}$

$\therefore \quad A = \cos^{-1}\sqrt{\dfrac{35}{41}}$.

Ex.2. Find the cosine of the angle between directions of the vectors $\vec{a} = 4\vec{i} + 3\vec{j} + \vec{k}$ and $\vec{b} = 2\vec{i} - \vec{j} + 2\vec{k}$. Also find unit vector perpendicular to both $\vec{a}$ and $\vec{b}$

Solution:

Here $\quad \vec{a} = 4\vec{i} + 3\vec{j} + \vec{k}, \quad \vec{b} = 2\vec{i} - \vec{j} + 2\vec{k}$

$\vec{a}.\vec{b} = (4\vec{i} + 3\vec{j} + \vec{k}).(2\vec{i} - \vec{j} + 2\vec{k}) = 8 - 3 + 2 = 7$

$|\vec{a}| = a = \sqrt{16+9+1} = \sqrt{26}, \quad |\vec{b}| = b = \sqrt{4+1+4} = 3$

We have $\vec{a}.\vec{b} = ab\cos\theta$ where θ be angle between two vectors $\vec{a}$ and $\vec{b}$

or $\quad 7 = 3\sqrt{26}\cos\theta, \quad$ or $\cos\theta = \dfrac{7}{3\sqrt{26}}$,

$\therefore \quad \theta = \cos^{-1}\left(\dfrac{7}{3\sqrt{26}}\right)$.

Also, we know that the vector $\vec{a} \times \vec{b}$ is perpendicular to both $\vec{a}$ and $\vec{b}$

Therefore

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & 3 & 1 \\ 2 & -1 & 2 \end{vmatrix} = 7\vec{i} - 6\vec{j} - 10\vec{k}$$

$|\vec{a} \times \vec{b}| = \sqrt{49+36+100} = \sqrt{185}$

The unit vector along $\vec{a} \times \vec{b}$ is

$$= \frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|} = \frac{1}{\sqrt{85}}(7\vec{i} - 6\vec{j} - 10\vec{k})$$

Ex. 3. Find the area of the parallelogram determined by the sides of vectors $\vec{i} + 2\vec{j} + 3\vec{k}$ and $3\vec{i} - 2\vec{j} + \vec{k}$

Solution:

Let $\quad \vec{a} = \vec{i} + 2\vec{j} + 3\vec{k}, \quad \vec{b} = 3\vec{i} - 2\vec{j} + \vec{k}$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 3 \\ 3 & -2 & 1 \end{vmatrix} = \vec{i}(2+6) - \vec{j}(1-9) + \vec{k}(-2-6)$$

$= 8\vec{i} + 8\vec{j} - 8\vec{k}$

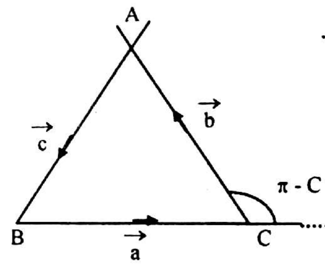
Area of the parallelogram $= |\vec{a} \times \vec{b}| = \sqrt{64+64+64}$

$= \sqrt{3 \times 64} = 8\sqrt{3}$ Sq. unit

Ex. 4. By the vector method, establish the relation between the sides and angles of a triangle. i.e. $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$

Solution:

Let ABC be a triangle with vectors $\overrightarrow{BC}, \overrightarrow{CA}$ and $\overrightarrow{AB}$ denoted by $\vec{a}, \vec{b}$ and $\vec{c}$ respectively.



Then

$\vec{a} + \vec{b} + \vec{c} = 0 \quad$ .... (1)

Taking cross product of $\vec{a}$ on both sides

$\vec{a} \times (\vec{a} + \vec{b} + \vec{c}) = 0$

or $\quad \vec{a} \times \vec{b} + \vec{a} \times \vec{c} = 0$

or $\quad \vec{a} \times \vec{b} = \vec{c} \times \vec{a} \quad$ .... (2)

Again taking cross product of $\vec{b}$ to (1)

$\vec{b} \times (\vec{a} + \vec{b} + \vec{c}) = 0$

or $\quad \vec{b} \times \vec{a} + \vec{b} \times \vec{c} = 0$

or $\quad \vec{a} \times \vec{b} = \vec{b} \times \vec{c} \quad$ .... (3)

From (2) and (3) $|\vec{a} \times \vec{b}| = |\vec{b} \times \vec{c}| = |\vec{c} \times \vec{a}|$

or $\quad ab\sin(\pi - C) = bc\sin(\pi - A) = ca\sin(\pi - B)$

$ab\sin C = bc\sin A = ca\sin B$

$\therefore \quad \dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$ Proved.

Exercise -19

1. Find the angles of a triangle whose vertices are $7\vec{j} + 10\vec{k}$, $-\vec{i} + 6\vec{j} + 6\vec{k}$ and $-4\vec{i} + 9\vec{j} + 6\vec{k}$ respectively

2. Find the unit vector perpendicular to each of the vectors $2\vec{i} - \vec{j} + \vec{k}$ and $3\vec{i} + 4\vec{j} - \vec{k}$ and calculate the sine of the angle between the two vectors

3. If $|\vec{a} + \vec{b}| = |\vec{a} - \vec{b}|$ then show that $\vec{a}$ and $\vec{b}$ are perpendicular.

4. If A, B, C and D are any four points, show that
$$\overrightarrow{AB}\,.\,\overrightarrow{CD} + \overrightarrow{BC}\,.\,\overrightarrow{AD} + \overrightarrow{CA}\,.\,\overrightarrow{BD} = 0$$

5. If $\hat{a}$ and $\hat{b}$ are unit vectors and θ is the angle between them, show that $\sin\frac{\theta}{2} = \frac{1}{2}|\hat{a} - \hat{b}|$

6. Prove vertically
   (i) $a = b\cos C + c\cos B$ (ii) $c = a\cos B + b\cos A$
   (iii) $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$ (iv) $\cos B = \frac{a^2 + c^2 - b^2}{2ac}$
   (v) $\cos(A + B) = \cos A\cos B - \sin A\sin B$

7. If $\vec{a}$ is any vector, prove that $(\vec{a}.\vec{i})\vec{i} + (\vec{a}.\vec{j})\vec{j} + (\vec{a}.\vec{k})\vec{k} = \vec{a}$

8. Find the angles vertically between the diagonals of a cube

9. If a line makes angle α, β, γ, δ with four diagonals of a cube prove that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = \frac{4}{3}$

10. Prove vectorically that $\sin(A + B) = \sin A\cos B + \cos A\sin B$

Answers

1. $\frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{2}$ 2. $\frac{1}{\sqrt{155}}(-3\vec{i} + 5\vec{j} + 11\vec{k}), \sin^{-1}\sqrt{\frac{155}{156}}$

8. $\cos^{-1}\left(\frac{1}{3}\right)$

5.3.3 Products of Three Vectors

The dot product of two vectors $\vec{a}$ and $\vec{b}$ is scalar quantity and the cross product of two vectors $\vec{a}$ and $\vec{b}$ is a vector quantity.

i.e. $\vec{a}.\vec{b} = a\,b\cos\theta$ is a scalar

and $\vec{a} \times \vec{b} = a\,b\sin\theta\,\hat{n}$ is a vector where $\hat{n}$ is a unit vector perpendicular to the plane of given vectors.

Here $\vec{a} \times \vec{b}$ is a vector which can be multiplied both scalarly and vertically by another vector $\vec{c}$.

i.e. $(\vec{a} \times \vec{b}).\,\vec{c} = \vec{c}.\,(\vec{a} \times \vec{b})$ is scalar triple product and

$(\vec{a} \times \vec{b}) \times \vec{c} = -\,\vec{c} \times (\vec{a} \times \vec{b})$ is vector triple product of the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$.

5.3.4 Scalar Triple Product

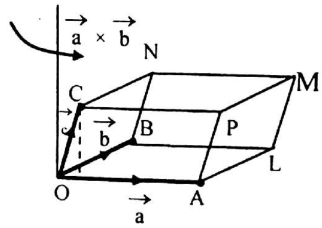
The scalar triple product is the scalar product of two vectors in which one of them must be vector product of two vectors.

If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are three vectors, then the scalar or dot product of $\vec{a} \times \vec{b}$ with $\vec{c}$ is called the *Scalar Product* of three vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ and is denoted by

$$\vec{a}.\vec{b} \times \vec{c} = [\vec{a}\ \vec{b}\ \vec{c}]$$

5.3.5 Geometrical Interperetation of Scalar Triple Product

Consider the parallelepiped whose concurrent edges OA, OB and OC represent in magnitude and direction of vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ respectively.



Scanned by CamScanner

Therefore $\vec{a} \times \vec{b}$ represents a vectors whose magnitude of the vector $\vec{a} \times \vec{b}$ is equal to the area of the parallelogram OALB whose direction is perpendicular to the plane of $\vec{a}$, $\vec{b}$ and that of $\vec{c}$.

If θ be the angle between the direction of $\vec{a} \times \vec{b}$ and $\vec{c}$, then

$$(\vec{a} \times \vec{b}).\vec{c} = |\vec{a} \times \vec{b}|\,|\vec{c}| \cos\theta$$

= (magnitude of $\vec{a} \times \vec{b}$) (projection of $\vec{c}$ on $\vec{a} \times \vec{b}$

= (area of the parallelogram OALB) (height)

= volume of the parallelopiped = V

Hence $(\vec{a} \times \vec{b}).\vec{c} = V$

In similar manner, we can show that $(\vec{b} \times \vec{c}).\vec{a}$ and $\vec{c} \times \vec{a}.\vec{b}$ also represent the volume of parallelopiped forming by the sides $\vec{a}, \vec{b}$ and $\vec{c}$.

$$\therefore (\vec{a} \times \vec{b}).\vec{c} = (\vec{b} \times \vec{c}).\vec{a} = (\vec{c} \times \vec{a}).\vec{b} = V$$

The value of the scalar triple product is positive if θ is acute i.e. $\vec{a}, \vec{b}, \vec{c}$ form a right-handed system of vectors.

5.3.6 Properties of Scalar Triple Product

(i) *The scalar triple product of unit vectors $\vec{i}$, $\vec{j}$ and $\vec{k}$ is unity*

Let $\vec{i}, \vec{j}, \vec{k}$ be three unit vectors, then

$$[\vec{i}\ \vec{j}\ \vec{k}] = \vec{i}.\vec{j} \times \vec{k} = \vec{i}.\vec{i} = 1 \qquad (\because\ \vec{j} \times \vec{k} = \vec{i})$$

$$\therefore [\vec{i}\ \vec{j}\ \vec{k}] = 1.$$

(ii) *Scalar triple product of vectors $\vec{a}$, $\vec{b}$ and is $\vec{c}$ can be expressed in determinant form*

Let the three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ can be written as

$$\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}, \quad \vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$$

$$\vec{c} = c_1\vec{i} + c_2\vec{j} + c_3\vec{k}$$ where $\vec{i}$, $\vec{j}$, $\vec{k}$ are three mutually perpendicular non-coplanar unit vectors.

Then $\vec{a} \times \vec{b} = (a_1\vec{i} + a_2\vec{j} + a_3\vec{k}) \times (b_1\vec{i} + b_2\vec{j} + b_3\vec{k})$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= (a_2b_3 - a_3b_2)\vec{i} + (a_3b_1 - a_1b_3)\vec{j} + (a_1b_2 - a_2b_1)\vec{k}$$

Now $\vec{a} \times \vec{b}.\vec{c} = \{(a_2b_3 - a_3b_2)\vec{i} + (a_3b_1 - a_1b_3)\vec{j} + (a_1b_2 - a_2b_1)\vec{k}\}.(c_1\vec{i} + c_2\vec{j} + c_3\vec{k})$

$$= c_1(a_2b_3 - a_3b_2) + c_2(a_3b_1 - a_1b_3) + c_3(a_1b_2 - a_2b_1)$$

$$= \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$\therefore [\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

(iii) *In a scalar triple product, the position of dot and cross can be interchanged without changing its value*

i.e. $(\vec{a} \times \vec{b}).\vec{c} = \vec{a}.(\vec{b} \times \vec{c})$

Let $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}, \quad \vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$

$\vec{c} = c_1\vec{i} + c_2\vec{j} + c_3\vec{k}$

Then we have

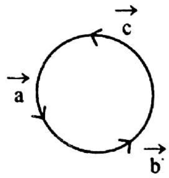
$$\vec{a} \times \vec{b}.\vec{c} = \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = - \begin{vmatrix} a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$\therefore \vec{a} \times (\vec{b}.\vec{c}) = \vec{a}.(\vec{b} \times \vec{c})$$

It shows that the position of dot and cross product can be inter charged without changing its value.

(iv) *The value of scalar triple product of three vectors remains unchanged if the vectors are in cyclic order otherwise the value of scalar triple product be changed*

In Art 5.3.3 ,we have shown that



$$\vec{a}.(\vec{b} \times \vec{c}) = \vec{b}.(\vec{c} \times \vec{a}) = \vec{c}.(\vec{a} \times \vec{b})$$

i.e. $[\vec{a}\ \vec{b}\ \vec{c}] = [\vec{b}\ \vec{c}\ \vec{a}] = [\vec{c}\ \vec{a}\ \vec{b}]$

Again, since the cross product is not commutative and we have

$$\vec{b} \times \vec{c} = -\vec{c} \times \vec{b}$$

Therefore

$$\vec{a}.(\vec{b} \times \vec{c}) = (\vec{b} \times \vec{c}).\vec{a} = -(\vec{c} \times \vec{b}).\vec{a} = -\vec{a}.(\vec{c} \times \vec{b})$$

$$[\vec{a}\ \ \vec{b}\ \ \vec{c}] = -[\vec{a}\ \ \vec{c}\ \ \vec{b}]$$

Similarly,

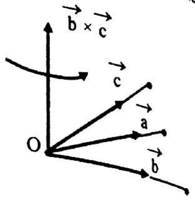
$$[\vec{b}\ \vec{c}\ \vec{a}] = -[\vec{b}\ \vec{a}\ \vec{c}] \text{ and } [\vec{c}\ \vec{a}\ \vec{b}] = -[\vec{c}\ \vec{b}\ \vec{a}]$$

(v) *Three vectors are coplanar if their scalar triple product be zero*

Let the three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ be coplanar. Now $\vec{b}\times\vec{c}$ represents a vector perpendicular to the plane of $\vec{b}$ and $\vec{c}$.

Since $\vec{a}$ lies in the plane of $\vec{b}$ and $\vec{c}$, so

$\vec{b}\times\vec{c}$ is also perpendicular to the plane of $\vec{a}$.



Hence

$$\vec{a}.(\vec{b}\times\vec{c}) = 0, \qquad [\vec{a}\ \vec{b}\ \vec{c}] = 0$$

Conversely,

let $[\vec{a}\ \vec{b}\ \vec{c}] = 0$ i.e. $\vec{a}.\vec{b}\times\vec{c} = 0$

It shows that $\vec{b}\times\vec{c}$ is a vector perpendicular to $\vec{a}$. But $\vec{b}\times\vec{c}$ is also a vector perpendicular to both plane of $\vec{b}$ and $\vec{c}$. Hence $\vec{a}$ should lie on the plane of $\vec{b}$ and $\vec{c}$. $\vec{a}$, $\vec{b}$, $\vec{c}$ are coplanar.

(vi) *The value of scalar triple product be zero when two of the vectors are equal*

Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be the three vectors in which $\vec{b} = \vec{a}$, then

$$[\vec{a}\ \vec{b}\ \vec{c}] = [\vec{a}\ \vec{a}\ \vec{c}]$$

$$= \vec{a}.\vec{a}\times\vec{c} = (\vec{a}\times\vec{a}).\vec{c} = 0$$

$$\therefore \quad [\vec{a}\ \vec{a}\ \vec{c}] = 0.$$

(vii) *The value of scalar triple product be zero when two of the vectors are parallel*

Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be the three vectors such that $\vec{a}$ and $\vec{b}$ are parallel, then there exist a scalar m in such that $\vec{b} = m\vec{a}$

So $[\vec{a}\ \vec{b}\ \vec{c}] = \vec{a}.(\vec{b}\times\vec{c}) = \vec{a}.(m\vec{a}\times\vec{c})$

$$= \vec{a}\times m\vec{a}.\vec{c} = m(\vec{a}\times\vec{a}).\vec{c} = 0$$

$$\therefore [\vec{a}\ \vec{b}\ \vec{c}] = 0.$$

(viii) *Scalar triple product of the three non-coplanar vectors*

If $\vec{l}, \vec{m}, \vec{n}$ be three non coplanar vectors $[\vec{l}\ \ \vec{m}\ \ \vec{n}] \neq 0$, then

$$[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} [\vec{l}\ \ \vec{m}\ \ \vec{n}]$$

We know that any vector can be expressed in terms of three non-coplanar vectors as

$$\vec{a} = a_1\vec{l} + a_2\vec{m} + a_3\vec{n}, \qquad \vec{b} = b_1\vec{l} + b_2\vec{m} + b_3\vec{n}$$

$$\vec{c} = c_1\vec{l} + c_2\vec{m} + c_3\vec{n}$$

Now

$$\vec{b}\times\vec{c} = (b_1\vec{l} + b_2\vec{m} + b_3\vec{n}) \times (c_1\vec{l} + c_2\vec{m} + c_3\vec{n})$$

$$= (b_2c_3 - b_3c_2)\ \vec{m}\times\vec{n} + (b_3c_1 - b_1c_3)\ \vec{n}\times\vec{l} + (b_1c_2 - b_2c_1)\ \vec{l}\times\vec{m}$$

Also

$$[\vec{a}\ \vec{b}\ \vec{c}] = \vec{a}.\vec{b}\times\vec{c}$$

$$= (a_1\vec{l} + a_2\vec{m} + a_3\vec{n}).\{(b_2c_3 - b_3c_2)\ \vec{m}\times\vec{n} + (b_3c_1 - b_1c_3)\ \vec{n}\times\vec{l} + (b_1c_2 - b_2c_1)\ \vec{l}\times\vec{m}\}$$

$$= a_1(b_2c_3 - b_3c_2)\ \vec{l}.(\vec{m}\times\vec{n}) + a_2(b_3c_1 - b_1c_3)\ \vec{m}.(\vec{n}\times\vec{l}) + a_3(b_1c_2 - b_2c_1)\ \vec{n}.\vec{l}\times\vec{m}$$

Writing $\vec{l}.\vec{m}\times\vec{n} = \vec{m}.\vec{n}\times\vec{l} = \vec{n}.\vec{l}\times\vec{m} = [\vec{l}\ \ \vec{m}\ \ \vec{n}]$

$$= [a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1)]\ [\vec{l}\ \ \vec{m}\ \ \vec{n}]$$

$$\therefore \quad [\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} [\vec{l}\ \vec{m}\ \vec{n}]$$

(ix) *Derivation of distributive law of cross product of the two vectors with the help of scalar triple product*

i.e. *If $\vec{a}, \vec{b}, \vec{c}$ be any three vectors, then show that by the help of scalar triple product* $\vec{a}\times(\vec{b}+\vec{c}) = \vec{a}\times\vec{b} + \vec{a}\times\vec{c}$

Let $\quad \vec{r} = \vec{a}\times(\vec{b}+\vec{c}) - \vec{a}\times\vec{b} - \vec{a}\times\vec{c}$

We have $\quad \vec{r}.[\vec{a}\times(\vec{b}+\vec{c}) - \vec{a}\times\vec{b} - \vec{a}\times\vec{c}]$

$= \vec{r}\,.\,\vec{a} \times (\vec{b} + \vec{c}) - \vec{r}\,.\,(\vec{a} \times \vec{b}) - \vec{r}\,.\,(\vec{a} \times \vec{c})$

Interchanging the scalar triple product of three vectors

$= \vec{r} \times \vec{a}\,.\,(\vec{b} + \vec{c}) - (\vec{r} \times \vec{a})\,.\,\vec{b} - (\vec{r} \times \vec{a})\,.\,\vec{c}$

$= (\vec{r} \times \vec{a}).\,[(\vec{b} + \vec{c}) - \vec{b} - \vec{c}]$

$= \vec{r} \times \vec{a}\,.\,\vec{0} = [\vec{r}\ \vec{a}\ \vec{0}] = 0$

Hence

$\vec{r}\,.\,[\vec{a} \times (\vec{b} \times \vec{c}) - \vec{a} \times \vec{b} - \vec{a} \times \vec{c}] = 0$

For all values of $\vec{r}$

$\vec{a} \times (\vec{b} + \vec{c}) - \vec{a} \times \vec{b} - (\vec{a} \times \vec{c}) = 0$

$\therefore\ \vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$

5.3.7 Vector Triple Product

Vector triple product is also cross product of two vectors in which one of them must be cross product of two vectors.

If $\vec{a}, \vec{b}, \vec{c}$ are three non-zero vectors then $\vec{b} \times \vec{c}$ is a vector perpendicular to the plane of vectors $\vec{b}$ and $\vec{c}$ in which $\vec{a}$ and $\vec{b} \times \vec{c}$ are both vectors. Thus vector triple product is defined by $\vec{a} \times (\vec{b} \times \vec{c})$. The cross product $\vec{a} \times (\vec{b} \times \vec{c})$ is again a vector.

Here the brackets are essential as $\vec{a} \times (\vec{b} \times \vec{c}) \neq (\vec{a} \times \vec{b}) \times \vec{c}$ expressing the fact that vector triple is not associative.

5.3.8 Geometrical Meaning of Vector Triple Product

If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are three non-coplanar vectors then their vector triple product defined by

$\vec{a} \times (\vec{b} \times \vec{c})$ is a vector perpendicular to both $\vec{a}$ and $\vec{b} \times \vec{c}$.

Hence, their scalar product is zero.

i.e. $\vec{a} \times (\vec{b} \times \vec{c}).\ \vec{b} \times \vec{c} = 0$

or $\quad \vec{u}\,.\,\vec{b} \times \vec{c} = 0$ where $\vec{u} = \vec{a} \times (\vec{b} \times \vec{c})$

or $\quad [\vec{u}\ \vec{b}\ \vec{c}] = 0.$

It shows that $\vec{u}$ is coplanar with $\vec{b}$ and $\vec{c}$.

Hence $\vec{a} \times (\vec{b} \times \vec{c})$ is coplanar with $\vec{b}$ and $\vec{c}$.

In particular, $\vec{a} \times (\vec{b} \times \vec{a})$ is a vector coplanar with $\vec{a}$ and $\vec{b}$.

5.3.9 Expression for Vector Triple Product

If $\vec{a}, \vec{b}$ and $\vec{c}$ are any three vectors, then

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a}.\vec{c})\,\vec{b} - (\vec{a}.\vec{b})\,\vec{c}$$

Let $A(a_1, a_2, a_3)$, $B(b_1, b_2, b_3)$ and $C(c_1, c_2, c_3)$ be the three points in space and O the origin. We know any vector can be expressed as linear combination of three mutually perpendicular unit vectors $\vec{i}, \vec{j}, \vec{k}$.

Then $\quad \vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}, \quad \vec{b} = b\,\vec{i} + b_2\vec{j} + b_3\vec{k}$

$\vec{c} = c_1\vec{i} + c_2\vec{j} + c_3\vec{k}$

$\vec{a}.\vec{b} = a_1b_1 + a_2b_2 + a_3b_3, \qquad \vec{b}.\vec{c} = b_1c_1 + b_2c_2 + b_3c_3$

$\vec{a}.\vec{c} = a_1c_1 + a_2c_2 + a_3c_3$

$\vec{b} \times \vec{c} = (b_1\vec{i} + b_2\vec{j} + b_3\vec{k}) \times (c_1\vec{i} + c_2\vec{j} + c_3\vec{k})$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \vec{i}(b_2c_3 - b_3c_2) + \vec{j}(b_3c_1 - b_1c_3) + \vec{k}(b_1c_2 - b_2c_1)$$

Now

$\vec{a} \times (\vec{b} \times \vec{c}) = (a_1\vec{i} + a_2\vec{j} + a_3\vec{k}) \times [(b_2c_3 - b_3c_2)\vec{i}$

$\qquad + (b_3c_1 - b_1c_3)\vec{j} + (b_1c_2 - b_2c_1)\vec{k}]$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_2c_3 - b_3c_2 & b_3c_1 - b_1c_3 & b_1c_2 - b_2c_1 \end{vmatrix}$$

$= [a_2(b_1c_2 - b_2c_1) - a_3(b_3c_1 - b_1c_3)]\,\vec{i}$

$\qquad + [a_3(b_2c_3 - b_3c_2) - a_1(b_1c_2 - b_2c_1)]\,\vec{j}$

$$+ [a_1(b_3c_1 - b_1c_3) - a_2(b_2c_3 - b_3c_2)]\vec{k}$$

$$= (a_2b_1c_2 + a_3b_1c_3)\ \vec{i} - (a_2b_2c_1 + a_3b_3c_1)\vec{i}$$

$$+ (a_3b_2c_3 + a_1b_2c_1)\vec{j} - (a_3b_3c_2 + a_1b_1c_2)\vec{j}$$

$$+ (a_1b_3c_1 + a_2b_3c_2)\ \vec{k} - (a_1b_1c_3 + a_2b_2c_3)\vec{k}$$

$$= (a_1b_1c_1 + a_2b_1c_2 + a_3b_1c_3)\vec{i} - (a_1b_1\, c_1 + a_2b_2c_1 + a_3b_3c_1)\ \vec{i}$$

$$+ (a_1b_2c_1 + a_2b_2c_2 + a_3b_2c_3)\vec{j} - (a_1b_1c_2 + a_2b_2c_2 + a_3b_3c_2)\vec{j}$$

$$+ (a_1b_3c_1 + a_2b_3c_2 + a_3b_3c_3)\vec{k} - (a_1b_1c_3 + a_2b_2c_3 + a_3b_3c_3)\vec{k}$$

$$= (a_1c_1 + a_2c_2 + a_3c_3)\ (b_1\vec{i} + b_2\vec{j} + b_3\vec{k})$$

$$- (a_1b_1 + a_2b_2 + a_3b_3)\ (c_1\vec{i} + c_2\vec{j} + c_3\vec{k})$$

$$= (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\ \vec{c}$$

Hence $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a}.\vec{c})\ \vec{b} - (\vec{a}.\vec{b})\ \vec{c}$

Alternative method

We know that the vector triple product of the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ is defined by $\vec{a} \times (\vec{b} \times \vec{c})$ is a vector which is coplanar with $\vec{b}$ and $\vec{c}$. Therefore, we can write as the linear combination of $\vec{b}$ and $\vec{c}$.

i.e. $\vec{a} \times (\vec{b} \times \vec{c}) = l\vec{b} + m\vec{c}$ ..(1)

where l are m are some scalars.

Multiplying both sides scalarly by $\vec{a}$, we get

$$\vec{a}.\ \vec{a} \times (\vec{b} \times \vec{c}) = l(\vec{a}.\vec{b}) + m\ (\vec{a}.\vec{c})$$

The scalar triple product on the left hand side is zero. Since two of its vectors are equal.

$$\therefore \quad l(\vec{a}.\vec{b}) + \vec{m}\ (\vec{a}.\vec{c}) = 0$$

or $$\frac{l}{\vec{a}.\vec{c}} = \frac{m}{-\vec{a}.\vec{b}} = n \text{ (say)}.$$

Then

$$l = n(\vec{a}.\vec{b}), \qquad m = -n(\vec{a}.\vec{b})$$

Substituting the values of l and m in (1), we get

$$\vec{a} \times (\vec{b} \times \vec{c}) = n[(\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c}] \qquad \text{....... (2)}$$

Evidently n is some numerical constant.

To find it take the special case,

$$\vec{a} = \vec{b} = \vec{i}, \quad \vec{c} = \vec{j}$$

Then (2) gives

$$\vec{i} \times (\vec{i} \times \vec{j}) = n\ [(\vec{i}\ .\ \vec{j})\vec{j} - (\vec{i}\ .\ \vec{i})\vec{j}]$$

or $$\vec{i} \times \vec{k} = -n\vec{j}$$

or $$-\vec{j} = -n\vec{j}.$$

This gives $n = 1$.

Hence (2) reduces to

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a}.\vec{c})\ \vec{b} - (\vec{a}.\vec{b})\ \vec{c}$$

Worked Out Examples

Ex.1. Find the volume of the parallelepiped whose edges are represented by $\vec{a} = 2\ \vec{i} - 3\vec{j} + 4\vec{k}$, $\vec{b} = \vec{i} + 2\vec{j} - \vec{k}$ $\vec{c} = 3\vec{i} - \vec{j} + 2\vec{k}$

Solution:

Given $\vec{a} = 2\vec{i} - 3\vec{j} + 4\vec{k}$, $\qquad \vec{b} = \vec{i} + 2\vec{j} - \vec{k}$

$\vec{c} = 3\vec{i} - \vec{j} + 2\vec{k}$

Volume of the parallelopiped with edges $\vec{a}$, $\vec{b}$ and $\vec{c}$ is given by

$$[\vec{a}\ \vec{b}\ \vec{c}] = \vec{a}.\vec{b} \times \vec{c} = \begin{vmatrix} 2 & -3 & 4 \\ 1 & 2 & -1 \\ 3 & -1 & 2 \end{vmatrix}$$

$$= 2(4 - 1) + 3(2 + 3) + 4(-1 - 6) = 6 + 15 - 28 = -7$$

$\therefore$ Volume $= 7$ cubic unit.

Ex.2. Prove that the four points $4\vec{i} + 5\vec{j} + \vec{k}$, $-\vec{j} - \vec{k}$, $3\vec{i} + 9\vec{j} + 4\vec{k}$ and $-4\vec{i} + 4\vec{j} + 4\vec{k}$ are coplanar.

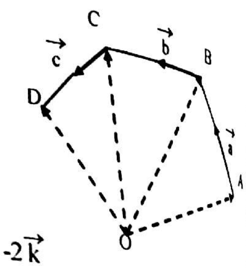
Solution:

Let $\overrightarrow{OA} = 4\vec{i} + 5\vec{j} + \vec{k}$

$\overrightarrow{OB} = -\vec{j} - \vec{k}$

$\overrightarrow{OC} = 3\vec{i} + 9\vec{j} + 4\vec{k}$

$\overrightarrow{OD} = -4\vec{i} + 4\vec{j} + 4\vec{k}$



Now

$\vec{a} = \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = -4\vec{i} - 6\vec{j} - 2\vec{k}$

$\vec{b} = \overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = 3\vec{i} + 10\vec{j} + 5\vec{k}$

$\vec{c} = \overrightarrow{CD} = \overrightarrow{OD} - \overrightarrow{OC} = -7\vec{i} - 5\vec{j}$

The four points are coplanar if the three vectors $\vec{a}, \vec{b}, \vec{c}$ are coplanar i.e. if

$[\vec{a}\ \vec{b}\ \vec{c}] = 0$

Now

$$[\vec{a}\ \vec{b}\ \vec{c}] = \vec{a}.\vec{b}\times\vec{c} = \begin{vmatrix} -4 & -6 & -2 \\ 3 & 10 & 5 \\ -7 & -5 & 0 \end{vmatrix}$$

$$= -4(0+25) + 6(0+35) - 2(-15+70)$$

$$= -100 + 210 - 110 = 0.$$

Hence the four points are coplanar.

Ex. 3. Show that $[\vec{a}+\vec{b}\quad \vec{b}+\vec{c}\quad \vec{c}+\vec{a}] = 2[\vec{a}\ \vec{b}\ \vec{c}]$ [2062 Jestha, B.E]

Solution:

We have

$[\vec{a}+\vec{b}\quad \vec{b}+\vec{c}\quad \vec{c}+\vec{a}]$

$$= (\vec{a}+\vec{b}).(\vec{b}+\vec{c})\times(\vec{c}+\vec{a})$$

$$= (\vec{a}+\vec{b})[\vec{b}\times\vec{c} + \vec{b}\times\vec{a} + \vec{c}\times\vec{c} + \vec{c}\times\vec{a}]$$

$$= (\vec{a}+\vec{b}).(\vec{b}\times\vec{c} + \vec{b}\times\vec{a} + \vec{c}\times\vec{a}) \quad (\because \vec{c}\times\vec{c} = 0)$$

$$= \vec{a}.[\vec{b}\times\vec{c} + \vec{b}\times\vec{a} + \vec{c}\times\vec{a}] + \vec{b}.[\vec{b}\times\vec{c} + \vec{b}\times\vec{a} + \vec{c}\times\vec{a}]$$

$$= \vec{a}.\vec{b}\times\vec{c} + \vec{a}.\vec{b}\times\vec{a} + \vec{a}.\vec{c}\times\vec{a} + \vec{b}.\vec{b}\times\vec{c} + \vec{b}.\vec{b}\times\vec{a} + \vec{b}.\vec{c}\times\vec{a}$$

$$= [\vec{a}\ \vec{b}\ \vec{c}] + 0 + 0 + 0 + [\vec{a}\ \vec{b}\ \vec{c}] = 2[\vec{a}\ \vec{b}\ \vec{c}]$$

$$\therefore [\vec{a}+\vec{b}\quad \vec{b}+\vec{c}\quad \vec{c}+\vec{a}] = 2[\vec{a}\ \vec{b}\ \vec{c}]$$

Ex. 4. Prove that $[\vec{l}\ \vec{m}\ \vec{n}]\ [\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} \vec{l}.\vec{a} & \vec{l}.\vec{b} & \vec{l}.\vec{c} \\ \vec{m}.\vec{a} & \vec{m}.\vec{b} & \vec{m}.\vec{c} \\ \vec{n}.\vec{a} & \vec{n}.\vec{b} & \vec{n}.\vec{c} \end{vmatrix}$

Solution:

We know that every vector can be expressed in terms of three mutually perpendicular unit vector $\vec{i}, \vec{j}, \vec{k}$.

Let $\vec{l} = l_1\vec{i} + l_2\vec{j} + l_3\vec{k}$, $\quad \vec{m} = m_1\vec{i} + m_2\vec{j} + m_3\vec{k}$

$\vec{n} = n_1\vec{i} + n_2\vec{j} + n_3\vec{k}$, $\quad \vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$,

$\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$, $\quad \vec{c} = c_1\vec{i} + c_2\vec{j} + b_3\vec{k}$

Now $[\vec{l}\ \vec{m}\ \vec{n}][\vec{a}\ \vec{b}\ \vec{c}]$

$$= \begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{vmatrix} \times \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \qquad \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} l_1a_1 + l_2a_2 + l_3a_3 & l_1b_1 + l_2b_2 + l_3b_3 & l_1c_1 + l_2c_2 + l_3c_3 \\ m_1a_1 + m_2a_2 + m_3a_3 & m_1b_1 + m_2b_2 + m_3b_3 & m_1c_1 + m_2c_2 + m_3c_3 \\ n_1a_1 + n_2a_2 + n_3a_3 & n_1b_1 + n_2b_2 + n_3b_3 & n_1c_1 + n_2c_2 + n_3c_3 \end{vmatrix}$$

$$= \begin{vmatrix} \vec{l}.\vec{a} & \vec{l}.\vec{b} & \vec{l}.\vec{c} \\ \vec{m}.\vec{a} & \vec{m}.\vec{b} & \vec{m}.\vec{c} \\ \vec{n}.\vec{a} & \vec{n}.\vec{b} & \vec{n}.\vec{c} \end{vmatrix}$$

$$\therefore \quad [\vec{l}\ \vec{m}\ \vec{n}][\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} \vec{l}.\vec{a} & \vec{l}.\vec{b} & \vec{l}.\vec{c} \\ \vec{m}.\vec{a} & \vec{m}.\vec{b} & \vec{m}.\vec{c} \\ \vec{n}.\vec{a} & \vec{n}.\vec{b} & \vec{n}.\vec{c} \end{vmatrix}$$

Ex. 5. If $\vec{a} = \vec{i} - 2\vec{j} - 3\vec{k}$, $\vec{b} = 2\vec{i} + \vec{j} - \vec{k}$ and $\vec{c} = \vec{i} + 3\vec{j} - 2\vec{k}$ find $\vec{a}\times(\vec{b}\times\vec{c})$, $(\vec{a}\times\vec{b})\times\vec{c}$ and $|\vec{a}\times(\vec{b}\times\vec{c})|$. Also prove that $\vec{a}\times(\vec{b}\times\vec{c}) \neq (\vec{a}\times\vec{b})\times\vec{c}$.

Solution:

Given $\vec{a} = \vec{i} - 2\vec{j} - 3\vec{k}$, $\quad \vec{b} = 2\vec{i} + \vec{j} - \vec{k}$

$\vec{c} = \vec{i} + 3\vec{j} - 2\vec{k}$

$$\vec{b} \times \vec{c} = (2\vec{i} + \vec{j} - \vec{k}) \times (\vec{i} + 3\vec{j} - 2\vec{k})$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & -1 \\ 1 & 3 & -2 \end{vmatrix} = \vec{i}\,(-2+3) - \vec{j}\,(-4+1) + \vec{k}\,(6-1)$$

$$= \vec{i} + 3\vec{j} + 5\vec{k}$$

Again, $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{i} - 2\vec{j} - 3\vec{k}) \times (\vec{i} + 3\vec{j} + 5\vec{k})$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -2 & -3 \\ 1 & 3 & 5 \end{vmatrix} = \vec{i}\,(-10+9) - \vec{j}\,(5+3) + \vec{k}\,(3+2)$$

$$\vec{a} \times (\vec{b} \times \vec{c}) = -\vec{i} - 8\vec{j} + 5\vec{k}$$

$$|\vec{a} \times (\vec{b} \times \vec{c})| = \sqrt{1 + 64 + 25} = 3\sqrt{10}$$

Again, $\vec{a} \times \vec{b} = (\vec{i} - 2\vec{j} - 3\vec{k}) \times (2\vec{i} + \vec{j} - \vec{k})$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -2 & -3 \\ 2 & 1 & -1 \end{vmatrix} = \vec{i}\,(2+3) - \vec{j}\,(-1+6) + \vec{k}\,(1+4)$$

$$= 5\vec{i} - 5\vec{j} + 5\vec{k}$$

Now $(\vec{a} \times \vec{b}) \times \vec{c} = (5\vec{i} - 5\vec{j} + 5\vec{k}) \times (\vec{i} + 3\vec{j} - 2\vec{k})$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 5 & -5 & 5 \\ 1 & 3 & -2 \end{vmatrix} = \vec{i}\,(10-15) - \vec{j}\,(-10-5) + \vec{k}\,(15+5)$$

$$= -5\vec{i} + 15\vec{j} + 20\vec{k}$$

Thus $\vec{a} \times (\vec{b} \times \vec{c}) \neq (\vec{a} \times \vec{b}) \times \vec{c}$

Ex. 6: Prove that $\vec{i} \times (\vec{a} \times \vec{i}) + \vec{j} \times (\vec{a} \times \vec{j}) + \vec{k} \times (\vec{a} \times \vec{k}) = 2\vec{a}$ where $\vec{i}, \vec{j}, \vec{k}$ are mutually unit vectors along the co-ordinates axes.

Solution:

We have $\vec{i} \times (\vec{a} \times \vec{i}) + \vec{j} \times (\vec{a} \times \vec{j}) + \vec{k} \times (\vec{a} \times \vec{k})$

$$= (\vec{i}.\vec{i})\vec{a} - (\vec{i}.\vec{a})\vec{i} + (\vec{j}.\vec{j})\vec{a} - (\vec{j}.\vec{a})\vec{j} + (\vec{k}.\vec{k})\vec{a} - (\vec{k}.\vec{a})\vec{k}$$

$$= \vec{a} - (\vec{i}.\vec{a})\vec{i} + \vec{a} - (\vec{j}.\vec{a})\vec{j} + \vec{a} - (\vec{k}.\vec{a})\vec{k}$$

$$= 3\vec{a} - [(\vec{i}.\vec{a})\vec{i} + (\vec{j}.\vec{a})\vec{j} + (\vec{k}.\vec{a})\vec{k}]$$

$$= 3\vec{a} - [(\vec{i}.(a_1\vec{i} + a_2\vec{j} + a_3\vec{k}))]\vec{i} + (\vec{j}.(a_1\vec{i} + a_2\vec{j} + a_3\vec{k}))\vec{j} + (\vec{k}.(a_1\vec{i} + a_2\vec{j} + a_3\vec{k})\,\vec{k}]$$

$$= 3\vec{a} - (a_1\vec{i} + a_2\vec{j} + a_3\vec{k}) = 3\vec{a} - \vec{a} = 2\vec{a}$$

$$\therefore \vec{i} \times (\vec{a} \times \vec{i}) + \vec{j} \times (\vec{a} \times \vec{j}) + \vec{k} \times (\vec{a} \times \vec{k}) = 2\vec{a}$$

Ex.7. Prove that $(\vec{a} \times \vec{b}) \times \vec{c} = \vec{a} \times (\vec{b} \times \vec{c})$ if an only if the vectors $\vec{a}$ and $\vec{c}$ are parallel.

King in the north.

Solution:

"$\Rightarrow$" Given that $(\vec{a} \times \vec{b}) \times \vec{c} = \vec{a} \times (\vec{b} \times \vec{c})$

$$\vec{a} \times (\vec{b} \times \vec{c}) - (\vec{a} \times \vec{b}) \times \vec{c} = 0$$

or $(\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c} + (\vec{c}.\vec{b})\vec{a} - (\vec{c}.\vec{a})\vec{b} = 0$

or $(\vec{b}.\vec{c})\vec{a} - (\vec{a}.\vec{b})\vec{c} = 0$

or $\vec{a} = \dfrac{(\vec{a}.\vec{b})}{(\vec{b}.\vec{c})}\vec{c}$.

or $\vec{a} = x\,\vec{c}$ where $x = \dfrac{(\vec{a}.\vec{b})}{(\vec{b}.\vec{c})}$ is scalar. It shows that $\vec{a}$ and $\vec{c}$ are parallel.

"$\Leftarrow$" Given that $\vec{a}$ and $\vec{c}$ are parallel, then there exists a scalar k such that $\vec{a} = k\vec{c}$, we have to show that $(\vec{a} \times \vec{b}) \times \vec{c} = \vec{a} \times (\vec{b} \times \vec{c})$.

Now

$$(\vec{a} \times \vec{b}) \times \vec{c} = (k\vec{c} \times \vec{b}) \times \vec{c} = -\vec{c} \times (k\vec{c} \times \vec{b})$$

$$= -(\vec{c}.\vec{b})k\vec{c} + (\vec{c}.k\vec{c})\vec{b}$$

$$= k[(\vec{c}.\vec{c})\vec{b} - (\vec{c}.\vec{b})\vec{c}] \qquad \text{....... (1)}$$

and $\vec{a} \times (\vec{b} \times \vec{c}) = k\vec{c} \times (\vec{b} \times \vec{c})$

$$= k[(\vec{c}.\vec{c})\vec{b} - (\vec{c}.\vec{b})\vec{c}] \qquad \text{...... (2)}$$

From (1) and (2), we obtain

$$(\vec{a} \times \vec{b}) \times \vec{c} = \vec{a} \times (\vec{b} \times \vec{c})$$

Ex.8. If $\vec{a}, \vec{b}, \vec{c}$ are three non-coplanar vectors, then expressed $\vec{a}$, $\vec{b}, \vec{c}$ in terms of $\vec{b} \times \vec{c}, \vec{c} \times \vec{a}, \vec{a} \times \vec{b}$ [2063, Kartik B.E.]

Solution:

Let
$$\vec{a} = x_1(\vec{b}\times\vec{c}) + x_2(\vec{c}\times\vec{a}) + x_3(\vec{a}\times\vec{b}) \quad \ldots(1)$$
$$\vec{b} = y_1(\vec{b}\times\vec{c}) + y_2(\vec{c}\times\vec{a}) + y_3(\vec{a}\times\vec{b}) \quad \ldots(2)$$
$$\vec{c} = z_1(\vec{b}\times\vec{c}) + z_2(\vec{c}\times\vec{a}) + z_3(\vec{a}\times\vec{b}) \quad \ldots(3)$$

where x_1, x_2, x_3; y_1, y_2, y_3; z_1, z_2, z_3 are scalars.

Multiplying both sides scalarly by $\vec{a}$ to (i), we get

$$\vec{a}.\vec{a} = x_1(\vec{a}.\vec{b}\times\vec{c}) + x_2(\vec{a}.\vec{c}\times\vec{a}) + x_3(\vec{a}.\vec{a}\times\vec{b})$$

or $$\vec{a}.\vec{a} = x_1[\vec{a}\ \vec{b}\ \vec{c}] + x_2[\vec{a}\ \vec{c}\ \vec{a}] + x_3[\vec{a}\ \vec{a}\ \vec{b}]$$

or $$\vec{a}.\vec{a} = x_1[\vec{a}\ \vec{b}\ \vec{c}] + 0 + 0$$

$$\therefore \quad x_1 = \frac{\vec{a}.\vec{a}}{[\vec{a}\ \vec{b}\ \vec{c}]}$$

Similarly, multiplying both sides of (1) scalarly by $\vec{b}$ and $\vec{c}$, we get

$$x_2 = \frac{\vec{a}.\vec{b}}{[\vec{a}\ \vec{b}\ \vec{c}]}, \quad x_3 = \frac{\vec{a}.\vec{c}}{[\vec{a}\ \vec{b}\ \vec{c}]}$$

Substituting these values of x_1, x_2, x_3 in (1), we get

$$\vec{a} = \frac{1}{[\vec{a}\vec{b}\vec{c}]}[(\vec{a}.\vec{a})(\vec{b}\times\vec{c}) + (\vec{a}.\vec{b})(\vec{c}\times\vec{a}) + (\vec{a}.\vec{c})(\vec{a}\times\vec{b})]$$

Again, multiplying both sides of (2) scalarly by $\vec{a}$, $\vec{b}$ and $\vec{c}$ respectively, we get

$$y_1 = \frac{\vec{a}.\vec{b}}{[\vec{a}\vec{b}\vec{c}]}, \quad y_2 = \frac{\vec{b}.\vec{b}}{[\vec{a}\vec{b}\vec{c}]}, \quad y_3 = \frac{\vec{c}.\vec{b}}{[\vec{a}\vec{b}\vec{c}]}$$

Substituting these values of y_1, y_2, y_3 in (2), we get

$$\vec{b} = \frac{1}{[\vec{a}\vec{b}\vec{c}]}[(\vec{a}.\vec{b})(\vec{b}\times\vec{c}) + (\vec{b}.\vec{b})(\vec{c}\times\vec{a}) + (\vec{c}.\vec{b})(\vec{a}\times\vec{b})]$$

Similarly, multiplying both sides of (3) scalarly by $\vec{a}$, $\vec{b}$ and $\vec{c}$ respectively, we get

$$z_1 = \frac{\vec{a}.\vec{c}}{[\vec{a}\vec{b}\vec{c}]}, \quad z_2 = \frac{\vec{b}.\vec{c}}{[\vec{a}\vec{b}\vec{c}]}, \quad z_3 = \frac{\vec{c}.\vec{c}}{[\vec{a}\vec{b}\vec{c}]}$$

Substituting these values of z_1, z_2, z_3 in (3), we get

$$\vec{c} = \frac{1}{[\vec{a}\ \vec{b}\ \vec{c}]}[(\vec{a}.\vec{c})(\vec{b}\times\vec{c}) + (\vec{b}.\vec{c})(\vec{c}\times\vec{a}) + (\vec{c}.\vec{c})(\vec{a}\times\vec{b})]$$

Exercise - 20

1. Compute the following scalar triple product
   (i) $(3\vec{i} - \vec{j} + 2\vec{k}).(2\vec{i} + \vec{j} - \vec{k}) \times (\vec{i} - 2\vec{j} + 2\vec{k})$
   (ii) $(2\vec{i} - 3\vec{j} + \vec{k}).(\vec{i} - \vec{j} + 2\vec{k}) \times (2\vec{i} + \vec{j} - \vec{k})$
   (iii) $(\vec{i} - 2\vec{j} + 3\vec{k}).(2\vec{i} + \vec{j} - \vec{k}) \times (\vec{j} + \vec{k})$
2. Find the volumes of the parallelepiped whose edges are represented by the vectors.
   (i) $\vec{i} + 2\vec{j} - \vec{k}$, $\vec{i} - \vec{j} + \vec{k}$ and $\vec{i} + \vec{j} + \vec{k}$
   (ii) $\vec{i} + 2\vec{j} + 3\vec{k}$, $\vec{i} - 2\vec{j} + 3\vec{k}$ and $3\vec{i} + 4\vec{j} - 5\vec{k}$
   (iii) $2\vec{i} + 3\vec{j} - 4\vec{k}$, $\vec{i} + \vec{j}$, $\vec{i} - \vec{j} + \vec{k}$
3. Prove that the following four points are coplanar.
   (i) $2\vec{i} + 3\vec{j} - \vec{k}$, $\vec{i} - 2\vec{j} + 3\vec{k}$, $3\vec{i} + 4\vec{j} - 2\vec{k}$, $\vec{i} - 6\vec{j} + 6\vec{k}$
   (ii) $4\vec{i} - 2\vec{j} + \vec{k}$, $5\vec{i} + \vec{j} + 6\vec{k}$, $2\vec{i} + 2\vec{j} - 5\vec{k}$ and $3\vec{i} + 5\vec{j}$
   (iii) $-\vec{i} + 4\vec{j} - 3\vec{k}$, $3\vec{i} + 2\vec{j} - 5\vec{k}$, $3\vec{i} + 8\vec{j} - 5\vec{k}$ and $-3\vec{i} + 2\vec{j} + \vec{k}$
   (iv) $-\vec{i} + 2\vec{j} - 4\vec{k}$, $2\vec{i} - \vec{j} + 3\vec{k}$, $6\vec{i} + 2\vec{j} - \vec{k}$ and $-12\vec{i} - \vec{j} + 3\vec{k}$
4. If $\vec{a} = \vec{i} - 2\vec{j} - 3\vec{k}$, $\vec{b} = 2\vec{i} + \vec{j} - \vec{k}$, $\vec{c} = \vec{i} + 3\vec{j} - 2\vec{k}$, find $\vec{a}.(\vec{b}\times\vec{c})$ and $(\vec{a}\times\vec{b}).\vec{c}$ and hence verify that $\vec{a}.(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b}).\vec{c}$.
5. Find 'p' such that the vectors $2\vec{i} - \vec{j} + \vec{k}$, $\vec{i} + 2\vec{j} + 3\vec{k}$ and $3\vec{i} + p\vec{j} + 5\vec{k}$ are coplanar. [2064PaushB.E.]
6. If $\vec{a}, \vec{b}, \vec{c}$ be three non-coplanar vectors such that $\vec{a} = a_1\vec{l} + a_2\vec{m} + a_3\vec{n}$, $\vec{b} = b_1\vec{l} + b_2\vec{m} + b_3\vec{n}$ and $\vec{c} = c_1\vec{l} + c_2\vec{m} + c_3\vec{n}$

   Prove that $[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} [\vec{l}\ \vec{m}\ \vec{n}]$ [2053 B.E.]
7. Prove that if $\vec{l}, \vec{m}, \vec{n}$ be three non-coplanar vectors then

$$[\vec{l}\;\vec{m}\;\vec{n}][\vec{a}\times\vec{b}] = \begin{vmatrix} \vec{l}.\vec{a} & \vec{l}.\vec{b} & \vec{l} \\ \vec{m}.\vec{a} & \vec{m}.\vec{b} & \vec{m} \\ \vec{n}.\vec{a} & \vec{n}.\vec{b} & \vec{n} \end{vmatrix}$$

9. If $\vec{a} = \vec{i} - 2\vec{j} + \vec{k}$, $\vec{b} = 2\vec{i} + \vec{j} + \vec{k}$ and $\vec{c} = \vec{i} + 2\vec{j} - \vec{k}$ then verify that $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c}$ [2069 Bhadra, B.E.]

10. Prove that $\vec{a}\times(\vec{b}\times\vec{a}) = (\vec{a}\times\vec{b})\times\vec{a}$ [2063Asadh, B.E]

11. Show that $\vec{a}\times(\vec{b}\times\vec{c})$, $\vec{b}\times(\vec{c}\times\vec{a})$, $\vec{c}\times(\vec{a}\times\vec{b})$ are coplanar. [2065, Kartik B.E]

12. Prove that $[\vec{b}\times\vec{c}\;\;\vec{c}\times\vec{a}\;\;\vec{a}\times\vec{b}] = [\vec{a}\;\vec{b}\;\vec{c}]^2$ [2063, Kartik B.E]

13. If $\vec{a}, \vec{b}, \vec{c}$ be three unit vectors such that $\vec{a}\times(\vec{b}\times\vec{c}) = \frac{\vec{b}}{2}$. Find the angles which $\vec{a}$ makes with $\vec{b}$ and $\vec{c}$, where $\vec{b}$ and $\vec{c}$ being non parallel vectors.

14. Prove that $b^2\vec{a} = (\vec{a}.\vec{b})\vec{b} + \vec{b}\times(\vec{a}\times\vec{b})$ where $\vec{a}, \vec{b}, \vec{c}$ are any three vectors.

15. If $\vec{a}, \vec{b}$ and $\vec{c}$ are any three vectors then prove that:

$$\vec{a}\times(\vec{b}\times\vec{c}) + \vec{b}\times(\vec{c}\times\vec{a}) + \vec{c}\times(\vec{a}\times\vec{b}) = 0$$

16. If $\vec{u}, \vec{v}, \vec{w}$ are three vectors defined by the relations

$$\vec{u} = \frac{\vec{b}\times\vec{c}}{[\vec{a}\;\vec{b}\;\vec{c}]},\quad \vec{v} = \frac{\vec{c}\times\vec{a}}{[\vec{a}\;\vec{b}\;\vec{c}]},\quad \vec{w} = \frac{\vec{a}\times\vec{b}}{[\vec{a}\;\vec{b}\;\vec{c}]} \text{ where } [\vec{a}\;\vec{b}\;\vec{c}]\neq 0$$

Prove that $\vec{a}\times\vec{u} + \vec{b}\times\vec{v} + \vec{c}\times\vec{w} = 0$

17. If $\vec{a}, \vec{b}, \vec{c}$ are three non-coplanar vectors then express $\vec{b}\times\vec{c}$, $\vec{c}\times\vec{a}$, $\vec{a}\times\vec{b}$ in terms of $\vec{a}, \vec{b}, \vec{c}$.

18. Show that the four points $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ are coplanar if

$$[\vec{b}\;\vec{c}\;\vec{d}] + [\vec{c}\;\vec{a}\;\vec{d}] + [\vec{a}\;\vec{b}\;\vec{d}] = [\vec{a}\;\vec{b}\;\vec{c}]$$

Answers

1. (i) -5 (ii) -14 (iii) 12
2. (i) 4 (ii) 56 (iii) 7

5. p = 2

13. $\frac{\pi}{2}, \frac{\pi}{3}$

5.3.10 Product of Four Vectors

Scalar product of four vectors

Scalar product of four vectors is also scalar product of two vectors in which both of them must be cross product of two vectors.

If $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ are four vectors, then scalar product of four vectors is $(\vec{a}\times\vec{b}).(\vec{c}\times d)$, where $\vec{a}\times\vec{b}$, $\vec{c}\times d$ are two vectors quantities.

Determination of $(\vec{a}\times\vec{b}).(\vec{c}\times d)$

Let $\vec{a}\times\vec{b} = \vec{u}$ so that

$$\begin{aligned}(\vec{a}\times\vec{b}).(\vec{c}\times d) &= \vec{u}.(\vec{c}\times\vec{d}) = \vec{u}\times\vec{c}.\vec{d} \\ &= (\vec{c}\times\vec{u}).\vec{d} = -\vec{c}\times(\vec{a}\times\vec{b}).\vec{d} \\ &= -[(\vec{c}.\vec{b})\vec{a} - (\vec{c}.\vec{a})\vec{b}].\vec{d} \\ &= [(\vec{c}.\vec{a})\vec{b} - (\vec{c}.\vec{b})\vec{a}].\vec{d} \\ &= (\vec{a}.\vec{c})(\vec{b}.\vec{d}) - (\vec{b}.\vec{c})(\vec{a}.\vec{d})\end{aligned}$$

$$\therefore\; (\vec{a}\times\vec{b}).(\vec{c}\times\vec{d}) = \begin{vmatrix} \vec{a}.\vec{c} & \vec{b}.\vec{c} \\ \vec{a}.\vec{d} & \vec{b}.\vec{d} \end{vmatrix}$$

Vector product of Four vectors:

The vector product of four vectors is also cross product of two vectors in which two of them must be again cross product of two vector.

If $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ are four vectors, then vector product of four vectors is $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})$ where $(\vec{a}\times\vec{b})$ and $(\vec{c}\times\vec{d})$ are two vectors quantities.

Determination of $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})$

The cross product of four vectors can be determined in two ways.

Let $\vec{a}\times\vec{b} = \vec{u}$, so that

$$(\vec a\times\vec b)\times(\vec c\times d) = \vec u\times(\vec c\times\vec d)$$
$$=(\vec u.\vec d)\vec c-(\vec u.\vec c)\vec d$$
$$=(\vec a\times\vec b.\vec d)\,\vec c-(\vec a\times\vec b).\vec c]\vec d$$
$$=[\vec a\ \vec b\ \vec d]\,\vec c-[\vec a\ \vec b\ \vec c]\,\vec d \qquad ...(1)$$

Again, let $\vec c\times\vec d=\vec v$ so that

$$(\vec a\times\vec b)\times(\vec c\times\vec d)=(\vec a\times\vec b)\times\vec v=-\vec v\times(\vec a\times\vec b)$$
$$=-[(\vec v.\vec b)\,\vec a-(\vec v.\vec a)\vec b]$$
$$=(\vec v.\vec a)\,\vec b-(\vec v.\vec b)\vec a$$
$$=(\vec c\times\vec d.\vec a)\,\vec b-(\vec c\times\vec d.\vec b)\,\vec a$$
$$=[\vec a\ \vec c\ \vec d]\,\vec b-[\vec b\ \vec c\ \vec d]\,\vec a \qquad ...(2)$$

Note 1:

If $\vec a,\vec b,\vec c,\vec d$ are coplanar, then we have

$$[\vec a\ \vec b\ \vec c]=0,\ [\vec a\ \vec b\ \vec d]=0.$$

Hence $\quad(\vec a\times\vec b)\times(\vec c\times\vec d)=0$

Note 2:

From (1) and (2), we have the following relations between the four vectors.

$$[\vec a\ \vec b\ \vec d]\,\vec c-[\vec a\ \vec b\ \vec c]\,\vec d=[\vec a\ \vec c\ \vec d]\vec b-[\vec b\ \vec c\ \vec d]\,\vec a$$

or $\quad[\vec b\ \vec c\ \vec d]\,\vec a-[\vec a\ \vec c\ \vec d]\,\vec b+[\vec a\ \vec b\ \vec d]\,\vec c-[\vec a\ \vec b\ \vec c]\,\vec d=0$

or $\quad[\vec a\ \vec b\ \vec c]\,\vec d=[\vec b\ \vec c\ \vec d]\,\vec a+[\vec c\ \vec a\ \vec d]\,\vec b+[\vec a\ \vec b\ \vec d]\,\vec c$

If $\vec a,\vec b,\vec c$ are non-coplanar then $[\vec a\ \vec b\ \vec c]\neq 0$, and we have

$$\vec a=\frac{[\vec b\ \vec c\ \vec d]\vec a}{[\vec a\ \vec b\ \vec c]}+\frac{[\vec c\ \vec a\ \vec d]\vec b}{[\vec a\ \vec b\ \vec c]}+\frac{[\vec a\ \vec b\ \vec d]}{[\vec a\ \vec b\ \vec c]}.$$

5.4. Reciprocal System of Vectors

Let $\vec a',\vec b',\vec c'$ be a set of a non -coplanar vectors so that $[\vec a\ \vec b\ \vec c]\neq 0$. Then the three vectors defined by the relations

$$\vec a'=\frac{\vec b\times\vec c}{[\vec a\ \vec b\ \vec c]},\quad \vec b'=\frac{\vec c\times\vec a}{[\vec a\ \vec b\ \vec c]},\quad \vec c'=\frac{\vec a\times\vec b}{[\vec a\ \vec b\ \vec c]}$$

respectively to the planes containing $\vec b$ and $\vec c$, $\vec c$ and $\vec a$, $\vec a$ and $\vec b$ are called *reciprocal system* to $\vec a,\vec b,\vec c$.

Properties of reciprocal system of vectors

Property 1

If $\vec a,\vec b,\vec c$ are $\vec a',\vec b',\vec c'$ are reciprocal system of vectors, then

$$\vec a.\vec a'=1,\quad \vec b.\vec b'=1,\quad \vec c.\vec c'=1$$

Proof

We have, the reciprocal system of the vectors are

$$\vec a'=\frac{\vec b\times\vec c}{[\vec a\ \vec b\ \vec c]},\quad \vec b'=\frac{\vec c\times\vec a}{[\vec a\ \vec b\ \vec c]},\quad \vec c'=\frac{\vec a\times\vec b}{[\vec a\ \vec b\ \vec c]}$$

Now

$$\vec a.\vec a'=\vec a.\frac{\vec b\times\vec c}{[\vec a\ \vec b\ \vec c]}=\frac{[\vec a\ \vec b\ \vec c]}{[\vec a\ \vec b\ \vec c]}=1$$

Similarly, $\quad\vec b.\vec b'=\vec b.\dfrac{\vec c\times\vec a}{[\vec a\ \vec b\ \vec c]}=\dfrac{[\vec a\ \vec b\ \vec c]}{[\vec a\ \vec b\ \vec c]}=1$

and $\quad\vec c.\vec c'=\vec c.\dfrac{\vec a\times\vec b}{[\vec a\ \vec b\ \vec c]}=\dfrac{[\vec a\ \vec b\ \vec c]}{[\vec a\ \vec b\ \vec c]}=1.$

Property 2

The dot product of any vector of one system with a vector of the other system, which does not correspond to it, is zero

i.e. $\vec a.\vec b'=0,\quad \vec b.\vec c'=0,\quad \vec c.\vec a'=0$

Proof

We have the relations

$$\vec a'=\frac{\vec b\times\vec c}{[\vec a\ \vec b\ \vec c]},\quad \vec b'=\frac{\vec c\times\vec a}{[\vec a\ \vec b\ \vec c]},\quad \vec c'=\frac{\vec a\times\vec b}{[\vec a\ \vec b\ \vec c]}$$

Now

$$\vec a.\vec b'=\vec a.\frac{\vec c\times\vec a}{[\vec a\ \vec b\ \vec c]}=\frac{[\vec a\ \vec c\ \vec a]}{[\vec a\ \vec b\ \vec c]}=0,$$

$$\vec b.\vec c'=\vec b.\frac{\vec a\times\vec b}{[\vec a\ \vec b\ \vec c]}=\frac{[\vec b\ \vec a\ \vec b]}{[\vec a\ \vec b\ \vec c]}=0$$

and $$\vec{c}\cdot\vec{a}' = \vec{c}\cdot\frac{\vec{b}\times\vec{c}}{[\vec{a}\ \vec{b}\ \vec{c}]} = \frac{[\vec{c}\ \vec{b}\ \vec{c}]}{[\vec{a}\ \vec{b}\ \vec{c}]} = 0$$

Property 3

The vectors $\vec{a}, \vec{b}, \vec{c}$ and $\vec{a}', \vec{b}', \vec{c}'$ are mutually reciprocal i.e. if

$$\vec{a}' = \frac{\vec{b}\times\vec{c}}{[\vec{a}\ \vec{b}\ \vec{c}]},\quad \vec{b}' = \frac{\vec{c}\times\vec{a}}{[\vec{a}\ \vec{b}\ \vec{c}]},\quad \vec{c}' = \frac{\vec{a}\times\vec{b}}{[\vec{a}\ \vec{b}\ \vec{c}]},\ \text{then}$$

$$\vec{a} = \frac{\vec{b}'\times\vec{c}'}{[\vec{a}'\ \vec{b}'\ \vec{c}']},\quad \vec{b} = \frac{\vec{c}'\times\vec{a}'}{[\vec{a}'\ \vec{b}'\ \vec{c}']}\ \text{and}\ \vec{c} = \frac{\vec{a}'\times\vec{b}'}{[\vec{a}'\ \vec{b}'\ \vec{c}']}$$

Proof

We have $\vec{a}\cdot\vec{b}' = 0,\quad \vec{a}\cdot\vec{c}' = 0,\quad \vec{b}\cdot\vec{c}' = 0$

It shows that $\vec{a}$ is perpendicular to both vectors $\vec{b}'$ and $\vec{c}'$ i.e. $\vec{a}$ is parallel to $\vec{b}'\times\vec{c}'$.

Hence $\vec{a} = d(\vec{b}'\times\vec{c}')$ where d is a scalar.

Taking scalar product of $\vec{a}'$ on both sides

$$\therefore\quad \vec{a}\cdot\vec{a}' = d(\vec{a}'\cdot\vec{b}'\times\vec{c}')$$

$$1 = d[\vec{a}'\ \vec{b}'\ \vec{c}']$$

or $$d = \frac{1}{[\vec{a}'\ \vec{b}'\ \vec{c}']}$$

Thus $$\vec{a} = \frac{\vec{b}'\times\vec{c}'}{[\vec{a}'\times\vec{b}'\ \vec{c}']}$$

Similarly, we can easily show that

$$\vec{b} = \frac{\vec{c}'\times\vec{a}'}{[\vec{a}'\ \vec{b}'\ \vec{c}']},\qquad \vec{c} = \frac{\vec{a}'\times\vec{b}'}{[\vec{a}'\ \vec{b}'\ \vec{c}']}.$$

Property 4

The scalar triple product $[\vec{a}\ \vec{b}\ \vec{c}]$ and $[\vec{a}'\ \vec{b}'\ \vec{c}']$ are reciprocally connected. i.e. $[\vec{a}'\ \vec{b}'\ \vec{c}'] = \dfrac{1}{[\vec{a}\ \vec{b}\ \vec{c}]}$

Proof

Now, $$[\vec{a}'\ \vec{b}'\ \vec{c}'] = \vec{a}'\cdot\vec{b}'\times\vec{c}'$$

$$= \frac{\vec{b}\times\vec{c}}{[\vec{a}\ \vec{b}\ \vec{c}]}\cdot\frac{(\vec{c}\times\vec{a})}{[\vec{a}\ \vec{b}\ \vec{c}]}\times\frac{(\vec{a}\times\vec{b})}{[\vec{a}\ \vec{b}\ \vec{c}]}$$

$$= \frac{1}{[\vec{a}\ \vec{b}\ \vec{c}]^3}[(\vec{b}\times\vec{c})\cdot(\vec{c}\times\vec{a})\times(\vec{a}\times\vec{b})]$$

$$= \frac{1}{[\vec{a}\ \vec{b}\ \vec{c}]}[(\vec{b}\times\vec{c})\cdot\vec{u}\times(\vec{a}\times\vec{b})] \quad \text{as } \vec{u} = \vec{c}\times\vec{a}$$

$$= \frac{1}{[\vec{a}\ \vec{b}\ \vec{c}]}[(\vec{b}\times\vec{c})\cdot\{(\vec{u}\cdot\vec{b})\vec{a} - (\vec{u}\cdot\vec{a})\vec{b}\}]$$

$$= \frac{1}{[\vec{a}\ \vec{b}\ \vec{c}]^3}(\vec{b}\times\vec{c})\cdot[(\vec{c}\times\vec{a}\cdot\vec{b})\vec{a} - (\vec{c}\times\vec{a}\cdot\vec{a})\vec{b}]$$

$$= \frac{1}{[\vec{a}\ \vec{b}\ \vec{c}]^3}(\vec{b}\times\vec{c})\cdot[\vec{a}\ \vec{b}\ \vec{c}]\,\vec{a}$$

$$= \frac{1}{[\vec{a}\ \vec{b}\ \vec{c}]^3}[\vec{a}\ \vec{b}\ \vec{c}](\vec{b}\times\vec{c}\cdot\vec{a})$$

$$= \frac{[\vec{a}\ \vec{b}\ \vec{c}]^2}{[\vec{a}\ \vec{b}\ \vec{c}]^3} = \frac{1}{[\vec{a}\ \vec{b}\ \vec{c}]}$$

$$\therefore\ [\vec{a}'\ \vec{b}'\ \vec{c}'] = \frac{1}{[\vec{a}\ \vec{b}\ \vec{c}]}.$$

Property 5

The orthonormal vector triad $(\vec{i}, \vec{j}, \vec{k})$ is itself reciprocal.

i.e. $\vec{i}' = \vec{i},\ \vec{j}' = \vec{j}' = \vec{j},\ \vec{k}' = \vec{k}$

Proof

We have

$$\vec{i}' = \frac{\vec{j}\times\vec{k}}{[\vec{i}\ \vec{j}\ \vec{k}]} = \frac{\vec{j}\times\vec{k}}{\vec{i}\cdot\vec{j}\times\vec{k}} = \frac{\vec{i}}{\vec{i}\cdot\vec{i}} = \vec{i}$$

Similarly, $$\vec{j}' = \frac{\vec{k}\times\vec{i}}{[\vec{i}\ \vec{j}\ \vec{k}]} = \frac{\vec{j}}{\vec{i}\cdot\vec{j}\times\vec{k}} = \frac{\vec{j}}{\vec{i}\cdot\vec{i}} = \vec{j}$$ and

$$\vec{k}' = \frac{\vec{i} \times \vec{j}}{[\vec{i}\ \ \vec{j}\ \ \vec{k}\,]} = \frac{\vec{k}}{\vec{i}\,.\,\vec{j} \times \vec{k}} = \frac{\vec{k}}{\vec{i}\,.\,\vec{i}} = \vec{k}$$

Property 6

Any vector $\vec{r}$ may be expressed as a linear combination of the non-coplanar vectors $\vec{a}, \vec{b}, \vec{c}$

i.e. $\vec{r} = (\vec{r}\,.\,\vec{a}\,')\,\vec{a} + (\vec{r}\,.\vec{b}\,')\,\vec{b} + (\vec{r}\,.\vec{c}\,')\,\vec{c}$

Proof

We can write $\vec{r} = x\vec{a} + y\vec{b} + z\vec{c}$ ....(1)

where, x, y, z are scalars and $\vec{a}, \vec{b}, \vec{c}$ are three non-coplanar vectors.

Multiplying both sides scalarly by $\vec{b} \times \vec{c}$, we get

$$\vec{r}\,.\vec{b} \times \vec{c} = x\vec{a}\,.\,\vec{b} \times \vec{c} + y\vec{b}\,.\vec{b} \times \vec{c} + z\vec{c}\,.\vec{b} \times \vec{c}$$

$$[\vec{r}\ \ \vec{b}\ \ \vec{c}\,] = x\,[\vec{a}\ \ \vec{b}\ \ \vec{c}\,]$$

$$\therefore \quad x = \frac{[\vec{r}\ \vec{b}\ \vec{c}\,]}{[\vec{a}\ \vec{b}\ \vec{c}\,]}$$

Similarly, $y = \dfrac{[\vec{r}\ \vec{c}\ \vec{a}\,]}{[\vec{a}\ \vec{b}\ \vec{c}\,]}$, $z = \dfrac{[\vec{r}\ \vec{a}\ \vec{b}\,]}{[\vec{a}\ \vec{b}\ \vec{c}\,]}$

Substituting these values of x, y, z in (1), we get

$$\vec{r} = \frac{[\vec{r}\ \vec{b}\ \vec{c}\,]}{[\vec{a}\ \vec{b}\ \vec{c}\,]}\vec{a} + \frac{[\vec{r}\ \vec{c}\ \vec{a}\,]}{[\vec{a}\ \vec{b}\ \vec{c}\,]}\vec{b} + \frac{[\vec{r}\ \vec{a}\ \vec{b}\,]}{[\vec{a}\ \vec{b}\ \vec{c}\,]}\vec{c}$$

$$= \left\{\vec{r}\,.\frac{\vec{b} \times \vec{c}}{[\vec{a}\ \vec{b}\ \vec{c}\,]}\vec{a} + \vec{r}\,.\frac{\vec{c} \times \vec{a}}{[\vec{a}\ \vec{b}\ \vec{c}\,]}\vec{b} + \vec{r}\,.\frac{\vec{a} \times \vec{b}}{[\vec{a}\ \vec{b}\ \vec{c}\,]}\vec{c}\right\}$$

$$= (\vec{r}\,.\vec{a}\,')\,\vec{a} + (\vec{r}\,.\vec{b}\,')\vec{b} + (\vec{r}\,.\vec{c}\,')\vec{c}$$

$$\therefore \ \vec{r} = (\vec{r}\,.\vec{a}\,')\vec{a} + (\vec{r}\,.\vec{b}\,') + (\vec{r}\,.\vec{b}\,')\vec{b} + (\vec{r}\,.\vec{c}\,')\vec{c}$$

Property 7

Any vector $\vec{r}$ may be expressed as linear combination of $\vec{a}\,', \vec{b}\,', \vec{c}\,'$

i.e. $\vec{r} = (\vec{r}.\vec{a})\vec{a}\,' + (\vec{r}\,.\,\vec{b})\vec{b}\,' + (\vec{r}\,.\,\vec{c})\vec{c}\,'$

Proof

We can write $\vec{r} = x\vec{a}\,' + y\vec{b}\,' + z\vec{c}\,'$ .....(1)

Where x, y, z are scalars and $\vec{a}\,', \vec{b}\,'$ and $\vec{c}\,'$ are reciprocal system of vectors.

Multiplying both sides scalarly by $\vec{b}\,' \times \vec{c}\,'$, we get

$$[\vec{r}\ \ \vec{b}\,'\ \ \vec{c}\,'] = x[\vec{a}\,'\ \ \vec{b}\,'\ \ \vec{c}\,']$$

$$x = \frac{[\vec{r}\,'\ \vec{b}\,'\ \vec{c}\,']}{[\vec{a}\,'\ \vec{b}\,'\ \vec{c}\,']}$$

or

Similarly, $y = \dfrac{[\vec{r}\ \vec{c}\,'\ \vec{a}\,']}{[\vec{a}\,'\ \vec{b}\,'\ \vec{c}\,']}$, $z = \dfrac{[\vec{r}\ \vec{a}\,'\ \vec{b}\,']}{[\vec{a}\,'\ \vec{b}\,'\ \vec{c}\,']}$

Substituting these values of x, y, z in (1), we get

$$\vec{r} = \frac{[\vec{r}\ \vec{b}\,'\ \vec{c}\,']}{[\vec{a}\,'\ \vec{b}\,'\ \vec{c}\,']}\vec{a}\,' + \frac{[\vec{r}\ \vec{c}\,'\ \vec{a}\,']}{[\vec{a}\,'\ \vec{b}\,'\ \vec{c}\,']}\vec{b}\,' + \frac{[\vec{r}\ \vec{a}\,'\ \vec{b}\,']}{[\vec{a}\,'\ \vec{b}\,'\ \vec{c}\,']}\vec{c}\,'$$

$$= \vec{r}\,.\frac{\vec{b}\,' \times \vec{c}\,'}{[\vec{a}\,'\ \vec{b}\,'\ \vec{c}\,']}\vec{a}\,' + \vec{r}\,.\frac{\vec{c}\,' \times \vec{a}\,'}{[\vec{a}\,'\ \vec{b}\,'\ \vec{c}\,']}\vec{b} + \vec{r}\,.\frac{\vec{a}\,' \times \vec{b}\,'}{[\vec{a}\,'\ \vec{b}\,'\ \vec{c}\,']}\vec{c}\,'$$

$$\vec{r} = (\vec{r}\,.\vec{a})\,\vec{a}\,' + (\vec{r}\,.\vec{b})\,\vec{b}\,' + (\vec{r}\,.\vec{c})\,\vec{c}\,'.$$

Worked Out Examples

Ex.1. If $\vec{a} = \vec{i} - 2\vec{j} + \vec{k}$, $\vec{b} = 2\vec{i} + \vec{j} + \vec{k}$, $\vec{c} = \vec{i} + 2\vec{j} - \vec{k}$, $\vec{d} = 5\vec{i} + 6\vec{j} - 7\vec{k}$ **then, find the value of** $(\vec{a} \times \vec{b}).(\vec{c} \times \vec{d})$ **and** $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}\,)$

Solution:

Here $\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -2 & 1 \\ 2 & 1 & 1 \end{vmatrix} = \vec{i}(-2-1) - \vec{j}\,(1-2) + \vec{k}\,(1+4)$

$= -3\vec{i} + \vec{j} + 5\vec{k}$

$\vec{c} \times \vec{d} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & -1 \\ 5 & 6 & -7 \end{vmatrix} = \vec{i}(-14+6) - \vec{j}\,(-7+5) + \vec{k}\,(6-10)$

$= -8\vec{i} + 2\vec{j} - 4\vec{k}$

$(\vec{a} \times \vec{d}).(\vec{c} \times \vec{d}) = (-3\vec{i} + \vec{j} + 5\vec{k}).(-8\vec{i} + 2\vec{j} - 4\vec{k})$
$= 24 + 2 - 20 = 6$

Also $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -3 & 1 & 5 \\ -8 & 2 & -4 \end{vmatrix}$

$= \vec{i}(-4-10) - \vec{j}(12+40) + \vec{k}(-6+8)$

$= -14\vec{k} - 52\vec{j} + 2\vec{k}$

Ex .2 Find $(\vec{a} \times \vec{b}).(\vec{c} \times \vec{d})$ and $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d})$ and verify the identity, $(\vec{a} \times \vec{b}).(\vec{c} \times \vec{d}) = (\vec{a}.\vec{c})(\vec{b}.\vec{d}) - (\vec{b}.\vec{c})(\vec{a}.\vec{d})$

where $\vec{a} = 2\vec{i} + \vec{j} - \vec{k}$, $\vec{b} = -\vec{i} + 2\vec{j} - 4\vec{k}$

$\vec{c} = \vec{i} + \vec{j} + \vec{k}$ and $\vec{d} = 5\vec{i} - 2\vec{j} + 3\vec{k}$

Solution:

Now $\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & -1 \\ -1 & 2 & -4 \end{vmatrix} = \vec{i}(-4+2) - \vec{j}(-8-1) + \vec{k}(4+1)$

$= -2\vec{i} + 9\vec{j} + 5\vec{k}$

$(\vec{c} \times \vec{d}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ 5 & -2 & 3 \end{vmatrix} = \vec{i}(3+2) - \vec{j}(3-5) + \vec{k}(-2-5)$

$= 5\vec{i} + 2\vec{j} - 7\vec{k}$

$(\vec{a} \times \vec{b}).(\vec{c} \times \vec{d}) = (-2\vec{i} + 9\vec{j} + 5\vec{k}).(5\vec{i} + 2\vec{j} - 7\vec{k})$
$= -10 + 18 - 35 = 27$

$(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & 9 & 5 \\ 5 & 2 & -7 \end{vmatrix} = \vec{i}(-63-10) - \vec{j}(14-25) + \vec{k}(-4-45)$

$= -73\vec{i} + 9\vec{j} - 49\vec{k}$

Here $\vec{a}.\vec{c} = (2\vec{i} + \vec{j} - \vec{k}).(\vec{i} + \vec{j} + \vec{k}) = 2 + 1 - 1 = 2$

$\vec{b}.\vec{d} = (-\vec{i} + 2\vec{j} - 4\vec{k}).(5\vec{i} - 2\vec{j} + 3\vec{k})$
$= -5 - 4 - 12 = -21$

$\vec{b}.\vec{c} = (-\vec{i} + 2\vec{j} - 4\vec{k}).(\vec{i} + \vec{j} + \vec{k})$
$= -1 + 2 - 4 = -3$

$\vec{a}.\vec{d} = (2\vec{i} + \vec{j} - \vec{k}).(5\vec{i} - 2\vec{j} + 3\vec{k}) = 10 - 2 - 3 = 5$

$\therefore (\vec{a}.\vec{c})(\vec{b}.\vec{d}) - (\vec{b}.\vec{c})(\vec{a}.\vec{d}) = 2 \times -21 - (-3).5$
$= -42 + 15 = -27$

Hence $(\vec{a} \times \vec{b}).(\vec{c} \times \vec{d}) = (\vec{a}.\vec{c})(\vec{b}.\vec{d}) - (\vec{b}.\vec{c})(\vec{a}.\vec{d})$

Ex.3 Show that $(\vec{b} \times \vec{c}).(\vec{b} \times \vec{a}) + (\vec{a}.\vec{b})(\vec{b}.\vec{c}) = b^2(\vec{a}.\vec{c})$ and $(\vec{a} \times \vec{b}) \times (\vec{a} \times \vec{c}) = [\vec{a}\ \vec{b}\ \vec{c}]\,\vec{a}$

Solution:

We have

$(\vec{b} \times \vec{c}).(\vec{b} \times \vec{a}) + (\vec{a}.\vec{b})(\vec{b}.\vec{c})$

$= \begin{vmatrix} \vec{b}.\vec{b} & \vec{b}.\vec{a} \\ \vec{c}.\vec{b} & \vec{c}.\vec{a} \end{vmatrix} + (\vec{a}.\vec{b})(\vec{b}.\vec{c})$

$= (\vec{b}.\vec{b})(\vec{c}.\vec{a}) - (\vec{b}.\vec{a})(\vec{c}.\vec{b}) + (\vec{a}.\vec{b})(\vec{b}.\vec{c}) = b^2(\vec{a}.\vec{c})$

Next, $(\vec{a} \times \vec{b}) \times (\vec{a} \times \vec{c})$

$= \vec{u} \times (\vec{a} \times \vec{c})$ as $\vec{u} = \vec{a} \times \vec{b}$

$= (\vec{u}.\vec{c})\vec{a} - (\vec{u}.\vec{a})\vec{c}$

$= (\vec{a} \times \vec{b}.\vec{c})\,\vec{a} - (\vec{a} \times \vec{b}.\vec{a})\,\vec{c}$

$= [\vec{a}\ \vec{b}\ \vec{c}]\,\vec{a} - [\vec{a}\ \vec{b}\ \vec{a}]\,\vec{c}$

$= [\vec{a}\ \vec{b}\ \vec{c}]\,\vec{a}$ $\quad (\because [\vec{a}\ \vec{b}\ \vec{a}] = 0)$

Ex. 4. Show that

$(\vec{b} \times \vec{c}).(\vec{a} \times \vec{d}) + (\vec{c} \times \vec{a}).(\vec{b} \times \vec{d}) + (\vec{a} \times \vec{b}).(\vec{c} \times \vec{d}) = 0$ and deduce that $\sin(\alpha + \beta)\sin(\alpha - \beta) = \sin^2\alpha - \sin^2\beta$

$\cos(\alpha + \beta)\cos(\alpha - \beta) = \cos^2\alpha - \sin^2\beta$

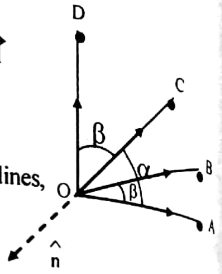
Solution:

Now $(\vec{b} \times \vec{c}).(\vec{a} \times \vec{d}) + (\vec{c} \times \vec{a}).(\vec{b} \times \vec{d}) + (\vec{a} \times \vec{d}).(\vec{c} \times \vec{d})$

$= \begin{vmatrix} \vec{b}.\vec{a} & \vec{b}.\vec{d} \\ \vec{c}.\vec{a} & \vec{c}.\vec{d} \end{vmatrix} + \begin{vmatrix} \vec{c}.\vec{b} & \vec{c}.\vec{d} \\ \vec{a}.\vec{b} & \vec{a}.\vec{d} \end{vmatrix} + \begin{vmatrix} \vec{a}.\vec{c} & \vec{a}.\vec{d} \\ \vec{b}.\vec{c} & \vec{b}.\vec{d} \end{vmatrix}$

$$= (\vec{b}.\vec{a})(\vec{c}.\vec{d}) - (\vec{b}.\vec{d})(\vec{c}.\vec{a}) + (\vec{c}.\vec{b})(\vec{a}.\vec{d}) - (\vec{c}.\vec{d})(\vec{a}.\vec{b})$$
$$+ (\vec{a}.\vec{c})(\vec{b}.\vec{d}) - (\vec{b}.\vec{c})(\vec{a}.\vec{d}) = 0$$

Let the vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ acting along coplanar lines OA, OB, OC, OD respectively.



i.e. $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$, $\overrightarrow{OC} = \vec{c}$, $\overrightarrow{OD} = \vec{d}$

Take $\angle AOC = \alpha$ and $\angle AOB = \angle COD = \beta$

So that $\angle AOD = \alpha + \beta$ and $\angle BOC = \alpha - \beta$

If $\hat{n}$ be a unit vector normal to the plane of these lines, then we have

$$(\vec{b} \times \vec{c}).(\vec{a} \times \vec{d}) = \begin{vmatrix} \vec{b}.\vec{a} & \vec{b}.\vec{d} \\ \vec{c}.\vec{a} & \vec{c}.\vec{d} \end{vmatrix}$$

$$(\vec{b} \times \vec{c}).(\vec{a} \times \vec{d}) = (\vec{b}.\vec{a})(\vec{c}.\vec{d}) - (\vec{c}.\vec{a})(\vec{b}.\vec{d})$$

or $bc \sin(\alpha - \beta)\,\hat{n}.\, ad \sin(\alpha + \beta)\,\hat{n} = ab \cos\beta\; cd \cos\beta - ca \cos\alpha\; bd \cos\alpha$

or $abcd \sin(\alpha - \beta) \sin(\alpha + \beta) = abcd(\cos^2\beta - \cos^2\alpha)$

or $\sin(\alpha - \beta)\sin(\alpha + \beta) = \cos^2\beta - \cos^2\alpha = 1 - \sin^2\beta - 1 + \sin^2\alpha$
$= \sin^2\alpha - \sin^2\beta$

Again, $(\vec{a} \times \vec{b}).(\vec{c} \times \vec{d}) = \begin{vmatrix} \vec{a}.\vec{c} & \vec{a}.\vec{d} \\ \vec{b}.\vec{c} & \vec{b}.\vec{d} \end{vmatrix}$

$$= (\vec{a}.\vec{c})(\vec{b}.\vec{d}) - (\vec{b}.\vec{c})(\vec{a}.\vec{d})$$

or $(ab \sin\beta)\,\hat{n}.(cd \sin\beta)\,\hat{n} = (ac \cos\alpha)(bd)\cos\alpha - bc\cos(\alpha - \beta)\, ad \cos(\alpha+\beta)$

or $\sin^2\beta = \cos^2\alpha - \cos(\alpha - \beta)\cos(\alpha + \beta)$

$\therefore$ $\cos(\alpha - \beta)\cos(\alpha + \beta) = \cos^2\alpha - \sin^2\beta$

Ex. 5 Prove that $(\vec{b} \times \vec{c}) \times (\vec{c} \times \vec{a}) = [\vec{a}\ \vec{b}\ \vec{c}]\,\vec{c}$ and deduce that $[\vec{b} \times \vec{c}\ \ \vec{c} \times \vec{a}\ \ \vec{a} \times \vec{b}] = [\vec{a}\ \vec{b}\ \vec{c}]^2$ *2067/069/070 Magh, B.E*

Solution:

For the first part

Let $\vec{b} \times \vec{c} = \vec{u}$. Then

$$(\vec{b} \times \vec{c}) \times (\vec{c} \times \vec{a}) = \vec{u} \times (\vec{c} \times \vec{a})$$

$$= (\vec{u}.\vec{a})\,\vec{c} - (\vec{u}.\vec{c})\,\vec{a}$$
$$= (\vec{b} \times \vec{c}.\vec{a})\,\vec{c} - (\vec{b} \times \vec{c}.\vec{c})\,\vec{a}$$
$$= [\vec{b}\ \vec{c}\ \vec{a}]\,\vec{c} - 0 = [\vec{a}\ \vec{b}\ \vec{c}]\,\vec{c}$$

For the second part

$$[\vec{b} \times \vec{c}\ \ \vec{c} \times \vec{a}\ \ \vec{a} \times \vec{b}] = (\vec{b} \times \vec{c}) \times (\vec{c} \times \vec{a}).(\vec{a} \times \vec{b})$$
$$= [\vec{a}\ \vec{b}\ \vec{c}]\,\vec{c}.(\vec{a} \times \vec{b})$$
$$= [\vec{a}\ \vec{b}\ \vec{c}][\vec{c}\ \vec{a}\ \vec{b}]$$
$$= [\vec{a}\ \vec{b}\ \vec{c}][\vec{a}\ \vec{b}\ \vec{c}] = [\vec{a}\ \vec{b}\ \vec{c}]^2$$

$\therefore$ $[\vec{b} \times \vec{c}\ \ \vec{c} \times \vec{a}\ \ \vec{a} \times \vec{b}] = [\vec{a}\ \vec{b}\ \vec{c}][\vec{a}\ \vec{b}\ \vec{c}] = [\vec{a}\ \vec{b}\ \vec{c}]^2$

Ex. 6. Prove that $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) + (\vec{a} \times \vec{c}) \times (\vec{d} \times \vec{b}) + (\vec{a} \times \vec{d}) \times (\vec{b} \times \vec{c})$ $= -2[\vec{b}\ \vec{c}\ \vec{d}]\,\vec{a}$ *2067, Mangsir B. E.*

Solution:

Let $\vec{a} \times \vec{b} = \vec{u}$, $\quad \vec{d} \times \vec{b} = \vec{v}$, $\quad \vec{b} \times \vec{c} = \vec{w}$

Now

$$(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) + (\vec{a} \times \vec{c}) \times (\vec{d} \times \vec{b}) + (\vec{a} \times \vec{d}) \times (\vec{b} \times \vec{c})$$
$$= \vec{u} \times (\vec{c} \times \vec{d}) + (\vec{a} \times \vec{c}) \times \vec{v} + (\vec{a} \times \vec{d}) \times \vec{w}$$
$$= (\vec{u}.\vec{d})\vec{c} - (\vec{u}.\vec{c})\vec{d} + (\vec{v}.\vec{a})\vec{c} - (\vec{v}.\vec{c})\vec{a} + (\vec{w}.\vec{a})\vec{d} - (\vec{w}.\vec{d})\vec{a}$$
$$= [\vec{a}\ \vec{b}\ \vec{d}]\,\vec{c} - [\vec{a}\ \vec{b}\ \vec{c}]\,\vec{d} + [\vec{d}\ \vec{b}\ \vec{a}]\,\vec{c} - [\vec{d}\ \vec{b}\ \vec{c}]\,\vec{a} + [\vec{b}\ \vec{c}\ \vec{a}]\,\vec{d} - [\vec{b}\ \vec{c}\ \vec{d}]\,\vec{a}$$
$$= [\vec{a}\ \vec{b}\ \vec{d}]\,\vec{c} - [\vec{a}\ \vec{b}\ \vec{c}]\,\vec{d} - [\vec{a}\ \vec{b}\ \vec{d}]\,\vec{c} - [\vec{b}\ \vec{c}\ \vec{d}]\,\vec{a} + [\vec{a}\ \vec{b}\ \vec{c}]\,\vec{d} - [\vec{b}\ \vec{c}\ \vec{d}]\,\vec{a}$$
$$= -2[\vec{b}\ \vec{c}\ \vec{d}]\,\vec{a}.$$

Ex.7. Prove that $\vec{b} \times \vec{c}$, $\vec{c} \times \vec{a}$, $\vec{a} \times \vec{b}$ are coplanar or non-coplanar according as $\vec{a}$, $\vec{b}$, $\vec{c}$ are coplanar or non coplanar *2068 Bhadra, B. E.*

Solution:

We have,

$$[\vec{b} \times \vec{c}\ \ \vec{c} \times \vec{a}\ \ \vec{a} \times \vec{b}] = (\vec{b} \times \vec{c}).\{(\vec{c} \times \vec{a}) \times (\vec{a} \times \vec{b})\}$$

Let $\vec{c} \times \vec{a} = \vec{u}$

$$= (\vec{b}\times\vec{c}) . \{\vec{u} \times (\vec{a}\times\vec{b})\}$$
$$= (\vec{b}\times\vec{c}) . \{(\vec{u}.\vec{b})\vec{a} - (\vec{u}.\vec{a})\vec{b}\}$$
$$= (\vec{b}\times\vec{c}) . \{(\vec{c}\times\vec{a}.\vec{b})\vec{a} - (\vec{c}\times\vec{a}.\vec{a})\vec{b}\}$$
$$= \vec{b}\times\vec{c} . \{[\vec{a}\ \vec{b}\ \vec{c}]\vec{a} - 0\}$$
$$= [\vec{a}\ \vec{b}\ \vec{c}](\vec{b}\times\vec{c}.\vec{a})$$
$$= [\vec{a}\ \vec{b}\ \vec{c}][\vec{a}\ \vec{b}\ \vec{c}] = [\vec{a}\ \vec{b}\ \vec{c}]^2$$
$$\therefore\ [\vec{b}\times\vec{c}\ \ \vec{c}\times\vec{a}\ \ \vec{a}\times\vec{b}] = [\vec{a}\ \vec{b}\ \vec{c}]^2$$

If $\vec{a}, \vec{b}, \vec{c}$ are coplanar ,then $[\vec{a}\ \vec{b}\ \vec{c}] = 0$ and hence

$$[\vec{b}\times\vec{c}\ \ \vec{c}\times\vec{a}\ \ \vec{a}\times\vec{b}] = 0 .$$

Therefore, the vectors $\vec{b}\times\vec{c}$, $\vec{c}\times\vec{a}$, $\vec{a}\times\vec{b}$ are coplanar.

If $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar ,then $[\vec{a}\ \vec{b}\ \vec{c}] \neq 0$ and hence

$$[\vec{b}\times\vec{c}\ \ \vec{c}\times\vec{a}\ \ \vec{a}\times\vec{b}] \neq 0 .$$

Therefore, the vectors $\vec{b}\times\vec{c}$, $\vec{c}\times\vec{a}$, $\vec{a}\times\vec{b}$ are non-coplanar.

Ex.8. Find the set of reciprocal system to the set of vectors $2\vec{i} + 3\vec{j} - \vec{k}, - \vec{i} + 2\vec{j} - 3\vec{k}$ and $3\vec{i} - 4\vec{j} + 2\vec{k}$ *2063 Asadh, B.E.*

Solution:

Let $\vec{a} = 2\vec{i} + 3\vec{j} - \vec{k}$, $\vec{b} = -\vec{i} + 2\vec{j} - 3\vec{k}$, $\vec{c} = 3\vec{i} - 4\vec{j} + 2\vec{k}$

$$[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} 2 & 3 & -1 \\ -1 & 2 & -3 \\ 3 & -4 & 2 \end{vmatrix} = 2(4 - 12) - 3(-2 + 9) - 1(4 - 6)$$
$$= -35$$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & -1 \\ -1 & 2 & -3 \end{vmatrix} = \vec{i}(-9 + 2) - \vec{j}(-6 - 1) + \vec{k}(4 + 3)$$
$$= -7\vec{i} + 7\vec{j} + 7\vec{k}$$

$$\vec{b}\times\vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 2 & -3 \\ 3 & -4 & 2 \end{vmatrix} = \vec{i}(4 - 12) - \vec{j}(-2 + 9) + \vec{k}(4 - 6)$$
$$= -8\vec{i} - 7\vec{j} - 2\vec{k}$$

$$\vec{c}\times\vec{a} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -4 & 2 \\ 2 & 3 & -1 \end{vmatrix} = \vec{i}(4 - 6) - \vec{j}(-3 - 4) + \vec{k}(9 + 8)$$
$$= -2\vec{i} + 7\vec{j} + 17\vec{k}$$

So the reciprocal system of the set of given vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are $\vec{a}'$, $\vec{b}'$ and $\vec{c}'$ respectively and defined by

$$\vec{a}' = \frac{\vec{b}\times\vec{c}}{[\vec{a}\ \vec{b}\ \vec{c}]} = \frac{-8\vec{i} - 7\vec{j} - 2\vec{k}}{-35}$$

$$\vec{b}' = \frac{\vec{c}\times\vec{a}}{[\vec{a}\ \vec{b}\ \vec{c}]} = \frac{-2\vec{i} + 7\vec{j} + 17\vec{k}}{-35}$$

$$\vec{c}' = \frac{\vec{a}\times\vec{b}}{[\vec{a}\ \vec{b}\ \vec{c}]} = \frac{-7\vec{i} + 7\vec{j} + 7\vec{k}}{-35} = \frac{-\vec{i} + \vec{j} + \vec{k}}{-5}.$$

Ex.9 If $\vec{a}, \vec{b}, \vec{c}$ and $\vec{a}', \vec{b}', \vec{c}'$ are the reciprocal system of vectors then prove that $\vec{a}\times\vec{a}' + \vec{b}\times\vec{b}' + \vec{c}\times\vec{c}' = 0$

and $\vec{a}'\times\vec{b}' + \vec{b}'\times\vec{c}' + \vec{c}'\times\vec{a} = \dfrac{\vec{a} + \vec{b} + \vec{c}}{[\vec{a}\ \vec{b}\ \vec{c}]}$, $[\vec{a}\ \vec{b}\ \vec{c}] \neq 0$

2071/072 Magh, B. E.

Solution:

For the first part;

$$\vec{a}\times\vec{a}' + \vec{b}\times\vec{b}' + \vec{c}\times\vec{c}'$$
$$= \vec{a}\times\frac{(\vec{b}\times\vec{c})}{[\vec{a}\ \vec{b}\ \vec{c}]} + \vec{b}\times\frac{(\vec{c}\times\vec{a})}{[\vec{a}\ \vec{b}\ \vec{c}]} + \vec{c}\times\frac{(\vec{a}\times\vec{b})}{[\vec{a}\ \vec{b}\ \vec{c}]}$$
$$= \frac{1}{[\vec{a}\ \vec{b}\ \vec{c}]}[(\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c} + (\vec{b}.\vec{a})\vec{c} - (\vec{b}.\vec{c})\vec{a} + (\vec{c}.\vec{b})\vec{a} - (\vec{c}.\vec{a})\vec{b}]$$
$$= 0$$
$$\therefore\ \vec{a}\times\vec{a}' + \vec{b}\times\vec{b}' + \vec{c}\times\vec{c}' = 0$$

For the second part,

$$\vec{a}'\times\vec{b}' + \vec{b}'\times\vec{c}' + \vec{c}'\times\vec{a}'$$

$(\vec b\ \vec c\ \vec a)\vec c + [\vec c\ \vec a\ \vec b]\vec a + [\vec a\ \vec b\ \vec c]\vec b$
$-(\vec a \ldots$

$$= \frac{(\vec b\times\vec c)}{[\vec a\ \vec b\ \vec c]}\times\frac{(\vec c\times\vec a)}{[\vec a\ \vec b\ \vec c]} + \frac{(\vec c\times\vec a)}{[\vec a\ \vec b\ \vec c]}\times\frac{(\vec a\times\vec b)}{[\vec a\ \vec b\ \vec c]} + \frac{(\vec a\times\vec b)}{[\vec a\ \vec b\ \vec c]}\times\frac{(\vec b\times\vec c)}{[\vec a\ \vec b\ \vec c]}$$

$$= \frac{1}{[\vec a\ \vec b\ \vec c]^2}[\vec u\times(\vec c\times\vec a)+\vec v\times(\vec a\times\vec b)+\vec w\times(\vec b\times\vec c)]$$

where $\vec u = \vec b\times\vec c,\quad \vec v = \vec c\times\vec a$ and $\vec w = \vec a\times\vec b$

$$= \frac{1}{[\vec a\ \vec b\ \vec c]^2}[(\vec u.\vec a)\vec c - (\vec u.\vec c)\vec a + (\vec v.\vec b)\vec a - (\vec v.\vec a)\vec b + (\vec w.\vec c)\vec b - (\vec w.\vec b)\vec c]$$

$$= \frac{1}{[\vec a\ \vec b\ \vec c]^2}[(\vec b\times\vec c.\vec a)\vec c - (\vec b\times\vec c.\vec c)\vec a + (\vec c\times\vec a.\vec b)\vec a - (\vec c\times\vec a.\vec a)\vec b + (\vec a\times\vec b.\vec c)\vec b - (\vec a\times\vec b.\vec b)\vec c]$$

$$= \frac{1}{[\vec a\ \vec b\ \vec c]^2}\{[\vec a\ \vec b\ \vec c]\vec c + [\vec a\ \vec b\ \vec c]\vec a + [\vec a\ \vec b\ \vec c]\vec b\}$$

$$= \frac{[\vec a\ \vec b\ \vec c]}{[\vec a\ \vec b\ \vec c]^2}[\vec c+\vec a+\vec b] = \frac{\vec a+\vec b+\vec c}{[\vec a\ \vec b\ \vec c]}$$

$$\therefore\ \vec a'\times\vec b' + \vec b'\times\vec c' + \vec c'\times\vec a' = \frac{\vec a+\vec b+\vec c}{[\vec a\ \vec b\ \vec c]}.$$

Exercise - 21

1. If $\vec a = 7\vec i + 6\vec j + 5\vec k$, $b = 4\vec i + 3\vec j + 2\vec k$, $\vec c = 2\vec i - 3\vec j + 4\vec k$ and $\vec d = 5\vec i + 6\vec j - 7\vec k$, then find the value of $(\vec a\times\vec b)\times(\vec c\times\vec d)$ and $(\vec a\times\vec b).(\vec c\times\vec d)$

2. Prove that $[\vec a\times(\vec b+\vec c)]\times[\vec b\times(\vec c+\vec a)] = [\vec a\ \vec b\ \vec c][\vec a+\vec b+\vec c]$.

3. Prove that $\vec a\times\{\vec a\times(\vec a\times\vec b)\} = (\vec a\,.\,\vec a)(\vec b\times\vec a)$.

4. Prove that $2(\vec c\times\vec d)\times(\vec a\times\vec b) = \begin{vmatrix} \vec a & a_1 & a_2 & a_3 \\ \vec b & b_1 & b_2 & b_3 \\ -\vec c & c_1 & c_2 & c_3 \\ -\vec d & d_1 & d_2 & d_3 \end{vmatrix}$

5. Prove that $\vec a\times\{\vec b\times(\vec c\times\vec d)\} = (\vec b.\vec d)(\vec a\times\vec c) - (\vec b.\vec c)(\vec a\times\vec d)$

$[\vec a\ \vec c\ \vec d]\vec b - (\vec a.\vec b)(\vec c\times\vec d)$

6. Show that $\vec c\begin{vmatrix} \vec a.\vec a & \vec a.\vec b \\ \vec a.\vec b & \vec b.\vec b\end{vmatrix} = \begin{vmatrix} \vec c.\vec a & \vec a.\vec b \\ \vec c.\vec b & \vec b.\vec b\end{vmatrix}\vec a + \begin{vmatrix} \vec a.\vec a & \vec c.\vec a \\ \vec a.\vec b & \vec c.\vec b\end{vmatrix}\vec b$.
where $\vec a, \vec b, \vec c$ are coplanar and $\vec a$ is not parallel to $\vec b$.

7. Prove that $[\vec a\times\vec b\ \ \vec c\times\vec d\ \ \vec e\times\vec f]$
$= [\vec a\ \vec b\ \vec d]\ [\vec c\ \vec e\ \vec f] - [\vec a\ \vec b\ \vec c][\vec d\ \vec e\ \vec f]$

8. Prove that $\vec a\times[\vec b\times\{\vec c\times(\vec d\times\vec e)\}]$
$= \{(\vec a.\vec d)(\vec c.\vec e) - (\vec c.\vec d)(\vec a.\vec e)\}\vec b + (\vec a.\vec b)\{(\vec c.\vec d)\vec e - (\vec c.\vec e)\vec d\}$

9. Find a set of vectors reciprocal to the following vectors:
$\vec a = 2\vec i - \vec j + 3\vec k, \vec b = -\vec i + 3\vec j + 3\vec k$
$\vec c = \vec i + \vec j - 2\vec k$ and verify that $[\vec a\ \vec b\ \vec c](\vec a'\ \vec b'\ \vec c'] = 1$

10. Find a set of vectors reciprocal to the following vectors
(i) $2\vec i - 3\vec j + 4\vec k, \vec i + 2\vec j - \vec k$ and $3\vec i - \vec j + 2\vec k$ **2072 Aswin, B. E.**
(ii) $2\vec i - 3\vec j,\ \vec i + \vec j + -\vec k$ and $3\vec i - \vec k$
(iii) $2\vec i + 3\vec j - \vec k,\ \vec i - \vec j - 2\vec k, -\vec i + 2\vec j + 2\vec k$
(iv) $-\vec i + \vec j + \vec k,\ \vec i - \vec j + \vec k,\ \vec i + \vec j - \vec k$ **2068 Bhadra, B. E.**

11. If $\vec a, \vec b, \vec c$ and $\vec a', \vec b', \vec c'$ are reciprocal system of vectors then show that
(i) $\vec a.\vec a' + \vec b.\vec b' + \vec c.\vec c' = 3$ **2062 Jestha, B.E.**
(ii) $(\vec a + \vec b).\vec a' + (\vec b + \vec c).\vec b' + (\vec c + \vec a).\vec c' = 3$

12. If $\vec a' = \vec u = \dfrac{\vec b\times\vec c}{[\vec a\ \vec b\ \vec c]}, \vec b' = \vec v = \dfrac{\vec c\times\vec a}{[\vec a\ \vec b\ \vec c]}, \vec c' = \vec w = \dfrac{\vec a\times\vec b}{[\vec a\ \vec b\ \vec c]}$ then
(i) Find $[\vec u\ \vec v\ \vec w]$
(ii) Show that $\vec a = \dfrac{\vec b'\times\vec c'}{[\vec a'\ \vec b'\ \vec c']}, \vec b = \dfrac{\vec c'\times\vec a'}{[\vec a'\ \vec b'\ \vec c']}, \vec c = \dfrac{\vec a'\times\vec b'}{[\vec a'\ \vec b'\ \vec c']}$

Answers

1. $132\vec{i}+45\vec{j}-42\vec{k}$, 132

10. (i) $\frac{-3\vec{i}+5\vec{j}+7\vec{k}}{7}, \frac{-2\vec{i}+8\vec{j}+7\vec{k}}{7}, \frac{5\vec{i}-6\vec{j}-7\vec{k}}{7}$

(ii) $\frac{-\vec{i}-2\vec{j}-3\vec{k}}{4}, \frac{-3\vec{i}-2\vec{j}-9\vec{k}}{4}, \frac{3\vec{i}+2\vec{j}+5\vec{k}}{4}$

(iii) $\frac{2\vec{i}+\vec{k}}{3}, \frac{-8\vec{i}+3\vec{j}-7\vec{k}}{3}, \frac{-7\vec{i}+3\vec{j}-5\vec{k}}{3}$

(iv) $\frac{\vec{j}+\vec{k}}{2}, \frac{\vec{i}+\vec{k}}{2}, \frac{\vec{i}+\vec{j}}{2}$

12. $\frac{1}{[\vec{a}\ \vec{b}\ \vec{c}]}$

5.5 Applications of vectors :Lines and Planes

In this section, we shall describe the lines and planes in the three-dimensional rectangular co-ordinate system by means of the vector concepts of parallel and orthogonal respectively.

5.5.1 Equation of a Line Through a Given a Point and Parallel to Given Vector

To find the parametric equation for a line through given a point and parallel to the given vector

The line through $P_1(x_1, y_1, z_1)$ parallel to the vector $\vec{a}$ is defined as the set of all points $P(x, y, z)$ such that $\overrightarrow{P_1P}$ is parallel to $\overrightarrow{OA}$.

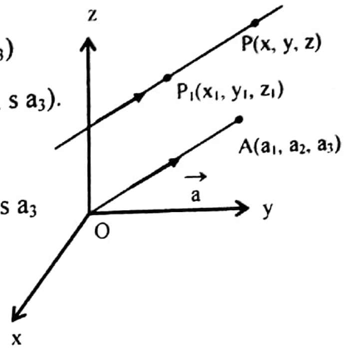
Let $\vec{a} = (a_1, a_2, a_3)$ be a non-zero given vector and $P_1(x_1, y_1, z_1)$ be any point on the line and let $\overrightarrow{OA} = \vec{a} = (a_1, a_2, a_3)$.

Take $P(x, y, z)$ any variable point on the line. Then

$\overrightarrow{P_1P} = s\ \overrightarrow{OA}$ for some scalar s

or $(x - x_1, y - y_1, z - z_1) = s\,(a_1, a_2, a_3)$

or $(x - x_1, y - y_1, z - z_1) = (s\,a_1, s\,a_2, s\,a_3)$.



This gives

$$x - x_1 = s\,a_1,\quad y - y_1 = s\,a_2,\quad z - z_1 = s\,a_3$$

for any real number s.

These are required parametric equations for the line.

5.5.2 Equation of a Line Through Given Two Points

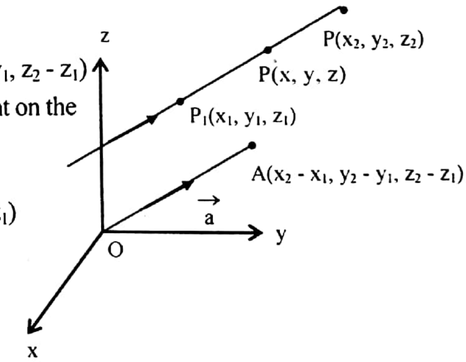
To find the parametric equations for the line through two given points

Let $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ be two given points on the line. Then the vector corresponds to $\overrightarrow{P_1P_2}$ is

$$\overrightarrow{OA} = \vec{a} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$$

Let $P(x, y, z)$ be any variable point on the line such that

$$\overrightarrow{P_1P} = (x - x_1, y - y_1, z - z_1)$$



We have

$\overrightarrow{P_1P} = s\ \overrightarrow{OA}$ for any real number s.

or $(x - x_1, y - y_1, z - z_1) = t\,(x_2 - x_1, y_2 - y_1, z_2 - z_1)$

Equating components, we get

$$x = x_1 + s\,(x_2 - x_1),\quad y = y_1 + s\,(y_2 - y_1),\quad z = z_1 + s\,(z_2 - z_1)$$

This is the required parametric equation.

Note:

If $s = 0$, then it gives the point P_1, $s = \frac{1}{2}$, it gives the mid-point of P_1P_2 and $s = 1$ it gives the point P_2.

5.5.3 Angle Between Two Lines

To find the angle between two lines parallel to the given vector

Let l_1 and l_2 be two lines parallel to the vectors $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$ respectively, then the angle between l_1 and l_2 are defined as θ and $\pi - \theta$.

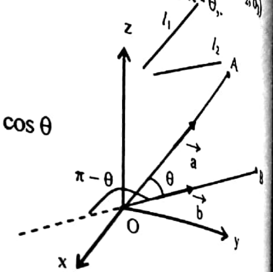
We have

$$\vec{a}.\vec{b} = ab \cos\theta$$

or $\quad a_1b_1 + a_2b_2 + a_3b_3$

$$= \sqrt{a_1^2 + a_2^2 + a_3^2}\ \sqrt{b_1^2 + b_2^2 + b_3^2}\ \cos\theta$$

or $\quad \cos\theta = \dfrac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\ \sqrt{b_1^2 + b_2^2 + b_3^2}}$



The lines are *orthogonal* if $\vec{a}.\vec{b} = 0$

Equivalently $\quad$ if $a_1b_1 + a_2b_2 + a_3b_3 = 0$

and the lines are *parallel* if $\vec{b} = s\vec{a}$ for some scalar s.

that is $\quad$ if $b_1 = sa_1, \quad b_2 = sa_2, \quad b_3 = sa_3$

$$\therefore \quad \frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$$

5.5.4 Equation of a Line in Symmetric Form

The equaiton of any line through a given point $P(x_1, y_1, z_1)$ and parallel to the vector $\vec{a} = (a_1, a_2, a_3)$ in parametric form is

$$x = x_1 + a_1s, \qquad y = y_1 + a_2s, \qquad z = z_1 + a_3s$$

If a_1, a_2, a_3 are different from zero, we may solve each equation for t. Then

$$\frac{x - x_1}{a_1} = s, \qquad \frac{y - y_1}{a_2} = s, \qquad \frac{z - z_1}{a_3} = s$$

It implies that a point P(x, y, z) lie on the line l if and only if

$$\frac{x - x_1}{a_1} = \frac{y - y_1}{a_2} = \frac{z - z_2}{a_3}$$

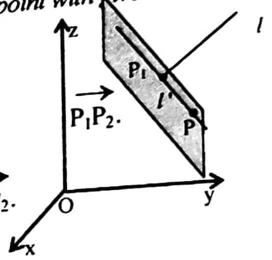
Note:

When a_1, a_2, a_3 are not all zeros, the lines may be described as the intersection of two planes.

5.5.5 The Plane

The plane through P_1 with normal line l is defined as the set of all points P such that $\overrightarrow{P_1P}$ is orthogonal to $\overrightarrow{P_1P_2}$.

1. *To find the equation of a plane through a given point with given normal vector.*



Let $P_1(x_1, y_1, z_1)$ be any point on the plane.

Choose another point P_2 on a line l with vector $\overrightarrow{P_1P_2}$.

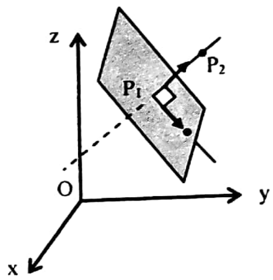
If $\vec{a} = (a_1, a_2, a_3)$ be a vector corresponds to $\overrightarrow{P_1P_2}$.

i.e. $\overrightarrow{P_1P_2} = (a_1, a_2, a_3)$.

Take any variable point P(x, y, z) on the plane

with $\quad \overrightarrow{P_1P} = (x - x_1, y - y_1, z - z_1)$.

Then by the definition of the plane



$\overrightarrow{P_1P}$ is orthogonal to $\overrightarrow{P_1P_2}$

or $\quad \overrightarrow{P_1P}.\overrightarrow{P_1P_2} = 0$

or $\quad (x - x_1, y - y_1, z - z_1).(a_1, a_2, a_3) = 0$

$\therefore \quad a_1(x - x_1) + a_2(y - y_1) + a_3(z - z_1) = 0$ is required equation of the plane.

2. *To prove that every linear equation $ax + by + cz + d = 0$ represents a plane with normal vector $\vec{a} = (a, b, c)$*

Given that

$$ax + by + cz + d = 0 \qquad \text{....(1)}$$

with a, b and c not all zero be the linear equation.

We choose that a point $P(x_1, y_1, z_1)$ on the plane (1). So the point $P(x_1, y_1, z_1)$ satisfies the equation of the plane (1)

$$ax_1 + by_1 + cz_1 + d = 0$$

$\therefore \quad d = - ax_1 - by_1 - cz_1$

Putting the value of d in (1), we get

$$ax + by + cz - ax_1 - by_1 - cz_1 = 0$$

or $$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$

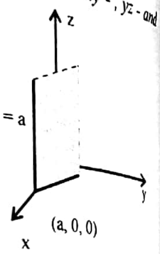
This is the equation of a plane through the $P(x_1, y_1, z_1)$ with normal vector $\vec{a} = (a, b, c)$. Hence the linear equation $ax + by + cz + d = 0$ with a, b, c are not all zero represents a plane with normal vector $\vec{a} = (a, b, c)$.

Definition:

Two planes are said to *parallel* if their normals vectors $\vec{a}$ and $\vec{b}$ are parallel and the planes are said to be *orthogonal* if their normals $\vec{a}$ and $\vec{b}$ are parallel.

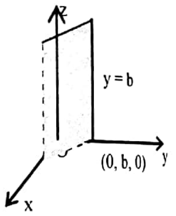
3. *To obtain the equation of a plane parallel and orthogonal to xy -, yz - and zx - planes*

Since the vector $\vec{i} = (1, 0, 0)$ is a normal vector for the yz - plane. A plane that has an equation x = a has also normal vector $\vec{i} = (1, 0, 0)$ and parallel to yz – plane.



Hence x = a is the equation of a plane parallel to yz - plane.

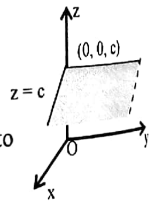
Similarly, y = b is a plane parallel to zx-plane with y intercept -b and z = c is a plane parallel to the xy - plane with z-intercept c which are shown as in the figure.



Since by + cz + d = 0 has normal vector

$$\vec{a} = (0, b, c) \text{ and}$$

$$\vec{a} \cdot \vec{i} = (0, b, c) . (1, 0, 0) = 0$$



So the equation of plane by + cz + d = 0 is the plane which is orthogonal to yz - plane. Similarly, ax + by + d = 0 and ax + cz + d = 0 are planes orthogonal to xy-plane and xz-plane, respectively.

Worked Out Examples

Ex. 1. Find parametric equations for the line through P(5, - 2, 4) and parallel to $\vec{a} = \left(\frac{1}{2}, 2, -\frac{2}{3}\right)$. Also find the point in which the line cuts z-axis

Solution:

We know parametric equation of a line through P(5, - 2, 4) and parallel to $\vec{a} = \left(\frac{1}{2}, 2, -\frac{2}{3}\right)$ are

$$x = 5 + \frac{1}{2}s, \quad y = -2 + 2s, \quad z = 4 - \frac{2s}{3}, \text{ where s in R.}$$

If the line cuts z-axis at R(x, y, z), then z = 0,

$$4 - \frac{2s}{3} = 0 \qquad \therefore \quad s = 6$$

So

$$x = 5 + \frac{1}{2} \times 6 = 8, \qquad y = -2 + 12 = 10$$

Hence R is the point with coordinates (8, 10, 0).

Ex. 2: Find parametric equations for the line through $P_1(3, 1, -2)$ and $P_2(-2, 7, -4)$

Solution:

Given two points are

$P_1(3, 1, -2)$ and $P_2(-2, 7, -4)$.

$\vec{OP_1} = (3, 1, -2)$ and $\vec{OP_2} = (-2, 7, -4)$

The vector $\vec{a}$ corresponds to $\vec{P_1P_2}$ is

$$\vec{a} = \vec{P_1P_2} = \vec{OP_2} - \vec{OP_1} = (-2 - 3,\ 7 - 1,\ -4 + 2)$$

$$\vec{a} = (-5, 6, -2)$$

So the equation of a line through $P_1(3, 1, -2)$ and parallel to $\vec{a} = (-5, 6, -2)$ is

$x = 3 - 5s, y = 1 + 6s, z = -2 - 2s$ where $s \in R$.

Ex. 3: Find the equation of the plane through the point

(5, - 2, 4) with normal vector $\vec{a} = (1, 2, 3)$

Solution:

We know that equation of a plane through the point (5, -2, 4) with normal vector $\vec{a} = (1, 2, 3)$ is

$$1(x - 5) + 2(x + 2) + 3(x - 4) = 0$$

$$\therefore \quad x + 2y + 3z - 13 = 0$$

Ex. 4: Find the equation of a plane through the points P(3, 2, 1), Q (- 1, 1, - 2) and R(3, - 4, 1)

Solution:

Let the points P(3, 2, 1), Q(- 1, 1, - 2) and R(3, - 4, 1) determine a plane that contains the triangle.

The vectors, $\vec{a}$ and $\vec{b}$ corresponding to $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are

$$\vec{a} = \overrightarrow{OQ} - \overrightarrow{OP} = (-1-3,\ 1-2,\ -2-1) = (-4, -1, -3)$$

and $\vec{b} = \overrightarrow{OR} - \overrightarrow{OP} = (3-3,\ -4-2,\ 1-1) = (0, -6, 0)$

The vector $\vec{a} \times \vec{b}$ is normal to the plane determined by P, Q and R so that

$$\vec{a} \times \vec{b} = (-4\vec{i} - \vec{j} - 3\vec{k}) \times (-6\vec{j})$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -4 & -1 & -3 \\ 0 & -6 & 0 \end{vmatrix} = -18\vec{i} + 24\vec{k}$$

We know that equation of plane through P(3, 2, 1) and normal to the vector (- 18, 0, 24) is

$$-18(x - 3) + 24(z - 1) = 0$$

$$\therefore \quad 3x - 4z + 5 = 0$$

Ex. 5: Find a symmetric form for the line through $P_1(3, 1, -2)$ and $P_2(-2, 7, -4)$

Solution:

Given points are

$P_1(3, 1, -2)$ and $P_2(-2, 7, -4)$.

So $\overrightarrow{OP_1} = (3, 1, -2)$ and $\overrightarrow{OP_2} = (-2, 7, -4)$.

Let a vector $\vec{a}$ corresponding to $\overrightarrow{P_1P_2}$ is

$$\vec{a} = \overrightarrow{PP_2} = \overrightarrow{OP_2} - \overrightarrow{OP_1} = (-2-3,\ 7-1,\ -4+2) = (-5, 6, -2).$$

The parametric equation for the line through $P_1(3, 1, -2)$ and parallel to the vector $\vec{a} = (-5, 6, -2)$ is

$$x = 3 - 5s,\ y = 1 + 6s,\ z = -2 - 2s;\ s \in R.$$

Solving each equation for t, we get

$$s = \frac{x-3}{5},\quad s = \frac{y-1}{6},\quad s = \frac{z+2}{-2}$$

$$\therefore \quad \frac{x-3}{5} = \frac{y-1}{6} = \frac{z+2}{-2}$$

Exercise - 22

1. Find the parametric equations for the line through P and parallel to $\vec{a}$
   - (i) P(4, 2, - 3), $\vec{a} = \left(\frac{1}{3}, 2, \frac{1}{2}\right)$
   - (ii) P(5, 0, - 2), $\vec{a} = (-1, -4, 1)$
   - (iii) P(0, 0, 0), $\vec{a} = (0, 1, 0)$
   - (iv) P(1, 2, 3), $\vec{a} = (1, 2, 3)$.
2. Find the parametric equations for the line through $P_1(5, -2, 4)$, $P_2(2, 6, 1)$. Also find the points at which the line intersects each of the coordinate planes
3. Find the equation of the plane passing through (- 11, 4, - 2) with normal vector $6\vec{i} - 5\vec{j} - \vec{k}$
4. Find the equation of plane through P(- 2, 5, - 8) with normal vector (a) $\vec{i}$ (b) $\vec{j}$ (c) $\vec{k}$
5. Find the equation of a plane through (2, 5, - 6) parallel to the plane 3x - y + 2z = 10 by the vector method
6. Find the equation of a plane through origin and the points P(0, 2, 5) and Q(1, 4, 0) by the vector method
7. Find the equation of a line in symmetric form through $P_1(5, -2, 4)$, $P_2(2, 6, 1)$ by the vector method

8. Find the equation of a line in symmetric form through $P_1(4, 2, -3)$ $P_2(-3, 2, 5)$ by the vector method

Answers

1. (i) $x = 4 + \frac{t}{3}$, $y = 2 + 2t$, $z = -3 + \frac{t}{2}$

   (ii) $x = 5 - t$, $y = -4t$, $z = -2 + t$

   (iii) $x = 0$, $y = t$, $z = 0$

   (iv) $x = 1 + t$, $y = 2 + 2t$, $z = 3 + 3t$, t in R

2. $x = 5 - 3t$, $y = -2 + 8t$, $z = 4 - 3t$

   $\left(1, \frac{26}{3}, 0\right), \left(\frac{17}{4}, 0, \frac{13}{4}\right), \left(0, \frac{34}{3}, -1\right)$

3. $6x - 5y - z + 84 = 0$

4. (i) $x + 2 = 0$ (ii) $y - 5 = 0$, (iii) $z + 8 = 0$

5. $3x - y + 2z + 11 = 0$ 6. $20x - 5y + 2z = 0$

7. $\frac{x-5}{3} = \frac{y+2}{-8} = \frac{z-4}{3}$ 8. $\frac{x-4}{-7} = \frac{z+3}{8}$, $y - 2 = 0$

5.6. Vector Calculus: Scalar and Vector fields

In Vector calculus, we shall be concerned with the vectors and vectors function of scalar variable in order to define limit continuity differentiability and integrability of vector function of scalar variable. Vector calculus consisting of vector differential calculus and vector integral calculus begins a discussion of vector function which represents vector fields. Vector functions are useful in studying curves and their applications. Physically and geometrically important of scalar and vector fields are gradient, divergence and curl which shall also be defined in this chapter.

5.6.1 Vector Function of a Scalar Variable

If $\vec{r}$ be a vector variable depends on scalar variable parameter t defined on the interval (a, b), then $\vec{r}$ is said to be a vector function of the scalar variable t defined in the interval (a, b) and we write $\vec{r} = \vec{r}(t)$ or $\vec{r} = \vec{f}\ (t)$.

If a particle moves along the curve with position $\vec{r}$ at any point on the curve with respect to origin at time t, then the vector $\vec{r}$ changes. Thus $\vec{r}$ may be regarded as a vector function of scalar variable whose domain is the set of real numbers and whose range is a set of vectors.

So $\vec{r} = \vec{f}$ (t) represents the vector equation of the curve. The coordinates of any point on the ellipse in parametric form is

$x = a\cos t$, $y = b \sin t$, the vector equation of an ellipse is

$\vec{r} = a\cos t\ \vec{i} + b\ \sin t\ \vec{j} + 0\vec{k}$, where $\vec{i}, \vec{j}, \vec{k}$ are three mutually perpendicular unit vectors. Therefore every vector equation of the curve

$\vec{r} = \vec{f}$ (t) can be expressed as

$\vec{r} = \vec{f}(t) = f_1(t)\ \vec{i} + f_2(t)\ \vec{j} + f_3(t)\ \vec{k}$ where $f_1(t)$, $f_2(t)$, $f_3(t)$

are called components of the vector $\vec{f}$ (t) along the coordinate axes.

5.6.2 Limit and Continuity of a Vector Function

Let $\vec{r} = \vec{f}$ (t) be a vector function of a scalar variable $t \in (a,b)$, then the vector function $\vec{f}$ (t) is said to have limit $\vec{L}$, a constant vector at $t = t_0$

$\in(a,b)$ if to any positive number ε we can find a positive number δ such that $|\vec{f}(t) - \vec{L}| < \varepsilon$ whenever $|t - t_0| < \delta$.

We write $\lim\limits_{t\to t_0} \vec{f}(t) = \vec{L}$

A vector function $\vec{r} = \vec{f}(t)$ is said to be continuous at $t = t_0 \in (a, b)$ if

$$\lim_{t\to t_0} \vec{f}(t) = \vec{f}(t_0)$$

5.6.3 Derivative of a Vector Function

If $\vec{r} = \vec{f}(t)$ be a vector function of a scalar variable t defined on the interval (a, b), then the derivative or differential coefficient of the function $\vec{f}(t)$ at a point t of the interval is defined as

$$\lim_{\delta t\to 0} \frac{\vec{f}(t+\delta t) - \vec{f}(t)}{\delta t} \text{ if it exists.}$$

We write $\dfrac{d\vec{r}}{dt} = \lim\limits_{\delta t\to 0} \dfrac{\vec{f}(t+\delta t) - \vec{f}(t)}{\delta t}$

A vector function $\vec{r} = \vec{f}(t)$ is said to be derivable at $t = t_0 \in (a, b)$ if

$$\lim_{t\to t_0} \frac{\vec{f}(t) - \vec{f}(t_0)}{t - t_0} \text{ exists.}$$

We write $\vec{f}\,'(t_0) = \lim\limits_{t\to t_0} \dfrac{\vec{f}(t) - \vec{f}(t_0)}{t - t_0}$

A vector function $\vec{f}(t)$ is said to be derivable or differentiable if it is differentiable for every value of $t \in (a, b)$.

5.6.4 Geometrical Meaning of $\dfrac{d\vec{r}}{dt}$

Let $\vec{r} = \vec{r}(t)$ be the equation of curve with position vector $\vec{r}$ at a point P with respect to the origin O. If P moves to Q i.e. as t changes from t to $t + \delta t$, then position of vector $\vec{r}$ changes to $\vec{r} + \delta\vec{r}$.

So $\vec{OP} = \vec{r}$ and $\vec{OQ} = \vec{r} + \delta\vec{r}$ so that

$\vec{PQ} = \vec{OQ} - \vec{OP} = \vec{r} + \delta r - \vec{r} = \delta\vec{r}$ is the change in $\vec{r}$ for the change δt in t.

$\dfrac{\delta\vec{r}}{\delta t} = \dfrac{\vec{PQ}}{\delta t}$ represents average rate of change of $\vec{r}$ with respect to t.

When $\delta t \to 0$, $Q \to P$ and the chord PQ tends to a tangent PT at P.

Therefore $\dfrac{d\vec{r}}{dt} = \lim\limits_{\delta t\to 0} \dfrac{\delta\vec{r}}{\delta t}$ represents a vector along the tangent to the curve at the point P in the sense of t increasing.

5.6.5 Higher Order Derivatives

Let $\vec{r} = \vec{f}(t)$ be vector function of scalar variable t in which

$\vec{f}(t) = f_1(t)\vec{i} + f_2(t)\vec{j} + f_3(t)\vec{k}$ if $f_1(t)$, $f_2(t)$, $f_3(t)$ are differentiable up to n^{th} order, then

$$\frac{d\vec{f}}{dt} = \frac{df_1}{dt}\vec{i} + \frac{df_2}{dt}\vec{j} + \frac{df_3}{dt}\vec{k}$$

$$\frac{d^2\vec{f}}{dt^2} = \frac{d^2f_1}{dt^2}\vec{i} + \frac{d^2f_2}{dt^2}\vec{j} + \frac{d^2f_3}{dt^2}\vec{k}$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$\frac{d^n\vec{f}}{dt^n} = \frac{d^nf_1}{dt^n}\vec{i} + \frac{d^nf_2}{dt^n}\vec{j} + \frac{d^nf_3}{dt^n}\vec{k}$$ are successive derivatives of the vector function $\vec{r} = \vec{f}(t)$.

5.6.6 Partial Derivatives of Vector Functions

Let a continuous vector function $\vec{r}$ depend on three independent scalar variables x, y, z i.e. $\vec{r}(x, y, z)$. As ordinary calculus this vector function may be differentiated partially with respect to any one of the variables regarding others are constants.

If $\delta\vec{r}$ be small increment of $\vec{r}$ as small increment of x is δx regarding y and z constant, then partial derivatives of the vector function $\vec{r}(x, y, z)$ with respect to x is defined as

$$\lim_{\delta x\to 0}\frac{\vec{r}(x+\delta x, y, z)-\vec{r}(x, y, z)}{\delta x}$$

provided this limit exist.

We write

$$\frac{\partial\vec{r}}{\partial x}=\lim_{\delta x\to 0}\frac{\vec{r}(x+\delta x, y, z)-\vec{r}(x, y, z)}{\delta x}$$

Similarly,

$$\frac{\partial\vec{r}}{\partial y}=\lim_{\delta y\to 0}\frac{\vec{r}(x, y+\delta y, z)-\vec{r}(x, y, z)}{\delta y}$$

$$\frac{\partial\vec{r}}{\partial z}=\lim_{\delta z\to 0}\frac{\vec{r}(x, y, z+\delta z)-\vec{r}(x, y, z)}{\delta z}$$

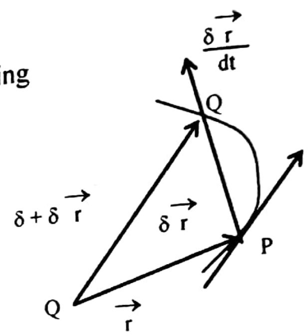
are called the partial derivatives of the vector function $\vec{r}(x, y, z)$ with respect to y and z respectively provided these limit exist.

5.6.7 Physical Interpretation of First and Second Derivatives of $\vec{r}=\vec{r}(t)$

Velocity and acceleration of a moving point on a curve $\vec{r}=\vec{r}(t)$ may be obtained by physical interpretation of first and second derivatives of $\vec{r}=\vec{r}(t)$. For this, let P be the position of the particle moving along the curve $\vec{r}=\vec{r}(t)$ at time t with position vector $\vec{r}$ relative to origin O so that

$\vec{OP}=\vec{r}$, let Q be the position of the particle at time t + δt with position vector $\vec{r}+\delta\vec{r}$ relative to origin O.

$\vec{OP}=\vec{r}+\delta\vec{r}-\vec{r}=\delta\vec{r}$. Since the vector $\vec{r}$ depend on the scalar variable t, $\delta\vec{r}$ be displacement $\vec{PQ}$ of the point P in time δt and therefore $\frac{\delta\vec{r}}{\delta t}$ is the average velocity of P during the interval δt along the direction PQ.



As δt→0, Q→P and the chord PQ tends to become the tangent to the curve at P.

$\lim_{\delta t\to 0}\frac{\delta\vec{r}}{dt}$ is the velocity of the particle P along PT and denoted by $\vec{v}=\frac{d\vec{r}}{dt}$

Similarly, if $\delta\vec{v}$ be increment of the velocity $\vec{v}$ in time δt, then $\frac{\delta\vec{v}}{\delta t}$ is average rate of change of $\delta\vec{v}$ during the interval δt. The limiting value of $\frac{\delta\vec{v}}{\delta t}$ as δt→0 is the *acceleration* of the particle at P and denoted by

$$\vec{f}=\frac{d\vec{v}}{dt}=\frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right)=\frac{d^2\vec{r}}{dt^2}$$

5.6.8 Derivative of a Constant Vector

We know that a vector has both magnitude and direction. Hence a vector will change when either its magnitude changes or its direction changes or both change.

A vector $\vec{r}$ is said to be constant if it does not change its magnitude and direction or both.

Let $\vec{r}$ be a constant vector function of scalar variable $\vec{t}$. So

$$\vec{r} = \vec{c}$$

where $\vec{c}$ is constant vector and $\delta\vec{r}$ be small increment of $\vec{r}$.

Then $\vec{r} + \delta\vec{r} = \vec{c}$ ....(1)

On subtraction (1) and (2),

$$\vec{r} + \delta\vec{r} - \vec{r} = \vec{c} - \vec{c}$$ ...(2)

Taking limit $\delta t \to 0$ on both sides

or $$\lim_{\delta t\to 0} \frac{\delta\vec{r}}{\delta t} = 0$$

or $$\frac{d\vec{r}}{dt} = 0$$

$$\therefore \quad \frac{d\vec{c}}{dt} = 0$$

5.6.9 Differentiation of Sum of Vector Functions

Let $\vec{r_1}$ and $\vec{r_2}$ be differentiable vector function of scalar variable, then

$$\vec{r} = \vec{r_1} + \vec{r_2}$$ ....(1)

Also let $\delta\vec{r}$, $\delta\vec{r_1}$ and $\delta\vec{r_2}$ be increments in $\vec{r}$, $\vec{r_1}$ and $\vec{r_2}$ respectively with increment δt in t.

Then $\vec{r} + \delta\vec{r} = \vec{r_1} + \delta\vec{r_1} + \vec{r_2} + \delta\vec{r_2}$ ...(2)

On subtraction (1) and (2), we get

$$\delta\vec{r} = (\vec{r_1} + \delta\vec{r_1}) + (\vec{r_2} + \delta\vec{r_2}) - (\vec{r_1} + \vec{r_2}) = \delta\vec{r_1} + \delta\vec{r_2}$$

Dividing both sides by δt and taking limit $\delta t\to 0$ on both sides, we get

$$\lim_{\delta t\to 0} \frac{\delta\vec{r}}{\delta t} = \lim_{\delta t\to 0} \left[\frac{\delta\vec{r_1}}{\delta t} + \frac{\delta\vec{r_2}}{\delta t}\right]$$

or $$\frac{d\vec{r}}{dt} = \frac{d\vec{r_1}}{dt} + \frac{d\vec{r_2}}{dt}$$

$$\therefore \quad \frac{d}{dt}(\vec{r_1} + \vec{r_2}) = \frac{d\vec{r_1}}{dt} + \frac{d\vec{r_2}}{dt}$$

5.6.10 Differentiation of Product of Vector Functions

(a) *Differentiation of the product of a scalar and vector function*

Let a and $\vec{r_1}$ be differentiable scalar and vector functions of scalar variable t.

Let $$\vec{r} = a\,\vec{r_1}$$ ....(1)

Let $\delta\vec{r}$, δa, $\delta\vec{r_1}$ be small increments of $\vec{r}$, a and $\delta\vec{r_1}$ respectively with increment δt in t, then

$$\vec{r} + \delta\vec{r} = (a+\delta a)(\vec{r_1} + \delta\vec{r_1})$$ .....(2)

On subtraction (1) and (2), we get

$$\delta\vec{r} = (a + \delta a)(\vec{r_1} + \delta\vec{r_1}) - a\,\vec{r}$$

$$= a\,\vec{r_1} + a\delta\vec{r_1} + \delta a\,\vec{r_1} + \delta a\delta\vec{r_1} - a\vec{r_1} = a\delta\vec{r_1} + \delta a\,\vec{r_1} + \delta a\,\delta\vec{r_1}$$

Dividing by δt and taking limit $\delta t \to 0$ on both sides, we get

$$\lim_{\delta t\to 0} \frac{\delta\vec{r}}{\delta t} = \lim_{\delta t\to 0}\left(a\frac{\delta\vec{r_1}}{\delta t} + \frac{\delta a}{\delta t}\vec{r_1} + \frac{\delta a}{\delta t}\delta\vec{r_1}\right)$$

$$\frac{d\vec{r}}{dt} = a\frac{d\vec{r_1}}{dt} + \frac{da}{dt}\vec{r_1}$$

$$\therefore \quad \frac{d}{dt}(a\vec{r_1}) = a\frac{d\vec{r_1}}{dt} + \frac{da}{dt}\vec{r_1}$$

Note:

If $\vec{r_1}$ is constant vector, then $\frac{d\vec{r_1}}{dt} = 0$ .So

$$\frac{d}{dt}(a\vec{r_1}) = \frac{da}{dt}\vec{r_1}.$$

(b) *Differentiation of scalar product of two vector functions*

Let $\vec{r_1}$ and r_2 be differentiable vector function of scalar variable t, then

$$r = \vec{r_1} \cdot \vec{r_2} \quad \text{.....(1)}$$

Let δr, $\delta\vec{r_1}$, $\delta\vec{r_2}$ be small increments of r, $\vec{r_1}$ and $\vec{r_2}$ respectively with increment δt in t.

$$r + \delta r = (\vec{r_1} + \delta\vec{r_1}) \cdot (\vec{r_2} + \delta\vec{r_2}) \quad \text{.....(2)}$$

On subtraction (1) and (2), we get

$$\delta r = (\vec{r_1} + \delta\vec{r_1}).(\vec{r_2} + \delta\vec{r_2}) - \vec{r_1}.\vec{r_2}$$

$$= \vec{r_1}.\vec{r_2} + \vec{r_1}.\delta\vec{r_2} + \delta\vec{r_1}.\vec{r_2} + \delta\vec{r_1}.\delta\vec{r_2} - \vec{r_1}.\vec{r_2}$$

$$= \vec{r_1}.\delta\vec{r_2} + \delta\vec{r}.\vec{r_2} + \delta\vec{r_1}.\delta\vec{r_2}$$

Dividing by δt and taking limit as $\delta t \to 0$, we get

$$\lim_{\delta t\to 0}\frac{\delta r}{\delta t} = \lim_{\delta t\to 0}\left[\vec{r_1}.\frac{\delta\vec{r_2}}{\delta t} + \vec{r_2}.\frac{\delta\vec{r_1}}{\delta t} + \frac{\delta\vec{r_1}.\delta\vec{r_2}}{\delta t}\right]$$

or $$\frac{dr}{dt} = \vec{r_1}.\frac{d\vec{r_2}}{dt} + \vec{r_2}.\frac{d\vec{r_1}}{dt}$$

$$\therefore \quad \frac{d}{dt}(\vec{r_1}.\vec{r_2}) = \vec{r_1}.\frac{d\vec{r_2}}{dt} + \vec{r_2}.\frac{d\vec{r_1}}{dt}$$

Note:

If $\vec{r_1} = \vec{r_2} = \vec{r}$, then

$$\frac{d}{dt}(\vec{r_1}.\vec{r_2}) = \vec{r_1}.\frac{d\vec{r_2}}{dt} + \vec{r_2}.\frac{d\vec{r_1}}{dt} \text{ becomes}$$

$$\frac{d}{dt}(\vec{r}.\vec{r}) = \vec{r}.\frac{d\vec{r}}{dt} + \vec{r}.\frac{d\vec{r}}{dt} = 2\vec{r}.\frac{d\vec{r}}{dt}$$

But $\vec{r}.\vec{r} = r^2$ where r is the length of the vector $\vec{r}$, then

$$\frac{d}{dt}(r^2) = 2\vec{r}.\frac{d\vec{r}}{dt}$$

$$\therefore \quad 2r\frac{dr}{dt} = 2\vec{r}.\frac{d\vec{r}}{dt}$$

(c) Differentiation of vector product of two vectors

Let $\vec{r_1}$ and $\vec{r_2}$ be two differentiable vector functions, then the cross product of two vector function is

$$\vec{r} = \vec{r_1} \times \vec{r_2} \quad \text{.....(1)}$$

Let $\delta\vec{r}$, $\delta\vec{r_1}$ and $\delta\vec{r_2}$ be small increments of $\vec{r}$, $\vec{r_1}$ and $\vec{r_2}$ respectively with increment δt in t. Then

$$\vec{r} + \delta\vec{r} = (\vec{r_1} + \delta\vec{r_1}) \times (\vec{r_2} + \delta\vec{r_2}) \quad \text{.....(2)}$$

On subtraction (1) and (2), we get

$$\delta\vec{r} = (\vec{r_1} + \delta\vec{r_1}) \times (\vec{r_2} + \delta\vec{r_2}) - \vec{r_1} \times \vec{r_2}$$

$$= \vec{r_1} \times \vec{r_2} + \vec{r_1} \times \delta\vec{r_2} + \delta\vec{r_1} \times \vec{r_2} + \delta\vec{r_1} \times \delta\vec{r_2} - \vec{r_1} \times \vec{r_2}$$

$$\delta\vec{r} = \vec{r_1} \times \delta\vec{r_2} + \delta\vec{r_1} \times \vec{r_2} + \delta\vec{r_1} \times \delta\vec{r_2}$$

Dividing by δt and taking limit as $\delta t \to 0$, we get

$$\lim_{\delta t\to 0}\frac{\delta\vec{r}}{\delta t} = \lim_{\delta t\to 0}\left[\vec{r_1} \times \frac{\delta\vec{r_2}}{\delta t} + \frac{\delta\vec{r_1}}{\delta t} \times \vec{r_2} + \frac{\delta\vec{r_1} \times \delta\vec{r_2}}{\delta t}\right]$$

or $$\frac{d\vec{r}}{dt} = \vec{r_1} \times \frac{d\vec{r_2}}{dt} + \frac{d\vec{r_1}}{dt} \times \vec{r_2}$$

$$\therefore \quad \frac{d}{dt}(\vec{r_1} \times \vec{r_2}) = \vec{r_1} \times \frac{d\vec{r_2}}{dt} + \frac{d\vec{r_1}}{dt} \times \vec{r_2}$$

Note:

If $\vec{r_1} = \vec{r}$ and $\vec{r_2} = \frac{d\vec{r}}{dt}$, then

$$\frac{d}{dt}(\vec{r_1} \times \vec{r_2}) = \vec{r_1} \times \frac{d\vec{r_2}}{dt} + \frac{d\vec{r_1}}{dt} \times \vec{r_2} \text{ becomes}$$

$$\frac{d}{dt}\left(\vec{r} \times \frac{d\vec{r}}{dt}\right) = \vec{r} \times \frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right) + \frac{d\vec{r}}{dt} \times \frac{d\vec{r}}{dt}$$

$$= \vec{r} \times \frac{d^2\vec{r}}{dt^2} + 0 = \vec{r} \times \frac{d^2\vec{r}}{dt^2}$$

5.6.11 Differentiation of Scalar Triple Product of Vector Functions

Let $\vec{r_1}$, $\vec{r_2}$ and $\vec{r_3}$ be three differentiable vector function of scalar variable t, then scalar triple product of these vector is

$$r = [\vec{r_1} \ \vec{r_2} \ \vec{r_3}]$$

$$\frac{d\vec{r}}{dt} = \frac{d}{dt}[\vec{r_1} \ \vec{r_2} \ \vec{r_3}] = \frac{d}{dt}\{\vec{r_1}.(\vec{r_2} \times \vec{r_3})\}$$

$$= \vec{r_1} \cdot \frac{d}{dt}(\vec{r_2} \times \vec{r_3}) + \frac{d\vec{r_1}}{dt} \cdot \vec{r_2} \times \vec{r_3}$$

$$= \vec{r_1} \cdot \left\{\vec{r_2} \times \frac{d\vec{r_3}}{dt} + \frac{d\vec{r_2}}{dt} \times \vec{r_3}\right\} + \frac{d\vec{r_1}}{dt} \cdot \vec{r_2} \times \vec{r_3}$$

$$= \vec{r_1} . \vec{r_2} \times \frac{d\vec{r_3}}{dt} + \vec{r_1} . \frac{d\vec{r_2}}{dt} \times \vec{r_3} + \frac{d\vec{r_1}}{dt} \cdot \vec{r_2} \times \vec{r_3}$$

$$= \left[\vec{r_1} \ \vec{r_2} \ \frac{d\vec{r_3}}{dt}\right] + \left[\vec{r_1} \ \frac{d\vec{r_2}}{dt} \ \vec{r_3}\right] + \left[\frac{d\vec{r_1}}{dt} \ \vec{r_2} \ \vec{r_3}\right]$$

$$\therefore \frac{d}{dt}[\vec{r_1} \ \vec{r_2} \ \vec{r_3}] = \left[\frac{d\vec{r_1}}{dt} \ \vec{r_2} \ \vec{r_3}\right] + \left[\vec{r_1} \ \frac{d\vec{r_2}}{dt} \ \vec{r_3}\right] + \left[\vec{r_1} \ \vec{r_2} \ \frac{d\vec{r_3}}{dt}\right]$$

5.6.12 Differentiation of Vector Triple Product of Vector Functions

Let $\vec{r_1}$, $\vec{r_2}$ and $\vec{r_3}$ be three differentiable vector functions of scalar variable t, then vector triple product of these vectors is

$$\vec{r} = \vec{r_1} \times (\vec{r_2} \times \vec{r_3})$$

$$\frac{d\vec{r}}{dt} = \frac{d}{dt}\{\vec{r_1} \times (\vec{r_2} \times \vec{r_3})\}$$

$$= \frac{d\vec{r_1}}{dt} \times (\vec{r_2} \times \vec{r_3}) + \vec{r_1} \times \frac{d}{dt}(\vec{r_2} \times \vec{r_3})$$

$$= \frac{d\vec{r_1}}{dt} \times (\vec{r_2} \times \vec{r_3}) + \vec{r_1} \times \left\{\vec{r_2} \times \frac{d\vec{r_3}}{dt} + \frac{d\vec{r_2}}{dt} \times \vec{r_3}\right\}$$

$$\therefore \frac{d}{dt}[\vec{r_1} \times (\vec{r_2} \times \vec{r_3})] = \frac{d\vec{r_1}}{dt} \times (\vec{r_2} \times \vec{r_3}) + \vec{r_1} \times \left(\frac{d\vec{r_2}}{dt} \times \vec{r_3}\right) + \vec{r_1} \times \left(\vec{r_2} \times \frac{d\vec{r_3}}{dt}\right)$$

Theorem 1

The necessary and sufficient condition for the vector function of a scalar variable t have constant magnitude is $\vec{a} \cdot \frac{d\vec{a}}{dt} = 0$

2067/069/070 Bhadra, B. E.

Proof

Necessary condition

Let a vector function $\vec{a}$ of scalar variable t be of constant magnitude

i.e. $|\vec{a}| = a$ (constant)

$$|\vec{a}|^2 = a^2$$

$$\vec{a} . \vec{a} = a^2$$

Differentiating, $\frac{d}{dt}(\vec{a} . \vec{a}) = 0$

or $\vec{a} . \frac{d\vec{a}}{dt} + \frac{d\vec{a}}{dt} . \vec{a} = 0$

or $2\vec{a} . \frac{d\vec{a}}{dt} = 0$

$\therefore$ $\vec{a} . \frac{d\vec{a}}{dt} = 0$. Hence the necessary condition.

It also shows that if $\vec{a}$ is vector function of constant magnitude, then $\vec{a}$ and $\frac{d\vec{a}}{dt}$ are perpendicular to each other.

Sufficient condition:

Let $\vec{a} . \frac{d\vec{a}}{dt} = 0$. We have to show that $\vec{a}$ is of constant magnitude.

We have

$$\vec{a} . \vec{a} = |\vec{a}|^2$$

Differentiating,

$$\frac{d}{dt}(\vec{a} . \vec{a}) = \frac{d}{dt}|\vec{a}|^2$$

or $\vec{a} . \frac{d\vec{a}}{dt} + \frac{d\vec{a}}{dt} . \vec{a} = 2|\vec{a}| \frac{d|\vec{a}|}{dt}$

or $\quad \vec{a} \cdot \dfrac{d\vec{a}}{dt} = |\vec{a}| \dfrac{d|\vec{a}|}{dt}$

or $\quad |\vec{a}| \dfrac{d|\vec{a}|}{dt} = 0$. But $|\vec{a}| \neq 0$

$\therefore \quad \dfrac{d|\vec{a}|}{dt} = 0$.

Hence $|\vec{a}| =$ constant i.e. $\vec{a}$ is of constant magnitude.

Theorem 2

The necessary and sufficient condition for the function $\vec{a}$ of scalar variable to have a constant direction is $\vec{a} \times \dfrac{d\vec{a}}{dt} = 0$ [2069/070/071/072 Aswin, B.E]

Proof

Necessary condition

Let $\quad \vec{a} = a\hat{a}$ ....(1)

where a denotes the magnitude of $\vec{a}$ and $\hat{a}$ is a unit vector whose magnitude is unity for all values of t along the direction of $\vec{a}$.

Differentiating (1), we get

$$\frac{d\vec{a}}{dt} = a\frac{d\hat{a}}{dt} + \frac{da}{dt}\hat{a}$$

Pre-multiplying both sides vectorically by $\vec{a}$, we get

$$\vec{a} \times \frac{d\vec{a}}{dt} = \vec{a} \times \left(\frac{ad\hat{a}}{dt} + \frac{da}{dt}\hat{a}\right)$$

$$= a\hat{a} \times \left(a\frac{d\hat{a}}{dt} + \frac{da}{dt}\hat{a}\right) \qquad (\because \vec{a} = a\hat{a})$$

$$= a^2\hat{a} \times \frac{d\hat{a}}{dt} + a\frac{da}{dt}(\hat{a} \times \hat{a})$$

or $\quad \vec{a} \times \dfrac{d\vec{a}}{dt} = a^2\hat{a} \times \dfrac{d\hat{a}}{dt} \qquad (\because \hat{a} \times \hat{a} = 0)$

Since $\hat{a}$ is a constant vector $\dfrac{d\hat{a}}{dt} = 0$

$\vec{a} \times \dfrac{d\vec{a}}{dt} = 0$. Hence the necessary condition.

Sufficient condition:

$\vec{a} \times \dfrac{d\vec{a}}{dt} = 0$. We have to prove that $\vec{a}$ has constant direction.

Let $\quad \vec{a} \times \dfrac{d\vec{a}}{dt} = 0$

Now $\quad a\hat{a} \times \dfrac{d(a\hat{a})}{dt} = 0$

or $\quad a^2\hat{a} \times \dfrac{d\hat{a}}{dt} = 0$

or $\quad \hat{a} \times \dfrac{d\hat{a}}{dt} = 0$ ....(1)

$\therefore$

Again, $\hat{a}$ is a vector function of constant magnitude and hence

$\hat{a} \cdot \dfrac{d\hat{a}}{dt} = 0$ ....(2)

From (1) and (2), we have

$\dfrac{d\hat{a}}{dt} = 0$. Thus $\vec{a}$ to have constant direction.

Worked out Examples

Ex. 1. If $\vec{r} = a\cos t\,\vec{i} + a\sin t\,\vec{j} + t\,\vec{k}$, find $\dfrac{d\vec{r}}{dt}$, $\dfrac{d^2\vec{r}}{dt^2}$ and $\left|\dfrac{d^2\vec{r}}{dt^2}\right|$

Solution:

Given that $\quad \vec{r} = a\cos t\,\vec{i} + a\sin t\,\vec{j} + t\vec{k}$

Differentiating with respect to t, we get

$$\frac{d\vec{r}}{dt} = -a\sin t\,\vec{i} + a\cos t\,\vec{j} + \vec{k}$$

Again differentiating with respect to t, we get

$$\frac{d^2\vec{r}}{dt^2} = -a\cos t\,\vec{i} - a\sin t\,\vec{j}$$

$$\left|\frac{d^2\vec{r}}{dt^2}\right| = \sqrt{a^2\cos^2 t + a^2\sin^2 t} = \sqrt{a^2(\cos^2 t + \sin^2 t)} = a.$$

Ex. 2. If $\vec{r} = t^2\,\vec{i} + (t+2)\,\vec{j} - (2t+1)\,\vec{k}$, find $\frac{d\vec{r}}{dt}$, $\left|\frac{d\vec{r}}{dt}\right|_{at\ t=1}$, $\left|\frac{d^2\vec{r}}{dt^2}\right|$

Solution:

Given that $\vec{r} = t^2\,\vec{i} + (t+2)\,\vec{j} - (2t+1)\,\vec{k}$

Differentiating with respect to t, we get

$$\frac{d\vec{r}}{dt} = 2t\,\vec{i} + \vec{j} - 2\vec{k}$$

Again differentiating with respect to t, we get

$$\frac{d^2\vec{r}}{dt^2} = 2\vec{i}$$

$$\left|\frac{d\vec{r}}{dt}\right| = \sqrt{4t^2+1+4}$$

$$\left|\frac{d\vec{r}}{dt}\right|_{at\ t=1} = \sqrt{4+1+4} = 3 \text{ and } \left|\frac{d^2\vec{r}}{dt^2}\right| = \sqrt{4} = 2.$$

Ex. 3. If $\vec{r_1} = (2t+1)\,\vec{i} - t^2\,\vec{j} + 3t^3\,\vec{k}$ and $\vec{r_2} = t^2\,\vec{i} + t\vec{j} - (t-1)\,\vec{k}$, find

(i) $\frac{d}{dt}(\vec{r_1}.\vec{r_2})$ (ii) $\frac{d}{dt}(\vec{r_1}\times\vec{r_2})$ and verify that

(iii) $\frac{d}{dt}(\vec{r_1}.\vec{r_2}) = \vec{r_1}.\frac{d\vec{r_2}}{dt} + \frac{d\vec{r_1}}{dt}.\vec{r_2}$

(iv) $\frac{d}{dt}(\vec{r_1}\times\vec{r_2}) = \vec{r_1}\times\frac{d\vec{r_2}}{dt} + \frac{d\vec{r_1}}{dt}\times\vec{r_2}$

Solution:

Given that $\vec{r_1} = (2t+1)\,\vec{i} - t^2\vec{j} + 3t^3\vec{k}$

and $\vec{r_2} = t^2\,\vec{i} + t\,\vec{j} - (t-1)\vec{k}$

$$\vec{r_1}.\vec{r_2} = t^2(2t+1) - t^3 - 3t^3(t-1) = 2t^3 + t^2 - t^3 - 3t^4 + 3t^3$$

$$\vec{r_1}.\vec{r_2} = -3t^4 + 4t^3 + t^2$$

$$\vec{r_1}\times\vec{r_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2t+1 & -t^2 & 3t^3 \\ t^2 & t & -(t-1) \end{vmatrix}$$

$$= (t^3 - t^2 - 3t^4)\vec{i} + (3t^5 + 2t^2 - t - 1)\vec{j} + (t^4 + 2t^2 + t)\vec{k}.$$

Now

$$\frac{d\vec{r_1}}{dt} = \frac{d}{dt}\{(2t+1)i - t^2\,\vec{j} + 3t^3\vec{k}\} = 2\,\vec{i} - 2t\,\vec{j} + 9t^2\vec{k}$$

$$\frac{d\vec{r_2}}{dt} = \frac{d}{dt}\{t^2\,\vec{i} + t\,\vec{j} - (t-1)\vec{k}\} = 2t\,\vec{i} + \vec{j} - \vec{k}.$$

Thus

$$\frac{d}{dt}(\vec{r_1}.\vec{r_2}) = \frac{d}{dt}(-3t^4 + 4t^3 + t^2) = -12t^3 + 12t^2 + 2t \text{ and}$$

$$\frac{d}{dt}(\vec{r_1}\times\vec{r_2}) = \frac{d}{dt}\{(t^3 - t^2 - 3t^4)\,\vec{i} + (3t^5 + 2t^2 - t - 1)\,\vec{j} + (t^4 + 2t^2 + t)\,\vec{k}\}$$

$$= (3t^2 - 2t - 12t^3)\,\vec{i} + (15t^4 + 4t - 1)\,\vec{j} + (4t^3 + 4t + 1)\,\vec{k}$$

$$\vec{r_1}.\frac{d\vec{r_2}}{dt} + \frac{d\vec{r_1}}{dt}.\vec{r_2} = \{(2t+1)\vec{i} - t^2\vec{j} + 3t^3\vec{k}\}.(2t\,\vec{i} + \vec{j} - \vec{k}) + (2\,\vec{i} - 2t\vec{j} + 9t^2\vec{k}).\{t^2\vec{i} + t\vec{j} - (t-1)\vec{k}\}$$

$$= (4t^2 + 2t - t^2 - 3t^3 + 2t^2 - 2t^2 - 9t^3 + 9t^2)$$

$$= -12t^3 + 12t^2 + 2t = \frac{d}{dt}(\vec{r_1}.\vec{r_2})$$

Hence $\frac{d}{dt}(\vec{r_1}.\vec{r_2}) = \vec{r_1}.\frac{d\vec{r_2}}{dt} + \frac{d\vec{r_1}}{dt}.\vec{r_2}$ is verified.

Also

$$\vec{r_1}\times\frac{d\vec{r_2}}{dt} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2t+1 & -t^2 & 3t^3 \\ 2t & 1 & -1 \end{vmatrix}$$

$$= (t^2 - 3t^3)\,\vec{i} + (6t^4 + 2t + 1)\vec{j} + (2t + 1 + 2t^3)\,\vec{k} \quad \text{and}$$

$$\frac{d\vec{r_1}}{dt}\times\vec{r_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -2t & 9t^2 \\ t^2 & t & -(t-1) \end{vmatrix}$$

$$= (2t^2 - 2t - 9t^3)\,\vec{i} + (9t^4 + 2t - 2)\vec{j} + (2t + 2t^3)\vec{k}$$

$$\vec{r_1}\times\frac{d\vec{r_2}}{dt} + \frac{d\vec{r_1}}{dt}\times\vec{r_2} = (3t^2 - 2t - 12t^3)\,\vec{i} + (15t^4 + 4t - 1)\vec{j} + (4t + 1 + 4t^3)\vec{k}$$

$$= \frac{d}{dt}(\vec{r_1}\times\vec{r_2})$$

Hence $\frac{d}{dt}(\vec{r_1}\times\vec{r_2}) = \vec{r_1}\times\frac{d\vec{r_2}}{dt} + \frac{d\vec{r_1}}{dt}\times\vec{r_2}$ is verified.

Ex. 4. If $\vec{r} = \vec{a}\cos\omega t + \vec{b}\sin\omega t$, show that

(i) $\vec{r}\times\frac{d\vec{r}}{dt} = \omega\,\vec{a}\times\vec{b}$ (ii) $\frac{d^2\vec{r}}{dt^2} = -\omega^2\vec{r}$ where $\vec{a}$ and $\vec{b}$ are constant vectors

Solution:

We have $\vec{r} = \vec{a}\cos\omega t + \vec{b}\sin\omega t$

Differentiating with respect to t, we get

$$\frac{d\vec{r}}{dt} = -\vec{a}\omega\sin\omega t + \vec{b}\omega\cos\omega t$$

$$\therefore \quad \vec{r}\times\frac{d\vec{r}}{dt} = \vec{a}\times\vec{b}\,\omega\cos^2\omega t - \vec{b}\times\vec{a}\,\omega\sin^2\omega t$$

$$= \vec{a}\times\vec{b}\,\omega\cos^2 wt - \vec{b}\times\vec{a}\,\omega\sin^2\omega t$$

$$= (\vec{a}\times\vec{b})\omega\cos^2\omega t + (\vec{a}\times\vec{b})\omega\sin^2\omega t$$

$$= (\vec{a}\times\vec{b})\omega(\cos^2\omega t + \sin^2\omega t) = (\vec{a}\times\vec{b})\,\vec{\omega}$$

Again differentiating (1) with respect to t, we get

$$\frac{d^2\vec{r}}{dt^2} = -\vec{a}\omega^2\cos\omega t - \vec{b}\omega^2\sin\omega t$$

$$= -\omega^2(\vec{a}\cos\omega t + \vec{b}\sin\omega t) = -\omega^2\vec{r}$$

$$\therefore \quad \frac{d^2\vec{r}}{dt^2} = -\omega^2\vec{r}.$$

Ex. 5. If $\vec{r} = a\cos t\,\vec{i} + a\sin t\,\vec{j} + at\tan\alpha\vec{k}$, find $\left|\frac{d\vec{r}}{dt}\times\frac{d^2\vec{r}}{dt^2}\right|$ and $\left[\vec{r}\ \ \frac{d\vec{r}}{dt}\ \ \frac{d^2\vec{r}}{dt^2}\right]$ [2072 Magh, B.E.]

Solution:

Given that

$$\vec{r} = a\cos t\,\vec{i} + a\sin t\,\vec{j} + a\,t\tan\alpha\,\vec{k}$$

Differentiating with respect to t, we get

$$\frac{d\vec{r}}{dt} = -a\sin t\,\vec{k} + a\cos t\,\vec{j} + a\tan\alpha\,\vec{k}$$

$$\frac{d^2\vec{r}}{dt^2} = -a\cos t\,\vec{i} - a\sin t\,\vec{j} + 0$$

Now

$$\frac{d\vec{r}}{dt}\times\frac{d^2\vec{r}}{dt^2} = (-a\sin t\,\vec{i} + a\cos t\,\vec{j} + a\tan\alpha\,\vec{k})\times(-a\cos t\,\vec{i} - a\sin t\,\vec{j})$$

$$= a^2\sin^2 t\,\vec{k} + a^2\cos^2 t\,\vec{k} - a^2\tan\alpha\cos t\,\vec{j} + a^2\sin t\tan\alpha\,\vec{i}$$

$$= a^2(\sin^2 t + \cos^2 t)\vec{k} + a^2\sin t\tan\alpha\,\vec{i} - a^2\tan\alpha\cos t\,\vec{j}$$

$$= a^2\tan\alpha\sin t\,\vec{i} - a^2\tan\alpha\cos t\,\vec{j} + a^2\vec{k}.$$

$$\left|\frac{d\vec{r}}{dt}\times\frac{d^2\vec{r}}{dt^2}\right| = \sqrt{a^4\tan^2\alpha\sin^2 t + a^4\cos^2 t\tan^2\alpha + a^4}$$

$$= \sqrt{a^4\tan^2\alpha(\sin^2 t + \cos^2 t) + a^4}$$

$$= \sqrt{a^4\tan^2\alpha + a^4} = \sqrt{a^4(\tan^2\alpha + 1)}$$

$$= \sqrt{a^4\sec^2\alpha} = a^2\sec\alpha$$

Again

$$\left[\vec{r}\ \ \frac{d\vec{r}}{dt}\ \ \frac{d^2\vec{r}}{dt^2}\right] = \vec{r}\cdot\frac{d\vec{r}}{dt}\times\frac{d^2\vec{r}}{dt^2}$$

$$= (a\cos t\,\vec{i} + a\sin t\,\vec{j} + a\tan\alpha\,\vec{k}).(a^2\tan\alpha\sin t\,\vec{i} - a^2\tan\alpha\cos t\,\vec{j} + a^2\vec{k})$$

$$= a^3\tan\alpha\sin t\cos t - a^3\tan\alpha\sin t\cos t + a^3\tan\alpha$$

$$= a^3\tan\alpha$$

$$\therefore \quad \left[\vec{r}\ \ \frac{d\vec{r}}{dt}\ \ \frac{d^2\vec{r}}{dt^2}\right] = a^3\tan\alpha$$

Ex. 6. If $\vec{r_1} = a\cos t\,\vec{i} + b\sin t\,\vec{j}$, $\vec{r_2} = -a\sin t\,\vec{i} + b\cos t\,\vec{j} + t\vec{k}$ and $\vec{r_3} = \vec{i} + 2\vec{j} + 3\vec{k}$, find $\frac{d}{dt}\{\vec{r_1}.(\vec{r_2}\times\vec{r_3})\}$ and $\frac{d}{dt}\{\vec{r_1}\times(\vec{r_2}\times\vec{r_3})\}$ at t = 0

Solution:

Given that

$$\vec{r_1} = a\cos t\,\vec{i} + b\sin t\,\vec{j},\quad \vec{r_2} = -a\sin t\,\vec{i} + b\cos t\,\vec{j} + t\vec{k}$$

$$\vec{r_3} = \vec{i} + 2\vec{j} + 3\vec{k}$$

Now

$$\vec{r_2}\times\vec{r_3} = (-a\sin t\,\vec{i} + b\cos t\,\vec{j} + t\vec{k})\times(\vec{i} + 2\vec{j} + 3\vec{k})$$

$$= (3b\cos t - 2t)\,\vec{i} + (3a\sin t + t)\,\vec{j} + (-2a\sin t - b\cos t)\,\vec{k}$$

$$\vec{r_1}\cdot\vec{r_2}\times\vec{r_3} = (a\cos t\,\vec{i} + b\sin t\,\vec{j})\cdot\{(3b\cos t - 2t)\,\vec{i} + (3a\sin t + t)\,\vec{j} + (-2a\sin t - b\cos t)\,\vec{k}$$

$$= 3ab\cos^2 t - 2at\cos t + 3ab\sin^2 t + bt\sin t = 3ab - 2at\cos t \ldots$$

$$\frac{d}{dt}\{\vec{r_1}\cdot(\vec{r_2}\times\vec{r_3})\} = \frac{d}{dt}(3ab - 2at\cos t + bt\sin t)$$

$$= 2at\sin t - 2a\cos t + bt\cos t + b\sin t$$

$$\frac{d}{dt}(\vec{r_1}\cdot\vec{r_2}\times\vec{r_3})_{at\ t=0} = -2a$$

$$\vec{r_1}\times(\vec{r_2}\times\vec{r_3}) = (a\cos t\,\vec{i} + b\sin t\,\vec{j})\times\{(3b\cos t - 2t)\,\vec{i} + (3a\sin t + t)\,\vec{j} + (-2a\sin t - b\cos t)\,\vec{k}\}$$

$$= (-2ab\sin^2 t - b^2\sin t\cos t)\,\vec{i} + (2a^2\sin t\cos t + ab\cos^2 t)\,\vec{j} + (3a^2\sin t\cos t + at\cos t - 3b^2\sin t\cos t + 2bt\sin t)\,\vec{k}$$

$$\frac{d}{dt}\{\vec{r_1}\times(\vec{r_2}\times\vec{r_3})\}$$

$$= \frac{d}{dt}\{(-2ab\sin^2 t - b^2\sin t\cos t)\,\vec{i} + (2a^2\sin t\cos t + ab\cos^2 t)\,\vec{j} + (3a^2\sin t\cos t + at\cos t - 3b^2\sin t\cos t + 2bt\sin t)\,\vec{k}\}$$

$$= (-4ab\sin t\cos t - b^2\cos^2 t + b^2\sin^2 t)\,\vec{i} + (4a^2\cos^2 t - 2a^2 - 2ab\ldots)\,\vec{j}$$

$$+ (6a^2\cos^2 t - 3a^2 + a\cos t - at\sin t - 6b^2\cos^2 t + 3b^2 + 2b\sin t + 2bt\cos t)\,\vec{k}$$

$$\frac{d}{dt}\{\vec{r_1}\times(\vec{r_2}\times\vec{r_3})\}_{at\ t=0} = -b^2\vec{i} + 2a^2\vec{j} + (3a^2 - 3b^2 + a)\vec{k}$$

Ex. 7. If $\vec{r} = e^{xy}\,\vec{i} + (x - 2y)\,\vec{j} + (x\sin y)\,\vec{k}$, calculate

$$\frac{\partial\vec{r}}{\partial x},\ \frac{\partial\vec{r}}{\partial y},\ \frac{\partial^2\vec{r}}{\partial x^2},\ \frac{\partial^2\vec{r}}{\partial y^2},\ \frac{\partial^2\vec{r}}{\partial y\partial x}$$

Solution:

Given that $\vec{r} = e^{xy}\,\vec{i} + (x - 2y)\,\vec{j} + (x\sin y)\,\vec{k}$

Partially differentiating (1) with respect to x, we get

$$\frac{\partial\vec{r}}{\partial x} = y\,e^{xy}\,\vec{i} + \vec{j} + \sin y\,\vec{k} \quad ....(2)$$

Again, partially differentiating it with respect to x, we get

$$\frac{\partial^2\vec{r}}{\partial x^2} = y^2 e^{xy}\,\vec{i}$$

Partially differentiating (1) with respect to y, we get

$$\frac{\partial\vec{r}}{\partial y} = x\,e^{xy}\,\vec{i} - 2\,\vec{j} + x\cos y\,\vec{k}$$

Again, partially differentiating it with respect to y, we get

$$\frac{\partial^2\vec{r}}{\partial y^2} = x^2 e^{xy}\,\vec{i} - x\sin y\,\vec{k}$$

Partially differentiating (2) with respect to y, we get

$$\frac{\partial^2\vec{r}}{\partial y\partial x} = (x\,y\,e^{xy} + e^{xy})\,\vec{i} + \cos y\,\vec{k}$$

Ex. 8. A particle moves along the curve x = a cost, y = a sin t, z = bt. Find the velocity and acceleration at t = 0, and t = π/2

Solution:

Let $\vec{r}$ be the position vector of the particle at the point (x, y, z) at time t, then

$$\vec{r} = x\,\vec{i} + y\,\vec{j} + z\,\vec{k} = a\cos t\,\vec{i} + a\sin t\,\vec{j} + b\,t\,\vec{k}$$

Differentiating it with respect to t, we get

$$\frac{d\vec{r}}{dt} = -a\sin t\,\vec{i} + a\cos t\,\vec{j} + b\,\vec{k}$$

or $\vec{v} = -a\sin t\,\vec{i} + a\cos t\,\vec{j} + b\,\vec{k}$

$$\frac{d^2\vec{r}}{dt^2} = -a\cos t\,\vec{i} - a\sin t\,\vec{j}$$

or $\vec{f} = -a\cos t\,\vec{i} - a\sin t\,\vec{j}$

At $t = 0$ $\quad \vec{v} = a\,\vec{j} + b\,\vec{k}$ and $\vec{f} = -a\,\vec{i}$

At $t = \pi/2$ $\quad \vec{v} = -a\,\vec{i} + b\,\vec{k}$ and $\vec{f} = -a\,\vec{j}$

The magnitude of the velocity and acceleration are

$$|\vec{v}| = \sqrt{a^2\sin^2 t + a^2\cos^2 t + b^2} = \sqrt{a^2 + b^2}$$

$$|\vec{f}| = \sqrt{a^2\cos^2 t + a^2\sin^2 t} = a$$

Exercise - 23

1. If $\vec{r} = t^2\vec{i} - t\vec{j} + (2t+1)\vec{k}$, find the value of

(i) $\frac{d\vec{r}}{dt} \cdot \frac{d^2\vec{r}}{dt^2}$ (ii) $\left|\frac{d\vec{r}}{dt}\right|$ at t = 0

(iii) $\left|\frac{d^2\vec{r}}{dt^2}\right|$ at t = 0 (iv) $\vec{r} \cdot \frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}$

2. If $\vec{r_1} = t^3\vec{i} + t^2\vec{j} + t\vec{k}$ and $\vec{r_2} = (t+1)\vec{i} + (t+2)\vec{j} - 3t\vec{k}$, find

(i) $\frac{d}{dt}(\vec{r_1} \cdot \vec{r_2})$ at t = 2 (ii) $\frac{d}{dt}(\vec{r_1} \times \vec{r_2})$ at t = 2

(iii) $\frac{d}{dt}(\vec{r_1} \cdot \vec{r_2}) = \vec{r_1} \cdot \frac{d\vec{r_2}}{dt} + \frac{d\vec{r_1}}{dt} \cdot \vec{r_2}$

(iv) $\frac{d}{dt}(\vec{r_1} \times \vec{r_2}) = \vec{r_1} \times \frac{d\vec{r_2}}{dt} + \frac{d\vec{r_1}}{dt} \times \vec{r_2}$.

3. (i) If $\vec{r} = \vec{a}e^{mt} + \vec{b}e^{nt}$ where $\vec{a}$ and $\vec{b}$ are constant vectors, show that

$$\frac{d^2\vec{r}}{dt^2} - (m+n)\frac{d\vec{r}}{dt} + mn\vec{r} = 0$$

(ii) If $\vec{r} = \vec{a}e^{ct} + \vec{b}e^{-ct}$ where $\vec{a}$ and $\vec{b}$ are constant vectors, show that

$$\frac{d^2\vec{r}}{dt^2} - c^2\vec{r} = 0$$

(iii) For the curve x = 3t, y = 3t², z = 2t³, prove that $\left[\dot{\vec{r}}\ \ddot{\vec{r}}\ \dddot{\vec{r}}\right] = 216$

4. (i) A particle moves along the curve x = 4 cost, y = 4 sint, z = 6t. Find the velocity and acceleration at time t = 0 and t = π/2. Also find their magnitudes

(ii) A particle moves along the curve $x = t^3 + 1$, $y = t^2$, z = 2t. Find the velocity and acceleration at time t = 1. Also find their magnitudes

(iii) A particle moves along the curve x = 4 cost, $y = t^2$, z = 2t. Find the velocity and acceleration at time t = 0 and π/2 [2070 Bhadra, B.E.]

(iv) The position vector of a moving particle at any point is given by $\vec{r} = (t^2+1)\vec{i} + (4t-3)\vec{j} + (2t^2-6)\vec{k}$. Find the velocity and acceleration at time t = 1. Also obtain the magnitude [2067 Bhadra, B.E.]

5.(a) If $\vec{r_1} = xyz\vec{i} + xz^2\vec{j} - y^3\vec{k}$ and $\vec{r_2} = x^3\vec{i} - xyz\vec{j} + x^2z\vec{k}$, then find the value of

(i) $\frac{\partial^2}{\partial x \partial y}(\vec{r_1} \times \vec{r_2})$ and $\frac{\partial^2}{\partial x \partial y}(\vec{r_2} \times \vec{r_1})$

(ii) $\frac{\partial^2 \vec{r_1}}{\partial y^2} \times \frac{\partial^2 \vec{r_2}}{\partial x^2}$ at the point (1, 1, 0)

(b) If $\vec{r_1} = x^2yz\vec{i} - 2xz^2\vec{j} + xz^2\vec{k}$ and $\vec{r_2} = 2z\vec{i} + y\vec{j} - x^2\vec{k}$, then find the values of $\frac{\partial^2}{\partial x \partial y}(\vec{r_1} \times \vec{r_2})$ and $\frac{\partial^2}{\partial x \partial y}(\vec{r_2} \times \vec{r_1})$. Also verify that $\frac{\partial^2}{\partial x \partial y}(\vec{r_1} \times \vec{r_2}) = -\frac{\partial^2}{\partial x \partial y}(\vec{r_2} \times \vec{r_1})$ at (1, 0, - 2).

6. A particle moves so that its position vector is given by $\vec{r} = \cos\omega t\,\vec{i} + \sin\omega t\,\vec{j}$, where ω is a constant, show that the velocity $\vec{v}$ is perpendicular to $\vec{r}$ and the acceleration $\vec{f}$ is directed towards the origin and has magnitude proportion to the distance from the origin.

7. Evaluate the derivatives of the following.

(i) $\hat{r} = \frac{\vec{r}}{r}$ (ii) $\vec{r}^{\,2} + \frac{1}{\vec{r}^{\,2}}$

(iii) $\vec{r} \times \left(\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}\right)$ (iv) $\frac{\vec{r} \times \vec{a}}{\vec{r} \cdot \vec{a}}$

(v) $\vec{r} \cdot \frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}$ (vi) $\frac{\vec{r} + \vec{a}}{\vec{r}^{\,2} + \vec{a}^{\,2}}$

8. If $\frac{d\vec{a}}{dt} = \vec{c} \times \vec{a}$, $\frac{d\vec{b}}{dt} = \vec{c} \times \vec{b}$, show that $\frac{d}{dt}(\vec{a} \times \vec{b}) = \vec{c} \times (\vec{a} \times \vec{b})$

9. If $\vec{r}$ is a unit vector, prove that $\left|\vec{r} \times \frac{d\vec{r}}{dt}\right| = \left|\frac{d\vec{r}}{dt}\right|$

10. Evaluate :

(i) $\frac{d}{dt}\left\{\left(\vec{r} \times \frac{d\vec{r}}{dt}\right) \times \left(\frac{d^2\vec{r}}{dt^2}\right)\right\}$

(ii) $\dfrac{d}{dt}\left(\vec{r_1} \times \dfrac{d\vec{r_2}}{dt} - \dfrac{d\vec{r_1}}{dt} \times \vec{r_2}\right)$

(iii) $\dfrac{d}{dt}\left(\vec{r_1} \cdot \dfrac{d\vec{r_2}}{dt} - \dfrac{d\vec{r_1}}{dt} \cdot \vec{r_2}\right)$

(iv) $\dfrac{d^2}{dt^2}\left\{\left(\vec{r} \times \dfrac{d\vec{r}}{dt}\right) \times \dfrac{d^2\vec{r}}{dt^2}\right\}$

Answers

1. (i) $4t$ (ii) $\sqrt{5}$ (iii) 2 (iv) $4t - 2$

2. (i) 52 (ii) $-42\vec{i} + 101\vec{j} + 40\vec{k}$

4. $4 + 6\vec{k}$, $-4\vec{i} + 6\vec{k}$ and $-4\vec{i}$, $-4\vec{j}$

5(a) (i) $-(4y^3z\vec{i} + (9x^2y^2 + 3x^2z^2)\vec{j} + 4xyz^2\vec{k})$;

$4y^3z\vec{i} + (9x^2y^2 + 3x^2z^2)\vec{j} + 4xyz^2\vec{k}$

(b) $-z^2\vec{i} + 4x^3z\vec{j} + 4xyz\vec{k}$; $z^2\vec{i} - 4x^3z\vec{j} - 4xyz\vec{k}$ (ii) $-36\vec{j}$

7. (i) $\dfrac{1}{r}\dfrac{d\vec{r}}{dt} - \dfrac{1}{r^2}\dfrac{dr}{dt}\vec{r}$ (ii) $\left(2r - \dfrac{2}{r^3}\right)\dfrac{dr}{dt}$

(iii) $\dfrac{d\vec{r}}{dt} \times \left(\dfrac{d\vec{r}}{dt} \times \dfrac{d^2\vec{r}}{dt^2}\right) + \vec{v} \times \left(\dfrac{d\vec{r}}{dt} \times \dfrac{d^3\vec{r}}{dt^3}\right)$

(iv) $\dfrac{1}{\vec{r}\cdot\vec{a}}\left\{\dfrac{d\vec{r}}{dt} \times \vec{a}\right\} - \dfrac{\vec{r} \times \vec{a}}{(\vec{r}\cdot\vec{a})^2}\left\{\dfrac{d\vec{r}}{dt} \cdot \vec{a}\right\}$

(v) $\vec{r} \cdot \dfrac{d\vec{r}}{dt} \times \dfrac{d^3\vec{r}}{dt^3}$ (vi) $\dfrac{1}{(r^2 + a^2)}\dfrac{d\vec{r}}{dt} - \dfrac{1}{(r^2 + a^2)^2}2\vec{r} \cdot \dfrac{d\vec{r}}{dt}(\vec{r} + \vec{a})$

10. (i) $\left(\vec{r} \times \dfrac{d^2\vec{r}}{dt^2}\right) \times \dfrac{d^2\vec{r}}{dt^2} + \left(\vec{r} \times \dfrac{d\vec{r}}{dt}\right) \times \dfrac{d^3\vec{r}}{dt^3}$

(ii) $\vec{r_1} \times \dfrac{d^2\vec{r_2}}{dt^2} - \dfrac{d^2\vec{r_1}}{dt^2} \times \vec{r_2}$ (iii) $\vec{r_1} \cdot \dfrac{d^2\vec{r_2}}{dt^2} - \dfrac{d^2\vec{r_1}}{dt^2} \cdot \vec{r_2}$

(iv) $\left(\dfrac{d\vec{r}}{dt} \times \dfrac{d^2\vec{r}}{dt^2}\right) + \left(\vec{r} \times \dfrac{d^3\vec{r}}{dt^2}\right) \times \dfrac{d^2\vec{r}}{dt^2} + 2\left(\vec{r} \times \dfrac{d^2\vec{r}}{dt^2}\right)$

$\times \dfrac{d^3\vec{r}}{dt^3} + \left(\vec{r} \times \dfrac{d\vec{r}}{dt}\right) \times \dfrac{d^4\vec{r}}{dt^4}$

5.6.13 Vector Integration

Let $\vec{f}(t)$ and $\vec{F}(t)$ are two vector functions of the scalar variable t such that $\dfrac{d}{dt}[\vec{f}(t)] = \vec{F}(t)$, then vector integration of the vector function $\vec{F}(t)$ with respect to the scalar variable t is $\vec{f}(t)$.

Mathematically,

$$\int \vec{F}(t)\, dt = \vec{f}(t).$$

Also we know that $\vec{c}$ is constant vector whose derivative is $\dfrac{d\vec{c}}{dt} = 0$.

Also

$$\frac{d}{dt}(\vec{f}(t) + \vec{c}) = \frac{d}{dt}\vec{f}(t) + \frac{d\vec{c}}{dt} = \vec{F}(t) \text{ where } \vec{c} \text{ is constant of}$$

integration and the vector function $\vec{f}(t) + \vec{c}$ is indefinite integral of $\vec{F}(t)$.

Furthermore, a definite integral of the vector function $\vec{F}(t)$ of scalar variable t between the limits $t = t_1$ and $t = t_2$ is defined as

$$\int_{t_1}^{t_2} \vec{F}(t)\, dt = \int_{t_1}^{t_2} \frac{d}{dt}(\vec{f}(t))\, dt = \int_{t_1}^{t_2} d\{\vec{f}(t)\}$$

$$= [\vec{F}(t)]_{t_1}^{t_2} = \vec{F}(t_2) - \vec{F}(t_1).$$

Let $F_1(t)$, $F_2(t)$, $F_3(t)$ are components of the vector function $\vec{F}(t)$, then

$$\vec{F}(t) = F_1(t)\,\vec{i} + F_2(t)\,\vec{j} + F_3(t)\,\vec{k}$$

$$\therefore \quad \int F(t)\, dt = \vec{i}\int F_1(t)\, dt + \vec{j}\int F_2(t)\, dt + \vec{k}\int F_3(t)\, dt.$$

5.6.14 Some Important Examples on Vector Integration

The vector integration is also considered as the inverse of differentiation so that various important may be obtained by using the standard formulas on differentiation. Some examples are defined as follows.

1. $\frac{d}{dt}(\vec{r_1}.\vec{r_2}) = \vec{r_1}.\frac{d\vec{r_2}}{dt} + \frac{d\vec{r_1}}{dt}.\vec{r_2}$

$\therefore \int\left(\vec{r_1}.\frac{d\vec{r_2}}{dt} + \frac{d\vec{r_1}}{dt}.\vec{r_2}\right)dt = \vec{r_1}.\vec{r_2} + \vec{c}$

where $\vec{c}$ is constant of integration.

2. $\frac{d}{dt}(\vec{r}.\vec{r}) = 2\vec{r}.\frac{d\vec{r}}{dt}$ (on putting $\vec{r_1} = \vec{r_2} = \vec{r}$)

$\therefore \int\left(2\vec{r}.\frac{d\vec{r}}{dt}\right)dt = \vec{r}^2 + \vec{c}$

3. $\frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right)^2 = 2\frac{d\vec{r}}{dt}.\frac{d^2\vec{r}}{dt^2}$

$\therefore \int\left(2\frac{d\vec{r}}{dt}.\frac{d^2\vec{r}}{dt^2}\right)dt = \left(\frac{dr}{dt}\right)^2 + \vec{c}$

4. $\frac{d}{dt}(\vec{r_1}\times\vec{r_2}) = \vec{r_1}\times\frac{d\vec{r_2}}{dt} + \frac{d\vec{r_1}}{dt}\times\vec{r_2}$

$\therefore \int\left(\vec{r_1}\times\frac{d\vec{r_2}}{dt} + \frac{d\vec{r_1}}{dt}\times\vec{r_2}\right)dt = \vec{r_1}\times\vec{r_2} + \vec{c}$

5. $\frac{d}{dt}(\vec{a}\times\vec{r}) = \vec{a}\times\frac{d\vec{r}}{dt}$ where $\vec{a}$ is constant vector

$\therefore \int\left(\vec{a}\times\frac{d\vec{r}}{dt}\right)dt = \vec{a}\times\vec{r} + \vec{c}$

6. $\frac{d}{dt}\left(\vec{r_1}\times\frac{d\vec{r_1}}{dt}\right) = \vec{r_1}\times\frac{d^2\vec{r_1}}{dt^2}$

$\therefore \int\left(\vec{r_1}\times\frac{d^2\vec{r_1}}{dt^2}\right)dt = \vec{r_1}\times\frac{d\vec{r_1}}{dt} + \vec{c}$

7. $\frac{d}{dt}\left(\frac{\vec{r}}{r}\right) = \left(\frac{1}{r}\frac{d\vec{r}}{dt} - \frac{1}{r^2}\frac{dr}{dt}\vec{r}\right)$

$\therefore \int\left(\frac{1}{r}\frac{d\vec{r}}{dt} - \frac{1}{r^2}\frac{dr}{dt}\vec{r}\right)dt = \frac{\vec{r}}{r} + \vec{c}$

8. $\frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right) = \frac{d^2\vec{r}}{dt^2}$

$\therefore \int\frac{d^2\vec{r}}{dt^2}\,dt = \frac{d\vec{r}}{dt} + \vec{c}$

9. $\frac{d}{dt}(\vec{a}.\vec{r}) = \vec{a}.\frac{d\vec{r}}{dt}$

where $\vec{a}$ is a constant vector

$\therefore \int\left(\vec{a}.\frac{d\vec{r}}{dt}\right)dt = \vec{a}.\vec{r} + \vec{c}$

10. $\frac{d}{dt}(\phi\,\vec{r}) = \left(\phi\frac{d\vec{r}}{dt} + \frac{d\phi}{dt}\vec{r}\right)$

$\therefore \int\left(\phi\frac{d\vec{r}}{dt} + \frac{d\phi}{dt}\vec{r}\right)dt = \phi\,\vec{r} + \vec{c}$

Worked out Examples

Ex. 1 Integrate $\int_1^3\left(\vec{r}.\frac{d\vec{r}}{dt}\right)dt$, **where** $\vec{r}(t) = 2t\vec{i} - t\vec{j} + 2t\vec{k}$

Solution:

Given that $\vec{r}(t) = 2t\vec{i} - t\vec{j} + 2t\vec{k}$

Differentiating it with respect to t, we get

$$\frac{d\vec{r}}{dt} = 2\vec{i} - \vec{j} + 2\vec{k}$$

Now $\vec{r}.\frac{d\vec{r}}{dt} = (2t\vec{i} - t\vec{j} + 2t\vec{k}).(2\vec{i} - \vec{j} + 2\vec{k})$

$= (4t + t + 4t) = 9t$

Then $\int_1^3 \left(\vec{r}\cdot\frac{d\vec{r}}{dt}\right) dt = \int_1^3 9t\, dt = \left[\frac{9t^2}{2}\right]_1^3 = \frac{9}{2}(9-1) = \frac{9}{2}\times 8 = 36$

Ex. 2 Evaluate

(i) $\int_0^1 \{t\vec{i} + (t^2-2t)\vec{j} + 3t^2\vec{k}\}\, dt$ **(ii)** $\int_1^2 (5t^2\vec{i} + t\vec{j} - t^3\vec{k})\, dt$

(iii) $\int_1^2 \left(\vec{r}\cdot\frac{d\vec{r}}{dt}\right) dt$ **and** $\int_1^2 \left(\vec{r}\times\frac{d^2\vec{r}}{dt^2}\right) dt$

where $\vec{r} = 5t^2 + t\vec{j} - t^3\vec{k}$

Solution:

(i) We have $\int_0^1 (t\vec{i} + (t^2-2t)\vec{j} + 3t^2\vec{k})\, dt$

$$= \left[\frac{t^2}{2}\vec{i} + \left(\frac{t^3}{3} - t^2\right)\vec{j} + t^3\vec{k}\right]_0^1$$

$$= \frac{1}{2}(1-0)\vec{i} + \left(\frac{1}{3} - 1 - 0\right)\vec{j} + (1-0)\vec{k} = \frac{1}{2}\vec{i} - \frac{2}{3}\vec{j} + \vec{k}$$

(ii) We have $\int_1^2 (5t^2\vec{i} + t\vec{j} - t^3\vec{k})\, dt$

$$= \left[\frac{5t^3}{3}\vec{i} + \frac{t^2}{2}\vec{j} - \frac{t^4}{4}\vec{k}\right]_1^2$$

$$= \frac{5}{3}(8-1)\vec{i} + \frac{1}{2}(4-1)\vec{j} - \frac{1}{4}(16-1)\vec{k}$$

$$= \frac{35}{3}\vec{i} + \frac{3}{2}\vec{j} - \frac{15}{4}\vec{k}$$

(iii) Given that $\vec{r} = 5t^2\vec{i} + t\vec{j} - t^3\vec{k}$

Differentiating it with respect to t, we get

$$\frac{d\vec{r}}{dt} = 10t\,\vec{i} + \vec{j} - 3t^2\vec{k} \qquad \text{...(I)}$$

Now $\vec{r}\cdot\frac{d\vec{r}}{dt} = (5t^2\vec{i} + t\vec{j} - t^3\vec{k})\cdot(10t\,\vec{i} + \vec{j} - 3t^2\vec{k})$

$$= 50t^3 + t + 3t^5$$

Then $\int_1^2 \left(\vec{r}\cdot\frac{d\vec{r}}{dt}\right) dt = \int_1^2 (50t^3 + t + 3t^5)\, dt = \left[\frac{50t^4}{4} + \frac{t^2}{2} + \frac{3t^6}{6}\right]_1^2$

$$= \frac{25}{2}(16-1) + \frac{1}{2}(4-1) + \frac{1}{2}(64-1)$$

$$= \frac{25}{2}\times 15 + \frac{3}{2} + \frac{1}{2}\times 63 = \frac{375}{2} + \frac{66}{2} = \frac{441}{2}$$

$\therefore \int_1^2 \left(\vec{r}\cdot\frac{d\vec{r}}{dt}\right) dt = \frac{441}{2}$

Differentiating (1) with respect to t, we get

$$\frac{d^2\vec{r}}{dt^2} = 10\vec{i} - 6t\vec{k}$$

$$\vec{r}\times\frac{d^2\vec{r}}{dt^2} = (5t^2\vec{i} + t\vec{j} - t^3\vec{k})\times(10\vec{i} - 6t\vec{k})$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 5t^2 & t & -t^3 \\ 10 & 0 & -6t \end{vmatrix}$$

$$= (-6t^2 - 0)\vec{i} - (-30t^3 + 10t^3)\vec{j} + (0 - 10t)\vec{k}$$

$$= -6t^2\vec{i} + 20t^3\vec{j} - 10t\vec{k}$$

Now $\int_1^2 \left(\vec{r}\times\frac{d^2\vec{r}}{dt^2}\right) dt = \int_1^2 (-6t^2\vec{i} + 20t^3\vec{j} - 10t\vec{k})\, dt$

$$= \left[-\frac{6t^3}{3}\vec{i} + \frac{20t^4}{4}\vec{j} - \frac{10t^2}{2}\vec{k}\right]_1^2$$

$$= -2(8-1)\vec{i} + 5(16-1)\vec{j} - 5(4-1)\vec{k}$$

$$= -14\vec{i} + 75\vec{j} - 15\vec{k}$$

Ex. 3 Solve $\frac{d^2\vec{r}}{dt^2} = \vec{a}t + \vec{b}$ where $\vec{a}$ and $\vec{b}$ are constant vectors and given that $\vec{r} = 0$, $\frac{d\vec{r}}{dt} = 0$ at $t = 0$,

Solution:

Given that $\frac{d^2\vec{r}}{dt^2} = \vec{a}t + \vec{b}$

$$\int \frac{d^2\vec{r}}{dt^2}\, dt = \int (\vec{a}t + \vec{b})\, dt$$

or $\frac{d\vec{r}}{dt} = \frac{\vec{a}t^2}{2} + \vec{b}t + \vec{c}$ where $\vec{c}$ is constant vector integration

When $t = 0$, $\frac{d\vec{r}}{dt} = 0$

$\therefore$ $\vec{c} = 0$.

Thus

$$\frac{d\vec{r}}{dt} = \frac{\vec{a}t^2}{2} + \vec{b}t$$

Again integrating with respect to t, we get

$$\int \left(\frac{d\vec{r}}{dt}\right) dt = \int \left(\frac{\vec{a}t^2}{2} + \vec{b}t\right) dt$$

$\vec{r} = \frac{\vec{a}t^3}{6} + \frac{\vec{b}t^2}{2} + \vec{c_1}$ where $\vec{c_1}$ is constant of integration.

When $t = 0$, $\vec{r} = 0$ gives $\vec{c_1} = 0$.

Thus $\vec{r} = \frac{1}{6}\vec{a}t^3 + \frac{1}{2}\vec{b}t^2$

Exercise - 24

1. Evaluate

(i) $\int_1^2 (2t^2\,\vec{i} + t\,\vec{j} - 3t^2\,\vec{k})\, dt$

(ii) $\int_1^2 \left(\vec{r} \times \frac{d^2\vec{r}}{dt^2}\right) dt$, where $\vec{r} = 2t^2\,\vec{i} + t\,\vec{j} - 3t^2\vec{k}$

(iii) $\int_2^3 \left(\vec{r} \cdot \frac{d\vec{r}}{dt}\right) dt$ where $\vec{r}(t) = 2\vec{i} - \vec{j} + 2\vec{k}$ when $t = 2$

$= 4\vec{i} - 2\vec{j} + 3\vec{k}$ when $t = 3$.

(iv) $\int_0^1 (t\,\vec{i} + e^t\,\vec{j} + e^{-2t})\, dt$

(v) $\int_0^2 (\vec{r_1} \times \vec{r_2})\, dt$, where $\vec{r_1} = 2\vec{i} + t\vec{j} - \vec{k}$

$\vec{r_2} = t\,\vec{i} + 2\vec{j} + 3\vec{k}$

(vi) $\int_0^1 \vec{r_1} \cdot (\vec{r_2} \times \vec{r_3})\, dt$ where $\vec{r_1} = 2t^2\,\vec{i} - t\,\vec{j} + 2t^2\vec{k}$

$\vec{r_2} = t^2\,\vec{i} + 2t\,\vec{j} - t^2\vec{k}$

$\vec{r_3} = t^2\,\vec{i} - t\,\vec{j} - t^2\,\vec{k}$.

2. Given that $\vec{r_1} = 2t^2\,\vec{i} + 3(t-1)\,\vec{j} + 4t^2\,\vec{k}$ and $\vec{r_2} = (t-1)\,\vec{i} + t^2\,\vec{j} + (t-2)\vec{k}$, *find*

(i) $\int_1^2 (\vec{r_1} \cdot \vec{r_2})\, dt$ (ii) $\int_1^2 \left(\vec{r_1} \cdot \frac{d\vec{r_2}}{dt}\right) dt$

(iii) $\int_1^2 \left(\vec{r_1} \times \frac{d^2\vec{r_2}}{dt^2}\right) dt$

3. Solve for $\vec{r}$ *to the equation*

(i) $\vec{a} \times \frac{d^2\vec{r}}{dt^2} = \vec{b}$ *where* $\vec{a}$ and $\vec{b}$ are constant vectors, given that $\vec{r} = 0, \frac{d\vec{r}}{dt} = 0$ at $t = 0$

(ii) $\frac{d^2\vec{r}}{dt^2} = e^t\,\vec{i} + e^{2t}\,\vec{j} + \vec{k}$

given that $\frac{d\vec{r}}{dt} = \vec{i} + \vec{j}$ at $t = 0$ and $\vec{r} = 0$ at $t = 0$.

4. If $\frac{d\vec{r}}{dt} = t^2\,\vec{i} + (6t+1)\,\vec{j} + 8t^3\vec{k}$ and $\vec{r}(0) = 2\vec{i} - 3\vec{j} + \vec{k}$, find $\vec{r}$.

5. If $\frac{d^2\vec{r}}{dt^2} = 6t\,\vec{i} - 12t^2\,\vec{j} + \vec{k}$ and $\vec{r}\,'(0) = \vec{i} + 2\vec{j} - 3\vec{k}$

$\vec{r}(0) = 7\vec{i} + \vec{k}$ find $\vec{r}$.

Answers

1. (i) $\frac{14}{3}\vec{i} + \frac{3}{2}\vec{j} - 7\vec{k}$ (ii) $-9\vec{i} - 6\vec{k}$ (iii) 10

(iv) $\frac{1}{2}\vec{i} + (e-1)\vec{j} + \frac{1}{2}(1-e^2)\vec{k}$ (v) $10\vec{i} - 14\vec{j} + \frac{16}{3}\vec{k}$ (vi) -2

2. (i) $\frac{41}{12}$ (ii) 19 (iii) $\frac{28}{3}(\vec{k} - 2\vec{i})$

3. (i) $\vec{a} \times \vec{r} = \frac{1}{2}t^2\,\vec{b}$ (ii) $\vec{r} = (e^t - 1)\vec{i} + \left(e^{2t} + \frac{t}{2} - \frac{1}{4}\right)\vec{j} + \frac{t^2}{2}\vec{k}$

4. $\vec{r} = \left(\frac{t^3}{3} + 2\right)\vec{i} + (3t^2 + t - 3)\vec{j} + (2t^4 + 1)\vec{k}$

5. $\vec{r} = (t^3 + t + 7)\,\vec{i} + (2t - t^4)\vec{j} + \left(\frac{t^2}{2} - 3t + 1\right)\vec{k}$

5.7. Gradient, Divergence and Curl

Gradient, divergence and curl are basic in connection with fields in which its application are most important in physics, mathematics and engineering field. To define these, we would like to define scalar and vector fields which would be needed in the discussion of the subject.

Point functions:

In the set of co-ordinates of a point in a region of space, a variable quantity which depends upon the co-ordinates of the point of region of space is called a ***point function*** and region in which it defines the quantity is known as a field. There are two types of functions.

(i) Scalar point function

A point function is said to be *scalar point function* in a region of space if corresponds to each point of the region of space a definite scalar $\phi(x, y, z)$ is assigned .The region in which it is defined is called a *scalar field.*

The temperature at any instant,density of a body potential due to gravitational matter are examples of scalar point function.

(ii) Vector point function

A point function is said to be *vector point function* in a region of space if corresponds to each point of the region of space a definite vector $\vec{v}\,(x, y, z)$ is assigned. The region in which it is defined is called *vector field.*

The velocity of a moving fluid at any instant, the gravitational or electrical intensity of force are examples of vector point functions.

Differentiation of vector and scalar point function follow the same rules as those of ordinary calculus.

5.7.1 Vector Differential Operator

The vector operator of the form $\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$ is called *vector differential operator* and denoted by ∇ (read as del or nabla) . We write

$$\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial x} + \vec{k}\frac{\partial}{\partial z}$$

5.7.2 Differntial Operator (del) Applied to Scalar Point Function

1. Gradient of Scalar Field

If ϕ (x, y, z) be scalar point function of the coordinates x, y, z defined and differentiable at each point (x, y, z) in space, then the gradient of the function $\phi(x, y, z)$ denoted by $\nabla\phi$ or grad ϕ is defined as $\left(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial z}+\vec{k}\frac{\partial}{\partial z}\right)\phi$.

We write $\quad \text{grad }\phi = \nabla\phi = \left(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}\right)\phi$

$$= \vec{i}\frac{\partial\phi}{\partial x}+\vec{j}\frac{\partial\phi}{\partial y}+\vec{k}\frac{\partial\phi}{\partial z}$$

Note:

Let ϕ (x, y, z) is scalar point function and $\nabla\phi$ is vector function whose components are

$$\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z} \text{ along the directions } \vec{i}, \vec{j} \text{ and } \vec{k}.$$

Theorem 1

The necessary and sufficient condition for a scalar point function to be constant is that $\nabla\phi = 0$

Proof

The condition is necessary

Let $\phi(x, y, z)$ be constant function , then

$$\frac{\partial\phi}{\partial x}=\frac{\partial\phi}{\partial y}=\frac{\partial\phi}{\partial z}=0$$

or $\quad \nabla\phi = \vec{i}\frac{\partial\phi}{\partial x}+\vec{j}\frac{\partial\phi}{\partial y}+\vec{k}\frac{\partial\phi}{\partial z}=0$

$\therefore \quad \nabla\phi = 0$

The condition is sufficient

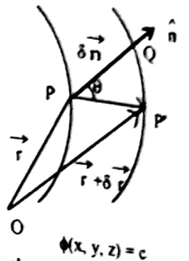
Let $\quad \nabla\phi = 0$

or $\quad \vec{i}\frac{\partial\phi}{\partial x}+\vec{j}\frac{\partial\phi}{\partial y}+\vec{k}\frac{\partial\phi}{\partial z}=0$

or $\quad \frac{\partial\phi}{\partial x}=0, \frac{\partial\phi}{\partial y}=0, \frac{\partial\phi}{\partial z}=0.$ Hence ,the function $\phi(x, y, z)$ is constant.

2. Geometrical Interpretation of Gradient

Consider the scalar point function $\phi(x, y, z)$ which is continuous in a given region of space. If a surface $\phi(x, y, z) = c$ be drawn through any point O such that at each point on it, the function has the same value as at P, then such a surface is called a **level surface** of the function ϕ through P. For example the height of the land, on the map equipotent surface at the point (x, y, z) represented by $\phi(x, y, z) = c$ gives the level surfaces.



Let ϕ be function at the point P(x, y, z) of the level surface and $\phi + \delta\phi$ be neighboring level surface of ϕ at the point $(x+\delta x, y+\delta y, z+\delta z)$.

Let $\vec{r}$ and $\vec{r}+\delta\vec{r}$ be the position vectors of P and P' relative to origin,, then we have $\delta\vec{r} = \vec{i}\,\delta x+\vec{j}\,\delta y+\vec{k}\,\delta z$

$$\nabla\phi.d\vec{r} = \left(\vec{i}\frac{\partial\phi}{\partial x}+\vec{j}\frac{\partial\phi}{\partial y}+\vec{k}\frac{\partial\phi}{\partial z}\right).(\vec{i}\,\delta x+\vec{j}\,\delta y+\vec{k}\,\delta z)$$

$$= \frac{\partial\phi}{\partial x}\delta x+\frac{\partial\phi}{\partial y}\delta y+\frac{\partial\phi}{\partial z}\delta z = \delta\phi$$

If P' lies on the same level surface as P ,so that $d\phi=0$, then $\nabla\phi.\delta\vec{r}=0$

This shows that $\nabla\phi$ is perpendicular to every $\delta\vec{r}$ lying on this surface. Thus $\nabla\phi$ is normal to the surface $\phi(x, y, z) = c$.

Therefore $\nabla\phi = |\nabla\phi|\,\hat{n}$ be a unit vector normal to this surface.

If $\delta n = PQ$ is the distance between the surfaces through P and P' ,then

$$\delta x = PQ = PP'\cos\theta = \delta r\cos\theta = \hat{n}.\delta\vec{r}$$

and rate of change of ϕ normal to the surface through P is

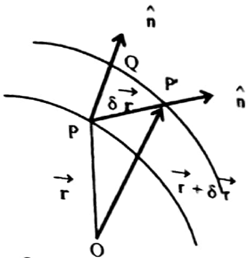
$$\delta n = \frac{\partial\phi}{\partial n}\delta n = \frac{\partial\phi}{\partial n}(\hat{n}.\delta\vec{r})$$

$$\nabla\phi.\delta\vec{r} = \left(\frac{\partial\phi}{\partial n}\hat{n}\right).\delta\vec{r}$$

$$\therefore \quad \nabla\phi = \frac{\partial\phi}{\partial n}\hat{n}$$

Thus grad ϕ is a vector normal to the surface $\phi(x, y, z)$ = constant and has a magnitude equal to the rate of change of ϕ along the normal.

5.8 Directional Derivatives



Let P(x, y, z) and P' (x + δx, y + δy, z + δz) are two points on the neighboring level surfaces with the scalar point functions ϕ and $\phi+\delta\phi$ respectively. The position of the vectors of P and P' relative to origin O are $\vec{r}$ and $\vec{r}+\delta\vec{r}$. If δr denotes the length PP' and $\hat{n}$ is a unit vector in the direction PP' then the limiting value of

$\lim_{\delta r \to 0} \frac{\delta\phi}{\delta r} = \frac{\partial\phi}{\partial r}$ is known as the *directional derivative* of ϕ at P along the direction PP'

Since $\delta r = \frac{\delta n}{\cos\alpha} = \frac{\delta n}{\hat{n}.\hat{n}}$

$$\therefore \quad \frac{\partial\phi}{\partial r} = \lim_{\delta r \to 0} \frac{\delta\phi}{\delta n}(\hat{n}.\hat{n}) = \frac{\partial\phi}{\partial n}\hat{n}.\hat{n}$$

$$\frac{\partial\phi}{\partial r} = \nabla\phi.\hat{n}.$$

Thus the directional derivative of ϕ in the direction of $\hat{n}$ is the resolved part of $\nabla\phi$ in the direction $\hat{n}$.

Also $\frac{\partial\phi}{\partial r} = \nabla\phi.\hat{n} = |\nabla\phi|\,|\hat{n}|\cos\alpha = |\nabla\phi|\cos\alpha \le |\nabla\phi|$

It shows that $|\nabla\phi|$ be maximum rate of change of ϕ.

5.8.1 Differential Operator(del) Applied to Vector Point Function

1. Divergence of a vector function

Let $\vec{v}$ (x, y, z) be continuously differentiable vector point function, then divergence

of $\vec{v}$ denoted by div $\vec{v}$ or $\nabla.\vec{v}$ and is defined by

$$\text{div}\,\vec{v} = \nabla.\vec{v} = \left(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{i}\frac{\partial}{\partial z}\right).\vec{v}$$

$$= \vec{i}.\frac{\partial\vec{v}}{\partial x}+\vec{j}.\frac{\partial\vec{v}}{\partial y}+\vec{k}.\frac{\partial\vec{v}}{\partial z}$$

If $\vec{v} = v_1\vec{i}+v_2\vec{j}+v_3\vec{k}$, then

$$\text{div}\,\vec{v} = \nabla.\vec{v} = \left(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{i}\frac{\partial}{\partial z}\right).(v_1\vec{i}+v_2\vec{j}+v_3\vec{k})$$

$$= \frac{\partial v_1}{\partial x}+\frac{\partial v_2}{\partial y}+\frac{\partial v_3}{\partial z}$$

2. Curl of a vector point function:

Let $\vec{v}$(x, y, z) be continuously differentiable vector point function, then curl of $\vec{v}$ denoted by curl $\vec{v}$ or $\nabla\times\vec{v}$ is defined by

$$\text{Curl}\,\vec{v} = \nabla\times\vec{v} = \left(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{i}\frac{\partial}{\partial z}\right)\times\vec{v}$$

$$= \vec{i}\times\frac{\partial\vec{v}}{\partial x}+\vec{j}\times\frac{\partial\vec{v}}{\partial y}+\vec{k}\times\frac{\partial\vec{v}}{\partial z}$$

If $\vec{v} = v_1\vec{i}+v_2\vec{j}+v_3\vec{k}$, then

$$\text{Curl}\,\vec{v} = \nabla\times\vec{v} = \left(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}\right)\times(v_1\vec{i}+v_2\vec{j}+v_3\vec{k})$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix}$$

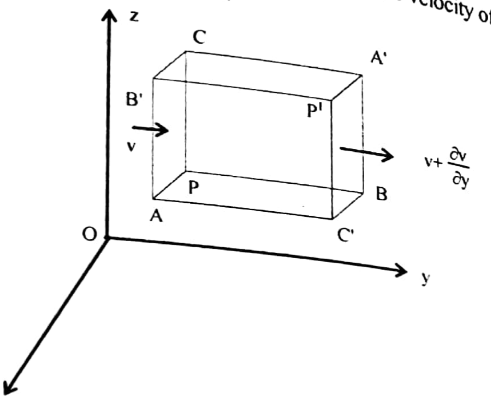
$$= \vec{i}\left(\frac{\partial v_3}{\partial y}-\frac{\partial v_2}{\partial z}\right)+\vec{j}\left(\frac{\partial v_1}{\partial z}-\frac{\partial v_3}{\partial x}\right)+\vec{k}\left(\frac{\partial v_2}{\partial x}-\frac{\partial v_1}{\partial y}\right)$$

3.Physical Interpretation of Divergence

For the physical concept of the divergence of a vector function we consider the case of fluid flow in a stream. Consider the motion of the fluid having velocity $\vec{v} = u\vec{i}+v\vec{j}+w\vec{k}$ at a point P(x, y, z). Consider a small parallelepiped with edges δx, δy, δz parallel to the coordinate axes in the mass of fluid with one of its corner at P.

We consider the flow of the fluid parallel to y-axis i.e. across the faces PCB'A and BA'P'C'.

In the component of $\vec{v}$ along the y-axis we find that the velocity of fluid perpendicular to the face APCB' = v.



Therefore the amount of fluid entering the face APCB' whose area is $\delta z\,\delta x$ in unit time $= v\,\delta z\,\delta x$.

The velocity of fluid across the face BA'P'C' be $v+\frac{\partial v}{\partial y}\delta y$.

Therefore the amount of fluid leaving the face BA' P'C' whose area is $\delta z\,\delta x$ in unit time $=\left(v+\frac{\partial v}{\partial y}\delta y\right)\delta z\,\delta x$.

Therefore ,the net decrease of the amount of fluid due to flow across these two faces from the elementary parallelepiped along the y-axis

$$=\left(v+\frac{\partial v}{\partial y}\delta y\right)\delta z\,\delta x - v\,\delta z\,\delta x = \frac{\partial v}{\partial y}\delta x\,\delta y\,\delta z$$

Similarly the contributions of other two pair of faces, we get the net decrease of the amount of fluid due to flow across other two pair of faces in which component of $\vec{v}$ along the z-axis and x-axis, are $\frac{\partial w}{\partial z}\delta x\,\delta y\,\delta z$ and $\frac{\partial u}{\partial x}\delta x\,\delta y\,\delta z$ respectively.

The total decrease of amount of fluid inside the parallelepiped per unit time =

$$\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)\delta x\,\delta y\,\delta z$$

Thus, the rate of loss of fluid per unit volume

$$=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=\left(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}\right).(u\vec{i}+v\vec{j}+w\vec{k})$$

$$=\operatorname{div}\vec{v}$$

Hence div $\vec{v}$ gives the rate of fluid throughout the rectangular parallelepiped per unit volume.

Solenoidal

A vector function $\vec{v}$ is said to be *Solenoidal* if the divergence of that vector function is zero i.e. div $\vec{v} = 0$

4. Physical interpretation of Curl

Consider the motion of a rigid body rotating about a fixed axis through O. If $\vec{\omega}$ be an angular velocity then $\vec{v}$ be velocity of any particle P(x, y, z) of the body is given by $\vec{v}=\vec{\omega}\times\vec{r}$ where

$$\vec{\omega}=\omega_1\vec{i}+\omega_2\vec{j}+\omega_3\vec{k} \text{ and } \vec{r}=\vec{i}x+\vec{j}y+\vec{k}z$$

$$\vec{v}=\vec{\omega}\times\vec{r}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \omega_1 & \omega_2 & \omega_3\\ x & y & z\end{vmatrix}$$

$$=\vec{i}(\omega_2 z-\omega_3 y)+\vec{j}(\omega_3 x-\omega z)+\vec{k}(\omega_1 y-\omega_2 x)$$

$$\text{Curl }\vec{v}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ \omega_2 z-\omega_3 y & \omega_3 x-\omega_1 z & \omega_1 y-\omega_2 x\end{vmatrix}$$

$$=\vec{i}(\omega_1+\omega_1)+\vec{j}(\omega_2+\omega_2)+\vec{k}(\omega_3+\omega_3)$$

$$=2(\omega_1\vec{i}+\omega_2\vec{j}+\omega_3\vec{k})=2\vec{\omega}$$

Hence the curl of the velocity vector equals to the twice the angular velocity of the body. In general the curl of any vector point function gives the measure of the angular velocity at any point of the vector field.

Irrotational

In any motion the velocity vector point function $\vec{v}$ is said to be *irrotational* if curl $\vec{v} = 0$.

5. Some important deductions involving ▯

Let $\phi(x, y, z)$ and $\psi(x, y, z)$ are scalar point functions and $\vec{a}(x,y,z)$ and $\vec{b}$ (x, y, z) are vector point functions, then

(i) Grad $(\phi \pm \Psi) = \nabla\phi \pm \nabla\Psi$

Proof:

We have

$$\nabla(\phi \pm \Psi) = \vec{i}\,\frac{\partial(\phi \pm \Psi)}{\partial x} + \vec{j}\,\frac{\partial(\phi \pm \psi)}{\partial y} + \vec{k}\,\frac{\partial(\phi \pm \psi)}{\partial z}$$

$$= \vec{i}\left(\frac{\partial\phi}{\partial x} \pm \frac{\partial\psi}{\partial x}\right) + \vec{j}\left(\frac{\partial\phi}{\partial y} \pm \frac{\partial\psi}{\partial y}\right) + \vec{k}\left(\frac{\partial\phi}{\partial z} \pm \frac{\partial\psi}{\partial z}\right)$$

$$= \left(\vec{i}\,\frac{\partial\phi}{\partial x} + \vec{j}\,\frac{\partial\phi}{\partial y} + \vec{k}\,\frac{\partial\phi}{\partial z}\right) \pm \left(\vec{i}\,\frac{\partial\psi}{\partial x} + \vec{j}\,\frac{\partial\psi}{\partial y} + \vec{k}\,\frac{\partial\psi}{\partial z}\right)$$

$$\therefore \quad \nabla(\phi \pm \Psi) = \nabla\phi \pm \nabla\psi.$$

(ii) Div $(\vec{a} \pm \vec{b})$ = div $\vec{a}$ ± div $\vec{b}$

i.e. $\nabla.(\vec{a} \pm \vec{b}) = \nabla.\vec{a} \pm \nabla.\vec{b}$

Proof:

We have

$$\nabla.(\vec{a} \pm \vec{b}) = \left(\vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z}\right).(\vec{a} \pm \vec{b})$$

$$= \vec{i}.\frac{\partial(\vec{a} \pm \vec{b})}{\partial x} + \vec{j}.\frac{\partial(\vec{a} \pm \vec{b})}{\partial y} + \vec{k}.\frac{\partial(\vec{a} \pm \vec{b})}{\partial z}$$

$$= \left(\vec{i}.\frac{\partial\vec{a}}{\partial x} + \vec{j}.\frac{\partial\vec{a}}{\partial y} + \vec{k}.\frac{\partial \vec{a}}{\partial z}\right) \pm \left(\vec{i}.\frac{\partial\vec{b}}{\partial x} + \vec{j}.\frac{\partial\vec{b}}{\partial y} + \vec{k}.\frac{\partial\vec{b}}{\partial z}\right)$$

$$= \nabla.\vec{a} \pm \nabla.\vec{b}.$$

$$\therefore \quad \nabla.(\vec{a} \pm \vec{b}) = \nabla.\vec{a} \pm \nabla.\vec{b}.$$

(iii) Curl $(\vec{a} \pm \vec{b})$ = curl $\vec{a}$ ± curl $\vec{b}$

i.e. $\nabla \times (\vec{a} \pm \vec{b}) = \nabla \times \vec{a} \pm \nabla \times \vec{b}$

Proof:

We have,

$$\nabla \times (\vec{a} \pm \vec{b}) = \left(\vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z}\right) \times (\vec{a} \pm \vec{b})$$

$$= \vec{i} \times \frac{\partial(\vec{a} \pm \vec{b})}{\partial x} + \vec{j} \times \frac{\partial(\vec{a} \pm \vec{b})}{\partial y} + \vec{k} \times \frac{\partial(\vec{a} \pm \vec{b})}{\partial z}$$

$$= \left(\vec{i} \times \frac{\partial\vec{a}}{\partial x} + \vec{j} \times \frac{\partial\vec{a}}{\partial y} + \vec{k} \times \frac{\partial\vec{a}}{\partial z}\right) \pm \left(\vec{i} \times \frac{\partial\vec{b}}{\partial x} + \vec{j} \times \frac{\partial\vec{b}}{\partial y} + \vec{k} \times \frac{\partial\vec{b}}{\partial z}\right)$$

$$= \left(\vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z}\right) \times \vec{a} \pm \left(\vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z}\right) \times \vec{b}$$

$$= \nabla \times \vec{a} \pm \nabla \times \vec{b}$$

$$\therefore \quad \nabla \times (\vec{a} \pm \vec{b}) = \nabla \times \vec{a} \pm \nabla \times \vec{b}$$

(iv) Grad $(\phi\Psi) = \phi$ grad $\psi + \psi$ grad ϕ

i.e. $\nabla(\phi\psi) = \phi\nabla\psi + \psi\nabla\phi$.

Proof:

We have

$$\nabla(\phi\Psi) = \left(\vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z}\right)(\phi\Psi)$$

$$= \vec{i}\,\frac{\partial(\phi\Psi)}{\partial x} + \vec{j}\,\frac{\partial(\phi\Psi)}{\partial y} + \frac{\partial(\phi\Psi)}{\partial z}$$

$$= \Sigma\,\vec{i}\left(\phi\frac{\partial\Psi}{\partial x} + \Psi\frac{\partial\phi}{\partial x}\right) = \phi\,\Sigma\,\vec{i}\,\frac{\partial\Psi}{\partial x} + \Psi\,\Sigma\,\vec{i}\,\frac{\partial\phi}{\partial x}$$

$$= \phi\nabla\Psi + \Psi\nabla\phi$$

$$\therefore \quad \nabla(\phi\Psi) = \phi\nabla\Psi + \Psi\nabla\phi$$

(v) $\nabla\left(\frac{\phi}{\Psi}\right) = \frac{\Psi\nabla\phi - \phi\nabla\Psi}{\Psi^2}$

Proof:

We have

$$\nabla\left(\frac{\phi}{\Psi}\right) = \left(\vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z}\right)\left(\frac{\phi}{\psi}\right)$$

$$= \vec{i}\,\frac{\partial}{\partial x}\left(\frac{\phi}{\psi}\right) + \vec{j}\,\frac{\partial}{\partial y}\left(\frac{\phi}{\psi}\right) + \vec{k}\,\frac{\partial}{\partial z}\left(\frac{\phi}{\psi}\right)$$

$$= \vec{i}\,\frac{\left(\psi\frac{\partial\phi}{\partial x} - \phi\frac{\partial\psi}{\partial x}\right)}{\psi^2} + \vec{j}\,\frac{\psi\frac{\partial\phi}{\partial y} - \phi\frac{\partial\psi}{\partial y}}{\psi^2} + \vec{k}\,\frac{\left(\psi\frac{\partial\phi}{\partial z} - \phi\frac{\partial\psi}{\partial z}\right)}{\psi^2}$$

$$= \frac{1}{\psi^2}\left[\psi\left(\vec{i}\frac{\partial\phi}{\partial x}+\vec{j}\frac{\partial\phi}{\partial y}+\vec{k}\frac{\partial\phi}{\partial z}\right)-\phi\left(\vec{i}\frac{\partial\psi}{\partial x}+\vec{j}\frac{\partial\psi}{\partial y}+\vec{k}\frac{\partial\psi}{\partial z}\right)\right]$$

$$= \frac{1}{\psi^2}\left[\psi\left(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}\right)\phi-\phi\left(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}\right)\psi\right]$$

$$= \frac{1}{\psi^2}\,(\psi\nabla\phi-\phi\nabla\psi)$$

$$\therefore \quad \nabla\left(\frac{\phi}{\psi}\right) = \frac{\psi\nabla\phi-\phi\nabla\psi}{\psi^2}$$

5.8.2 Differential Operator (del) Applied to Products of Point Functions

Let $\phi(x, y, z)$ be a scalar point function and $\vec{a}(x, y, z)$ and $\vec{b}(x, y, z)$ are vector point functions, then products of these functions are defined as $\phi\vec{a}$, $\vec{a}.\vec{b}$; $\vec{a}\times\vec{b}$ where $\phi\vec{a}$ and $\vec{a}\times\vec{b}$ are vector functions but $\vec{a}.\vec{b}$ is a scalar function. So ,we have the following identities .

1. Div $(\phi\vec{a}) = \phi$ div $\vec{a} + \vec{a}.$(grad ϕ)

i.e. $\nabla.(\phi\vec{a}) = \phi(\nabla.\vec{a}) + \vec{a}.(\nabla\phi)$.

Proof:

We have

$$\text{div}(\phi\vec{a}) = \nabla.(\phi\vec{a})$$

$$= \left(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}\right).(\phi\vec{a})$$

$$= \vec{i}.\frac{\partial(\phi\vec{a})}{\partial x}+\vec{j}.\frac{\partial(\phi\vec{a})}{\partial y}+\vec{k}.\frac{\partial(\phi\vec{a})}{\partial z}$$

$$= \vec{i}.\left(\phi\frac{\partial\vec{a}}{\partial x}+\vec{a}\frac{\partial\phi}{\partial x}\right)+\vec{j}.\left(\phi\frac{\partial\vec{a}}{\partial y}+\vec{a}\frac{\partial\phi}{\partial y}\right)+\vec{k}.\left(\phi\frac{\partial\vec{a}}{\partial z}+\vec{a}\frac{\partial\phi}{\partial z}\right)$$

$$= \phi\left(\vec{i}.\frac{\partial\vec{a}}{\partial x}+\vec{j}.\frac{\partial\vec{a}}{\partial y}+\vec{k}.\frac{\partial\vec{a}}{\partial z}\right)+\vec{a}.\left(\vec{i}\frac{\partial\phi}{\partial x}+\vec{j}\frac{\partial\phi}{\partial y}+\vec{k}\frac{\partial\phi}{\partial z}\right)$$

$$= \phi\left(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}\right).\vec{a}+\vec{a}.\left(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}\right)\phi$$

$$= \phi(\nabla.\vec{a})+\vec{a}.(\nabla\phi)$$

$$\therefore \quad \text{div}(\phi\vec{a}) = \phi(\nabla.\vec{a})+\vec{a}.(\nabla\phi)$$

2. Curl $(\phi\vec{a}) = \phi$ Curl $\vec{a}$ + (grad ϕ) $\times\vec{a}$

i.e. $\nabla\times(\phi\vec{a}) = \phi(\nabla\times\vec{a}) + (\nabla\phi)\times\vec{a}$

Proof:

We have,

$$\nabla\times(\phi\vec{a}) = \left(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}\right)\times(\phi\vec{a})$$

$$= \vec{i}\times\frac{\partial(\phi\vec{a})}{\partial x}+\vec{j}\times\frac{\partial(\phi\vec{a})}{\partial y}+\vec{k}\times\frac{\partial(\phi\vec{a})}{\partial z}$$

$$= \vec{i}\times\left(\phi\frac{\partial\vec{a}}{\partial x}+\vec{a}\frac{\partial\phi}{\partial x}\right)+\vec{j}\times\left(\phi\frac{\partial\vec{a}}{\partial y}+\vec{a}\frac{\partial\phi}{\partial y}\right)+\vec{k}\times\left(\phi\frac{\partial\vec{a}}{\partial z}+\vec{a}\frac{\partial\phi}{\partial z}\right)$$

$$= \phi\left(\vec{i}\times\frac{\partial\vec{a}}{\partial x}+\vec{j}\times\frac{\partial\vec{a}}{\partial y}+\vec{k}\times\frac{\partial\vec{a}}{\partial z}\right)+\left(\vec{i}\frac{\partial\phi}{\partial x}\times\vec{a}+\vec{j}\frac{\partial\phi}{\partial y}\times\vec{a}+\vec{k}\frac{\partial\phi}{\partial z}\times\vec{a}\right)$$

$$= \phi\left(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}\right)\times\vec{a}+\left\{\left(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}\right)\phi\right\}\times\vec{a}$$

$$= \phi(\nabla\times\vec{a})+(\nabla\phi)\times\vec{a}$$

$$\therefore \nabla\times(\phi\vec{a}) = =\phi(\nabla\times\vec{a})+(\nabla\phi)\times\vec{a}$$

3. Div $(\vec{a}\times\vec{b}) = \vec{b}.$(Curl $\vec{a}$) - $\vec{a}.$(Curl $\vec{b}$)

i.e. $\nabla.(\vec{a}\times\vec{b}) = (\nabla\times\vec{a}).\vec{b} - (\nabla\times\vec{b}).\vec{a}$

Proof:

We have

$$\nabla.(\vec{a}\times\vec{b}) = \left(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}\right).(\vec{a}\times\vec{b})$$

$$= \vec{i}.\frac{\partial(\vec{a}\times\vec{b})}{\partial x}+\vec{j}.\frac{\partial(\vec{a}\times\vec{b})}{\partial y}+\vec{k}.\frac{\partial(\vec{a}\times\vec{b})}{\partial z}$$

$$= \vec{i}.\left(\vec{a}\times\frac{\partial\vec{b}}{\partial x}+\frac{\partial\vec{a}}{\partial x}\times\vec{b}\right)+\vec{j}.\left(\vec{a}\times\frac{\partial\vec{b}}{\partial y}+\frac{\partial\vec{a}}{\partial y}\times\vec{b}\right)$$

$$+\vec{k}.\left(\vec{a}\times\frac{\partial\vec{b}}{\partial z}+\frac{\partial\vec{a}}{\partial z}\times\vec{b}\right.$$

$$= \left(\vec{i}.\vec{a}\times\frac{\partial\vec{b}}{\partial x}+\vec{j}.\vec{a}\times\frac{\partial\vec{b}}{\partial y}+\vec{k}.\vec{a}\times\frac{\partial\vec{b}}{\partial z}\right)$$
$$+\left(\vec{i}.\frac{\partial\vec{a}}{\partial x}\times\vec{b}+\vec{j}.\frac{\partial\vec{a}}{\partial y}\times\vec{b}+\vec{k}.\frac{\partial\vec{a}}{\partial z}\times\vec{b}\right)$$
$$= -\left(\vec{i}\times\frac{\partial\vec{b}}{\partial x}.\vec{a}+\vec{j}\times\frac{\partial\vec{b}}{\partial y}.\vec{a}+\vec{k}\times\frac{\partial\vec{b}}{\partial z}.\vec{a}\right)+$$
$$+\left(\vec{i}\times\frac{\partial\vec{a}}{\partial x}.\vec{b}+\vec{j}\times\frac{\partial\vec{a}}{\partial y}.\vec{b}+\vec{k}\times\frac{\partial\vec{a}}{\partial z}.\vec{b}\right)$$
$$= -\left\{\left(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}\right)\times\vec{b}\right\}.\vec{a}$$
$$+\left\{\left(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial j}+\vec{k}\frac{\partial}{\partial z}\right)\times\vec{a}\right\}.\vec{b}$$
$$= -(\nabla\times\vec{b}).\vec{a}+(\nabla\times\vec{a}).\vec{b}$$
$$= (\nabla\times\vec{a}).\vec{b}-(\nabla\times\vec{b}).\vec{a}$$
$$\therefore\quad \nabla.(\vec{a}\times\vec{b}) = (\nabla\times\vec{a}).\vec{b}-(\nabla\times\vec{b}).\vec{a}$$

4. Curl $(\vec{a}\times\vec{b}) = (\vec{b}.\nabla)\vec{a}-(\vec{a}.\nabla)\vec{b}+\vec{a}\text{ div }\vec{b}-\vec{b}\text{ div }\vec{a}$

i.e. $\nabla\times(\vec{a}\times\vec{b}) = (\vec{b}.\nabla)\vec{a}-(\vec{a}.\nabla)\vec{b}+a(\nabla.\vec{b})-\vec{b}(\nabla.\vec{a})$

Proof:

We have

$$\text{Curl}(\vec{a}\times\vec{b}) = \nabla\times(\vec{a}\times\vec{b})$$
$$= \Sigma\,\vec{i}\times\frac{\partial}{\partial x}(\vec{a}\times\vec{b}) = \vec{i}\times\left(\vec{a}\times\frac{\partial\vec{b}}{\partial x}+\frac{\partial\vec{a}}{\partial x}\times\vec{b}\right)$$
$$= \Sigma\,\vec{i}\times\left(\vec{a}\times\frac{\partial\vec{b}}{\partial x}\right)+\Sigma\,\vec{i}\times\left(\frac{\partial\vec{a}}{\partial x}\times\vec{b}\right)$$
$$= \Sigma\left\{\left(\vec{i}.\frac{\partial\vec{b}}{\partial x}\right)\vec{a}-(\vec{i}.\vec{a})\frac{\partial\vec{b}}{\partial x}\right\}+\Sigma\left\{(\vec{i}.\vec{b})\frac{\partial\vec{a}}{\partial x}-\left(\vec{i}.\frac{\partial\vec{a}}{\partial x}\right)\vec{b}\right\}$$

But $\Sigma(\vec{i}.\vec{a})\frac{\partial\vec{b}}{\partial x} = \Sigma(\vec{a}.\vec{i})\frac{\partial\vec{b}}{\partial x} = \Sigma\left(\vec{a}.\,\vec{i}\frac{\partial}{\partial x}\right)b = (\vec{a}.\nabla)\vec{b}$

and $\Sigma(\vec{i}.\vec{b})\frac{\partial\vec{a}}{\partial x} = (\vec{b}.\nabla)\vec{a}$

Also $\Sigma\left(\vec{i}.\frac{\partial\vec{a}}{\partial x}\right)\vec{b} = \vec{b}\Sigma\left(\vec{i}.\frac{\partial\vec{a}}{\partial x}\right) = \vec{b}\Sigma\left(\vec{i}\frac{\partial}{\partial x}\right).\vec{a} = \vec{b}(\nabla.\vec{a})$

and $\Sigma\left(\vec{i}.\frac{\partial\vec{b}}{\partial x}\right)\vec{a} = \vec{a}(\nabla.\vec{b})$

Hence $\nabla\times(\vec{a}\times\vec{b}) = \vec{a}(\nabla.\vec{b})-(\vec{a}.\nabla)\vec{b}+(\vec{b}.\nabla)\vec{a}-\vec{b}(\nabla.\vec{a})$

$$= (\vec{b}.\nabla)\vec{a}-(\vec{a}.\nabla)\vec{b}+\vec{a}\text{ div }\vec{b}-\vec{b}\text{ div }\vec{a}$$

5.8.3. Second Order Differential Operators

If $\phi(x, y, z)$ and $\vec{v}(x, y, z)$ are scalar and vector point functions, then $\nabla\phi$ and $\nabla\times\vec{v}$ being vector point functions. It can be formed their divergence and curl .Also, $\nabla.\vec{v}$ being a scalar point function we can have its gradient only. Thus we have the following five formulae.

1. **div(grad ϕ)= $\nabla.(\nabla\phi) = \frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}$**
2. **curl (grad ϕ) = $\nabla\times\nabla\phi = 0$**
3. **div (curl $\vec{v}$) = $\nabla.(\nabla\times\vec{v}) = 0$**
4. **curl (curl $\vec{v}$) = $\nabla\times(\nabla\times\vec{v})$ = grad (div $\vec{v}$) - $\nabla^2\vec{v}$**
5. **grad (div $\vec{v}$) = $\nabla\times(\nabla\times\vec{v})+\nabla^2\vec{v}$**

Proofs:

1. We have

$$\text{div (grad }\phi) = \nabla.(\nabla\phi)$$
$$= \left(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}\right).\left(\vec{i}\frac{\partial\phi}{\partial x}+\vec{j}\frac{\partial\phi}{\partial y}+\vec{k}\frac{\partial\phi}{\partial z}\right)$$
$$= \frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2} = \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\phi = \nabla^2\phi$$
$$\text{div (grad }\phi) = \frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2} = \nabla^2\phi$$

Where $\nabla^2 = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$ is called the *Laplacian Operator* and $\nabla^2\phi = 0$ is called *Laplace's equations.*

2. Curl (grad ϕ) = $\nabla \times (\nabla \phi)$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \times \left(\vec{i}\frac{\partial \phi}{\partial x} + \vec{j}\frac{\partial \phi}{\partial y} + \vec{k}\frac{\partial \phi}{\partial z}\right)$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{vmatrix}$$

$$= \vec{i}\left(\frac{\partial^2 \phi}{\partial y \partial z} - \frac{\partial^2 \phi}{\partial z \partial y}\right) + \vec{j}\left(\frac{\partial^2 \phi}{\partial z \partial x} - \frac{\partial^2 \phi}{\partial x \partial z}\right) + \vec{k}\left(\frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \phi}{\partial y \partial x}\right) = 0$$

$\therefore$ Curl (grad ϕ) = 0.

3. Let $\vec{v} = u\,\vec{i} + v\,\vec{j} + w\,\vec{k}$

$$\text{div (curl } \vec{v}) = \nabla \cdot \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \times (u\,\vec{i} + v\,\vec{j} + w\,\vec{k})$$

$$= \nabla \cdot \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix}$$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \cdot \left\{\vec{i}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + \vec{j}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + \vec{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\right\}$$

$$= \frac{\partial^2 w}{\partial x \partial y} - \frac{\partial^2 v}{\partial x \partial z} + \frac{\partial^2 u}{\partial y \partial z} - \frac{\partial^2 w}{\partial y \partial x} + \frac{\partial^2 v}{\partial z \partial x} - \frac{\partial^2 u}{\partial z \partial y} = 0$$

$\therefore$ div (curl $\vec{v}$) = 0

4. We have

$$\text{Curl (curl } \vec{v}) = \nabla \times (\nabla \times \vec{v})$$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \times \left\{\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\vec{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\vec{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\vec{k}\right\}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} & \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} & \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \end{vmatrix}$$

$$= \vec{i}\left\{\left(\frac{\partial^2 v}{\partial y \partial x} + \frac{\partial^2 w}{\partial z \partial x}\right) - \left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)\right\}$$
$$+ \vec{j}\left\{\left(\frac{\partial^2 w}{\partial z \partial y} + \frac{\partial^2 u}{\partial x \partial y}\right) - \left(\frac{\partial^2 v}{\partial z^2} + \frac{\partial^2 v}{\partial x^2}\right)\right\}$$
$$+ \vec{k}\left\{\left(\frac{\partial^2 u}{\partial x \partial z} + \frac{\partial^2 v}{\partial y \partial z}\right) - \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)\right\}$$

$$= \vec{i}\left\{\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y \partial x} + \frac{\partial^2 w}{\partial z \partial x}\right) - \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)\right\}$$
$$+ \vec{j}\left\{\left(\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 w}{\partial z \partial y}\right) - \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right)\right\}$$
$$+ \vec{k}\left\{\left(\frac{\partial^2 u}{\partial x \partial z} + \frac{\partial^2 v}{\partial y \partial z} + \frac{\partial^2 w}{\partial z^2}\right) - \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right)\right\}$$

$$= \vec{i}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + \vec{j}\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)$$
$$+ \vec{k}\frac{\partial}{\partial z}\left(\frac{\partial u}{dx} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) - \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)(u\,\vec{i} + v\,\vec{j} + w\,\vec{k})$$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) - \nabla^2 \vec{v}$$

$$= \nabla(\nabla \cdot \vec{v}) - \nabla^2 \vec{v}$$

Hence Curl (Curl $\vec{v}$) = grad (div $\vec{v}$) - $\nabla^2 \vec{v}$. Proved.

5. It is directly obtained from (4),

i.e. grad (div $\vec{v}$) = $\nabla \times (\nabla \times \vec{v}) + \nabla^2 \vec{v}$

Worked out Examples

Ex. 1. Find a unit normal to the surface $xy^3z^2 = 4$ at the point (- 1, - 1, 2)

Solution:

Here the given surface is

$$\phi = x\,y^3 z^2 - 4 = 0$$

A vector normal to the given surface is

$$\nabla \phi = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial y}\right)(xy^3z^2 - 4)$$

$$= \vec{i}\,\frac{\partial(xy^3z^2-4)}{\partial x} + \vec{j}\,\frac{\partial(xy^3z^2-4)}{\partial y} + \vec{k}\,\frac{\partial(xy^3z^2-4)}{\partial z}$$

$$= y^3z^2\,\vec{i} + 3xy^2z^2\,\vec{j} + 2xy^3z\,\vec{k}$$

$$= -4\,\vec{i} - 12\,\vec{j} + 4\vec{k} \text{ at the point } (-1, -1, 2)$$

Hence, the unit normal to the surface

$$= \frac{-4\vec{i} - 12\vec{j} + 4\vec{k}}{\sqrt{(-4)^2+(-12)^2+(4)^2}} = -\frac{1}{\sqrt{11}}(\vec{i} + 3\vec{j} - \vec{k})$$

Ex. 2. Find the directional derivative of $\phi(x, y, z) = xy^2 + yz^3$ at the point (2, - 1, 1) in the direction of vector $\vec{i} + 2\vec{j} + 2\vec{k}$

[2072 Magh, B. E.]

Solution:

Given surface is $\phi(x, y, z) = xy^2 + yz^3$

$$\nabla\phi = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(xy^2 + yz^3)$$

$$= y^2\,\vec{i} + \vec{j}\,(2xy + z^3) + \vec{k}\,(3yz^2)$$

$$= \vec{i} - 3\vec{j} - 3\vec{k} \text{ at the point } (2, -1, 1)$$

$\therefore$ Directional derivative of ϕ in the direction $\vec{i} + 2\,\vec{j} + 2\vec{k}$ is

$$= (\vec{i} - 3\vec{j} - 3\vec{k}).\frac{(\vec{i} + 2\vec{j} + 2\vec{k})}{\sqrt{(1)^2 + (2)^2 + (2)^2}}$$

$$= \frac{1-6-6}{3} = -\frac{11}{3} = -3\frac{2}{3}.$$

Ex. 3. Find the divergence and curl of $\vec{F}$ where $\vec{F} = \text{grad}\,(x^3 + y^3 + z^3 - 3xyz)$

Solution:

Here $\vec{F} = \nabla(x^3 + y^3 + z^3 - 3xyz)$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(x^3 + y^3 + z^3 - 3xyz)$$

$$= \vec{i}(3x^2 - 3yz) + \vec{j}(3y^2 - 3xz) + \vec{k}(3z^2 - 3xy)$$

$$\text{div}\,\vec{F} = \nabla.\vec{F}$$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right).\{(3x^2-3yz)\,\vec{i} + (3y^2-3xz)\,\vec{j} + (3z^2-3xy)\,\vec{k}\}$$

$$= \frac{\partial(3x^2-3yz)}{\partial x} + \frac{\partial(3y^2-3xz)}{\partial y} + \frac{\partial(3z^2-3xy)}{\partial z}$$

$$= 6x + 6y + 6z = 6(x + y + z)$$

$$\text{Curl}\,\vec{F} = \nabla\times\vec{F}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2-3yz & 3y^2-3xz & 3z^2-3xy \end{vmatrix}$$

$$= \vec{i}(-3x + 3x) + \vec{j}(-3y + 3y) + \vec{k}(-3z + 3z) = 0$$

$\therefore$ $\text{Curl}\,\vec{F} = 0.$

Ex. 4. Find angle between the normals to the surface at $xy = z^2$ at the points (1, 4, 2) and (-3, - 3, 3).

Solution:

Here the equation of the surface is

$$\phi = xy - z^2 = 0$$

Normal to the surface

$$= \text{grad}\,\phi = \nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$= \vec{i}\,\frac{\partial(xy-z^2)}{\partial x} + \vec{j}\,\frac{\partial(xy-z^2)}{\partial y} + \vec{k}\,\frac{\partial(xy-z^2)}{\partial z}$$

$$= \vec{i}\,y + x\,\vec{j} - 2z\vec{k}$$

Unit normal to the surface

$$= \frac{y\,\vec{i} + x\,\vec{j} - 2z\vec{k}}{\sqrt{y^2 + x^2 + 4z^2}}$$

$\hat{n}_1$ = Unit normal to the surface at the point (1, 4, 2)

$$= \frac{4\vec{i} + \vec{j} - 4\vec{k}}{\sqrt{16+1+16}} = \frac{4\vec{i} + \vec{j} - 4\vec{k}}{\sqrt{33}}$$

$\hat{n}_2$ = Unit normal to the surface at the point (- 3, - 3, 3)

$$= \frac{-3\vec{i} - 3\vec{j} - 6\vec{k}}{\sqrt{9+9+36}} = \frac{-1}{\sqrt{6}}(\vec{i} + \vec{j} + 2\vec{k})$$

If θ be angle between the normals, then $\cos\theta = \hat{n}_1.\hat{n}_2$

$$= \frac{1}{\sqrt{33}}(4\vec{i} + \vec{j} - 4\vec{k}).\frac{-1}{\sqrt{6}}(\vec{i} + \vec{j} + 2\vec{k}) = \frac{-1}{\sqrt{22}}$$

$$\therefore \theta = \cos^{-1}\left(\frac{1}{\sqrt{22}}\right)$$

Ex. 5. Find the angle between the surface $x^2 + y^2 + z^2 = 9$ and $z = x^2 + y^2 - 3$ at the point $(2, -1, 2)$ [2069 Poush, B. E.]

Solution:

We know that angle between two surfaces $x^2 + y^2 + z^2 = 9$ and $z = x^2 + y^2 - 3$ is the angle between their normals at $(2, -1, 2)$

Let $\phi_1 = x^2 + y^2 + z^2 - 9 = 0, \qquad \phi_2 = z - x^2 - y^2 + 3 = 0$

Normal to the surface ϕ_1 is

$$\text{grad } \phi_1 = \nabla \phi_1 = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\phi_1$$

$$= \vec{i}\frac{\partial \phi_1}{\partial x} + \vec{j}\frac{\partial \phi}{\partial y} + \vec{k}\frac{\partial \phi_1}{\partial z}$$

$$= \vec{i}\frac{\partial(x^2+y^2+z^2-9)}{\partial x} + \vec{j}\frac{\partial(x^2+y^2+z^2-9)}{\partial y} + \vec{k}\frac{\partial(x^2+y^2+z^2-9)}{\partial z}$$

$$= 2x\,\vec{i} + 2y\,\vec{j} + 2z\,\vec{k}$$

$\hat{n}_1$ = Unit normal to the surface ϕ_1 at $(2, -1, 2)$

$$= \frac{4\vec{i} - 2\vec{j} + 4\vec{k}}{\sqrt{16+4+16}} = \frac{1}{3}(2\vec{i} - \vec{j} + 2\vec{k})$$

Normal to the surface ϕ_2 is

$$\text{grad } \phi_2 = \nabla \phi_2$$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\phi_2 = \vec{i}\frac{\partial \phi_2}{\partial x} + \vec{j}\frac{\partial \phi_2}{\partial y} + \vec{k}\frac{\partial \phi_2}{\partial z}$$

$$= \vec{i}\frac{\partial(z-x^2-y^2+3)}{\partial x} + \vec{j}\frac{\partial(z-x^2-y^2+3)}{\partial y} + \vec{k}\frac{\partial(z-x^2-y^2+3)}{\partial z}$$

$$= -2x\,\vec{i} - 2y\,\vec{j} + \vec{k}$$

$\hat{n}_2$ = Unit normal to the surface ϕ_2 at $(2, -1, 2)$

$$= \frac{-4\vec{i} + 2\vec{j} + \vec{k}}{\sqrt{16+4+1}} = \frac{1}{\sqrt{21}}(-4\vec{i} + 2\vec{j} + \vec{k})$$

If θ be angle between their normals, then $\cos\theta = \hat{n}_1 . \hat{n}_2$

$$= \frac{1}{3}(2\vec{i} - \vec{j} + 2\vec{k}).\frac{1}{\sqrt{21}}(-4\vec{i} + 2\vec{j} + \vec{k}) = \frac{-8}{3\sqrt{21}}$$

$$\cos\theta = \frac{-8}{3\sqrt{21}}$$

$$\therefore \theta = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right).$$

Ex. 6. If $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ and $|\vec{r}| = r = \sqrt{x^2+y^2+z^2}$, then prove that

(i) $\nabla.\vec{r} = 3$ **(ii)** $\nabla \times \vec{r} = 0$

(iii) $\nabla\left(\frac{1}{r}\right) = -\frac{\vec{r}}{r^3}$ **(iv)** $\nabla(\vec{r}^2) = 2\vec{r}$

(v) $\nabla.(r^3\vec{r}) = 6r^3$ **(vi)** $\nabla(\log r) = \frac{\vec{r}}{r^2}$

(vii) $\nabla(f(r)) = f'(r)\nabla r$

Solution:

Given that

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k},\ |\vec{r}| = r = \sqrt{x^2+y^2+z^2}$$

(i) We have

$$\nabla.\vec{r} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right).(x\vec{i} + y\vec{j} + z\vec{k})$$

$$= 1 + 1 + 1 = 3$$

$$\therefore \nabla.\vec{r} = 3$$

(ii) We have

$$\nabla \times \vec{r} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \times (x\vec{i} + y\vec{j} + z\vec{k})$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \vec{i}(0-0) + \vec{j}(0-0) + \vec{k}(0-0) = 0$$

(iii) We have

$$\nabla\left(\frac{1}{r}\right) = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\left(\frac{1}{r}\right)$$

$$= \left(-\vec{i}\frac{1}{r^2}\frac{\partial r}{\partial x} - \vec{j}\frac{1}{r^2}\frac{\partial r}{\partial y} - \vec{k}\frac{1}{r^2}\frac{\partial r}{\partial z}\right)$$

$$= -\frac{1}{r^2}\left(\vec{i}\frac{\partial r}{\partial x} + \vec{j}\frac{\partial r}{\partial y} + \vec{k}\frac{\partial r}{\partial z}\right)$$

But $\qquad r^2 = x^2 + y^2 + z^2$

Partially differentiating with respect to x, y and z respectively, we get

$$r\frac{\partial r}{\partial x} = x,\quad r\frac{\partial r}{\partial y} = y \quad \text{and} \quad r\frac{\partial r}{\partial z} = z$$

So $$\frac{\partial r}{\partial x} = \frac{x}{r},\quad \frac{\partial r}{\partial y} = \frac{y}{r} \quad \text{and} \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\therefore \quad \nabla\left(\frac{1}{r}\right) = -\frac{1}{r^3}(x\vec{i} + y\vec{j} + z\vec{k}) = -\frac{\vec{r}}{r^3}$$

(iv) We have

$$\nabla(\vec{r}^2) = \nabla(r^2) = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) r^2$$

$$= \vec{i}\frac{\partial r^2}{\partial x} + \vec{j}\frac{\partial r^2}{\partial y} + \vec{k}\frac{\partial r^2}{dz}$$

$$= 2r\left(\vec{i}\frac{\partial r}{\partial x} + 2j\frac{r\partial r}{\partial y} + 2\vec{k}\frac{r\partial r}{\partial z}\right)$$

$$= 2(x\vec{i} + y\vec{j} + z\vec{k}) = 2\vec{r}$$

$$\therefore \nabla(\vec{r}^2) = 2\vec{r}$$

(v) We have

$$\nabla.(r^3\vec{r}) = r^3(\nabla.\vec{r}) + \vec{r}.\nabla r^3$$

$$= r^3\left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right).(x\vec{i} + y\vec{j} + z\vec{k})$$

$$+ \vec{r}.\left(\vec{i}\frac{\partial r^3}{\partial x} + \vec{j}\frac{\partial r^3}{\partial y} + \vec{k}\frac{\partial r^3}{\partial z}\right)$$

$$= r^3(1 + 1 + 1) + (\vec{x}\vec{i} + y\vec{j} + z\vec{k})$$

$$\left(3\vec{i}\frac{r^2\partial r}{\partial x} + 3\vec{j}\frac{r^2\partial r}{\partial y} + 3\vec{k}\frac{r^2\partial r}{\partial z}\right)$$

$$= 3r^3 + 3r^2\left(x\frac{\partial r}{\partial x} + y\frac{\partial r}{\partial y} + z\frac{\partial r}{\partial z}\right)$$

$$= 3r^3 + 3r^2\left(x.\frac{x}{r} + y.\frac{y}{r} + z.\frac{z}{r}\right)$$

$$= 3r^3 + 3r^2.\frac{1}{r}(x^2 + y^2 + z^2) = 3r^3 + 3r.r^2 = 3r^3 + 3r^3 = 6r^3$$

$$\therefore \quad \nabla.(r^3\vec{r}) = 6r^3$$

(iv) We have

$$\nabla(\log r) = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(\log r)$$

$$= \vec{i}\frac{\partial(\log r)}{\partial x} + \vec{j}\frac{\partial(\log r)}{\partial y} + \vec{k}\frac{\partial(\log r)}{\partial z}$$

$$= \vec{i}\frac{1}{r}\frac{\partial r}{\partial x} + \vec{j}\frac{1}{r}\frac{\partial r}{\partial y} + \vec{k}\frac{1}{r}\frac{\partial r}{\partial z}$$

$$= \frac{1}{r}\left(\vec{i}\frac{x}{r} + \vec{j}\frac{y}{r} + \vec{k}\frac{z}{r}\right) = \frac{1}{r^2}(x\vec{i} + y\vec{j} + z\vec{k}) = \frac{\vec{r}}{r^2}$$

$$\therefore \quad \nabla(\log r) = \frac{\vec{r}}{r^2}$$

(vii) We have

$$\nabla(f(r)) = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(f(r))$$

$$= \vec{i}\frac{\partial f(r)}{\partial x} + \vec{j}\frac{\partial f(r)}{\partial y} + \vec{k}\frac{\partial f(r)}{\partial z}$$

$$= \left(\vec{i}\, f'(r)\frac{\partial r}{\partial x} + \vec{j}\, f'(r)\frac{\partial r}{\partial y} + \vec{k}\, f'(r)\frac{\partial r}{\partial z}\right)$$

$$= f'(r)\left(\vec{i}\frac{\partial r}{\partial x} + \vec{j}\frac{\partial r}{\partial y} + \vec{k}\frac{\partial r}{\partial z}\right)$$

$$= f'(r)\left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) r = f'(r)\,\text{grad}\, r$$

$$= f'(r)\nabla r$$

$$\therefore \quad \nabla(f(r)) = f'(r)\nabla r$$

Ex. 7. If $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ and $\vec{a}, \vec{b}$ are constant vectors, prove that

(i) grad $[\vec{r}\ \ \vec{a}\ \ \vec{b}] = \vec{a}\times\vec{b}$

(ii) div $[(\vec{r}\times\vec{a})\times\vec{b}] = -2\vec{a}.\vec{b}$

(iii) curl $[(\vec{r}\times\vec{a})\times\vec{b}] = \vec{b}\times\vec{a}$

(iv) curl $[\vec{r}\times(\vec{a}\times\vec{b})] = 2(\vec{b}\times\vec{a})$

(v) grad $[(\vec{r}\times\vec{a}).(\vec{r}\times\vec{b})] = (\vec{b}\times\vec{r})\times\vec{a} + (\vec{a}\times\vec{r})\times\vec{b}$

Solution:

(i) Let $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$, $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$

$\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$,

then $$\vec{r}.(\vec{a}\times\vec{b}) = \begin{vmatrix} x & y & z \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= x(a_2b_3 - a_3b_2) + y(a_3b_1 - a_1b_3) + z(a_1b_2 - a_2b_1)$$

$$\text{grad}\,[\vec{r}\ \vec{a}\ \vec{b}] = \nabla\{\vec{r}.(\vec{a}\times\vec{b})\}$$

$$= \left(\vec{i}\frac{\partial}{dx}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}\right)\{\vec{r}.(\vec{a}\times\vec{b})\}$$

$$= \left(\vec{i}\frac{\partial}{dx}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}\right)\{x(a_2b_3-a_3b_2)+y(a_3b_1-a_1b_3)+z(a_1b_2-a_2b_1)\}$$

$$= \vec{i}(a_2b_3-a_3b_2)+\vec{j}(a_3b_1-a_1b_3)+\vec{k}(a_1b_2-a_2b_1)$$

$$= \vec{i}(a_2b_3-a_3b_2)-\vec{j}(a_1b_3-a_3b_1)+\vec{k}(a_1b_2-a_2b_1)$$

$$= \begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix} = (a_1\vec{i}+a_2\vec{j}+a_3\vec{k})\times(b_1\vec{i}+b_2\vec{j}+b_3\vec{k})$$

$$= \vec{a}\times\vec{b}$$

$$\text{grad}\,[\vec{r}\ \vec{a}\ \vec{b}] = \vec{a}\times\vec{b}.$$

(ii) We have

$$\text{div}[(\vec{r}\times\vec{a})\times\vec{b}] = \text{div}\,[(\vec{b}.\vec{r})\,\vec{a}-(\vec{b}.\vec{a})\,\vec{r}]$$

$$= \text{div}\,(\vec{b}.\vec{r})\vec{a} - \text{div}\,(\vec{b}.\vec{a})\vec{r}$$

$$= (\vec{b}.\vec{r})(\nabla.\vec{a})+\vec{a}.\nabla(\vec{b}.\vec{r})-(\vec{b}.\vec{a})(\nabla.\vec{r})-\vec{r}.\nabla(\vec{b}.\vec{a})$$

Since $\vec{b}$ and $\vec{a}$ are constant vectors, so that

$$\nabla(\vec{b}.\vec{a})=0,\quad \nabla.\vec{a}=0 \text{ and } \nabla.\vec{b}=0$$

So, $\text{div}\,[(\vec{r}\times\vec{a})\times\vec{b}] = \vec{a}.\nabla(\vec{b}.\vec{r})-(\vec{b}.\vec{a})(\nabla.\vec{r})$

$$= \vec{a}.\left(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}\right)(\vec{b}.\vec{r})$$

$$-(\vec{b}.\vec{a})\left(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}\right).\vec{r}$$

$$= \vec{a}.\left(\vec{i}\,\vec{b}.\frac{\partial\vec{r}}{\partial x}+\vec{j}\,\vec{b}.\frac{\partial\vec{r}}{\partial y}+\vec{k}\,\vec{b}.\frac{\partial\vec{r}}{\partial z}\right)$$

$$-(\vec{b}.\vec{a})\left(\vec{i}.\frac{\partial\vec{r}}{\partial x}+\vec{j}.\frac{\partial\vec{r}}{\partial y}+\vec{k}.\frac{\partial\vec{r}}{\partial z}\right)$$

$$= \vec{a}.\{\vec{i}(\vec{b}.\vec{i})+\vec{j}(\vec{b}.\vec{j})+\vec{k}(\vec{b}.\vec{k})\}-(\vec{b}.\vec{a})(\vec{i}.\vec{i}+\vec{j}.\vec{j}+\vec{k}.\vec{k})$$

$$= \vec{a}.\{\vec{i}b_1+\vec{j}b_2+\vec{k}b_3\}-(\vec{b}.\vec{a})(1+1+1)$$

$$= \vec{a}.\vec{b}-3\vec{a}.\vec{b} = -2\vec{a}.\vec{b}$$

$$\therefore\ \text{div}\,[(\vec{r}\times\vec{a})\times\vec{b}] = -2\vec{a}.\vec{b}$$

(iii) We have

$$\text{curl}\,[(\vec{r}\times\vec{a})\times\vec{b}]$$

$$= \text{curl}\,\{(\vec{b}.\vec{r})\,\vec{a}-(\vec{b}.\vec{a})\vec{r}\} = \text{curl}\,(\vec{b}.\vec{r})\,\vec{a}-\text{curl}\,(\vec{b}.\vec{a})\,\vec{r}$$

$$= \nabla\times\{(\vec{b}.\vec{r})\,\vec{a}\}-\nabla\times\{(\vec{b}.\vec{a})\,\vec{r}\}$$

$$= (\vec{b}.\vec{r})(\nabla\times\vec{a})+\nabla(\vec{b}.\vec{r})\times\vec{a}-(\vec{b}.\vec{a})(\nabla\times\vec{r})-\nabla(\vec{b}.\vec{a})\times\vec{r}$$

Since $\vec{a}$, $\vec{b}$ being constant vectors, so that $\nabla\times\vec{a}=0$ and $\nabla(\vec{b}.\vec{a})=0.$

$$= 0+\nabla(\vec{b}.\vec{r})\times\vec{a}-(\vec{b}.\vec{a})(\nabla\times\vec{r})-0$$

$$= \left\{\vec{i}\frac{\partial(\vec{b}.\vec{r})}{\partial x}+\vec{j}\frac{\partial(\vec{b}.\vec{r})}{\partial y}+\vec{k}\frac{\partial(\vec{b}.\vec{r})}{\partial z}\right\}\times\vec{a}-(\vec{b}.\vec{a})(\nabla\times\vec{r})$$

$$= \left[\vec{i}\left(\vec{b}.\frac{\partial\vec{r}}{\partial x}\right)+\vec{j}\left(\vec{b}.\frac{\partial\vec{r}}{\partial y}\right)+\vec{k}\left(\vec{b}.\frac{\partial\vec{r}}{\partial z}\right)\right]\times\vec{a}-(\vec{b}.\vec{a})(\nabla\times\vec{r})$$

But $\nabla\times\vec{r} = \begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ x & y & z\end{vmatrix} = \vec{i}\left(\frac{\partial z}{\partial y}-\frac{\partial y}{\partial z}\right)-\vec{j}\left(\frac{\partial z}{\partial x}-\frac{\partial x}{\partial z}\right)+\vec{k}\left(\frac{\partial y}{\partial x}-\frac{\partial x}{\partial y}\right)=0$

and $\vec{r} = x\vec{i}+y\vec{j}+z\vec{k}$, $\frac{\partial\vec{r}}{\partial x}=\vec{i}$, $\frac{\partial\vec{r}}{\partial y}=\vec{j}$, $\frac{\partial\vec{r}}{\partial z}=\vec{k}$

So $\text{curl}\,[(\vec{r}\times\vec{a})\times\vec{b}] = \left[\vec{i}\left(\vec{b}.\frac{\partial\vec{r}}{\partial x}\right)+\vec{j}\left(\vec{b}.\frac{\partial\vec{r}}{\partial y}\right)+\vec{k}\left(\vec{b}.\frac{\partial\vec{r}}{\partial z}\right)\right]\times\vec{a}$

$$= \{\vec{i}(\vec{b}.\vec{i})+\vec{j}(\vec{b}.\vec{j})+\vec{k}(\vec{b}.\vec{k})\}\times\vec{a}$$

$$= \{\vec{i}b_1+\vec{j}b_2+\vec{k}b_3\}\times\vec{a} = \vec{b}\times\vec{a}$$

$$\therefore\ \text{curl}\,[(\vec{r}\times\vec{a})\times\vec{b}] = \vec{b}\times\vec{a}$$

(iv) We have

$$\text{curl}[\vec{r}\times(\vec{a}\times\vec{b})] = \text{curl}\,[(\vec{r}.\vec{b})\vec{a}-(\vec{r}.\vec{a})\vec{b}]$$

$$= \text{curl}\,\{(\vec{r}.\vec{b})\vec{a}\}-\text{Curl}\,\{(\vec{r}.\vec{a})\vec{b}\}$$

By using the formula, we get

$$= (\vec{r}.\vec{b})(\nabla\times\vec{a})+\nabla(\vec{r}.\vec{b})\times\vec{a}-\nabla(\vec{r}.\vec{a})(\nabla\times\vec{b})-\nabla(\vec{r}.\vec{a})\times\vec{b}$$

Since $\vec{a}$, $\vec{b}$ being constant vectors, so that $\nabla\times\vec{a}=0$ and $(\nabla\times\vec{b})=0$

So $\text{curl}\,(\vec{r}\times(\vec{a}\times\vec{b})) = 0+\nabla(\vec{r}.\vec{b})\times\vec{a} - 0 - \nabla(\vec{r}.\vec{a})\times\vec{b}$

$$= \left\{\vec{i}\frac{\partial(\vec{r}.\vec{b})}{\partial x} + \vec{j}\frac{\partial}{\partial y}(\vec{r}.\vec{b}) + \vec{k}\frac{\partial}{\partial z}(\vec{r}.\vec{b})\right\}\times\vec{a} - \left\{\vec{i}\frac{\partial(\vec{r}.\vec{a})}{\partial x} + \vec{j}\frac{\partial}{\partial y}(\vec{r}.\vec{a}) + \vec{k}\frac{\partial}{\partial z}(\vec{r}.\vec{a})\right\}\times\vec{b}$$

$$= \left\{\vec{i}\left(\frac{\partial\vec{r}}{\partial x}.\vec{b}\right) + \vec{j}\left(\frac{\partial\vec{r}}{\partial y}.\vec{b}\right) + \vec{k}\left(\frac{\partial\vec{r}}{\partial z}.\vec{b}\right)\right\}\times\vec{a} - \left\{\vec{i}\left(\frac{\partial\vec{r}}{\partial x}.\vec{a}\right) + \vec{j}\left(\frac{\partial\vec{r}}{\partial y}.\vec{a}\right) + \vec{k}\left(\frac{\partial\vec{r}}{\partial z}.\vec{a}\right)\right\}\times\vec{b}$$

$$= \{\vec{i}(\vec{i}.\vec{b}) + \vec{j}(\vec{j}.\vec{b}) + \vec{k}(\vec{k}.\vec{b})\}\times\vec{a} - \{\vec{i}(\vec{i}.\vec{a}) + \vec{j}(\vec{j}.\vec{a}) + \vec{k}(\vec{k}.\vec{a})\}\times\vec{b}$$

$$= \{\vec{i}\,b_1 + \vec{j}\,b_2 + \vec{k}\,b_3\}\times\vec{a} - \{\vec{i}\,a_1 + \vec{j}\,a_2 + \vec{k}\,a_3\}$$

$$= \vec{b}\times\vec{a} - \vec{a}\times\vec{b} = \vec{b}\times\vec{a} + \vec{b}\times\vec{a} = 2\,\vec{b}\times\vec{a}$$

$\therefore\ \text{curl}\,(\vec{r}\times(\vec{a}\times\vec{b})) = 2\,\vec{b}\times\vec{a}$

(v) We have

$$\text{grad}[(\vec{r}\times\vec{a}).(\vec{r}\times\vec{b})] = \nabla[(\vec{r}.\vec{r})(\vec{a}.\vec{b}) - (\vec{a}.\vec{r})(\vec{r}.\vec{b})]$$
$$= \nabla\{(\vec{r}.\vec{r})(\vec{a}.\vec{b})\} - \nabla\{(\vec{a}.\vec{r})(\vec{r}.\vec{b})\}$$

By using formula, we get

$$= (\vec{r}.\vec{r})\nabla(\vec{a}.\vec{b}) + (\vec{a}.\vec{b})\nabla(\vec{r}.\vec{r}) - (\vec{a}.\vec{r})\nabla(\vec{r}.\vec{b}) - (\vec{r}.\vec{b})\nabla(\vec{a}.\vec{r})$$

Since $\vec{a}.\vec{b}$ being constant vectors so that

$\nabla(\vec{a}.\vec{b}) = 0,\ \vec{r}.\vec{r} = x^2+y^2+z^2$

Hence $\text{grad}[(\vec{r}\times\vec{a}).(\vec{r}\times\vec{b})]$

$$= 0 + (\vec{a}.\vec{b})\left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(\vec{r}.\vec{r}) - (\vec{a}.\vec{r})\nabla(\vec{r}.\vec{b}) - (\vec{r}.\vec{b})\nabla(\vec{a}.\vec{r})$$

$$= (\vec{a}.\vec{b})(2x\,\vec{i} + 2y\,\vec{j} + 2z\,\vec{k}) - (\vec{a}.\vec{r})\left\{\vec{i}\frac{\partial}{\partial x}(\vec{r}.\vec{b}) + \vec{j}\frac{\partial}{\partial y}(\vec{r}.\vec{b}) + \vec{k}\frac{\partial}{\partial z}(\vec{r}.\vec{b})\right\}$$

$$- (\vec{r}.\vec{b})\left\{\left(\vec{i}\frac{\partial(\vec{a}.\vec{r})}{\partial x} + \vec{j}\frac{\partial(\vec{a}.\vec{r})}{\partial y} + \vec{k}\frac{\partial(\vec{a}.\vec{r})}{\partial z}\right)\right\}$$

$$= 2(\vec{a}.\vec{b})\vec{r} - (\vec{a}.\vec{r})\{\vec{i}(\vec{i}.\vec{b}) + \vec{j}(\vec{j}.\vec{b}) + \vec{k}(\vec{k}.\vec{b})\} - (\vec{r}.\vec{b})\{\vec{i}(\vec{i}.\vec{a}) + \vec{j}(\vec{j}.\vec{a}) + \vec{k}(\vec{k}.\vec{a})\}$$

$$= 2(\vec{a}.\vec{b})\,\vec{r} - (\vec{a}.\vec{r})\,\vec{b} - (\vec{r}.\vec{b})\,\vec{a}$$

$$= (\vec{a}.\vec{b})\,\vec{r} - (\vec{a}.\vec{r})\,\vec{b} + (\vec{a}.\vec{b})\,\vec{r} - (\vec{r}.\vec{b})\,\vec{a}$$

$\therefore\ \text{grad}[(\vec{r}\times\vec{a}).(\vec{r}\times\vec{b})] = (\vec{b}\times\vec{r})\times\vec{a} + (\vec{a}\times\vec{r})\times\vec{b}$

Ex. 8. If $\vec{a}$ is a constant vector and $\vec{r}$ be the position of vector, then prove that

(i) $\nabla\times(\vec{a}\times\vec{r}) = 2\vec{a}$ [2071/072 Aswin, B. E.]

(ii) $\nabla.(\vec{a}\times\vec{r}) = 0$ **(iii) $\nabla(\vec{a}.\vec{r}) = \vec{a}$**

(iv) $(\vec{a}\times\nabla)\times\vec{r} = -2\vec{a}$ **(v) $(\vec{a}\times\nabla).\vec{r} = 0$**

Solution:

(i) We have

$$\nabla\times(\vec{a}\times\vec{r}) = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\times(\vec{a}\times\vec{r})$$

$$= \vec{i}\times\frac{\partial(\vec{a}\times\vec{r})}{\partial x} + \vec{j}\times\frac{\partial(\vec{a}\times\vec{r})}{\partial y} + \vec{k}\times\frac{\partial(\vec{a}\times\vec{r})}{\partial z}$$

$$= \vec{i}\times\left(\vec{a}\times\frac{\partial\vec{r}}{\partial x}\right) + \vec{j}\times\left(\vec{a}\times\frac{\partial\vec{r}}{\partial y}\right) + \vec{k}\times\left(\vec{a}\times\frac{\partial\vec{r}}{\partial z}\right)$$

$$= \vec{i}\times(\vec{a}\times\vec{i}) + \vec{j}\times(\vec{a}\times\vec{j}) + \vec{k}\times(\vec{a}\times\vec{k})$$

$$= (\vec{i}.\vec{i})\vec{a} - (\vec{i}.\vec{a})\vec{i} + (\vec{j}.\vec{j})\vec{a} - (\vec{j}.\vec{a})\vec{j} + (\vec{k}.\vec{k})\vec{a} - (\vec{k}.\vec{a})\vec{k}$$

$$= 3\vec{a} - [(\vec{i}.\vec{a})\,\vec{i} + (\vec{j}.\vec{a})\vec{j} + (\vec{k}.\vec{a})\,\vec{k}]$$

$$= 3\vec{a} - [a_1\vec{i} + a_2\vec{j} + a_3\vec{k}] = 3\vec{a} - \vec{a} = 2\vec{a}$$

$\therefore\ \nabla\times(\vec{a}\times\vec{r}) = 2\vec{a}$

(ii) We have

$$\nabla.(\vec{a}\times\vec{r}) = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right).(\vec{a}\times\vec{r})$$

$$= \vec{i}.\frac{\partial(\vec{a}\times\vec{r})}{\partial x} + \vec{j}.\frac{\partial(\vec{a}\times\vec{r})}{\partial y} + \vec{k}.\frac{\partial(\vec{a}\times\vec{r})}{\partial z}$$

$$= \vec{i}\cdot\left(\vec{a}\times\frac{\partial \vec{r}}{\partial x}\right)+\vec{j}\cdot\left(\vec{a}\times\frac{\partial \vec{r}}{\partial y}\right)+\vec{k}\cdot\left(\vec{a}\times\frac{\partial \vec{r}}{\partial z}\right)$$

$$= \vec{i}\cdot(\vec{a}\times\vec{i})+\vec{j}\cdot(\vec{a}\times\vec{j})+\vec{k}\cdot(\vec{a}\times\vec{k})$$

$$= 0+0+0=0$$

$\therefore \nabla\cdot(\vec{a}\times\vec{r})=0$

(iii) We have

$$\nabla(\vec{a}\cdot\vec{r}) = \left(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}\right)(\vec{a}\cdot\vec{r})$$

$$= \vec{i}\frac{\partial(\vec{a}\cdot\vec{r})}{\partial x}+\vec{j}\frac{\partial(\vec{a}\cdot\vec{r})}{\partial y}+\vec{k}\frac{\partial(\vec{a}\cdot\vec{r})}{\partial z}$$

$$= \vec{i}\left(\vec{a}\cdot\frac{\partial \vec{r}}{\partial x}\right)+\vec{j}\left(\vec{a}\cdot\frac{\partial \vec{r}}{\partial y}\right)+\vec{k}\left(\vec{a}\cdot\frac{\partial \vec{r}}{\partial z}\right)$$

$$= \vec{i}(\vec{a}\cdot\vec{i})+\vec{j}(\vec{a}\cdot\vec{j})+\vec{k}(\vec{a}\cdot\vec{k})$$

$$= a_1\vec{i}+a_2\vec{j}+a_3\vec{k} = \vec{a}$$

$\therefore \quad \nabla(\vec{a}\cdot\vec{r}) = \vec{a}$

(iv) Let $\vec{a}=a_1\vec{i}+a_2\vec{j}+a_3\vec{k}$ and $\vec{r}=x\vec{i}+y\vec{j}+z\vec{k}$

We have

$$\vec{a}\times\nabla = (a_1\vec{i}+a_2\vec{j}+a_3\vec{k})\times\left(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}\right)$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{vmatrix}$$

$$= \vec{i}\left(a_2\frac{\partial}{\partial z}-a_3\frac{\partial}{\partial y}\right)+\vec{j}\left(a_3\frac{\partial}{\partial x}-a_1\frac{\partial}{\partial z}\right)+\vec{k}\left(a_1\frac{\partial}{\partial y}-a_2\frac{\partial}{\partial x}\right)$$

$$(\vec{a}\times\nabla)\times\vec{r} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_2\frac{\partial}{\partial z}-a_3\frac{\partial}{\partial y} & a_3\frac{\partial}{\partial x}-a_1\frac{\partial}{\partial z} & a_1\frac{\partial}{\partial y}-a_2\frac{\partial}{\partial x} \\ x & y & z \end{vmatrix}$$

$$= \vec{i}(-a_1-a_1)-\vec{j}(a_2+a_2)+\vec{k}(-a_3-a_3)$$

$$= -2a_1\vec{i}-2a_2\vec{j}-2a_3\vec{k} = -2(a_1\vec{i}+a_2\vec{j}+a_3\vec{k}) = -2\vec{a}$$

$\therefore (\vec{a}\times\nabla)\times\vec{r} = -2\vec{a}$

(v) Let $\vec{a}=a_1\vec{i}+a_2\vec{j}+a_3\vec{k}$ and $\vec{r}=x\vec{i}+y\vec{j}+z\vec{k}$

Here, we have

$$(\vec{a}\times\nabla)\cdot\vec{r} = \vec{a}\cdot\nabla\times\vec{r}$$

$$= \vec{a}\cdot\left(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}\right)\times\vec{r}$$

$$= \vec{a}\cdot\left(\vec{i}\times\frac{\partial\vec{r}}{\partial x}+\vec{j}\times\frac{\partial\vec{r}}{\partial y}+\vec{k}\times\frac{\partial\vec{r}}{\partial z}\right)$$

But $\vec{r}=x\vec{i}+y\vec{j}+z\vec{k}$ so that $\frac{\partial\vec{r}}{\partial x}=\vec{i}$, $\frac{\partial\vec{r}}{\partial y}=\vec{j}$ and $\frac{\partial\vec{r}}{\partial z}=\vec{k}$

Thus

$$(\vec{a}\times\nabla)\cdot\vec{r} = \vec{a}\cdot(\vec{i}\times\vec{i}+\vec{j}\times\vec{j}+\vec{k}\times\vec{k})$$

$$= \vec{a}\cdot(0+0+0)=0$$

$\therefore \quad (\vec{a}\times\nabla)\cdot\vec{r}=0$

Ex. 9. If $\vec{r}$ be the position vectors and $\vec{a}$ is constant vector then prove that

(i) $\nabla\cdot\left(\frac{\vec{a}\times\vec{r}}{r}\right)=0$ [2072 Magh, B.E.] (ii) $\vec{a}\cdot\nabla\left(\frac{1}{r}\right)=\left(-\frac{\vec{a}\cdot\vec{r}}{r^3}\right)$

(iii) $\nabla\left(\frac{\vec{a}\cdot\vec{r}}{r^n}\right)=\frac{\vec{a}}{r^n}-\frac{n(\vec{a}\cdot\vec{r})}{r^{n+2}}\vec{r}$

(iv) $\nabla\times\left(\frac{\vec{a}\times\vec{r}}{r^3}\right)=-\frac{\vec{a}}{r^3}+\frac{3(\vec{a}\cdot\vec{r})}{r^5}\vec{r}$

Solution:

(i) We have

$$\mathrm{div}\left(\frac{\vec{a}\times\vec{r}}{r}\right)=\nabla\cdot\left(\frac{\vec{a}\times\vec{r}}{r}\right)$$

$$=\left(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}\right)\cdot\left(\frac{\vec{a}\times\vec{r}}{r}\right)=\Sigma\,\vec{i}\frac{\partial}{\partial x}\cdot\left(\frac{\vec{a}\times\vec{r}}{r}\right)$$

$$=\Sigma\,\vec{i}\cdot\frac{\partial}{\partial x}\left(\frac{\vec{a}\times\vec{r}}{r}\right)=\Sigma\,\vec{i}\cdot\frac{r\left(\vec{a}\times\frac{\partial\vec{r}}{\partial x}\right)-(\vec{a}\times\vec{r})\frac{\partial r}{\partial x}}{r^2}$$

$$=\Sigma\,\vec{i}\cdot\frac{r(\vec{a}\times\vec{i})}{r^2}-\Sigma\,\vec{i}\cdot(\vec{a}\times\vec{r})\frac{\partial r}{\partial x}$$

$$=\Sigma\vec{i}.\frac{1}{r}(\vec{a}\times\vec{i})-\frac{1}{r^2}\Sigma\vec{i}.(\vec{a}\times\vec{r})\frac{x}{r}$$

$$=\frac{1}{r}\Sigma\vec{i}.(\vec{a}\times\vec{i})-\frac{1}{r^2}\Sigma x\vec{i}.(\vec{a}\times\vec{r})$$

$$=0-\frac{1}{r^2}\vec{r}.(\vec{a}\times\vec{r})=\frac{1}{r^3}[\vec{r}\ \ \vec{a}\ \ \vec{r}]=0$$

$$\therefore\ \text{div}\left(\frac{\vec{a}\times\vec{r}}{r}\right)=0$$

(ii) We have

$$\vec{a}.\nabla\left(\frac{1}{r}\right)=\vec{a}.\left(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}\right)\left(\frac{1}{r}\right)$$

$$=\vec{a}.\Sigma\vec{i}\frac{\partial}{\partial x}\left(\frac{1}{r}\right)=\vec{a}.\Sigma\vec{i}\left(-\frac{1}{r^2}\right)\frac{\partial r}{\partial x}$$

$$=-\frac{1}{r^2}\vec{a}.\Sigma\vec{i}\frac{\partial r}{\partial x}=-\frac{1}{r^2}\vec{a}.\Sigma\vec{i}\frac{x}{r}$$

$$=-\frac{1}{r^3}\vec{a}.\Sigma x\vec{i}=-\frac{1}{r^3}\{\vec{a}.(x\vec{i}+y\vec{j}+z\vec{k})\}=-\frac{1}{r^3}(\vec{a}.\vec{r})$$

$$\therefore\ \vec{a}.\nabla\left(\frac{1}{r}\right)=-\frac{1}{r^3}(\vec{a}.\vec{r})$$

(iii) We have

$$\nabla\left(\frac{\vec{a}.\vec{r}}{r^n}\right)=\nabla\left(\frac{\vec{a}.\vec{r}}{r^n}\right)$$

$$=\left(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}\right)\left(\frac{\vec{a}.\vec{r}}{r^n}\right)=\Sigma\vec{i}\frac{\partial}{\partial x}\left(\frac{\vec{a}.\vec{r}}{r^n}\right)$$

$$=\Sigma\vec{i}\left[\frac{r^n\left(\vec{a}.\frac{\partial\vec{r}}{\partial x}\right)-(\vec{a}.\vec{r})n\,r^{n-1}\frac{\partial r}{\partial x}}{r^{2n}}\right]$$

$$=\Sigma\vec{i}\,\frac{r^n(\vec{a}.\vec{i})-(\vec{a}.\vec{r})n\,r^{n-1}\frac{x}{r}}{r^{2n}}$$

$$=\Sigma\vec{i}\frac{(\vec{a}.\vec{i})}{r^n}-\frac{n}{r^{n+2}}\Sigma\vec{i}\,x(\vec{a}.\vec{r})$$

$$=\frac{1}{r^n}\Sigma a_1\vec{i}-\frac{n}{r^{n+2}}(\vec{a}.\vec{r})\Sigma\vec{i}x$$

$$=\frac{1}{r^n}(a_1\vec{i}+a_2\vec{j}+a_3\vec{k})-\frac{n}{r^{n+2}}(\vec{a}.\vec{r})(x\vec{i}+y\vec{j}+z\vec{k})$$

$$=\frac{1}{r^n}\vec{a}-\frac{n}{r^{n+2}}(\vec{a}.\vec{r})\vec{r}$$

$$\therefore\ \text{grad}\left(\frac{\vec{a}.\vec{r}}{r^n}\right)=\frac{\vec{a}}{r^n}-\frac{n}{r^{n+2}}\vec{r}(\vec{a}.\vec{r})$$

(iv) We have

$$\nabla\times\left(\frac{\vec{a}\times\vec{r}}{r^3}\right)=\left(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}\right)\times\left(\frac{\vec{a}\times\vec{r}}{r^3}\right)$$

$$=\Sigma\vec{i}\frac{\partial}{\partial x}\times\left(\frac{\vec{a}\times\vec{r}}{r^3}\right)=\Sigma\vec{i}\times\frac{\partial}{\partial x}\left(\frac{\vec{a}\times\vec{r}}{r^3}\right)$$

$$=\Sigma\vec{i}\times\left\{\frac{r^3\left(\vec{a}\times\frac{\partial\vec{r}}{\partial x}\right)-(\vec{a}\times\vec{r})3r^2\frac{\partial r}{\partial x}}{r^6}\right\}$$

$$=\Sigma\vec{i}\times\frac{1}{r^3}\left(\vec{a}\times\frac{\partial\vec{r}}{\partial x}\right)-\Sigma\vec{i}\times\frac{3}{r^4}(\vec{a}\times\vec{r})\frac{\partial r}{\partial x}$$

$$=\frac{1}{r^3}\Sigma\vec{i}\times(\vec{a}\times\vec{i})-\frac{3}{r^5}\Sigma\vec{i}\times(\vec{a}\times\vec{r})\frac{x}{r}$$

$$=\Sigma[(\vec{i}.\vec{i})\vec{a}-(\vec{i}.\vec{a})\vec{i}]-\frac{3}{r^5}\Sigma x\vec{i}\times(\vec{a}\times\vec{r})$$

$$=\frac{1}{r^3}\Sigma(\vec{i}.\vec{i})\vec{a}-\frac{1}{r^3}\Sigma a_1\vec{i}-\frac{3}{r^5}\vec{r}\times(\vec{a}\times\vec{r})$$

$$=\frac{1}{r^3}3\vec{a}-\frac{1}{r^3}\vec{a}-\frac{3}{r^5}[(\vec{r}.\vec{r})\vec{a}-(\vec{r}.\vec{a})\vec{r}]$$

$$=\frac{3\vec{a}}{r^3}-\frac{\vec{a}}{r^3}-\frac{3}{r^5}[r^2\vec{a}-(\vec{r}.\vec{a})\vec{r}]$$

$$=\frac{2\vec{a}}{r^3}-\frac{3\vec{a}}{r^3}+\frac{3}{r^5}(\vec{r}.\vec{a})\vec{r}=-\frac{\vec{a}}{r^3}+\frac{3}{r^5}(\vec{r}.\vec{a})\vec{r}$$

$$\therefore\ \nabla\times\left(\frac{\vec{a}\times\vec{r}}{r^3}\right)=-\frac{\vec{a}}{r^3}+\frac{3}{r^5}(\vec{r}.\vec{a})\vec{r}$$

Ex. 10 . Show that $\nabla^2(r^m)=m(m+1)r^{m-2}$

Solution:

We have $\nabla^2(r^m)=\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)(r^m)=\Sigma\frac{\partial^2}{\partial x^2}(r^m)$

$$=\Sigma\frac{\partial}{\partial x}\left(m\,r^{m-1}\frac{\partial r}{\partial x}\right)=\Sigma\frac{\partial}{\partial x}\left(m\,r^{m-1}\frac{x}{r}\right)$$

$$=\Sigma\frac{\partial}{\partial x}(m\,r^{m-2}x)=m\Sigma\left(r^{m-2}+(m-2)r^{m-3}\frac{\partial r}{\partial x}x\right)$$

$$=m\Sigma r^{m-2}+m\Sigma\left((m-2)r^{m-3}\frac{x}{r}x\right)$$

$= 3mr^{m-2} + m(m-2)\sum r^{m-4} x^2$

$= 3mr^{m-2} + m(m-2)\, r^{m-4} \sum x^2$

$= 3mr^{m-2} + m(m-2)\, r^{m-4}\, r^2$

$= 3mr^{m-2} + m(m-2)\, r^{m-2} = mr^{m-2}(3 + m - 2)$

$= mr^{m-2}(m+1) = m(m+1)r^{m-2}$

$\therefore \quad \nabla^2(r^m) = m(m+1)r^{m-2}$

Exercise - 25

1. Find the gradient, divergence and curl (whichever possible) of the following scalar and vector point functions.
   (i) $\vec{v} = 3x^2\,\vec{i} + 5xy^2\,\vec{j} + xyz^3\,\vec{k}$ at the point (1, 2, 3)
   (ii) $\vec{v} = xyz\,\vec{i} + 3x^2y\,\vec{j} + (xz^2 - y^2z)\,\vec{k}$
   (iii) $\vec{v} = \dfrac{x\vec{i} + y\vec{j} + z\vec{k}}{\sqrt{x^2 + y^2 + z^2}}$
   (iv) $\phi = \log(x^2 + y^2 + z^2)$
   (v) $\phi = \log\sin^{-1}(r)$ where $r = |\vec{r}|$, $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$
   (vi) $\vec{v} = x^2yz\,\vec{i} + xy^2z\,\vec{j} + xyz^2\,\vec{k}$
   (vii) $\vec{v} = x^2z\,\vec{i} - 2y^3z^2\,\vec{j} + xy^2z\,\vec{k}$ at the point (1, - 1, 1)
   (viii) $\phi = 3x^2y - y^3z^2$ at the point (1, - 2, - 1)
2. (i) Find the directional derivative of $\phi = x^2yz + 4xz^2$ at the point (1, - 2, - 1) in the direction $2\vec{i} - \vec{j} - 2\vec{k}$

   2071/072 Aswin, B. E.

   (ii) In what direction from the point (3, 1, -2) is the directional derivative of $\phi = x^2y^2z^4$ maximum ? Find also the magnitude of this maximum
   (iii) Find the unit vector normal to the surface $z = x^2 + y^2$ at the point (- 1, - 2, 5)
   (iv) Find the angle between the normals to the surfaces $x \log z = y^2 - 1$ and $x^2y + z = 2$ at the point (1, 1, 1)

   2069 Bhadra, B. E.

   (v) Find a unit normal to the surface $x^3 + y^3 + 3xyz = 3$ at the point (1, 2, - 1)

3. (i) If $\phi = \log(x^2+y^2+z^2)$, find div (grad ϕ) and curl (grad ϕ).
   (ii) If $\vec{v} = x^2y\,\vec{i} - 2xz\,\vec{j} + 2yz\,\vec{k}$, find div $\vec{v}$, curl $\vec{v}$ and curl curl $\vec{v}$
   (iii) If $f = x + y + z$, $g = x^2 + y^2 + z^2$ $h = xy + yz + zx$ show that [grad f grad g grad h] = 0

   2068 Bhadra, B. E.

   (iv) Find the constant a so that the vector $(ax^2y+yz)\,\vec{i} + (xy^2-xz^2) + (2xyz - 2x^2y^2)\,\vec{k}$ is solenoidal.
   (v) Find the constants a, b, c, so that the vector $\vec{v} = (x+2y + az)\,\vec{i} + (bx - 3y - z)\,\vec{j} + (4x + cy + 2z)\,\vec{k}$ is irrotational.
   (vi) Find n so that $r^n\vec{r}$ is solenoidal

   2070 Magh, B. E.

   (vii) Show that the vector $3y^4z^2\,\vec{i} + 4x^3z^2\,\vec{j} + 3x^2y^2\,\vec{k}$ is solenoidal
   (viii) If $\phi = \log(xy + yz + zx)$ and $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$, then find $\vec{r}.\nabla\phi$ and $\vec{r} \times \nabla\phi$
4. If $\vec{r}$ be the position vector of a point in space and $\vec{a}$ is constant vector then prove that
   (i) $\text{div}\,\{\vec{r} \times (\vec{a} \times \vec{r})\} = -2\,\vec{r}.\vec{a}$
   (ii) $\text{curl}\,\{\vec{r} \times (\vec{a} \times \vec{r})\} = 3\,\vec{r} \times \vec{a}$
   (iii) $\nabla \times \left(\dfrac{\vec{a} \times \vec{r}}{r}\right) = \dfrac{\vec{a}}{r} + \dfrac{\vec{a}.\vec{r}}{r^3}\vec{r}$
   (iv) $\nabla \times \{(\vec{a}.\vec{r})\,\vec{a}\} = 0$
   (v) $\nabla.\left(\dfrac{\vec{a} \times \vec{r}}{r^n}\right) = 0$
   (vi) $\nabla \times \{(\vec{a} \times \vec{r})\,r^n\} = (n+2)\,r^n\,\vec{a} - nr^{n-2}(\vec{a}.\vec{r})\,\vec{r}$
   (vii) $\nabla \times ((\vec{a}.\vec{r})\,\vec{r}) = \vec{a} \times \vec{r}$
   (viii) $\nabla \times \left(\dfrac{\vec{a} \times \vec{r}}{r^n}\right) = \dfrac{-(n-2)}{r^n}\vec{a} + n\dfrac{(\vec{a}.\vec{r})}{r^{n+2}}\vec{r}$
   (ix) $\nabla.(\hat{r}) = \dfrac{2}{r}$
   (x) $\nabla\left(\nabla.\dfrac{\vec{r}}{r}\right) = -\dfrac{2\vec{r}}{r^3}$

5. If $\vec{a}$ and $\vec{b}$ are constant vectors, prove that

   (i) $\nabla \{(\vec{b}.\vec{r})\,\vec{a}\} = \vec{a}.\vec{b}$

   (ii) $\nabla \times \{\vec{a} \times (\vec{b} \times \vec{r})\} = \vec{a} \times \vec{b}$

   (iii) $\nabla . \{\vec{r} \times (\vec{a} \times \vec{b})\} = 0$

   (iv) $\nabla \times \left\{\vec{k} \times \nabla\left(\frac{1}{r}\right)\right\} + \nabla\left\{\vec{k}.\nabla\left(\frac{1}{r}\right)\right\} = 0$

   (v) $\vec{a}.\nabla\left\{\vec{b}.\nabla\left(\frac{1}{r}\right)\right\} = \frac{3(\vec{a}.\vec{r})(\vec{b}.\vec{r})}{r^5} - \frac{\vec{a}.\vec{b}}{r^3}$

Answers

1. (i) $80, 27\vec{i} - 54\vec{j}$

   (ii) $yz+3x^2+2xz-y^2; -2yz\,\vec{i} + (xy-z^2)\,\vec{j} + (6xy - xz)\,\vec{k}$

   (iii) $\frac{2}{\sqrt{x^2+y^2+z^2}}; 0$

   (iv) $\frac{2x}{x^2+y^2+z^2}\vec{i} + \frac{2y}{x^2+y^2+z^2}\vec{j} + \frac{2z}{x^2+y^2+z^2}\vec{k}$

   (v) $\frac{1}{\sin^{-1} r\sqrt{1-r^2}}\,\frac{\vec{r}}{r}$

   (vi) $6xyz; x(z^2-y^2)\,\vec{i} + y(x^2-z^2)\,\vec{j} + z(y^2-x^2)\,\vec{k}$

   (vii) $-3; 2\vec{i} - 2\vec{j}$ (viii) $-12\vec{i} - 9\vec{j} - 16\vec{k}$

2. (i) $\frac{31}{3}$ (ii) $96(\vec{i} + 3\vec{j} - 3\vec{k}); \frac{96}{\sqrt{19}}$

   (iii) $\frac{2\vec{i} + 4\vec{j} + \vec{k}}{\sqrt{21}}$ (iv) $\cos^{-1}\left(-\frac{1}{\sqrt{30}}\right)$ (v) $\frac{-\vec{i} + 3\vec{j} + 2\vec{k}}{\sqrt{14}}$

3. (i) $\frac{2}{x^2+y^2+z^2}; 0$ (ii) $2y(x+1); 2(x+z)\,\vec{i} - (x^2+2z)\,\vec{k}; 2(x+1)\,\vec{j}$

   (iv) $\frac{14}{3}$ (v) $a = 4, b = 2, c = -1$ (vi) $n = -3$

   (viii) $2; \frac{x+y+z}{xy+yz+zx}[(y-z)\,\vec{i} + (z-x)\,\vec{j} + (x-y)\,\vec{k}]$

Chapter - 6

Infinite Series

- Introduction
- Infinite Sequence
- Convergent and Divergent Sequence
- Infinite Series
- P- Series or Harmonic Series
- Quotient or Limit Comparison Test
- De'Alembert's Ratio Test
- Root Test
- Integral Test
- Alternating Series
- Radius and Interval of Convergence

Chapter – 6

Infinite Series

6.1 Introduction:

Many problems in applied mathematics require use of infinite series to represent a function. The expansions of a function into an infinite series does not require necessarily mean that it is representative of the function unless the series is convergent. Numerical solution of many important physical problems, solution in series of some differential equations such as Bessel's equation and Legendre' equation can be easily solved by the use of Infinite series.

6.1.1 Infinite Sequence

A sequence is a function whose domain is the set of natural numbers and range a set of real numbers. So that the ordered set of real number u_1, u_2, ..., u_n is called a sequence and is denoted by $\{u_n\}$. If the number of terms is unlimited, then the sequence is said to be *Infinite Sequence* and u_n is its general term.

For example

1, 3, 5, 7,......... ,(2n - 1), ... is infinite sequence whose general term $u_n = 2n - 1$ but the sequence $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}$ is finite sequence.

(i) $1, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}, \ldots\ldots\ldots, \frac{1}{2^n}, \ldots$

(ii) $2, \left(\frac{3}{2}\right)^2, \left(\frac{4}{3}\right)^3, \ldots\ldots\ldots, \left(\frac{n+1}{n}\right)^n, \ldots\ldots$

(iii) $1, 1+\frac{1}{2}, 1+\frac{1}{2}+\frac{1}{3}, \ldots\ldots, 1+\frac{1}{2}+\frac{1}{3}+\ldots\ldots+\frac{1}{n}, \ldots.$

These sequences can be symbolically expressed as

(i) $\{u_n\} = \frac{1}{2^n}$ (ii) $\{u_n\} = 1+\frac{1}{2}+\frac{1}{3}+\ldots\ldots+\frac{1}{n}$

(iii) $\{u_n\} = \left(1+\frac{1}{n}\right)^n$

Limit

A sequence $\{u_n\}$ is said to have *limit* l if for every $\varepsilon > 0$ there exist positive integer N such that $|u_n - l| < \varepsilon$ whenever $n \geq N$.

We write

$$\lim_{n \to \infty} u_n = l \quad \text{or} \quad \text{Simply } u_n \to l \text{ as } n \to \infty.$$

6.1.2. Convergent and Divergent Sequence

If a sequence $\{u_n\}$ has finite limit l, then the sequence is called convergence sequence.

If such number l does not exist, then the sequence has no limit or divergence and is denoted by

$$\lim_{n \to \infty} u_n = \infty .$$

This means that for every chosen positive real number P, there exist a positive integer N such that $u_n > P$ for all $n \geq N$.

For examples

The sequences $1, 3, 5, 7, \ldots\ldots, (2n-1), \ldots\ldots$ and

$1, -1, 1, -1, \ldots\ldots, (-1)^{n-1}, \ldots\ldots$ are divergent

but the sequence $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\ldots, \frac{1}{n}, \ldots\ldots$ is convergent.

Bounded Sequence

The sequence $\{u_n\}$ is said to be *bounded sequence* if there exist a real number K such that $u_n < K$ for every natural number n.

Monotonic Sequence

The sequence $\{u_n\}$ is said to be increasing and decreasing strictly according as $u_n \leq u_{n+1}$ and $u_{n+1} \leq u_n$ for all values of n. Both increasing and decreasing sequences are called *monotonic sequences*.

A sequence, which is monotonic and bounded, is *convergent*.

In an infinite sequence if $\{u_n\}$, $\{v_n\}$, $\{w_n\}$ are infinite and such that $u_n \leq v_n \leq w_n$ for every n and $\lim_{n \to \infty} u_n = l = \lim_{n \to \infty} w_n$ then $\lim_{n \to \infty} v_n = l$.

Worked Out Examples

Ex. 1: Determine the sequence $\left\{\frac{4n^2 + n - 1}{3n^2 + 2n + 1}\right\}$ converges or diverges

Solution:

Given sequence is

$$\left\{\frac{4n^2 + n - 1}{3n^2 + 2n + 1}\right\}$$

Its general term is

$$u_n = \frac{4n^2 + n - 1}{3n^2 + 2n + 1}$$

Taking limit $n \to \infty$, we get

$$\lim_{n \to \infty} u_n = \lim_{n \to \infty} \frac{4n^2 + n - 1}{3n^2 + 2n + 1}$$

$$= \lim_{n \to \infty} \frac{\left(4 + \frac{1}{n} - \frac{1}{n^2}\right)}{\left(3 + \frac{2}{n} + \frac{1}{n^2}\right)} = \frac{4}{3}$$

$$\therefore \lim_{n \to \infty} u_n = \frac{4}{3}.$$

Hence, the sequence is convergent.

Ex. 2: If $u_n = 1 + \frac{1}{n}$, determine $\{u_n\}$ converge or diverges.

Solution:

Given sequence is

$$\left\{1 + \frac{1}{n}\right\}$$

Its general term is

$$u_n = 1 + \frac{1}{n}$$

Taking limit $n \to \infty$, we get

$$\lim_{n \to \infty} u_n = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right) = 1.$$

It shows that the sequence $\{u_n\}$ is convergent and $\{u_n\}$ converges to 1.

Ex.3: If $u_n = \frac{n^2 + 2n + 3}{2n + 7}$ determine $\{u_n\}$ converges or diverges

Solution:

Given sequence is

$$\left\{\frac{u^2+2n+3}{2n+7}\right\}$$

Its general term is

$$u_n = \frac{u^2+2n+3}{2n+7}$$

Taking limit $n \to \infty$, we obtain

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{n^2+2n+3}{2n+7} = \lim_{n\to\infty} \frac{n\left(1+\frac{2}{n}+\frac{3}{n^2}\right)}{\left(2+\frac{7}{n}\right)} = \infty$$

Hence, the sequence $\{u_n\}$ diverges to ∞.

Ex: 4. If $u_n = n^3 e^{-n}$ determine the sequence $\{u_n\}$ diverges or converges

Solution:

Here, the general term of the sequence is

$$u_n = n^3 e^{-n}$$

Taking limit $n \to \infty$, we get

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{n^3}{e^n} \quad \left(\frac{0}{0}\text{ form}\right)$$

Using L- Hospital's rule

$$= \lim_{n\to\infty} \frac{3!}{e^n} = \frac{3!}{\infty} = 0$$

It shows that the sequence $\{u_n\}$ converges to zero.

Ex: 5. Determine the sequence $\left\{\frac{\cos^2 n}{3^n}\right\}$ converges or diverges.

Solution:

We know that the relation

$0 < \cos^2 n < 1$ for every positive integer n

$$0 < \frac{\cos^2 n}{3^n} \le \frac{1}{3^n}$$

Here, let $u_n = 0$, $v_n = \frac{\cos^2 n}{3^n}$ and $w_n = \frac{1}{3^n}$

Now $\lim_{n\to\infty} w_n = \lim_{n\to\infty} \frac{1}{3^n} = 0$

Since $\{u_n\}$ and $\{w_n\}$ converges to zero such that $u_n \le v_n \le w_n$

$$\lim_{n\to\infty} v_n = \lim_{n\to\infty} \frac{\cos^2 n}{3^n} = 0$$

Hence, $\{v_n\}$ converges to zero.

6.2 Infinite Series

A series is the sum of the terms of a sequence.

Thus, if $u_1, u_2, \ldots\ldots\ldots, u_n, \ldots\ldots\ldots$ is a sequence, then the sum $u_1 + u_2 + \ldots\ldots + u_n + \ldots\ldots$ of all terms is known as *infinite series* and is denoted by $\sum_{n=1}^{\infty} u_n$ or simply by $\sum u_n$.

The sum of the first n terms of the series is called a partial sum and denoted by s_n.

Thus,

$$s_n = u_1 + u_2 + \ldots\ldots\ldots + u_n = \sum_{i=1}^{n} u_i$$

The sequence of the partial sums s_n are

$$s_1 = u_1$$
$$s_2 = u_1 + u_2$$
$$s_3 = u_1 + u_2 + u_3$$
$$s_n = u_1 + u_2 + u_3 + \ldots\ldots + u_n.$$

6.3 Convergent and Divergent Infinite Series

An infinite series $\sum_{n=1}^{\infty} u_n$ is said to be convergent, divergent or oscillatory according as its sequence $\{s_n\}$ of partial sum $s_1, s_2, s_3, \ldots\ldots s_n$ converges, diverges or oscillates.

(i) If s_n tends to finite limit as $n \to \infty$, then series $\sum u_n$ is convergent.

(ii) If s_n tends to $\pm\infty$ as $n \to \infty$, then the series $\sum u_n$ is divergent.

(iii) If s_n doesn't tends to a unique limit as $n \to \infty$, then the series $\sum u_n$ is oscillatory.

Special infinite series occur frequently in solutions of applied problems. One of the most important is the geometric series

$$a + ar + ar^2 + ar^3 + ar^4 + \ldots\ldots + ar^{n-1} \ldots, a \ne 0$$

Theorem 1

The geometric series is $a + ar + ar^2 + \ldots\ldots + ar^{n-1} + \ldots\ldots$, *$a \ne 0$ converges and has sum $\frac{a}{1-r}$ if $|r| < 1$ and diverges if $|r| \ge 1$*

Proof

Let r = 1, then partial sum of the series is

$$S_n = a + a + a + \ldots\ldots + a = na$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} na = \infty.$$

Hence S_n diverges for r = 1.

Let r = - 1, then partial sum of the series $S_n = a$ if n is *odd* and $S_n = 0$ if n is *even*. Hence the sequence of two partial sum oscillates between a and 0 hence the series diverges.

Let $r \neq 1$, then

$$s_n = a + ar + ar^2 + \ldots\ldots + ar^{n-1}$$

and $$rS_n = ar + ar^2 + ar^3 + \ldots\ldots + ar^n$$

Subtracting corresponding sides of this equation, we get

$$(1 - r)S_n = a\ ar^n$$

or $$S_n = \frac{a(1 - r^n)}{1 - r}$$

or $$S_n = \frac{a}{1 - r} - \frac{ar^n}{1 - r}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a}{1 - r} - \lim_{n\to\infty} \frac{ar^n}{1 - r}$$

$$= \frac{a}{1 - r} - \frac{a}{1 - r} \lim_{n\to\infty} r^n$$

If $|r| < 1$, then $r^n \to 0$ as $n \to \infty$ and hence $\lim_{n\to\infty} S_n = \frac{a}{1 - r}$.

Thus the series converges for $|r| < 1$. If $|r| > 1$, then $r^n \to \infty$ and hence $\lim_{n\to\infty} S_n = \infty$. Thus the series diverges for $|r| > 1$.

Theorem

If an infinite series Σu_n is convergent, then $\lim_{n\to\infty} u_n = 0$

Proof

Given that Σu_n is convergent so that partial sums of the series also convergent.

$$\lim_{n\to\infty} S_n = S \text{ and } \lim_{n\to\infty} S_{n-1} = S_n$$

Now we have

$$u_n = S_n - S_{n-1}$$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = S - S = 0$$

$$\lim_{n\to\infty} u_n = 0$$

$\therefore$

But the converse is not true, for ,taking theor example of the series

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots\ldots + \frac{1}{\sqrt{n}} + \ldots\ldots \infty$$

Now

$$s_n = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots\ldots + \frac{1}{\sqrt{n}}$$

$$> \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n}} + \ldots\ldots \frac{1}{\sqrt{n}}$$

$$= \frac{n}{\sqrt{n}} = \sqrt{n}$$

$$\therefore \lim_{n\to\infty} s_n = \lim_{n\to\infty} \sqrt{n} = \infty.$$

The series is divergent even though

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0.$$

Hence Σu_n is convergent and $\lim_{n\to\infty} u_n = 0$

If $\lim_{n\to\infty} u_n = 0$, it is not necessary that Σu_n is convergent. It may be divergent. If follows that the following test for divergence.

Note:

If $\lim_{n\to\infty} u_n \neq 0$, then the series Σu_n must be divergent

Worked Out Examples

Ex: 1. Prove that the infinite series $3 + \frac{3}{4} + \frac{3}{4^2} + \ldots\ldots + \frac{3}{4^{n-1}} + \ldots\ldots$ converges and find its sum

Solution:

Given series is

$$3 + \frac{3}{4} + \frac{3}{4^2} + \ldots\ldots + \frac{3}{4^{n-1}} + \ldots\ldots$$

Its nth partial sum of the series is

$$S_n = 3 + \frac{3}{4} + \frac{3}{4^2} + \ldots\ldots\ldots\ldots + \frac{3}{4^{n-1}}$$

$$= 3\left(1 + \frac{1}{4} + \frac{1}{4^2} + \ldots\ldots + \frac{1}{4^{n-1}}\right) = \frac{3\left[1 - \left(\frac{1}{4}\right)^n\right]}{1 - \frac{1}{4}}$$

$$S_n = 4\left(1 - \frac{1}{4^n}\right)$$

$$\lim_{n \to \infty} S_n = \lim_{n \to \infty} \left(4 - \frac{4}{4^n}\right) = 4.$$

The series converges and has sum 4.

Ex: 2. Prove that the infinite series $\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \ldots + \frac{1}{n(n+1)} + \ldots$ converges and find its sum

Solution:

Given series is

$$\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \ldots\ldots\ldots\ldots\ldots + \frac{1}{n(n+1)} + \ldots\ldots\ldots\ldots$$

The n^{th} partial sum of the series is

$$S_n = \frac{1}{1.2} + \frac{1}{2.3} + \ldots\ldots\ldots\ldots\ldots\ldots + \frac{1}{n(n+1)}$$

$$= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \ldots\ldots\ldots + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= 1 - \frac{1}{n+1} = \frac{n}{n+1}$$

$$\lim_{n \to \infty} S_n = \lim_{n \to \infty} \frac{n}{n+1} = 1.$$

Hence the series converges and has its sum 1.

Ex: 3: Test whether the series $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \ldots\ldots\ldots\ldots\ldots\ldots$ is convergent or divergent

Solution:

Here the series is

$$\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \ldots\ldots\ldots\ldots\ldots\ldots + \frac{n}{n+1} + \ldots\ldots$$

Its general term is

$$u_n = \frac{n}{n+1}$$

$$\lim_{n \to \infty} u_n = \lim_{n \to \infty} \frac{n}{n+1} = \lim_{n \to \infty} \frac{1}{1 + \frac{1}{n}} = 1$$

$\therefore \lim_{n \to \infty} u_n = 1 \neq 0.$ Hence the series is divergent.

Ex:4. Test whether the series $\frac{2}{1} + \left(\frac{3}{2}\right)^2 + \left(\frac{4}{3}\right)^2 + \ldots\ldots\ldots + \left(\frac{n+1}{n}\right)^n + \ldots\ldots$ is convergent or divergent

Solution:

Here the series is

$$\frac{2}{1} + \left(\frac{3}{2}\right)^2 + \left(\frac{4}{3}\right)^2 + \ldots\ldots\ldots\ldots + \left(\frac{n+1}{n}\right)^n + \ldots\ldots\ldots\ldots.$$

Its general term is

$$u_n = \left(\frac{n+1}{n}\right)^n$$

$$\lim_{n \to \infty} u_n = \lim_{n \to \infty} \left(\frac{n+1}{n}\right)^n = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e \neq 0$$

$\therefore$ The series is divergent.

Exercise - 26

Determine the following series are convergent or divergent if it converges find its sum

1. $3 + \frac{3}{(-4)} + \frac{3}{(-4)^2} + \ldots\ldots\ldots\ldots\ldots + \frac{3}{(-4)^{n-1}} + \ldots\ldots\ldots\ldots$
2. $\frac{1}{4.5} + \frac{1}{5.6} + \frac{1}{6.7} + \ldots\ldots\ldots\ldots\ldots + \frac{1}{(n+3)(n+4)} + \ldots\ldots\ldots\ldots$
3. $\frac{5}{1.2} + \frac{5}{2.3} + \frac{5}{3.4} + \ldots\ldots\ldots\ldots\ldots + \frac{5}{n(n+1)} + \ldots\ldots\ldots\ldots$

4. $3 + \frac{3}{2} + \frac{3}{3} + \ldots\ldots + \frac{3}{n} + \ldots\ldots$

5. $\frac{3.1}{4} + \frac{3.2}{9} + \frac{3.3}{14} + \ldots\ldots + \frac{3n}{5n-1} + \ldots\ldots$

6. $\Sigma\left(\frac{1}{8^n} + \frac{1}{n(n+1)}\right)$

7. $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots\ldots + \frac{1}{n} + \ldots\ldots$

8. $\frac{1}{3} + \frac{2}{5} + \frac{3}{7} + \ldots\ldots$

9. Show that the series $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\ldots$ is convergent.

10. Show that $\Sigma\left(1 + \frac{1}{2n}\right)^n$ is divergent.

Answers

1. Convergent ; $\frac{12}{5}$ 2. Convergent; $\frac{1}{4}$ 3. Convergent; 5
4. Divergent 5. Divergent 6. Convergent; $\frac{8}{7}$
7. Divergent 8. Divergent

6.3.1 p- Series or Harmonic Series

If $f(n) = \frac{1}{n^p}$ for $p > 0$, then the series $\Sigma f(x)$ is of the form

$$\frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \frac{1}{5^p} + \ldots\ldots$$ is called the *p series* or *Harmonic series.*

Theorem: The infinite series $\frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \ldots\ldots + \frac{1}{n^p} \ldots$ i convergent if $p > 1$ and divergent if $p \le 1$.

Proof:

Let $p > 1$, then

$$\Sigma\frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \frac{1}{5^p} + \ldots\ldots$$

By grouping the terms as

$$= \frac{1}{1^p} + \left(\frac{1}{2^p} + \frac{1}{3^p}\right) + \left(\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p}\right)$$

$$+ \left(\frac{1}{8^p} + \frac{1}{9^p} + \frac{1}{10^p} + \frac{1}{11^p} + \frac{1}{12^p} + \frac{1}{13^p} + \frac{1}{14^p} + \frac{1}{15^p}\right) + \ldots$$

$$< 1 + \left(\frac{1}{2^p} + \frac{1}{2^p}\right) + \left(\frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p}\right) + \left(\frac{1}{8^p} + \frac{1}{8^p} + \frac{1}{8^p} + \ldots \frac{1}{8^p}\right) + \ldots$$

$$= 1 + \frac{2}{2^p} + \frac{4}{4^p} + \frac{8}{8^p} + \ldots\ldots.$$

This is geometric series with common ratio $\frac{1}{2^{p-1}} < 1$. Hence the series is convergent for $p > 1$.

Let $p = 1$, then

$$\Sigma\frac{1}{n^p} = \Sigma\frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots\ldots + \frac{1}{n} + \ldots\ldots$$

Its n^{th} partial sum is

$$s_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots\ldots + \frac{1}{n}$$

$$= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)$$

$$+ \left(\frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16}\right) + \ldots \text{ to terms, } n > m$$

$$> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right)$$

$$+ \left(\frac{1}{16} + \frac{1}{16} + \frac{1}{17} + \ldots \frac{1}{16}\right) + \ldots \text{tom terms, } n > m.$$

$$= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \ldots\ldots \text{ to m terms } = 1 + \frac{m}{2}$$

$$s_n > 1 + \frac{m}{2} \quad \text{for } n > m.$$

It follows that s_n can be made as large as desired by taking n sufficiently large, i.e. $\lim_{n \to \infty} s_n = \infty$. Hence, $\{s_n\}$ diverges.

If $\{s_n)$ diverges, then the infinite series $\Sigma\frac{1}{n^p}$ diverges for $p = 1$.

Let $p < 1$, then

$\frac{1}{2^p} > \frac{1}{2}, \frac{1}{3^p} > \frac{1}{3}, \frac{1}{4^p} > \frac{1}{4}$ and so on.

So that $\quad \Sigma \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \dots\dots\dots + \frac{1}{n^p} + \dots\dots .$

Its n^{th} partial sum is

$$s_n = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \dots\dots\dots\dots\dots\dots\dots + \frac{1}{n^p}$$

$$> 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots\dots\dots\dots\dots + \frac{1}{n}.$$

The series of RHS is divergent so that s_n is also divergent.

Hence, $\Sigma \frac{1}{n^p}$ is divergent for $p < 1$.

Example1

The series $1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \dots\dots\dots$ is convergent because $p = 3 > 1$.

Example 2

The series $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \dots\dots$ is divergent because $p = \frac{1}{2} < 1$.

6.3.2 Quotient or Limit Comparison Test

Theorem

If Σu_n and Σv_n are the series of positive terms, then Σu_n and Σv_n are either both convergent or divergent if $\lim_{n \to \infty} \frac{u_n}{v_n}$ is finite and non-zero

Proof

Let $\lim_{n \to \infty} \frac{u_n}{v_n} = l$ finite and non-zero .Hence for a given positive numbers ε however small , we can find a number N such that

$$\left| \frac{u_n}{v_n} - l \right| < \varepsilon \text{ for all } n > N.$$

or $\quad -\varepsilon < \frac{u_n}{v_n} - l < \varepsilon, \quad l - \varepsilon < \frac{u_n}{v_n} < l + \varepsilon$

Thus $\quad \frac{u_n}{v_n} < l + \varepsilon$

and $\quad \frac{u_n}{v_n} > l - \varepsilon \quad \dots\dots\dots\dots (1)$

Form (1), $\quad \dots\dots\dots\dots (2)$

$$u_n < (l + \varepsilon)\, v_n$$

If Σv_n is convergent, it implies that $\Sigma(l + \varepsilon)\, v_n$ is also convergent and Σu_n is term by term less than $\Sigma\,(l + \varepsilon)\, v_n$ except for finite number of terms hence Σu_n is also convergent.

Form (2), $\quad u_n > (l - \varepsilon)\, v_n$

If Σv_n is divergent, it implies that $\Sigma\,(l - \varepsilon)\, v_n$ is also divergent and Σu_n is term by term greater than $\Sigma\,(l - \varepsilon)\, v_n$ except for a finite number of terms. Hence Σu_n is divergent.

It is necessary that the auxiliary series Σv_n to be chosen in such a way that $\lim_{n \to \infty} \frac{u_n}{v_n}$ is non-zero and finite.

Note that $\quad \Sigma \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots\dots\dots\dots\dots\dots\dots \infty$ is divergent and

$\Sigma \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots\dots\dots\dots\dots \infty$ is convergent series.

Worked Out Examples

Ex: 1. Test for convergence $\Sigma\,(\sqrt{n^2 + 1} - n)$

Solution:

The general term of the given series is

$$u_n = \sqrt{n^2 + 1} - n$$

$$= \frac{(\sqrt{n^2 + 1} - n)\,(\sqrt{n^2 + 1} + n)}{(\sqrt{n^2 + 1} + n)}$$

$$u_n = \frac{1}{\sqrt{n^2 + 1} + n}$$

Taking the series $\quad \Sigma v_n = \Sigma \frac{1}{n}$ and its general term is

$$v_n = \frac{1}{n}$$

So $\quad \lim_{n \to \infty} \frac{u_n}{v_n} = \lim_{n \to \infty} \frac{n}{\sqrt{n^2 + 1} + n}$

$\quad = \lim_{n \to \infty} \frac{1}{\sqrt{1 + \frac{1}{n^2}} + 1} = \frac{1}{2}$ is finite and non-zero.

However, we know that Σv_n is divergent. Therefore, by the comparison test Σu_n is also divergent.

Ex:2. Determine whether the following series are convergent or divergent.

(i) $1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\ldots\ldots\ldots$

(ii) $\sqrt{\frac{1}{2^3}}+\sqrt{\frac{2}{3^3}}+\sqrt{\frac{3}{4^3}}+\ldots\ldots\ldots$

(iii) $1+\frac{1}{2^2}+\frac{2^2}{3^3}+\frac{3^3}{4^4}+\frac{4^4}{5^5}+\ldots\ldots\ldots$

(i) Solution:

Given series is

$$1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\ldots\ldots\ldots$$

Its general term is

$$u_n=\frac{1}{2n-1}$$

Taking the series $\Sigma\, v_n=\Sigma\frac{1}{n}$ and its general term is

$$v_n=\frac{1}{n}$$

Now $\lim\limits_{n\to\infty}\frac{u_n}{v_n}=\lim\limits_{n\to\infty}\frac{n}{2n-1}$

$=\lim\limits_{n\to\infty}\frac{1}{2-\frac{1}{n}}=\frac{1}{2}$ is finite and non-zero.

But we know that $\Sigma\, v_n=\Sigma\frac{1}{n}$ is divergent.

Therefore, the given series is $\Sigma\, u_n=\Sigma\frac{1}{2n-1}$ is also divergent.

(ii) Solution:

Given series is

$$\sqrt{\frac{1}{2^3}}+\sqrt{\frac{2}{3^3}}+\sqrt{\frac{3}{4^3}}+\ldots\ldots\ldots$$

Its general term is

$$u_n=\sqrt{\frac{n}{(n+1)^3}}$$

Taking the series $\Sigma\, v_n=\Sigma\frac{1}{n}$ and its general term is

$$v_n=\frac{1}{n}$$

Now $\lim\limits_{n\to\infty}\frac{u_n}{v_n}=\lim\limits_{n\to\infty}\sqrt{\frac{n}{(n+1)^3}}\cdot n$

$=\lim\limits_{n\to\infty}\sqrt{\frac{n^3}{(n+1)^3}}=\lim\limits_{n\to\infty}\sqrt{\frac{1}{\left(1+\frac{1}{n}\right)^3}}$

$=1$ is finite and non-zero.

But $\Sigma\frac{1}{n}$ is divergent. Therefore, the series Σu_n is also divergent.

(iii) Solution:

Given series is

$$1+\frac{1}{2^2}+\frac{2^2}{3^3}+\frac{3^3}{4^4}+\frac{4^4}{5^5}+\ldots\ldots\ldots$$

Its general term after first term is

$$u_n=\frac{n^n}{(n+1)^{n+1}}$$

Taking the series $\Sigma\frac{1}{n+1}$ and its general term is

$$v_n=\frac{1}{n+1}$$

Now

$\lim\limits_{n\to\infty}\frac{u_n}{v_n}=\lim\limits_{n\to\infty}\frac{n^n(n+1)}{(n+1)^{n+1}}$

$=\lim\limits_{n\to\infty}\frac{n^n}{(n+1)^n}=\lim\limits_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^n}$

$=\frac{1}{e}$ is finite and non-zero.

Since $\Sigma v_n=\Sigma\frac{1}{n+1}$ is divergent. Therefore, the series $\Sigma\, u_n$ is also divergent.

Exercise - 27

Determine whether the following series are convergent or divergent

1. $1+\frac{1}{2}+\frac{1}{5}+\frac{1}{10}+\ldots\ldots+\frac{1}{(n^2+1)}+\ldots\ldots$ $V_n=\frac{1}{n^2}$

2. $\frac{1}{1.2.3}+\frac{3}{2.3.4}+\frac{5}{3.4.5}+\ldots\ldots$ $\frac{(2n-1)}{n(n+1)(n+2)}$ $V_n=\frac{1}{n^2}$ **2062 Bhadra B.E.**

3. $\frac{2}{1^p}+\frac{3}{2^p}+\frac{4}{3^p}+\frac{5}{4^p}+\ldots\ldots$ $\frac{(n+1)}{n^p}$ $V_n=\frac{1}{n^{p-1}}$ **2067 Chaitra, B.E.**

4. $\frac{1}{1.2}+\frac{2}{3.4}+\frac{3}{5.6}+......\frac{n}{(2n-1)(2n)}$ $V_n = \frac{1}{n}$

5. $\Sigma \frac{\sqrt{n}}{n^2+1}$ $V_n = \frac{1}{n^{3/2}}$

6. $\cancel{\Sigma(\sqrt[3]{n^3+1} - n)}$ $\boxed{\text{2071 Bhadra, B. E.}}$

7. $\Sigma \frac{n}{1+n\sqrt{n+1}}$ $\boxed{\text{2072 Magh, B. E.}}$

8. $\Sigma(\sqrt{n^4-1} - n^2)$

9. $\sum_{n=0}^{\infty} \frac{2n^3+5}{4n^5+1}$ $V_n = \frac{1}{n^2}$

10. $\Sigma\{\sqrt{(n+1)} - \sqrt{n}\}$

Answers

1. Convergent
2. Convergent
3. Convergent for $p > 2$ and divergent $p \le 2$
4. Divergent
5. Convergent
6. Convergent
7. Divergent
8. Convergent
9. Convergent
10. Divergent

6.3.3 De'Alembert's Ratio Test

It is the method for determining the series may be divergent or convergent when other tests are not applicable.

Theorem

Let Σu_n be the series of positive term. If $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = l$, then the series is convergent for $l < 1$ and divergent for $l > 1$

Proof

Here we have $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = l$

When $l < 1$

By definition of a limit, we can find a positive number $r(<1)$ such that

$\frac{u_{n+1}}{u_n} < r$ for all n.

Given series is

$u_1 + u_2 + u_3 + ...$ and also given that

$\frac{u_2}{u_1} < r, \frac{u_3}{u_2} < r, \frac{u_4}{u_3} < r.....................$ and so on.

Therefore, the given sries can be written as

$$u_1+u_2+u_3+ ..... = u_1\left(1+\frac{u_2}{u_1}+\frac{u_3}{u_2}\cdot\frac{u_2}{u_1}+\frac{u_4}{u_3}\cdot\frac{u_3}{u_2}\frac{u_2}{u_1}+......\right)$$

$$< u_1(1+r+r^2+r^3+ .... = \frac{u_1}{1-r} \quad [\because r<1]$$

This is finite quantity. Hence the series Σu_n is convergent.

When $l > 1$

We have there is an integer m such that $\frac{u_{n+1}}{u_n} > 1$ for all $n > m$.

So that

$$\frac{u_2}{u_1} > 1, \frac{u_3}{u_2} > 1, \frac{u_4}{u_3} > 1 \text{ .... and so on.}$$

Therefore the n^{th} partial sum of the given series is

$$s_n = u_1 + u_2 + u_3 + ... + u_n$$

$$= u_1\left(1+\frac{u_2}{u_1}+\frac{u_3}{u_2}\frac{u_2}{u_1}+..... .... + \frac{u_n}{u_{n-1}}\frac{u_{n-1}}{u_{n-2}}\cdot\frac{u_2}{u_1}\right)$$

$$> u_1(1+1+1+1+1 \text{ .... to n terms}) = nu_1$$

$\lim_{n\to\infty} s_n > \lim_{n\to\infty} nu_1$ which tends to infinity and hence Σu_n is divergent for $l > 1$.

When $l = 1$ the ratio test fails. We cannot say whether the series is convergent and divergent. Then a different test must be employed.

Worked Out Examples

Ex: 1 Determine if each series is convergent or divergent.

(i) $\sum_{n=1}^{\infty} \frac{3n+1}{2^n}$ (ii) $\sum_{n=1}^{\infty} \frac{5^n}{n\,3^{n+1}}$

(i) Solution:

Given series is

$$\sum_{n=1}^{\infty} \frac{3n+1}{2^n}$$

Its general term is

$$u_n = \frac{3n+1}{2^n}, \qquad u_{n+1} = \frac{3n+4}{2^{n+1}}$$

Now by ratio test,

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{(3n+4)}{2^{n+1}} \frac{2^n}{(3n+1)} = \frac{1}{2} \lim_{n\to\infty} \frac{3+\frac{4}{n}}{3+\frac{1}{n}} = \frac{1}{2}$$

$\therefore \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \frac{1}{2} < 1.$ Hence, the series Σu_n is convergent.

(ii) Solution:

Given series is

$$\sum_{n=1}^{\infty} \frac{5^n}{n\, 3^{n+1}}$$

Here $u_n = \frac{5^n}{n . 3^{n+1}}$ and $u_{n+1} = \frac{5^{n+1}}{(n+1)\, 3^{n+2}}$

Now by using the ratio test

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{n\, 5^{n+1}}{(n+1)\, 3^{n+2}} \frac{3^{n+1}}{5^n}$$

$$= \frac{5}{3} \lim_{n\to\infty} \frac{n}{n+1} = \frac{5}{3} \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = \frac{5}{3}$$

$$\therefore \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \frac{5}{3} > 1.$$

Hence, the series Σu_n is divergent.

Ex: 2. Test for convergence the series $\frac{x}{1.2} + \frac{x^2}{3.4} + \frac{x^3}{5.6} + \ldots\ldots, x > 0.$

Solution:

Here

$$u_n = \frac{x^n}{(2n-1)\, 2n} \text{ and } u_{n+1} = \frac{x^{n+1}}{(2n+1)\, 2\,(n+1)}$$

Applying the ratio test

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{x^{n+1}}{2(2n+1)(n+1)} \frac{(2n-1)\, 2n}{x^n}$$

$$= x \lim_{n\to\infty} \frac{n\,(2n-1)}{(2n+1)(n+1)}$$

$$= x \lim_{n\to\infty} \frac{2-\frac{1}{n}}{\left(2+\frac{1}{n}\right)\left(1+\frac{1}{n}\right)} = x$$

Here if $x < 1$, then the series Σu_n is convergent and if $x > 1$ then the series is divergent.

If $x = 1$, then $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = 1$. The test fails so that further test is necessary.

Ex: 3. Test for convergence the series

(i) $\frac{1}{2\sqrt{1}} + \frac{x^2}{3\sqrt{2}} + \frac{x^4}{4\sqrt{3}} + \frac{x^6}{5\sqrt{4}} + \ldots\ldots\ldots\ldots \infty$

(ii) $1 + \frac{2}{5}x + \frac{6}{9}x^2 + \frac{14}{17}x^3 + \ldots\ldots\ldots\ldots \frac{2^n - 2}{2^n + 1}x^{n-1} + \ldots\ldots(x > 0)\ldots\ldots$

(i) Solution:

Given series is

$$\frac{1}{2\sqrt{1}} + \frac{x^2}{3\sqrt{2}} + \frac{x^4}{4\sqrt{3}} + \frac{x^6}{5\sqrt{4}} + \ldots\ldots\ldots\ldots \infty$$

We have $u_n = \frac{x^{2n-2}}{(n+1)\sqrt{n}}$ and $u_{n+1} = \frac{x^{2n}}{(n+2)\sqrt{n+1}}$

So

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{x^{2n}}{(n+2)\sqrt{n+1}} \frac{(n+1)\sqrt{n}}{x^{2n-2}}$$

$$= x^2 \lim_{n\to\infty} \frac{(n+1)\sqrt{n}}{(n+2)\sqrt{n+1}}$$

$$= x^2 \lim_{n\to\infty} \frac{\left(1+\frac{1}{n}\right)}{\left(1+\frac{2}{n}\right)\left(\sqrt{1+\frac{1}{n}}\right)} = x^2.$$

Hence the series Σu_n converges if $x^2 < 1$ and diverges if $x^2 > 1$.

If $x^2 = 1$, then $\quad u_n = \frac{1}{(n+1)\sqrt{n}} = \frac{1}{n^{3/2}} \frac{1}{\left(1+\frac{1}{n}\right)}$

Taking the series $\Sigma \frac{1}{n^{3/2}}$ and its general term is

$$v_n = \frac{1}{n^{3/2}}$$

By using the comparison test

$$\lim_{n \to \infty} \frac{u_n}{v_n} = \lim_{n \to \infty} \frac{1}{1+\frac{1}{n}} = 1 \text{ (finite quantity)}$$

Clearly, the series $\Sigma v_n = \Sigma \frac{1}{n^{3/2}}$, being p-series with $p = \frac{3}{2} > 1$ is convergent. By the comparison test the series Σu_n is also convergent.

Hence the given series converges if $x^2 \le 1$ and diverges if $x^2 > 1$.

(ii) Solution:

Given series is

$$1 + \frac{2}{5}x + \frac{6}{9}x^2 + \frac{14}{17}x^3 + \ldots\ldots\ldots + \frac{2^n - 2}{2^n + 1}x^{n-1} + \ldots\ldots (x>0)$$

Here $\quad u_n = \frac{2^n - 2}{2^n + 1}x^{n-1}$ and $u_{n+1} = \frac{2^{n+1} - 2}{2^{n+1} + 1}x^n$

Now by ratio test

$$\lim_{n \to \infty} \frac{u_{n+1}}{u_n} = \lim_{n \to \infty} \frac{2^{n+1} - 2}{2^{n+1} + 1} x^n \frac{2^n + 1}{(2^n - 2)\, x^{n-1}}$$

$$= x \lim_{n \to \infty} \frac{\left(2 - \frac{2}{2^n}\right)\left(1 + \frac{1}{2^n}\right)}{\left(2 + \frac{1}{2^n}\right)\left(1 - \frac{2}{2^n}\right)}$$

$$= x \frac{2-0}{2+0} \cdot \frac{1+0}{1-0} = x$$

Hence Σu_n converges if $x < 1$ and diverges if $x > 1$ and the ratio test fails for $x = 1$.

When $x = 1$

$$\lim_{n \to \infty} u_n = \lim_{n \to \infty} \frac{2^n - 2}{2^n + 1} = \lim_{n \to \infty} \frac{1 - \frac{2}{2^n}}{1 + \frac{1}{2^n}} = 1$$

$$\therefore \lim_{n \to \infty} u_n = 1 \ne 0$$

So Σu_n is divergent for $x = 1$

Hence the given series converges for $x < 1$ and diverges for $x \ge 1$.

Ex: 4. Test for convergence the following series.

(i) $1 + \frac{x}{2} + \frac{x^2}{5} + \frac{x^3}{10} + \ldots\ldots\ldots + \frac{x^n}{n^2+1} + \ldots\ldots\ldots$

(ii) $2x + \frac{3x^2}{8} + \frac{4x^3}{27} + \ldots\ldots\ldots + \frac{(n+1)}{n^3}x^n + \ldots\ldots$ [2061 Bhadra, B.E.]

(iii) $1 + \frac{2^2}{2!} + \frac{2^2}{3!} + \frac{4^2}{4!} + \ldots\ldots\ldots$

(i) Solution:

Given series is $1 + \frac{x}{2} + \frac{x^2}{5} + \frac{x^3}{10} + \ldots\ldots\ldots + \frac{x^n}{n^2+1} + \ldots\ldots$

Here $\quad u_n = \frac{x^n}{n^2+1}$ and $u_{n+1} = \frac{x^{n+1}}{(n+1)^2+1}$

Now

$$\lim_{n \to \infty} \frac{u_{n+1}}{u_n} = \lim_{n \to \infty} \frac{x^{n+1}}{n^2+2n+2} \frac{(n^2+1)}{x^n}$$

$$= x \lim_{n \to \infty} \frac{n^2+1}{n^2+2n+2} = x \lim_{n \to \infty} \frac{\left(1 + \frac{1}{n^2}\right)}{\left(1 + \frac{2}{n} + \frac{2}{n^2}\right)}$$

$$= x$$

Hence Σu_n converges for $x < 1$ and diverges for $x > 1$.

When $x = 1$, the general term of the series is

$$u_n = \frac{1}{n^2+1}$$

Taking $v_n = \frac{1}{n^2}$

Now $\lim_{n \to \infty} \frac{u_n}{v_n} = \lim_{n \to \infty} \frac{1}{1 + \frac{1}{n^2}} = 1$

$\lim_{n \to \infty} \frac{u_n}{v_n} = 1$, a finite quantity.

But we know that $\Sigma v_n = \Sigma \frac{1}{n^2}$, being p-series with $p > 1$, is convergent.

Hence by the comparison test Σu_n is also convergent for $x = 1$.

Therefore Σu_n converges for $x \le 1$ and diverges for $x > 1$.

(ii) Solution:

Given series is

$$2x + \frac{3x^2}{8} + \frac{4x^3}{27} + \dots\dots\dots + \frac{(n+1)\,x^n}{n^3} + \dots\dots$$

Here $u_n = \frac{(n+1)}{n^3} x^n$ and $u_{n+1} = \frac{n+2}{(n+1)^3} x^{n+1}$

Now

$$\lim_{n \to \infty} \frac{u_{n+1}}{u_n} = \lim_{n \to \infty} \frac{(n+2)}{(n+1)^3} x^{n+1} \frac{n^3}{(n+1)\,x^n}$$

$$= x \lim_{n \to \infty} \frac{(n+2)\,n^3}{(n+1)^3 (n+1)}$$

$$= x \lim_{n \to \infty} \frac{\left(1 + \frac{2}{n}\right)}{\left(1 + \frac{1}{n}\right)^3 \left(1 + \frac{1}{n}\right)} = x$$

By the ratio test Σu_n converges for $x < 1$ and diverges for $x > 1$.

When $x = 1$

The general term of the series is

$$u_n = \frac{n+1}{n^3}$$

Taking the series whose general term is

$$v_n = \frac{1}{n^2}$$

Now $\lim_{n \to \infty} \frac{u_n}{v_n} = \lim_{n \to \infty} \frac{(n+1)}{n^3} \times n^2 = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)$

$= 1$, a finite quantity.

But we know that $\Sigma v_n = \Sigma \frac{1}{n^2}$, being p-series with $p > 1$, is convergent.

Hence, the series Σu_n is convergent for $x = 1$. Therefore, the series Σu_n is convergent for $x \le 1$ and divergent for $x > 1$.

(iii) Solution:

Given series is

$$1 + \frac{2^2}{2!} + \frac{3^2}{3!} + \frac{4^2}{4!} + \dots\dots\dots\dots$$

The general term of the series is

$$u_n = \frac{n^2}{n!} \quad \text{and} \quad u_{n+1} = \frac{(n+1)^2}{(n+1)!}$$

Now

$$\lim_{n \to \infty} \frac{u_{n+1}}{u_n} = \lim_{n \to \infty} \frac{(n+1)^2\, n!}{(n+1)!\, n^2}$$

$$= \lim_{n \to \infty} \frac{(n+1)^2}{(n+1)\, n^2} = \lim_{n \to \infty} \frac{n+1}{n^2} = 0 < 1.$$

Hence, Σu_n is convergent.

6.3.4 Root Test

Theorem

The series Σu_n of positive terms is convergent if $\lim_{n \to \infty} \sqrt[n]{u_n} < 1$ and divergent if $\lim_{n \to \infty} \sqrt[n]{u_n} > 1$ and if $\lim_{n \to \infty} \sqrt[n]{u_n} = 1$, then test fails.

Proof

Let $\sqrt[n]{u_n} < r < 1$, then $u_n < r^n$ for all values of n.

So

$$u_1 + u_2 + u_3 + \ldots\ldots + u_n + \ldots\ldots$$
$$< r + r^2 + r^3 + \ldots\ldots + r^n + \ldots\ldots$$

or $\Sigma u_n < \frac{r}{1-r}$ which is fixed number hence Σu_n is convergent.

Let $\sqrt[n]{u_n} > r > 1$, then $u_n > r^n$

$$u_1 + u_2 + \ldots\ldots + u_n + \ldots\ldots$$
$$> r + r^2 + r^3 + \ldots\ldots + r^n \ldots\ldots$$

or $\Sigma u_n > \Sigma r^n$

Since Σr^n is geometric series for $r > 1$, Σr^n diverges hence Σu_n diverges.

If $\sqrt[n]{u_n} = 1$ the ratio test fails then the further tests should be applied.

Worked Out Examples

Ex: 1. Test for convergence the series $1 + \frac{x}{2} + \frac{x^2}{3^2} + \frac{x^3}{4^3} + \ldots\ldots (x > 0)$.

Solution:

General term of the series is

$$u_n = \frac{x^n}{(n+1)^n}$$

So that

$$\lim_{n\to\infty} \sqrt[n]{u_n} = \lim_{n\to\infty} \sqrt[n]{\frac{x^n}{(n+1)^n}}$$
$$= \lim_{n\to\infty} \frac{x}{n+1} = 0$$

$\therefore \lim_{n\to\infty} \sqrt[n]{u_n} = 0 < 1$. Hence, the series is convergent.

Ex: 2. Test the convergence of the series $\frac{1}{2} + \frac{2}{3}x + \left(\frac{3}{4}\right)^2 x^2 + \left(\frac{4}{5}\right)^3 x^3 + \ldots$ for positive values of x

Solution: *2067 Mangsir, B.E.*

Given series is

$$\frac{1}{2} + \frac{2}{3}x + \left(\frac{3}{4}\right)^2 x^2 + \left(\frac{4}{5}\right)^3 x^3 + \ldots\ldots$$

Omitting the first term, general term is $u_n = \left(\frac{n+1}{n+2}\right)^n x^n$

By applying root test

$$\lim_{n\to\infty} \sqrt[n]{u_n} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{n+1}{n+2}\right)^n x^n}$$

$$= x \lim_{n\to\infty} \frac{n+1}{n+2} = x \lim_{n\to\infty} \frac{1+\frac{1}{n}}{1+\frac{2}{n}} = x$$

Therefore the series is convergent if $x < 1$ and divergent if $x > 1$.

If $x = 1$, then $\lim_{n\to\infty} \sqrt[n]{u_n} = 1$ the root test fails.

Therefore

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \left(\frac{n+1}{n+2}\right)^n$$

$$= \lim_{n\to\infty} \frac{\left(1+\frac{1}{n}\right)^n}{\left(1+\frac{2}{n}\right)^n} = \lim_{n\to\infty} \frac{\left(1+\frac{1}{n}\right)^n}{\left[\left(1+\frac{2}{n}\right)^{\frac{n}{2}}\right]^2} = \frac{e}{e^2}$$

$$= \frac{1}{e} \neq 0.$$

Thus the series is divergent for $x = 1$.

Ex: 3. Test for convergence the series $\Sigma \frac{1}{\left(1+\frac{1}{\sqrt{n}}\right)^{n^{3/2}}}$

Solution:

Given that the series is

$$\Sigma \frac{1}{\left(1+\frac{1}{\sqrt{n}}\right)^{n^{3/2}}}$$

So $\lim_{n\to\infty} \sqrt[n]{u_n} = \lim_{n\to\infty} \left[\frac{1}{\left(1+\frac{1}{\sqrt{n}}\right)^{n\sqrt{n}}}\right]^{1/n} = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{\sqrt{n}}\right)^{\sqrt{n}}}$

$= \frac{1}{e} < 1$. Hence, the given series is convergent.

Ex.4. Test the convergence of the series

$$1+\frac{1^2}{2^2}+\frac{2^2}{3^3}+\frac{3^3}{4^4}+\frac{4^4}{5^5}+\ldots$$

2072 Aswin, B. E.

Solution:

Here, the given series is,

$$1+\frac{1}{2^2}+\frac{2^2}{3^3}+\frac{3^3}{4^4}+\ldots\ldots\ldots\ldots$$

Omitting the first term, its general term is $u_n = \frac{n^n}{(n+1)^{n+1}}$

We have, by the root test

$$\lim_{n\to\infty} \sqrt[n]{u_n} = \lim_{n\to\infty} \sqrt[n]{\frac{n^n}{(n+1)^{n+1}}} = \lim_{n\to\infty} \sqrt[n]{\frac{n^n}{(n+1)^n(n+1)}}$$

$$= \lim_{n\to\infty} \sqrt[n]{\frac{n^n}{(n+1)^n(n+1)}} = \lim_{n\to\infty} \frac{n}{n+1} \frac{1}{(n+1)^{1/n}}$$

$$= \lim_{n\to\infty} \frac{n}{n\left(1+\frac{1}{n}\right)} \frac{1}{(n+1)^{1/n}}$$

$$= \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)} \frac{1}{(n+1)^{1/n}} = \frac{1}{(1+0)}\frac{1}{e} = \frac{1}{e}$$

$$\therefore \lim_{n\to\infty} \sqrt[n]{u_n} = \frac{1}{e} < 1.$$

Hence, by the root test, the series is convergent.

6.3.5 Further Test for Convergence

If $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = 1$, then the ratio test fails. We apply the higher ratio test or logarithmic ratio test which are explained as follows without proofs:

I. Higher ratio test (Raabe's test)

In a positive term series $\sum u_n$, if $\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = l$, then the series convergence for $l > 1$ and diverges for $l < 1$ but test fails for $l = 1$.

II. Logarithmic Ratio test

In the positive term series $\sum u_n$, if $\lim_{n\to\infty} n\left(\log\frac{u_n}{u_{n+1}}\right) = l$, then the series converges for $l > 1$ and diverges for $l < 1$ but the test fails for $l = 1$.

Worked Out Examples

Test for convergence of the series

Ex.1. $\frac{x}{1}+\frac{1}{2}\cdot\frac{x^2}{3}+\frac{1.3}{2.4}\frac{x^3}{5}+\frac{1.3.5}{2.4.6}\frac{x^4}{7}+\ldots\ldots\ldots\ldots$

Solution:

Omitting the first term, the general term of the series is

$$u_n = \frac{1.3.5\ldots\ldots(2n-1)}{2.4.6\ldots\ldots 2n}\frac{x^{n+1}}{2n+1}$$

and $$u_{n+1} = \frac{1.3.5\ldots\ldots(2n-1)(2n+1)}{2.4.6\ldots\ldots 2n(2n+2)}\frac{x^{n+2}}{(2n+3)}$$

$$\frac{u_{n+1}}{u_n} = \frac{(2n+1)(2n+1)}{(2n+2)(2n+3)}x$$

$$\therefore \lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\frac{\left(2+\frac{1}{n}\right)^2}{\left(2+\frac{2}{n}\right)\left(2+\frac{3}{n}\right)}x = x$$

We conclude that the series is convergent if $x<1$ and divergent if $x>1$

If $x = 1$, then

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \lim_{n\to\infty} n\left(\frac{(2n+2)(2n+3)}{(2n+1)^2} - 1\right)$$

$$= \lim_{n\to\infty}\frac{n(6n+5)}{(2n+1)^2} = \lim_{n\to\infty}\frac{\left(6+\frac{5}{n}\right)}{\left(2+\frac{1}{n}\right)^2} = \frac{3}{2} > 1$$

Hence the series is convergent for x = 1. Thus, the series is convergent for $x \le 1$ and divergent for $x > 1$.

Ex: 2. Test the convergence of the series

$$\frac{1^2}{4^2}+\frac{1^2.5^2}{4^2.8^2}+\frac{1^2.5^2.9^2}{4^2.8^2.12^2}+\frac{1^2.5^2.9^2.13^2}{4^2.8^2.12^2.16^2}+\ldots\ldots$$

Solution:

Given series is

$$\frac{1^2}{4^2}+\frac{1^2.5^2}{4^2.8^2}+\frac{1^2.5^2.9^2}{4^2.8^2.12^2}+\frac{1^2.5^2.9^2.13^2}{4^2.8^2.12^2.16^2}+\ldots\ldots$$

Here $u_n = \dfrac{1^2.5^2.9^2.13^2\ldots\ldots(4n-3)^2}{4^2.8^2.12^2.16^2\ldots\ldots(4n)^2}$

and $u_{n+1} = \dfrac{1^2.5^2.9^2.13^2\ldots\ldots(4n-3)^2(4n+1)^2}{4^2.8^2.12^2.16^2\ldots\ldots(4n)^2(4n+4)^2}$

Here $\dfrac{u_{n+1}}{u_n} = \dfrac{(4n+1)^2}{(4n+4)^2}$

or $\lim\limits_{n\to\infty}\dfrac{u_{n+1}}{u_n} = \lim\limits_{n\to\infty}\dfrac{\left(4+\frac{1}{n}\right)^2}{\left(4+\frac{4}{n}\right)^2}$

$\therefore \lim\limits_{n\to\infty}\dfrac{u_{n+1}}{u_n} = 1.$

The ratio test fails, so applying the higher ratio test.

Now

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) = \lim_{n\to\infty} n\left(\frac{(4n+4)^2}{(4n+1)^2}-1\right) = \lim_{n\to\infty}\frac{n(24n+15)}{(4n+1)^2}$$

$$= \lim_{n\to\infty}\frac{\left(24+\frac{15}{n}\right)}{\left(4+\frac{1}{n}\right)^2} = \frac{3}{2} > 1.$$

Hence, the series is convergent.

Ex: 3. Test for convergence of the series $1+\frac{x}{2}+\frac{2!}{3^2}x^2+\frac{3!}{4^3}x^3+\ldots$

Solution: 2070 Bhadra, B. E.

Given series is

$$1+\frac{x}{2}+\frac{2!}{3^2}x^2+\frac{3!}{4^3}x^3+\ldots\ldots$$

Omitting the first term, the general term of the series is

$u_n = \dfrac{n!}{(n+1)^n}x^n$

$u_n = \quad x^n$ and $u_{n+1} = \dfrac{(n+1)!}{(n+2)^{n+1}}x^{n+1}$

Now $\dfrac{u_{n+1}}{u_n} = \dfrac{(n+1)(n+1)^n}{(n+2)^{n+1}}x$

or $\lim\limits_{n\to\infty}\dfrac{u_{n+1}}{u_n} = \lim\limits_{n\to\infty}\dfrac{(n+1)^{n+1}}{(n+2)^{n+1}}x$

$$= \lim_{n\to\infty}\frac{\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n}\right)^n}{\left(1+\frac{2}{n}\right)\left[\left(1+\frac{2}{n}\right)^{\frac{n}{2}}\right]^2}x = \frac{x}{e}$$

The series is convergent if $\frac{x}{e} < 1$ and divergent if $\frac{x}{e} > 1$.

When $\frac{x}{e} = 1$, i. e. $x = e$.

Then the ratio test fails we use logarithmic ratio test. For this

$$\frac{u_n}{u_{n+1}} = \left(\frac{n+2}{n+1}\right)^n\frac{(n+2)}{(n+1)e}$$

or $\log\dfrac{u_n}{u_{n+1}} = \log\left(\dfrac{n+2}{n+1}\right)^n + \log\left(\dfrac{n+2}{n+1}\right) - \log e$

$$= n\log\frac{\left(1+\frac{2}{n}\right)}{1+\frac{1}{n}} + \log\frac{\left(1+\frac{2}{n}\right)}{\left(1+\frac{1}{n}\right)} - 1$$

$$= n\log\left(1+\frac{2}{n}\right) - n\log\left(1+\frac{1}{n}\right) + \log\left(1+\frac{2}{n}\right) - \log\left(1+\frac{1}{n}\right) - 1$$

$$= n\left(\frac{2}{n}-\frac{4}{2n^2}+\frac{8}{3n^3}-\ldots\ldots\right) - n\left(\frac{1}{n}+\frac{1}{2n^2}+\frac{1}{3n^3}+\ldots\ldots\right)$$

$$+\left(\frac{2}{n}-\frac{4}{2n^2}+\ldots\ldots\right)-\left(\frac{1}{n}-\frac{1}{2n^2}+\ldots\ldots\right)-1$$

$$= -\frac{1}{n}+\frac{5}{6n^2}+\ldots\ldots$$

$$\therefore \lim_{n\to\infty} n\log\frac{u_n}{u_{n-1}} = \lim_{n\to\infty}\left(-1+\frac{5}{6n}+\ldots\ldots\right) = -1 < 1$$

Hence the series is divergent for $x = e$.

Exercise - 28

Test the following series for convergence by applying ratio test

1. $x + 2x^2 + 3x^3 + 4x^4 + \dots\dots, x>0$ $\quad n x^n$
2. $1 + 3x + 5x^2 + 7x^3 + \dots\dots, x>0$ $\quad (2n-1)\, x^{n-1}$
3. $1 + \frac{2^p}{2!} + \frac{3^p}{3!} + \frac{4^p}{4!} + \dots\dots$ to ∞ $\quad \frac{n^p}{n!}$
4. $2x + \frac{3x^2}{8} + \frac{4x^3}{27} + \dots\dots, x>0$ $\quad \frac{(n+1)\, x^n}{n^3}$ $\quad$ 2068 Bhadra, B. E.
5. $x + \frac{3}{5}x^2 + \frac{8}{10}x^3 + \frac{15}{17}x^4 + \dots + \frac{n^2-1}{n^2+1}x^n + \dots, x>0$ $\quad$ 2070 Magh, B. E.
6. $\frac{1^2}{3} + \frac{2^2}{3^2} + \frac{3^2}{3^3} + \dots$ $\quad \frac{n^2}{3^n}$
7. $\frac{x}{1.2} + \frac{x^2}{2.3} + \frac{x^3}{3.4} + \dots\dots, x>0$ $\quad \frac{x^n}{n(n+1)}$ $\quad$ 2069 Bhadra, B. E.
8. $1 + \frac{2!}{2^2} + \frac{3!}{3^3} + \frac{4!}{4^4} + \dots\dots$ $\quad \frac{n!}{n^n}$
9. $3 + \frac{3^2}{2!} + \frac{3^3}{3!} + \frac{3^4}{4!} + \dots\dots$ $\quad \frac{3^n}{n!}$
10. $1 + \frac{2^2}{2!} + \frac{3^3}{3!} + \frac{4^4}{4!} + \dots\dots + \frac{n^n}{n!} + \dots$
11. $2x + \frac{9x^2}{8} + \frac{64x^3}{81} + \dots\dots + \frac{(n+1)^n x^n}{n^{n+1}} + \dots\dots$ $\quad$ 2069 Poush, B. E.

Test the following series for convergence by applying root test

12. $x + \frac{3}{5}x^2 + \frac{8}{10}x^3 + \frac{15}{17}x^4 + \dots\dots + \frac{n^2-1}{n^2+1}x^n + \dots$ $\quad$ 2070 Magh, B. E.
13. $\Sigma\left(1 + \frac{1}{n}\right)^{-n^2}$
14. $\Sigma\left(\frac{nx}{1+n}\right)^n$
15. $\Sigma\frac{1}{n^n}$
16. $\Sigma\frac{1}{(\log n)^n}$

Answers

1. Convergent for $x < 1$; divergent for $x \geq 1$
2. Convergent for $x < 1$, divergent for $x \geq 1$
3. Convergent for all values of p.
4. Convergent for $x \leq 1$ and divergent for $x > 1$
5. Convergent for $x < 1$ and divergent for $x \geq 1$
6. Convergent
7. Convergent for $x \leq 1$, divergent for $x > 1$
8. Convergent
9. Convergent
10. Divergent
11. Convergent for $x < 1$ and divergent for ≥ 1
12. Convergent for $x < 1$, divergent for $x \geq 1$
13. Convergent
14. Convergent for $x < 1$ and divergent for $x > 1$
15. Convergent
16. Convergent

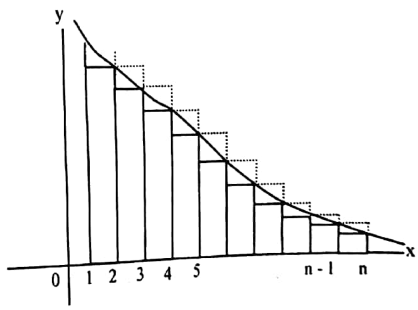
6.3.6 Integral Test

Theorem

If f(x) is positive, continuous and decreasing for $x \geq 1$, then the infinite series $f(1) + f(2) + \dots + f(n) + \dots$ converges or diverges according as $\int_1^\infty f(x)\,dx$ is finite or infinite.

Proof

Let $y = f(x)$ be the curve. The area between any two ordinates lies between set of inscribed and described rectangles formed by the ordinates at $x = 1, 2, 3, \dots\dots, n$ as shown as in the figure.



The area of the inscribed rectangular polygon is

$$f(2) + f(3) + \ldots\ldots\ldots\ldots + f(n).$$

Similarly, the area of the circumscribed rectangular polygon is

$$f(1) + f(2) + \ldots\ldots\ldots\ldots + f(n-1).$$

Since $\int_1^n f(x)\,dx$ is the area under the graph f(x) from 1 to n.

$$\therefore \quad f(2) + f(3) + f(4) + \ldots\ldots + f(n) \leq \int_1^n (f(x)\,dx$$

$$\leq f(1) + f(2) + \ldots\ldots + f(n-1)$$

If $S_n = f(1) + f(2) + \ldots\ldots\ldots + f(n)$, then

$$S_n - f(1) \leq \int_1^n f(x)\,dx \leq S_{n-1}$$

Therefore $S_n - f(1) \leq \int_1^n f(x)\,dx$ .... (1)

and $\int_1^n f(x)\,dx \leq S_{n-1}$ ....(2)

Form (1), if $\lim_{n\to\infty} \int_1^n f(x)\,dx$ is finite and equal to k, then

$\lim_{n\to\infty} S_n \leq k + f(1)$ for every positive integer n.

Hence, S_n *converges* for $\lim_{n\to\infty} \int_1^n f(x)\,dx$ is *finite*.

Form (2), if $\lim_{n\to\infty} \int_1^n f(x)\,dx$ is infinite i.e. $\lim_{n\to\infty} \int_1^n f(x)\,dx = \infty$, then

$\lim_{n\to\infty} S_{n-1} \geq \infty$.

Hence the series S_n *diverges* for $\lim_{n\to\infty} \int_1^n f(x)\,dx$ is *infinite*.

Worked Out Examples

Ex: 1. Show that p series $\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \ldots\ldots\ldots \infty$ converges for p > 1 and diverges for p ≤ 1

Solution:

Since $f(x) = \frac{1}{x^p}$ for $x \geq 1$ is decreasing continuous function.

If $p \neq 1$, then

$$\lim_{n\to\infty} \int_1^n f(x)\,dx = \lim_{n\to\infty} \int_1^n \frac{1}{x^p}\,dx$$

$$= \lim_{n\to\infty} \left[\frac{x^{1-p}}{1-p}\right]_1^n = \lim_{n\to\infty} \frac{n^{1-p} - 1}{1-p}$$

$$= \frac{1}{p-1} \text{ for } p > 1 \to \infty \text{ for } p < 1.$$

If p = 1, then

$$\lim_{n\to\infty} \int_1^n f(x)\,dx = \lim_{n\to\infty} \int_1^n \frac{1}{x}\,dx = \lim_{n\to\infty} \int_1^n \frac{1}{x}\,dx$$

$$= \lim_{n\to\infty} [\log x]_1^n = \lim_{n\to\infty} \log n = \infty$$

Therefore, given series is convergent for p > 1 and divergent for p ≤ 1.

Ex: 2. Test the convergence series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n} + \sqrt{n+1}}$

Solution:

General term of the series is

$$f(n) = \frac{1}{\sqrt{n} + \sqrt{n+1}}$$

Since $\quad f(x) = \frac{1}{\sqrt{x} + \sqrt{x+1}}$ for $x \geq 1$, f is decreasing for $x \geq 1$.

Therefore

$$\lim_{n\to\infty}\int_1^n f(x)\,dx = \lim_{n\to\infty}\int_1^n \frac{1}{\sqrt{x}+\sqrt{x+1}}\,dx$$

$$= \lim_{n\to\infty}\int_1^n \frac{\sqrt{x}-\sqrt{x+1}}{-1}\,dx$$

$$= \lim_{n\to\infty}\left[\frac{2}{3}(x+1)^{3/2} - \frac{2}{3}x^{3/2}\right]_1^n$$

$$= \frac{2}{3}\lim_{n\to\infty}\left[(n+1)^{\frac{3}{2}} - n^{\frac{3}{2}} - 1\right] = \infty$$

$\therefore \lim_{n\to\infty}\int_1^n f(x)\,dx \to \infty$ the given series is divergent.

Ex: 3. Test the convergence or divergence of the series

$$\frac{1}{1.2}+\frac{2}{3.4}+\frac{3}{5.6}+\ldots\ldots\ldots\ldots$$

Solution:

General term of the series is

$$f(n) = \frac{n}{(2n-1)\,2n} = \frac{1}{2\,(2n-1)}$$

Replacing n by x for $x \ge 1$, we get

$$f(x) = \frac{1}{2\,(2x-1)}$$

Now

$$\lim_{n\to\infty}\int_1^n f(x)\,dx = \lim_{n\to\infty}\int_1^n \frac{1}{2(2x-1)}\,dx$$

$$= \frac{1}{4}\lim_{n\to\infty}\left[\log(2x-1)\right]_1^n$$

$$= \frac{1}{4}\lim_{n\to\infty}\log(2n-1) = \infty$$

$\therefore \lim_{n\to\infty}\int_1^n f(x)\,dx \to \infty$,by the integral test, the given series is divergent.

6.4 Alternating Series

A series whose terms are alternatively positive and negative is called an alternating series.

$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots\ldots\ldots\ldots$ is an example of alternating series.

Leibnitz Test

Theorem

The alternating series $u_1 - u_2 + u_3 - u_4 + \ldots\ldots\ldots$ converges if $u_{n+1} \le u_n$ numerically for all n and $\lim_{n\to\infty} u_n = 0$. If $\lim_{n\to\infty} u_n \ne 0$,then the series is oscillatory

Proof

Given series is

$$u_1 - u_2 + u_3 - u_4 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$u_1 > u_2 > u_3 > u_4 \ldots\ldots\ldots > u_{n-1} > \ldots\ldots\ldots\ldots\ldots\ldots$$

and $\lim_{n\to\infty} u_n = 0.$

Consider the n^{th} partial sum

$S_1 = u_1$ $\quad S_2 = u_1 - u_2$

$S_3 = u_1 - u_2 + u_3$ $\quad S_4 = u_1 - u_2 + u_3 - u_4$

$S_5 = u_1 - u_2 + u_3 - u_4 + u_5$ $\quad S_6 = u_1 - u_2 + u_3 - u_4 + u_5 - u_6$

and so on.

It follows that

$S_5 = S_4 + u_5$ and $S_6 = u_1 - (u_2 - u_3) - (u_4 - u_5) - u_6$

In general,

$S_{2n+1} = S_{2n} + u_{2n+1}$ and $S_{2n} = u_1 - (u_2 - u_3) - (u_4 - u_5) - \ldots\ u_{2n} < u_1$

$$\lim_{n\to\infty} S_{2n+1} = \lim_{n\to\infty} S_{2n} + \lim_{n\to\infty} u_{2n+1}$$

$$\lim_{n\to\infty} S_{2n+1} = \lim_{n\to\infty} S_{2n} + 0$$

Since $\{S_{2n}\}$ is bounded by u_1.

$$\therefore \lim_{n\to\infty} S_n = S = \lim_{n\to\infty} S_{2n+1}$$

Hence, the given series is convergent.

Worked Out Examples

Ex:1. Discuss the convergence of the series $1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \ldots$

Solution:

Given series is

$$1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \ldots\ldots\ldots$$

Here $u_n = \frac{1}{\sqrt{n}}$ Clearly the terms of this series are alternatively positive and negative and each term is numerically less than its preceding terms and $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$

Hence, by the Leibnitz test the given series is convergent.

Ex: 2. Discuss the convergence of the series $\frac{2}{1^3} - \frac{3}{2^3} + \frac{4}{3^3} - \frac{5}{4^3} + \ldots\ldots$

Solution:

Given series is

$$\frac{2}{1^3} - \frac{3}{2^3} + \frac{4}{3^3} - \frac{5}{4^3} + \ldots\ldots\ldots$$

Here general term of the series is

$$u_n = \frac{n+1}{n^3}.$$

Clearly the terms are alternately positive and negative and each term is numerically less than its preceding term and

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{n+1}{n^3} = \lim_{n\to\infty} \frac{1+\frac{1}{n}}{n^2} = 0$$

Hence, the series is convergent.

Ex: 3. Examine the character of the series $\sum_{n=1}^{\infty} \frac{n(-1)^{n-1}}{2n-1}$

Solution:

The terms of the series is

$$\sum \frac{(-1)^{n-1}n}{2n-1}$$

This is alternately positive and negative and each terms is numerically less than its preceding term and

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{n}{2n-1} = \lim_{n\to\infty} \frac{1}{2-\frac{1}{n}} = \frac{1}{2} \neq 0$$

Hence, the given series is oscillatory.

Ex: 5. Discuss the convergence of the series

$$\left(\frac{1}{2} - \frac{1}{\log 2}\right) - \left(\frac{1}{2} - \frac{1}{\log 3}\right) + \left(\frac{1}{2} - \frac{1}{\log 4}\right) - \left(\frac{1}{2} - \frac{1}{\log 5}\right) + \ldots\ldots\ldots$$

Solution:

The terms of the series is

$$\left(\frac{1}{2} - \frac{1}{\log 2}\right) - \left(\frac{1}{2} - \frac{1}{\log 3}\right) + \left(\frac{1}{2} - \frac{1}{\log 4}\right) - \left(\frac{1}{2} - \frac{1}{\log 5}\right) + \ldots\ldots\ldots$$

This is alternately positive and negative and each term is numerically less than its preceding term but

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \left(\frac{1}{2} - \frac{1}{\log(n+1)}\right)$$

$$= \frac{1}{2} - \lim_{n\to\infty} \left(\frac{n}{\log(n+1)}\right)\frac{1}{n} = \frac{1}{2} - 0 = \frac{1}{2} \neq 0$$

Hence, the given series is oscillatory.

6.4.1 Absolute Convergence

A series Σu_n is said to be *convergent absolutely* if the series $\Sigma |u_n| = |u_1| + |u_2| + \ldots\ldots\ldots + |u_n| + \ldots\ldots\ldots$ is convergent.

If Σu_n is convergent but $\Sigma |u_n|$ is divergent then Σu_n is said to be *conditionally convergent.*

For example,

The series

$$1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \ldots\ldots\ldots\ldots \text{ is convergent absolutely.}$$

Since the series

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \ldots\ldots\ldots\ldots \text{ is known to be convergent series}$$

and the series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \ldots\ldots\ldots\ldots \text{ is convergent .}$$

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots\ldots\ldots\ldots \text{ is known to be divergent.}$$

So that the series

$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \ldots\ldots\ldots\ldots$ is *conditionally convergent* or semi-convergent.

Note 1:

The convergent absolutely series is necessarily convergent but not conversely.

Let Σu_n be absolutely convergent and we have

$$u_1 + u_2 + u_3 + \ldots + u_n + \ldots \le |u_1| + |u_2| + |u_3| + \ldots\ldots + |u_n| + \ldots\ldots$$

This is convergent.

$\therefore$ the series Σu_n is convergent.

Note 2:

The series Σu_n is absolutely convergent if $\lim\limits_{n \to \infty} \left|\frac{u_{n+1}}{u_n}\right| < 1$ and divergent if $\lim\limits_{n \to \infty} \left|\frac{u_{n+1}}{u_n}\right| > 1$. The test fails if $\lim\limits_{n \to \infty} \left|\frac{u_{n+1}}{u_n}\right| = 1$.

Ex.6: Determine the series $\frac{\sin 1}{1} + \frac{\sin 2}{2^2} + \frac{\sin 3}{3^2} + \ldots\ldots + \frac{\sin n}{n^2} + \ldots\ldots$ is convergent or divergent.

Solution:

Given series is

$$\frac{\sin 1}{1} + \frac{\sin 2}{2^2} + \frac{\sin 3}{3^2} + \ldots\ldots\ldots\ldots + \frac{\sin n}{n^2} + \ldots\ldots$$

Now absolute values of the series is

$$\sum_{n=1}^{\infty} \left|\frac{\sin n}{n^2}\right| = \sum_{n=1}^{\infty} \left|\frac{\sin n}{n^2}\right|$$

Since $\left|\frac{\sin n}{n^2}\right| \le \frac{1}{n^2}$ and the series $\Sigma \frac{1}{n^2}$, being p-series with $p = 2 > 1$, is convergent hence $\Sigma \frac{|\sin n|}{n^2}$ is also convergent. Therefore, the given series is convergent. This implies that the series is convergent.

Exercise - 30

Apply integral test to test the convergence of the series.

1. $\sum\limits_{n=1}^{\infty} \frac{1}{n^2 + n}$

2. $\sum\limits_{n=1}^{\infty} \frac{1}{n^2 + 1}$

3. $\sum\limits_{n=1}^{\infty} \frac{1}{(3 + 2n)^2}$

4. $\sum\limits_{n=1}^{\infty} \frac{1}{4n + 7}$

5. $\sum\limits_{n=1}^{\infty} \frac{\log n}{n}$

6. $\sum\limits_{n=1}^{\infty} \frac{1}{\sqrt[3]{2n + 1}}$

7. $\Sigma\, n\, e^{-n^2}$

8. $\sum\limits_{n=1}^{\infty} \frac{\tan^{-1} n}{1 + n^2}$

9. $\sum\limits_{n=2}^{\infty} \frac{1}{n \sqrt[3]{\log n}}$

10. $\sum\limits_{n=1}^{\infty} n e^{-n}$

Determine each alternating series is convergent conditionally convergent or divergent.

11. $\sum\limits_{n=1}^{\infty} (-1)^n \frac{1}{n^2}$

12. $\sum\limits_{n=1}^{\infty} (-1)^n \frac{n^2 + 4}{2^n}$

13. $\sum\limits_{n=1}^{\infty} (-1)^{n-1} \frac{1}{\sqrt{2n + 1}}$

14. $\sum\limits_{n=1}^{\infty} (-1)^{n-1} \frac{1}{\log(n + 1)}$

15. $\sum\limits_{n=2}^{\infty} (-1)^n \frac{n}{\log n}$

16. $\sum\limits_{n=1}^{\infty} (-1)^n \frac{5}{n^3 + 1}$

Answers

1. Convergent
2. Convergent
3. Convergent
4. Divergent
5. Divergent
6. Divergent
7. Convergent
8. Convergent
9. Divergent
10. Convergent
11. Convergent
12. Convergent
13. Conditionally Convergent
14. Conditionally Convergent
15. Divergent
16. Convergent

6.5 Radius and Interval of Convergence

A series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots\ldots\ldots + a_n x^n + \ldots\ldots$$

where the numbers a_n dependent on n but not dependent on x, is called *Power series* in x. The important facts about the power series

(i) If it converges for no value of x other than x = 0, then it is called *no where convergent.*

(ii) If it converges for all values of x, then it is called *everywhere convergent.*

(iii) If it converges for some values of x and diverges for other values of x ,then the total point of x for which it converges is called its *region of convergence.*

If the power series $\Sigma a_n x^n$ is absolutely convergent through the interval $\left(-\frac{1}{l}, \frac{1}{l}\right)$ and diverges outside of the closed interval $\left[-\frac{1}{l}, \frac{1}{l}\right]$, then the number $\frac{1}{l}$ is called *radius of convergence* whereas interval of convergence is one of the following

$$\left(-\frac{1}{l}, \frac{1}{l}\right), \left(\frac{1}{l}, \frac{1}{l}\right], \left[-\frac{1}{l}, \frac{1}{l}\right) \text{ or } \left[-\frac{1}{l}, \frac{1}{l}\right]$$

In the power series $a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$ general term is $u_n = a_n x^n$ and $u_{n+1} = a_{n+1} x^{n+1}$

$$\therefore \lim_{n \to \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \to \infty} \left| \frac{a_{n+1} x^{n+1}}{a_n x^n} \right| = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} x \right|$$

If $\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = l$, then by the ratio test, the series converges when $lx < 1$.

i.e. $\quad |x| < \frac{1}{l}, \quad -\frac{1}{l} < x < \frac{1}{l}.$

Hence interval of convergence of the power series is $\left(-\frac{1}{l}, \frac{1}{l}\right)$.

Hence the totality of numbers for which a power series converges is called *interval of convergence.*

6.5.1 Convergence of Exponential Series

The exponential series $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots + \frac{x^n}{n!} + \ldots\ldots$ is convergent for all values of x

Here

$$u_n = \frac{x^n}{n!} \quad \text{and} \quad u_{n+1} = \frac{x^{n+1}}{(n+1)!}$$

Now

$$\lim_{n \to \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \to \infty} \left| \frac{x^{n+1}}{(n+1)!} \frac{n!}{x^n} \right|$$

$$= \lim_{n \to \infty} \left| \frac{x}{n+1} \right| = 0 < 1$$

Hence, the series is convergent for all value of x. The interval of convergent is $(-\infty, \infty)$ and radius of convergence is $r = \infty$.

6.5.2 Convergence of Logarithmic Series

Find the radius and interval of convergence of logarithmic series

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots\ldots\ldots\ldots (-1)^n \frac{x^n}{n} + \ldots\ldots\ldots$$

Here

$$u_n = \frac{(-1)^n x^n}{n} \quad \text{and} \quad u_{n+1} = \frac{(-1)^{n+1} x^{n+1}}{(n+1)}$$

$$\therefore \lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{n+1}\frac{n}{x^n}\right| = \lim_{n\to\infty}\left|\frac{n}{n+1}x\right|$$

$$= |x|\lim_{n\to\infty}\left|\frac{1}{1+\frac{1}{n}}\right| = |x|$$

The series is convergent if $|x| < 1$ and divergent if $|x| > 1$.

When $x = 1$, then the series becomes

$1 - \frac{1}{2} + \frac{1}{3} - \dots\dots\dots$, being alternating series , is convergent.

When $x = -1$, then the series becomes

$-\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots\dots\dots\right)$, being p-series with $p = 1$, is divergent.

Hence the series is convergent for $|x| < 1$ and $x = 1$ i.e. $-1 < x \le 1$.

So that the interval of convergence is

$-1 < x \le 1$ and radius of convergent is 1.

Convergence of binomial series

The binomial series

$$1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots\dots\dots$$

$$+ \frac{n(n+1)\dots\dots\dots(n-r+1)}{r!}x^r + \dots\dots\dots\dots\dots \text{converges for } |x| < 1$$

Here $u_r = \frac{n(n-1)\dots\dots\dots(n-r+1)}{r!}x^r$ and

$$u_{r+1} = \frac{n(n-1)\dots\dots\dots(n-r+1)(n-r)}{(r+1)!}x^{r+1}$$

Now

$$\lim_{r\to\infty}\left|\frac{u_{r+1}}{u_r}\right| = \lim_{r\to\infty}\left|\frac{(n-r)}{(r+1)!}x^{r+1}\frac{r!}{x^r}\right|$$

$$= \lim_{r\to\infty}\left|\frac{n-r}{r+1}x\right|$$

$$= \lim_{r\to\infty}\left|\left(\frac{n}{r+1} - \frac{1}{1+\frac{1}{r}}\right)x\right| = |x|$$

Hence the series converges if $|x| < 1$ and diverges if $|x| > 1$.

Convergence of Power series $\sum_{n=0}^{\infty} a_n (x-c)^n$

Let c be a real number. A power series in $(x - c)$ is of the form

$$\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \dots\dots + a_n(x-c)^n + \dots\dots$$

Where each a_n is real number. We can assume that one of the following is true.

(i) The series *converges* for $x - c = 0$ i.e. $x = c$.

(ii) The series is *absolutely convergent* for every values of x.

(iii) The series is *absolutely convergent* if $|x - c| < r$ and *divergent* if $|x - c| > r$.

Worked Out Examples

Ex:1. Find the interval and radius of convergence of the power series

$$\sum \frac{x^n}{\sqrt{n}}$$

2060 Bhadra, B.E.

Solution:

Here $u_n = \frac{x^n}{\sqrt{n}}$ and $u_{n+1} = \frac{x^{n+1}}{\sqrt{n+1}}$

Now

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{\sqrt{n+1}}\frac{\sqrt{n}}{x^n}\right| = \lim_{n\to\infty}\left|\frac{\sqrt{n}}{\sqrt{n+1}}x\right|$$

$$= \lim_{n\to\infty}\left|\frac{x}{\sqrt{1+\frac{1}{n}}}\right| = |x|$$

The series is convergent if $|x| < 1$. i.e. $-1 < x < 1$ and divergent if $|x| > 1$.

When x = 1, then the series becomes

$$\sum_{n=1}^{\infty} \frac{(1)^n}{\sqrt{n}} = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots\ldots\ldots\ldots + \frac{1}{\sqrt{n}} + \ldots\ldots\ldots$$

Here $p = \frac{1}{2} < 1$. Hence, it is divergent p-series.

When x = - 1, then the series becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} = -1 + \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}} + \ldots\ldots\ldots\ldots + \frac{(-1)^n}{\sqrt{n}} + \ldots\ldots\ldots\ldots$$

This is alternating series in which each term is numerically less than its preceding term and $u_n = \frac{1}{\sqrt{n}}$.

$$\lim_{n \to \infty} u_n = 0$$

Therefore, it is convergent.

Hence the power series is convergent for $-1 \le x < 1$, the interval of convergence of power series is $[-1, 1)$ and radius is 1.

Ex. 2. Find the radius of convergence and the interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n.2^n}$ 2070/072 Magh, B. E.

Solution:

Given series is

$$\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n.2^n}$$

Its general term is

$$u_n = \frac{(-1)^n x^n}{n.2^n} \text{ and } u_{n+1} = \frac{(-1)^{n+1} x^{n+1}}{(n+1).2^{n+1}}$$

By the ratio test

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^{n+1}}{(n+1).2^{n+1}} \times \frac{n.2^n}{(-1)^n x^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{x.n}{2(n+1)} \right|$$

$$= \left|\frac{x}{2}\right| \lim_{n\to\infty} \left|\frac{n}{n+1}\right|$$

$$= \left|\frac{x}{2}\right| \lim_{n\to\infty} \left|\frac{n}{n\left(1+\frac{1}{n}\right)}\right|$$

$$= \left|\frac{x}{2}\right| \left|\frac{1}{(1+0)}\right| = \left|\frac{x}{2}\right|$$

$$\therefore \lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{x}{2}\right|$$

If $\left|\frac{x}{2}\right| < 1$, then $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1$ and by the ratio test the series is convergent.

Thus the series is convergent for $\left|\frac{x}{2}\right| < 1$

i.e. $-1 < \frac{x}{2} < 1$.

i.e. $-2 < x < 2$

When x = -2, the given series becomes,

$$\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n.2^n} = \sum_{n=1}^{\infty} \frac{(-1)^n(-2)^n}{n.2^n} = \sum_{n=1}^{\infty} \frac{(-1)^n(-1)^n 2^n}{n.2^n}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots\ldots\ldots\ldots\ldots$$

This is p - series with p = 1, is divergent.

When x = 2 the given series becomes,

$$\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n.2^n} = \sum_{n=1}^{\infty} \frac{(-1)^n 2^n}{n.2^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

$$= -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \ldots\ldots\ldots\ldots\ldots$$

This is alternating series with $u_{n+1} < u_n$ numerically, and $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{n} = 0$.

Hence, the given series is convergent for x = 2.

Therefore, the given series is convergent in the interval $-2 < x \le 2$

So, the interval of convergence is $(2, 2]$.

Radius = 2 with centre = 0.

Ex. 3: Find the interval and radius of convergence of the power series
$$\frac{1}{1.2}(x-2)+\frac{1}{2.3}(x-2)^2+\frac{1}{3.4}(x-2)^3 ......+\frac{1}{n(n+1)}(x-2)^n+......$$

2059Magh, B.E.

Solution:

Here the general term of the series is

$$u_n=\frac{1}{n(n+1)}(x-2)^n \text{ and } u_{n+1}=\frac{1}{(n+1)(n+2)}(x-2)^{n+1}$$

Now
$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|=\lim_{n\to\infty}\left|\frac{(x-2)^{n+1}}{\cancel{(n+1)}(n+2)}\cdot\frac{n\cancel{(n+1)}}{(x-2)^n}\right|$$
$$=\lim_{n\to\infty}\left|\frac{n(x-2)}{(n+2)}\right|=|x-2|\lim_{n\to\infty}\left|\frac{1}{1+\frac{2}{n}}\right|$$
$$=|x-2|.$$

The series is convergent if $|x-2|<1$

i.e. $-1<x-2<1$ and divergent if $|x-2|>1$.

When $x-2=1$ i.e. $x=3$.

The series becomes

$$\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\frac{1}{4.5}+.....+\frac{1}{n(n+1)}+......$$

Here the series can be written as

$$S_n=1-\frac{1}{2}+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+....................+\left(\frac{1}{n}-\frac{1}{n+1}\right)$$

or $$S_n=1-\frac{1}{n+1}$$

$$\therefore \lim_{n\to\infty}S_n=\lim_{n\to\infty}\left(1-\frac{1}{n+1}\right)=1, \text{ which is finite.}$$

Hence the series is convergent for $x-2=1$.

When $x-2=-1$ i.e. $x=1$, then the series becomes

$$-\frac{1}{1.2}+\frac{1}{2.3}-\frac{1}{3.4}+........................$$

$1 \le x \le 3$
$1-2 \le x-2 \le 3-2$
$-1 \le x-2 \le 1$

This is alternating series and hence the series is convergent for $x=1$. Hence the power series converges in the interval $1\le x\le 3$. Therefore the interval of convergence is [1, 3] and radius of convergence is 1.

Ex. 4. Find the radius and interval of convergence of the power series
$$\sum_{n=0}^{\infty}\frac{(-1)^n(x-3)^n}{n+1}$$

2068/070/072 Aswin, B. E.

Solution:

Given series is

$$\sum_{n=0}^{\infty}\frac{(-1)^n(x-3)^n}{n+1}$$

Its general term is,

$$u_n=\frac{(-1)^n(x-3)^n}{n+1}$$

and
$$u_{n+1}=\frac{(-1)^{n+1}(x-3)^{n+1}}{n+2}$$

By the ratio test

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|=\lim_{n\to\infty}\left|\frac{(-1)^{n+1}(x-3)^{n+1}}{n+2}\times\frac{n+1}{(-1)^n(x-3)^n}\right|$$
$$=|x-3|\lim_{n\to\infty}\left|\frac{n+1}{n+2}\right|$$
$$=|x-3|\lim_{n\to\infty}\left|\frac{n\left(1+\frac{1}{n}\right)}{n\left(1+\frac{2}{n}\right)}\right|=|x-3|\,\frac{1+0}{1+0}$$

$$\therefore \lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|=|x-3|$$

If $|x-3|<1$, then $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|<1$ and by the ratio test, the series is convergent.

Thus the series is convergent for $|x-3|<1$

i.e. $-1<x-3<1$.

i.e. $2<x<4$

i.e. $2<x<4$

When $x=2$, the given series becomes

$$\sum_{n=0}^{\infty}\frac{(-1)^n(x-3)^n}{n+1}=\sum_{n=0}^{\infty}\frac{1}{n+1}=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+.........$$

Clearly this is p-series with $p=1$ is divergent.

When $x=4$, the given series becomes

$$\sum_{n=0}^{\infty}\frac{(-1)^n(x-3)^n}{n+1}=\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+.........$$

This is alternating series with $u_{n+1} < u_n$ numerically for all n and

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{n+1} = 0.$$

Thus the series is convergent for x = 4.

Hence the given series is convergent in the interval $2 < x \le 4$

So the interval of convergent is $(2, 4]$ $\quad 2 < x \le 4$

Radius = 1 with centre = 3. $\quad 2-3 \le x \le 4-3 \quad -1 < x \le 1$

Ex. 5. Find the radius and interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$

Solution:

Given series is $\sum_{n=1}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$.

Its general term is $u_n = \frac{(-3)^n x^n}{\sqrt{n+1}}$ and

$$u_{n+1} = \frac{(-3)^{n+1} x^{n+1}}{\sqrt{n+2}}$$

Applying the ratio test

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(-3)^{n+1}x^{n+1}}{\sqrt{n+2}} \times \frac{\sqrt{n+1}}{(-3)^n x^n}\right|$$

$$= 3|x| \lim_{n\to\infty}\left|\frac{\sqrt{n+1}}{\sqrt{n+2}}\right|$$

$$= 3|x| \lim_{n\to\infty}\left|\sqrt{\frac{n\left(1+\frac{1}{n}\right)}{n\left(1+\frac{2}{n}\right)}}\right| = 3|x| \frac{1+0}{1+0}$$

$$\therefore \lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = 3|x|$$

If $3|x| < 1$, then $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1$ and by the ratio test, the series is convergent. Thus the series is convergent for $3|x| < 1$

i. e. $\quad -1 < 3x < 1.$

i. e. $\quad -\frac{1}{3} < x < \frac{1}{3}$

When $x = -\frac{1}{3}$, the given series becomes

$$\sum_{n=1}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \dots\dots\dots$$

This is p - series with $p = \frac{1}{2}$, is divergent.

When $x = \frac{1}{3}$, the given series becomes

$$\sum_{n=1}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+1}} = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \dots\dots\dots$$

This is alternating series with $u_{n+1} < u_n$ numerically for all n and

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{\sqrt{n+1}} = 0.$$

Thus the series is convergent for $x = \frac{1}{3}$.

Hence the given series is convergent in the interval $-\frac{1}{3} < x \le \frac{1}{3}$

So the interval of convergent is $\left(-\frac{1}{3}, \frac{1}{3}\right]$

Radius = $\frac{1}{3}$ with centre = 0.

Exercise - 31

Find the interval of convergence of the power series:

1. $\sum_{n=1}^{\infty} \frac{2^n (x-3)^n}{n+3}$ [2071 Bhadra, B. E.]
2. $\sum_{n=1}^{\infty} \frac{n x^n}{5^n}$
3. $\sum_{n=1}^{\infty} \frac{x^n}{n+4}$
4. $\sum_{n=1}^{\infty} \frac{n^2 x^n}{2^n}$
5. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{\sqrt{n}}$
6. $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \dots\dots$
7. $x - \frac{x^2}{2^2} + \frac{x^3}{3^2} - \frac{x^4}{4^2} + \dots\dots$
8. $\sum_{n=1}^{\infty} \frac{n(x+2)^n}{3^{n+1}}$
9. $\sum_{n=1}^{\infty} \frac{(x+2)^n}{3^n n}$
10. $\sum_{n=1}^{\infty} \frac{nx^n}{(n+1)(n+2)}$ $(x > 0)$

11. $\sum_{n=1}^{\infty} \frac{(-1)^n (x-1)^n}{\sqrt{n}}$ $\boxed{\textit{2069 Bhadra, B.E.}}$

12. $\sum_{n=1}^{\infty} \frac{(n+1)(x-4)^n}{10^n}$

13. $\sum_{n=1}^{\infty} \frac{3^{2n} (x-2)^n}{n+1}$

14. $\sum_{n=1}^{\infty} \frac{n^2 (x+4)^n}{2^{3n}}$

15. $\sum_{n=1}^{\infty} \frac{(-1)^n (x+1)^n}{2^n}$ $\boxed{\textit{2066 Kartik, B.E.}}$

16. $1 + \frac{1}{5}x + \frac{2}{5^2}x^2 + \frac{3}{5^3}x^3 + \ldots$

17. $\sum_{n=0}^{\infty} \frac{3^n x^n}{(n+1)^2}$ $\boxed{\textit{2065 Kartik, B.E.}}$

18. $\sum_{n=0}^{\infty} \frac{x^n}{2^n}$ $\boxed{\textit{2060 Chaitra, B.E}}$

19. $1 + 2x + 3x^2 + 4x^3 + \ldots\ldots\ldots$

20. $1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^4}{4} + \ldots\ldots\ldots\ldots\ldots$

21. $\sum \frac{nx^n}{(n+1)^2}$

22. $\sum \frac{2^n x^n}{n!}$ $\boxed{\textit{2069 Bhadra, B. E.}}$

23. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} (x-1)^n}{n}$

24. $\sum_{n=1}^{\infty} \frac{n^n x^n}{n!}$

Answers

1. $\left[\frac{5}{2}, \frac{7}{2}\right)$	2. (-5, 5)	3. [-1, 1)	4. (-2, 2)
5. (- 1, 1]	6. (-1, 1]	7. [- 1, 1]	8. (-5, 1)
9. [- 5, 1)	10. [- 1, 1)	11. (0, 2]	12. (-6, 14)
13. $\left[\frac{17}{9}, \frac{19}{9}\right)$	14. (-12, 4)	15. (-3, 1)	16. (-5, 5)
17. $\left[\frac{-1}{3}, \frac{1}{3}\right]$	18. (-2,2)	19. (-1, 1); 1	20. (-1, 1]; 1
21. [-1, 1);1	22. $(-\infty, \infty)$; ∞	23. (0, 2];1	24. $\frac{1}{e}$

Bibliography

Bali, N.P. and Iyengar, N. Ch. Naryana (2004): *A Text Book of Engineering Mathematics* (6th Edition)

ERWIN KREYSZIG, JOHN WILEY & SONS,INC.
New york Chichester Brisbane Tornoto Singapore:
Advanced Engineering Mathematics (8th Edition)

Dass, H.K.: *Advanced Engineering Mathematics*. S. Chand & Company Ltd
Laxmi Publication (P) Ltd., New Delhi

Sastry, S.S. (2002): *Engineering Mathematics* Vol. I (2nd Edition) Prentice, Hall of India Private Limited New Delhi - 110001

SPIEGEL, MURRAY R. (1984): *Theory and Problems of Advanced Calculus*, Mc Graw-Hill Book Company.

Thomas George B. Jr. and Finney Ross L. (2003): *Calculus and Analytic Geometry* (9th Edition). Pearson Education Pvt. Ltd. Singapore.

2nd G