A Complete Solution of

ENGINEERING MATHEMATICS

Volume - II

For B. E. First Year / Second Part

Civil, Electronics, Computer, Electrical and Mechanical, Agriculture, Industrial and B. Arch

with Examination Questions & Solutions

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Calculus of Several Variables

1

Partial derivative

Let u = f(x, y) be function of two independent variables x and y. If we keep y constant and x varies then u becomes a function of x only. The derivative of u with respect to x, keeping, y as constant is called partial derivative of u w.r.t. x and it is denoted as

$$\frac{\partial u}{\partial x}, \frac{\partial f}{\partial x}, f_x(x, y); f_x \text{ . etc.}$$

$$\text{Then } \frac{\partial u}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x, y)}{\Delta x}$$

The process of finding the partial differential coefficient of u w.r.t. 'x' is that of ordinary differentiation but with the only difference that we treat y as constant.

Similarly, the partial derivative of u w.r.t. y keeping x as constant is denoted by

$$\frac{\partial u}{\partial y}, \frac{\partial f}{\partial y}, f_y(x, y), f_y \text{ etc.}$$

$$\text{Then, } \frac{\partial u}{\partial y} = \lim_{\Delta x \to 0} \frac{f(x, y+\Delta y) - f(x, y)}{\Delta y}$$

Partial derivative of higher orders

Let u = f(x, y) then $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ being the function of x and y and so these can be differentiated further partially with respect to x and y.

Symbolically,

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial x^2} \quad \text{or, } \frac{\partial^2 f}{\partial x^2} \quad \text{or, } f_{xx}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial y \partial x} \quad \text{or, } \frac{\partial^2 f}{\partial y \partial x} \quad \text{or, } f_{yx}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial^2 u}{\partial y^2} \quad \text{or, } \frac{\partial^2 f}{\partial y^2} \quad \text{or, } f_{yy}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial^2 u}{\partial x \partial y} \quad \text{or, } \frac{\partial^2 f}{\partial x \partial y} \quad \text{or, } f_{xy}$$

Note : $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$

Homogeneous function

A function $u = f(x, y)$ of two independent variables x and y is said to be homogeneous function of degree n. If it can be expressed either of the form

$$u = x^n \phi\left(\frac{y}{x}\right) \text{ or } u = y^n \phi\left(\frac{x}{y}\right)$$

For example,

Consider the function

$$f(x, y) = ax^2 + 2hxy + by^2$$

$$= x^2\left[a + 2h\left(\frac{y}{x}\right) + b\frac{y^2}{x^2}\right]$$

$$= x^2\left[a\left(\frac{y}{x}\right)^0 + 2h\left(\frac{y}{x}\right) + b\left(\frac{y}{x}\right)^2\right]$$

$$= x^2\phi\left(\frac{y}{x}\right)$$

Hence $f(x, y) = ax^2 + 2hxy + by^2$ is homogeneous function of degree two.

Euler's theorem on homogeneous function of two independent variables.

Statement

If $u = f(x, y)$ be a homogeneous function of two independent variables x and y of degree n then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\,u.$$

Euler's theorem on homogeneous function of three independent variables.

Statement

If $u = f(x, y, z)$ be a homogeneous function of three independent variables x, y, and z of degree n then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu.$$

Total differential coefficient

1. If $u = f(x, y)$ and $x = \phi(t)$, $y = \psi(t)$.
   Then,
   $$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$$
2. If $u = f(x, y)$. The the total differential of u is defined as
   $$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy.$$
3. If $u = f(x, y) = 0$ be implicity function then
   $$\frac{\partial y}{\partial x} = \frac{f_x}{f_y},\ f_y \neq 0$$

Increments

If $u = f(x, y)$ be a function of two variables x and y and let Δx and Δy be increments of x and y respectively, then the increment Δu of $u = f(x, y)$ is

$$\Delta u = f(x + \Delta x, y + \Delta y) - f(x, y).$$

Maxima and minima

A function $f(x, y)$ is said to have a maximum or minimum at (a, b) according as

$$f(a + h, b + k) < f(a, b)$$
$$\text{or, } f(a + h, b + k) > f(a, b)$$

for all positive or negative values of h and k.

- **Conditions for maxima and minima of the function f(x, y) of two independent variable at a point (a, b)**

1. For maximum value at (a, b)
   (a) $f_x = 0, f_y = 0$
   (b) $f_{xx} f_{yy} - (f_{xy})^2 > 0$
   and $f_{xx} < 0$
2. For minimum value at (a, b)
   (a) $f_x = 0, f_y = 0$
   (b) $f_{xx} f_{yy} - (f_{xy})^2 > 0$ and $f_{xx} > 0$
3. No maxima. no minima or saddle points
   (a) $f_x = 0,\ f_y = 0$
   (b) $f_{xx} f_{yy} - (f_{xy}) < 0$
4. No information
   (a) $f_x = 0, f_y = 0$
   (b) $f_{xx} f_{yy} - (f_{xy})^2 = 0$

- **Condition for maxima and minima of the function f(x, y, z) of three independent variable at a point (a, b, c)**

1. For maximum value at (a, b, c)
   (a) $f_x = 0, f_y = 0, f_z = 0$
   (b) $f_{xx} < 0, f_{xx} f_{yy} - (f_{xy})^2 > 0$

and

$$\begin{vmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{vmatrix} < 0$$

2. For minimum value at (a, b, c)

(a) $f_x = 0, f_y = 0, f_z = 0$

(b) $f_{xx} > 0, f_{xx} f_{yy} - (f_{xy})^2 > 0$

and $\begin{vmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{vmatrix} > 0$

Exercise - 1

1. Evaluate f_x and f_y of the following function f(x, y).

(i) $x^2y - x \sin xy$

Solution: Let $f(x, y) = x^2y - x \sin (xy)$ ..... (i)

Differentiating (i) p. w. r. t. x,

$\therefore f_x = 2xy - xy \cos (xy) - 1. \sin (xy)$

$= 2xy - xy \cos (xy) - \sin (xy)$ Ans.

Differentiating (i) p. w. r. t. y

$\therefore f_y = x^2 - x. x \cos (xy)$

$= x^2 \{1 - \cos (xy)\}$ Ans.

(ii) $\log (x^2 + y^2)$

Solution: Let $f(x, y) = \log (x^2 + y^2)$ ..... (i)

Differentiating (i) partially w. r. t. x

$$\therefore f_x = \frac{1}{(x^2+y^2)} . 2x = \frac{2x}{(x^2+y^2)} \text{ Ans.}$$

Differentiating (i) partially w. r. t. y

$$\therefore f_y = \frac{1}{(x^2+y^2)} . 2y = \frac{2y}{(x^2+y^2)} \text{ Ans.}$$

(iii) $\tan^{-1} \dfrac{x^2+y^2}{x+y}$

Solution: Let $f(x, y) = \tan^{-1} \dfrac{x^2+y^2}{x+y}$ ..... (i)

Differentiating (i) p. w. r. t. 'x'

$$\therefore f_x = \frac{1}{1+\left(\frac{x^2+y^2}{x+y}\right)^2} . \frac{(x+y) . 2x - (x^2+y^2) . 1}{(x+y)^2}$$

$$= \frac{x^2+2xy-y^2}{(x+y)^2 + (x^2+y^2)^2} \text{ Ans.}$$

Differentiating (i) p. w. r. t. 'y'

$$\therefore f_y = \frac{1}{1+\left(\frac{x^2+y^2}{x+y}\right)^2} . \frac{(x+y) . 2y - (x^2+y^2) . 1}{(x+y)^2}$$

$$= \frac{y^2+2xy-x^2}{(x+y)^2 + (x^2+y^2)^2} \text{ Ans.}$$

(iv) $\dfrac{x^2-y^2}{x^2+y^2}$

Solution: Lat $f(x, y) = \dfrac{x^2-y^2}{x^2+y^2}$ ..... (i)

Differentiating (i) p. w. r. t. 'x'

$$\therefore f_x = \frac{(x^2+y^2) . 2x - (x^2-y^2) . 2x}{(x^2+y^2)^2}$$

$$= \frac{2x^3+2xy^2-2x^3+2xy^2}{(x^2+y^2)^2} = \frac{4xy^2}{(x^2+y^2)^2} \text{ Ans.}$$

Differentiating (i) p. w. r. t. 'x'

$$\therefore f_y = \frac{(x^2+y^2)(-2y) - (x^2-y^2) . 2y}{(x^2+y^2)^2}$$

$$= \frac{-2x^2y-2y^3-2x^2y+2y^3}{(x^2+y^2)^2} = \frac{-4x^2y}{(x^2+y^2)^2} \text{ Ans.}$$

(v) $\sin^{-1} (ax + by)$

Solution : Let $f(x, y) = \sin^{-1} (ax + by)$ ..... (i)

Differentiating (i) p. w. r. 'x'

$$\therefore f_x = \frac{1}{\sqrt{1-(ax+by)^2}} . a = \frac{a}{\sqrt{1-(ax+by)^2}} \text{ Ans.}$$

Differentiating (i) p. w. r. t. 'y'

$$\therefore f_y = \frac{1}{\sqrt{1-(ax+by)^2}} \cdot b = \frac{b}{\sqrt{1-(ax+by)^2}} \text{ Ans.}$$

2. Find the partial derivative of second orders of the following functions f(x, y).

(i) $x^3 + 3x^2y + 3xy^2 + y^3$

Solution: Let $f(x, y) = x^3 + 3x^2y + 3xy^2 + y^3$ ..... (i)

Differentiating (i) partially we get,

$\therefore f_x = 3x^2 + 6xy + 3y^2$

$\therefore f_{xx} = 6x + 6y = 6(x + y)$ Ans.

$\therefore f_{yx} = 6x + 6y = 6(x + y)$ Ans.

$\therefore f_y = 3x^2 + 6xy + 3y^2$

$\therefore f_{yy} = 6x + 6y = 6(x + y)$ Ans.

$\therefore f_{xy} = 6x + 6y = 6(x + y)$ Ans.

(ii) $e^{x^2+xy+y^2}$

Solution: Let $f(x, y) = e^{x^2+xy+y^2}$ ..... (i)

Differentiating (i) partially we get,

$\therefore f_x = e^{x^2+xy+y^2} \cdot (2x + y)$

$\therefore f_{xx} = e^{x^2+xy+y^2} \cdot 2 + (2x + y)\, e^{x^2+xy+y^2} \cdot (2x + y)$

$= e^{x^2+xy+y^2} \{2 + (2x + y)^2\}$ Ans.

$\therefore f_{yx} = e^{x^2+xy+y^2} \cdot 1 + (2x + y)\, e^{x^2+xy+y^2} \cdot (x + 2y)$

$= e^{x^2+xy+y^2} \{1 + (2x + y)(x + 2y)\}$ Ans.

$\therefore f_y = e^{x^2+xy+y^2} \cdot (x + 2y)$

$\therefore f_{yy} = e^{x^2+xy+y^2} \cdot 2 + e^{x^2+xy+y^2} \cdot (x + 2y)(x + 2y)$

$= e^{x^2+xy+y^2} \{2 + (x + 2y)^2\}$ Ans.

$\therefore f_{xy} = e^{x^2+xy+y^2} \cdot 1 + (x + 2y)\, e^{x^2+xy+y^2} (2x + y)$

$= e^{x^2+xy+y^2} \{1 + (x + 2y)(2x + y)\}$ Ans.

(iii) $\log(x^2y + xy^2)$

Solution: Let $f(x, y) = \log(x^2y + xy^2)$ ..... (i)

Differentiating (i) partially we get,

$$\therefore f_x = \frac{1}{(x^2y+xy^2)} \cdot (2xy + y^2)$$

$$= \frac{y(2x+y)}{xy(x+y)} = \frac{(2x+y)}{x(x+y)} = \frac{2x+y}{x^2+xy}$$

$$\therefore f_{xx} = \frac{(x^2+xy) \cdot 2 - (2x+y)(2x+y)}{x^2(x+y)^2}$$

$$= \frac{2x^2+2xy-4x^2-4xy-y^2}{x^2(x+y)^2}$$

$$= \frac{-2x^2-2xy-y^2}{x^2(x+y)^2} = \frac{-x^2-(x^2+2xy+y^2)}{x^2(x+y)^2}$$

$$\therefore f_{xx} = -\frac{1}{x^2} - \frac{1}{(x+y)^2} \text{ Ans.}$$

$$\therefore f_{yx} = \frac{(x^2+xy) \cdot 1 - (2x+y) \cdot x}{x^2(x+y)^2}$$

$$= \frac{x^2+xy-2x^2-xy}{x^2(x+y)^2} = \frac{-x^2}{x^2(x+y)^2} \quad \therefore f_{yx} = -\frac{1}{(x+y)^2}$$

Ans.

$$\therefore f_y = \frac{1}{(x^2y+xy^2)} \cdot (x^2 + 2xy)$$

$$= \frac{x(x+2y)}{xy(x+y)} = \frac{(x+2y)}{y(x+y)}$$

$$\therefore f_{yy} = \frac{(xy+y^2) \cdot 2 - (x+2y) \cdot (x+2y)}{y^2(x+y)^2}$$

$$= \frac{2xy+2y^2-x^2-4xy-4y^2}{y^2(x+y)^2} = \frac{-x^2-2xy-2y^2}{y^2(x+y)^2}$$

$$= \frac{-(x^2+2xy+y^2)-y^2}{y^2(x+y)^2} = \frac{-(x+y)^2-y^2}{y^2(x+y)^2}$$

$$\therefore f_{yy} = -\frac{1}{(x+y)^2} - \frac{1}{y^2} \text{ Ans.}$$

$$\therefore f_{xy} = \frac{(xy+y^2) \cdot 1 - (x+2y) \cdot y}{y^2(x+y)^2}$$

$$= \frac{xy+y^2-xy-2y^2}{y^2(x+y)^2} = \frac{-y^2}{y^2(x+y)^2} \quad \therefore f_{xy} = \frac{1}{(x+y)^2} \text{ Ans.}$$

(iv) $\tan^{-1}\dfrac{2xy}{x^2-y^2}$

Solution: Let $f(x, y) = \tan^{-1}\dfrac{2xy}{x^2-y^2}$ ..... (i)

Differentiating (i) partially we get,

$$\therefore f_x = \frac{1}{1+\left(\frac{2xy}{x^2-y^2}\right)^2}\cdot\frac{(x^2-y^2).2y-2xy.2x}{(x^2-y^2)^2}$$

$$= \frac{(x^2-y^2)^2}{(x^2-y^2)^2+4x^2y^2}\cdot\frac{2x^2y-2y^3-4x^2y}{(x^2-y^2)^2}$$

$$= \frac{-2y^3-2x^2y}{(x^2+y^2)^2} = \frac{-2y(x^2+y^2)}{(x^2+y^2)^2}$$

$$\therefore f_x = \frac{-2y}{(x^2+y^2)^2} \text{ Ans.}$$

$$\therefore f_{xx} = -\frac{-2y\,.\,2x}{(x^2+y^2)^2} = \frac{4xy}{(x^2+y^2)^2} \text{ Ans.}$$

$$\therefore f_{yx} = \frac{-2\left[(x^2+y^2)\,.\,1-y\,.\,2y\right]}{(x^2+y^2)^2}$$

$$= -2\,\frac{\left[x^2+y^2-2y^2\right]}{(x^2+y^2)^2} = -2\,\frac{(x^2-y^2)}{(x^2+y^2)^2}$$

$$\therefore f_{yx} = \frac{2(y^2-x^2)}{(x^2+y^2)^2} \text{ Ans.}$$

$$\therefore f_y = \frac{1}{1+\left(\frac{2xy}{x^2-y^2}\right)^2}\cdot\frac{(x^2-y^2)\,.\,2x-2xy\,(-2y)}{(x^2-y^2)^2}$$

$$= \frac{(x^2-y^2)^2}{(x^2-y^2)^2+4x^2y^2}\cdot\frac{2x^3-2xy^2+4xy^2}{(x^2-y^2)^2}$$

$$= \frac{2x^3+2xy^2}{(x^2+y^2)^2} = \frac{2x(x^2+y^2)}{(x^2+y^2)^2}$$

$$\therefore f_y = \frac{2x}{(x^2+y^2)}$$

$$\therefore f_{yy} = \frac{2x\,.\,-2y}{(x^2+y^2)^2} = -\frac{4xy}{(x^2+y^2)^2} \text{ Ans.}$$

Similarly, $f_{xy} = \dfrac{2(y^2-x^2)}{(x^2+y^2)^2}$ Ans.

3. (i) If $u = \sin^{-1}\dfrac{x}{y} + \tan^{-1}\dfrac{y}{x}$, then show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 0$.

Solution: Given $u = \sin^{-1}\dfrac{x}{y} + \tan^{-1}\dfrac{y}{x}$ ..... (i)

Differentiating (i) partially we get,

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}}\cdot\frac{1}{y} + \frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot -\frac{y}{x^2}$$

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{y^2-x^2}} - \frac{y}{x^2+y^2} \text{ ..... (ii)}$$

Differentiating (i) p. w. r. t. 'y'

$$\frac{\partial u}{\partial y} = \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}}\cdot\left(-\frac{x}{y^2}\right) + \frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\frac{1}{x}$$

$$\frac{\partial u}{\partial y} = \frac{-x}{y\sqrt{y^2-x^2}} + \frac{x}{x^2+y^2} \text{ ..... (iii)}$$

Multiplying equation (ii) and (iii) by x & y resp. we get,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{x}{\sqrt{y^2-x^2}} - \frac{xy}{x^2+y^2} - \frac{xy}{y\sqrt{y^2-x^2}} + \frac{xy}{(x^2+y^2)}$$

$$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0 \text{ Proved.}$$

(ii) If $u = x^2y + y^2z + z^2x$, then show $u_x + u_y + u_z = (x + y + z)^2$.

Solution: Given, $u = x^2y + y^2z + z^2x$ ..... (i)

Differentiating (i) partially we get,

$\therefore u_x = 2xy + z^2$ ..... (ii)

$\therefore u_y = x^2 + 2yz$ ..... (iii)

$\therefore u_z = y^2 + 2zx$ ..... (iv)

Adding (ii), (iii) & (iv)

$u_x + u_y + u_z = 2xy + z^2 + x^2 + 2yz + y^2 + 2zx$

$= x^2 + y^2 + z^2 + 2xy + 2yz + 2zx$

$\therefore u_x + y_y + u_z = (x + y + z)^2$ Proved.

(iii) If $u = \log \dfrac{x^2+y^2}{x+y}$

Then prove that $x\dfrac{\partial^2 u}{\partial x} + y\dfrac{\partial u}{\partial y} = 1$ **(BE 2070 Bhadra)**

Solution: Given, $u = \log \dfrac{x^2+y^2}{x+y}$ ..... (i)

Differentiating (i) partially w. r. to x & y we get,

$$\frac{\partial u}{\partial x} = \frac{1}{\dfrac{x^2+y^2}{x+y}} \cdot \frac{(x+y)\,.\,2x - (x^2+y^2)}{(x+y)^2}$$

$$= \frac{2x^2+2xy-x^2-y^2}{(x+y)\,(x^2+y^2)}$$

$$\therefore \frac{\partial u}{\partial x} = \frac{x^2+2xy-y^2}{(x+y)\,(x^2+y^2)} \text{ ..... (ii)}$$

$$\frac{\partial u}{\partial y} = \frac{1}{\dfrac{x^2+y^2}{x+y}} \cdot \frac{(x+y)\,.\,2y - (x^2+y^2)}{(x+y)^2}$$

$$= \frac{2xy+2y^2-x^2-y^2}{(x+y)\,(x^2+y^2)}$$

$$\therefore \frac{\partial u}{\partial x} == \frac{y^2+2xy-x^2}{(x+y)\,(x^2+y^2)} \text{ ..... (iii)}$$

Multiplying equation (ii) and (iii) by x & y resp. and adding

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{x^3+2x^2y-xy^2}{(x+y)\,(x^2+y^2)} + \frac{y^3+2xy^2-yx^2}{(x+y)\,(x^2+y^2)}$$

$$= \frac{x^3+2x^2y-xy^2+y^3+2xy^2-yx^2}{(x+y)\,(x^2+y^2)}$$

$$= \frac{x^3+y^3+x^2y+y^2x}{(x+y)\,(x^2+y^2)} = \frac{(x+y)\,(x^2-xy+y^2)+xy(x+y)}{(x+y)\,(x^2+y^2)}$$

$$= \frac{(x+y)\,(x^2-xy+y^2+xy)}{(x+y)\,(x^2+y^2)} = \frac{(x^2+y^2)}{(x^2+y^2)}$$

$$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 1 \text{ proved.}$$

(iv) If $u = \log(x^2 + y^2 + z^2)$, show that $x\,u_{yz} = y\,u_{zx} = z\,u_{xy}$

Solution: Given, $u = \log(x^2 + y^2 + z^2)$ ..... (i)

Differentiating (i) partially we get,

$$\therefore u_x = \frac{1}{(x^2+y^2+z^2)}\,.\,2x$$

$$\therefore u_{zx} = \frac{-1}{(x^2+y^2+z^2)^2}\,.\,2z\,.\,2x = \frac{-4zx}{(x^2+y^2+z^2)^2} \text{ ..... (ii)}$$

$$\therefore u_y = \frac{1}{(x^2+y^2+z^2)}\,.\,2y$$

$$\therefore u_{xy} = \frac{-1}{(x^2+y^2+z^2)^2}\,.\,2x\,.\,2y = \frac{-4xy}{(x^2+y^2+z^2)^2} \text{ ..... (iii)}$$

$$\therefore u_z = \frac{1}{(x^2+y^2+z^2)}\,.\,2z$$

$$\therefore u_{yz} = \frac{-1}{(x^2+y^2+z^2)^2}\,.\,2y\,.\,2z = \frac{-4yz}{(x^2+y^2+z^2)^2} \text{ ..... (iv)}$$

Multiplying equation (iv), (ii) and (iii) by x, y, z resp. then we get, $x\,u_{yz} = y\,u_{zx} = z\,u_{xy}$ Proved.

4. If $u = \log(x^3 + y^3 + z^3 - 3xyz)$, then show that,

(i) $$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3}{x+y+z}$$

Solution: Given, $u = \log(x^3 + y^3 + z^3 - 3xyz)$ ..... (i)

Differentiating (i) partially w. r. t. x, y, & z resp. we get,

$$\therefore \frac{\partial u}{dx} = \frac{1}{(x^3+y^3+z^3-3xyz)}\,.\,(3x^2-3yz)$$

$$= \frac{3(x^2 - yz)}{(x^3 + y^3 + z^3 - 3xyz)} \quad \text{..... (ii)}$$

$$\therefore \frac{\partial u}{dy} = \frac{3(y^2 - zx)}{(x^3 + y^3 + z^3 - 3xyz)} \quad \text{..... (iii)}$$

$$\therefore \frac{\partial u}{dz} = \frac{3(z^2 - xy)}{(x^3 + y^3 + z^3 - 3xyz)} \quad \text{..... (iv)}$$

Adding (ii), (iii) and (iv) we get,

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3(x^2 - yz + y^2 - zx + z^2 - xy)}{(x^3 + y^3 + z^3 - 3xyz)}$$

$$= \frac{3(x^2 + y^2 + z^2 - xy - yz - zx)}{(x + y + z)\ (x^2 + y^2 + z^2 - xy - yz - zx)}$$

$$= \frac{3}{x + y + z} \quad \text{Proved.}$$

(ii) $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial y^2} = -\frac{3}{(x + y + z)^2}$$

Solution: Given, $u = \log (x^3 + y^3 + z^3 - 3xyz)$ ..... (i)

Differentiating (i) partially w. r. t. x, y, & z resp. we get,

$$\therefore \frac{\partial u}{dx} = \frac{1}{(x^3 + y^3 + z^3 - 3xyz)} \cdot (3x^2 - 3yz)$$

$$= \frac{3(x^2 - yz)}{(x^3 + y^3 + z^3 - 3xyz)} \quad \text{..... (ii)}$$

$$\therefore \frac{\partial u}{dy} = \frac{3(y^2 - zx)}{(x^3 + y^3 + z^3 - 3xyz)} \quad \text{..... (iii)}$$

$$\therefore \frac{\partial u}{dz} = \frac{3(z^2 - xy)}{(x^3 + y^3 + z^3 - 3xyz)} \quad \text{..... (iv)}$$

$$\therefore \frac{\partial^2 u}{dx^2} = \frac{(x^3 + y^3 + z^3 - 3xyz) \cdot 6x - 3(x^2 - yz)\ 3(x^2 - yz)}{(x^3 + y^3 + z^3 - 3xyz)^2}$$

$$= \frac{6x^4 + 6xy^3 + 6xz^3 - 18x^2yz - 9x^4 + 18x^2yz - 9y^2z^2}{(x^3 + y^3 + z^3 - 3xyz)^2}$$

$$= \frac{-3x^4 + 6xy^3 + 6xz^3 - 8y^2z^2}{(x^3 + y^3 + z^3 - 3xyz)^2}$$

$$= -\frac{3(x^4 - 2xy^3 - 2xz^3 + 3y^2z^2)}{(x^3 + y^3 + z^3 - 3xyz)^2} \quad \text{..... (v)}$$

Similarly,

$$\therefore \frac{\partial^2 u}{dy^2} = -\frac{3(y^4 - 2xy^3 - 2yz^3 + 3z^2x^2)}{(x^3 + y^3 + z^3 - 3xyz)^2} \quad \text{..... (vi)}$$

$$\therefore \frac{\partial^2 u}{dz^2} = -\frac{3(z^4 - 2zx^3 - 2zy^3 + 3x^2y^2)}{(x^3 + y^3 + z^3 - 3xyz)^2} \quad \text{..... (vii)}$$

Adding (v), (vi) and (vii) we get,

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{-3\ (x^4 - 2xy^3 - 2xz^3 + 3y^2z^2 \quad y^4 - 2yx^3 - 2yz^3 + 3z^2x^2 \quad z^4 - 2zx^3 - 2zy^3 + 3x^2y^2\ )}{(x^3 + y^3 + z^3 - 3xyz)^2}$$

$$= \frac{-3\ (x^4 + y^4 + z^4 - 2xy^3 - 2xz^3 + 3y^2z^2 - 2yz^3 - 2yz^3 + 3z^2x^2 - 2zx^3 - 3zy^3 + 3x^2y^2)}{(x + y + z)^2\ (x^2 + y^2 + z^2 - xy - yz - zx)^2}$$

$$= -\frac{3(x^2 + y^2 + z^2 - xy - yz - zx)^2}{(x + y + z)^2\ (x^2 + y^2 + z^2 - xy - yz - zx)^2}$$

$$= -\frac{3}{(x + y + z)^2} \quad \text{Proved.}$$

(iii) $$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = -\frac{9}{(x + y + z)^2}$$

Solution: Now, L. H. S. $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right) u$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right)$$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right) \frac{3}{(x + y + z)} \quad \text{[from (i)]}$$

$$= -\frac{3}{(x+y+z)^2} - \frac{3}{(x+y+z)^2} - \frac{3}{(x+y+z)^2}$$

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x+y+z)^2} \text{ Proved.}$$

5. (i) If $u = f\left(\frac{y}{x}\right)$, then show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$

Solution: Given, $u = f\left(\frac{y}{x}\right)$ ..... (i)

Differentiating (i) partially w. r. t. x we get,

$$\frac{\partial u}{\partial x} = f'\left(\frac{y}{x}\right) \cdot \left(-\frac{y}{x^2}\right)$$

$$= -\frac{y}{x^2} f'\left(\frac{y}{x}\right) \text{ ..... (ii)}$$

Differentiating (i) partially w. r. t. y we get,

$$\frac{\partial u}{\partial y} = f'\left(\frac{y}{x}\right) \cdot \left(\frac{1}{x}\right)$$

$$= -\frac{1}{x} f'\left(\frac{y}{x}\right) \text{ ..... (iii)}$$

Multiplying equation (ii) by x and equation (iii) by y and adding we get,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -\frac{y}{x} \cdot f'\left(\frac{y}{x}\right) + \frac{y}{x} f'\left(\frac{y}{x}\right)$$

$$\therefore\ x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0 \text{ Proved.}$$

(ii) If $V = z\tan^{-1}\left(\frac{y}{x}\right)$, then show that

$$x\frac{\partial^2 V}{\partial x^2} + y\frac{\partial^2 V}{\partial y^2} + z\frac{\partial^2 V}{\partial z^2} = 0$$

Solution: Given, $V = z\tan^{-1}\left(\frac{y}{x}\right)$ ..... (i)

Differentiating (i) partially w. r. t. x we get,

$$\therefore\ \frac{\partial V}{\partial x} = z \cdot \frac{1}{1+\frac{y^2}{x^2}}\left(-\frac{y}{x^2}\right)$$

$$= \frac{-zy}{x^2+y^2}$$

$$\therefore\ \frac{\partial^2 V}{\partial x^2} = \frac{2xyz}{(x^2+y^2)^2} \text{ ..... (ii)}$$

Differentiating (i) partially w. r. t. y we get,

$$\therefore\ \frac{\partial V}{\partial y} = z \cdot \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \left(\frac{1}{x}\right) = \frac{zx}{x^2+y^2}$$

$$\therefore\ \frac{\partial^2 V}{\partial y^2} = \frac{-2xyz}{(x^2+y^2)^2} \text{ ..... (iii)}$$

Differentiating (i) partially w. r. t. 'z' we get,

$$\therefore\ \frac{\partial V}{\partial z} = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\therefore\ \frac{\partial^2 V}{\partial z^2} = 0 \text{ ..... (iv)}$$

Adding (ii), (iii) and (iv) we get,

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = \frac{2xyz}{(x^2+y^2)^2} + \frac{-2xyz}{(x^2+y^2)^2} + 0$$

$$\therefore\ \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \text{ Proved.}$$

(iii) If $u = \tan(y + ax) - (y - ax)^{3/2}$ then show that,

$$\frac{\partial^2 u}{\partial x^2} = a^2 \frac{\partial^2 u}{\partial y^2}.$$

Solution: Given, $u = \tan(y + ax) - (y - ax)^{3/2}$ ..... (i)

Differentiating (i) partially w. r. t. 'x' we get,

$$\therefore\ \frac{\partial u}{\partial x} = a \cdot \sec^2(y + ax) + \frac{3}{2} a (y - ax)^{\frac{1}{2}}$$

$$\therefore\ \frac{\partial^2 u}{\partial x^2} = 2a^2 \sec^2(y + ax)\tan(y + ax) - \frac{3a^2}{4}(y - ax)^{-\frac{1}{2}} \text{ ... (ii)}$$

Differentiating (i) partially w. r. t. 'y' we get,

$$\therefore\ \frac{\partial u}{\partial y} = \sec^2(y + ax) - \frac{3}{2}(y - ax)^{\frac{1}{2}}$$

$$\therefore\ \frac{\partial^2 u}{\partial y^2} = 2\sec^2(y + ax)\tan(y + ax) - \frac{3}{4}(y - ax)^{-\frac{1}{2}} \text{ ... (iii)}$$

Multiplying equation (iii) by a^2 and subtracting (iii) from (ii) we get

$$\frac{\partial^2 u}{\partial x^2} - a^2 \frac{\partial^2 u}{\partial y^2} = 0 \Rightarrow \frac{\partial^2 u}{\partial x^2} = a^2 \frac{\partial^2 u}{\partial y^2} \quad \text{Proved.}$$

(iv) If $z = \phi(x + ay) + \psi(x - ay)$ then show that $a^2 \frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial y^2}$.

Solution: Given, $z = \phi(x + ay) + \psi(x - ay)$ ..... (i)

Differentiating (i) partially w. r. t. 'x' we get,

$$\therefore \frac{\partial z}{\partial x} = \phi'(x + ay) + \psi'(x - ay)$$

$$\therefore \frac{\partial^2 z}{\partial x^2} = \phi''(x + ay) + \psi''(x - ay) \text{ ..... (ii)}$$

Differentiating (i) partially w. r. t. 'y' we get,

$$\therefore \frac{\partial z}{\partial y} = a\,\phi'(x + ay) - a\,\psi'(x - ay)$$

$$\therefore \frac{\partial^2 z}{\partial y^2} = a^2 \phi''(x + ay) + a^2 \psi''(x - ay) \text{ ..... (iii)}$$

Multiplying equation (ii) by a^2 and subtracting (iii) from (ii) we get,

$$a^2 \frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = 0 \Rightarrow a^2 \frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial y^2} \quad \text{Proved.}$$

6. (i) If $u = f(xyz)$ show that $x^2 u_{xy} = y^2 u_{yy} = z^2 u_{zz}$.

Solution: Given $u = f(xyz)$ ..... (i)

Differentiating (i) partially w. r. t. 'x' we get,

$\therefore u_x = yz\, f'(xyz)$

$\therefore u_{xx} = y^2 z^2 f''(xyz)$ ..... (ii)

Similarly,

$u_{yy} = x^2 z^2 f''(xyz)$ ..... (iii)

and $u_{zz} = x^2 y^2 f''(xyz)$ ..... (iv)

Multiplying equation (ii), (iii) and (iv) by x^2, y^2 and z^2 resp. we get,

$x^2 u_{xx} = y^2 u_{yy} = z^2 u_{zz}$ Proved.

(ii) If $u = \frac{1}{\sqrt{x^2 + y^2 + z^2}}$ proved that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

Solution: Given, $u = \frac{1}{\sqrt{x^2 + y^2 + z^2}}$ ..... (i)

Differentiating (i) partially w. r. t. 'x' we get,

$$\therefore \frac{\partial^2 u}{\partial x^2} = \frac{-1}{2}\left(x^2 + y^2 + z^2\right)^{-\frac{3}{2}} . 2x$$

$$= \frac{-x}{\left(x^2 + y^2 + z^2\right)^{\frac{3}{2}}}$$

$$\therefore \frac{\partial^2 u}{\partial x^2} = \frac{\left(x^2 + y^2 + z^2\right)^{\frac{3}{2}} .1 - x \,.\, \frac{3}{2}\left(x^2 + y^2 + z^2\right)^{\frac{1}{2}} .2x}{\left\{\left(x^2 + y^2 + z^2\right)^{\frac{3}{2}}\right\}^2}$$

$$= \frac{\left(x^2 + y^2 + z^2\right)^{\frac{3}{2}} - 3x^2\left(x^2 + y^2 + z^2\right)^{\frac{1}{2}}}{\left(x^2 + y^2 + z^2\right)^3} \text{ ..... (ii)}$$

Similarly,

$$\therefore \frac{\partial^2 u}{\partial y^2} = \frac{\left(x^2 + y^2 + z^2\right)^{\frac{3}{2}} - 3y^2\left(x^2 + y^2 + z^2\right)^{\frac{1}{2}}}{\left(x^2 + y^2 + z^2\right)^3} \text{ ....(iii)}$$

$$\text{and } \frac{\partial^2 u}{\partial z^2} = \frac{\left(x^2 + y^2 + z^2\right)^{\frac{3}{2}} - 3z^2\left(x^2 + y^2 + z^2\right)^{\frac{1}{2}}}{\left(x^2 + y^2 + z^2\right)^3} \text{ ..... (iv)}$$

Adding (ii), (iii) and (iv) we get,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

$$= \frac{3\left(x^2 + y^2 + z^2\right)^{\frac{3}{2}} - 3\left(x^2 + y^2 + z^2\right)^{\frac{1}{2}}\left(x^2 + y^2 + z^2\right)}{\left(x^2 + y^2 + z^2\right)^3}$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \quad \text{Proved.}$$

7. If $x = r\cos\theta$. $y = r\sin\theta$, then show that

(i) $\frac{\partial r}{\partial x} = \frac{\partial x}{\partial r}$

Solution: Given $x = r\cos\theta$ ..... (i) $\quad y = r\sin\theta$ ..... (ii)

Squaring (i), (ii) and adding

$x^2 + y^2 = r^2(\cos^2\theta + \sin^2\theta)$

or, $x^2 + y^2 = r^2$ ..... (iii)

Differentiating (i) partially w. r. t. 'x' we get,

$$2x + 0 = 2r\frac{\partial r}{\partial x}$$

$$\therefore \frac{\partial r}{\partial x} = \frac{x}{r} = \frac{r\cos\theta}{r} = \cos\theta$$

$$\therefore \frac{\partial r}{\partial x} = \cos\theta \text{ ..... (iv)}$$

Differentiating (i) partially w. r. t. 'r' we get,

$$\frac{\partial x}{\partial r} = \cos\theta \text{ ..... (v)}$$

From (iv) and (v) we get,

$$\frac{\partial r}{\partial x} = \frac{\partial x}{\partial r} \quad \text{Proved.}$$

(ii) $\dfrac{1}{r}\dfrac{\partial x}{\partial \theta} = \dfrac{\partial \theta}{\partial x}$

Solution: Given $x = r\cos\theta$ ..... (i) $\quad y = r\sin\theta$ ..... (ii)

Squaring (i), (ii) and adding

$x^2 + y^2 = r^2(\cos^2\theta + \sin^2\theta)$

or, $x^2 + y^2 = r^2$ ..... (iii)

Differentiating (i) partially w. r. t. 'θ' we get,

$$\frac{\partial x}{\partial \theta} = -r\sin\theta$$

$$\text{or, } \frac{1}{r}\frac{\partial x}{\partial \theta} = -\sin\theta \text{ ..... (vi)}$$

Dividing (ii) by (i) we get,

$$\tan\theta = \frac{y}{x} \text{ ..... (vii)}$$

Differentiating (vii) partially w. r. t. 'x' we get,

$$\sec^2\theta\frac{\partial \theta}{\partial x} = -\frac{y}{x^2} = \frac{-r\sin\theta}{r\cos^2\theta}$$

$$\text{or, } \frac{\partial \theta}{\partial x} = -\frac{\sin\theta}{r\cos^2\theta \,.\, \sec^2\theta}$$

$$\therefore r\frac{\partial \theta}{\partial x} = -\sin\theta \text{ ..... (viii)}$$

From (vi) & (viii) we get,

$$\frac{1}{r}\frac{\partial x}{\partial \theta} = r\frac{\partial \theta}{\partial x} \quad \text{Proved.}$$

(iii) $\dfrac{\partial^2\theta}{\partial x^2} + \dfrac{\partial^2\theta}{\partial y^2} = 0$

Solution: Given $x = r\cos\theta$ ..... (i) $\quad y = r\sin\theta$ ..... (ii)

Squaring (i), (ii) and adding

$x^2 + y^2 = r^2(\cos^2\theta + \sin^2\theta)$

or, $x^2 + y^2 = r^2$ ..... (iii)

From (iii) of (ii)

$$r\frac{\partial \theta}{\partial x} = -\sin\theta$$

$$\frac{\partial \theta}{\partial x} = -\frac{\sin\theta}{r} = \frac{r\sin\theta}{r^2} = -\frac{y}{(x^2+y^2)}$$

Differentiating (i) partially w. r. t. 'x' we get,

$$\frac{\partial^2\theta}{\partial x^2} = \frac{2xy}{(x^2+y^2)^2} \text{ ..... (iv)}$$

Again $\tan\theta = \dfrac{y}{x}$

$$\text{Similarly, } \frac{\partial \theta}{\partial y} = \frac{x}{(x^2+y^2)}$$

$$\therefore \frac{\partial^2\theta}{\partial y^2} = \frac{-2xy}{(x^2+y^2)^2} \text{ ..... (v)}$$

Adding (iv) and (v) we get,

$$\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} = \frac{2xy}{(x^2+y^2)^2} - \frac{2xy}{(x^2+y^2)^2}$$

$$\therefore \frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} = 0$$

8. If $u = e^{r\cos\theta}\cos(r\sin\theta)$, $V = e^{r\cos\theta}\sin(r\sin\theta)$, then prove that,

$$\frac{\partial^2 u}{\partial r} = \frac{1}{r}\frac{\partial^2 v}{\partial r} \text{ and } -\frac{1}{r}\frac{\partial^2 u}{\partial \theta}.$$

Solution: Given $u = e^{r\cos\theta}\cos(r\sin\theta)$ ..... (i)

$V = e^{r\cos\theta}\sin(r\sin\theta)$ ..... (ii)

Differentiating (i) and (ii) partially w. r. t. 'r' and 'θ' resp. we get,

$$\therefore \frac{\partial u}{\partial r} = e^{r\cos\theta} \,.\, -\sin(r\sin\theta) \,.\, \sin\theta + \cos(r\sin\theta) \,.\, e^{r\cos\theta} \,.\, \cos\theta$$

$$= e^{r\cos\theta}[-\sin\theta \,.\, \sin(r\sin\theta) + \cos\theta\cos(r\sin\theta)] \text{ ..... (iii)}$$

$$\therefore \frac{\partial V}{\partial \theta} = e^{r\cos\theta} \cos(r\sin\theta) \,.\, r\cos\theta + \sin(r\sin\theta) \,.\, e^{r\cos\theta}(-r\sin\theta)$$

$$= r\, e^{r\cos\theta} [\cos\theta \cos(r\sin\theta) - \sin\theta \sin(r\sin\theta)]$$

$$\frac{1}{r}\frac{\partial V}{\partial \theta} = e^{r\cos\theta} [-\sin\theta \sin(r\sin\theta) + \cos\theta \cos(r\sin\theta)] \text{ ..... (iv)}$$

From (iii) and (iv)

$$\frac{\partial U}{\partial r} = \frac{1}{r}\frac{\partial V}{\partial \theta} \quad \text{Proved.}$$

2nd Part

$$\frac{\partial u}{\partial \theta} = e^{r\cos\theta} - \sin(r\sin\theta) \,.\, r\cos\theta + \cos(r\sin\theta) \,.\, e^{r\cos\theta}(-r\sin\theta)$$

$$= -r\, e^{r\cos\theta} [\cos\theta \sin(r\sin\theta) + \sin\theta \cos(r\sin\theta)]$$

$$\therefore -\frac{1}{r}\frac{\partial u}{\partial \theta} = e^{r\cos\theta} [\cos\theta \sin(r\sin\theta) + \sin\theta \cos(r\sin\theta) \text{ .... (v)}$$

$$\therefore \frac{\partial v}{\partial r} = e^{r\cos\theta} \cos(r\sin\theta) \,.\, \sin\theta + \sin(r\sin\theta) \,.\, e^{r\cos\theta} \,.\, \cos\theta$$

$$= e^{r\cos\theta} [\sin\theta \cos(r\sin\theta) + \cos\theta \sin(r\sin\theta)] \text{ ..... (vi)}$$

From (v) and (vi)

$$\frac{\partial V}{\partial r} = -\frac{1}{r}\frac{\partial V}{\partial \theta} \quad \text{Proved.}$$

9. If u = f(r), then prove that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r} f'(r).$$

Solution. $u = f(r)$ ..... (i)

Differentiating (i) partially w. r. t. 'x' we get,

$$\therefore \frac{\partial u}{\partial x} = f'(r)\frac{\partial r}{\partial x}$$

$$\therefore \frac{\partial^2 u}{\partial x^2} = f'(r)\frac{\partial^2 r}{\partial x^2} + \frac{\partial r}{\partial x} \,.\, f''(r)\left(\frac{\partial r}{\partial x}\right)$$

$$= \left(\frac{\partial r}{\partial x}\right)^2 f''(r) + \frac{\partial^2 r}{\partial x^2} f'(r) \text{ ..... (ii)}$$

Similarly,

$$\frac{\partial^2 u}{\partial y^2} = \left(\frac{\partial r}{\partial y}\right)^2 f''(r) + \frac{\partial^2 r}{\partial y^2} f'(r) \text{ ..... (iii)}$$

Adding (ii) and (iii) we get,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \left(\frac{\partial r}{\partial x}\right)^2 f''(r) + \frac{\partial^2 r}{\partial x^2} f'(r)$$

$$\left(\frac{\partial r}{\partial y}\right)^2 f''(r) + \frac{\partial^2 r}{\partial y^2} f'(r)$$

$$= \left[\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2\right] f''(r) + \left[\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2}\right] f'(r) \text{ ..... (iv)}$$

We have, $x = r\cos\theta$, $y = r\sin\theta$

$\therefore r^2 = x^2 + y^2$

or, $2r \,.\, \frac{\partial r}{\partial x} = 2x$

$$\therefore \frac{\partial r}{\partial x} = \frac{x}{r}$$

Similarly, $\frac{\partial r}{\partial y} = \frac{x}{r}$

$$\therefore \frac{\partial^2 r}{\partial x^2} = \frac{r \,.\, 1 - x \,.\, \frac{\partial r}{\partial x}}{r^2}$$

$$= \frac{r - x.\frac{x}{r}}{r^2} = \frac{r^2 - x^2}{r^2 \,.\, r} = \frac{x^2 + y^2 - x^2}{r^3}$$

$$= \frac{y^2}{r^3} \text{ .... (v)}$$

Similarly, $\frac{\partial^2 r}{\partial y^2} = \frac{x^2}{r^3}$ ..... (vi)

Adding (v) and (vi) we get,

$$\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} = \frac{x^2}{r^3} + \frac{y^3}{r^3} = \frac{1}{r}\left[\frac{x^2}{r^2} + \frac{y}{r^2}\right]$$

$$= \frac{1}{r}\left[\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2\right]$$

Using this equation (iv) we get,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \left[\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2\right] f''(r) + \frac{1}{r} f'(r)$$

$$= \left[\frac{x^2}{r^2} + \frac{y^2}{r^2}\right] f''(r) + \frac{1}{r} f'(r)$$

$$= \left(\frac{x^2 + y^2}{r^2}\right) f''(r) + \frac{1}{r} f'(r)$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r} f'(r) \text{ Hence proved.}$$

Exercise - 2

1. Verify Euler's theorem for the following functions.

(i) $\mathbf{u = x\, f\left(\frac{y}{x}\right)}$

Solution: We have, $u = x\, f\left(\frac{y}{x}\right)$ ..... (i)

This equation is homogeneous function of degre one. Then by Euler's theorems,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = U$$

Differentiating (i) partially w. r. t. x & y resp. we get,

$$\therefore \frac{\partial u}{\partial x} = x\, f'\left(\frac{y}{x}\right) \cdot \left(-\frac{y}{x^2}\right) + f\left(\frac{y}{x}\right) \cdot 1$$

$$= -\frac{y}{x} f'\left(\frac{y}{x}\right) + f\left(\frac{y}{x}\right) \text{ ..... (ii)}$$

$$\therefore \frac{\partial u}{\partial y} = x\, f'\left(\frac{y}{x}\right) \cdot \frac{1}{x}$$

$$= f'\left(\frac{y}{x}\right) \text{ ..... (iii)}$$

Multiplying equation (ii) and (iii) by x & y resp. and adding we get,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -y\, f'\left(\frac{y}{x}\right) + x\, f\left(\frac{y}{x}\right) + y\, f'\left(\frac{y}{x}\right)$$

$$= x\, f\left(\frac{y}{x}\right) = u \quad \therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u$$

Hence Euler's theorem verify.

(ii) $\mathbf{u = \sin^{-1}\frac{x}{y} + \tan^{-1}\frac{y}{x}}$

Solution: We have, $u = \sin^{-1}\frac{x}{y} + \tan^{-1}\frac{y}{x}$

Since U is a homogeneous function of degree 0. Then by Euler's theorem

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0 \,.\, U = 0$$

Differentiating (i) partially w. r. t. x & y resp. we get,

$$\therefore \frac{\partial u}{\partial x} = \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}} \cdot \frac{1}{y} + \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \left(-\frac{y}{x^2}\right)$$

$$= \frac{1}{\sqrt{y^2 - x^2}} - \frac{y}{x^2 + y^2} \text{ ..... (ii)}$$

$$\therefore \frac{\partial u}{\partial y} = \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}} \cdot \left(\frac{x}{-y^2}\right) + \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{1}{x}$$

$$= -\frac{x}{y\sqrt{y^2 - x^2}} + \frac{x}{x^2 + y^2} \text{ ..... (iii)}$$

Multiply equation (ii) and (iii) x & y resp. and adding we get,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{x}{\sqrt{y^2 - x^2}} - \frac{xy}{(x^2 + y^2)} - \frac{x}{\sqrt{y^2 - x^2}} + \frac{xy}{(x^2 + y^2)}$$

$$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$$

Hence Euler's theorem verify.

(iii) $\mathbf{u = x^n \tan^{-1}\left(\frac{y}{x}\right)}$.

Solution: We have, $u = x^n \tan^{-1}\left(\frac{y}{x}\right)$ ..... (i)

This equation is homogeneous function of degree n then by Euler's theorem.

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$$

Differentiating (i) partially w. r. t. x & y resp. we get,

$$\therefore \frac{\partial u}{\partial x} = x^n \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \left(\frac{-y}{x^2}\right) + nx^{n-1}\tan^{-1}\frac{y}{x}$$

$$= -\frac{x^n y}{x^2 + y^2} + nx^{n-1}\tan^{-1}\frac{y}{x} \text{ ..... (ii)}$$

$$\therefore \frac{\partial u}{\partial y} = x^n \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{1}{x} = \frac{x^{n+1}}{x^2 + y^2} \text{ ..... (iii)}$$

Multiplying equation (ii) & (iii) by x & y resp. and adding we get,

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{-x^{n+1}y}{x^2+y^2}+n\,x^n\tan^{-1}\frac{y}{x}+\frac{x^{n+1}y}{x^2+y^2}.$$

$$\therefore\ x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=n\,u$$

Hence Eluler's theorem verify.

(iv) $\mathbf{u=x^n\sin\left(\frac{y}{x}\right)}$

Solution: We have, $u=x^n\sin\left(\frac{y}{x}\right)$ ..... (i)

This is homogeneous function of degree n then by Euler's theorem

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu$$

Differentiating (i) partially w. r. t. x & y resp. we get,

$$\therefore\ \frac{\partial u}{\partial x}=x^n\cos\left(\frac{y}{x}\right)\left(\frac{-y}{x^2}\right)+n\,x^{n-1}\sin\left(\frac{y}{x}\right)$$

$$=-x^{n-2}y\cos\left(\frac{y}{x}\right)+n\,x^{n-1}\sin\left(\frac{y}{x}\right)$$

$$\therefore\ x\frac{\partial u}{\partial x}=-x^{n-1}y\cos\left(\frac{y}{x}\right)+n\,x^n\sin\left(\frac{y}{x}\right)\ \text{..... (ii)}$$

$$\therefore\ \frac{\partial u}{\partial y}=x^n\cos\left(\frac{y}{x}\right)\cdot\frac{1}{x}=x^{n-1}\cos\left(\frac{y}{x}\right)$$

$$\therefore\ y\frac{\partial u}{\partial y}=x^{n-1}y\cos\left(\frac{y}{x}\right)\ \text{..... (iii)}$$

Adding (ii) and (iii) we get,

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=n\,x^n\sin\left(\frac{y}{x}\right)$$

$$\therefore\ x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu$$

Hence the Euler's theorem verify.

(v) $\mathbf{u=\frac{x^2y^2}{x^3+y^3}}$

Solution: We have, $u=\frac{x^2y^2}{x^3+y^3}=\frac{x^4\left(\frac{y}{x}\right)^2}{x^3\left[1+\left(\frac{y}{x}\right)^3\right]}$

$$u=x\,\phi\left(\frac{y}{x}\right)\ \text{..... (i)}$$

Hence, equation (i) is homogeneous equation of degree one then by Euler's theorem.

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=1\,.\,U$$

Differentiating (i) partially w. r. t. x & y resp. we get,

$$\therefore\ x\frac{\partial u}{\partial x}=x\,\phi'\left(\frac{y}{x}\right)\cdot\left(\frac{y}{-x^2}\right)+1\,.\,\phi\left(\frac{y}{x}\right)$$

$$=\frac{-y}{x}\phi'\left(\frac{y}{x}\right)+\phi\left(\frac{y}{x}\right)$$

$$\therefore\ x\frac{\partial u}{\partial x}=-\phi\left(\frac{y}{x}\right)+x\phi\left(\frac{y}{x}\right)\ \text{..... (ii)}$$

$$\therefore\ \frac{\partial u}{\partial y}=x\,\phi'\left(\frac{y}{x}\right)\cdot\frac{1}{x}=\phi'\left(\frac{y}{x}\right)$$

$$\therefore y\frac{\partial u}{\partial y}=y\,\phi'\left(\frac{y}{x}\right)\ \text{..... (iii)}$$

Adding (ii) and (iii) we get,

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=x\,\phi\left(\frac{y}{x}\right)$$

$$=u$$

$$\therefore\ x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=V$$

Hence the Eciler's theorem verify.

(vi) $\mathbf{u=\frac{x^3y}{x^2+y^2}}$

Solution: We have, $u=\frac{x^3y}{x^2+y^2}=\frac{x^4\left(\frac{y}{x}\right)}{x^2\left[1+\left(\frac{y}{x}\right)^2\right]}=x^2\,\phi\left(\frac{y}{x}\right)$ .... (i)

Hence equation (i) is homogeneous function of degree two then by Euler's theorem.

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2u$$

Differentiating (i) partially w. r. t. x & y resp. we get,

$$\therefore\ \frac{\partial u}{\partial x}=x^2\,\phi'\left(\frac{y}{x}\right)\cdot\left(\frac{y}{-x^2}\right)+2x\,\phi\left(\frac{y}{x}\right)$$

$$= -y\,\phi'\left(\frac{y}{x}\right) + 2x\,\phi\left(\frac{y}{x}\right)$$

$$\therefore \frac{\partial u}{\partial x} = -xy\,\phi'\left(\frac{y}{x}\right) + 2x^2\,\phi\left(\frac{y}{x}\right) \text{ ..... (ii)}$$

$$\therefore \frac{\partial u}{\partial y} = x^2\,\phi'\left(\frac{y}{x}\right) \cdot \frac{1}{x} = x\,\phi'\left(\frac{y}{x}\right)$$

$$\therefore y\frac{\partial u}{\partial y} = xy\,\phi'\left(\frac{y}{x}\right) \text{ ..... (iii)}$$

Adding (ii) and (iii) we gat,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2x^2\,\phi\left(\frac{y}{x}\right) = 2u$$

$$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u$$

Hence, Euler's theorem verify.

2. (i) If $u = \sin^{-1}\dfrac{x^2y^2}{x+y}$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 3\tan U$

Solution: We have, $u = \sin^{-1}\dfrac{x^2y^2}{x+y}$

$$\therefore \sin u = \frac{x^2y^2}{x+y} = \frac{x^4\left(\frac{y}{x}\right)^2}{x\left[1+\left(\frac{y}{x}\right)\right]}$$

$$\text{Since, } x^3\,\phi\left(\frac{y}{x}\right) \text{ ..... (i)}$$

Thus, sin u is homogeneous function of degree three then by Euler's theorem.

$$x\partial\frac{\sin u}{\partial x} + y\partial\frac{\sin u}{\partial y}\ 3\sin u.$$

Differentiating (i) partially w. r. t. x we get,

$$\therefore \cos u\frac{\partial u}{\partial x} = x^3\,\phi'\left(\frac{y}{x}\right) \cdot \left(\frac{-y}{x^2}\right) + \phi\left(\frac{y}{x}\right) \cdot 3x^2$$

$$= -xy\,\phi'\left(\frac{y}{x}\right) + 3x^2\,\phi\left(\frac{y}{x}\right) \text{ ..... (ii)}$$

$$\therefore \text{Cos } u\frac{\partial u}{\partial y} = x^3\,\phi'\left(\frac{y}{x}\right) \cdot \frac{1}{x} = x^2\phi'\left(\frac{y}{x}\right) \text{ ..... (iii)}$$

Multiplying equation (ii) by x & (iii) by y and adding we get,

$$x\cos u\frac{\partial u}{\partial x} = -x^2y\,\phi'\left(\frac{y}{x}\right) + 3x^3\,\phi\left(\frac{y}{x}\right) + x^2y\,\phi'\left(\frac{y}{x}\right)$$

$$= 3x^3\,\phi\left(\frac{y}{x}\right)$$

$$\text{Cos } u\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) = 3\sin u \text{ from (i)}$$

$$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3\frac{\sin u}{\cos u} = 3\tan u \text{ hence proved.}$$

(ii) If $u = \log x$ show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 3$.

Solution: We have, $u = \log x$

$e^u = x$

$$= y\left(\frac{x}{y}\right) = y\,\phi\,e^u\left(\frac{x}{y}\right) \text{ ..... (i)}$$

Thus e^u is homogeneous function of degree one then by Ealer's theorem.

$$x\frac{\partial e^u}{\partial x} + y\frac{\partial e^u}{\partial y} = 1 \cdot e^u$$

Differentiating (i) partially w. r. t. x & y resp. we get,

$$\therefore e^u\frac{\partial u}{\partial x} = y\,\phi'\frac{x}{y} \cdot \frac{1}{y} = \phi'\frac{x}{y} \text{ ..... (ii)}$$

$$\therefore e^u\frac{\partial u}{\partial y} = y\,\phi'\frac{x}{y} \cdot \left(\frac{-x}{y^2}\right) + 1 \cdot \phi\left(\frac{x}{y}\right)$$

$$= -\frac{x}{y}\phi'\frac{x}{y} + \phi\left(\frac{x}{y}\right) \text{ ..... (iii)}$$

Multiplying equation (ii) & (iii) by x & y resp. and adding we get,

$$xe^u\frac{\partial u}{\partial x} + y\,e^u\frac{\partial u}{\partial y} = x\,\phi'\frac{x}{y} - x\,\phi'\frac{x}{y} + y\,\phi\left(\frac{x}{y}\right)$$

$$= y\,\phi\left(\frac{x}{y}\right)$$

$$e^u\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) = e^u \text{ from (i)}$$

$$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{e^u}{e^u}$$

Hence, $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 1$ Proved.

(iii) If $\sin u = \dfrac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}$. Then show that $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=0$

(BE 2069, Poush, Bhadra, 2068 Magh)

Solution: We have,

$$\text{Sin } u = \frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}} = \frac{x^{\frac{1}{2}}-y^{\frac{1}{2}}}{x^{\frac{1}{2}}+y^{\frac{1}{2}}} = \frac{x^{\frac{1}{2}}\left[1+\left(\frac{y}{x}\right)^{\frac{1}{2}}\right]}{x^{\frac{1}{2}}\left[1+\left(\frac{y}{x}\right)^{\frac{1}{2}}\right]}$$

$$\text{Sin } u = \phi\left(\frac{y}{x}\right) \text{ ..... (i)}$$

Thus sin u is homogeneous of function of degree zero. Then by Euler's thorem.

$$x\frac{\partial \sin u}{\partial x}+y\frac{\partial \sin u}{\partial y} = 0 \,.\, u$$

Differentiating (i) partially w. r. t. x & y resp. we get,

$$\text{Cos}u\,\frac{\partial u}{\partial x} = \phi'\left(\frac{y}{x}\right)\left(\frac{y}{-x^2}\right) = -\frac{y}{x^2}\,\phi'\left(\frac{y}{x}\right) \text{ ..... (ii)}$$

$$\text{Cos } u\,\frac{\partial u}{\partial y} = \phi'\left(\frac{y}{x}\right)\cdot\frac{1}{x} = \frac{1}{x}\,\phi'\left(\frac{y}{x}\right) \text{ ..... (iii)}$$

Multiplying equation (ii) & (iii) by x & y resp. and adding

$$x\cos u\,\frac{\partial u}{\partial x}+y\cos u\,\frac{\partial u}{\partial y} = -\frac{y}{x}\phi'\left(\frac{y}{x}\right)+\frac{y}{x}\phi'\left(\frac{y}{x}\right)$$

$$= \cos u\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right) = 0$$

$$\therefore\ x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = \frac{0}{\cos u} = 0$$

Hence, $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y} = 0$ Proved.

(iv) If $u = \tan^{-1}\dfrac{x^3+y^3}{x-y}$, show that $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y} = \sin 2u$.

Solution: We have, $u = \tan^{-1}\dfrac{x^3+y^3}{x-y}$

$$\therefore \tan u = \frac{x^3+y^3}{x-y} = \frac{x^3\left\{1+\left(\frac{y}{x}\right)^3\right\}}{x\left(1-\frac{y}{x}\right)}$$

$$\therefore \text{ Hence, } x^2\,\phi\left(\frac{y}{x}\right) \text{ ..... (i)}$$

Thus tan U is homogeneous function of degree two. Then by Euler's theorem.

$$x\frac{\partial \tan u}{\partial x}+y\frac{\partial \tan u}{\partial y} = 2\tan u$$

Differentiating (i) partially w. r. t. x & y resp. we get,

$$\text{Sec}^2u\,\frac{\partial u}{\partial x} = x^2\,\phi'\left(\frac{y}{x}\right)\cdot\left(\frac{y}{-x^2}\right)+2x\,\phi\left(\frac{y}{x}\right)$$

$$= -y\,\phi'\left(\frac{y}{x}\right)+2x\,\phi\left(\frac{y}{x}\right) \text{ ..... (ii)}$$

$$\text{Sec}^2u\,\frac{\partial u}{\partial y} = x^2\,\phi'\left(\frac{y}{x}\right)\cdot\frac{1}{x}$$

$$= x\,\phi'\left(\frac{y}{x}\right) \text{ ..... (iii)}$$

Multiplying equation (ii) by x and equation (iii) by y and adding we get,

$$x\sec^2u\,\frac{\partial u}{\partial x}+y\sec^2u\,\frac{\partial u}{\partial y}$$

$$= -xy\,\phi'\left(\frac{y}{x}\right)+2x^2\,\phi\left(\frac{y}{x}\right)+xy\,\phi'\left(\frac{y}{x}\right) = 2x^2\,\phi\left(\frac{y}{x}\right)$$

$$\text{Sec}^2u\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right) = 2\tan u \text{ from (i)}$$

$$\therefore\ x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = 2\,\frac{\sin u}{\cos u\,.\,\sec^2u} = 2\sin u\,\cos u = \sin 2u$$

Hence, $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y} = \sin 2u$. Proved.

(v) If $u = \text{cosec}^{-1}\dfrac{x^{\frac{1}{2}}+y^{\frac{1}{2}}}{x^{\frac{1}{3}}+y^{\frac{1}{3}}}$.

Show that $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y} = -\dfrac{1}{6}\tan u$.

Solution: We have, $u = \text{cosec}^{-1}\dfrac{x^{\frac{1}{2}}+y^{\frac{1}{2}}}{x^{\frac{1}{3}}+y^{\frac{1}{3}}}$

$$\text{Cosec } u = \frac{x^{\frac{1}{2}} + y^{\frac{1}{2}}}{x^{\frac{1}{3}} + y^{\frac{1}{3}}} = \frac{x^{\frac{1}{2}}\left[1+\left(\frac{y}{x}\right)^{\frac{1}{2}}\right]}{x^{\frac{1}{3}}\left[1+\left(\frac{y}{x}\right)^{\frac{1}{3}}\right]} = x - \frac{1}{6}\,\phi\left(\frac{y}{x}\right).$$

Thus cosec u is homogeneous function of degree $\frac{1}{6}$. Then by Euler's theorem.

$$x\frac{\partial \cos ecu}{\partial x} + y\frac{\partial \cos ecu}{\partial y} = \frac{1}{6}\,.\,\text{cosec } u$$

Differentiating (i) partially w. r. t. x & y resp. we get,

$$-\text{cosec}u\,\text{cot}u\,\frac{\partial u}{\partial x} = x\frac{1}{6}\,\phi'\left(\frac{y}{x}\right).\left(\frac{y}{-x^2}\right) + \frac{1}{6}\,x^{\frac{-5}{6}}\,\phi\left(\frac{y}{x}\right)$$

$$= -\,x^{-\frac{11}{6}}\,y\,\phi'\left(\frac{y}{x}\right) + \frac{1}{6}\,x^{\frac{-5}{6}}\,\phi\left(\frac{y}{x}\right) \text{ ..... (ii)}$$

$$-\quad \text{Cosec}u\,\text{cot } u\,\frac{\partial u}{\partial y} = x^{\frac{1}{6}}\,\phi'\left(\frac{y}{x}\right).\frac{1}{x}$$

$$= x^{\frac{-5}{6}}\,\phi'\left(\frac{y}{x}\right) \text{ ..... (iii)}$$

Multiplying equation (ii) & (iii) by x & y rep. and adding

$$= -\,x^{\frac{-5}{6}}\,\phi'\left(\frac{y}{x}\right) + \frac{1}{6}\,x^{\frac{1}{6}}\,\phi\left(\frac{y}{x}\right) + x^{-\frac{5}{6}}\,y\,\phi'\left(\frac{y}{x}\right)$$

$$-\,x\,\text{cosec}u\,\text{cot } u\,\frac{\partial u}{\partial x} - y\,\text{cosec}u\,\text{cot } u\,\frac{\partial u}{\partial y}$$

$$\text{or, } -\,\text{cosec } u\,\text{cot } u\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) = \frac{1}{6}\,x^{\frac{1}{6}}\,\phi\left(\frac{y}{x}\right)$$

$$= \frac{1}{6}\,\text{cosec } u \text{ from (i)}$$

$$\therefore\ x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{6}\,\frac{\cos ecu}{-\cos ecu\,\cot u}$$

$$= \frac{1}{-6}\,\frac{1}{\cot u} = -\,\frac{1}{6}\,\tan u.$$

Hence, $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -\,\frac{1}{6}\,\tan u$ Proved.

) If $u = \cos^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$,

then show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -\frac{1}{2}\,\text{Cot}u.$ **(BE 2068 Bhadra)**

Solution: We have, $u = \cos^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$

$$\therefore \cos u = \frac{x+y}{\sqrt{x}+\sqrt{y}} = \frac{x+y}{x^{\frac{1}{2}}+y^{\frac{1}{2}}} = \frac{x\left[x+\frac{y}{x}\right]}{x^{\frac{1}{2}}\left[1+\left(\frac{y}{x}\right)^{\frac{1}{2}}\right]}$$

$$\therefore \cos u\ x^{\frac{1}{2}}\ \phi\left(\frac{y}{x}\right) \text{ ..... (i)}$$

Thus cos u is homogeneous function of degree $\frac{1}{2}$. Then by Euler's theorem.

$$x\frac{\partial \cos u}{\partial x} + y\frac{\partial \cos u}{\partial y} = \frac{1}{2}\,\cos u.$$

Differentiating (i) partially w. r. t. x & y resp. we get,

$$-\sin u\,\frac{\partial u}{\partial x} = x^{\frac{1}{2}}\,\phi'\left(\frac{y}{x}\right)\left(\frac{y}{-x^2}\right) + \frac{1}{2}\,x^{-\frac{1}{2}}\,\phi\left(\frac{y}{x}\right)$$

$$= -\,x^{\frac{-3}{2}}\,y\,\phi'\left(\frac{y}{x}\right) + \frac{1}{2}\,x^{-\frac{1}{2}}\,\phi\left(\frac{y}{x}\right) \text{ ..... (ii)}$$

$$-\sin u\,\frac{\partial u}{\partial y} = x^{\frac{1}{2}}\,\phi'\left(\frac{y}{x}\right).\frac{1}{x}$$

$$= x^{-\frac{1}{2}}\,\phi'\left(\frac{y}{x}\right) \text{ ..... (iii)}$$

Multiplying equation (ii) & (iii) by x & y resp. then we get,

$$-\,x\sin u\,\frac{\partial u}{\partial x} - y\sin u\,\frac{\partial u}{\partial y} = -\,x^{-\frac{1}{2}}\,\phi'\left(\frac{y}{x}\right)$$

$$+\,\frac{1}{2}\,x^{\frac{1}{2}}\,\phi\left(\frac{y}{x}\right) + x^{-\frac{1}{2}}\,\phi'\left(\frac{y}{x}\right)$$

$$-\,\text{Sin } u\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) = \frac{1}{2}\,x^{\frac{1}{2}}\,\phi\left(\frac{y}{x}\right)$$

$$= \frac{1}{2}\,\cos u \text{ from (i)}$$

$$\therefore\ x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=-\frac{1}{2}\ \frac{\cos u}{\sin u}=-\frac{1}{2}\cot u.$$

Hence, $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=-\frac{1}{2}\cot u$. Proved.

(vii) If $u=\log(x^2+y^2+z^2)$. Then prove that $x\frac{\partial^2 u}{\partial y\partial z}=y\frac{\partial^2 u}{\partial z\partial x}$.

Solution: We have, $u=\log(x^2+y^2+z^2)$ ..... (i)

Differentiating (i) partially w. r. t. 'x' we get,

$$\therefore\ \frac{\partial u}{\partial x}=\frac{1}{(x^2+y^2+z^2)}\cdot 2x=\frac{2x}{(x^2+y^2+z^2)}\ \text{..... (ii)}$$

Differentiating (ii) partially w. r. t. 'z' we get,

$$\frac{\partial^2 u}{\partial z\partial x}=\frac{-2x}{(x^2+y^2+z^2)^2}\cdot 2z=\frac{-4zx}{(x^2+y^2+z^2)^2}\ \text{..... (iii)}$$

Again, Differentiating (i) partially w. r. t. 'z' we get,

$$\frac{\partial u}{\partial z}=\frac{1}{(x^2+y^2+z^2)}\cdot 2z=\frac{2z}{(x^2+y^2+z^2)}\ \text{..... (iv)}$$

Differentiating (iv) partially w. r. t. 'y' we get,

$$\frac{\partial^2 u}{\partial y\partial z}=-\frac{-2z}{(x^2+y^2+z^2)^2}\cdot 2y=\frac{-4yz}{(x^2+y^2+z^2)}\ \text{..... (v)}$$

Multiplying equation (v) by x and (iii) by y we get,

$$x\frac{\partial^2 u}{\partial y\partial z}=y\frac{\partial^2 u}{\partial z\partial x}.\ \text{Proved.}$$

3. Find $\frac{dy}{dx}$ in the following functions.

(i) $x^{\frac{2}{3}}+y^{\frac{2}{3}}=1$

Solution: We have, $x^{\frac{2}{3}}+y^{\frac{2}{3}}=1$ ..... (i)

Differentiating (i) partially w. r. t. 'x' we get,

$$\frac{2}{3}x^{-\frac{1}{3}}+\frac{2}{3}y^{-\frac{1}{3}}\frac{dy}{dx}=0$$

$$\text{or, }\frac{dy}{dx}=-\frac{x^{-\frac{1}{3}}}{y^{-\frac{1}{3}}}\Rightarrow\frac{dy}{dx}=-\left(\frac{x}{y}\right)^{-\frac{1}{3}}$$

$$\Rightarrow\frac{dy}{dx}=-\left(\frac{x}{y}\right)^{-\frac{1}{3}}\quad\therefore\ \frac{dy}{dx}=-\left(\frac{y}{x}\right)^{-\frac{1}{3}}\ \text{Ans.}$$

(ii) $2x^3+x^2y+y^3=1$

Solution: We have, $2x^3+x^2y+y^3=1$ .....(i)

Differentiating (i) partially w. r. t. 'x' we get,

$$6x^2+x^2\frac{dy}{dx}+2xy+3y^2\frac{dy}{dx}=0$$

$$(x^2+3y^2)\frac{dy}{dx}=-(6x^2+2xy)\ \therefore\ \frac{dy}{dx}=-\left(\frac{6x^2+2xy}{x^2+3y^2}\right)\ \text{Ans.}$$

(iii) $(\tan x)^y+(y)^{\tan x}=0$

Solution: $(\tan x)^y+(y)^{\tan x}=0$

Let $f=(\tan x)^y+(y)^{\tan x}$ ..... (i)

Differentiating (i) partially w. r. t. 'x' and 'y' resp. we get,

$$\frac{\partial f}{\partial x}=y(\tan x)^{y-1}\cdot\sec^2 x+y^{\tan x}\cdot\log y\cdot\sec^2 x$$

$$\text{Also, }\frac{\partial f}{\partial x}=(\tan x)^y\log\tan x+\tan x\,y^{\tan x-1}$$

$$\left[\text{Since, }\frac{d}{dx}a^x=a^x\log a\right]$$

$$\text{We know, }\frac{dy}{dx}=\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$$

$$=-\left[\frac{y(\tan x)^{y-1}\sec^2 x+y^{\tan x}.\log y\sec^2 x}{(\tan x)^y\log\tan x+\tan x\,y^{\tan x-1}}\right]$$

$$=-\sec^2 x\,\frac{\left[y(\tan x)^{y-1}+\log y\cdot y^{\tan x}\right]}{(\tan x)^y\log\tan x+\tan x\cdot y^{\tan x-1}}$$

(iv) $x^3+y^3=3axy$

Solution: We have, $x^3+y^3=3xay$ ..... (i)

Differentiating (i) partially w. r. t. 'x' we get,

$$3x^2+3y^2\frac{dy}{dx}=3ax\frac{dy}{dx}+3ay\cdot 1$$

$$\text{or, }\frac{dy}{dx}(3y^2-3ax)=(3ay-3x^2)$$

$$\therefore\ \frac{dy}{dx}=\frac{3(ay-x^2)}{3(y^2-ax)}$$

$$\frac{dy}{dx}=\frac{ay-x^2}{y^2-ax}\ \text{Ans.}$$

(v) $x^py^q=(x+y)^{p+q}$

Solution: We have, $x^py^q=(x+y)^{p+q}$

Taking log on both sides,

$\log(x^p y^q) = \log(x+y)^{p+q}$

or, $\log x^p + \log y^q = (p+q)\log(x+y)$

or, $p\log x + q\log y = (p+q)\log(x+y)$

Differentiating (i) partially w. r. t. 'x' we get,

$$p \cdot \frac{1}{x} + q \cdot \frac{1}{y}\frac{dy}{dx} = \frac{p+q}{x+y} \cdot \left(1 + \frac{dy}{dx}\right)$$

$$\frac{dy}{dx}\left[\frac{q}{y} - \frac{p+q}{x+y}\right] = \frac{p+q}{x+y} - \frac{p}{x}$$

$$\frac{dy}{dx}\left[\frac{qx+qy-py-qy}{y(x+y)}\right] = \left[\frac{px+qx-px-py}{x(x+y)}\right]$$

or, $\frac{dy}{dx} = \frac{(qx-py)}{x} \cdot \frac{y}{(qx-py)} \Rightarrow \frac{dy}{dx} = \frac{y}{x}$ Ans.

(vi) $x^y + y^x = a^b$

Solution: We have, $x^y + y^x = a^b$

$x^y + y^x - a^b = 0$

Let, $f = x^y + y^x - a^b$ ..... (i)

Differentiating (i) partially w. r. t. 'x' & 'y' resp. we get,

$$\frac{\partial f}{\partial x} = y\,x^{y-1} + y^x \log y$$

Also, $\frac{\partial f}{\partial y} = x^y \log x - x\,y^{x-1}$

We know,

$$\frac{dy}{dx} = \frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} \quad \therefore \frac{dy}{dx} = \frac{y\,x^{y-1} + y^x \log y}{x^y \log x + xy^{x-1}} \text{ Ans.}$$

(vii) $e^x + e^y = 2xy$

Solution: We have, $e^x + e^y = 2xy$ ..... (i)

Differentiating (i) partially w. r. t. 'x' we get,

$$e^x + e^y\frac{dy}{dx} = 2x\frac{dy}{dx} + 2y \cdot 1$$

$$(e^y - 2y)\frac{dy}{dx} = 2x - e^x$$

$$\therefore \frac{dy}{dx} = \frac{2y - e^x}{e^y - 2x} \text{ Ans.}$$

4. Find $\frac{dz}{df}$ in the following functions.

(i) $z = x^3 - y^3,\ x = \frac{1}{t+1},\ y = \frac{t}{t+1}$

Solution: We have, $z = x^3 - y^3,\ x = \frac{1}{t+1},\ y = \frac{t}{t+1}$

Differentiating w. r. t. 't' we get,

$$\frac{dz}{dt} = 3x^2\frac{dx}{dt} - 3y^2\frac{dy}{dt}$$

$$\frac{dx}{dt} = -\frac{1}{(t+1)^2}$$

and $\frac{dy}{dt} = \frac{(t+1) \cdot 1 - t}{(t+1)^2} = \frac{1}{(t+1)^2}$

$$\therefore \frac{dz}{dt} = 3\frac{1}{(t+1)^2} \cdot \frac{-1}{(t+1)^2} - 3\frac{t^2}{(t+1)^2} \cdot \frac{1}{(1+t)^2}$$

$$= -3\left[\frac{(1+t^2)}{(t+1)^4}\right] \text{ Ans.}$$

(ii) $z = \log(x+y),\ x = e^{-2t},\ y = t^3 - t^2$

Solution: We have, $z = \log(x+y),\ x = e^{-2t},\ y = t^3 - t^2$

Differentiating w. r. t. 't' we get,

$$\frac{dz}{dt} = \frac{1}{x+y} \cdot \left(\frac{dx}{dt} + \frac{dy}{dt}\right);\ \frac{dx}{dt} = -2e^{-2t} \text{ and } \frac{dy}{dt} = 3t^2 - 2t$$

$$\therefore \frac{dz}{dt} = \frac{1}{e^{-2t} + t^3 - t^2} \cdot (-2e^{-2t} + 3t^2 - 2t)$$

$$= \frac{3t^2 - 2t - 2e^{-2t}}{t^3 - t^2 + e^{-2t}} \text{ Ans.}$$

Find $\frac{dz}{dx}$ in the following functions.

(i) $z = x\cos y + y\sin x,\ y = x^2 + 1$

Solution: We have, $z = x\cos y + y\sin x,\ y = x^2 + 1$

Differentiating w. r. t. 'x' we get,

$$\frac{dz}{dx} = x(-\sin y)\frac{dy}{dx} + \cos y \cdot 1 + y\cos x + \sin x\frac{dy}{dx}$$

and $\frac{dy}{dx} = 2x + 1$

$$\therefore \frac{dz}{dx} = -x\sin y\frac{dy}{dx} + \cos y + y\cos x + \sin x\frac{dy}{dx}$$

$$= -x\sin(x^2+1)(2x+1) + \cos(x^2+1) + (x^2+1)\cos x + \sin x \cdot (2x+1)$$

$$\frac{dz}{dt} = \cos(x^2+1) + (x^2+1)\cos x - (2x^2+x)\sin(x^2+1) + (2x+1)\sin x \text{ Ans.}$$

(ii) $z = (y + x)\, e^{xy},\ y = \dfrac{1}{x^2}$

Solution: We have, $z = (y + x)\, e^{xy},\ y = \dfrac{1}{x^2}$

Differentiating w. r. t. 'x' we get,

$$\frac{dz}{dx} = (y+x)\, e^{xy}\left(x\frac{dy}{dz} + y\right) + e^{xy}\left(1 + \frac{dy}{dx}\right) \text{ and } \frac{dy}{dx} = -\frac{2}{x^3}$$

$$\therefore \frac{dz}{dx} = \left(\frac{1}{x^2} + x\right) e^{x.\frac{1}{x^2}} \left\{x\left(\frac{-2}{x^3}\right) + \frac{1}{x^2}\right\}$$

$$+ e^{x.\frac{1}{x^2}} \left\{1 + \left(\frac{-2}{x^3}\right.\right.$$

$$= e^{\frac{1}{x}} \left[\left(\frac{1}{x^2} + x\right)\left\{\left(-\frac{2}{x^2}\right) + \frac{1}{x^2}\right\} + 1 - \frac{2}{x^3}\right]$$

$$= e^{\frac{1}{x}} \left(-\frac{2}{x^4} - \frac{2}{x} + \frac{1}{x^4} + \frac{1}{x} + 1 - \frac{2}{x^3}\right)$$

$$\therefore \frac{dz}{dx} = e^{\frac{1}{x}} \left[1 - \frac{1}{x} - \frac{2}{x^3} - \frac{1}{x^4}\right] \text{ Ans.}$$

6. if $u = f(x, y) = x^2 - 2xy + 3y$, find the differential du and chan Δu in u if f (x, y) changes from (1, 2) to (1, 03, 1.99).

Solution: Given $u = f(x, y) = x^2 - 2xy + 3y$ ..... (i)

Differentiating (i) partially w. r. t. x & y resp. we get,

$$\frac{\partial u}{\partial x} = 2x - 2y \text{ and } \frac{\partial u}{\partial y} = -2x + 3$$

Here $x = 1, y = 2$

$dx = \Delta x = 1.03 - 1 = 0.03$

$dy = \Delta y = 1.99 - 2 = -0.01$

Now,

$$du = \frac{\partial u}{\partial x}\, dx + \frac{\partial u}{\partial y}\, dy$$

$= (2x - 2y)\, dx + (-2x + 3)\, dy$

$= (2 \times 1 - 2 \times 2)\ (0.03) + (-2 \times 1 + 3)\,(-0.01)$

$= -0.06 - 0.01$

$= -0.07$ Ans.

and,

$\Delta u = f(x + \Delta x, y + \Delta y) - f(x, y) = f(1.3, 1.99) - f(1, 2)$

$= (1.03)^2 - 2\,(1.03)\,(1.99) + 3\,(1.99) - [1^2 - 2 \times 1 \times 2 + 3 \times 2]$

$= 2.9315 - 3$

$= -0.0685$ Ans.

7. If $u = f(x, y) = x^3 + y^3$ find the differential du and change Δu in u if f (x, y) changes from (10, 10) to (10.1, 10.1).

Solution: Given $u = f(x, y) = x^3 + y^3$ ..... (i)

Differentiating (i) partially w. r. t. x & y resp. we get,

$$\therefore \frac{\partial u}{\partial x} = 3x^2 \text{ and } \frac{\partial u}{\partial y} = 3y^2$$

Here, $x = 10, \quad y = 10$

$dx = \Delta x = 10.1 - 10 = 0.1$

$dy = \Delta y = 10.1 - 10 = 0.1$

Now,

$$du = \frac{\partial u}{\partial x}\, dx + \frac{\partial u}{\partial y}\, dy$$

$= 3x^2dx + 3y^2dy = 3 \times (10)^2 \times (0.1) + 3 \times (10)^2 \times (0.1)$

$= 3 \times 100 \times 0.1 + 3 \times 100 \times 0.1 = 30 + 30 = 60$ Ans.

And,

$\Delta u = f(x + \Delta x, y + \Delta y) - f(x, y) = f(10.1, 10.1) - f(10.10)$

$= (10.1)^3 + (10.1)^3 - \{(10)^3 + (10)^3\}$

$= 1030\,.\,30.1 + 1030\,.\,301 - 2000$

$= 60.602$ Ans.

8. If $u = f(x, y) = x^2 - 3x^3y^2 + 4x - 2y^3 + 6$. Find the differential du and change Δu in u if f (x, y) changes from (– 2, 3) to (– 2.02, 3.01)

Solution: Given $u = f(x, y) = x^2 - 3x^3y^2 + 4x - 2y^3 + 6$

Differentiating (i) partially w. r. t. x & y resp. we get,

$$\therefore \frac{\partial u}{\partial x} = 2x - 9x^2y^2 + 4 \text{ and } \frac{\partial u}{\partial y} = -6x^3y - 6y^2$$

Here, $x = -2, y = 3$

$dx = \Delta x = -2.02 - (-2) = -0.02$

$dy = \Delta y = 3.01 - 3 = 0.01$

Now, $du = \dfrac{\partial u}{\partial x}\, dx + \dfrac{\partial u}{\partial y}\, dy$

$= (2x - 9x^2y^2 + 4)\, dx + (-6x^3y - 6y^2)\, dy$

$= [2 \times (-2) - 9 \times (-2)^2 \times 3^2 + 4] \times (0.02)$

$- [6 \times (-2)^3 \times 3 - 6 \times e^2] \times (0.01)$

$= 6.32 + 1.98 = 8.3$ Ans.

And,

$\Delta u = f(x + \Delta x, y + \Delta y) - f(x, y)$

$= f(-0.02, 0.01) - f(-2, 3)$

$= \{(-0.02)^2 - 3\,(-0.02)^3\,(0.01)^2 + 4\,(-0.02) - 2\,(0.01)^3 + 6\}$

$- \{(-2)^2 - 3\,(-2)^3\,(3)^2 + 4\,(-2)\,(3)^3 + 6\}$

$= 6.08 - (164)$

$= -157.92$ ans.

9. The radius and altitude of a rigid circular cylinder and measured as 3 inches and 8 inches respectively with possible error in measurement of 0.05 inches find the maximum error in the calculated volume of the cylinder.

Solution: Let the radius of the cylinder = r inches, altitutde of the cylinder = h inches

Volume of the cylinder = V cublic inches.

We know that

$V = \pi r^2 h$ ..... (i)

Differentiating (i) partially w. r. t. r & h resp. we get,

$\frac{\partial v}{\partial r} = 2\pi rh$ and $\frac{\partial v}{\partial h} = \pi r^2$

Also, given,

r = 3 and h = 8

dr = ± 0.05 and dh = ± 0.05

Now, $dv = \frac{\partial v}{\partial r} dr + \frac{\partial v}{\partial h} dh = 2\pi rh\, dr + \pi r^2\, dh$

$= 2\pi (3)(8)(\pm 0.05) + \pi (3)^2 (\pm 0.05)$

$= \pm 2.4\pi \pm 0.45\pi = \pm 2.85\pi$

Thus the maximum

Changes = $2.85\pi = 2.85 \times 3.14 = 8.95$ cubic inches

10. If the dimension in inches of a rectangular box change from 9, 6 and 4 to 9.02, 5.97 and 4.01 respectively. Find the change in volume.

Solution: Let l, b, h be the length , breath and height of the box in inches.

Let V be its volume

Then,

$V = lbh$ ..... (i)

Differentiating (i) partially w. r. t. r and h resp. we get,

$\frac{\partial v}{\partial r} = bh$, $\frac{\partial v}{\partial b} = lh$ and $\frac{\partial v}{\partial h} = lb$

Also given,

$l = 9$, b = 6, h = 4

$dl = \Delta l = 9.02 - 9 = 0.02$

$db = \Delta b = 5.97 - 6 = -0.03$

$dh = \Delta h = 4.01 - 4 = 0.01$

Now, $dv = \frac{\partial v}{\partial \ell} dl + \frac{\partial v}{\partial b} db + \frac{\partial v}{\partial h} dh$

$= bh\,(dl) + lh\,(db) + lb\,(dh)$

$= 6 \times 4 \times (0.02) + 9 \times 4 \times (-0.03) + 9 \times 6 \times (0.01)$

$= -0.06$

Thus the volume decreases by 0.06 cubic inches.

Important IOE Questions and Answers

1. State Euler's Theorem for a homogeneous function of two independent variables and verify it for the function.

(BE 2067, Mangsir)

$$u = \frac{x^{\frac{1}{4}} + y^{\frac{1}{4}}}{x^{\frac{1}{5}} + y^{\frac{1}{5}}}$$

Sol^n. **First part :** Statement of Euler's theorem on homogeneous function of two independent variables-

If U = f(x, y) be a homogeneous function of two independent variables x and y of degxen then,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = xu$$

Second part : Given $U(x, y) = \frac{x^{\frac{1}{4}} + y^{\frac{1}{4}}}{x^{\frac{1}{5}} + y^{\frac{1}{5}}}$ ..... (i)

$$\text{or, } u(x, y) = \frac{x^{\frac{1}{4}}\left[1 + \left(\frac{y}{x}\right)^{\frac{1}{4}}\right]}{x^{\frac{1}{5}}\left[1 + \left(\frac{y}{x}\right)^{\frac{1}{5}}\right]} \Rightarrow U = x^{\frac{1}{20}}\,\phi\left(\frac{y}{x}\right)$$

Hence U is homogeneous function of degree $\frac{1}{20}$. Then by Euler's theorem

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{20}U$$

Diff. (i) partially w. r. t. x and y resp.

$$\frac{\partial u}{\partial x} = \frac{\left(x^{\frac{1}{5}} + y^{\frac{1}{5}}\right)\left(\frac{1}{4}\right)x^{-\frac{3}{4}} - \left(x^{\frac{1}{4}} + y^{\frac{1}{4}}\right)\left(\frac{1}{5}\right)y^{-\frac{4}{5}}}{\left(x^{\frac{1}{5}} + y^{\frac{1}{5}}\right)^2} \quad \text{..... (iii)}$$

Multiplying (ii) and (iii) by x and y resp. and adding.

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=x\frac{\left(x^{\frac{1}{5}}+y^{\frac{1}{5}}\right)\left(\frac{1}{4}\right)x^{-\frac{3}{4}}-\left(x^{\frac{1}{4}}+y^{\frac{1}{4}}\right)\left(\frac{1}{5}\right)x^{-\frac{4}{5}}}{\left(x^{\frac{1}{5}}+y^{\frac{1}{5}}\right)^2}$$

$$+y\frac{\left(x^{\frac{1}{5}}+y^{\frac{1}{5}}\right)\left(\frac{1}{4}\right)x^{-\frac{3}{4}}-\left(x^{\frac{1}{4}}+y^{\frac{1}{4}}\right)\left(\frac{1}{5}\right)x^{-\frac{4}{5}}}{\left(x^{\frac{1}{5}}+y^{\frac{1}{5}}\right)^2}$$

$$=\frac{\left(x^{\frac{1}{5}}+y^{\frac{1}{5}}\right)\left(\frac{1}{4}\right)x^{\frac{1}{4}}-\left(x^{\frac{1}{4}}+y^{\frac{1}{4}}\right)\left(\frac{1}{5}\right)x^{\frac{1}{5}}}{\left(x^{\frac{1}{5}}+y^{\frac{1}{5}}\right)^2}+\frac{\left(x^{\frac{1}{5}}+y^{\frac{1}{5}}\right)\left(\frac{1}{4}\right)y^{\frac{1}{4}}\left(x^{\frac{1}{4}}+y^{\frac{1}{4}}\right)\left(\frac{1}{5}\right)y^{\frac{1}{5}}}{\left(x^{\frac{1}{5}}+y^{\frac{1}{5}}\right)^2}$$

$$\frac{\frac{1}{4}\left(x^{\frac{1}{5}}+y^{\frac{1}{5}}\right)\left(x^{\frac{1}{4}}+y^{\frac{1}{4}}\right)-\frac{1}{5}\left(x^{\frac{1}{4}}+y^{\frac{1}{4}}\right)\left(x^{\frac{1}{5}}+y^{\frac{1}{5}}\right)}{\left(x^{\frac{1}{5}}+y^{\frac{1}{5}}\right)^2}$$

$$=\frac{1}{20}\frac{\left(x^{\frac{1}{5}}+y^{\frac{1}{5}}\right)\left(x^{\frac{1}{4}}+y^{\frac{1}{4}}\right)}{\left(x^{\frac{1}{5}}+y^{\frac{1}{5}}\right)^2}=\frac{1}{20}\frac{x^{\frac{1}{4}}+y^{\frac{1}{4}}}{x^{\frac{1}{5}}+y^{\frac{1}{5}}}=\frac{1}{20}\text{ U.}$$

$\therefore\ x\frac{\partial u}{\partial x}+y\frac{\partial u}{y}=\frac{1}{20}u$ Hence the Euler's theorem is verigied.

Exercise - 3

1. Examine and find the maximum and minimum.

(i) x^3+y^3-3axy

Solution: Let $f(x, y) = x^3 + y^3 - 3axy$ ..... (i)

Differentiating (i) partially w. r. t. x & y resp. we get,

$\therefore f_x = 3x^2 - 3ay$ $\qquad \therefore f_y = 3y^2 - 3ax$

$\therefore f_{xx} = 6x$ $\qquad \therefore f_{yy} = 6y$

$\therefore f_{xy} = -3a$

For the stationary points

$f_x = 0, \quad f_y = 0$

i.e. $3x^2 - 3ay = 0 \quad \Rightarrow x^2 - ay = 0$ ..... (ii)

and $3y^2 - 3ax = 0 \quad \Rightarrow y^2 - ax = 0$ ..... (iii)

Solving (ii) and (iii)

$x = 0,\ y = 0$ and $x = a, y = a$

$\therefore$ The stationary points are (0, 0) and (a, a)

At (0, 0) $\quad f_{xx} = 0, \quad f_{yy} = 0, \quad f_{xy} = -3a$

Now, $f_{xx}f_{yy} - (f_{xy})^2 = -9a^2 < 0$

Hence, there is value at (0, 0) no extreme

At, (a, a)

$f_{xx} = 6a, \quad f_{yy} = 6a, \quad f_{xy} = -3a$

Now, $f_{xx}f_{yy} - (f_{xy})^2 = 36a^2 - 9a^2 = 27a^2 > 0$

and $f_{xx} = 6a > 0$ if $a > 0$

and $f_{xx} = 6a < 0$ if $a < 0$

Hence, given function has a maximum or a minimum value at (a, a) if $a < 0$ or $a > 0$ and the maximum or minimum value,

$f(a, a) = a^3 + a^3 - 3a^3$

$= -a^3$ Ans.

(ii) $x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$

Solution: Let $f(x, y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$ ..... (i)

Differentiating (i) partially w. r. t. x & y resp. we get,

$\therefore f_x = 3x^2 + 3y^2 - 30x + 72$

$\therefore f_y = 6xy - 30y$

$\therefore f_{xx} = 6x - 30$

$\therefore f_{yy} = 6x - 30$

$\therefore f_{xy} = 6y$

For extreme value $f_x = 0, f_y = 0$

i.e. $3x^2 + 3y^2 - 30x + 72 = 0$ ..... (ii)

and $6xy - 30y = 0$ ..... (iii)

From (iii),

$6xy - 30y = 0$

or, $6y(x-5) = 0$

$\therefore\ y = 0,\ x = 5$

If $y = 0$ then equation (ii) gives,

$3x^2 + 0 - 30x + 72 = 0$

$\Rightarrow\ x^2 - 10x + 24 = 0$

$\Rightarrow\ (x-4)(x-6) = 0$

$\therefore\ x = (4, 6)$

$\therefore\ (4, 0)\ (6, 0)$

Again, if $x = 5$ then equation (ii) gives,

$3.25 + 3y^2 - 30.5 + 72 = 0$

$3y^2 - 3 = 0$

$\therefore\ y = \pm 1$

$\therefore\ (5, \pm 1)$

$\therefore$ The stationary points are,

$(4, 0)\ (6, 0)$ and $(5, \pm 1)$

At, $(4, 0)$

$f_{xx} = 6.4 - 30 = -6$

$f_{yy} = 6.4 - 30 = -6$

$f_{xy} = 6.0 = 0$

Now, $f_{xx}f_{yy} - (f_{xy}) = -6 . -6 = 36 > 0$

and $f_{xx} = -6 < 0$

Thus the given function has maximum value at $(4, 0)$ and maximum value is,

$f(4, 0) = 64 + 0 - 15 \times 16 - 0 + 72 .4$

$= 64 - 240 + 288 = 112$ Ans.

At, $(6, 0)$

$f_{xx} = 6.6 - 30 = 6$

$f_{yy} = 6.6 - 30 = 6$

$f_{xy} = 6.0 = 0$

Now, $f_{xx}f_{yy} - (f_{xy})^2 = 6.6 - 0$

$= 36 > 0$

and $f_{xx} = 6 > 0$

Thus the given function $f(x, y)$ has minimum value at $(6, 0)$ and the minimum value is,

$f(6, 0) = 6^3 + 0 - 15.36 - 0 + 72 .6$

$= 216 - 540 + 432 = 108$

At $(5, \pm 1)$

$f_{xx} = 6.5 - 30 = 0$

$f_{yy} = 6.5 - 30 = 0$

$f_{xy} = 6 \times \pm 1 = \pm 6$

Now,

$f_{xx}f_{yy} - (f_{xy})^2 = 0 - (\pm 6)^2 = -36 < 0$

and $f_{xx} = 0$

Thus $f_{xx}f_{yy} - (f_{xy}) < 0$ and $f_{xx} = 0$ so the given function has heither maxima nor minima at $(5, \pm 1)$ Ans.

(iii) $xy(a - x - y)$

Solution: Let $f(x, y) = xy(a - x - y)$

$= axy - x^2y - xy^2$ ..... (i)

Differentiating (i) partially w. r. t. x & y resp. we get,

$\therefore\ f_x = ay - 2xy - y^2$ $\qquad \therefore\ f_y = ax - x^2 - 2xy$

$\therefore\ f_{xx} = -2y$ $\qquad \therefore\ f_{yy} = -2x$

$\therefore\ f_{xy} = a - 2x - 2y$

For extreme value, $f_x = 0$, $f_y = 0$

i.e. $ay - 2xy - y^2 = 0$ ..... (ii)

and $ax - x^2 - 2xy = 0$ ..... (iii)

From (ii)

$y(a - 2x - y) = 0$

$\Rightarrow\ y = 0,\ a - 2x - y = 0$ ..... (iv)

From (iii)

$x(a - x - 2y) = 0$

$\Rightarrow\ x = 0,\ a - x - 2y = 0$ ..... (v)

Solving (iv) and (v)

$$\begin{array}{r} 2a - 4x - 2y = 0 \\ a - x - 2y = 0 \\ - \quad + \quad + \\ \hline a - 3x = 0 \end{array}$$

$\Rightarrow\ x = \dfrac{a}{3}$

Put $x = \dfrac{a}{3}$ in (iv)

$a - 2 . \dfrac{a}{3} - y = 0$

$\dfrac{3a - 2a}{3} - y = 0 \qquad \Rightarrow y = \dfrac{a}{3}$

$\therefore$ The stationary points are $(0, 0)$ and $\left(\dfrac{a}{3}, \dfrac{a}{3}\right)$

At, $(0, 0)$

$f_{xx} = 0,\ f_{yy} = 0,\ f_{xy} = a$

Now, $f_{xx}f_{yy} - (f_{xy})^2 = -a^2 < 0$

Hence, there is no extreme value at $(0, 0)$

At $\left(\dfrac{a}{3}, \dfrac{a}{3}\right)$

$f_{xx} = -\dfrac{2a}{3}, \qquad f_{yy} = -\dfrac{2a}{3}$

$$f_{xy} = a - \frac{2a}{3} - \frac{2a}{3} = \frac{3a - 2a - 2a}{3} = \frac{a}{3}$$

Now, $f_{xx}; f_{yy} - (f_{xy})^2 = \left(-\frac{2a}{3}\right) \cdot \left(-\frac{2a}{3}\right) - \left(\frac{a}{3}\right)^2$

$$= \frac{4a^2}{9} - \frac{a^2}{9} = \frac{3a^2}{9} = \frac{a^2}{3} > 0$$

and, $f_{xx} = -\frac{2a}{3} > 0$ if $a < 0$

$f_{xx} = -\frac{2a}{3} < 0$ if $a > 0$

Hence the given function has a maximum or a minimum value at $\left(\frac{a}{3}, \frac{a}{3}\right)$ if $a > 0$ or $a < 0$ and maximum or minimum value is,

$$f\left(\frac{a}{3}, \frac{a}{3}\right) = \frac{a}{3} \cdot \frac{a}{3}\left(a - \frac{a}{3} - \frac{a}{3}\right)$$

$$= \frac{a^2}{9}\left(\frac{3a - a - a}{3}\right) = \frac{a^3}{27} \text{ Ans.}$$

(iv) $2(x-y)^2 - x^4 - y^4$

Solution: Let $f(x, y) = 2(x-y)^2 - x^4 - y^4$

Differentiating (i) partially w. r. t. x & y resp. we get,

$\therefore f_x = 4(x-y) - 4x^3$ $\quad\therefore f_y = -4(x-y) - 4y^3$

$\therefore f_{xx} = 4 - 12x^2$ $\quad\therefore f_{yy} = 4 - 12y^2$

$\therefore f_{xy} = -4$

Now, for extreme value $f_x = 0$, $f_y = 0$

i.e. $\left.\begin{aligned} 4x - 4y - 4x^3 &= 0 \\ -4x + 4y - 4y^3 &= 0 \end{aligned}\right\}$

$\Rightarrow \left.\begin{aligned} x - y - x^3 &= 0 \\ -x + y - y^3 &= 0 \end{aligned}\right\}$

$\Rightarrow \left.\begin{aligned} y &= x - x^3 \\ x - y + y^3 &= 0 \end{aligned}\right\}$

$\therefore x - (x - x^3) + (x - x^3)^3 = 0$

$\Rightarrow x^3 + x^3(1 - x^2)^3 = 0$

$\Rightarrow x^3\{1 + (1 - x^2)^3\} = 0$

Either $x^3 = 0$ $\quad$ or, $1 + (1 - x^2)^3 = 0$

$(1 - x^2)^3 = -1$

$(1 - x^2)^3 = (-1)^3$

$\Rightarrow 1 - x^2 = -1$

$\therefore x^2 = 2$

$\therefore x = \pm\sqrt{2}$

Taken from equation $y = x - x^3$

$= x(1 - x^2) = \pm\sqrt{2}\,(1 - 2) = \pm\sqrt{2}$

$\therefore$ The stationary points are $(0, 0)$, $(\sqrt{2}, -\sqrt{2})$ and $(-\sqrt{2}, \sqrt{2})$

At $(0, 0)$: $f_{xx} = 4$, $f_{yy} = 4$ & $f_{xy} = -4$

Now, $f_{xx} f_{yy} - (f_{xy})^2 = 4.4 - (-4)^2 = 16 - 16 = 0$

Hence, the given function has no information at $(0, 0)$

At, $(\pm\sqrt{2}, \pm\sqrt{2})$

$f_{xx} = 4 - 24 = -20$

$f_{yy} = 4 - 24 = -20$

$f_{xy} = -4$

Now, $f_{xx}f_{yy} - (f_{xy})^2$

$= (-20).(-20) - (-4)^2 = 400 - 16 = 384 > 0$

and $f_{xx} = -20 < 0$

Thus the given function $f(x, y)$ has maximum value at $(\pm\sqrt{2}, \pm 2)$ and the maximum value is,

$f(\pm\sqrt{2}, \pm\sqrt{2}) = 2(\pm\sqrt{2} \pm\sqrt{2})^2 - (\pm\sqrt{2})^4 - (\pm\sqrt{2})^4$

$= 2(\pm 2\sqrt{2})^2 - 4 - 4 = 16 - 8 = 8$ Ans.

(v) $16 - (x+2)^2 - (y-2)^2$

Solution: Let $f(x, y) = 16 - (x+2)^2 - (y-2)^2$ ..... (i)

Differentiating (i) partially w. r. t. x & y resp. we get,

$\therefore f_x = -2(x+2)$ $\quad\therefore f_y = -2(y-2)$

$\therefore f_{xx} = -2$ $\quad\therefore f_{yy} = -2$

$\therefore f_{xy} = 0$

Now, for extreme value,

$f_x = 0$, $f_y = 0$

i.e. $-2x - 4 = 0 \quad \Rightarrow x = -2$

$-2y + 4 = 0 \quad \Rightarrow y = 2$

$\therefore$ The stationary points is $(-2, 2)$

Now, At $(-2, 2)$

$f_{xx}f_{yy} - (f_{xy})^2 = -2 . -2 - (0)^2$

$= 4 > 0$

and $f_{xx} = -2 < 0$

Hence the given function $f(x, y)$ has maximum value at $(-2, 2)$ and the maximum value is

$f(-2, 2) = 16 - (-2+2)^2 - (2-2)^2$

$= 16$ Ans.

(vi) $8 - 4x + 4y - x^2 - y^2$

Solution: Let $f(x, y) = 8 - 4x + 4y - x^2 - y^2$ ..... (i)

Differentiating (i) partially w. r. t. x & y resp. we get,

$\therefore f_x = -4 - 2x$ $\quad\therefore f_y = 4 - 2y$

$\therefore f_{xx} = -2$ $\quad\therefore f_{yy} = -2$

$\therefore f_{xy} = 0$

Now, for extreme value,

$$f_x = 0, \quad f_y = 0$$

i.e. $-4 - 2x = 0 \quad \Rightarrow x = -2$

$4 - 2y = 0 \quad \Rightarrow y = 2$

$\therefore$ The stationary points is $(-2, 2)$

Now,

At $(-2, 2)$

$$f_{xx}f_{yy} - (f_{xy})^2 = (-2)(-2) - (0)^2$$
$$= 4 > 0$$

and, $f_{xx} = -2 < 0$

Hence the given function f(x, y) is maximum value at $(-2, 2)$ and maximum value is,

$f(-2, 2) = 8 - 4(-2) + 4.2 - (-2)^2 - (2)^2$

$= 8 + 8 + 8 - 4 - 4 = 16$ Ans.

(vii) $2x^2 - 3xy + 4y^2 + 5$

Solution: Let $f(x, y) = 2x^2 - 3xy + 4y^2 = 5$ ..... (i)

Differentiating (i) partially w. r. t. x & y resp. we get,

$\therefore f_x = 4x - 3y \qquad \therefore f_y = -3x + 8y$

$\therefore f_{xx} = 4 \qquad \therefore f_{yy} = 8$

$\therefore f_{xy} = -3$

Now, for extreme value

$$f_x = 0, \quad f_y = 0$$

i.e. $4x - 3y = 0$ and $-3x + 8y = 0$

Solving these, we gat $x = 0$ and $y = 0$.

$\therefore$ The stationary points is $(0, 0)$

Now,

At $(0, 0)$

$$f_{xx}\, f_{yy} - (f_{xy})^2 = 4.8 - (-3)^2 = 32 - 9 \qquad = 23 > 0$$

and $f_{xx} = 4 > 0$

Hence the given function f(x, y) is minimum value at $(0, 0)$ and minimum value is

$f(0, 0) = 0 - 0 + 0 + 5 = 5$ Ans.

2. Examine and find the maximum and minimum of.

(i) $y^2 + 2z^2 - 5x^4 + 4x^5$

Solution: Let $f(x, y, z) = y^2 + 2z^2 - 5x^4 + 4x^5$ ..... (i)

Differentiating (i) partially we get,

$\therefore f_x = -20x^3 + 20x^4,\ f_y = 2y,\ f_z = 4z$

$\therefore f_{xx} = -60x^2 + 80x^3;\ f_{yy} = 2,\ f_{zz} = 4$

$\therefore f_{yx} = 0,\ f_{zx} = 0 \qquad f_{zy} = 0,\ f_{xy} = 0,\ f_{zx} = 0,\ f_{yz} = 0$

For extreme value,

$f_x = 0,\ f_y = 0,\ f_z = 0$

i.e. $-20x^3 + 20x^4 = 0$

$2y = 0$

$4z = 0$

$\Rightarrow 20x^3(x - 1) = 0$

$y = 0$

$z = 0$

$\Rightarrow x = 0$ and $x = 1$

$y = 0$

$z = 0$

$\therefore$ The stationary point are, $(0, 0, 0)$ $(1, 0, 0)$

At $(1, 0, 0)$

$f_{xx} = 80.1^3 - 60.1^3 = 80 - 60 = 20 > 0$

Also, $f_{xx}f_{yy} - (f_{xy})^2 = 20 \times 2 - (0)^2 = 40 > 0$

and

$$\begin{vmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{vmatrix} = \begin{vmatrix} 20 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{vmatrix} = 160 > 0$$

Hence the given function f (x, y, z) is minimum value at $(1, 0, 0)$ and minimum value is,

$f(1, 0, 0) = 0 + 0 - 5.1^4 + 4.1^3 = -5 + 4 = -1$ Ans.

$-3x^2 + 6xy + 4y - 2y^2 - 6z^2$

Solution: Let $f(x, y, z) = -3x^2 + 6xz + 4y - 2y^2 - 6z^2$ ..... (i)

Differentiating (i) partially we get,

$\therefore f_x = -6x + 6z, \qquad f_y = 4 - 4y, \qquad f_z = 6x - 12z$

$\therefore f_{xx} = -6 \qquad f_{yy} = -4 \qquad f_{zz} = -12$

$\therefore f_{yx} = 0 \qquad f_{xy} = 0 \qquad f_{xz} = 6$

$= f_{zx} = 6 \qquad f_{zy} = 0 \qquad f_{yz} = 0$

For extreme value,

$f_x = 0,\ f_y = 0,\ f_z = 0$

$$\text{i.e.} \left.\begin{aligned} -6x + 6z &= 0 \\ 4 - 4y &= 0 \\ 6x - 12z &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} -x + z &= 0 \\ y &= 1 \\ x - 2z &= 0 \end{aligned}$$

Solving these, we get,

$x = 0,\ y = 1, \qquad z = 0$

$\therefore$ The stationary points is $(0, 1, 0)$

At $(0, 1, 0)$

$f_{xx} = -6 < 0$

Also, $f_{xx}\, f_{yy} - (f_{xy})^2 = (-6)(-4) - (0)^2$

$= 24 > 0$

and $$\begin{vmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{vmatrix} = \begin{vmatrix} -6 & 0 & 6 \\ 0 & -4 & 0 \\ 6 & 0 & -12 \end{vmatrix}$$

$$= \begin{vmatrix} -6 & 0 & 6 \\ 0 & -4 & 0 \\ 0 & 0 & -6 \end{vmatrix} = -6 \times 24 = -144 < 0$$

Hence the given function f(x, y, z) is minimum value at (0, 1, 0) a
the minimum value is,

$$f(0, 1, 0) = 0 + 0 + 4.1 - 2.1^2 - 0$$
$$= 2 \text{ Ans.}$$

(iii) $35 - (2x + 3)^2 - (y - 4)^2 - (z + 1)^2$

Solution: Let $f(x, y, z) = 35 - (2x + 3)^2 - (y - 4)^2 - (z + 1)^2$ ..... (i)

Differentiating (i) partially we get,

$\therefore f_x = -2(2x + 3)$, $\quad f_y = -2(y - 4)$, $\quad f_z = -2(z + 1)$

$\therefore f_{xx} = -4 \quad f_{yy} = -2 \quad f_{zz} = -2$

$\therefore f_{yx} = 0 \quad f_{xy} = 0 \quad f_{xz} = 0$

$\therefore f_{zx} = 0 \quad f_{zy} = 0 \quad f_{yz} = 0$

For extreme points,

$$f_x = 0, \; f_y = 0, \; f_z = 0$$

i.e. $-2(2x + 3) = 0 \quad \Rightarrow x = -\dfrac{3}{2}$

$-2(y - 4) = 0 \quad \Rightarrow y = 4$

$-2(z + 1) = 0 \quad \Rightarrow z = -1$

$\therefore$ The stationary points is $\left(-\dfrac{3}{2}, 4, -1\right)$

Now, At $\left(-\dfrac{3}{2}, 4, -1\right)$

$$f_{xx} = -4 < 0$$

Also, $f_{xx}f_{yy} - (f_{xy})^2 = -4 . -2 - (0)^2$
$= 8 > 0$

and $\begin{vmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{vmatrix} = \begin{vmatrix} -4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{vmatrix} = -16 < 0$

Thus the given function f(x, y, z) is maximum value at $\left(-\dfrac{3}{2}, 4, -1\right)$ and the maximum value is,

$$f\left(-\frac{3}{2}, 4, -1\right) = 35 - \left(2.-\frac{3}{2} + 3\right)^2 - (4 - 4)^2 - (-1 + 1)^2$$
$$= 35 - 0 - 0 - 0$$
$$= 35 \text{ Ans.}$$

(iv) $8z + 2x^2 + 3y^2 + 4z^2 - 3xy$

Solution: Let $f(x, y, z) = 8z + 2x^2 + 3y^2 + 4z^2 - 3xy$ ..... (i)

Differentiating (i) partially we get,

$\therefore f_x = 4x - 3y \quad f_y = 6y - 3x \quad f_z = 8 + 8z$

$\therefore f_{xx} = 4 \quad f_{yy} = 6 \quad f_{zz} = 8$

$\therefore f_{yx} = -3 \quad f_{xy} = -3 \quad f_{xz} = 0$

$\therefore f_{zx} = 0 \quad f_{zy} = 0 \quad f_{yz} = 0$

For extreme points,

$$f_x = 0, \quad f_y = 0, \quad f_z = 0$$

i.e. $\left.\begin{aligned} 4x - 3y &= 0 \\ 6y - 3x &= 0 \\ 8 + 8z &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} 4x - 3y &= 0 \\ 2y - x &= 0 \\ z &= -1 \end{aligned}$

Solving these x = 0, y = 0, & z = – 1

$\therefore$ The stationary points is (0, 0, – 1)

Now,

A (0, 0, – 1); $\quad f_{xx} = 4 > 0$

Also, $f_{xx}f_{yy} - (f_{xy})^2 = 4.6 - (-3)^2$
$= 24 - 9$
$= 15 > 0$

and $\begin{vmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{vmatrix} = \begin{vmatrix} 4 & -3 & 0 \\ -3 & 6 & 0 \\ 0 & 0 & 8 \end{vmatrix}$

$$= 8 \begin{vmatrix} 4 & -3 \\ -3 & 6 \end{vmatrix} = 8(24 - 9)$$
$$= 8 \times 15 = 120 > 0$$

Hence the given function f(x, y, z) is minimum value at (0, 0, – 1) and the minimum value is,

$$f(0, 0, -1) = 8(-1) + 0 + 0 + 4 . (-1)^2 - 0$$
$$= -8 + 4$$
$$= -4 \text{ Ans.}$$

(v) $20 - x^2 - y^2 - z^2$

Solution: Let $f(x, y, z) = 20 - x^2 - y^2 - z^2$ ..... (i)

Differentiating (i) partially we get,

$\therefore f_x = -2x, \quad f_y = -2y \quad f_z = -2z$

$\therefore f_{xx} = -2 \quad f_{yy} = -2 \quad f_{zz} = -2$

$\therefore f_{yx} = 0 \quad f_{xy} = 0 \quad f_{xz} = 0$

$\therefore f_{zx} = 0 \quad f_{zy} = 0 \quad f_{yz} = 0$

For the extreme value

$$f_x = 0, \; f_y = 0, \; f_z = 0$$

i.e. $\left.\begin{aligned} -2x &= 0 \\ -2y &= 0 \\ -2z &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} x &= 0 \\ y &= 0 \\ z &= 0 \end{aligned}$

$\therefore$ The stationary point is (0, 0, 0)

Now,

At (0, 0, 0)

$$f_{xx} = -2 < 0$$

Also, $f_{xx}f_{yy} - (f_{xy})^2 = (-2)(-2) - (0)^2$

and $\begin{vmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{vmatrix} = \begin{vmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{vmatrix} = -8 < 0$

Hence the given function f(x, y, z) is minimum value at (0, 0, 0) and the minimum value is $f(0, 0, 0) = 20 - 0 - 0 - 0$

$= 20$ Ans.

3. Show that following functions have a minimum at the points indicated.

(i) $x^2 + y^2 + z^2 + 2xyz$ at (0, 0, 0)

Solution: Let $f(x, y, z) = x^2 + y^2 + z^2 + 2xyz$ ..... (i)

Differentiating (i) partially we get,

$\therefore f_x = 2x + 2yz$ $\quad f_y = 2y + 2xz$ $\quad f_z = 2z + 2xy$

$\therefore f_{xx} = 2$ $\quad f_{yy} = 2$ $\quad f_{zz} = 2$

$\therefore f_{yy} = 2z$ $\quad f_{xy} = 2z$ $\quad f_{xz} = 2y$

$\therefore f_{zx} = 2y$ $\quad f_{zy} = 2x$ $\quad f_{yz} = 2x$

Now,

At (0, 0, 0)

$$f_{xx} = 2 > 0$$

Also, $f_{xx}f_{yy} - (f_{xy})^2 = 2.2 - 0$

$= 4 > 0$

and, $\begin{vmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{vmatrix} = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 8 > 0$

Hence, the given function is minimum value at (0, 0, 0)

(ii) $x^4 + y^4 + z^4 - 4xyz$ at (1, 1, 1)

Solution: Let $f(x, y, z) = x^4 + y^4 + z^4 - 4xyz$ ..... (i)

Differentiating (i) we get,

$\therefore f_x = 4x^3 - 4xyz,$ $\quad f_y = 4y^3 - 4zx$ $\quad f_z = 4z^3 - 4xy$

$\therefore f_{xx} = 12x^2$ $\quad f_{yy} = 12y^2$ $\quad f_{zz} = 12z^2$

$\therefore f_{yx} = -4z$ $\quad f_{xy} = -4z$ $\quad f_{xz} = -4y$

$\therefore f_{zx} = -4y$ $\quad f_{xy} = -4x$ $\quad f_{yz} = -4x$

Now,

At, (1, 1, 1)

$$f_{xx} = 12.1^2 = 12 > 0$$

Also, $f_{xx}f_{yy} - (f_{xy})^2 = 12\,(1)^2 \times 12\,(1)^2 - (4 \times 1)^2$

$= 144 - 4$

$= 140 > 0$

Hence the given function f(x, y, z) is minimum value at (1, 1, 1) Proved.

Find the extreme value for the function $x^2 + y^2$ under the condition $x + 4y = 2$.

Solution: Let $f(x, y) = x^2 + y^2$ ..... (i)

Under the condition

$x + 4y = 2$

or, $x = 2 - 4y$ ..... (ii)

From (i) and (ii)

$f(x) = (2 - 4y)^2 + y^2$ ..... (iii)

Differentiating (iii) w. r. t. 'y' we get

$$\frac{df}{dy} = 2\,(2 - 4y)\,.\,(-4) + 2y = -16 + 32y + 2y = -16 + 34y$$

$$\therefore \frac{d^2f}{dy^2} = +34 > 0$$

For the extreme value

$$\frac{df}{dy} = 0$$

i.e. $-16 + 34y = 0$

$$\Rightarrow y = -\frac{16}{34} = -\frac{8}{17}$$

From (ii)

$$x = 2 - 4\,.\,\frac{8}{17} = \frac{34-32}{17} = \frac{2}{17}$$

$\therefore$ The stationary points is $\left(\frac{2}{17}, \frac{8}{17}\right)$

Now, At, $\left(\frac{2}{17}, \frac{8}{17}\right)$; $\frac{d^2f}{dy^2} = 34 > 0$

Hence the given function f (x, y) has minimum value at $\left(\frac{2}{17}, \frac{8}{17}\right)$

and the minimum value is,

$$f\left(\frac{2}{17}, \frac{8}{17}\right) = \left(\frac{2}{17}\right)^2 + \left(\frac{8}{17}\right)^2$$

$$= \frac{4}{289} + \frac{64}{289} = \frac{4+64}{289} = \frac{68}{289} = \frac{4}{17} \text{ Ans.}$$

5. Find the extreme value of $48 - (x - 5)^2 - 3\,(y - 4)^2$ Subject to the condition $x + 3y = 9$.

Solution: Let $f(x, y) = 48 - (x - 5)^2 - 3\,(y - 4)^2$ ..... (i)

Subject to the condition

$x + 3y = 9$

or, $x = 9 - 3y$ ..... (ii)

From (i) and (ii)

$$f(y) = 48 - (9 - 3y - 5)^2 - 3(y - 4)^2$$
$$= 48 - (4 - 3y)^2 - 3(y - 4)^2 \text{ ..... (iii)}$$

Differentiating (iii) w. r. t. 'y' we get

$$\frac{df}{dy} = -2(4 - 3y).(-3) - 3.2(y - 4)$$
$$= 24 - 18y - 6y + 24 = 48 - 24y$$
$$\therefore \frac{d^2f}{dy^2} = -24 < 0$$

For the extreme value, $\frac{df}{dy} = 0$

i.e. $48 - 24y = 0$

or, $y = 2$

From (ii)

$$x = 9 - 3.2 = 3$$

Now, At (3, 2)

$$\frac{d^2f}{dy^2} = -24 < 0$$

Hence the given function f(x, y) had maximum value at (3, 2) and the maximum value is

$f(3, 2) = 48 - (3 - 5)^2 - 3(2 - 4)^2 = 48 - 4 - 12 = 32$ Ans.

6. Find the extreme value of $x^2 + y^2 + z^2$ Subject to the condition $x + y + z = 1$.

Solution: Let $f(x, y, z) = x^2 + y^2 + z^2$ ..... (i)

Subject to the condition $x + y + z = 1$

$$\Rightarrow z = 1 - x - y \text{ ..... (ii)}$$

From (i) and (ii)

$$f(x, y) = x^2 + y^2 + (1 - x - y)^2 \text{ ..... (iii)}$$

Differentiating (iii) partially w. r. t. 'x' & 'y' restp. we get

$$\therefore f_x = 2x + 2(1 - x - y).(-1)$$
$$= 2x - 2 + 2x + 2y$$
$$= 4x + 2y - 2$$

$\therefore f_{xx} = 4$ $\qquad \therefore f_{yx} = 2$

$$\therefore f_y = 2y + 2(1 - x - y).(-1)$$
$$= 2y - 2 + 2x + 2y$$
$$= 4y + 2x - 2$$

$\therefore f_{yy} = 4$ $\qquad \therefore f_{xy} = 2$

For extreme value is,

$$f_x = 0, \quad f_y = 0$$

i.e. $\left.\begin{array}{l} 4x + 2y - 2 = 0 \\ 4y + 2x - 2 = 0 \end{array}\right\} \Rightarrow \begin{array}{l} 2x + y - 1 = 0 \\ x + 2y - 1 = 0 \end{array}$

Solving these,

$$\begin{array}{l} 2x + y - 1 = 0 \\ 2x + 4y - 2 = 0 \\ \underline{-\quad -\quad +} \\ -3y + 1 = 0 \end{array}$$

$$y = \frac{1}{3}$$

$$\therefore 2x + \frac{1}{3} - 1 = 0$$

$$2x = \frac{3-1}{3} = \frac{2}{3}$$

$$\therefore x = \frac{1}{3}$$

From (ii)

$$z = 1 - \frac{1}{3} - \frac{1}{3} = \frac{3-1-1}{3} = \frac{1}{3}$$

$\therefore$ The stationary point is $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$

Now, at, $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$

$$f_{xx} = 4 > 0$$

and $f_{xx}f_{yy} - (f_{xy})^2 = 4.4 - (2)^2$

$$= 16 - 4 = 12 > 0$$

Hence the given function f(x, y, z) has minimum value at $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ and the minimum value is,

$$f\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) = \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2$$
$$= \frac{3}{9} = \frac{1}{3} \text{ Ans.}$$

7. Find the maximum and minimum values of xy^2 under the condition $x + y = 1$.

Solution: Let $f(x, y) = xy^2$ ..... (i)

Under the condition $x + y = 1$

$$x = 1 - y \text{ ..... (ii)}$$

From (i) and (ii)

$$f(y) = (1 - y)y^2$$
$$\text{or, } f(y) = y^2 - y^3 \text{ ..... (iii)}$$

Differentiating (iii) partially w. r. t. 'y' we get

$$\frac{df}{dy} = 2y - 3y^2$$

and $\frac{d^2f}{dy^2} = 2 - 6y$

For extreme value is,

$$2y - 3y^2 = 0$$
$$\text{or, } y(2 - 3y) = 0$$
$$\therefore\ y = 0,\ y = \frac{2}{3}$$

From (ii) when $y = 0$, $x = 1$

When $y = \frac{2}{3}$, $x = 1 - \frac{2}{3} = \frac{1}{3}$

$\therefore$ The stationary points are, $(1, 0)$ & $\left(\frac{1}{3}, \frac{2}{3}\right)$.

At, $(1, 0)$

$$\frac{d^2f}{dy^2} = 2 - 6.0 = 2 > 0$$

Hence the given function $f(x, y)$ has minimum value at $(1, 0)$ and the minimum value is $f(1, 0) = 0$

At $\left(\frac{1}{3}, \frac{2}{3}\right)$

$$\frac{d^2f}{dy^2} = 2 - 6 \cdot \frac{2}{3} = \frac{6-12}{3} = -\frac{6}{3} = -2 < 0$$

Hence the given function $f(x, y)$ has maximum value at $\left(\frac{1}{3}, \frac{2}{3}\right)$

and the maximum values is,

$$f\left(\frac{1}{3}, \frac{2}{3}\right) = \frac{1}{3} \cdot \left(\frac{2}{3}\right)^2 = \frac{4}{27} \text{ Ans.}$$

8. Find the maximum value of xyz under the condition $x + y + z = 8$.

Solution: Let $f(x, y, z) = xyz$ ..... (i)

Under the condition,

$$x + y + z = 8$$
$$\text{or, } z = 8 - x - y \text{ ..... (ii)}$$

From (i) and (ii)

$$f(x, y) = xy(8 - x - y)$$
$$= 8xy - x^2y - xy^2 \text{ ..... (iii)}$$

Differentiating (iii) partially we get

$\therefore f_x = 8y - 2xy - y^2$ $\quad$ $f_y = 8x - x^2 - 2xy$

$\therefore f_{xx} = -2y$ $\quad$ $f_{yy} = -2x$

$\therefore f_{yx} = 8 - 2x - 2y$ $\quad$ $f_{xy} = 8 - 2x - 2y$

For extreme value,

$f_x = 0,$ $\quad$ $f_y = 0$

i.e. $\left.\begin{aligned} 8y - 2xy - y^2 &= 0 \\ 8x - x^2 - 2xy &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} y(8 - 2x - y) &= 0 \\ x(8 - x - 2y) &= 0 \end{aligned}$

$\Rightarrow 8 - 2x - y = 0$

$8 - x - 2y = 0$

Solving these,

$$\begin{array}{l} 8 - 2x - y = 0 \\ 16 - 2x - 4y = 0 \\ \underline{-\ \ +\ \ \ +\ \ \ -} \\ -8 \qquad +3y = 0 \end{array}$$

$$\Rightarrow y = \frac{8}{3}$$

$$\therefore\ 8 - 2x - \frac{8}{3} = 0$$

$$\text{or, } 2x = \frac{24-8}{3} = \frac{16}{3} \Rightarrow x = \frac{8}{3}$$

From (ii) $\quad z = 8 - \frac{8}{3} - \frac{8}{3} = \frac{8}{3}$

$\therefore$ The stationary point is $\left(\frac{8}{3}, \frac{8}{3}, \frac{8}{3}\right)$

At $\left(\frac{8}{3}, \frac{8}{3}, \frac{8}{3}\right)$

$$f_{xx} = -2 \cdot \frac{8}{3} = -\frac{16}{3} < 0$$

and $f_{xx}f_{yy} - (f_{xy})^2$

$$= -2\left(\frac{8}{3}\right) \times -2\left(\frac{8}{3}\right) - \left\{8 - 2\left(\frac{8}{3}\right) - 2\left(\frac{8}{3}\right)\right\}^2$$

$$= \frac{256}{3} - \left(8 - \frac{16}{3} - \frac{16}{3}\right)^2$$

$$= \frac{256}{3} - \left(\frac{24-16-16}{3}\right)^2 = \frac{256}{3} + \frac{8}{9} > 0$$

Hence the given function $f(x, y, z)$ has maximum value at $\left(\frac{8}{3}, \frac{8}{3}, \frac{8}{3}\right)$ and maximum value is

$$f\left(\frac{8}{3}, \frac{8}{3}, \frac{8}{3}\right) = \frac{8}{3}, \frac{8}{3}, \frac{8}{3} = \frac{512}{27} \text{ Ans.}$$

9. Find the minimum value of $x^2 + y^2 + z^2$ Subject to the condition $x + y + z - 1 = 0$ and $xyz + 1 = 0$

(BE 2068 Magh)

Solution: Let $f(x, y, z) = x^2 + y^2 + z^2$ ..... (i)

Subject to the condition

$$x + y + z - 1 = 0$$

$$\text{or, } y + z = 1 - x \text{ ..... (ii)}$$

and $xyz + 1 = 0$

$$yz = -\frac{1}{x} \text{ ..... (iii)}$$

Now from (i)

$$f(x, y, z) = x^2 + y^2 + z^2$$
$$= x^2 + (y + z)^2 - 2yz$$

Using (ii) and (iii) we get,

$$\therefore f(x) = x^2 + (1 - x)^2 + \frac{2}{x} \text{ ..... (iv)}$$

Differentiating (iv) partially w. r. t. 'x' restp. we get

$$\therefore \frac{df}{dx} = 2x + 2(1 - x).(-1) - \frac{2}{x^2}$$

$$= 2x - 2 + 2x - \frac{2}{x^2} = -\frac{2}{x^2} + 4x - 2$$

$$\therefore \frac{d^2f}{dx^2} = +\frac{4}{x^3} + 4$$

For extrime value is,

$$\frac{df}{dx} = 0$$

i.e. $-\frac{2}{x^2} + 4x - 2 = 0$

or, $-2 + 4x^3 - 2x^2 = 0$

or, $2(-1 + 2x^3 - x^2) = 0$

or, $2x^3 - x^2 - 1 = 0$

or, $2x^3 - 2x^2 + x^2 - x + x - 1 = 0$

or, $2x^2(x - 1) + x(x - 1) + 1(x - 1) = 0$

or, $(x - 1)(2x^2 + x + 1) = 0$

$\therefore x - 1 = 0$

$\Rightarrow x = 1$

Using $x = 1$ in (ii) and (iii)

$$y + z = 0$$
$$\Rightarrow y = -z \text{ ..... (v)}$$
$$yz = -1$$

Using (v)

$$y.(-y) = -1$$
$$y^2 = 1$$
$$\therefore y = \pm 1$$

From (v) $z = \pm 1$

$\therefore$ The stationary points are,

$$(1, \pm 1, \pm 1)$$

At $(1, \pm 1, \pm 1)$

$$\frac{d^2f}{dx^2} = \frac{1}{1^3} + 4 = 8 > 0$$

Hence the given function f(x, y, z) has minimum value at $(1, \pm 1, \pm 1)$ & the minimum value is

$$f(1, \pm 1, \pm 1) = (1)^2 + (\pm 1)^2 + (\pm 1)^2 = 3 \text{ Ans.}$$

10. Find the minimum value $x^2 + y^2 + z^2$ when $x + y + z = 3a$

Solution: Let $f(x, y, z) = x^2 + y^2 + z^2$ ..... (i)

and the condition is,

$$x + y + z = 3a$$
$$\text{or, } z = 3a - x - y \text{ ..... (ii)}$$

From (i) and (ii)

$f(x, y) = x^2 + y^2 + (3a - x - y)^2$ ..... (iii)

Differentiating (iii) partially w. r. t. 'x' & 'y' restp. we get

$\therefore f_x = 2y + 2(3a - x - y).(-1)$

$= 2y - 6a + 2x + 2y = 4x - 6a + 2y$

$\therefore f_{xx} = 4 \qquad \therefore f_{yx} = 2$

$\therefore f_y = 2y + 2(3a - x - y).(-1)$

$= 2y - 6a + 2x + 2y = 2x + 4y - 6a$

$\therefore f_{yy} = 4$

$\therefore f_{xy} = 2$

For the extreme values: $f_x = 0$ and $f_y = 0$

$$\text{i.e. } \left.\begin{aligned} 4x + 2y - 6a = 0 \\ 2x + 4y - 6a = 0 \end{aligned}\right\} \Rightarrow \begin{aligned} 2(2x + y - 3a) = 0 \\ 2(x + 2y - 3a) = 0 \end{aligned}$$

$\therefore 2x + y - 3a = 0$ and $x + 2y - 3a = 0$

Solving these,

$$\begin{array}{l} 2x + y - 3a = 0 \\ 2x + 4y - 6a = 0 \\ - \quad - \quad + \\ \hline -3y + 3a = 0 \end{array}$$

$$\therefore y = a$$

$\therefore 2x + a - 3a = 0$

$$\therefore x = a$$

From (i) $z = 3a - a - a = a$

$\therefore$ The stationary point is (a, a, a)

At (a, a, a)

$$f_{xx} = 4 > 0$$

and $f_{xx}f_{yy} - (f_{xy})^2$

$= 4.4 - (2)^2 = 16 - 4 = 12 > 0$

Hence the given function f(x, y, z) has minimum value at (a, a, a) and minimum value is

$$f(a, a, a) = a^2 + a^2 + a^2 = 3a^2 \text{ Ans.}$$

Important IOE Questions and Answers

1. Find the extreme value of $\phi = x^2 + y^2 + z^2$ connected by the relation $ax + by + cz = p$ (BE, 2067 Mangsir, 2070 Bhadra)

Sol^n. Let $f(x, y, z) = x^2 + y^2 + z^2$ ..... (i)

Which are connected by the relation $ax + by + cz = p$

i.e. $z = \dfrac{p - ax - by}{c}$ ..... (ii)

With the help of (ii), the function (i) reduces to.

$$f(x, y) = x^2 + y^2 + \frac{1}{c^2}(p - ax - by)^2$$

Diff. partially,

$$f_x = 2x - \frac{2a}{c^2}(p - ax - by), \quad f_y = 2y - \frac{2b}{c^2}(p - ax - by)$$

$$f_{xx} = 2 + \frac{2a^2}{c^2}, \; f_{yy} = 2 + \frac{2b^2}{c^2}, \; f_{xy} = \frac{2ab}{c^2}$$

For extreme values $f_x = 0, f_y = 0$

i.e. $(a^2 + c^2)x + aby - ap = 0 \Rightarrow \quad abx + (b^2 + c^2)y - bp = 0$

Solving these two equations, we obtain,

$$x = \frac{ap}{a^2 + b^2 + c^2}, \; y = \frac{bp}{a^2 + b^2 + c^2}$$

Putting these to (ii)

$$z = \frac{cp}{a^2 + b^2 + c^2}$$

$\therefore$ The stationary points

$$\left(\frac{ap}{a^2 + b^2 + c^2}, \frac{bp}{a^2 + b^2 + c^2}, \frac{cp}{a^2 + b^2 + c^2}\right)$$

At $\left(\dfrac{ap}{a^2 + b^2 + c^2}, \dfrac{bp}{a^2 + b^2 + c^2}, \dfrac{cp}{a^2 + b^2 + c^2}\right)$; $f_{xx} = 2 + \dfrac{2a^2}{c^2} > 0$

$$f_{xx}f_{yy} - (f_{xy})^2 = 4\left(1 + \frac{a^2}{c^2}\right)\left(1 + \frac{b^2}{c^2}\right) - \frac{4a^2b^2}{c^4} = 4\left(1 + \frac{a^2}{c^2} + \frac{b^2}{c^2}\right) > 0$$

Hence the given function in minimum at

$\left(\dfrac{ap}{a^2 + b^2 + c^2}, \dfrac{bp}{a^2 + b^2 + c^2}, \dfrac{cp}{a^2 + b^2 + c^2}\right)$ and minimum values is,

$$f = \frac{a^2p^2 + b^2p^2 + c^2p^2}{\left(a^2 + b^2 + c^2\right)^2} = \frac{p^2}{a^2 + b^2 + c^2} \text{ Ans.}$$

2. Obtain the maximum value of xyz such that $x + y + z = 24$. (BE 2069 Bhadra)

Sol^n. Let $f(x, y, z) = xyz$ ..... (i)

Such that $x + y + z = 24$ or, $z = 24 - x - y$ ..... (ii)

From (i) and (ii)

$f(x, y) = xy(24 - x - y) \Rightarrow f(x, y) = 24xy - x^2y - xy^2$ ..... (iii)

Differentiating (iii) partially, we get,

$\therefore f_x = 24y - 2xy - y^2$; $f_y = 24x - 2xy - x^2$

$\therefore f_{xx} = -2y$; $f_{yy} = -2x$

$\therefore f_{yx} = 24 - 2x - 2y$; $f_{xy} = 24 - 2x - 2y$

For extreme value $f_x = 0, f_y = 0$

i.e. $24y - 2xy - y^2 = 0 \Rightarrow y(24 - 2x - y)$

$24x - 2xy - x^2 = 0 \Rightarrow x(24 - x - 2y)$

$\Rightarrow 24 - 2x - y = 0$

$24 - x - 2y = 0$

Solving them,

$$\begin{array}{l} 24 - 2x - y = 0 \\ 48 - 2x - 4y = 0 \\ - \quad + \quad + \\ \hline -24 = 3y \Rightarrow y = 8 \end{array}$$

$\therefore 24 - 2x - 8 = 0 \Rightarrow x = 8$

From (ii) $z = 24 - 8 - 8 = 8$ $\therefore$ The stationary point is (8, 8, 8)

At (8, 8, 8); $f_{xx} = -2.8 = -16 < 0$

and $f_{xx}f_{yy} - (f_{xy})^2 = \{-2(8)\} \times \{-2(8)\} - \{24 - 2 \times 8 - 2 \times 8)\}$
$= 256 - 64 = 192 > 0$

Hence the given function is maximum at (8, 8, 8) and the maximum value is $f(8, 8, 8) = 512$ Ans.

3. Find the minimum value of $x^2 + xy + y^2 + 3z^2$ under the condition $x + 2y + 4z = 60$. (BE 2068 Bhadra)

Sol^n. Let $f(x, y, z) = x^2 + xy = y^2 + 3z^2$ ..... (i)

$\phi(x, y, z) = x + 2y + 4z - 60 = 0$ ..... (ii)

We defined, $F = f(x, y, z) + \lambda(\phi(x, y, z))$

Where λ is hagrange's multiplies.

$F = x^2 + xy + y^2 = 3z^2 + \lambda(x + 2y + 4z - 60)$

Differentiating partially,

$f_x = 2x = y + \lambda,\ f_y = x + 2y = 2\lambda,\ f_3 = 6z + 4\lambda$

For stationary points, $f_x = 0,\ f_y = 0,\ f_z = 0$

Solving these three equation and (ii) we have,

$$\lambda = -\frac{90}{7},\ x = 0,\ y = \frac{90}{7},\ z = \frac{60}{7}$$

$\therefore$ The stationary point is $\left(0, \frac{90}{7}, \frac{60}{7}\right)$

Differentiating (i) and (ii) partially,

$f_x = 2x + y,\ f_y = x + 2y,\qquad f_z = 6z$

$f_{xx} = 2 \qquad f_{yy} = 2 \qquad f_{zz} = 6$

$f_{yx} = 1 \qquad f_{zy} = 0 \qquad f_{xz} = 0$

$\phi_x = 1 \qquad \phi_y = 2 \qquad \phi_z = 4$

Now, at, $\left(0, \frac{90}{7}, \frac{60}{7}\right)$

$$\begin{vmatrix} 0 & \phi_x & \phi_y \\ \phi_x & f_{xx} & f_{xy} \\ \phi_y & f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 0 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 2 \end{vmatrix} = -8 < 0 \ \&\ \begin{vmatrix} 0 & \phi_x & \phi_y & \phi_z \\ \phi_x & f_{xx} & f_{xy} & f_{xz} \\ \phi_y & f_{yx} & f_{yy} & f_{yz} \\ \phi_z & f_{zx} & f_{zy} & f_{zz} \end{vmatrix} = \begin{vmatrix} 0 & 1 & 2 & 4 \\ 1 & 2 & 1 & 0 \\ 2 & 1 & 2 & 0 \\ 4 & 0 & 0 & 6 \end{vmatrix} = -12 < 0$$

Hence the given function is minimum at $\left(0, \frac{90}{7}, \frac{60}{7}\right)$ and the minimum value if $f\left(0, \frac{90}{7}, \frac{60}{7}\right) = \frac{8100}{49} + \frac{10800}{49} = \underline{2700}$ Ans.

4. Find the minimum value of the function $F(x, y, z) = x^2 + y^2 + z^2$ when $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$ **(BE 2069 Poush)**

Sol^n. Let $f(x, y, z) = x^2 + y^2 + z^2$ ..... (i)

Subject to the relation $\phi = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} - 1 = 0$ ..... (ii)

Then, $F = f(x, y, z) + \lambda\phi\,(x, y, z)$ where λ is a Lagrange's multiplier.

$\therefore F = x^2 + y^2 + z^2 + \lambda\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z} - 1\right)$ ..... (iii)

Differentiating (iii) primarily w. r. t. x, y and z respectively we gets,

$$f_x = 2x - \frac{\lambda}{y^2},\ F_y = 2y - \frac{\lambda}{y^2},\ F_z = 2z - \frac{\lambda}{z^2}$$

For extreme point, $F_x = 0,\ F_y = 0,\ F_z = 0$

i.e. $2x - \frac{\lambda}{y^2} = 0,\ 2y - \frac{\lambda}{y^2} = 0,\ 2z - \frac{\lambda}{z^2} = 0$

Solving there, $x = \left(\frac{\lambda}{2}\right)^{\frac{1}{3}},\ y = \left(\frac{\lambda}{2}\right)^{\frac{1}{3}},\ z = \left(\frac{\lambda}{2}\right)^{\frac{1}{3}}$.

Substituting these value of x, y, z in (ii)

$$\left(\frac{\lambda}{2}\right)^{\frac{1}{3}} + \left(\frac{\lambda}{2}\right)^{\frac{1}{3}} + \left(\frac{\lambda}{2}\right)^{\frac{1}{3}} - 1 = 0 \text{ or, } 3\left(\frac{2}{\lambda}\right)^{\frac{1}{3}} = 1,\ \Rightarrow \lambda = 54$$

Then $x = 3,\ y = 3$ and $z = 3$

$\therefore$ The stationery point (3, 3, 3)

Differentiating (i) and (ii) partially,

$\therefore f_x = 2x,\ f_y = 2y,\ f_z = 2z \qquad \therefore f_{xx} = 2,\ f_{yy} = 2,\ f_{zz} = 2$

$\therefore f_{yx} = 0,\ f_{xy} = 0,\ f_{yz} = 0 \qquad \therefore f_{zx} = 0,\ f_{zy} = 0,\ f_{xz} = 0$

$\therefore \phi_x = -\frac{1}{x^2},\ \phi_y = -\frac{1}{y^2},\ \phi_z = -\frac{1}{z^2}$

At (3, 3, 3);
$$\begin{vmatrix} 0 & \phi_x & \phi_y \\ \phi_x & f_{xx} & f_{xy} \\ \phi_y & f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 0 & -\frac{1}{9} & -\frac{1}{9} \\ -\frac{1}{9} & 2 & 0 \\ -\frac{1}{9} & 0 & 2 \end{vmatrix} = -\frac{4}{81} < 0$$

and
$$\begin{vmatrix} 0 & \phi_x & \phi_y & \phi_z \\ \phi_x & f_{xx} & f_{xy} & f_{xz} \\ \phi_y & f_{yx} & f_{yy} & f_{yz} \\ \phi_z & f_{zx} & f_{zy} & f_{zz} \end{vmatrix} = \begin{vmatrix} 0 & -\frac{1}{9} & -\frac{1}{9} & -\frac{1}{9} \\ -\frac{1}{9} & 2 & 0 & 0 \\ -\frac{1}{9} & 0 & 2 & 0 \\ -\frac{1}{9} & 0 & 0 & 2 \end{vmatrix} = -\frac{4}{27} < 0$$

Hence the given function is minimum at (3, 3, 3) and the minimum value is,

$f(3, 3, 3) = 3^2 + 3^2 + 3^2 = 27$ Ans.

Multiple Integral

2

Double integrals

Properties of double integrals

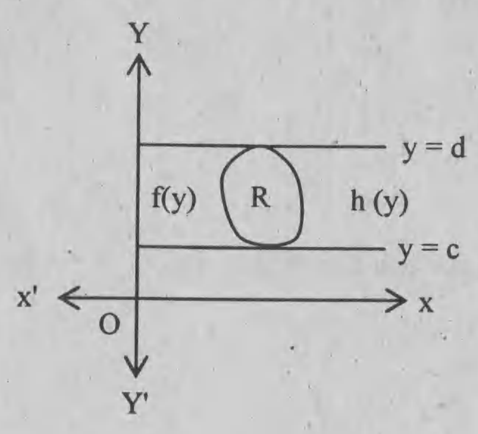
If f(x, y) and g (x, y) are defined, bounded and continuous in a region R of the (x, y) plane and if the region R may be subdivided into two regions R_1 and R_2, then,

1. $\iint_R k\, f(x, y)\, dxdy = k \iint_R f(x, y)\, dx\, dy$
   Where k is constant.
2. $\iint_R [f(x, y) \pm g(x, y)]\, dxdy = \iint_R f(x, y)\, dx\, dy \pm \iint_R g(x, y)\, dxdy$
3. $\iint_R f(x, y)\, dxdy = \iint_{R_1} f(x, y)\, dxdy + \iint_{R_2} f(x, y)\, dxdy$

Concept of double integrals

Suppose that the region R is described by the inequalities

$C \le y \le d$ and $g(y) \le x \le h(y)$



Then,

$$\iint_R f(x, y)\, dxdy = \int_{y=c}^{y=d}\left[\int_{x=g(y)}^{x=h(y)} f(x,y)\, dxdy\right] dy$$

Here, the inner integral is first evaluated with respect to x.

...uble integrals in polar co-ordinates

...double integral in the eartesian form

$\iint_R f(x, y)\, dx\, dy$ ..... (i)

Transformed to polar coordintes (i) becomes

$\iint_R f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$

...le integrals

$$...(x, y, z)\, dv = \int_{x_1}^{x_2}\int_{y_1(x)}^{y_2(x)}\int_{z_2(x,y)}^{z_2(x,y)} f(x,y,z)\ dzdydx$$

...itchlel's integrals

...is a region by $x \ge 0$, $y \ge 0$ and $z \ge 0$ and $x + y + z \le 1$ then,

$$\iiint_v x^{\ell-1}\, y^{m-1}\, z^{n-1}\, dx\, dy\, dz = \frac{\Gamma\ell\, \Gamma m\, \Gamma n}{\Gamma(\ell + m + n + 1)}$$

Area = $\iint_A dxdy$

Volume = $\iiint_v dxdydz$

Exercise - 4

Evaluate the following integrals.

$\int_0^3\int_1^2 xy\,(x + y)\, dxdy$

...tion: Let I = $\int_0^3\int_1^2 xy\,(x + y)\, dxdy$

$$= \int_0^3\left[\int_1^2\left(x^2y + xy^2\right)dx\right]dy$$

$$= \int_0^3\left[\frac{x^3y}{3} + \frac{x^2y^2}{2}\right]_1^2 dy$$

$$= \int_0^3\left[\frac{8}{3}y + \frac{4y^2}{2} - \frac{1}{3}y - \frac{1}{2}y^2\right]dy$$

$$= \int_0^3\left(\frac{7}{3}y + \frac{3y^2}{2}\right)dy$$

$$= \left[\frac{7}{3}\ \frac{y^2}{2} + \frac{3y^3}{2.3}\right]_0^3$$

$$= \frac{7}{3} \cdot \frac{9}{2} + \frac{1}{2} \cdot 27 - 0$$

$$= \frac{63}{6} + \frac{27}{2}$$

$$= \frac{63+81}{6}$$

$$= \frac{144}{6}$$

$\therefore$ I = 24 Ans.

(ii) $\int_0^3 \int_0^1 (x^2 + 3y^2)\, dy\, dx$

Solution: Let $I = \int_0^3 \left[\int_0^1 (x^2 + 3y^2) dy \right] dx$

$$= \int_0^3 \left[x^2 y + \frac{3y^3}{3} \right]_0^1 dx$$

$$= \int_0^3 (x^2 \cdot 1 + 1^3)\, dx$$

$$= \int_0^3 (x^2 + 1)\, dx$$

$$= \left[\frac{x^3}{3} + x \right]_0^3$$

$$= \frac{27}{3} + 3 - 0$$

$$= 9 + 3$$

$\therefore$ I = 12 Ans.

(iii) $\int_0^2 \int_0^{\sqrt{(2x-x^2)}} x\, dx\, dy$

Solution: Let $I = \int_0^2 \int_0^{\sqrt{(2x-x^2)}} x\, dx\, dy$

$$= \int_0^2 \left[\int_0^{\left(\sqrt{2x-x^2}\right)} x\, dy \right] dx$$

$$= \int_0^2 x \cdot [y]_0^{\sqrt{(2x-x^2)}}\, dx$$

$$= \int_0^2 x\sqrt{2x - x^2}\, dx$$

$$= -\frac{1}{2} \int_0^2 (2 - 2x - 2)\sqrt{2x - x^2}\, dx$$

$$= -\frac{1}{2} \int_0^2 (2 - 2x)\sqrt{2x - x^2}\, dx + \frac{1}{2}.2 \int_0^2 \sqrt{2x - x^2}\, dx$$

$$= -\frac{1}{2} \int_0^2 \left(2x - x^2\right)^{\frac{1}{2}} d\,(2x - x^2) + \int_0^2 \sqrt{1 - 1 + 2x - x^2}\, dx$$

$$= -\frac{1}{2} \left[\frac{\left(2x - x^2\right)^{\frac{1}{2}+1}}{\frac{1}{2}+1} \right]_0^2 + \int_0^2 \sqrt{1 - (1 - 2x + x^2)}\, dx$$

$$= -\frac{1}{2} \cdot \frac{2}{3}\ (4-4)^{\frac{3}{2}} + \int_0^2 \sqrt{1 - (1-x)^2}\, dx$$

$$= 0 + \left[\frac{(1-x)\sqrt{1-(1-x)^2}}{2} + \frac{1}{2}\sin^{-1}(1-x) \right]_0^2$$

$$= 0 + \frac{1}{2}\sin^{-1}(-1) - \frac{1}{2}\sin^{-1}(1)$$

$$= \frac{1}{2} \cdot \frac{3\pi}{2} - \frac{1}{2} \cdot \frac{\pi}{2}$$

$$= \frac{3\pi - \pi}{4}$$

$$= \frac{2\pi}{4}$$

$\therefore$ I = $\frac{\pi}{2}$ Ans.

(iv) $\int_0^1 \int_x^{\sqrt{x}} (x^2 + y^2)\, dx\, dy$

Solution: Let $I = \int_0^1 \int_x^{\sqrt{x}} \left(x^2 + y^2\right) dx\, dy$

$$= \int_0^1 \left[\int_x^{\sqrt{x}} \left(x^2 + y^2\right) dy \right] dx$$

$$= \int_0^1 \left[x^2 \cdot y + \frac{y^3}{3} \right]_x^{\sqrt{x}} dx$$

$$= \int_0^1 \left[x^2 . x^{\frac{1}{2}} + \frac{\left(x^{\frac{1}{2}}\right)^3}{3} - x^2 . x - \frac{x^3}{3} \right] dx$$

$$= \int_0^1 \left(x^{\frac{5}{2}} + \frac{x^{\frac{3}{2}}}{3} - x^3 - \frac{x^3}{3} \right) dx$$

$$= \int_0^1 \left(x^{\frac{5}{2}} + \frac{x^{\frac{3}{2}}}{3} - \frac{4}{3} x^3 \right) dx$$

$$= \left[\frac{x^{\frac{5}{2}+1}}{\frac{5}{2}+1} + \frac{x^{\frac{3}{2}+1}}{3\left(\frac{3}{2}+1\right)} - \frac{4}{3} \frac{x^4}{4} \right]_0^1$$

$$= \frac{2}{7} . (1)^{\frac{7}{2}} + \frac{2}{3.5}(1)^{\frac{5}{2}} - \frac{1}{3}(1)^4$$

$$= \frac{2}{7} + \frac{2}{15} - \frac{1}{3}$$

$$= \frac{30+14-35}{105}$$

$$= \frac{9}{105}$$

$$\therefore I = \frac{3}{35} \text{ Ans.}$$

(v) $\int_0^1 \int_0^{x^2} e^{\frac{y}{x}} \, dy \, dx$

Solution: Let $I = \int_0^1 \int_0^{x^2} e^{\frac{y}{x}} \, dy \, dx$

$$= \int_0^1 \left[\int_0^{x^2} e^{\frac{y}{x}} \, dy \right] dx$$

$$= \int_0^1 \left[\frac{e^{\frac{y}{x}}}{\frac{1}{x}} \right]_0^{x^2} dx$$

$$= \int_0^1 x \left(e^{\frac{x^2}{x}} - e^0 \right) dx$$

$$= \int_0^1 x \, (e^x - 1) \, dx$$

$$= \int_0^1 (xe^x - x) \, dx$$

$$= \left[x \, e^x - e^x - \frac{x^2}{2} \right]_0^1$$

$$= 1 . e^1 - e^1 - \frac{1}{2} + e^o$$

$$= -\frac{1}{2} + 1$$

$$\therefore I = \frac{1}{2} \text{ Ans.}$$

(vi) $\int_0^a \int_0^{\sqrt{a^2-y^2}} dx \, dy$

Solution: Let $I = \int_0^a \left[\int_0^{\sqrt{a^2-y^2}} dx \right] dy$

$$= \int_0^a [x]_0^{\sqrt{a^2-y^2}} \, dy$$

$$= \int_0^a \sqrt{a^2 - y^2} \, dy$$

$$= \left[\frac{y\sqrt{a^2-y^2}}{2} + \frac{a^2}{2} \sin^{-1} \frac{y}{a} \right]_0^a$$

$$= 0 + \frac{a^2}{2} \sin^{-1}(1) - \frac{a^2}{2} \sin^{-1}(0)$$

$$= \frac{a^2}{2} . \frac{\pi}{2} - 0$$

$$\therefore I = \frac{\pi a^2}{4} \text{ Ans.}$$

(vii) $\int_0^1 \int_0^x e^{\frac{y}{x}} \, dy \, dx$

Solution: Let $I = \int_0^1 \left[\int_0^x e^{\frac{y}{x}} \, dy \right] dx$

$$= \int_0^1 \left[\frac{e^{\frac{y}{x}}}{\frac{1}{x}} \right]_0^x dx$$

$$= \int_0^1 x\,(e^1 - e^0)\,dx$$

$$= \int_0^1 x\,(e^1 - 1)\,dx$$

$$= (e-1) \int_0^1 x\,dx$$

$$= (e-1) \left[\frac{x^2}{2} \right]_0^1$$

$$\therefore I = \frac{1}{2}(e - 1 \text{ Ans.}$$

(viii) $\int_1^{\log 8} \int_0^{\log y} e^{x+y}\,dx\,dy$

Solution : Let $I = \int_1^{\log 8} \int_0^{\log y} e^{x+y}\,dx\,dy$

$$= \int_1^{\log 8} \left[\int_0^{\log y} e^x \,.\, e^y\,dx \right] dy$$

$$= \int_1^{\log 8} e^y \left[e^x \right]_0^{\log y} dy$$

$$= \int_1^{\log 8} e^y \left[e^{\log y} - e^0 \right] dy$$

$$= \int_1^{\log 8} (y \,.\, e^y - e^y)\,dy$$

$$= \left[y\,e^y - e^y - e^y \right]_1^{\log 8}$$

$$= \log 8\; e^{\log 8} - e^{\log 8} - e^{\log 8} - 1 \,.\, e^1 + e^1 + e^1$$

$$= 8 \log 8 - 8 - 8 + e$$

$$\therefore I = 8 \log 8 - 16 + e \text{ Ans.}$$

(ix) $\int_0^1 \int_y^{\sqrt{y}} (x^2 + y^2)\,dy\,dx$

Solution: Let $I = \int_0^1 \int_y^{\sqrt{y}} (x^2 + y^2)\,dy\,dx$

$$= \int_0^1 \left[\int_y^{\sqrt{y}} \left(x^2 + y^2 \right) dx \right] dy$$

$$= \int_0^1 \left[\frac{x^3}{3} + y^2.x \right]_y^{\sqrt{y}} dy$$

$$= \int_0^1 \left(\frac{y^{\frac{3}{2}}}{3} + y^{\frac{5}{2}} - \frac{y^3}{3} - y^3 \right) dy$$

$$= \int_0^1 \left(\frac{y^{\frac{3}{2}}}{3} + y^{\frac{5}{2}} - \frac{4y^3}{3} \right) dy$$

$$= \left[\frac{y^{\frac{3}{2}+1}}{3\left(\frac{3}{2}+1\right)} + \frac{y^{\frac{5}{2}+1}}{\frac{5}{2}+1} - \frac{4y^4}{3.4} \right]_0^1$$

$$= \frac{2(1)^{\frac{5}{2}}}{15} + \frac{2(1)^{\frac{7}{2}}}{7} - \frac{1}{3}(1)^4$$

$$= \frac{2}{15} + \frac{2}{7} - \frac{1}{3}$$

$$= \frac{14 + 30 - 35}{105}$$

$$= \frac{9}{105}$$

$$\therefore I = \frac{3}{35} \text{ Ans.}$$

(x) $\int_0^a \int_0^{\sqrt{a^2-y^2}} \sqrt{a^2 - x^2 - y^2}\,dy\,dx$

Solution: Let $I = \int_0^a \int_0^{\sqrt{a^2-y^2}} \sqrt{a^2 - x^2 - y^2}\,dy\,dx$

$$= \int_0^a \left[\int_0^{\sqrt{a^2-y^2}} \left\{ \sqrt{\left(a^2 - y^2\right)^2} - x^2 \right\} dx \right] dy$$

$$= \int_0^a \left[\frac{x\sqrt{a^2 - y^2 - x^2}}{2} + \frac{\left(\sqrt{a^2 - y^2}\right)^2}{2} \sin^{-1} \frac{x}{\sqrt{a^2 - y^2}} \right]_0^{\sqrt{a^2-y^2}} dy$$

$$= \int_0^a \left(\frac{a^2-y^2}{2}\right) . \sin^{-1}(1)\, dy$$

$$= \frac{1}{2} \int_0^a \left(a^2-y^2\right) \frac{\pi}{2} dy$$

$$= \frac{\pi}{4} \left[a^2 y - \frac{y^3}{3}\right]_0^a$$

$$= \frac{\pi}{4} \left(a^2 a - \frac{a^3}{3}\right)$$

$$= \frac{\pi}{4} \frac{\left(3a^3 - a^3\right)}{3}$$

$$= \frac{\pi}{4} . \frac{2a^3}{3}$$

$$\therefore I = \frac{\pi a^3}{6} \text{ Ans.}$$

2. Evaluate $\iint (x + y)^2\, dx\, dy$ over the area bounded by the ellipse $\frac{x^2}{y^2} + \frac{y^2}{b^2} = 1$.

Solution: Let $I = \iint (x + y)^2\, dx\, dy$

Here the area of integration A is bounded by the ellipse

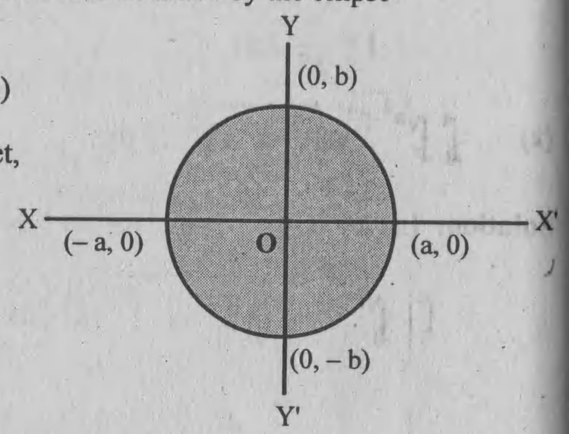
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ ..... (i)}$$

When $y = 0$ in (i) we get,

$$x = \pm a$$

and $\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$

$$\therefore y = \pm \frac{b}{a} \sqrt{a^2 - x^2}$$



Thus,

$I = \int_A (x + y)^2\, dx\, dy$

$$= \int_{x=-a}^{x=a} \left[\int_{y=-\frac{b}{a}\sqrt{a^2-x^2}}^{y=\frac{b}{a}\sqrt{a^2-x^2}} \left(x^2 + 2xy + y^2\right) dy \right] dx$$

$$= \int_{-a}^{a} \left[2 \int_0^{\frac{b}{a}\sqrt{a^2-x^2}} \left(x^2 + y^2\right) dy \right] dx$$

$$= 2 \int_{-a}^{a} \left(x^2 y + \frac{y^3}{3} \right)_0^{\frac{b}{a}\sqrt{a^2-x^2}} dx$$

$$= 2 \int_{-a}^{a} \left[x^2 . \frac{b}{a} \left(a^2 - x^2\right)^{\frac{1}{2}} + \frac{1}{3} \frac{b^3}{a^3} . \left(a^2 - x^2\right)^{\frac{3}{2}} \right] dx$$

$$= \frac{4b}{a} \int_0^a x^2 \left(a^2 - x^2\right)^{\frac{1}{2}} dx + \frac{4}{3} \frac{b^3}{a^3} \int_0^a \left(a^2 - x^2\right)^{\frac{3}{2}} dx$$

Put $x = a \sin\theta$

$dx = a \cos\theta\, d\theta$

When $x \to (0, a)$ then $\theta \to \left(0, \frac{\pi}{2}\right)$

$$\therefore I = \frac{4b}{a} \int_0^{\frac{\pi}{2}} a^2 \sin^2\theta \left(a^2 - a^2 \sin^2\theta\right)^{\frac{1}{2}} a \cos\theta\, d\theta$$

$$+ \frac{4}{3} \frac{b^3}{a^3} \int_0^{\frac{\pi}{2}} \left(a^2 - a^2 \sin^2\theta\right)^{\frac{3}{2}} a \cos\theta\, d\theta$$

$$= \frac{4b}{a} . a^4 \int_0^{\frac{\pi}{2}} \sin^2\theta . \cos^2\theta . d\theta + \frac{4}{3} \frac{b^3}{a^3} . a^4 \int_0^{\frac{\pi}{2}} \cos^4\theta . d\theta$$

$$= 4ba^3 \frac{1}{2} \frac{\Gamma\left(\frac{2+1}{2}\right) \Gamma\left(\frac{2+1}{2}\right)}{\Gamma\left(\frac{2+2+2}{2}\right)} + \frac{4}{3} ab^3 \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{4+1}{2}\right)}{\Gamma\left(\frac{4+2}{2}\right)}$$

$$= 2ba^3 \frac{\Gamma\left(\frac{3}{2}\right) . \Gamma\left(\frac{3}{2}\right)}{\Gamma 3} + \frac{2}{3} ab^3 \sqrt{\pi} \frac{\Gamma\left(\frac{5}{2}\right)}{\Gamma 3}$$

$$= 2ba^3 . \frac{\frac{1}{2} . \sqrt{\pi} . \frac{1}{2} . \sqrt{\pi}}{2!} + \frac{2}{3} ab^3 \frac{\sqrt{\pi} . \frac{3}{2} . \frac{1}{2} \sqrt{\pi}}{2!}$$

$$= \frac{ba^3\pi}{4} + \frac{ab^3\pi}{4}$$

$$\therefore I = \frac{\pi ab}{4} \left(a^2 + b^2\right) \text{ Ans.}$$

3. Evaluate $\iint xy\,(x + y)\, dx\, dy$ over the area between $y = x^2$ and $y = x$. (BE 2069 Bhadra)

Solution: Let $I = \iint xy\,(x + y)\, dx\, dy$

Here, the area of integration A is bounded by the curve $y = x^2$ and $y = x$.

Solving these, we get

$$x = 0,\ x = 1$$

Hence,

$$I = \iint_A (x^2y + xy^2)\, dx\, dy$$

$$= \int_0^1 \left[\int_{x^2}^{x} \left(x^2y + xy^2 \right) dy \right] dx$$

$$= \int_0^1 \left[\frac{x^2y^2}{2} + \frac{xy^3}{3} \right]_{x^2}^{x} dx$$

$$= \int_0^1 \left(\frac{x^4}{2} + \frac{x^4}{3} - \frac{x^6}{2} - \frac{x^7}{3} \right) dx$$

$$= \int_0^1 \left(\frac{5x^4}{6} - \frac{x^6}{2} - \frac{x^7}{3} \right) dx$$

$$= \left[\frac{5x^5}{6.5} - \frac{x^7}{2.7} - \frac{x^8}{3.8} \right]_0^1$$

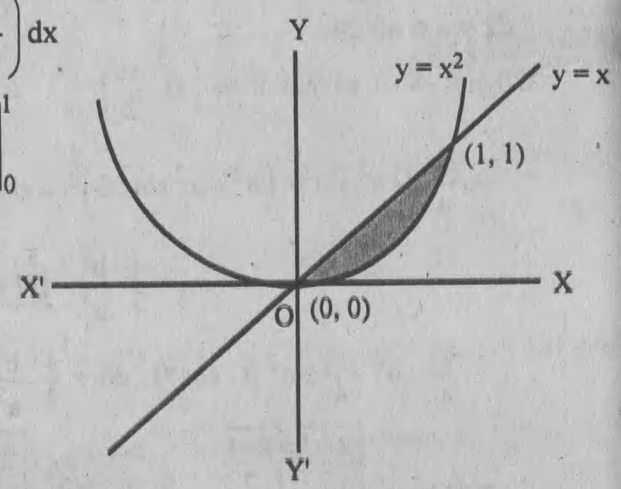
$$= \frac{1}{6} - \frac{1}{14} - \frac{1}{24}$$

$$= \frac{28-12-7}{168}$$

$$= \frac{28-19}{168}$$

$$= \frac{9}{168}$$

$$\therefore I = \frac{3}{56} \text{ Ans.}$$



4. Evaluate $\iint_R xy\, dx\, dy$ where R is the region bounded by the x-axis, the ordinate x = 2a and the curve $x^2 = 4ay$.

Solution: Let $I = \iint_R xy\, dxdy$

Here, the region of the integration. R is bounded by the x-axis and the ordinate x = 2a and the curve $x^2 = 4ay$

$$\Rightarrow y = \frac{x^2}{4a}$$

on x-axis y = 0 which gives x = 0

Hence, $I = \iint_R xy\, dxdy$

$$= \int_0^{2a} \left[\int_0^{\frac{x^2}{4a}} xy\, dy \right] dx$$

$$= \int_0^{2a} \left[x\frac{y^2}{2} \right]_0^{\frac{x^2}{4a}} dx$$

$$= \int_0^{2a} \frac{x}{2} \left(\frac{x^2}{4a} \right)^2 dx$$

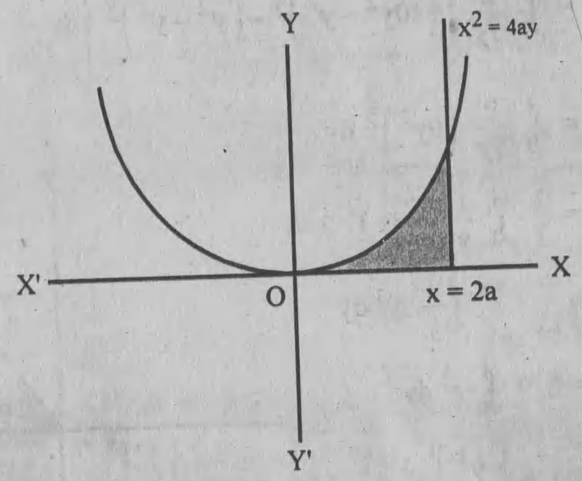
$$= \frac{1}{32a^2} \int_0^{2a} x^5\, dx$$

$$= \frac{1}{32a^2} \left[\frac{x^6}{6} \right]_0^{2a}$$

$$= \frac{1}{32a^2} \left[\frac{(2a)^6}{6} - 0 \right]$$

$$= \frac{1}{32a^2} \cdot \frac{64a^6}{6}$$

$$\therefore I = \frac{a^4}{6} \text{ Ans.}$$



5. Evaluate $\iint_R \sqrt{xy - y^2}\, dxdy$ where R is the triangle with vertices (0, 0), (10, 1) and (1, 1)

Solution: Let $I = \iint_R \sqrt{xy - y^2}\, dx\, dy$

Here, region of the integration R, is bounded by the traingle with vertices (0, 0), (10, 1) and (1, 1).

Let the gien vertices be 0, A, and B respectively equation of OA is

$$y - 0 = \frac{1-0}{1-0}(x - 0)$$

or, y = x ..... (i)

Equation of OB is

$$y - 0 = \frac{1-0}{10-0}(x - 0)$$

or, 10y = x ..... (ii)

Hence, $I = \iint_R \left(xy - y^2 \right)^{\frac{1}{2}} dx\, dy$

$$= \int_0^1 \left[\int_y^{10y} \left(xy - y^2 \right)^{\frac{1}{2}} dx \right] dy$$

$$= \int_0^1 \left[\frac{\left(xy - y^2 \right)^{\frac{1}{2}+1}}{\left(\frac{1}{2}+1 \right) . y} \right]_y^{10y} dy$$

$$= \int_0^1 \frac{2}{3y}\left[\left(10y^2 - y^2\right)^{\frac{3}{2}} - \left(y^2 - y^2\right)^{\frac{3}{2}}\right] dy$$

$$= \int_0^1 \frac{2}{3y}\left(9y^2\right)^{\frac{3}{2}} dy$$

$$= \frac{2}{3}\int_0^1 \frac{1}{y} \cdot (3y)^{2\times\frac{3}{2}} dy$$

$$= \frac{2}{3} \cdot 3^3 \int_0^1 \frac{1}{y} \cdot y^3 dy$$

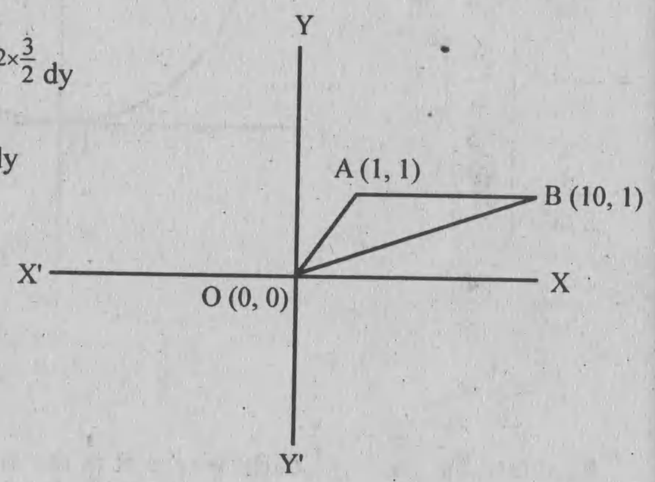
$$= 2.9 \int_0^1 y^2 dy$$

$$= 2.9 \left[\frac{y^3}{3}\right]_0^1$$

$$= 2.9 \cdot \frac{1}{3}$$

$$= 2 \cdot 3$$

$\therefore I = 6$ Ans.



6. Evaluate $\iint_R xy\, dx\, dy$. Where R is the region over the area of the ellipes $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in the first quadrant.

[B.E. 2063, 2067 Mangsir, 2070 Magh]

Solution: Let $I = \iint_R xy\, dx\, dy$

Here, the region of the integration R is bounded by the area of the ellipse

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in the first quadrant. (or positive quadrant)

when $y = 0$ in (i)

$x = \pm a$

$\Rightarrow x = a$ (for the first quadrant)

and $\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$

$\therefore y = \pm \frac{b}{a}\sqrt{a^2 - x^2}$

$\Rightarrow y = \frac{b}{a}\sqrt{a^2 - x^2}$ (for the first quadrant)

Hence, x is verries from 0 to a

and y is verries from 0 to $\frac{b}{a}\sqrt{a^2 - x^2}$

Hence, $I = \iint_R xy\, dx\, dy$

$$= \int_0^a \left[\int_0^{\frac{b}{a}\sqrt{a^2-x^2}} xy\, dy\right] dx$$

$$= \int_0^a \left[\frac{xy^2}{2}\right]_0^{\frac{b}{a}\sqrt{a^2-x^2}} dx$$

$$= \frac{1}{2}\int_0^a x \cdot \frac{b^2}{a^2}(a^2 - x^2)\, dx$$

$$= \frac{b^2}{2a^2}\int_0^a \left(a^2x - x^3\right) dx$$

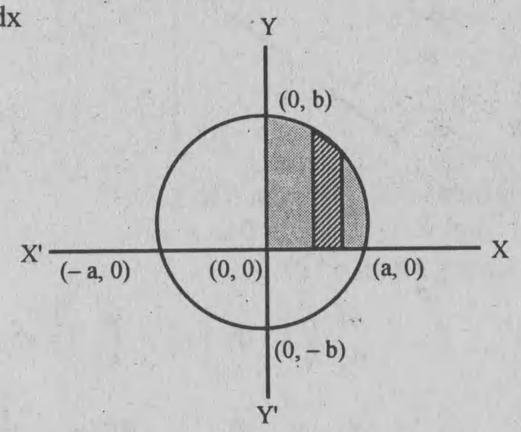
$$= \frac{b^2}{2a^2}\left[a^2\frac{x^2}{2} - \frac{x^4}{4}\right]_0^a$$

$$= \frac{b^2}{2a^2}\left[\frac{a^4}{2} - \frac{a^4}{4}\right]$$

$$= \frac{b^2}{2a^2}\left(\frac{2a^4 - a^4}{4}\right)$$

$$= \frac{b^2a^4}{8a^2}$$

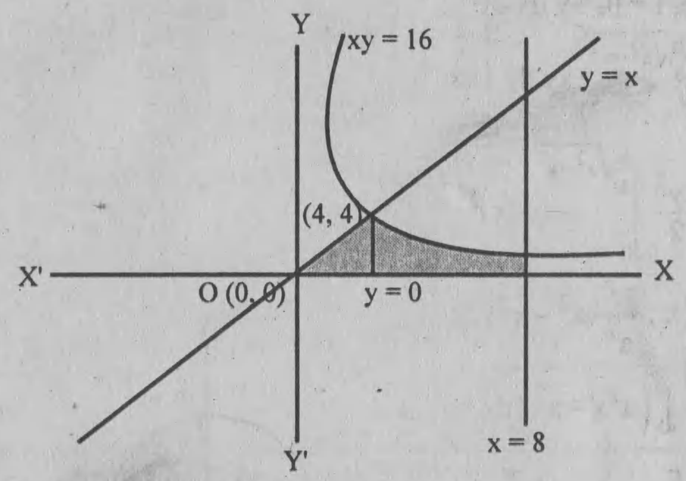
$\therefore I = \frac{b^2a^2}{8}$ Ans.



7. Evaluate $\iint_R x^2\, dx\, dy$ where R is the region in the first quadrant bounded by the hyperbola xy = 16 and the line y = x, y = 0 and x = 8

Solution: Let $I = \iint_R x^2\, dx\, dy$

Here, the region of the integration R is bounded by the hyperbola xy = 16, and the line y = x, y = 0, and x = 8 in the first quadrant i.e. pointive quadrant.



Hence, x verries from 0 to 4
and y verries from 0 to x

Hence, $I = \iint_R x^2\, dx\, dy$

$$= \int_0^4 \left[\int_0^x x^2\, dy \right] dx + \int_4^8 \left[\int_0^{\frac{16}{x}} x^2\, dy \right] dx$$

$$= \int_0^4 \left[x^2 . y \right]_0^x dx + \int_4^8 \left[x^2 . y \right]_0^{\frac{16}{x}} dx$$

$$= \int_0^4 x^3\, dx + 16 \int_4^8 x\, dx$$

$$= \left[\frac{x^4}{4} \right]_0^4 + 16 \left[\frac{x^2}{2} \right]_4^8$$

$$= \frac{1}{4} 4^4 + 16 \frac{\left(8^2 - 4^2\right)}{2}$$

$$= 64 + 8 . 48$$

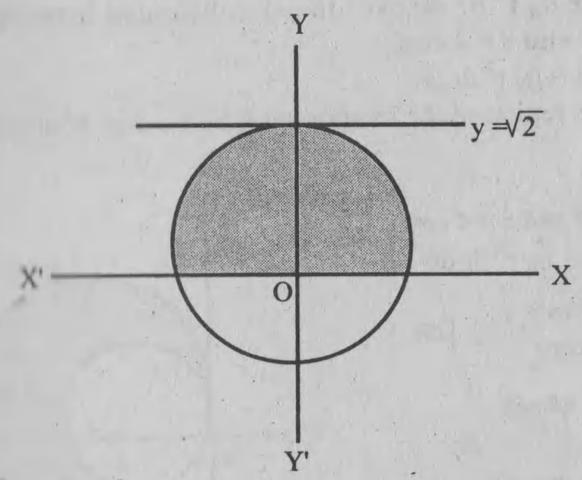
$$= 64 + 384$$

$\therefore I = 448$ Ans.

8. Evaluate $\iint_R y\, dy\, dx$ where R is the region bounded by $x^2 + 2y^2 = 4$ and the ordinate $y = 0$, $y = \sqrt{2}$.

Solution: Let $I = \iint_R y\, dy\, dx$

Here, the region of the integration is bounded by $x^2 + 2y^2 = 4$ and the ordinate $y = 0$, $y = \sqrt{2}$



Now, $x^2 = 4 - 2y^2$

$$x = \pm \sqrt{4 - 2y^2}$$

Hence, $I = \iint_R y\, dy\, dx$

$$= \int_0^{\sqrt{2}} \left[\int_{-\sqrt{4-2y^2}}^{\sqrt{4-2y^2}} y\, dx \right] dy$$

$$= \int_0^{\sqrt{2}} y\, [x]_{-\sqrt{4-2y^2}}^{\sqrt{4-2y^2}}\, dy$$

$$= \int_0^{\sqrt{2}} y \left(\sqrt{4-2y^2} + \sqrt{4-2y^2} \right) dy$$

$$= 2 \int_0^{\sqrt{2}} y \sqrt{4-2y^2}\, dy$$

$$= 2 . \frac{-1}{4} \int_0^{\sqrt{2}} (-4y) \sqrt{4-2y^2}\, dy$$

$$= -\frac{1}{2} \int_0^{\sqrt{2}} \left(4-2y^2\right)^{\frac{1}{2}} d\left(4-2y^2\right)$$

$$= -\frac{1}{2} \left[\frac{\left(4-2y^2\right)^{\frac{1}{2}+1}}{\frac{1}{2}+1} \right]_0^{\sqrt{2}}$$

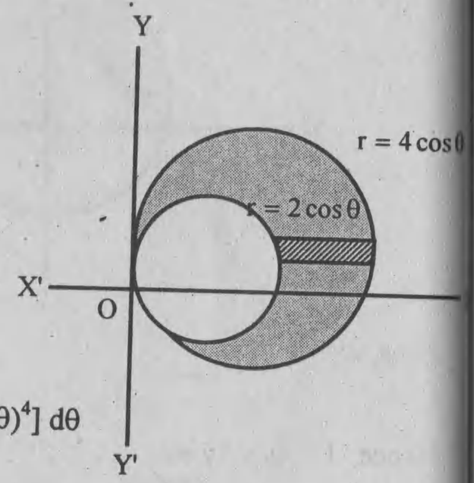
$$= -\frac{1}{2} . \frac{2}{3} \left[(4-4)^{\frac{3}{2}} - (4-0)^{\frac{3}{2}} \right] = -\frac{1}{3} \left[0 - \left(2^2\right)^{\frac{3}{2}} \right]$$

$\therefore I = \frac{8}{3}$ Ans.

9. Evaluate $\iint_R r^3\, dr\, d\theta$ over the area bounded between the circl = 2 cosθ and r = 4 cosθ.

Solution: Let $I = \iint_R r^3\, dr\, d\theta$

Here, the region of the integration R is the area bounded between circles



$r = 2\cos\theta$ and $r = 4\cos\theta$

Hence, $I = \iint_R r^3\, dr\, d\theta$

$$= \int_{\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\int_{2\cos\theta}^{4\cos\theta} r^3\, dr \right] d\theta$$

$$= \int_{\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\frac{r^4}{4} \right]_{2\cos\theta}^{4\cos\theta} d\theta$$

$$= \frac{1}{4} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} [(4\cos\theta)^4 - (2\cos\theta)^4]\, d\theta$$

$$= \frac{1}{4} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(256\cos^4\theta - 16\cos^4\theta\right) d\theta$$

$$= \frac{1}{4}\, 240 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4\theta\, d\theta$$

$$= 60 \times 2 \int_0^{\frac{\pi}{2}} \cos^4\theta\, d\theta$$

$$= 120 \cdot \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{4+1}{2}\right)}{\Gamma\left(\frac{4+2}{2}\right)}$$

$$= 60\sqrt{\pi}\, \frac{\Gamma\left(\frac{5}{2}\right)}{\Gamma 3}$$

$$= 60\sqrt{\pi} \cdot \frac{\frac{3}{2} \cdot \frac{1}{2} \cdot \sqrt{\pi}}{2!}$$

$$= \frac{60\sqrt{\pi} \cdot 3\sqrt{\pi}}{8}$$

$$\therefore I = \frac{45\pi}{2} \text{ Ans.}$$

10. Evaluate $\iint_R r^2 \sin\theta\, dr\, d\theta$ where R is the semi-cicle r = 2a cos above the initial line.

Solution: Let $I = \iint_R r^2 \sin\theta\, dr\, d\theta$

Here, the region of the integration R is bounded by the area of the semi-circle $r = 2a\cos\theta$ above the initial line.

Hence, $I = \iint_R r^2 \sin\theta\, dr\, d\theta$

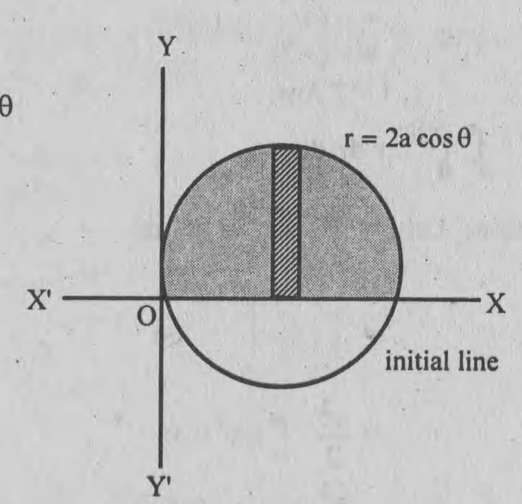
$$= \int_0^{\frac{\pi}{2}} \left[\int_0^{2a\cos\theta} r^2 \sin\theta\, dr \right] d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left[\frac{r^3}{3} \sin\theta \right]_0^{2a\cos\theta} d\theta$$

$$= \frac{1}{3} \int_0^{\frac{\pi}{2}} (2a\cos\theta)^3 \sin\theta\, d\theta$$

$$= \frac{8a^3}{3} \int_0^{\frac{\pi}{2}} \sin\theta \cos^3\theta\, d\theta$$

$$= \frac{8a^3}{3} \cdot \frac{\Gamma\left(\frac{1+1}{2}\right)\Gamma\left(\frac{3+1}{2}\right)}{2\,\Gamma\left(\frac{1+3+2}{2}\right)}$$



$$= \frac{8a^3}{3} \frac{\Gamma 1\, \Gamma 2}{2\, \Gamma 3}$$

$$= \frac{8a^3}{3} \cdot \frac{1 \,.\, 1!}{2 \,.\, 2!}$$

$$I = \frac{2a^3}{3} \text{ Ans.}$$

Evaluate the following integrals.

$$\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \sin(\theta + \phi)\, d\theta\, d\phi$$

Solution: Let $I = \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \sin(\theta + \phi)\, d\theta\, d\phi$

$$= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} (\sin\theta\cos\phi + \cos\theta\sin\phi)\, d\theta\, d\phi$$

$$= \int_0^{\frac{\pi}{2}} \left[\int_0^{\frac{\pi}{2}} (\sin\theta\cos\phi + \cos\theta\sin\phi)\, d\theta \right] d\phi$$

$$= \int_0^{\frac{\pi}{2}} [\cos\theta\cos\phi - \sin\theta\sin\phi]_0^{\frac{\pi}{2}}\, d\phi$$

$$= \left[\int_0^{\frac{\pi}{2}} \cos\frac{\pi}{2}\cos\phi - \sin\frac{\pi}{2}\sin\phi - \cos 0\cos\phi + \sin 0 . \sin\phi \right] d\phi$$

$$= \int_0^{\frac{\pi}{2}} [0 \,.\, \cos\phi - 1 \,.\, \sin\phi - 1 \,.\, \cos\phi - 0 \,.\, \sin\phi]\, d\phi$$

$$= -\int_0^{\frac{\pi}{2}} [\sin\phi + \cos\phi)\, d\phi$$

$$= -\left[\cos\phi - \sin\phi\right]_0^{\frac{\pi}{2}}$$

$$= -\left[\cos\frac{\pi}{2} - \sin\frac{\pi}{2} - \cos 0 - \sin 0\right]$$

$$= -(0 - 1 - 1 - 0)$$

$$= -(-2)$$

$$\therefore I = 2 \text{ Ans.}$$

(ii) $\int_0^{\pi}\int_0^{a\sin\theta} r\, dr\, d\theta$

Solution: Let $I = \int_0^{\pi}\left[\int_0^{a\sin\theta} r\, dr\right] d\theta$

$$= \int_0^{\pi}\left[\frac{r^2}{2}\right]_0^{a\sin\theta} d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi} \sin^2\theta\, d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi}\left(\frac{1-\cos 2\theta}{2}\right) d\theta$$

$$= \frac{a^2}{4}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi}$$

$$= \frac{a^2}{4}\left[\pi + \frac{\sin 2\pi}{2} - 0 - \frac{\sin 0}{2}\right] = \frac{a^2\pi}{4} + 0$$

$$\therefore I = \frac{\pi a^2}{4} \text{ Ans.}$$

(iii) $\int_0^{\pi}\int_0^{a\theta} r^3\, dr\, d\theta$

Solution: Let $I = \int_0^{\pi}\left[\int_0^{a\theta} r^3\, dr\right] d\theta$

$$= \int_0^{\pi}\left[\frac{r^4}{4}\right]_0^{a\theta} d\theta$$

$$= \frac{1}{4}\int_0^{\pi}\left(a^4\theta^4 - 0\right) d\theta$$

$$= \frac{a^4}{4}\cdot\left[\frac{\theta^5}{5}\right]_0^{\pi} = \frac{a^4}{4}\cdot\frac{1}{5}(\pi^5 - 0)$$

$$\therefore I = \frac{\pi^5 a^4}{20} \text{ Ans.}$$

) $\iint_R \frac{r\,d\theta\, dr}{\sqrt{a^2+r^2}}$ over one loop $r^2 = a^2\cos 2\theta$.

Solution: Let $I = \iint_R \frac{r}{\sqrt{a^2+r^2}}\, d\theta dr$

Here, the region of the integration R is the bounded by the area over one loop $r^2 = a^2\cos^2\theta$ or, $r = \pm a\sqrt{\cos 2\theta}$

Hence, $I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\left[\int_0^{a\sqrt{\cos 2\theta}} \frac{r}{\sqrt{a^2+r^2}}\, dr\right] d\theta$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\left[\int_0^{a\sqrt{\cos 2\theta}} \frac{1}{2}\left(\frac{2r}{\sqrt{a^2+r^2}}\right) dr\right] d\theta$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\left[\frac{1}{2}\cdot 2\sqrt{a^2+r^2}\right]_0^{a\sqrt{\cos 2\theta}} d\theta$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\left[\sqrt{a^2 + a^2\cos 2\theta} - a\right] d\theta$$

$$= 2\int_0^{\frac{\pi}{4}}\left\{a(1+\cos 2\theta)^{\frac{1}{2}} - a\right\} d\theta$$

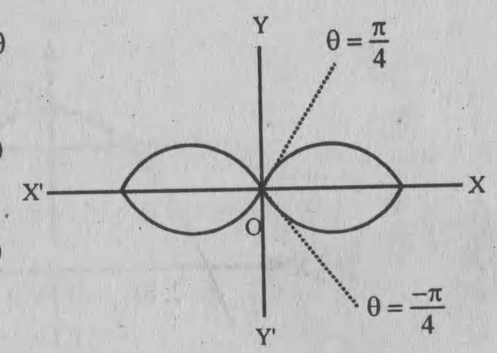
$$= 2a\int_0^{\frac{\pi}{4}}\left\{(1+\cos 2\theta)^{\frac{1}{2}} - 1\right\} d\theta$$

$$= 2a\int_0^{\frac{\pi}{4}}\left(\sqrt{2}\cos\theta - 1\right) d\theta$$

$$= 2a\left[\sqrt{2}\sin\theta - \theta\right]_0^{\frac{\pi}{4}}$$

$$= 2a\left(\sqrt{2}\cdot\frac{1}{\sqrt{2}} - \frac{\pi}{4}\right) = 2a\left(\sqrt{2}\cdot\sin\frac{\pi}{4} - \frac{\pi}{4}\right)$$

$$\therefore I = a\left(2 - \frac{\pi}{2}\right) \text{ Ans.}$$



Exercise - 5

1. Change the order of integration and hence evaluate the same

(i) $\int_0^3 \int_1^{\sqrt{4-y}} (x+y)dxdy$

Solution:

Let $I = \int_0^3 \left[\int_1^{\sqrt{4-y}} (x+y)dx \right] dy$

Here the integration is first integrated with respect to x along horizontal strip. Which varies from $x = 1$ to $x = \sqrt{4-y}$

i. e. $x^2 = 4 - y$

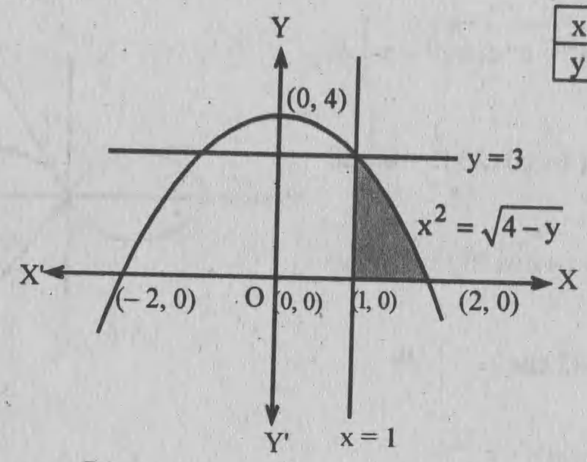
Those of y-varies from $y = 0$ to $y = 3$

Changing the order of integration. We first integrated with respe y along the vertical strip. Which varies from

$y = 0$ to $y = 4 - x^2$ and those x varies from

$x = 1$ to $x = 2$

x	0	±2
y	4	0



Now, $I = \int_1^2 \left[\int_0^{4-x^2} (x+y)dy \right] dx$

$= \int_1^2 \left[xy + \frac{y^2}{2} \right]_0^{4-x^2} dx$

$= \int_1^2 \left[x(4-x^2) + \frac{(4-x^2)^2}{2} \right] dx$

$= \int_1^2 \left[4x - x^3 + \frac{16-8x^2+x^4}{2} \right] dx$

$= \frac{1}{2} \int_1^2 (8x - 2x^3 + 16 - 8x^2 + x^4)dx$

$= \frac{1}{2} \left[\frac{8x^2}{2} - \frac{2x^4}{4} + 16x - \frac{8x^3}{3} + \frac{x^5}{5} \right]_1^2$

$= \frac{1}{2} \left[4(4-1) - \frac{1}{2}(16-1) + 16(2-1) - \frac{8}{3}(8-1) + \frac{1}{5}(32-1) \right]$

$= \frac{1}{2} \left(12 - \frac{15}{2} + 16 - \frac{56}{3} + \frac{31}{5} \right)$

$= \frac{1}{2} \left[\frac{360 - 225 + 480 - 560 + 186}{30} \right]$

$= \frac{1}{2} \left(\frac{241}{30} \right)$

$\therefore I = \frac{241}{60}$ Ans.

(ii) $\int_{-a}^{a} \int_0^{\sqrt{a^2-y^2}} dx\ dy.$

Solution:

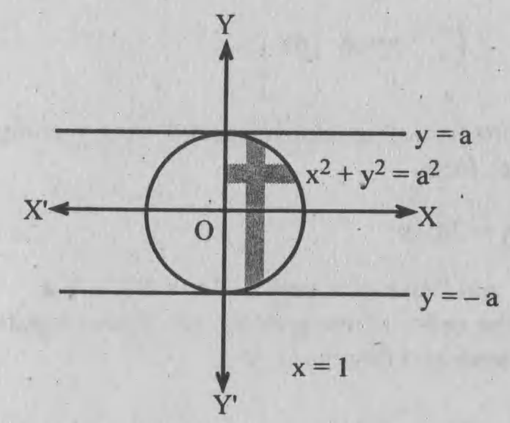
Let $I = \int_{-a}^{a} \left[\int_0^{\sqrt{a^2-y^2}} dx \right] dy$

Here the integration is the first integrated w.r.t. to x along the horizontal strip. Which varies from $x = 0$ to $x = \sqrt{a^2 - y^2}$

i.e. $x^2 = a^2 - y^2$

$x^2 + y^2 = a^2$

Those of y-varies from $y = -a$ to $y = a$



Changing the order of integration we first integrate w.r.t y along the vertical strip. Which varies from

$y = -\sqrt{a^2 - x^2}$ to $y = \sqrt{a^2 - x^2}$

and those x varies from x = – O to x = a.

Now,

$$I = \int_O^a \left[\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dy \right] dx$$

$$= \int_O^a [y]_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dx.$$

$$= \int_O^a \left[\sqrt{a^2 - x^2}\, dx + \sqrt{a^2 - x^2} \right] dx$$

$$= 2 \int_O^a \sqrt{a^2 - x^2}\, dx$$

$$= 2 \left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} \right]_0^a$$

$$= 2 \left[0 + \frac{a^2}{2} \text{Sin}^{-1} 1 - 0 - 0 \right]$$

$$= 2 . \frac{a^2}{2} . \frac{\pi}{2} = \frac{\pi a^2}{2}$$

$$\therefore I = \frac{\pi a^2}{2}$$

(iii) $\int_0^a \int_{\frac{x^2}{a}}^{2a-x} xy\, dy\, dx.$

Solution:

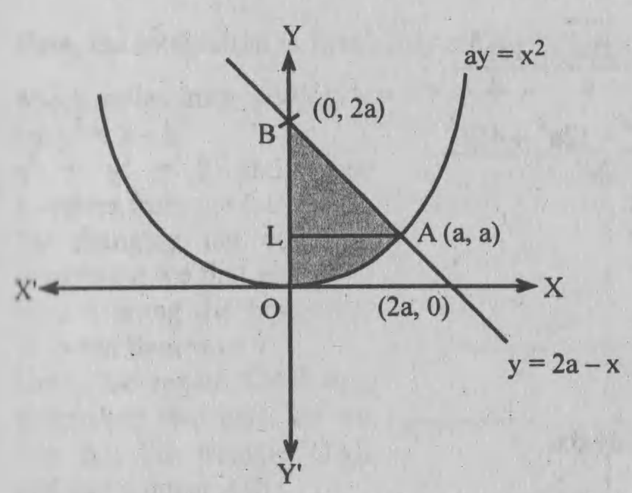
$$\text{Let } I = \int_0^a \left[\int_{\frac{x^2}{a}}^{2a-x} xy\, dy \right] dx$$

Here, the integration is first integrand w.r.t y along the vertical s which varies from

$y = \frac{x^2}{a}$ to $y = 2a - x$

i.e. $ay = x^2$ and those of x varies of x = 0 to x = a

Changing the order of integration. we first integrate w.r.t x along horizontal strip and than w.r.t y



Here the region OAB splits up into two parts in into two parts by the line AL the curve-linear OAL and triangle ABL.

For the region OAL, x varies from x = 0 to x = $\sqrt{ay}$ and those for y varies from y = 0 to y = a

For the region ALB, x varies from x = 0 to x = 2a – y and those for y varies from y = a to y = 2a

Now,

$$I = \int_0^a \left[\int_{\frac{x^2}{a}}^{2a-x} xy\, dy \right] dx$$

$$= \iint_{\text{Region OAL}} xy\, dx\, dy + \iint_{\text{Region ABL}} xy\, dx\, dy$$

$$= \int_0^a \left[\int_a^{\sqrt{ay}} xy\, dx \right] dy + \int_a^{2a} \left[\int_0^{2a-y} xy\, dx \right] dy$$

$$= \int_0^a \left[\frac{x^2 y}{2} \right]_0^{\sqrt{ay}} dy + \int_0^{2a} \left[\frac{x^2 y}{2} \right]_0^{2a-y} dy$$

$$= \int_0^a \frac{ay^2}{2} dy + \int_a^{2a} \frac{(2a-y)^2 . y}{2} dy$$

$$= \frac{1}{2} \left[\int_0^a ay^2 dy + \int_a^{2a} (4a^2 y - 4ay^2 + y^3) \right] dy$$

$$= \frac{1}{2} \left[\frac{ay^3}{3} \right]_0^a + \frac{1}{2} \left[\frac{4a^2 y^2}{2} - \frac{4ay^3}{3} + \frac{y^4}{4} \right]_a^{2a}$$

$$= \frac{1}{2} \left(\frac{a^4}{3} - 0 \right) + \frac{1}{2} \left\{ (8a^4 - 2a^4) - \left(\frac{32a^4}{3} - \frac{4a^4}{3} \right) + \frac{(16a^4 - a^4)}{4} \right\}$$

$$= \frac{a^4}{6} + 3a^4 - \frac{28a^4}{6} + \frac{15a^4}{8}$$

$$= \frac{4a^4 + 72a^4 - 12a^4 + 45a^4}{24}$$

$$= \frac{9a^4}{24} = \frac{3a^4}{8}$$

$$\therefore \quad I = \frac{3a^4}{8} \text{ Ans.}$$

(iv) $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\, dy\, dx.$

Solution:

$$\text{Let } I = \int_0^\infty \left[\int_x^\infty \frac{e^{-y}}{y} dy \right] dx$$

Here, the integration is first integrand w.r.t. y along the vertical strip which varies from $y = x$ to $y = \infty$
and those x varies from $x = 0$ to $x = \infty$
By changing the order of integration, we first integrand w.r.t. x along the horizontal strip which varies from
$x = y$ to $x = \infty$ and those y varies from $y = 0$ to $y = \infty$

Now,

$$I = \int_0^\infty \left[\int_y^\infty \frac{e^{-y}}{y} dx \right] dy$$

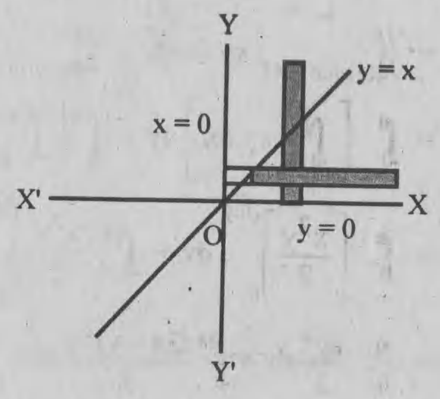
$$= \int_0^\infty \left[\frac{e^{-y}}{y} . x \right]_y^\infty dy$$

$$= \int_0^\infty e^{-y}\, dy$$

$$= \left[-e^{-y} \right]_0^\infty = e^{-\infty} + e^{-0}$$

$$= 0 + 1 = 1$$

$$\therefore \quad I = 1$$

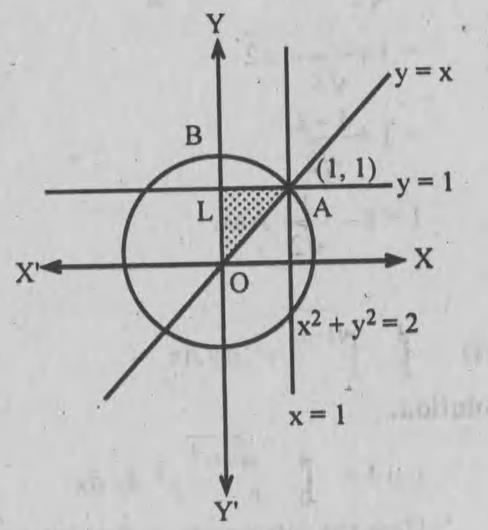


(v) $\int_0^1 \int_x^{\sqrt{2-x^2}} \frac{x}{\sqrt{x^2+y^2}}\, dy\, dx$

Solution:

$$\text{Let } I = \int_0^1 \left[\int_x^{\sqrt{2-x^2}} \frac{x}{\sqrt{x^2+y^2}} dy \right] dx$$

Here, the integration is first integrand w.r.t. y along the vertical strip which varies from $y = x$ to $y = \sqrt{2-x^2}$
i.e. $y^2 = 2 - x^2$
or, $x^2 + y^2 = 2$ and those x -varies from $x = 0$ to $x = 1$
By changing the order of integration we first integrand w.r.t x along the horizontal strip and then w.r.t y
Here, the region OAB into splits upto two parts by the line AL the triangle OAL and curve linear ABL.



For the region OAL x varies from $x = 0$ to $x = y$ and those for y varies from $y = 0$ to $y = 1$
For the region ALB x varies from $x = 0$ to $x = \sqrt{2-y^2}$ and those for y varies from $y = 1$ to $y = \sqrt{2}$

Now,

$$I = \int_0^1 \left[\int_x^{\sqrt{2-x^2}} \frac{x}{\sqrt{x^2+y^2}} dy \right] dx$$

$$= \iint_{\text{Region OAL}} \frac{x}{\sqrt{x^2+y^2}} dy\, dx + \iint_{\text{Region ALB}} \frac{x}{\sqrt{x^2+y^2}} dy dx.$$

$$= \int_0^1 \left[\int_0^y \frac{x}{\sqrt{x^2+y^2}} dx \right] dy + \int_1^{\sqrt{2}} \left[\int_0^{\sqrt{2-y^2}} \frac{x}{\sqrt{x^2+y^2}} dx \right] dy$$

$$= \int_0^1 \left[\sqrt{x^2+y^2} \right]_0^y dy + \int_1^{\sqrt{2}} \left[\sqrt{x^2+y^2} \right]_0^{\sqrt{2-y^2}} dy$$

$$= \int_0^1 \left(\sqrt{y^2+y^2} - \sqrt{y^2} \right) dy + \int_1^{\sqrt{2}} \left[\sqrt{2-y^2+y^2} - \sqrt{y^2} \right] dy$$

$$= \int_0^1 \left(\sqrt{2} - 1 \right) y\, dy + \int_1^{\sqrt{2}} \left(\sqrt{2} - y \right) dy$$

$$= \left[\left(\sqrt{2} - 1 \right) \frac{y^2}{2} \right]_0^1 + \left[\sqrt{2}\, y - \frac{y^2}{2} \right]_1^{\sqrt{2}}$$

$$= \left(\sqrt{2}-1\right).\frac{1}{2}+(2-1)-\left(\sqrt{2}-\frac{1}{2}\right)$$

$$= \frac{1}{\sqrt{2}}-\frac{1}{2}+1-\sqrt{2}+\frac{1}{2}$$

$$= 1+\frac{1}{\sqrt{2}}-\sqrt{2}$$

$$= 1+\frac{1-2}{\sqrt{2}}$$

$\therefore \quad I = 1-\frac{1}{\sqrt{2}}$ Ans.

(vi) $\int_0^1 \int_0^{\sqrt{1-x^2}} y^2 \, dy \, dx$

Solution:

Let $I = \int_0^1 \int_0^{\sqrt{1-x^2}} y^2 \, dy \, dx$

Here the integration is first integrated w.r.t y along the vertical strip which varies from $y = 0$ to $y = \sqrt{1-x^2}$

i.e. $y^2 = 1 - x^2$

or $\quad x^2 + y^2 = 1$

and those x varies from $x = 0$ to $x = 1$.

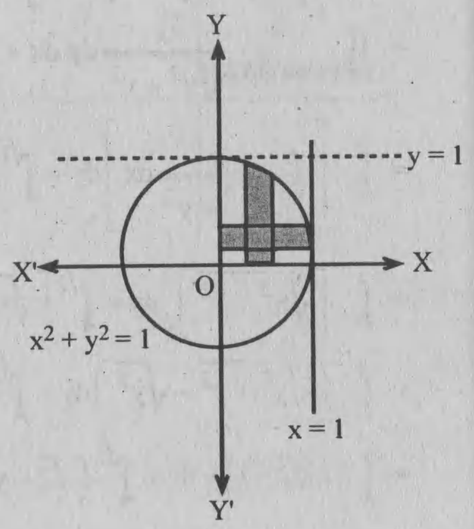
By changing the order of integration we first integrated w.r.t. x along the horizontal strip which varies from

Figure

$x = 0$ to $x = \sqrt{1-y^2}$

and those y varies from

$y = 0$ to $y = 1$



Now,

$$I = \int_0^1 \left[\int_0^{\sqrt{1-y^2}} y^2 \, dx \right] dy$$

$$= \int_0^1 \left[y^2 x \right]_0^{\sqrt{1-y^2}} dy$$

$$= \int_0^1 y^2 \sqrt{1-y^2} \, dy$$

Put $y = \sin\theta$

$\therefore \quad dy = \cos\theta \, d\theta$

When $y = 0$ then $\theta = 0$

$y = 1$ Then $\theta = \frac{\pi}{2}$

$$I = \int_0^{\frac{\pi}{2}} \sin^2\theta . \sqrt{1-\sin^2\theta} \cos\theta d\theta$$

$$= \int_0^{\frac{\pi}{2}} \sin^2\theta \cos^2\theta \, d\theta .$$

$$= \frac{\Gamma\left(\frac{2+1}{2}\right)\Gamma\left(\frac{2+1}{2}\right)}{2\Gamma\left(\frac{2+2+2}{2}\right)}$$

$$= \frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{3}{2}\right)}{2.\Gamma 3}$$

$$= \frac{\frac{1}{2}.\sqrt{\pi}.\frac{1}{2}\sqrt{\pi}}{2.2!}$$

$$= \frac{\pi}{16}$$

$$I = \frac{\pi}{16}$$

(vii) $\int_a^b \int_a^x xy \, dy \, dx$

Solution:

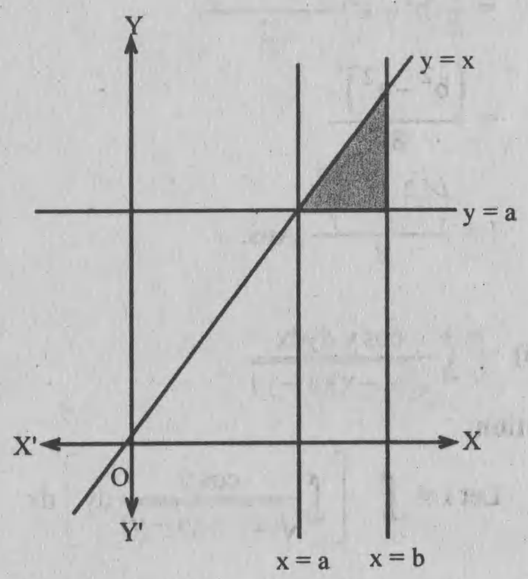
Let $I = \int_a^b \int_a^x xy \, dy \, dx$

Here, the integration is first integrated w.r.t y along the vertical strip which varies from $y = a$ to $y = x$ and those x varies from $x = a$ to $x = b$

Changing the order of integration, we first integrated w.r.t x along the horizontal strip which varies from

$x = y$ to $x = b$

and those y – varies from



y = a to y = b
From figure

Now,

$$I = \int_a^b \left[\int_y^b xy\,dx\right]dy$$

$$= \int_a^b \left[y\frac{x^2}{2}\right]_y^b dy$$

$$= \int_a^b y\left(\frac{b^2-y^2}{2}\right)dy$$

$$= \frac{1}{2}\int_a^b \left(b^2y-y^3\right)dy$$

$$= \frac{1}{2}\left[b^2\frac{y^2}{2}-\frac{y^4}{4}\right]_a^b$$

$$= \frac{1}{2}\left[\frac{b^2}{2}\left(b^2-a^2\right)-\frac{1}{4}\left(b^4-a^4\right)\right]$$

$$= \frac{1}{4}(b^2-a^2)\left[b^2-\frac{1}{2}(b^2+a^2)\right]$$

$$= \frac{1}{4}(b^2-a^2)\left(\frac{2b^2-b^2-a^2}{2}\right)$$

$$= \frac{1}{4}(b^2-a^2)\frac{\left(b^2-a^2\right)}{2}$$

$$= \frac{\left(b^2-a^2\right)^2}{8}$$

$$\therefore \quad I = \frac{\left(b^2-a^2\right)^2}{8} \text{ Ans.}$$

(viii) $\int_0^a \int_0^x \frac{\cos y\,dydx}{\sqrt{(a-x)(a-y)}}$ **(BE 2068, Bhadra)**

Solution:

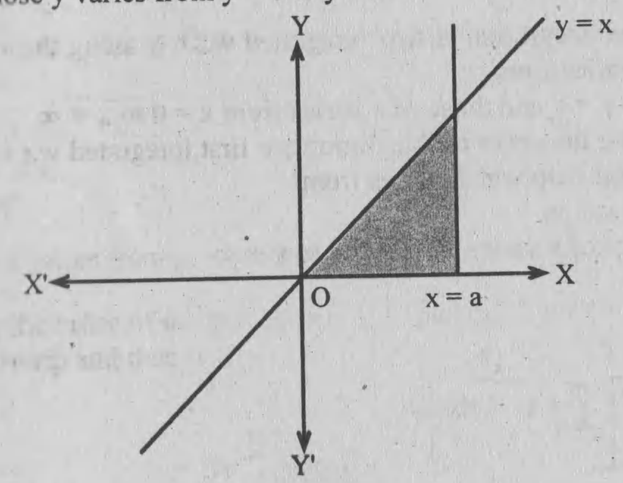
$$\text{Let } I = \int_0^x \left[\int_0^x \frac{\cos y}{\sqrt{(a-x)(a-y)}}dy\right]dx$$

Here the integration is first integrated w.r.t y along the vertical strip which yaries from y = 0 to y = x and those x varies from x = 0 to x = a.

Changing the order of integration, we first integrated w.r.t . x along the horizontal strip which varies from

x = y to y = a

and those y varies from y = 0 to y = a



Now,

$$I = \int_0^a \left[\int_y^a \frac{\cos y}{\sqrt{(a-x)(a-y)}}dx\right]dy$$

$$= \int_0^a \left[\frac{\cos y}{\sqrt{a-y}}\frac{2\sqrt{a-x}}{-1}\right]_y^a dy$$

$$= \int_0^a \frac{\cos y}{\sqrt{a-y}}2\sqrt{a-y}\,dy$$

$$= 2\int_0^a \cos y\,dy$$

$$= 2\left[\sin y\right]_0^a$$

$$= 2(\sin a - \sin 0)$$

$$= 2\sin a$$

$$I = 2\sin a$$

(ix) $\int_0^\infty \int_0^x x e^{-\frac{x^2}{y}}\, dy\, dx.$

Solution: Let $I = \int_0^\infty \left[\int_0^x x\, e^{-\frac{x^2}{y}}\, dy \right] dx.$

Here the integration is first integrated w.r.t. y along the vertical strip which varies from

$y = 0$ to $y = x$ and those of x varies from $x = 0$ to $x = \infty$

Changing the order of integration we first integrated w.r.t x along the horizontal strip which varies from

$x = y$ to $x = \infty$

and those of y varies from $y = 0$ to $y = \infty$

Now,

$$I = \int_0^\infty \left[\int_y^\infty x\, e^{-\frac{x^2}{y}}\, dx \right] dy.$$

$$= \int_0^\infty \left[\int_y^\infty \frac{1}{2} 2x\, e^{\frac{-x^2}{y}}\, dx \right] dy$$

$$= \frac{1}{2} \int_0^\infty \left[\int_y^\infty e^{\frac{-x^2}{y}}\, dx^2 \right] dy$$

$$= \frac{1}{2} \int_0^\infty \left[\frac{e^{-\frac{x^2}{y}}}{\frac{-1}{y}} \right]_y^\infty dy$$

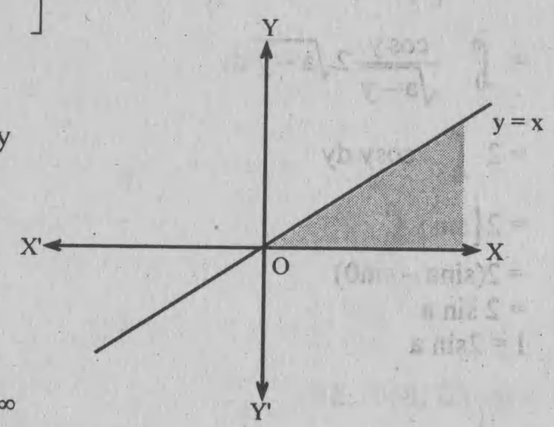
$$= \frac{1}{2} \int_0^\infty -y\, e^{-y}\, dy$$

$$= -\frac{1}{2} \int_0^\infty y\, e^{-y}\, dy$$

$$= -\frac{1}{2} \left[-y\, e^{-y} + e^{-y} \right]_0^\infty$$

$$= -\frac{1}{2}(0 - 1) = \frac{1}{2}$$

$$\therefore I = \frac{1}{2}$$



$$\int_0^{\frac{a}{\sqrt{2}}} \int_x^{\sqrt{a^2-x^2}} y^2\, dy\, dx$$

Solution:

Let $I = \int_0^{\frac{a}{\sqrt{2}}} \int_x^{\sqrt{a^2-x^2}} y^2\, dy\, dx$

Here the integration is first w.r.t y along the vertical strip, which varies from

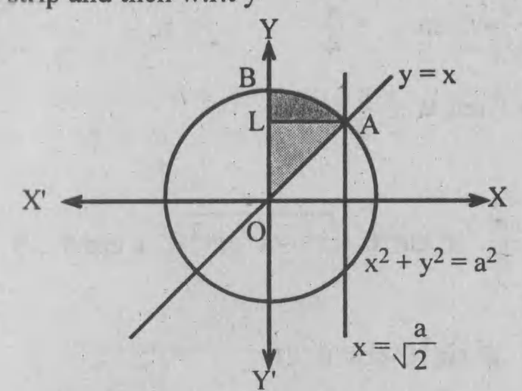
$y = x$ to $y = \sqrt{a^2 - x^2}$

i.e. $y^2 = a^2 - x^2$

or, $x^2 + y^2 = a^2$

and those x varies from $x = 0$ to $x = \frac{a}{\sqrt{2}}$

Changing the order of integration we first integrated w.r.t x along the horizontal strip and then w.r.t y



Here, the region OAB into splits upto two parts by the line AL the triangle OAL and curve-linear ABL

For the region OAL, x varies from $x = 0$ to $x = y$ and those y varies from $y = 0$ to $y = \frac{a}{\sqrt{2}}$ For the region ABL, x varies from $x = 0$ to $x = \sqrt{a^2 - y^2}$ and those

y varies from $y = \frac{a}{\sqrt{2}}$ to $y = a$

Now,

$$I = \int_0^{\frac{a}{\sqrt{2}}} \int_x^{\sqrt{a^2-x^2}} y^2\, dy\, dx$$

$$= \iint_{\text{Region OAL}} y^2 dy\, dx + \iint_{\text{Region ABL}} y^2 dy\, dx$$

$$= \int_0^{\frac{a}{\sqrt{2}}} \left[\int_0^{y} y^2\, dx \right] dy + \int_{\frac{a}{\sqrt{2}}}^{a} \left[\int_0^{\sqrt{a^2-y^2}} y^2\, dx \right] dy$$

$$= \int_0^{\frac{a}{\sqrt{2}}} \left[y^2 x \right]_0^{y} dy + \int_{\frac{a}{\sqrt{2}}}^{a} \left[y^2 x \right]_0^{\sqrt{a^2-y^2}} dy$$

$$= \int_0^{\frac{a}{\sqrt{2}}} y^3\, dy + \int_{\frac{a}{\sqrt{2}}}^{a} y^2 \sqrt{a^2-y^2}\, dy$$

$$= \left[\frac{y^4}{4} \right]_0^{\frac{a}{\sqrt{2}}} + \int_{\frac{a}{\sqrt{2}}}^{a} y^2 \sqrt{a^2-y^2}\, dy$$

Put $y = a \sin\theta$

$dy = a\cos\theta\, d\theta$

when $y = \frac{a}{\sqrt{2}}$ then $\theta = \frac{\pi}{4}$

when $y = a$ Then $\theta = \frac{\pi}{2}$

$$\therefore I = \frac{\left(\frac{a}{\sqrt{2}}\right)^4}{4} + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} a^2 \sin^2\theta \sqrt{a^2 - a^2\sin^2\theta}\,.\, a\cos\theta\, d\theta$$

$$= \frac{a^4}{16} + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} a^4 \sin^2\theta \cos^2\theta\, d\theta.$$

$$= \frac{a^4}{16} + \frac{a^4}{4} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (2\sin\theta\cos\theta)^2\, d\theta$$

$$= \frac{a^4}{16} + \frac{a^4}{4} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\sin 2\theta)^2\, d\theta$$

$$= \frac{a^4}{16} + \frac{a^4}{4} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{(1-\cos 4\theta)}{2} d\theta$$

$$= \frac{a^4}{16} + \frac{a^4}{8}\left[\theta - \frac{\sin 4\theta}{4} \right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= \frac{a^4}{16} + \frac{a^4}{8}\left[\left(\frac{\pi}{2} - \frac{\pi}{4}\right) - (0-0) \right]$$

$$= \frac{a^4}{16} + \frac{a^4\pi}{32}$$

$$= \frac{a^4}{16}\left[1 + \frac{\pi}{2} \right]$$

$$= \frac{(2+\pi)}{16} a^4$$

$$I = \frac{(2+\pi)}{16} a^4$$

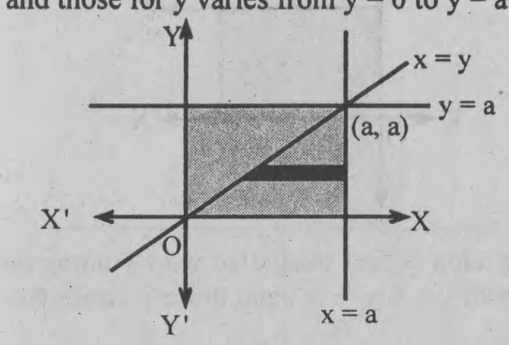
Evaluate the following integrals by changing to polar co-ordinates.

i) $\int_0^a \int_y^a \frac{x}{x^2+y^2} dx\, dy$

Solution:

Let $I = \int_0^a \int_y^a \frac{x}{x^2+y^2} dx\, dy$

Here, the integration is first integrated w.r.t x which varies from x = y to x = a and those for y varies from y = 0 to y = a



To change it into polar coordinates, put $x = r\cos\theta$, $y = r\sin\theta$

$x^2 + y^2 = r^2$ and $dx\, dy = r\, dr\, d\theta$ in given integral.

Clearly r varies from r = 0 to $r = r\sec\theta$ and those θ varies from $\theta = 0$ to $\theta = \frac{\pi}{4}$

Now,

$$I = \int_0^{\frac{\pi}{4}} \int_0^{a\sec\theta} \frac{r\cos\theta}{r^2} r\, dr\, d\theta.$$

$$= \int_0^{\frac{\pi}{4}} \left[\int_0^{a\sec\theta} \cos\theta\, dr \right] d\theta.$$

$$= \int_0^{\frac{\pi}{4}} \left[\cos\theta\, r \right]_0^{a} \sec\theta\, d\theta$$

$$= \int_0^{\frac{\pi}{4}} \cos\theta \,.\, a\sec\theta \; d\theta$$

$$= \int_0^{\frac{\pi}{4}} a\, d\theta \,.$$

$$= [a\theta]_0^{\frac{\pi}{4}}$$

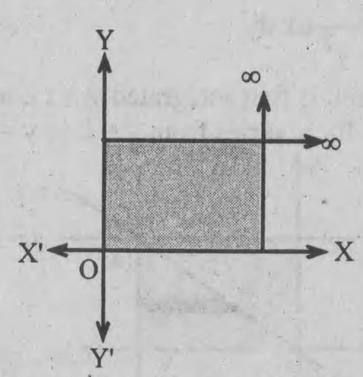
$$= \frac{a\pi}{4}$$

$$\therefore \quad I = \frac{a\pi}{4}$$

(ii) $\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)}\, dy\, dx$ **(BE 2068 Magh**

Solution:

Let $I = \int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)}\, dy\, dx$



Here, the integration is first integrated w.r.t y along the vertical st
which varies from y = 0 y = ∞, and those x varies from x = 0 to
∞

To change it into polar co-ordinates put $x = r\cos\theta$, $y = r\sin\theta$

$\therefore$ $r^2 = x^2 + y^2$ and

$dy\, dx = r dr\, d\theta$ in the given integral.

Clearly r varies from r = o to r = 1 and those θ varies from $\theta = 0$

$\theta = \frac{\pi}{2}$

Now,

$$I = \int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)}\, dy\, dx$$

$$= \int_0^{\frac{\pi}{2}}\int_0^{\infty} e^{-r^2} r dr d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \left[\int_0^{\infty} e^{-r^2} d(r^2)\right] d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \left[\frac{e^{-r^2}}{-1}\right]_0^{\infty} d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}} -(0-1)\, d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}} d\theta$$

$$= \frac{1}{2}[\theta]_0^{\frac{\pi}{2}} = \frac{\pi}{4} \quad \therefore I = \frac{\pi}{4}$$

(i) $\int_0^{a}\int_0^{\sqrt{a^2-x^2}} \sqrt{x^2+y^2}\, dy\, dx.$

Solution:

Let $I = \int_0^{a}\int_0^{\sqrt{a^2-x^2}} \sqrt{x^2+y^2}\, dy\, dx.$

Here, the integration is first integrand w.r.t y along the vertical strip.

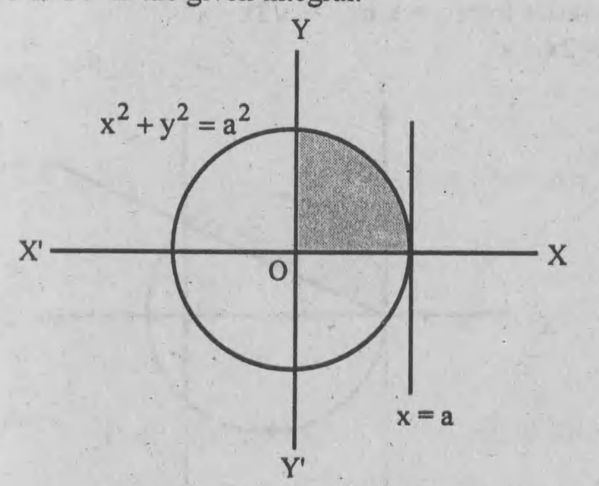
Which varies from y = 0 to $y = \sqrt{a^2 - x^2}$

i.e. $y^2 = a^2 - x^2$

$x^2 + y^2 = a^2$ and those x varies from x = 0 to x = a change it into polar co-ordinates $x = r\cos\theta$, $y = r\sin\theta$

$x^2 + y^2 = r^2$ and

$dy\, dx = r\, dr\, d\theta$ in the given integral.



Cleary r varies from r = 0 to r = a and those θ varies from θ = o to

$\theta = \frac{\pi}{2}$

Now,

$$I = \int_0^{\frac{\pi}{2}}\int_0^{a} r.\, r\, dr\, d\theta$$

$$= \int_0^{\frac{\pi}{2}}\int_0^{a} r^2\, dr\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left[\frac{r^3}{3}\right]_0^a d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{a^3}{3} d\theta$$

$$= \frac{a^3}{3}[\theta]_0^{\frac{\pi}{2}} = \frac{a^3}{3} \cdot \frac{\pi}{2} = \frac{\pi a^3}{6}$$

$$\therefore \quad I = \frac{\pi a^3}{6}$$

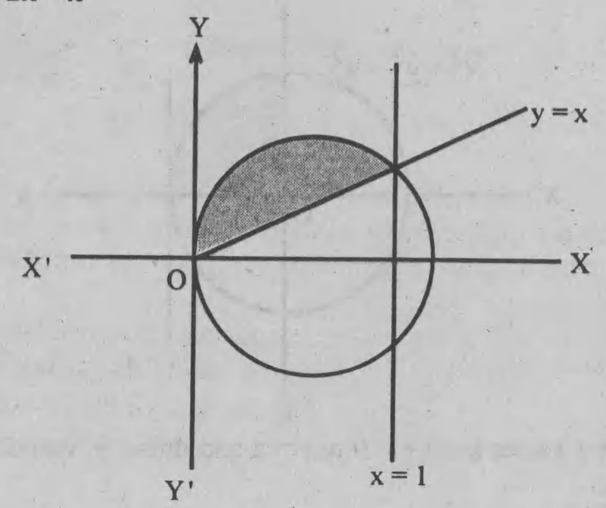
(iv) $\int_0^1 \int_x^{\sqrt{2x-x^2}} \left(x^2+y^2\right) dy\, dx$ **[B.E. 2**

Solution:

Let $I = \int_0^1 \int_x^{\sqrt{2x-x^2}} \left(x^2+y^2\right) dy\, dx$

Here, the integration is first integrand w.r.t y along the vertical which varies from $y = x$ to $y = \sqrt{2x-x^2}$

i.e. $y^2 = 2x - x^2$



or, $x^2 + y^2 = 2x$

and those x varies from $x = 0$ to $x = 1$

To change it into polar co-ordinates put $x = r\cos\theta$ and $y = r\sin\theta$

$\therefore$ $x^2 + y^2 = r^2$ and $dy\, dx = r dr d\theta$ in the given integral.

clearly, r varies from $r = 0$ to $r = 2\cos\theta$ and those θ varies from $\frac{\pi}{4}$ to

$\theta = \frac{\pi}{2}$

$$I = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{2\cos\theta} r^2.r\, dr\, d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{2\cos\theta} r^3\, dr\, d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left[\frac{r^4}{4}\right]_0^{2\cos\theta} d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{(2\cos\theta)^4}{4} d\theta = \frac{1}{4}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos^4\theta\, d\theta.$$

$$= 4\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{(1+\cos 2\theta)^2}{4} d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (1+2\cos 2\theta + \cos^2 2\theta)\, d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left(1+2\cos 2\theta + \frac{1+\cos 4\theta}{2}\right) d\theta$$

$$= \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (3+4\cos 2\theta + \cos 4\theta)\, d\theta$$

$$= \frac{1}{2}\left[3\theta + 4\frac{\sin 2\theta}{2} + \frac{\sin 4\theta}{4}\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= \frac{1}{2}\left[3\left(\frac{\pi}{2}-\frac{\pi}{4}\right) + 2\left(\sin\pi - \sin\frac{\pi}{2}\right) + \frac{1}{4}\left(\sin 2\pi - \sin\pi\right)\right]$$

$$= \frac{1}{2}\left[3\frac{\pi}{4} + 2(0-1) + \frac{1}{4}(0-0)\right] = \frac{3\pi}{8} - 1$$

$$\therefore I = \frac{3\pi}{8} - 1$$

(v) $\int_0^{2a} \int_0^{\sqrt{2ax-x^2}} \left(x^2+y^2\right) dy\, dx$ **(BE 2067 Mangsir)**

Solution: Let $I = \int_0^{2a} \int_0^{\sqrt{2ax-x^2}} \left(x^2+y^2\right) dy\, dx$

Here, the integration is first integrand w.r.t.y along the vertical strip which varies from

$y = 0$ to $y = \sqrt{2ax-x^2}$

i.e. $y^2 = 2ax - x^2$

or, $x^2 + y^2 = 2ax$.

and those x varies from $x = 0$ to $x = 2a$

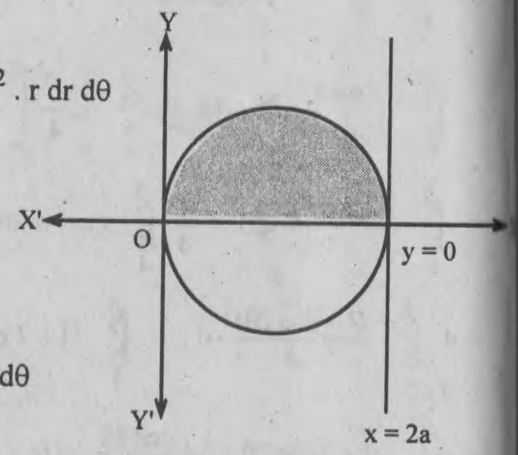
To change it into polar co-ordinates

Put $x = r\cos\theta$ and $y = r\sin\theta$

$\therefore$ $x^2 + y^2 = r^2$ and

$dy\,dx = r\,dr\,d\theta$ in the given integral.

Clearly r varies from $r = 0$ to $r = 2a\cos\theta$ and those θ varies from $= 0$ to $\theta = \frac{\pi}{2}$



Now, $I = \int_0^{\frac{\pi}{2}} \int_0^{2a\cos\theta} r^2 . r\,dr\,d\theta$

$= \int_0^{\frac{\pi}{2}} \int_0^{2a\cos\theta} r^3\,dr\,d\theta$

$= \int_0^{\frac{\pi}{2}} \left[\frac{r^4}{4}\right]_0^{2a\cos\theta} d\theta$

$= \frac{1}{4} \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} (2a\cos\theta)^4\,d\theta$

$= \frac{16a^4}{4} \int_0^{\frac{\pi}{2}} \cos^4\theta\,d\theta$

$= 4a^4 \frac{\sqrt{\pi}\,\Gamma\left(\frac{4+1}{2}\right)}{2\,\Gamma\left(\frac{4+2}{2}\right)} = 2a^4\sqrt{\pi}\,\frac{\Gamma\left(\frac{5}{2}\right)}{\Gamma 3} = 2a^4 . \frac{\sqrt{\pi} . \frac{3}{2} . \frac{1}{2}\sqrt{\pi}}{2!} = \frac{3\pi a^4}{4}$

$\therefore I = \frac{3\pi a^4}{4}$ Ans.

(vi) $\int_0^a \int_0^{\sqrt{a^2-y^2}} \left(x^2+y^2\right) dx\,dy$

Solution: Let $I = \int_0^a \int_0^{\sqrt{a^2-y^2}} \left(x^2+y^2\right) dx\,dy$

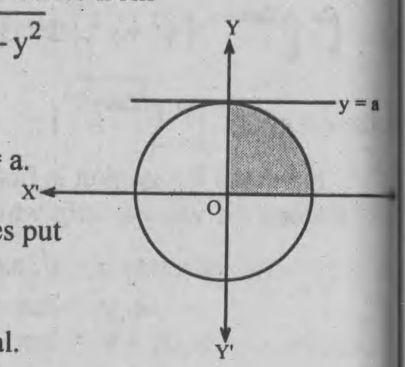
Here, the integration is first integrand w.r.t. x along the horizontal strip. Which varies from

$x = 0$ to $x = \sqrt{a^2-y^2}$

i.e. $x^2 = a^2 - y^2$

or, $x^2 + y^2 = a^2$

and those y vries from $y = 0$ to $y = a$.



To change it into polar co-ordinates put

$x = r\cos\theta$, $y = r\sin\theta$

$\therefore x^2 + y^2 = r^2$ and

$dy\,dx = r\,dr\,d\theta$ in the given integral.

Clearly r varies from $r = 0$ to $r = a$

and θ varies from $\theta = 0$ to $\theta = \frac{\pi}{2}$

Now, $I = \int_0^{\frac{\pi}{2}} \int_0^a r^2 . r\,dr\,d\theta$

$= \int_0^{\frac{\pi}{2}} \int_0^a r^3\,dr\,d\theta$

$= \int_0^{\frac{\pi}{2}} \left[\frac{r^4}{4}\right]_0^a d\theta$

$= \int_0^{\frac{\pi}{2}} \frac{a^4}{4}\,d\theta$

$= \frac{a^4}{4}\,[\theta]_0^{\frac{\pi}{2}} = \frac{a^4}{4} . \frac{\pi}{2} = \frac{\pi a^4}{8}$

$\therefore I = \frac{\pi a^4}{8}$ Ans.

(vii) $\int_0^a \int_0^{\sqrt{a^2-x^2}} y^2\sqrt{x^2+y^2}\,dy\,dx$ **(BE 2070 Magh)**

Solution: Let $I = \int_0^a \int_0^{\sqrt{a^2-x^2}} y^2\sqrt{x^2+y^2}\,dy\,dx$

Here, the integrating is first integrated w.r.t. y along the vertical strip which varies from.

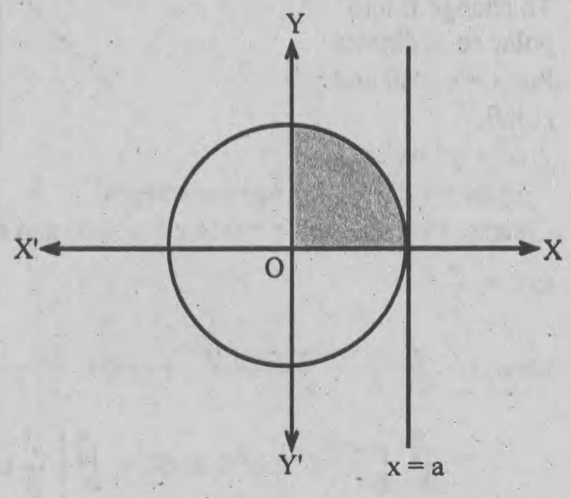
$y = 0$ to y

$= \sqrt{a^2-x^2}$ i.e.

$y^2 = a^2 - x^2$

or, $x^2 + y^2 = a^2$

and those of x varies from $x = 0$ to $x = a$.



To change it into polar coordinates

Put $x = r\cos\theta$

and $y = r\sin\theta$

$\therefore x^2 + y^2 = r^2$ and

$dydx = rdrd\theta$ in the given integral.

Clearly, r varies from $r = 0$ to $r = a$ and those θ varies from $\theta = 0$ to $\frac{\pi}{2}$.

Now, $I = \int_0^{\frac{\pi}{2}} \int_0^a r^2\sin^2\theta . r . r\,drd\theta$

$$= \int_0^{\frac{\pi}{2}} \int_0^a r^4 \sin^2\theta \, dr \, d\theta = \int_0^{\frac{\pi}{2}} \left[\sin^2\theta \frac{r^5}{5} \right]_0^a d\theta = \frac{a^5}{5} \int_0^{\frac{\pi}{2}} \sin^2\theta \, d\theta$$

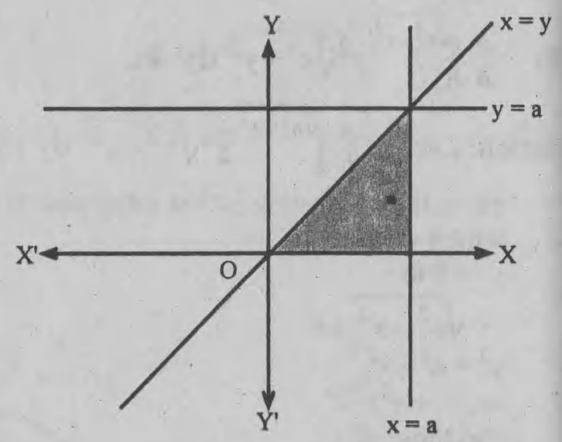
$$= \frac{a^5}{5} \int_0^{\frac{\pi}{2}} \left(\frac{1-\cos 2\theta}{2} \right) d\theta = \frac{a^5}{10} \left[\theta - \frac{\sin 2\theta}{2} \right]_0^{\frac{\pi}{2}} = \frac{a^5}{10} \left[\frac{\pi}{2} - 0 \right] = \frac{\pi a^5}{20}$$

$\therefore I = \frac{\pi a^5}{20}$ Ans.

(viii) $\int_0^a \int_y^a \frac{x^2}{\sqrt{x^2+y^2}} \, dx \, dy$

Solution: Let $I = \int_0^a \int_y^a \frac{x^2}{\sqrt{x^2+y^2}} \, dx \, dy$

Here, the integration is first integrand w.r.t. x along the horizontal strip which varies from x = y to x = a and those y varies from y = 0 to y = a



To change it into polar co-ordinates Put $x = r\cos\theta$ and $y = r\sin\theta$

$\therefore x^2 + y^2 = r^2$ and

$dydx = r \, dr \, d\theta$ in the given integral.

Clearly, r varies from r = 0 to $r = a\sec\theta$ and those θ varies from θ = 0 to $\theta = \frac{\pi}{4}$

Now, $I = \int_0^{\frac{\pi}{4}} \int_0^{a\sec\theta} \frac{r^2\cos^2\theta}{r} \, . \, r \, dr \, d\theta$

$$= \int_0^{\frac{\pi}{4}} \int_0^{a\sec\theta} r^2\cos^2\theta \, dr \, d\theta = \int_0^{\frac{\pi}{4}} \left[\frac{r^3}{3}\cos^2\theta \right]_0^{a\sec\theta} d\theta$$

$$= \frac{1}{3} \int_0^{\frac{\pi}{4}} (a\sec\theta)^3 \, . \cos^2\theta \, d\theta$$

$$= \frac{1}{3} \int_0^{\frac{\pi}{4}} a^3 \sec^3\theta \cos^2\theta \, d\theta = \frac{a^3}{3} \int_0^{\frac{\pi}{4}} \sec\theta \, d\theta$$

$$= \frac{a^3}{3} \left[\log(\sec\theta + \tan\theta) \right]_0^{\frac{\pi}{4}}$$

$$= \frac{a^3}{3} \left[\log\left(\sec\frac{\pi}{4} + \tan\frac{\pi}{4} \right) - \log(\sec 0 + \tan 0) \right]$$

$$= \frac{a^3}{3} \log(\sqrt{2} + 1) - \log 1$$

$$= \frac{a^3}{3} \log(\sqrt{2} + 1) - 0 \qquad \text{since } \log 1 = 0$$

$$= \frac{a^3}{3} \log(\sqrt{2} + 1)$$

$\therefore I = \frac{a^3}{3} \log(\sqrt{2} + 1)$ Ans.

(ix) $\int_0^2 \int_0^{\sqrt{2x-x^2}} (x^2+y^2) \, dy \, dx$

Solution: Let $I = \int_0^2 \int_0^{\sqrt{2x-x^2}} (x^2+y^2) \, dy \, dx.$

Here, the integration is first integrated w.r.t. y along the vertical strip. Which varies from

$y = 0$ to $y = \sqrt{2x - x^2}$

i. e. $y^2 = 2x - x^2$

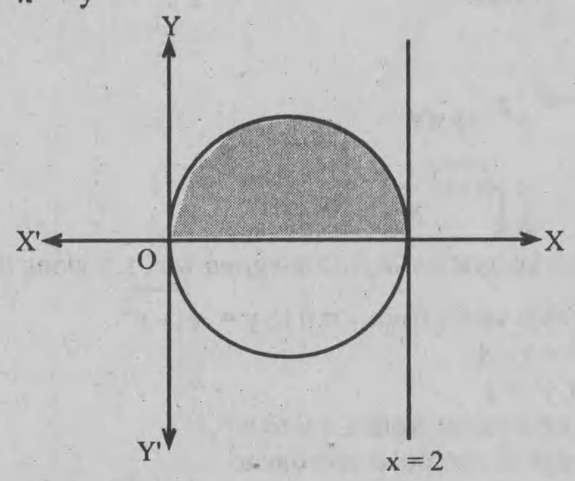
or, $x^2 + y^2 = 2x$

and those x varies from x = 0 to x = 2

To change it into polar co-ordinate

Put $x = r\cos\theta$ and $y = r\sin\theta$

$\therefore r^2 = x^2 + y^2$



and $dy \, dx = r \, dr \, d\theta$

Clearly, r varies from $r = 0$ to $r = 2\cos\theta$

and those θ varies from θ = 0 to $\theta = \frac{\pi}{2}$.

Now, $I = \int_0^{\frac{\pi}{2}} \int_0^{2\cos\theta} r^2\, r\, dr\, d\theta$

$$= \int_0^{\frac{\pi}{2}} \int_0^{2\cos\theta} r^3\, dr\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left[\frac{r^4}{4}\right]_0^{2\cos\theta} d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{(2\cos\theta)^4}{4}\, d\theta$$

$$= \frac{16}{4} \int_0^{\frac{\pi}{2}} \cos^4\theta\, d\theta$$

$$= 4 \cdot \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{4+1}{2}\right)}{\Gamma\left(\frac{4+2}{2}\right)}$$

$$= 2\sqrt{\pi}\, \frac{\Gamma\left(\frac{5}{2}\right)}{\Gamma 3}$$

$$= 2\, \frac{\sqrt{\pi} \cdot \frac{3}{2} \cdot \frac{1}{2} \cdot \sqrt{\pi}}{2!}$$

$$= \frac{3\pi}{4}$$

$\therefore I = \frac{3\pi}{4}$ Ans.

(x) $\int_0^1 \int_0^{\sqrt{1-x^2}} y^2\, dy\, dx$

Solution: Let $I = \int_0^1 \int_0^{\sqrt{1-x^2}} y^2\, dy\, dx$

Here the integration in frist integrted w.r . t. y along the vertical strip which varies from $y = 0$ to $y = \sqrt{1-x^2}$

i.e., $y^2 = 1 - x^2$

or, $x^2 + y^2 = 1$

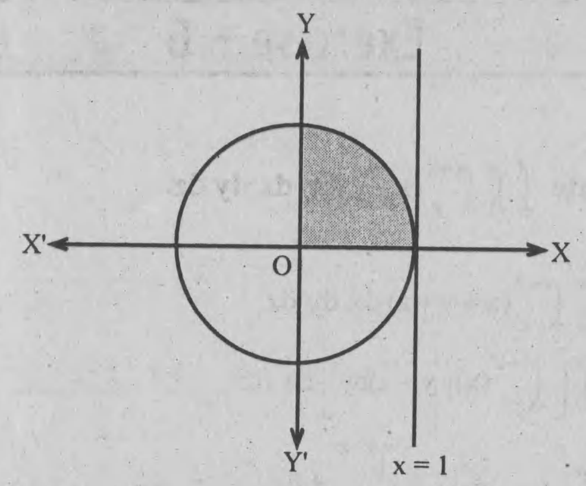
and those x varies from $x = 0$ to $x = 1$

To change in into polar coordinate

Put $x = r\cos\theta$ & $y = r\sin\theta$

$\therefore r^2 = x^2 + y^2$

and $dy\, dx = r\, dr\, d\theta$



Clearly, r varies from $r = 0$ to $r = 1$ and those θ varies from $\theta = 0$ to $\theta = \frac{\pi}{2}$

Now, $I = \int_0^{\frac{\pi}{2}} \int_0^1 r^2 \sin^2\theta \,.\, r\, dr\, d\theta$

$$= \int_0^{\frac{\pi}{2}} \int_0^1 r^3 \sin^2\theta\, dr\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left[\frac{r^4}{4} \sin^2\theta\right]_0^1 d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{4} \sin^2\theta\, d\theta$$

$$= \frac{1}{4} \int_0^{\frac{\pi}{2}} \sin^2\theta\, d\theta$$

$$= \frac{1}{4}\, \frac{\sqrt{\pi}}{2}\, \frac{\Gamma\left(\frac{2+1}{2}\right)}{\Gamma\left(\frac{2+2}{2}\right)}$$

$$= \frac{1}{4} \cdot \frac{\sqrt{\pi}}{2} \cdot \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma 2}$$

$$= \frac{\sqrt{\pi}}{8} \cdot \frac{1}{2} \cdot \frac{\sqrt{\pi}}{1!}$$

$$= \frac{\pi}{16}$$

$\therefore I = \frac{\pi}{16}$

Exercise - 6

1. Evaluate $\int_0^1\int_0^z\int_{x-z}^{x+z}(x+y+z)\,dx\,dy\,dz$.

Solution:

Let $I=\int_0^1\int_0^z\int_{x-z}^{x+z}(x+y+z)\,dx\,dy\,dz$

$$=\int_0^1\int_0^z\left[\int_{x-z}^{x+z}(x+y+z)dy\right]dx\,dz$$

$$=\int_0^1\int_0^z\left[xy+\frac{y^2}{2}+zy\right]_{x-z}^{x+z}dx\,dz$$

$$=\int_0^1\int_0^z x(x+z-x+z)+\frac{(x+z)^2-(x-z)^2}{2}+z(x+z-x+z)\,dx\,dz$$

$$=\int_0^1\int_0^z\left[2xz+2xz+2z^2\right]dx\,dz$$

$$=\int_0^1\int_0^z(4xz+2z^2)\,dx\,dz$$

$$=\int_0^1\left[\frac{4x^2z}{2}+2z^2.x\right]_0^z dz=\int_0^1(2z^3+2z^3)dz=\int_0^1 4z^3dz$$

$$=4\left[\frac{z^4}{4}\right]_0^1=\frac{4}{4}[1^4-0^4]=1$$

$\therefore\quad I=1.$

2. Evaluate $\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}}xyz\,dz\,dy\,dx$

Solution:

Let $I=\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}}xyz\;dz\,dy\,dx$

$$=\int_0^1\int_0^{\sqrt{1-x^2}}\left[\int_0^{\sqrt{1-x^2-y^2}}xyz\,dz\right]dy\,dx$$

$$=\int_0^1\int_0^{\sqrt{1-x^2}}\left[xy\frac{z^2}{2}\right]_0^{\sqrt{1-x^2-y^2}}dy\,dx$$

$$=\frac{1}{2}\int_0^1\int_0^{\sqrt{1-x^2}}xy(1-x^2-y^2)\,dy\,dx.$$

$$=\frac{1}{2}\int_0^1\left[\int_0^{\sqrt{1-x^2}}(xy-x^3y-xy^3)dy\right]dx.$$

$$=\frac{1}{2}\int_0^1\left[\frac{xy^2}{2}-\frac{x^3y^2}{2}-\frac{xy^4}{4}\right]_0^{\sqrt{1-x^2}}dx.$$

$$=\frac{1}{2}\int_0^1\left[\frac{x(1-x^2)}{2}-\frac{x^3(1-x^2)}{2}-\frac{x(1-x^2)^2}{4}\right]dx.$$

$$=\frac{1}{2}\int_0^1\left[\frac{2x-2x^3-2x^3+2x^5-x+2x^3-x^5}{4}\right]dx.$$

$$=\frac{1}{8}\int_0^1(x-2x^3+x^5)\,dx.$$

$$=\frac{1}{8}\left[\frac{x^2}{2}-\frac{2x^4}{4}+\frac{x^6}{6}\right]_0^1=\frac{1}{8}\left(\frac{1}{2}-\frac{2}{4}+\frac{1}{6}\right)=\frac{1}{8}\left(\frac{1}{2}-\frac{1}{2}+\frac{1}{6}\right)=\frac{1}{48}$$

$\therefore\quad I=\frac{1}{48}$

3. Evaluate $\int_0^1\int_0^{1-x}\int_0^{1-x-y}\frac{1}{(x+y+z+1)^3}dx\,dy\,dz$

(BE 2069 Bhadra)

Solution:

Let $I=\int_0^1\int_0^{1-x}\int_0^{1-x-y}\frac{1}{(x+y+z+1)^3}dx\,dy\,dz$

$$=\int_0^1\int_0^{1-x}\left[\int_0^{1-x-y}(x+y+z+1)^3dz\right]dy\,dx.$$

$$=\int_0^1\int_0^{1-x}\left[\frac{(x+y+z+1)^{-2}}{-2}\right]_0^{1-x-y}dy\,dx.$$

$$=-\frac{1}{2}\int_0^1\int_0^{1-x}\left[\frac{1}{(x+y+z+1)^2}\right]_0^{1-x-y}dy\,dx$$

$$=-\frac{1}{2}\int_0^1\int_0^{1-x}\left[\frac{1}{(x+y+1-x-y+1)^2}-\frac{1}{(x+y+1)^2}\right]dy\,dx$$

$$= -\frac{1}{2}\int_0^1\int_0^{1-x}\left[\frac{1}{4}-\frac{1}{(x+y+1)^2}\right]dy\,dx$$

$$= -\frac{1}{2}\int_0^1\left[\frac{y}{4}-\frac{(x+y+1)^{-1}}{-1}\right]_0^{1-x}dx$$

$$= -\frac{1}{2}\int_0^1\left[\frac{y}{4}+\frac{1}{(x+y+1)}\right]_0^{1-x}dx$$

$$= -\frac{1}{2}\int_0^1\left[\frac{1-x}{4}+\frac{1}{(x+1-x+1)}-\frac{1}{x+1}\right]dx$$

$$= -\frac{1}{2}\int_0^1\left(\frac{1-x}{4}+\frac{1}{2}-\frac{1}{x+1}\right)dx.$$

$$= -\frac{1}{2}\left[\frac{(1-x)^2.(-1)}{4.2}+\frac{1}{2}x-\log(x+1)\right]_0^1$$

$$= -\frac{1}{2}\left\{(0+\frac{1}{8})+\frac{1}{2}(1-0)-\log 2+\log 1\right\}$$

$$= -\frac{1}{2}\left(\frac{1}{8}+\frac{1}{2}-\log 2\right) = -\frac{1}{2}\left(\frac{5}{8}-\log 2\right) = \frac{1}{2}\left(\log 2-\frac{5}{8}\right)$$

$$\therefore\ I = \frac{1}{2}\left(\log 2-\frac{5}{2}\right).$$

4. Evaluate $\int_0^4\int_0^{2\sqrt{z}}\int_0^{\sqrt{4z-x^2}} dz\,dx\,dy$

Solution:

Let $I = \int_0^4\int_0^{2\sqrt{z}}\int_0^{\sqrt{4z-x^2}} dz\,dx\,dy$

$$= \int_0^4\int_0^{2\sqrt{z}}\left[\int_0^{\sqrt{4z-x^2}} dy\right]dx\,dz$$

$$= \int_0^4\int_0^{2\sqrt{z}}[y]_0^{\sqrt{4z-x^2}}\,dx\,dz$$

$$= \int_0^4\int_0^{2\sqrt{z}}\left(4z-x^2\right)^{\frac{1}{2}}dx\,dz$$

$$= \int_0^4\left[\int_0^{2\sqrt{z}}\sqrt{(2\sqrt{z})^2-x^2}\,dx\right]dz$$

$$= \int_0^4\left[\frac{x\sqrt{4z-x^2}}{2}+\frac{4z}{2}\sin^{-1}\frac{x}{2\sqrt{z}}\right]_0^{2\sqrt{z}}dz$$

$$= \int_0^4\left[0-0+\frac{4z}{2}\{\sin^{-1}(1)-\sin^{-1}0\}\right]dz$$

$$= \int_0^4 2z\sin^{-1}(1)dz = \int_0^4 2.z.\frac{\pi}{2}dz = \pi\int_0^4 z\,dz = \pi\left[\frac{z^2}{2}\right]_0^4 = \pi\left[\frac{4^2}{2}\right]$$

$$= \pi.\frac{16}{2} = 8\pi$$

$I = 8\pi$.

Evaluate $\iiint_V (x-2y+z)\,dx\,dy\,dz$ where v is the region bounded by
$0 \le x \le 1, 0 \le y \le x^2, 0 \le z \le x+y$.

Solution:

We have, $\iiint_V (x-2y+z)\,dx\,dy\,dz$ bounded by $0 \le x \le 1, 0 \le y \le x^2, 0 \le z \le x+y$.

$$I = \int_0^1\int_0^{x^2}\left[\int_0^{x+y}(x-2y+z)dz\right]dy\,dx$$

$$= \int_0^1\int_0^{x^2}\left[xz-2yz+\frac{z^2}{2}\right]_0^{x+y}dy\,dx.$$

$$= \int_0^1\int_0^{x^2}\left[x(x+y)-2y(x+y)+\frac{(x+y)^2}{2}\right]dy\,dx.$$

$$= \int_0^1\int_0^{x^2}\left[x^2+xy-2xy-2y^2+\frac{x^2+2xy+y^2}{2}\right]dy\,dx$$

$$= \int_0^1\int_0^{x^2}\left[x^2-xy-2y^2+\frac{x^2+2xy+y^2}{2}\right]dy\,dx$$

$$= \frac{1}{2}\int_0^1\int_0^{x^2}(2x^2-2xy-4y^2+x^2+2xy+y^2)\,dy\,dx.$$

$$= \frac{1}{2}\int_0^1\int_0^{x^2}(3x^2-3y^2)\,dy\,dx.$$

$$= \frac{1}{2}\int_0^1\left[3x^2y-\frac{3y^3}{3}\right]_0^{x^2}dx.$$

$$= \frac{1}{2}\int_0^1 \left[3x^2.x^2 - (x^2)^3\right] dx. = \frac{1}{2}\int_0^1 \left(3x^4 - x^6\right) dx.$$

$$= \frac{1}{2}\left[\frac{3x^5}{5} - \frac{x^7}{7}\right]_0^1 = \frac{1}{2}\left(\frac{3}{5} - \frac{1}{7}\right) = \frac{1}{2}\left(\frac{21-5}{35}\right) = \frac{1}{2}.\frac{16}{35} = \frac{8}{35}$$

$\therefore \quad I = \frac{8}{35}.$

6. Evaluate $\iiint_V x^2$ dx dy dz over the region v bounded by the planes x = 0, y = 0, z = 0 and x + y + z = a

Solution:

We have, $\iiint_V x^2 \, dx\, dy\, dz$

bounded by the planes x = 0, y = 0, z = 0 and x + y + z = a

Let $I = \iiint_V x^2 \, dx\, dy\, dz$

$$= \int_0^a \int_0^{a-x} \int_0^{a-x-y} x^2 \, dy\, dz\, dx = \int_0^a \int_0^{a-x} \left[x^2 z\right]_0^{a-x-y} dy\, dx.$$

$$= \int_0^a \int_0^{a-x} x^2(a-x-y)\, dy\, dx. = \int_0^a \int_0^{a-x} (ax^2 - x^3 - x^2 y)\, dy\, dx$$

$$= \int_0^a \left[ax^2 y - x^3 y - \frac{x^2 y^2}{2}\right]_0^{a-x} dy\, dx.$$

$$= \int_0^a \left[ax^2(a-x) - x^3(a-x) - x^2\frac{(a-x)^2}{2}\right] dx.$$

$$= \int_0^a \left[a^2x^2 - ax^3 - ax^3 + x^4 - x^2\frac{(a^2 - 2ax + x^2)}{2}\right] dx.$$

$$= \frac{1}{2}\int_0^a \left[2a^2x^2 - 4ax^3 + 2x^4 - x^2a^2 + 2ax^3 - x^4\right] dx.$$

$$= \frac{1}{2}\int_0^a \left[a^2x^2 - 2ax^3 + x^4\right] dx.$$

$$= \frac{1}{2}\left[\frac{a^2x^3}{3} - \frac{2ax^4}{2} + \frac{x^5}{5}\right]_0^a = \frac{1}{2}\left(a^2\frac{a^3}{3} - \frac{aa^4}{2} + \frac{a^5}{5}\right)$$

$$= \frac{1}{2}\left(\frac{a^5}{3} - \frac{a^5}{2} + \frac{a^5}{5}\right) = \frac{1}{2}\left[\frac{10a^5 - 15a^5 + 6a^5}{30}\right] = \frac{1}{2}.\frac{a^5}{30} = \frac{a^5}{60}$$

$\therefore \quad I = \frac{a^5}{60}.$

Evaluate $\iiint_V$ dx dy dz over the region bounded by the planes x = 0, y = 0, z = 0, y = 8 and cylinder $x^2 + z^2 = 9$.

Solution:

We have, $\iiint_V$ dx dy dz bounded by the planes x = 0, y = 0, z = 0 y = 8 and cylinder $x^2 + z^2 = 9$

$I = \iiint_V dx\, dy\, dz$

$$= \int_0^3 \int_0^8 \left[\int_0^{\sqrt{9-z^2}} dx\right] dy\, dz = \int_0^3 \int_0^8 [x]_0^{\sqrt{9-z^2}} dy\, dz$$

$$= \int_0^3 \left[\int_0^8 \sqrt{9-z^2}\, dy\right] dz. = \int_0^3 \left[\sqrt{9-z^2}\, y\right]_0^8 dz.$$

$$= \int_0^3 8\sqrt{9-z^2}\, dz. = 8\left[\frac{z}{2}\sqrt{9-z^2} + \frac{9}{2}\sin^{-1}\frac{z}{3}\right]_0^3$$

$$= 8\left[0 + \frac{9}{2}.\frac{\pi}{2}\right] = 8.\frac{9\pi}{4} = 18\pi$$

$I = 18\pi.$

Evaluate $\iiint_V$ xdv where v is the region bounded by the plane x + y + z = 1 [B.E. 2060]

Solution:

We have, $\iiint_V$ xdv bounded by the plane x + y + z = 1

$$I = \iiint_V x\,dv = \int_0^1 \int_0^{1-x} \left[\int_0^{1-x-y} x\, dz\right] dy\, dx$$

$$= \int_0^1 \int_0^{1-x} [xz]_0^{1-x-y} dy\, dx$$

$$= \int_0^1 \int_0^{1-x} x(1-x-y)\, dy\, dx$$

$$= \int_0^1 \int_0^{1-x} (x - x^2 - xy)\, dy\, dx$$

$$= \int_0^1 \left[xy - x^2y - \frac{xy^2}{2}\right]_0^{1-x} dx.$$

$$= \int_0^1 \left[x(1-x) - x^2(1-x) - \frac{x(1-x)^2}{2}\right] dx.$$

$$= \int_0^1 \left(x - x^2 - x^2 + x^3 - \frac{x(1-2x+x^2)}{2} \right) dx$$

$$= \frac{1}{2}\int_0^1 \left(2x - 4x^2 + 2x^3 - x + 2x^2 - x^3\right) dx$$

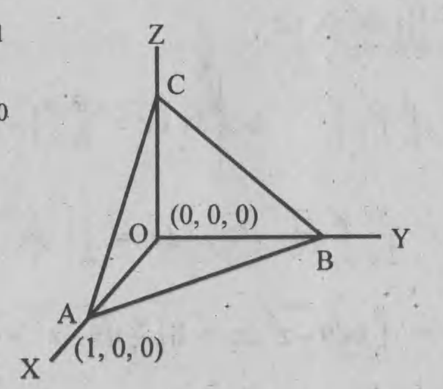
$$= \frac{1}{2}\int_0^1 \left(x - 2x^2 + x^3\right) dx .$$

$$= \frac{1}{2}\left[\frac{x^2}{2} - \frac{2x^3}{3} + \frac{x^4}{4}\right]_0^1$$

$$= \frac{1}{2}\left[\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right]$$

$$= \frac{1}{2}\left(\frac{6-8+3}{12}\right)$$

$$= \frac{1}{2}\cdot\frac{1}{12} = \frac{1}{24}$$



$\therefore \quad I = \frac{1}{24}.$

9. Evaluate $\iiint_v x\, dx\, dy\, dz$ where v is region in the first oc

bounded by surface $x^{2/3} + y^{2/3} + z^{2/3} = 1$. [B.E 2

Solution:

Let $I = \iiint_v x\, dx\, dy\, dz$ where v is region bounded by the octant bounded by the surface $x^{2/3} + y^{2/3} + z^{2/3} = 1$.

So put $x^{2/3} = u$; $y^{2/3} = v$; $z^{2/3} = w$

$\therefore x = u^{3/2}$; $y = v^{3/2}$; $z = w^{3/2}$

$dx = \frac{3}{2}u^{1/2}\, du$; $dy = \frac{3}{2}v^{\frac{1}{2}}\, dv$ and $dz = \frac{3}{2}\, w^{\frac{1}{2}}\, dw$

Now, (i) becomes,

$$I = \iiint_v u^{\frac{3}{2}} \left(\frac{3}{2}u^{\frac{1}{2}}\right)\left(\frac{3}{2}v^{\frac{1}{2}}\right)\left(\frac{3}{2}w^{\frac{1}{2}}\right) du\, dv\, dw$$

$$= \frac{27}{8} \iiint_v u^2\, v^{\frac{1}{2}}\, w^{\frac{1}{2}}\, du\, dv\, dw$$

where, $u + v + w \le 1$

$$= \frac{27}{8} \iiint_v u^{3-1}\, v^{\frac{3}{2}-1}\, w^{\frac{3}{2}-1}\, du\, dv\, dw$$

So by using dritchlet's theorem.

$$I = \frac{27}{8}\, \frac{\Gamma 3\, \Gamma\frac{3}{2}\, \Gamma\frac{3}{2}}{\Gamma\left(3 + \frac{3}{2} + \frac{3}{2} + 1\right)}$$

$$= \frac{27}{8}\, \frac{2!\, \frac{1}{2}\Gamma\frac{1}{2} \cdot \frac{1}{2}\Gamma\frac{1}{2}}{\Gamma 7!} = \frac{27\sqrt{\pi}\sqrt{\pi}}{8\times 2\times 6!} = \frac{27\pi}{16\times 6\times 5\times 4\times 3\times 2\times 1} = \frac{3\pi}{640}$$

$\therefore I = \frac{3\pi}{1280}.$

10. Evaluate $\iiint_v x^{\alpha-1} y^{\beta-1} z^{y-1}\, dx\, dy\, dz$

where v denotes the region in the first octant bounded by the surface,

$x^2 + y^2 + z^2 = 1$.

Solution: $\iiint_v x^{\alpha-1} y^{\beta-1} z^{y-1}\, dx\, dy\, dz$ where v denotes the region in the first octant bounded by the surface,

$x^2 + y^2 + z^2 = 1$.

So put $x^2 = v$, $y^2 = v$, $z^2 = w$

$\therefore x = u^{\frac{1}{2}}; y = v^{\frac{1}{2}}; z = w^{\frac{1}{2}}$

$dx = \frac{1}{2}u^{-\frac{1}{2}}du$; $dy = \frac{1}{2}v^{-\frac{1}{2}}dv$, $dz = \frac{1}{2}w^{-\frac{1}{2}}dw$

Now eq^n (i) becomes.

$$I = \iiint_v (u)^{\frac{\alpha-1}{2}} (v)^{\frac{\beta-1}{2}} (w)^{\frac{\gamma-1}{2}} \cdot \frac{1}{2}\left(u^{-\frac{1}{2}}\right) \cdot \frac{1}{2}\left(v^{-\frac{1}{2}}\right) \cdot \frac{1}{2}\left(w^{-\frac{1}{2}}\right) . du\, dv\, dw$$

$$= \frac{1}{8}\iiint_v (u)^{\frac{\alpha-1}{2}-\frac{1}{2}} (v)^{\frac{\beta-1}{2}-\frac{1}{2}} (w)^{\frac{\gamma-1}{2}-\frac{1}{2}}\, du\, dv\, dw$$

$$= \frac{1}{8}\iiint_v u^{\frac{\alpha-2}{2}} (v)^{\frac{\beta-2}{2}} (w)^{\left(\frac{\gamma-2}{2}\right)}\, du\, dv\, dw$$

$$= \frac{1}{8}\iiint_v (u)^{\frac{\alpha}{2}-1} (v)^{\frac{\beta}{2}-1} (w)^{\frac{\gamma}{2}-1}\, du\, dv\, dw$$

So by using dritchlet's theorem.

$$\therefore I = \frac{1}{8}\, \frac{\Gamma\frac{\alpha}{2}\, \Gamma\frac{\beta}{2}\, \Gamma\frac{\gamma}{2}}{\Gamma\left(\frac{\alpha}{2} + \frac{\beta}{2} + \frac{\gamma}{2} + 1\right)}$$

Exercise - 7

1. Find, by double integration, the area bounded by parabola $y = x^2$ and the line $y = 2x + 3$.

Solution: Given parabola $y = x^2$ and $y = 2x + 3$

Solving these.

$x^2 = 2x + 3$

or, $x^2 - 2x - 3 = 0$

or, $x - 3x + x - 3 = 0$

or, $x(x - 3) + 1(x - 3) = 0$

or $(x - 3)(x + 1) = 0$

$\Rightarrow x = -1, x = 3$

and $y = 1, y = 9$

Now taking vertical strip in which y varies from

$y = x^2$ to $y = 2x + 3$ and x various from -1 to 3

So the area of shaded region is the required Area $= \int_{-1}^{3}\int_{x^2}^{2x+3} dy\, dx.$

$= \int_{-1}^{3}[y]_{x^2}^{2x+3} dx.$

$= \int_{-1}^{3}(2x + 3 - x^2)dx.$

$= \left[\frac{2x^2}{2} + 3x - \frac{x^3}{3}\right]_{-1}^{3}$

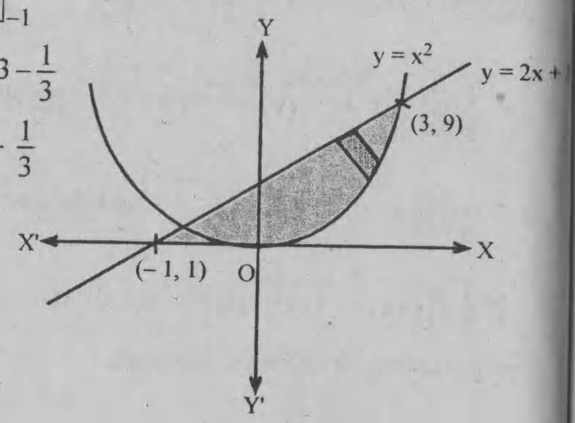
$= 9 + 9 - \frac{27}{3} - 1 + 3 - \frac{1}{3}$

$= 18 - 9 - 1 + 3 - \frac{1}{3}$

$= 11 - \frac{1}{3}$

$= \frac{33-1}{3}$

$= \frac{32}{3}$ square units.



2. Find, by double integration, the area enclosed by the pair curves $y = 2 - x$ and $y^2 = 2(2 - x)$

Solution:- Given, the pair of curves

$y = 2 - x$ and

$y^2 = 2(2 - x)$

Solving these

$(2 - x)^2 = 2(2 - x)$

$4 - 4x + x^2 = 4 - 2x$

$x^2 - 2x = 0$

$x(x - 2) = 0$

$x = 0, x = 2$

The vertex of the parabola is at (2, 0) the parabola meet the y – axis at $x = 0$ so $y^2 = 4$

$y = \pm 2$

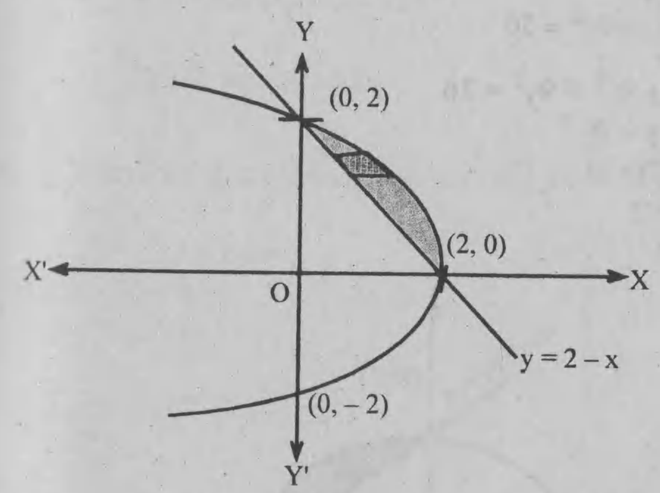
The line meet x – axis at (2, 0) and y – axis at (0, 2)

Now taking vertical strip. In which y varies from $y = 2 - x$ to $y = \sqrt{4 - 2x}$ and x varies from 0 to 2. The area of shaded region is the required area.

$= \int_0^2\int_{2-x}^{\sqrt{4-2x}} dy\, dx.$

$= \int_0^2 [y]_{2-x}^{\sqrt{4-2x}} dy\, dx.$

$= \int_0^2 \left[\sqrt{4-2x} - (2-x)\right] dx.$



$= \left[\frac{2}{3}\frac{[4-2x]^{\frac{3}{2}}}{-2} - 2x + \frac{x^2}{2}\right]_0^2$

$= \frac{2}{-6}(4-4)^{\frac{3}{2}} - 4 + \frac{4}{2} + \frac{2}{6}.(4)^{\frac{3}{2}} + 0 + 0$

$$= -\frac{1}{3}\times 0 - 4 + \frac{1}{2} + \frac{1}{3}\times 8$$

$$= -2 + \frac{8}{3}$$

$$= \frac{-6+8}{3}$$

$$= \frac{2}{3} \text{ square units.}$$

3. Find the double integration, the smaller of the ar bounded by the ellipse $4x^2 + 9y^2 = 36$ and the straight. $2x + 3y = 6$

Solution: we have,

The ellipse is $4x^2 + 9y^2 = 36$ ..... (i)

$$\Rightarrow \frac{x^2}{9} + \frac{y^2}{4} = 1$$

and the straight line $2x + 3y = 6$ .....(ii)

$$\Rightarrow x = \frac{6-3y}{2}$$

Solving these

$$4\left(\frac{6-3y}{2}\right)^2 + 9y^2 = 36$$

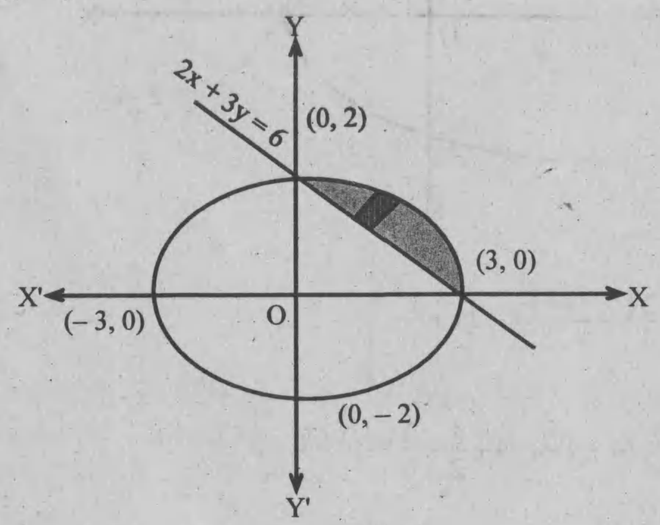
or, $36 - 36y + 9y^2 + 9y^2 = 36$

or, $18y^2 - 36y = 0$

or, $18y\,(y-2) = 0$

$\Rightarrow$ $y = 0, y = 2$

and $x = 3, x = 0$



The required area is the area of shaded region.

$$= \int_0^3 \left[\int_{\frac{6-2x}{3}}^{\frac{1}{3}\sqrt{36-4x^2}} dy \right] dx.$$

$$= \int_0^3 [y]_{\frac{6-2x}{3}}^{\frac{1}{3}\sqrt{36-4x^2}} dx$$

$$= \int_0^3 \left[\frac{1}{3}\sqrt{36-4x^2} - \frac{6-2x}{3} \right] dx.$$

$$= \int_0^3 \left[\frac{2}{3}\sqrt{\left(\frac{6}{2}\right)^2 - x^2} - \frac{6-2x}{3} \right] dx.$$

$$= \left[\frac{2.x}{3.2}\sqrt{\left(\frac{36}{2}\right)^2 - x^2} + \frac{2}{3}.\frac{36}{4}.\frac{1}{2}\sin^{-1}\left(\frac{x}{\frac{6}{2}}\right) - 2x + \frac{x^2}{3} \right]_0^3$$

$$= \left[\frac{x\sqrt{36-4x^2}}{6} + 3\sin^{-1}\frac{2x}{6} - 2x + \frac{x^2}{3} \right]_0^3$$

$$= 0 + 3.\frac{\pi}{2} - 6 + \frac{9}{3} = \frac{3\pi}{2} - 3$$

$$= \frac{3}{2}(\pi - 2) \text{ square units.}$$

4. Find, by double integration, the area of the region bounded by

$y^2 = x^2$ and $y = x$

Solution: we have,

$y^2 = x^3$ and

$y = x$.

Solving these,

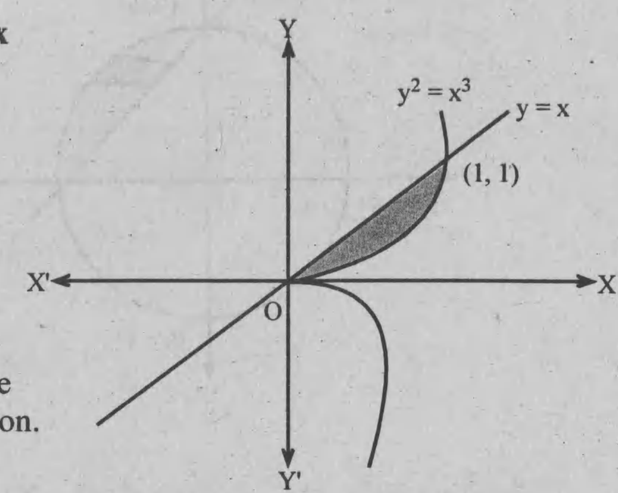
$x^2 = x^3$

or, $x^2 - x^3 = 0$

or, $x^2\,(1-x) = 0$

$\Rightarrow x = 0, x = 1$

The required area is the area of the shaded region.



$$= \int_0^1 \int_{x^{\frac{3}{2}}}^{x} dy\ dx. = \int_0^1 [y]_{x^{\frac{3}{2}}}^{x} dx = \int_0^1 \left(x - x^{\frac{3}{2}} \right) dx = \left[\frac{x^2}{2} - \frac{x^{\frac{5}{2}}}{\frac{5}{2}} \right]_0^1$$

$$= \frac{1}{2} - \frac{2}{5}. = \frac{5-4}{10} = \frac{1}{10} \text{ square units}$$

5. Find, by double integration, the area of the region enclosed by cures $x^2 + y^2 = a^2$, $x + y = a$ in first quadrant.

Solution: we have,

$x^2 + y^2 = a^2$

$x + y = a$

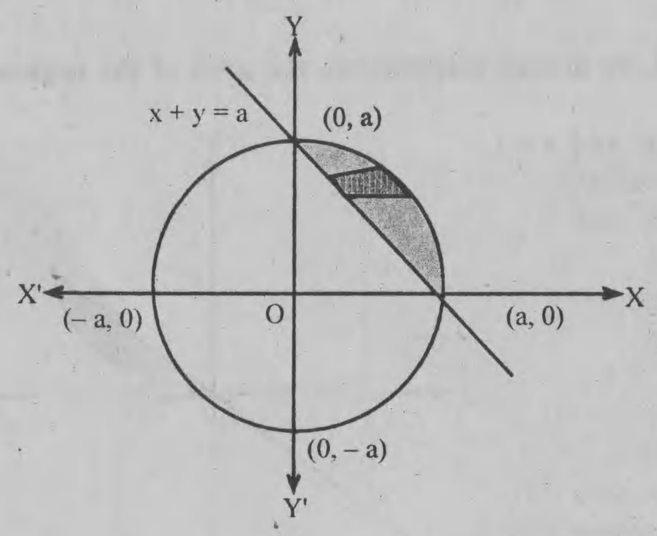
The required Area is the area of the shaded region

$$= \int_0^a \int_{a-x}^{\sqrt{a^2-x^2}} dy\, dx.$$

$$= \int_0^a [y]_{a-x}^{\sqrt{a^2-x^2}} dx.$$

$$= \int_0^a \left[\sqrt{a^2-x^2} - (a-x) \right] dx.$$

$$= \left[\frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} - ax + \frac{x^2}{2} \right]_0^a$$



$$= \frac{a^2}{2}.\frac{\pi}{2} - a^2 + \frac{a^2}{2} = \frac{\pi a^2}{4} - \frac{a^2}{2} = \frac{a^2}{4}[\pi - 2] \text{ square units.}$$

6. Find, by double integration, the area of the ellipse.

Solution: We have, Equation of the ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

The required area is the area of the shaded region

$$= 4 \int_0^a \int_0^{\frac{b}{a}\sqrt{a^2-x^2}} dy\ dx.$$

$$= 4 \int_0^a [y]_0^{\frac{b}{a}\sqrt{a^2-x^2}} dx.$$

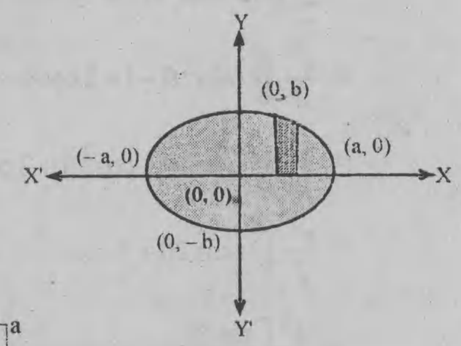
$$= 4 \int_0^a \frac{b}{a}\sqrt{a^2-x^2}\, dx.$$

$$= \frac{4b}{a} \int_0^a \sqrt{a^2-x^2} dx.$$

$$= \frac{4b}{a}\left[\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} \right]_0^a$$

$$= \frac{4b}{a}\left(0 + \frac{a^2}{2}.\frac{\pi}{2} \right)$$

$= \pi ab$. square units.



7. Show that by double integration that area lying inside the circle $r = a \sin\theta$ and outside the cardioid $r = a(1-\cos\theta)$ is $\frac{a^2}{4}(4-\pi)$.

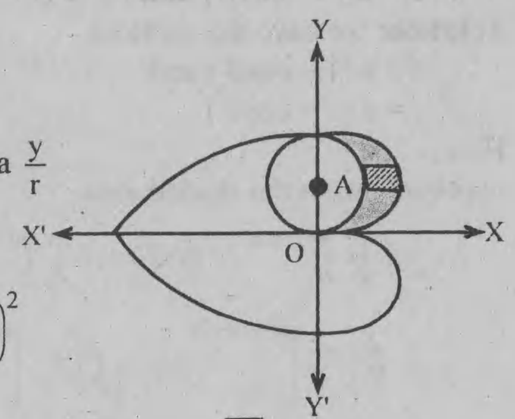
Solution:- We have,

The circle $r = a \sin\theta = a\ \frac{y}{r}$

$\Rightarrow r^2 = ay$

$\Rightarrow x^2 + y^2 - ay = 0$

$\Rightarrow x^2 + \left(y - \frac{a}{2}\right)^2 - \left(\frac{a}{2}\right)^2$



Circle with centre at $(o, \frac{a}{2})$ radious $= \sqrt{\frac{a^2}{4}} = \frac{a}{2}$ and outside the cardioid.

$r = a(1 - \cos\theta)$

Here, the required Area is the shaded

$A\left(o, \frac{a}{2}\right)$

$$\text{Area} = \int_0^{\frac{\pi}{2}} \int_{a(1-\cos\theta)}^{a\sin\theta} r\,dr\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left[\frac{r^2}{2}\right]_{a(1-\cos\theta)}^{a\sin\theta} d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} [a^2\sin^2\theta - a^2(1-\cos\theta)^2]\,d\theta$$

$$= \frac{a^2}{2}\int_0^{\frac{\pi}{2}} (\sin^2\theta - 1 + 2\cos\theta - \cos^2\theta)d\theta$$

$$= \frac{a^2}{2}\int_0^{\frac{\pi}{2}} (\sin^2\theta - \cos^2\theta + 2\cos\theta - 1)\,d\theta\,.$$

$$= \frac{a^2}{2}\int_0^{\frac{\pi}{2}} (\cos 2\theta + 2\cos\theta - 1)\,d\theta$$

$$= \frac{a^2}{2}\left[\frac{\sin 2\theta}{2} + 2\sin\theta - \theta\right]_0^{\frac{\pi}{2}}$$

$$= \frac{a^2}{2}\left(2 - \frac{\pi}{2}\right)$$

$$= \frac{a^2}{4}(4-\pi) \text{ square units.}$$

8. Find, by double integration, the area common the cardiods $r = a(1-\cos\theta)$ and $r = a(1+\cos\theta)$ [B.E.2061]

Solution: we have the cardioid

$r = a(1-\cos\theta)$ and

$r = a(1+\cos\theta)$

Here,

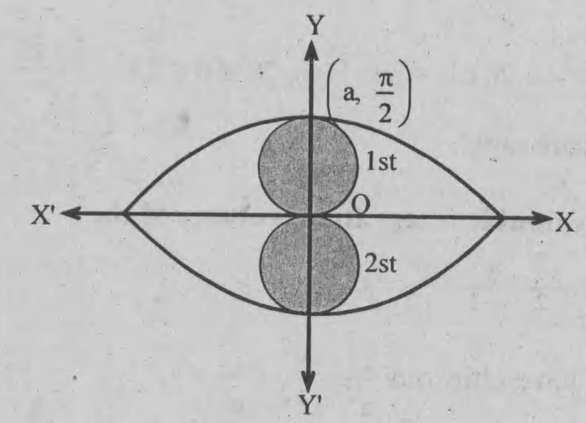
required area is the shaded area

$$= 2\int_0^{\frac{\pi}{2}}\int_0^{a(1-\cos\theta)} r\,dr\,d\theta + 2\int_{\frac{\pi}{2}}^{\pi}\int_0^{a(1+\cos\theta)} r\,dr\,d\theta$$

$$= \int_0^{\frac{\pi}{2}}\left[\frac{r^2}{2}\right]_0^{a(1-\cos\theta)} d\theta + 2\int_{\frac{\pi}{2}}^{\pi}\left[\frac{r^2}{2}\right]_0^{a(1+\cos\theta)} d\theta$$

$$= a^2\left[\int_0^{\frac{\pi}{2}}(1-2\cos\theta+\cos^2\theta)d\theta + \int_{\frac{\pi}{2}}^{\pi}(1+2\cos\theta+\cos^2\theta)\right]d\theta$$

$$= a^2\left[\theta - 2\sin\theta + \frac{1}{2}\left(\theta + \frac{\sin 2\theta}{2}\right)\right]_0^{\frac{\pi}{2}} + a^2\left[\theta + 2\sin\theta + \frac{1}{2}\theta + \frac{\sin\theta}{2}\right]_{\frac{\pi}{2}}^{\pi}$$



$$= a^2\left[\frac{\pi}{2} + \frac{\pi}{4} - 2\right] + a^2\left[\pi + \frac{\pi}{2} - \frac{\pi}{2} - \frac{\pi}{4} - 2\right]$$

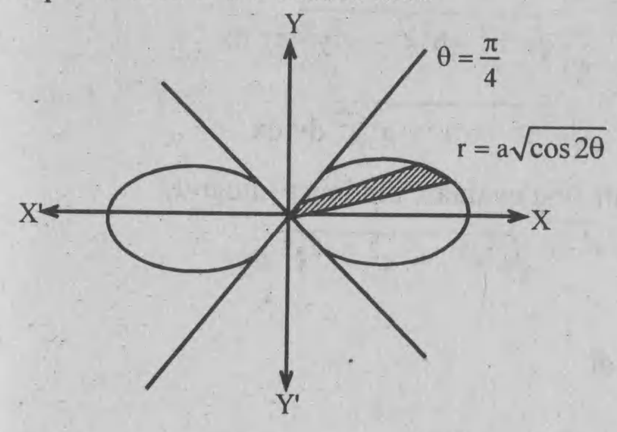
$$= a^2\left(\frac{2\pi+\pi-8}{4}\right) + a^2\left(\frac{4\pi-\pi-8}{4}\right) = \frac{a^2}{4}(3\pi - 8 + 3\pi - 8)$$

$$= \frac{a^2}{4}(6\pi - 16) = \frac{3\pi a^2}{2} - 4a^2 = \left[\frac{3\pi}{2} - 4\right]a^2 \text{ square unit.}$$

9. Find the area of the curve $r^2 = a^2\cos 2\theta$ by double integration.

Solution: We have curve $r^2 = a^2\cos 2\theta$

Here required area is the shaded area



$$= 4\int_0^{\frac{\pi}{4}}\int_0^{a\sqrt{\cos 2\theta}} r\,dr\,d\theta$$

$$= 4\int_0^{\frac{\pi}{4}}\left[\frac{r^2}{2}\right]_0^{a\sqrt{\cos 2\theta}} d\theta$$

$$= \frac{4}{2}\int_0^{\frac{\pi}{4}} a^2 \cos 2\theta \, d\theta = 2a^2 \int_0^{\frac{\pi}{4}} \cos 2\theta \; d\theta = 2a^2 \left[\frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{4}} = 2a^2 . \frac{1}{2}$$

$= a^2$ square units.

10. Find by double integration, volume of the ellpsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

Solution: we have ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

or, $\frac{z^2}{c^2} = 1-\frac{x^2}{a^2}-\frac{y^2}{b^2} = \frac{a^2b^2-b^2x^2-a^2y^2}{a^2b^2}$

$$z = \frac{c}{ab}\sqrt{a^2b^2-b^2x^2-a^2y^2}$$

Since ellipsoid is symmetry we compute the volume in the first octant only

volume $V = 8\iint_R z\,dx\,dy$.

Where R is the region bounded by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ in the first quadrant.

$$\therefore\ V = 8\int_0^a \int_0^{\frac{b}{a}\sqrt{a^2-x^2}} \frac{c}{ab}\sqrt{a^2b^2-b^2x^2-a^2y^2}\,dy\,dx$$

$$= \frac{8c}{ab}\int_0^{\frac{b}{a}\sqrt{a^2-x^2}} \sqrt{a^2b^2-b^2x^2-a^2y^2}\,dy\,dx$$

Now we shall first evaluate the inner integral,

Let $I_1 = \int_0^{\frac{b}{a}\sqrt{a^2-x^2}} \sqrt{a^2b^2-b^2x^2-a^2y^2}\,dy$

Put $ay = t$

Then $ady = dt$

$\therefore\ dy = \frac{1}{a}dt.$

when $y = 0$ then $t = 0$

$y = \frac{b}{a}\sqrt{a^2-x^2}$ then $t = b\sqrt{a^2-x^2}$

Now $I_1 = \frac{1}{a}\int_0^{b\sqrt{a^2-x^2}} \sqrt{a^2b^2-b^2x^2-t^2}\,dt$

$$= \left[\frac{t\sqrt{a^2b^2-b^2x^2-t^2}}{2} + \frac{1}{a}\frac{a^2b^2-b^2x^2}{2}\sin^{-1}\frac{t}{\sqrt{a^2b^2-b^2x^2}}\right]_0^{b\sqrt{a^2-x^2}}$$

$$= \frac{1}{a}\frac{a^2b^2-b^2x^2}{2}.\frac{\pi}{2}$$

$$\therefore \text{Volume } V = \frac{8c}{ab}.\frac{1}{a}.\frac{\pi}{2}.\frac{1}{2}\int_0^a (a^2b^2-b^2x^2)dx$$

$$= \frac{2c\pi}{a^2b}\left[a^2b^2x - \frac{b^2x^3}{3}\right]_0^a = \frac{2\pi c}{a^2b}\left(a^3b^2 - \frac{a^3b^2}{3}\right)$$

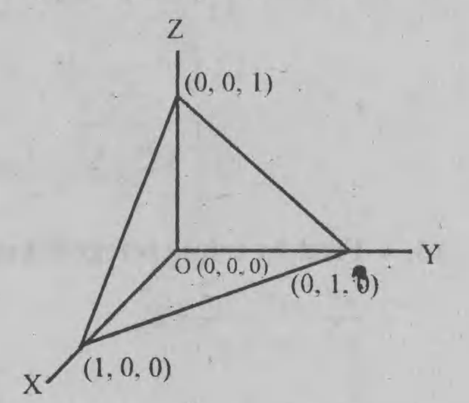
$$= \frac{2\pi c}{a^2b}(3a^3b^2 - a^3b^2)$$

$$= \frac{2\pi c}{a^2b}\left(\frac{2a^3b^2}{3}\right) = \frac{4\pi a^3b^2c}{3a^2b} = \frac{4\pi a}{3}\text{ abc square units.}$$

11. Calculate the volume of the solid bounded by the surface x = 0, y = 0, x + y + z = 1 and z = 0.

Solution: We have, the surface $x = 0$, $y = 0$, $x + y + z = 1$ and $z = 0$

Volume of the solid



$$V = \iiint_V dv$$

$$= \int_0^1\int_0^{1-x}\int_0^{1-x-y} dx\,dy\,dz$$

$$= \int_0^1\int_0^{1-x}[z]_0^{1-x-y}\,dy\,dx$$

$$= \int_0^1\int_0^{1-x}(1-x-y)\,dy\,dx$$

$$= \int_0^1\left[y - xy - \frac{y^2}{2}\right]_0^{1-x} dx$$

$$= \int_0^1\left[(1-x) - x(1-x) - \frac{(1-x)^2}{2}\right]dx$$

$$= \left[x - \frac{x^2}{2} - \frac{x^2}{2} + \frac{x^3}{3} + \frac{(1-x)^3}{2.3}\right]_0^1$$

$$= 1 - \frac{1}{2} - \frac{1}{2} + \frac{1}{3} - \frac{1}{6} = \frac{1}{3} - \frac{1}{6} = \frac{2-1}{6} = \frac{1}{6}\text{ cube units.}$$

12. Find the volume of the solid bounded by the surface $z = 0$, $x^2 + y^2 = 1$ $x + y + z = 3$.

Solution: we have the surface $z = 0$, $x^2 + y^2 = 1$, $x + y + z = 3$.

$\therefore$ Volume $V = \iiint_V dv$

$$= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_0^{3-x-y} dx\, dy\, dz$$

$$= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{x-x^2}}\left[\int_0^{3-x-y} dz\right] dy\, dx.$$

$$= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{x-x^2}}[z]_0^{3-x-y}\, dy\, dx.$$

$$= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{x-x^2}}(3-x-y)\, dy\, dx.$$

$$= \int_{-1}^{1}[3y-xy]_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\, dy\, dx.$$

$$= \int_{-1}^{1}3(\sqrt{1-x^2}+\sqrt{1-x^2})-x\left(\sqrt{1-x^2}+\sqrt{1-x^2}\right)dx$$

$$= \int_{-1}^{1}6\sqrt{1-x^2}\, dx - \int_{-2x}^{1}\sqrt{1-x^2}\, dx$$

$$= 12\int_0^1\sqrt{1-x^2}\, dx - 0$$

$$= 12\left[\frac{x\sqrt{1-x^2}}{2}+\frac{1}{2}\sin^{-1}\frac{x}{1}\right]_0^1 = \frac{12}{2}.\frac{\pi}{2} = 3\pi \text{ cube units.}$$

13. Find by triple integration, volume of ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

[B.E. 2057, 2058, 2059]

Solution: We have ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1. \text{.....(i)}$$

i.e $\frac{z^2}{c^2}=1-\frac{x^2}{a^2}-\frac{y^2}{b^2}$

$\therefore \quad z = \pm\frac{c}{ab}\sqrt{b^2a^2-b^2x^2-a^2y^2}$

In xy plane $z = 0$ so,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

or, $\frac{y^2}{b^2}=1-\frac{x^2}{a^2}$

or, $y = \pm\frac{b}{a}\sqrt{a^2-x^2}$

Since ellipsoid (i) is symmetry so we compute the volume in the first octant only

If v is the total volume,

Then $V = 8\iiint_V dv$

$$= 8\iiint_V dx\, dy\, dz$$

$$= 8\int_0^a\int_0^{\frac{b}{a}\sqrt{a^2-x^2}}\left[\int_0^{\frac{c}{ab}\sqrt{b^2a^2-b^2x^2-a^2y^2}} dz\right] dy\, dx$$

$$= 8\int_0^a\int_0^{\frac{b}{a}\sqrt{a^2-x^2}}[z]_0^{\frac{c}{ab}\sqrt{b^2a^2-b^2x^2-a^2y^2}}\, dy\, dx$$

$$= 8\int_0^a\int_0^{\frac{b}{a}\sqrt{a^2-x^2}}\frac{c}{ab}\sqrt{b^2a^2-b^2x^2-a^2y^2}\, dy\, dx.$$

$$= \frac{8c}{ab}\int_0^a\int_0^{\frac{b}{a}\sqrt{a^2-x^2}}\sqrt{a^2b^2-b^2x^2-a^2y^2}\, dy\, dx.$$

Now we shall first evaluate the inner integral.

Let $I_1 = \int_0^{\frac{b}{a}\sqrt{a^2-x^2}}\sqrt{a^2b^2-b^2x^2-a^2y^2}dy$

Put $ay = t$

Then $ady = dt$

$dy = \frac{1}{a}dt$.

when $y = 0$ then $t = 0$

$y = \frac{b}{a}\sqrt{a^2-x^2}$ then $t = b\sqrt{a^2-x^2}$

Now,

$$I_1 = \frac{1}{a}\int_0^{b\sqrt{a^2-x^2}}\sqrt{a^2b^2-b^2x^2-t^2}\, dt.$$

$$= \left[\frac{t\sqrt{a^2b^2-b^2x^2-t^2}}{2}+\frac{1}{a}\frac{\left(a^2b^2-b^2x^2\right)}{2}\sin^{-1}\frac{t}{\sqrt{a^2b^2-b^2x^2}}\right]_0^{b\sqrt{a^2-x^2}}$$

$$= \frac{1}{a}\frac{a^2b^2 - b^2x^2}{2}.\frac{\pi}{2}$$

$\therefore$ volume $v = \frac{8c}{ab}.\frac{1}{a}.\frac{\pi}{2}.\frac{1}{2}\int_0^a (a^2b^2 - b^2x^2)dx.$

$$= \frac{2c\pi}{a^2b}\left[a^2b^2x - \frac{b^2x^3}{3}\right]_0^a = \frac{2\pi c}{a^2b}\left(a^3b^2 - \frac{a^3b^2}{3}\right)$$

$$= \frac{2\pi c}{a^2b}(3a^3b^2 - a^3b^2) = \frac{2\pi c}{a^2b}\left(\frac{2a^3b^2}{3}\right) = \frac{4\pi a^3b^2c}{3a^2b}$$

$$= \frac{4}{3}\pi abc \text{ cube units.}$$

14. Find, by triple integration, volume of the sphere $x^2 + y^2 + z^2 = a^2$ **[B.E. 2063, 2069 Poush (OR**

Solution: We have, the sphere $x^2 + y^2 + z^2 = a^2$ .....(i)

$z^2 = a^2 - x^2 - y^2$

$\therefore \quad z = \pm\sqrt{a^2 - x^2 - y^2}$

In xy plane z = 0 so,

$x^2 + y^2 = a^2$

or, $\quad y^2 = a^2 - x^2$

$\therefore \quad y = \pm\sqrt{a^2 - x^2}$

Since sphere (1) is symmetry so we compute the volume in first octant only. If v is the total volume,

Then $V = 8\iiint_v dv$

$$= 8\iiint_v dx\,dy\,dz$$

$$= 8\int_0^a\int_0^{\sqrt{a^2-x^2}}\left[\int_0^{\sqrt{a^2-x^2-y^2}} dz\right]dy\,dx$$

$$= 8\int_0^a\int_0^{\sqrt{a^2-x^2}}[z]_0^{\sqrt{a^2-x^2-y^2}}\,dy\,dx.$$

$$= 8\int_0^a\int_0^{\sqrt{a^2-x^2}}\sqrt{a^2-x^2-y^2}\,dy\,dx.$$

$$= 8\int_0^a\left[\frac{y\sqrt{a^2-x^2-y^2}}{2} + \frac{a^2-x^2}{2}\sin^{-1}\frac{y}{\sqrt{a^2-x^2}}\right]_0^{\sqrt{a^2-x^2}} dx$$

$$= 8\int_0^a \frac{a^2-x^2}{2}.\frac{\pi}{2}dx. = 4\pi\int_0^a\left(\frac{a^2-x^2}{2}\right)dx. = 2\pi\int_0^a(a^2-x^2)dx$$

$$= 2\pi\left[a^2x - \frac{x^3}{3}\right]_0^a = 2\pi\left(a^3 - \frac{a^3}{3}\right) = 2\pi\frac{(3a^3-a^3)}{3} = 2\pi\left(\frac{2a^3}{3}\right)$$

$$= \frac{4\pi a^3}{3} \text{ cube units.}$$

Show that the area in the first quadrant enclosed by the curve.

$$\left(\frac{x}{a}\right)^\alpha + \left(\frac{y}{b}\right)^\beta = 1, \alpha > 0, \beta > 0$$

is given by $\frac{ab}{\alpha+\beta}\frac{\Gamma\left(\frac{1}{\alpha}\right)\Gamma\left(\frac{1}{\beta}\right)}{\Gamma\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)}$

Solution: we have the curve

$$\left(\frac{x}{a}\right)^\alpha + \left(\frac{y}{b}\right)^\beta = 1, \alpha > 0, \beta > 0$$

$$\left(\frac{x}{\alpha}\right)^\alpha = u$$

$x = u^{\frac{1}{\alpha}} \quad \therefore dx = \frac{a}{\alpha}u^{\frac{1}{\alpha}-1}du$ and $\quad \left(\frac{y}{b}\right)^\beta = v \Rightarrow y = bv^{\frac{1}{\beta}}$

$$dy = \frac{b}{\beta}v^{\frac{1}{\beta}-1}dv$$

The required area $A = \iint_R dy\,dx.$

$$= \iint_R \frac{b}{\beta}v^{\frac{1}{\beta}-1}\frac{a}{\alpha}u^{\frac{1}{\alpha}-1}du\,dv. = \frac{ab}{\alpha\beta}\iint_R u^{\frac{1}{\alpha}-1}v^{\frac{1}{\beta}-1}du\,dv.$$

$$= \frac{ab}{\alpha\beta}\frac{\Gamma\left(\frac{1}{\alpha}\right)\Gamma\left(\frac{1}{\beta}\right)}{\Gamma\left(\frac{1}{\alpha}+\frac{1}{\beta}+1\right)} \text{ by Drichlet's inteqral.}$$

$$= \frac{ab}{\alpha\beta}\frac{\Gamma\left(\frac{1}{\alpha}\right)\Gamma\left(\frac{1}{\beta}\right)}{\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)\Gamma\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)} = \frac{ab}{\alpha+\beta}\frac{\Gamma\left(\frac{1}{\alpha}\right)\Gamma\left(\frac{1}{\beta}\right)}{\Gamma\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)} \text{ Proved.}$$

Important IOE Questions and Answers

1. Evaluate $\iint r^3\, dr\, d\theta$ over the area included between the circles $\sin\theta$ and $r = 4\sin\theta$. **(BE 2069 Pou**

Sol^n. Given circles are $r = 2\sin\theta$ and $r = 4\sin\theta$. We first integrate w.r.t. θ in which the limits of r from $r = 2\sin\theta$ to $r = 4\sin\theta$ and θ from $\theta = 0$ to π

$$\text{Thus } \iint r^3 dr d\theta = \int_0^{\pi}\left[\int_{2\sin\theta}^{4\sin\theta} r^3 dr\right] d\theta = \int_0^{\pi}\left[\frac{r^4}{4}\right]_{2\sin\theta}^{4\sin\theta} d\theta$$

$$= 60\int_0^{\pi}\sin^4\theta\, d\theta = 120\int_0^{\frac{\pi}{2}}\sin^4\theta\, d\theta = 120\,\frac{\sqrt{\pi}\,\Gamma\left(\frac{4+}{}\right.}{2\Gamma\left(\frac{4+}{2}\right.}$$

$$= 120\,\frac{\sqrt{\pi}\,\frac{3}{2}\times\frac{1}{2}\times\sqrt{\pi}}{2\times2\times1} = \frac{45\pi}{2} \text{ Ans.}$$

2. Evaluate $\int_1^e\int_1^{\log y}\int_1^{ex} \log z\, dz\, dx\, dy$ **(BE 2069 Pou**

Sol^n. Let $I = \int_1^e\int_1^{\log y}\int_1^{e^x} \log z\, dz\, dxdy = \int_1^e\int_1^{\log y}\left[z\log z - z\right]_1^{e^x} dxdy$

$$= \int_1^e\int_1^{\log y}\left(xe^x - e^x + 1\right) dxdy = \int_1^e\left[xe^x - e^x - e^x + x\right]_1^{\log y}$$

$$= \int_1^e [\,y\log y - 2y + 1\log y) - 2 + 2e - 1]\, dy$$

$$= \int_1^e (\,y\log y - 2y + 1\log y + e - 1)\, dy$$

$$= \left[\frac{y^2}{z}\log y - \frac{y^2}{4} - y^2 + y\log y - y + e^y - y\right]_1^e$$

$$= \frac{e^2}{2} - \frac{e^2}{4} - e^2 + e - e + e^2 - e + \frac{1}{4} + 1 + 1 - e + 1$$

$$= \frac{e^2}{4} - 2e + \frac{13}{4} = \frac{1}{4}\left(e^2 - 8e + 13\right)$$

3. Prove that $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sqrt{x^2+y^2+z^2}\; e^{-\left(x^2+y^2+z^2\right)}\, dxdydz =$ **(BE 2068 M**

Sol^n. $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sqrt{x^2+y^2+z^2}\; e^{-\left(x^2+y^2+z^2\right)}\, dxdydz$

Put $x = \cos\theta\sin\phi$, $y = \sin\theta\sin\phi$, $z = \cos\theta$

$x^2 + y^2 + z^2 = r^2 = r^2\sin\phi\, dr d\theta d\phi$

So, r vories from $r = 0$ to $v = \infty$

θ veries from $\theta = -\pi$ to $\theta = \pi$

ϕ veries from $\phi = -\pi$ to $\phi = \pi$

We have, $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sqrt{x^2+y^2+z^2}\; e^{-\left(x^2+y^2+z^2\right)}\, dxdydz$

$$= \int_0^{\infty}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} re^{-r^2}\, r^2\,(\sin\phi\, d\phi)\, d\theta\, dr = \int_0^{\infty}\int_0^{\pi} re^{-r^2}\, r^2\left[\frac{\cos\phi}{-1}\right]_{-\pi}^{\pi} d\theta\, dr$$

$$= \int_0^{\infty}\int_{-\pi}^{\pi} re^{-r^2}\, r^2\,(1+1)\, d\theta\, dr = 2\int_0^{\infty} re^{-r^2} r^2\left[\theta\right]_{-\pi}^{\pi} dr = 2\pi\int_0^{\infty} r^2 e^{-r^2}\left(2rdr\right)$$

Put $r^2 = t$, $2rdr = dt$

When, $r = 0$, $t = 0$ when $r = \infty$, $t = \infty$

$$= 2\pi\int_0^{\infty} te^{-t}dt = 2\pi\int_0^{\infty} te^{-t}dt = 2\pi\left[\frac{te^{-t}}{-1} - 1\times\frac{e^{-t}}{(-1)^2}\right]_0^{\infty} = 2\pi\,(0+1) = 2\pi$$

$$\therefore \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sqrt{x^2+y^2+z^2}\; e^{-\left(x^2+y^2+z^2\right)}\, dxdydz = 2\pi$$

Find by double integration, the volume bounded by the plane $z = 0$, surface $z = x^2 + y^2 + 2$ and the cylinder $x^2 + y^2 = 4$. **(BE 2068 Bhadra)**

The surface $z = x^2 + y^2 + 2$ is to be integrated over the circle $x^2 + y^2 = 4$ in the xy-plane.

So the x veries from $x = -\sqrt{4-y^2}$ to $x = \sqrt{4-y^2}$ and thuse for y-varies from $y = -2$ to 2.

So the required volume $\int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} z\, dxdy = \int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}}(x^2+y^2+2)\, dx\, dy$

$$= 2\int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}}(x^2+y^2+2)\, dxdy = 2\int_{-2}^{2}\left[\frac{x^3}{3} + \left(y^2+2\right)x\right]_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}}$$

$$= 2\int_{-2}^{2}\left[\frac{1}{3}\left(4-y^2\right)^{\frac{3}{2}} + \left(y^2+2\right)\sqrt{4-y^2}\right] dy$$

$$= \frac{4}{3}\int_0^{2}\left(4-y^2\right)^{\frac{3}{2}} dy + 4\int_0^{2} y^2\sqrt{4-y^2}\, dy + 8\int_0^{2}\sqrt{4-y^2}\, dy$$

Put $y = 2\sin\theta$, $dy = 2\cos\theta d\theta$

$$= \frac{64}{3}\int_0^{\frac{\pi}{2}}\cos^4\theta d\theta + 6^4\int_0^{\frac{\pi}{2}}\sin^2\theta d\theta + 3^2\int_0^{\frac{\pi}{2}}\cos^2\theta d\theta$$

Using gamma function,

$$= \frac{64}{3}\frac{\sqrt{\pi}\,\Gamma\left(\frac{4+1}{2}\right)}{2\Gamma\left(\frac{4+2}{2}\right)} + 64\frac{\Gamma\left(\frac{2+1}{2}\right)\Gamma\left(\frac{2+1}{2}\right)}{2\Gamma\left(\frac{2+2+2}{2}\right)} + 3^2\frac{\sqrt{\pi}\,\Gamma\left(\frac{2+1}{2}\right)}{2\Gamma\left(\frac{2+2}{2}\right)}$$

$$= \frac{64}{3}\frac{\sqrt{\pi}\,\frac{3}{2}\times\frac{1}{2}\sqrt{}}{2\times2\times1} + 64\frac{\frac{1}{2}\sqrt{\pi}\times\frac{1}{2}\sqrt{\pi}}{2\times2\times1} + 32\frac{\sqrt{\pi}\,.\,\frac{1}{2}\,.\,\sqrt{\pi}}{2\times1}$$

$= 4\pi + 4\pi + 8\pi = 16\pi$ Ans.

5. **Evaluate:** $\iiint x^{l-1}\,.\,y^{m-1}.\,z^{n-1}\,dxdydz$

(BE 2067 Mangsir, 2070 Magh

Where x, y, z are all positive but $\left(\frac{x}{a}\right)^p + \left(\frac{y}{b}\right)^q + \left(\frac{z}{c}\right)^r \le 1$.

Sol^n. Let $I = \iiint x^{l-1}\,y^{m-1}\,z^{n-1}\,dxdydz$

Put $\left(\frac{x}{a}\right)^p = u, \left(\frac{y}{b}\right)^q = v, \left(\frac{z}{c}\right)^r = w, x = au^{\frac{1}{p}}, y = bv^{\frac{1}{q}}, z = cw^{\frac{1}{r}}$

$dx = \frac{a}{p}u^{\frac{1}{p}-1}du, dy = \frac{1}{q}bv^{\frac{1}{q}-1}dv, dz = \frac{1}{r}cw^{\frac{1}{r}-1}dw$

So, $I = \iiint x^{l-1}\,y^{m-1}\,z^{n-1}\,dxdydz$

$$= \iiint\left(au^{\frac{1}{p}}\right)^{l-1}\left(bv^{\frac{1}{q}}\right)^{m-1}\left(cw^{\frac{1}{r}}\right)^{n-1}$$

$$\left(\frac{a}{p}\right)u^{\frac{1}{p}-1}\left(\frac{b}{q}\right)v^{\frac{1}{q}-1}\left(\frac{c}{r}\right)w^{\frac{1}{r}-1}\,dudvdw$$

$$= \frac{a^l b^m c^n}{pqr}\iiint u^{\frac{l}{p}-1}\,v^{\frac{m}{q}-1}\,w^{\frac{n}{r}-1}\,dudvdw$$

Where $u + v + w \le 1$

So, by using dritchlet's theorem,

$$I = \frac{a^l b^m c^n}{pqr}\frac{\Gamma\left(\frac{l}{p}\right)\Gamma\left(\frac{m}{q}\right)\Gamma\left(\frac{n}{r}\right)}{\Gamma\left(\frac{l}{p}+\frac{m}{q}+\frac{n}{r}+1\right)}$$

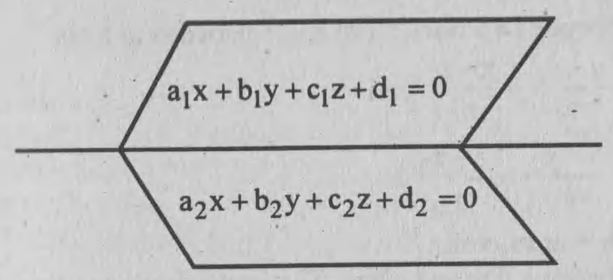
hree Dimensional Solid eometry

3

finition

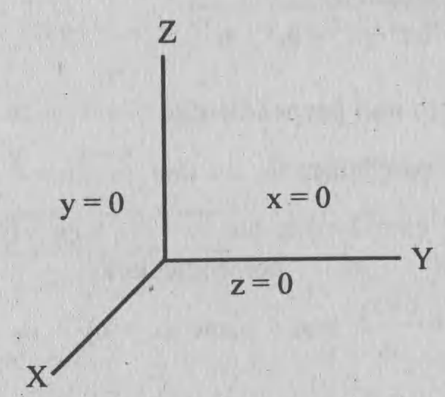
e locus of common points of two intersecting planes is called a straight . Thus, the equation of straight line in general form is $a_1x + b_1y + c_1z + = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$.

e equation is also called the equation of straight line is non-symmetrical m.



rectangular axes, x-axis, y-axis and z-axis are the intersection of the two . Thus, the x-axis is a straight line whose equation is

$y = 0, z = 0$



ilarly, the y-axis is straight line whose equation is

$y = 0, z = 0$

e z-axis is straight line whose equation is

$x = 0, y = 0$

ne important formulas

The equation of straight line in general form is

$$\boxed{\begin{aligned} a_1x + b_1y + c_1z + d_1 &= 0 \\ a_2x + b_2y + c_2z + d_2 &= 0 \end{aligned}}$$

2. The equation of straight line through the given point $(x_1, y_1$ having direction cosines l, m, n is given by

$$\frac{x-x_1}{\ell} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$$

3. The equation of straight line through the point (x_1, y_1, z_1) ha direction ratios a, b, c is given by

$$\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$$

4. The equation of a line through two the points (x_1, y_1, z_1) and $(x_2$ $z_2)$ is given by

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$$

5. **Angle between two lines :** The angle between to lines

$$\frac{x-x_1}{\ell_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1} \text{ and}$$

$$\frac{x-x_2}{\ell_2} = \frac{y-y_2}{m_2} = \frac{z-z_2}{n_2} \text{ is}$$

$\cos\theta = l_1l_2 + m_1m_2 + n_1n_2$

6. Angle between a line and plane. The angle between a line

$\frac{x-x_1}{\ell} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$ and a plane $ax + by + cz + d = 0$ is given

$$\sin\theta = \frac{a\ell + bm + cn}{\sqrt{a^2+b^2+c^2}\sqrt{\ell^2+m^2+n^2}}$$

Condition of parellelism and perpendicular

1. The condition of parellelism for the line $\frac{x-x_1}{\ell} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$ a plane $ax + by + cz + d = 0$ is that, $\boxed{al + bm + cn = 0}$
2. The condition of perpendicularity for the $\frac{x-x_1}{\ell} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$ and a plane $ax + by + cz + d = 0$ is gi that $\frac{a}{\ell} = \frac{b}{m} = \frac{c}{n}$
3. The equation of a plane containing the line

$\frac{x-x_1}{\ell} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$ is given by

$a(x-x_1) + b(y-y_1) + c(z-z_1) = 0$

Where $al + bm + cn = 0$

4. The equation of a plane containing the line

$a_1x + b_1y + c_1z + d_1 = 0$

$a_2x + b_2y + c_2z + d_2$ is given by

$a_1x + b_1y + c_1z + d_1 + \lambda(a_1x + b_2y + c_2z + d_2) = 0$

Condition that two lines will be co-planer

The condition for two lines

$\frac{x-x_1}{\ell_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1}$ and

$\frac{x-x_2}{\ell_2} = \frac{y-y_2}{m_2} = \frac{z-z_2}{n_2}$ are to be co-planer is given by

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ \ell_1 & m_1 & n_1 \\ \ell_2 & m_2 & n_2 \end{vmatrix}$$

Shortest distance

The shortest distance between two skew lines is the length of the line segment which is the right angles to both of them.

length of S. D. $= (x_2-x_1)\,l + (y_2-y_1)\,m + (z_2-z_1)\,n$

Where, l, m, n, be the d, c's of the line of S. D.

The equation of the line of shertest distance is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ \ell_1 & m_1 & n_1 \\ \ell_2 & m_2 & n_2 \end{vmatrix} = 0 = \begin{vmatrix} x-x_2 & y-y_2 & z-z_2 \\ \ell_1 & m_1 & n_1 \\ \ell_2 & m_2 & n_2 \end{vmatrix}$$

Some important formula

Sphere

1. The equation of sphere with centre (α, β, γ) and radius ' r is given by $(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2 = r^2$
2. The equation of sphere with centre origin and radius 'a' is given by $x^2 + y^2 + z^2 = a^2$
3. The equation of a sphere in general from is given by

$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$

It's centre $(-u, -v, -w)$

Its radius $R = \sqrt{u^2+v^2+w^2-d}$

4. The equation of a sphere on the line joining two points (x_1, y_1, z_1) and (x_2, y_2, z_2) as diameter is given by

$(x-x_1)(x-x_2) + (y-y_1)(y-y_2) + (z-z_1)(z-z_2) = 0$

5. Equation of tangent plane at (x_1, y_1, z_1) to the sphere

$x^2 + y^2 + z^2 = a^2$ is given by
$xx_1 + yy_1 + zz_1 = a^2$

6. The equation of a tangent plane at (x_1, y_1, z_1) to the sphere $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ is given by
$xx_1 + yy_1 + zz_1 + u(x + x_1) + v(y + y_1) + w(z + z_1) + d = 0$

7. The equation of a circle represented by the intersection of sphere with plane is

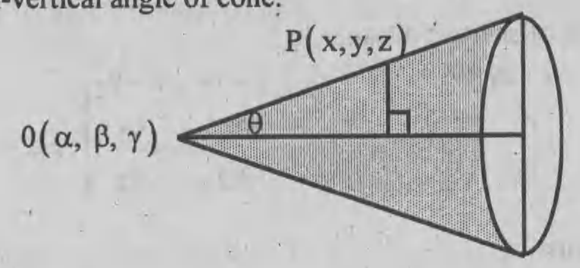
$$\boxed{\begin{array}{l} x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \\ lx + my + nz = p \end{array}}$$

Cone and cylinder

A cone is a surface generated by a variable line, which passess through a fixed point and intersects (or touch) a given curve (surface).
The variable line is called generator, the fixed point is called vertex and the given curve or the surface is called guiding curve of the cone.

Right circular cone

A right circular cone is a surface generated by a line which passes through a fixed point and makes a constant angle with fixed line through the fixed point.
The fixed point is called the vertex, fixed line, the axis and the fixed angle, the semi-vertical angle of cone.

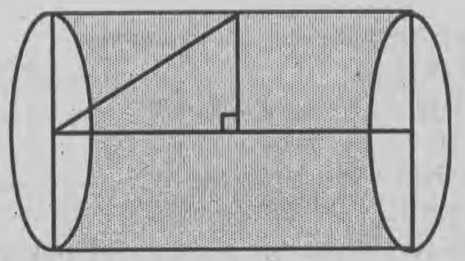


Cylinder

A cylinder is a surface generated by a straight line which is parallel to a fixed line and satisfies one or more condition, i.e. it may intersect or touch a given curve.
The given curve is called guiding curve and straight line in any position is called generateor and the fixed line, the axis of the cylinder.

Right circular cylinder

A right circular cylinder is a surface generated by a straight line which is parallel to a fixed line and is at a constant distance from it.
The constant distance is called radius of the cylinder.



$$\frac{x-\alpha}{\ell} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$$

Equation of right circular cylinder

The equation of right circular cylinder whose axis is
$\frac{x-\alpha}{\ell} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$ and radius r is given by
$(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2 + [(x-\alpha)l + (y-\beta)m + (z-\gamma)n]^2 = r^2$

Exercise - 8

1. **Find the equation of a plane passing through the points (1, 1, 0), (3, – 1, 1) and (– 1, 0, 2).**

Solution: The equation of a palne passing through the point (1, 1, 0) is
$a(x-1) + b(y-1) + c(z-0) = 0$ ..... (i)
Then, other two points passing through the (i)
$\therefore a(3-1) + b(-1-1) + c.1 = 0$
and $a(-1-1) + b(0-1)\ c(2-0) = 0$
i.e., $2a - 2b + c = 0$ ..... (ii)
$-2a - b + 2c = 0$ ..... (iii)
Solving (ii) and (iii) by the rule of cross multiplication, we get

$$\frac{a}{-4+2} = \frac{b}{-2-4} = \frac{c}{-2-4}$$

or, $\frac{a}{-3} = \frac{b}{-6} = \frac{c}{-6} = k$

$\therefore a = -3k, b = -6k, c = -6k$
Then from equation (i), we get,
$-3k(x-1) - 6k(y-1) - 5k(z-0) = 0$
or, $-3x + 3 - 6y + 6 - 6z = 0$
or, $3x + 6y + 6z - 9 = 0$
$\Rightarrow x + 2y + z - 9 = 0$

2. **Show that four points (0, – 1, 0), (2, 1, – 1), (1, 1, 1) and (3, 3, 0) and coplaner and find the equation of plane through them.**

Solution: The equation of the plane passing through the point (0, – 1, 0) is
$a(x-0) + b(y+1) + c(z-0) = 0$ ..... (i)

Then equation (i) passess through the points
(2, 1, – 1) and (3, 3, 0)
$\therefore$ a (2 – 0) + b (1 + 1) + c (– 1 – 0) = 0
2a + 2b – c = 0 ..... (ii)
and a (3 – 0) + b (3 + 1) + c (0 – 0) = 0
3a + 4b + 0 c = 0 ..... (iii)
Solving (ii) and (iii) by the rule of cross multipliying

$$\frac{a}{0-4} = \frac{b}{-3-0} = \frac{c}{8-6} = 0$$

$$\frac{a}{-4} = \frac{b}{-3} = \frac{c}{2} = k$$

$\therefore$ a = – 4k, b = – 3k, c = 2k
Then from equation (i), we get
– 4k (x – 0) – 3k (y + 1) + 2k (z – 0) = 0
– 4x – 3y – 3 + 2z = 0
– 4x – 3y + 2z – 3 = 0
$\therefore$ 4x + 3y – 2z + 3 = 0

3. Find the equation of the plane through origin with direction cosines proportional to 2, 1, – 2 and 5, 2, – 3.

Solution: The equation of plane through the origin (0, 0, 0) is a (x – 0) + b (y – 0) + c (z – 0) = 0
or, ax + by + cz = 0 ..... (i)
The direction ratio of the plane (i) is

$x_2 - x_1$,	$y_2 - y_1$,	$z_2 - z_1$
5 – 2	2 – 1	– 3 – (– 2)
3	1	– 1

Clearly, a = 3, b = 1 and c = – 1
Then from equation (i), we have the equation of the plane is 3x + y – z = 0

4. Find the length and direction cosines of the normal from the origin to the plane 6x – 3y + 2z = 14

Solution: The length of the normal from the point (0, 0, 0) to the plane 6x – 3y + 2z = 14 is

$$\left| \frac{6\times 0 - 30 + 2\times(0) - 14}{\sqrt{36+9+4}} \right|$$

$$= \frac{14}{7}$$

= 2
$\therefore$ length = 2
Clearly, the direction ratio of the plane is
a = 6, b = – 3, c = 2

$$\therefore l = \frac{a}{\sqrt{a^2+b^2+c^2}}, \quad m = \frac{b}{\sqrt{a^2+b^2+c^2}}, \quad n = \frac{c}{\sqrt{a^2+b^2+c^2}}$$

$$= \frac{6}{\sqrt{36+9+4}} \qquad = \frac{-3}{\sqrt{36+9+4}} \qquad = \frac{2}{\sqrt{36+9+4}}$$

$$= \frac{6}{7} \qquad = \frac{-3}{7} \qquad = \frac{2}{7}$$

$\therefore$ Direction cosines are, $\frac{6}{7}, \frac{-3}{7}, \frac{2}{7}$

5. Find the equation of the plane through the point (2, 1, 0) and perpendicular to the planes 2x – y – z = 5 and x + 2y – 3z = 3

Solution: The equation of the plane through the point (2, 1, 0) is,
a (x – 2) + b (y – 1) + c (z – 0) = 0 ..... (i)
The equation plane (i) is given by
2x – y – z = 5 and x + 2y – 3z = 3
So, by condition of 1^r to we get
2a – b – c = 0
a + 2b – 3c = 0
Solving by the rule of cross multiplication, we get

$$\frac{a}{3+2} = \frac{b}{-1+6} = \frac{c}{4+1}$$

$$\frac{a}{5} = \frac{b}{5} = \frac{c}{5} \Rightarrow \frac{a}{1} = \frac{b}{1} = \frac{c}{1} = k$$

$\therefore$ a = k, b = k, c = k
Then from equation (i), we get
k (x – 2) + k (y – 1) + k (z – 0) = 0
$\therefore$ x – 2 + y – 1 + z = 0
x + y + z – 3 = 0

6. Find the equation of the plane through the points (2, 2, 1) and (9, 3, 6) and perpendicular to the plane 2x + 6y + 6z = 9

Solution: Any plane through the point (2, 2, 1) is
a (x – 2) + b (y – 2) + c (z – 1) = 0 ..... (i)
It this plane passess through (9, 3, 6)
a (9 – 2) + b (3 – 2) + c (6 – 1) = 0
or, 7a + b + 5c = 0 ..... (ii)
This plane (i) will be perpendicular to the plane
2x + 3y + 6z = 9
2a + 3b + 6c = 0 ..... (iii)
Solving (ii) and (iii), we get

$$\frac{a}{6-15} = \frac{b}{10-42} = \frac{c}{21-2}$$

or, $\frac{a}{-9} = \frac{b}{-32} = \frac{c}{19}$

Then from equation (i) we get

$-9(x-2) - 32(y-2) + 19(z-1) = 0$

or, $9x + 32y - 19z - 63 = 0$

7. Find the equation of the plane through the points (2, 2, 1), (1, – 2, 3) and parallel to the line joining the point (2, 1, – 3) and (– 1, 5, – 8)

Solution: Any plane through (2, 2, 1) is

$a(x-2) + b(y-2) + c(z-1) = 0$ ..... (i)

If it passess through (1, – 2, 3)

$a(1-2) + b(-2-2) + c(3-1) = 0$

$-a - 4b + 2c = 0$ ..... (ii)

The direction ratio of line joining

(2, 1, – 3) and (– 1, 5, – 8) are

$-1-2, 5-1, -8+3$

or, $-3, 4, -5$

Clearly, $-3a + 4b - 5c = 0$ ..... (ii)

Eliminating a, b, c form (i), (ii) and (iii)

the required equation of the plane is,

$$\begin{vmatrix} x-2 & y-2 & z-1 \\ -1 & -4 & 2 \\ -3 & 4 & -5 \end{vmatrix} = 0$$

$\Rightarrow (x-2)(20.8) - (y-2)(3+6) + (z-1)(-4-12) = 0$

or, $12x + 11y - 16z + 14 = 0$

8. Find the equation of the plane through the intersection of the plane 2x + 3y + 10z = 8, 2x – 3y + 7z = 2 and normal to 3x – 2y + 4z = 5.

Solution: Any plane through the intersection of the planes $2x + 3y + 10z = 8$ and $2x - 3y + 7z = 2$ is

$2x + 3y + 10z - 8 + k(2x - 3y + 7z - 2) = 0$

or, $(2+2k)x + (3-3k)y + (10+7k)z - 8 - 2k = 0$ ..... (i)

The direction ratios of normal to the plane are

$2+2k, 3-3k, 10+7k$

∴ The direction ratios of a normal to the plane

$3x - 2y + 4z = 5$ ..... (ii)

are 3, – 2, 4

If the plane (i) is normal to the plane (ii)

$3(2+2k) - 2(3-3k) + 4(10+7k) = 0$

or, $6 + 6k + 28k + 40 = 0$

or, $40k + 40 = 0$

$k = -\frac{40}{40} = -1$

Putting the value of k in (i), the required plane is,

$(2-2)x + (3+3)y + (10-7)z - 8 + 2 = 0$

$6y + 3z - 6 = 0$

or, $2y + z - 2 = 0$

9. Find the angle between the following planes

x + 2y + 3z = 4 and 3x = 4y + 5z = 10

Solution: The given planes are,

$x + 2y + 3z = 4$ ..... (i)

$3x + 4y + 5z = 10$ ..... (ii)

If θ be the angle between the plane (i) and (ii)

$$\cos\theta = \frac{1.3 + 2.4 + 3.5}{\sqrt{1^2+2^2+3^2}\sqrt{3^2+4^2+5^2}}$$

$$= \frac{26}{\sqrt{14}\sqrt{50}} = \frac{26}{\sqrt{700}} = \frac{26}{10\sqrt{7}} = \frac{13}{5\sqrt{7}}$$

$$\therefore \theta = \cos^{-1}\left(\frac{13}{5\sqrt{7}}\right)$$

(i) x – 2y – z = 5 and 2x – y + z = 2

Solution: The given planes are

$x - 2y - z = 5$ ..... (i)

$2x - y + z = 2$ ..... (ii)

If θ be the angle between the planes (i) and (ii)

$$\cos\theta = \frac{1.2 + 2.1 - 1.1}{\sqrt{1^2+2^2+1^2}\sqrt{2^2+1^2+1^2}}$$

$$= \frac{3}{\sqrt{6}\sqrt{6}} = \frac{1}{2} = \frac{1}{2}$$

$$\therefore \theta = \cos^{-1}\frac{1}{2}$$

10. Find the equation of a plane through the points (1, 2, 3) and (7, 5, 6) parallel to x-axis.

Solution: The equation of the plane parallel to x-axis is of the form

$by + cz + d = 0$ ..... (ii)

If the plane passess through the points (1, 2, 3) and (7, 5, 6)

$2b + 3c + d = 0$ ..... (iii)

and $5b + 6c + d = 0$ ..... (iv)

eliminating b, c, d from (ii), (iii) and (iv), the required equation of the plane is,

$$\begin{vmatrix} y & z & 1 \\ 2 & 3 & 1 \\ 5 & 6 & 1 \end{vmatrix} = 0$$

$y(3-6) - z(2-5) + 1(12-15) = 0$

or, $-3y + 3z - 3 = 0$

or, $-3(y - z + 1) = 0$

or, $y - z + 1 = 0$

11. A plane meets the coordinates axes at A, B, C such that the centroid of triangle ABC is the point (a, b, c). Show that the equation of the plane is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 3$.

Solution: Let the coordinates of the point A, B, C be respectively $(x_1, 0, 0)$, $(0, y_1, 0)$ and $(0, 0, z_1)$

The coordinates of centroid of the triangle ABC are

$$\left(\frac{x_1+0+0}{3}, \frac{0+y_1-0}{3}, \frac{0+0+z_1}{3}\right)$$

$$= \left(\frac{x_1}{3}, \frac{y_1}{3}, \frac{z_1}{3}\right)$$

Let p (a, b, c) be the point on the centroid,

$\therefore \frac{x_1}{3} = a, \frac{y_1}{3} = b, \frac{z_1}{3} = c$

$\therefore x_1 = 3a, y_1 = 3b$ and $z_1 = 3c$

$\therefore$ The equation of plane in intercept from is,

$$\frac{x}{x_1} + \frac{y}{y_1} + \frac{z}{z_1} = 1$$

$$\frac{x}{3a} + \frac{y}{3b} + \frac{z}{3c} = 1$$

$$\therefore \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 3$$

12. Find the equation of the planes bisecting the angle between the planes $x + 2y + 2z - 9 = 0$, $4x - 3y + 12 + 13 = 0$ and specify the one which bisects the acute angle.

Solution: The given planes are,

$x + 2y = 2z - 9 = 0$ ..... (i)

$4x - 3y + 12z + 12 = 0$ ..... (ii)

The equations of the bisectors of the angles between the plane (i) and (ii) are

$$\frac{x+2y-2z-9}{\sqrt{1+4+4}} = \pm \frac{4x-3y+12z+12}{\sqrt{16+9+144}}$$

or, $\frac{x+2y+2z-9}{3} = \pm \frac{4x-3y+12z+12}{13}$

or, $13x + 26y + 26z - 117$

$= \pm(12x - 9y + 36z + 36)$

Taking +ve sign

$13x + 20y + 26z - 117$

$= 12x - 9y + 36z + 36$

or, $x + 35y - 10z - 153 = 0$ ..... (iii)

Taking – ve sign

$13x + 26y + 26z - 117$

$= -(12x - 9y + 36z + 36)$

or, $25x + 17y + 62z - 81 = 0$ ..... (iv)

Let θ be the angle between the plane (i) and angle bisector plane (iii), then

$$\cos\theta = \frac{1.1 + 2.35 + 2(-10)}{\sqrt{1+4+4}\ \sqrt{1+1225+100}}$$

or, $\cos\theta = \frac{1+70-20}{3 \times 36.4} = 0.47$

$\therefore \theta = \cos^{-1} 0.47 = 61.97^\circ > 45^\circ$

Thus plane (iii) is the obtuse angle bisector.

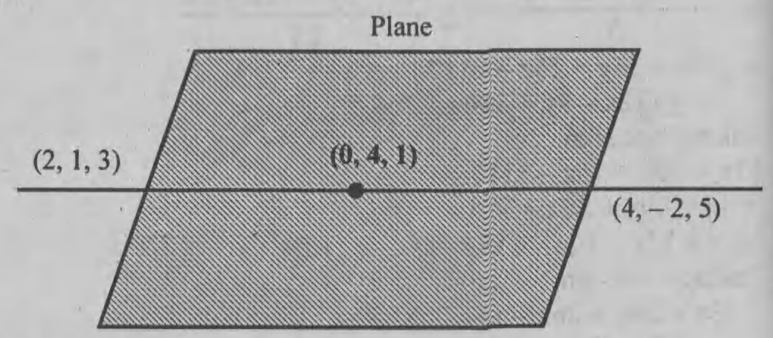
Hence, the actue angle bisector is the plane (iv) i.e.,

$25x + 17y + 62z - 81 = 0$

Exercise - 9

1. Find the point where the line joining (2, 1, 3), (4, – 2, 5) cuts the plane $2x + y - z - 3 = 0$

Solution:



The equation of the line passing through the points (2, 1, 3) and (4, – 2, 5) is

$$\frac{x-2}{4-2}=\frac{y-1}{-2-1}=\frac{z-3}{5-3}$$

or, $\frac{x-2}{2}=\frac{y-1}{-3}=\frac{z-3}{2}$

Let $\frac{x-2}{2}=\frac{y-1}{-3}=\frac{z-3}{2}$ r (say)

then the co-ordinates of any point on this line are $(2r + 2, -3r + 1, 2r + 3)$

If this point lies on the plane $2x + y - 2 - -3 = 0$

then, $2(2r + 2) + (-3r + 1) - (2r + 3) - -3 = 0$

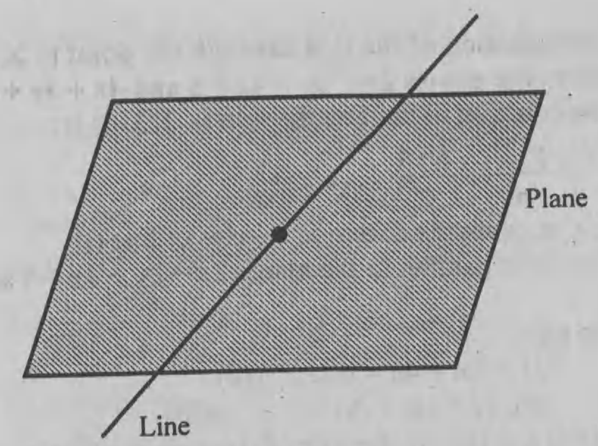
or, $4r + 4 - 3r + 1 - 2r - 3 - 3 = 0$

or, $r = -1$

∴ The co-ordinates of the point on intersection of the line joining two given points and given plane is (0, 4, 1)

2. Find where the line $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z+3}{4}$ meets the plane $2x + 4y - z + 1 = 0$

Solution: Let $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z+3}{4}=r$ (say)



Then the co-ordinates of any points on this line are $(2r + 1, -3r + 2, 4r - 3)$ ..... (i)

If this point lies on the plane this point satisfies the plane i.e.

$2(2r + 1) + 4(-3r - 2) - (4r - 3) + 1 = 0$

or, $4r + 2 - 12r - 8 - 4r + 3 + 1 = 0$

or, $-12r - 2 = 0$

$\therefore r = -\frac{1}{6}$

Substituting the value of r in equation (i) we get,

$$2\left(-\frac{1}{6}\right)+1, -3\left(-\frac{1}{6}\right)-2, 4\left(-\frac{1}{6}\right)-3$$

i.e., $\left(-\frac{1}{3}+1, \frac{1}{2}-2, -\frac{2}{3}-3\right)$

i.e., $\left(\frac{2}{3}, -\frac{3}{2}, -\frac{11}{3}\right)$ Ans.

Show that the equation of straight line through (a, b, c) parallel to the x-axis are y = b, z = c

Solution: The equation of the line through the point (a, b, c) is

$$\frac{x-a}{\ell}=\frac{y-b}{m}=\frac{z-c}{n} \text{ ..... (i)}$$

Where, l, m, n are the direction cosines of the line (i)

The direction cosines of the x-axis are (1, 0, 0)

Since, the line (i) is parallel to the x-axis so,

$$\frac{\ell}{1}=\frac{m}{0}=\frac{n}{0} \text{ ..... (ii)}$$

Eliminating l, m, n from (i) and (ii), we have,

$$\frac{x-a}{1}=\frac{y-b}{0}=\frac{z-c}{0}$$

From which we get y = b and z = c

4. Find the equation of the line through the point (– 2, 3, 4) and parallel to the planes 2x + 3y + 4z = 5 and 4x + 3y + 5z = 6.

Solution: The equation of the line through (– 2, 3, 4) is

$$\frac{x+2}{\ell}=\frac{y-3}{m}=\frac{z-4}{n} \text{ ..... (i)}$$

Where, l, m, n are the direction cosines of line (i)

If the line (i) is parallel to the planes 2x + 3y + 4z = 5 and 3x + 4y + 5z = 6

Then we have,

$2l + 3m + 4n = 0$ ..... (ii)

and $3l + 4m + 5n = 0$ ..... (iii)

Solving (ii) and (iii) by the rule of cross multi plication, we get,

$$\frac{\ell}{15-16}=\frac{m}{12-10}=\frac{n}{8-9}$$

$$\Rightarrow \frac{\ell}{-1}=\frac{m}{2}=\frac{n}{-1}=k(\text{say})$$

$\therefore\ l = -k,\ m = 2k,\ n = -k$

Hence,

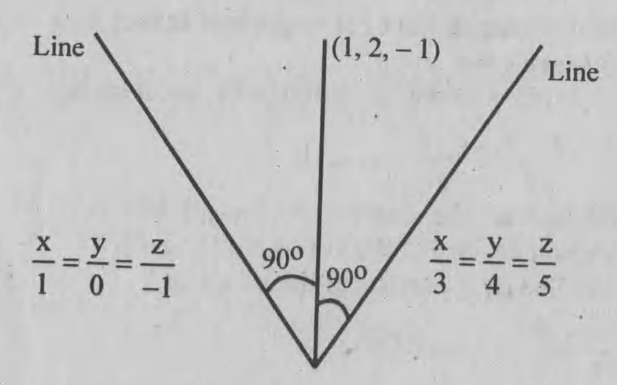
From equation (i), required equation of the line is,

$$\frac{x+2}{-k}=\frac{y-3}{2k}=\frac{z-4}{-k}$$

or, $$\frac{x+2}{1}=\frac{y-3}{-2}=\frac{z-4}{1}$$

5. Find the equation of the line through (1, 2, – 1) and perpendicular to each of the lines $\frac{x}{1}=\frac{y}{0}=\frac{z}{-1}$ and $\frac{x}{3}=\frac{y}{4}=\frac{z}{5}$.

Solution:



The equation of the line through the point (1, 2, – 1) is,

$$\frac{x-1}{\ell}=\frac{y-2}{m}=\frac{z+1}{n} \text{ ..... ()}$$

Where, l, m, n are dcs of the line (i). If the line (i) is perpendicular to the lines

$\frac{x}{1}=\frac{y}{0}=\frac{z}{-1}$ and $\frac{x}{3}=\frac{y}{4}=\frac{z}{5}$ then, we have,

$1.\,l + 0.\,m - n = 0$ ..... (ii)

$3l + 4m + 5n = 0$ ..... (iii)

Solving (ii) and (iii) by the rule of cross-multiplication,

$$\frac{\ell}{0+4}=\frac{m}{-3-5}=\frac{n}{4-0}$$

$$\therefore\ \frac{\ell}{4}=\frac{m}{-8}=\frac{n}{4}=k\ (\text{say})$$

$l = 4k,\ m = -8k,\ n = 4k$

Then from equation (i) $\frac{x-1}{4k}=\frac{y-2}{-8k}=\frac{z+1}{4k}$

$$\therefore\ \frac{x-1}{1}=\frac{y-2}{-2}=\frac{z+1}{1}$$

Which is the required equation of the line.

Prove that the lines x = ay + b, z = cy + d and x = a'y + b', z = c'y + d' are perpendicular if aa' + cc' + 1 = 0

Solution: Given, first line is x = ay + b, z = cy + d

$$\therefore\ \frac{x-b}{a}=y,\ \frac{z-d}{c}=y$$

$$\Rightarrow \frac{x-b}{a}=\frac{y-b}{1}=\frac{z-d}{c} \text{ ..... (i)}$$

$\therefore$ Direction ratios of the first line are a, 1, c

Again, given second line is,

$x = a'y + b',\qquad z = c'y + d'$

$$\therefore\ \frac{x-b'}{a'}=y,\ \frac{z-d'}{c'}=y$$

$$\therefore\ \frac{x-b'}{a'}$$

$$=\frac{y-b}{1}=\frac{z-d'}{c'} \text{ ..... (ii)}$$

Direction ratio's of the second line are a', 1, c'

Two lines (i) and (ii) are perpendicular if

$a_1b_2 + b_1b_2 + c_1c_2 = 0$

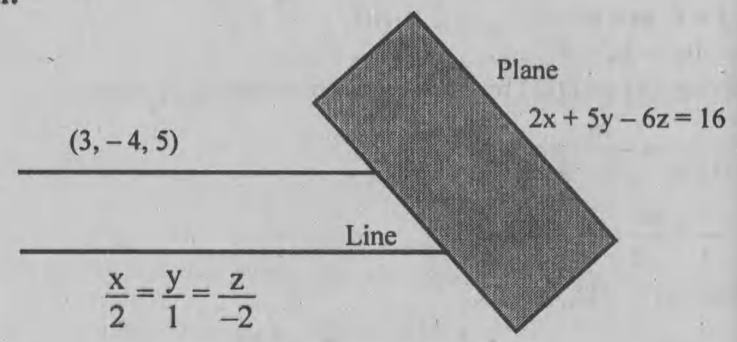
or, $aa' + 1.1 + c.c' = 0$

or, $aa' + cc' + 1 = 0$

Which is required equation proved.

7. **Find the distance of the points (3, – 4, 5) from the plane $2x + 5y - 6z = 16$ measured parallel to the line $\frac{x}{2} = \frac{y}{1} = \frac{z}{-2}$.**

Solution:



We have the given line is

$$\frac{x-3}{2} = \frac{y+4}{1} = \frac{z-5}{-2} = r \text{ (say)} \text{ ..... (i)}$$

Any point on the line (i) are

$(2r + 3, r - 4, -2r + 5)$ ..... (ii)

If the line (i) meets the given plane at the point (ii) then $2x + 5y - 6z = 16$

i.e., $2(2r + 3) + 5(r - 4) - 6(-2r + 5) = 16$

or, $21r + 6 - 20 - 30 = 16$

or, $21r - 44 = 16$ $\quad \therefore r = \frac{60}{21} = \frac{20}{7}$

Hence the line passing through the point (3, – 4, 5) meet the given plane at the point $(2r + 3, r - 4, -2r + 5)$

$$\left(2 \times \frac{61}{7} + 3,\ \frac{-8}{7} - 3,\ -2 \times \left(\frac{-5}{7}\right) + 5\right)$$

Hence the required distance is given by

$$\sqrt{\left(3 - \frac{61}{7}\right)^2 + \left(-4 + \frac{8}{7}\right)^2 + \left(5 + \frac{5}{7}\right)^2}$$

$$= \sqrt{\left(\frac{-40}{7}\right)^2 + \left(\frac{-20}{7}\right)^2 + \left(\frac{40}{7}\right)^2}$$

$$= \frac{1}{7}\sqrt{1600 + 400 + 1600}$$

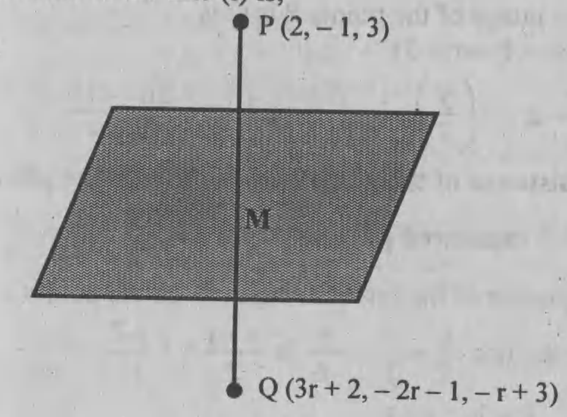
$$= \frac{1}{7}\sqrt{3600}$$

$$= \frac{60}{7}$$

Find the image of the point (2, – 1, 3) in the plane $3x - 2y - z - 9 = 0$ **(BE 2069 Bhadra)**

Solution: The given plane is $3x - 2y - z - 9 = 0$ ..... (i)

Now, the equation of a straignt line through p (2, – 1, 3) and perpendicular to the line (i) is,



$$\frac{x-2}{3} = \frac{y+1}{-2} = \frac{z-3}{-1}$$

Let, $\frac{x-2}{3} = \frac{y+1}{-2} = \frac{z-3}{-1} = r$ (say)

Now the co-ordinates of any point on the line (ii) is

$(3r + 2, -2r - 1, -r + 3)$

Hence, the co-ordinates of the point Q of the form

$(3r + 2, -2r - 1, -r + 3)$, the line PQ bisected by the point M in the plane (i).

Hence, M is the mid-point of the point PQ lies on the plane.

Hence, Co-ordinates of the point M is

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$$

i.e, $\left(\frac{3r + 2 + 2}{2}, \frac{-2r - 1 - 1}{2}, \frac{-r + 3 + 3}{2}\right)$

i.e., $\left(\frac{3r + 4}{2}, \frac{-2r - 2}{2}, \frac{-r + 6}{2}\right)$

i.e. M $\left(\frac{3r + 4}{2}, -r - 1\ \frac{-r + 6}{2}\right)$

But this point lies on the plane $2x - y + z + 3 = 0$

$\therefore 3\left(\frac{3r + 4}{2}\right) - 2(-r - 1) - \left(\frac{-r + 6}{2}\right) - 9 = 0$

or, $\frac{9r + 12}{2} + 2r + 2 + \frac{r - 6}{2} - 9 = 0$

or, $9r + 12 + 4r + 4 + r - 6 - 18 = 0$

or, $14r = 8$

or, $r = \frac{8}{14}$

$\therefore r = \frac{4}{7}$

Hence, The image of the points P is Q is

$(3r + 2, -2r - 1, -r + 3)$

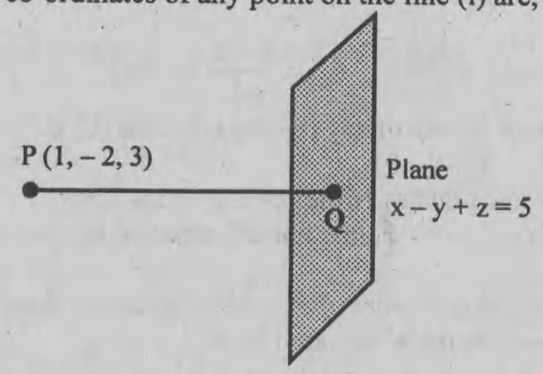
$$= \left(3\left(\frac{4}{7}\right)+2, -2\left(\frac{y}{7}\right)-1, -\left(\frac{4}{7}\right)+3\right) = \left(\frac{26}{7}, \frac{-15}{7}, \frac{17}{7}\right)$$

9. Find the distance of the point (1, – 2, 3) from the plane $x - y + z = 5$ measured parallel to the line $\frac{x}{2} = \frac{y}{3} = \frac{z}{-6}$.

Solution: The equation of the line passing through the point (1, – 2, 3) and parallel to the line $\frac{x}{2} = \frac{y}{3} = \frac{z}{-6}$ is $\frac{x-1}{2} = \frac{y+2}{3} = \frac{z-3}{-6}$ ..... (i)

Let, $\frac{x-1}{2} = \frac{y+2}{3} = \frac{z-3}{-6} = r$ (say)

Then the co-ordinates of any point on the line (i) are,



$(2r + 1, 3r - 2, -6r + 3)$ ..... (*)

If the line (i) meets the plane $x - y + z = 5$ at this point (*) then, we have

$$(2r + 1) - (3r - 2) + (-6r + 3) = 5$$

or, $-7r + 6 = 5$ $\quad \therefore -7r = -1 \quad \therefore r = \frac{1}{7}$

So the required point of intersection of the line (i) and the given plane is,

$$\left(2.\frac{1}{7}+1, 3-\frac{1}{7}-2, -6-\frac{1}{7}+3\right)$$

i.e. $Q\left(\frac{9}{7}, \frac{-11}{7}, \frac{15}{7}\right)$

Hence, the required distance (PQ)

$$= \sqrt{\left(\frac{9}{7}-1\right)^2+\left(\frac{-11}{7}+2\right)^2+\left(\frac{15}{7}-3\right)^2}$$

$$= \sqrt{\left(\frac{2}{7}\right)^2+\left(\frac{3}{7}\right)^2+\left(\frac{-6}{7}\right)^2} = \sqrt{\frac{4}{49}+\frac{9}{49}+\frac{36}{49}} = \sqrt{\frac{49}{49}} = 1$$

10. Express the equation of a line $x + y + z + 1 = 0$, $4x + y - 2z + 2 = 0$ in symmetrical form.

Solution: Let ℓ, m, n be the direction cosines of the line of intersection of two planes $x + y + z + 1 = 0$, $4x + y - 2z + 2 = 0$

The direction rasios'of the normal to the planes are 1, 1, 1 and 4, 1, – 2.

Since, the line is perpendicular to both normals of the plane, using the condition of perpendicularity, we have,

$$\ell + m + n = 0$$
$$4\ell + m - 2n = 0$$

Solving this, we get

$$\frac{\ell}{-2-1} = \frac{m}{4+2} = \frac{n}{1-4}$$

$$= \frac{\ell}{-3} = \frac{m}{6} = \frac{n}{-3} \Rightarrow \frac{\ell}{1} = \frac{m}{-2} = \frac{n}{1}$$

Thus, the direction cosines of the line of intersection of the given planes are proportions to 1, – 2, 1

To, find a point on the line, putting $z = 0$ in the given equation of the planes we have,

$x + y + 1 = 0$ ..... (i)

and $4x + y + 2 = 0$ ..... (ii)

Solving these two equation we get

$$\begin{array}{r} x + y = 1 = 0 \\ -4x + y + 2 = 0 \\ \hline -3x = 1 \end{array}$$

$$x = -\frac{1}{3}$$

Then, from equation (i) $\frac{-1}{3} + y + 1 = 0$

$$\therefore y = \frac{-2}{3}$$

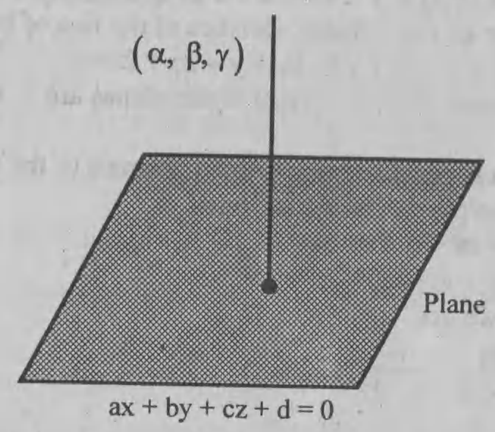
The point on the line is $\left(-\frac{1}{3}, \frac{-2}{3}, 0\right)$

Thus, the equation of the line in symmetrical from is,

$$\frac{x+\frac{1}{3}}{1} = \frac{y+\frac{2}{3}}{-2} = \frac{z-0}{1}$$

11. Show that the equation of the perpendicular from the point (α, β, γ) to the plane ax + by + cz + d = 0 $\frac{x-\alpha}{a}=\frac{y-\beta}{b}=\frac{z-\gamma}{c}$ and also find the perpendicular distance of the point ((α, β, γ) from the plane.

Solution: Given plane is ax + by + cz + d = 0 ..... (i)



Now, the direction ratio's of the normal to plane is a, b, c.
Hence the equation of straight line through the point (a, b, c) and perpendicular to the given plane (i) is,

$$\frac{x-\alpha}{a}=\frac{y-\beta}{b}=\frac{z-\gamma}{z} \text{ Ans.}$$

Also, the perpendicular distance of the point (α, β, γ) from the plane (i) is,

$$P=\frac{a\alpha+b\beta+c\gamma+d}{a^2+b^2+c^2} \text{ Ans.}$$

12. Find the distance from the point (3, 4, 5) to the point where the line $\frac{x-3}{1}=\frac{y-4}{2}=\frac{z-5}{2}$ meets the plane x + y + z = 2.

Solution: The given line is,

$$\frac{x-3}{1}=\frac{y-4}{2}=\frac{z-5}{2}=r \text{ (say) ..... (i)}$$

The co-ordinates of any point on the line (i) are,

(r + 3, 2r + 4, 2r + 5) ..... (*)

If the line (i) meets the plane x + y + z = 2 at the point (*), then we have,

(r + 3) + (2r + 4) + (2r + 5) = 2

or, 5r + 12 = 3

∴ r = – 2

Hence, the line (i) meets the plane x + y + z = 2, at the point (r + 3, 2r + 4, 2r + 5)

i.e. (1, 0, 1)

Hence the required distance = $\sqrt{(1-3)^2+(0-4)^2+(1-5)^2}$

$$=\sqrt{4+16+16}=\sqrt{36}=6$$

13. Find the angle between the lines x + 2y + z – 5 = 0 x + y – 2z – 3 and 2x – y – z = 0 = 7x + 10y – 8z.

Solution: The equation of the line are

x + 2y + z – 5 = 0, x + y – 2z – 3 = 0 ..... (i)

and 2x – y – z = 0 = 7x + 10y – 8z = 0 ..... (ii)

Let l_1, m_1, n_1 and l_2, m_2, n_2 be the direction cosines of the line of intersections of (i) and (ii)

Then by the condition of perpendicularity,

$l_1 + 2m_1 + n_1 = 0$

$l_1 + m_1 - 2n_1 = 0$

$$\frac{\ell_1}{-4-1}=\frac{m_1}{1+2}=\frac{n_1}{1-2}$$

$$\text{or, } \frac{\ell_1}{-5}=\frac{m_1}{3}=\frac{n_1}{-1}=\frac{\sqrt{\ell_1^2+m_1^2+n_1^2}}{\sqrt{25+9+1}}=\frac{1}{\sqrt{35}}$$

$$l_1=\frac{-5}{\sqrt{35}},\ m_1=\frac{3}{\sqrt{35}},\ n_1=\frac{1}{\sqrt{35}}$$

Similarly, $2l_2 - m_2 - n_2 = 0$

$7l_2 + 10m_2 - 8n_2 = 0$

$$\frac{\ell_2}{8+10}=\frac{m_2}{-7+16}=\frac{n_2}{20+7}$$

$$\frac{\ell_2}{18}=\frac{m_2}{9}=\frac{n_2}{27}$$

$$\frac{\ell_2}{2}=\frac{m_2}{1}=\frac{n_2}{3}=\frac{\sqrt{\ell_2^2+m_2^2+n_2^2}}{\sqrt{2^2+1^2+3^2}}=\frac{1}{\sqrt{14}}$$

$$l_2=\frac{2}{\sqrt{14}},\ m_2=\frac{1}{\sqrt{14}},\ n_2=\frac{3}{\sqrt{14}}$$

If θ be the angle between the lines,

$\cos\theta = l_1l_2 + m_1m_2 + n_1n_2$

$$=\frac{-5}{\sqrt{35}}\cdot\frac{2}{\sqrt{14}}+\frac{3}{\sqrt{35}}\cdot\frac{1}{\sqrt{14}}+\left(\frac{-1}{\sqrt{35}}\right)\cdot\frac{3}{\sqrt{14}}$$

$$=\frac{-10+3-3}{\sqrt{490}}=\frac{-10}{\sqrt{490}}=\frac{-10}{\sqrt{125}}$$

$$\therefore \theta=\left(\frac{-5}{\sqrt{125}}\right)$$

16. Find the equation of the plane containing the line $a_1x + b_1y + c_1z + d_1 = 0$, $a_2x + b_2y + c_2z + d_2 = 0$ and parallel to the line $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$.

Solution: The given lines is $a_1x + b_1y + c_1z + d_1 = 0$
and $a_2x + b_2y + c_2z + d_2 = 0$

The equation of the plane through the given line is,

$a_1x + b_1y + c_1z + d_1 + \lambda (a_2x + b_2y + c_2z + d_2) = 0$ ..... (i)

or, $(a_1 + \lambda a_2) x + (b_1 + \lambda b_2)y + (c_1 + \lambda c_2) z + (d_1 + \lambda d_2) = 0$

If this line is parallel to the line $\frac{x}{\ell} = \frac{y}{m} = \frac{z}{n}$ ..... (ii)

Then, $(a_1 + \lambda a_2) \ell + (b_1 + \lambda b_2) m + (c_1 + \lambda c_2) n = 0$

$$\text{or, } \lambda = -\frac{(a_1\ell + b_1m + c_1n)}{(a_2\ell + b_2m + c_2n)}$$

Substituting the value of λ in equation (i)
the required equation of the plane is,

$$(a_1x + b_1y + c_1z + d_1) - \frac{(a_1\ell + b_1m + c_1n)}{a_2\ell + b_2m + c_2n}(a_2x + b_2y + c_2z + d_2) = 0$$

or, $(a_1x + b_1y + c_1z + d_1)(a_2\ell + b_2m + c_2n)$
$= (a_2x + b_2y + c_2z + d_2)(a_1\ell + b_1m + c_1n)$

$$\therefore \frac{a_1x + b_1y + c_1z + d_1}{a_1\ell + b_1m + c_1n} = \frac{a_2x + b_2y + c_2z + d_2}{a_2\ell + b_2m + c_2n} \text{ Ans.}$$

17. Prove that the plane through the point (α, β, γ) and the line $x = py + q = rz + s$ is $\begin{vmatrix} x & py+q & rz+s \\ \alpha & p\beta+q & r\gamma+s \\ 1 & 1 & 1 \end{vmatrix} = 0$

(BE 2068 Bhadra)

Solution: The given line is,

$x = py + q = rz + s$

$$\text{or, } x = p\left(y + \frac{q}{p}\right) = r\left(z + \frac{s}{r}\right)$$

$$\text{or, } \frac{x-0}{1} = \frac{y + \frac{q}{p}}{\frac{1}{p}} = \frac{z + \frac{s}{r}}{\frac{1}{r}} \text{ ..... (i)}$$

The equation of the plane that passess through the line (i) is,

$$a(x - 0) + b\left(y + \frac{q}{p}\right) + c\left(z + \frac{s}{r}\right) = 0 \text{ ..... (ii)}$$

Where a, b, c are constant satisfying the condition,

$$a \cdot 1 + b \cdot \frac{1}{p} + c \cdot \frac{1}{r} = 0 \text{ ..... (iii)}$$

But the plane (ii) passess through (α, β, γ)

$$\therefore a(\alpha - 0) + b\left(\beta + \frac{q}{p}\right) + c\left(\gamma + \frac{s}{r}\right) = 0 \text{ ..... (iv)}$$

Eliminating a, b, c from (ii), (iv) and (iii)
We get the required equation of plane,

$$\text{or, } \begin{vmatrix} x & \frac{py+q}{p} & \frac{rz+s}{r} \\ \alpha & \frac{p\beta+q}{p} & \frac{r\gamma+s}{r} \\ 1 & \frac{1}{p} & \frac{1}{r} \end{vmatrix} = 0$$

Taking common $\frac{1}{p}$ from c_2 and $\frac{1}{r}$ from c_1 we get

$$\begin{vmatrix} x & py+q & rz+s \\ \alpha & p\beta+q & r\gamma+s \\ 1 & 1 & 1 \end{vmatrix} = 0$$

Hence proved.

18. Obtain equation of the plane passing through the line of intersection of two planes $7x - 4y + 7c + 16 = 0$ and $4x + 3y - 2z + 13 = 0$ and perpendicular to the plane $x - y - 2z + 5 = 0$

Solution : The equation of plane through the given line is

$7x - 4y + 7z + 16 + k(4x + 3y - 2z + 13) = 0$ ..... (i)

or, $(7 + 4k) x + (-4 + 3k) y + (7 - 2k) z + (16 + 13k) = 0$

Since, The plane (i) is perpendicular to the plane
$x - y - 2z + 5 = 0$, so by condition of perpendicularity,

$(7 + 4k)(1) - (-4 + 3k) \cdot (-1) + (7 - 2k) \cdot (-2) = 0$

or, $7 + 4k + 4 - 3k - 14 + 4k = 0$

or, $5k = 3$

$\therefore k = \frac{3}{5}$

Substitute the value of K in equation (i) we get,

$$7x - 4y + 7z + 16 + \frac{3}{5}(4x + 3y - 2z + 13) = 0$$

$\therefore 47x - 11y + 29z + 119 = 0$

19. Find the equation of the plane through the line $2x + 3y - 5z - 4 = 0$, $3x - 4y + 5z - 6 = 0$ and parallel the co-ordinates axes.

(BE 2070 Magh)

Solution: The equation of the plane through the given lines is

$(2x + 3y - 5z - 4) + k(3x - 4y + 5z - 6) = 0$ ..... (i)

Where, k is some constant.
or, $(2 + 3k)x + (3 - 4k)y + (5k - 5)z - (4 + 6k) = 0$
Case I : If this planes (i) is parallel to x – axis

i.e., $\frac{x}{1} = \frac{y}{0} = \frac{z}{0}$

Then, $(2 + 3k) . 1 \ (3 - 4k) . 0 + (5k - 5) . 0 = 0$
or, $2 + 3k = 0$

or, $k = -\frac{2}{3}$

∴ The required equation of the plane through the given line and parallel to x – axis is,

$(2x + 3y - 5z - 4) - \frac{2}{3}(3x - 4y + 5z - 6) = 0$

or, $6x + 9y - 15z - 12 - 6x + 8y - 10z + 12 = 0$
or, $17y - 25z = 0$ Ans.

20. Find the equation of the plane through the points (1, 0, – 1) and (3, 2, 2) and parallel to the line $\frac{x-1}{1} = \frac{1-y}{2} = \frac{z-2}{3}$.

Solution: The given line is $\frac{x-1}{1} = \frac{1-y}{2} = \frac{z-2}{3}$ ..... (i)

The equation of the plane through the point (1, 0, – 1) is $a(x - 1) + b(y - 0) + c(z + 1) = 0$ ..... (ii)
Where a, b, c are the direction ratios of the normal to the plane. If the plane (ii) passess through the point (3, 2, 2) and parallel to the line (i), then, we have,
$a(3 - 1) + b(2 - 0) + c(2 + 1) = 0$
and $a . 1 + b(-2) + c . 3 = 0$
$\Rightarrow 2a + 2b + 3c = 0$ ..... (iii)
and the plane (ii) is parallel to the line (i)
So, $a - 2b + 3c = 0$ ..... (iv)

Solving (iii) and (iv) by cross-multiplication, we have

$$\frac{a}{6+6} = \frac{b}{3-6} = \frac{c}{-4-2}$$

$\Rightarrow \frac{a}{12} = \frac{b}{-3} = \frac{c}{-6}$ $\quad \Rightarrow \frac{a}{4} = \frac{b}{-1} = \frac{c}{-2} = k$ (say)

$\Rightarrow a = 4k, \ b = -k, \ c = -2k$
From equation (ii), the required equation of plane is
$4k(x - 1) - k(y - 0) - 2k(z + 1) = 0$
or, $4x - y - 2z = 6$ Ans.

Exercise - 10

1. Show that the lines $\frac{x-5}{4} = \frac{y-7}{4} = \frac{z+3}{-5}$ $\frac{x-8}{7} = \frac{y-4}{1} = \frac{z-5}{3}$ are coplaner, Find their point of intersection and equation of plane in which they lie.

(BE 2070 Magh)

Solution:

Let $\frac{x-5}{4} = \frac{y-7}{4} = \frac{z+3}{-5} = r$

Any point on this line is $(4r + 5, 4r + 7, -5r - 3)$

and $\frac{x-8}{7} = \frac{y-4}{1} = \frac{z-5}{3} = r^1$

$(7r^1 + 8, r^1 + 4, 3r^1 + 5)$ respectively.
The lines will be co-planer if they meet at a single point. For the point of intersection, for suitable values of r and r^1, we have,
$4r + 5 = 7r^1 + 8 \Rightarrow 4r - 7r^1 = 3$ .....(1)
$4r + 7 = r^1 + 4 \Rightarrow 4r - r^1 = -3$ .....(2)
$-5r - 3 = 3r^1 + 5, \Rightarrow 5r + 3r^1 = -8$ ..... (3)
Solving (1) and (2), we get
$r^1 = -1$ and $r = -1$
As $r^1 = -1$ and $r = -1$ satisfy the equation (1) Hence, two given lines are coplaner. Then the point of intersection are $(4r + 5, 4r + 7, -5r - 3)$
$\therefore (4(-1) + 5, 4(-1) + 7, -5(-1) - 3) = (-4 + 5, -4 + 7, 5 - 3)$
$= (1, 3, 2)$

The equation of the plane through them is $\begin{vmatrix} x-5 & y-7 & z+3 \\ 4 & 4 & -5 \\ 7 & 1 & 3 \end{vmatrix} = 0$

or, $(x - 5)(12 + 5) - (y - 7)(12 + 35) + (z + 3)(4 - 28) = 0$
or, $17x - 85 - 47y + 329 - 24z - 72 = 0$
or, $17x - 47y - 24z + 172 = 0$

2. Prove that the lines $x = \frac{y-2}{2} = \frac{z+3}{3}$ and $\frac{x-2}{2} = \frac{y-6}{3} = \frac{z-3}{4}$ are co-planer and find their plane and point of intersection.

Solution:

Let $\frac{x-0}{1} = \frac{y-2}{2} = \frac{z+3}{3} = r$ and $\frac{x-2}{2} = \frac{y-6}{3} = \frac{z-3}{4} = r^1$

Then the co-ordinates of points on the given two lines are
$(r, 2r + 2, 3r - 3)$ and $(2r^1 + 2, 3r^1 + 6, 4r^1 + 3)$ respectively.

Then, two lines will be co-planer if they meet a single point, for the point of intersection.

$\therefore$ For suitable value of r and r^1 we have $r = 2r^1 + 2 \Rightarrow 2r^1 - r + 2 = 0$ .....(i)

$2r + 2 = 3r^1 + 6 \Rightarrow 3r^1 - 2r + 4 = 0$ .....(ii)

and $3r - 3 = 4r^1 + 3$ .....(iii)

solving (i) and (ii), we get

$r^1 = 0$ and $r = 2$

As $r = 2$ and $r^1 = 0$ satisfy the equation (3) Hence, the given lines are co-planer. Then the co-ordinates of point of intersection is $(r, 2r + 2, 3r - 3)$

or, (2, 6, 3)

Again, the equation of the plane through them is

$$\begin{vmatrix} x-0 & y-2 & z+3 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{vmatrix} = 0$$

or, $x(8-9) - (y-z)(4-6) + (z+3)(3-4) = 0$

or, $-x + 2y - 4 - z - 3 = 0$

or, $x - 2y + z + 7 = 0$

3. Prove that the lines $\frac{x-a}{a^1} = \frac{y-b}{b^1} = \frac{z-c}{c^1}$ and $\frac{x-a^1}{a} = \frac{y-b^1}{b} = \frac{z-c^1}{c}$ intersect and fine the point of intersection and the equation of the plane in which they lie.

Solution:

Let $\frac{x-a}{a^1} = \frac{y-b}{b^1} = \frac{z-c}{c^1} = r$

and $\frac{x-a^1}{a} = \frac{y-b^1}{b} = \frac{z-c^1}{c} = r'$

Then the co-ordinates of any points on the given the lines are $(ra^1 + a, rb^1 + b, rc^1 + c)$ and $(ar^1 + a^1, br^1 + b^1, cr^1 + c^1)$ respectively.

Two lines will be intersect if they meet at a point. For the point of intersection, we must have,

$ra^1 + a = ar^1 + a^1 \quad \therefore a^1r - ar^1 = a^1 - a$ .....(1)

$rb^1 + b = br^1 + b^1 \quad b^1r - br^1 = b^1 - b$ .....(2)

$rc^1 + c = cr^1 + c^1 \quad c^1r - cr^1 = c^1 - c$ ..... (3)

Solving (1) and (2), we have

Multiplying (2) by a^1 and (1) by b^1 and subtracting,

$a^1b^1r - ab^1r^1 = a^1b^1 - ab^1$

$a^1b^1r - a^1br^1 = a^1b^1 - a^1b$

$\quad - \quad + \quad - \quad +$

$(a^1b - ab^1)\, r^1 = a^1b - ab^1$

$r = 1,\ r^1 = 1$

As $r = 1$, $r^1 = 1$ satisfy the equation (3) Hence the two given lines intersect.

The point of intersection is

$(ra^1 + a, rb^1 + b, rc^1 + c)$

or, $(a^1 + a, b^1 + b, c^1 + c)$

The equation of the plane in which they lie is $\begin{vmatrix} x-a & y-b & z-c \\ a^1 & b^1 & c^1 \\ a & b & c \end{vmatrix} = 0$

or, $\begin{vmatrix} x & y & z \\ a^1 & b^1 & c^1 \\ a & b & c \end{vmatrix} = 0 \quad (\because R_1 \to R_1 + R_3)$

or, $(b^1c - bc^1)\,x + (c^1a - ca^1)\,y + (a^1b - ab^1)\,z = 0$

Prove that the lines $\frac{x-a+d}{\alpha-\delta} = \frac{y-a}{\alpha} = \frac{z-a-d}{\alpha+\delta}$ and $\frac{x-b+c}{\beta-\gamma} = \frac{y-b}{\beta} = \frac{z-b-c}{\beta+\gamma}$ are co-planer and find the equation to the plane in which they lie.

Solution:

The given lives are

$\frac{x-(a-d)}{\alpha-\delta} = \frac{y-a}{\alpha} = \frac{z-(a+d)}{\alpha+\delta}$ .....(1)

and $\frac{x-(b-c)}{\beta-\gamma} = \frac{y-b}{\beta} = \frac{z-(b+c)}{\beta+\gamma}$ .....(2)

The two lines (1) and (2) are coplaner if

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

$$\therefore \text{L.H.S} = \begin{vmatrix} b-c-a+d & b-a & b+c-a-d \\ \alpha-\delta & \alpha & \alpha+\delta \\ \beta-\gamma & \beta & \beta+\gamma \end{vmatrix}$$

Now, $C_1 \to C_1 + C_3$

$$\begin{vmatrix} 2(b-a) & b-a & b+c-a-d \\ 2a & \alpha & \alpha+\delta \\ 2\beta & \beta & \beta+\partial \end{vmatrix}$$

$$=2\begin{vmatrix} (b-a) & b-a & b+c-a-d \\ \alpha & \alpha & \alpha+\delta \\ \beta & \beta & \beta+\partial \end{vmatrix}$$

$=2\times 0=0$ $(\because C_1=C_3)$

Hence, the given two lines are co-planer Again, the equation of is which they lie is

$$\begin{vmatrix} x-x_1 & y-y & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0$$

$$=\begin{vmatrix} x-a+d & y-a & z-a-d \\ \alpha-\delta & \alpha & \alpha+\delta \\ \beta-\gamma & \beta & \beta+\gamma \end{vmatrix}=0$$

Now, $c_1 \to C_1-C_2$, $c_3 \to C_3-C_2$

$$\begin{vmatrix} x+d-y & y-a & z-d-y \\ -\delta & \alpha & \delta \\ -\gamma & \beta & \gamma \end{vmatrix}=0$$

or, $(x+d-y)(\alpha\gamma-\beta\delta)+(y-a)(-\delta y+\delta y)$

$+(z-d-y)(-\delta\beta+\gamma\alpha)$

or, $(x+d-y)(\alpha\gamma-\beta\delta)+(z-d-y)(-\delta\beta+\gamma\alpha)=0$

or, $(\delta\gamma-\beta\delta)[x+d-y+z-d-y]=0$

or, $x-y+z-y=0$

or, $2y=x+z$ which is the required equation of the plane.

5. Show that $\frac{x+4}{3}=\frac{y+6}{5}=\frac{z-1}{-2}$ and $3x-2y+z+5=2x+3y+4$ 4 are co-planer and Find the point of intersection.

Solution:

Let $\frac{x+4}{3}=\frac{y+6}{5}=\frac{z-1}{-2}=r$

Then the co-ordinates of any point on the first line are $(3r-4, 5r-6, -2r+1)$. The two given lines will be co-planer if the intersect at a point. But the second line is obtained by the intersecti of two plane

$3x-y+z+5=0$, $2x+3y+4z-4=0$

If the two given lines intersect, then there must be a common point between the first line and two planes.

The first line meets the plane

$3x-2y+z+5=0$ if

$3(3r-4)-2(5r-6)+(-2r+1)+5=0$

or, $9r-12-10r+12-2r+1+5=0$

or, $-3r=-6$

or, $r=2$

$\therefore$ The first line meets the plane $3x-2y+z+5=0$ at the point $(2, 4, -3)$

$\therefore$ The point $(2, 4, -3)$ clearly.

Satisfies the equation of the plane $2x+3y+4z-4=0$ Thus, two lines intersect at the point $(2, 4, -3)$ and hence they are coplanar.

Prove that the lines $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ and $4x-3y+1=0=5x-3z+2$ are coplaner. Also, Find their point of intersection.

Solution:

Let $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}=r$

Then the co-ordinates of any point on the first line are $(2r+1, 3r+2, 4r+3)$. The two given lines will be co-planer if they intersect at a point. But the second line is obtained by the intersection of two planes $4x-3y+1=0$ and $5x-3z+2=0$, If the two given lines are intersect, then there must be a common point between the first line and two planes. The first line meets the plane $4x-3y+1=0$ if $4(2r+1)-3(3r+2)+1=0$ or $r=-1$

The first line meets the plane $4x-3y+1=0$ at the point $(-1, -1, -1)$. The point $(-1, -1, -1)$ Clearly satisfies the equation of the plane $5x-3z+2=0$. Thus, two lines intersect at the point $(-1, -1, -1)$ and hence they are co-planer.

Prove that the line $x+y+z-3=0$, $2x+3y+4z-5$ is co-planer with the line $4x-y+5z-7=0$, $2x-5y-z-3=0$. Obtain the equation of plane in which they lie.

Solution:

One of the given line is $x+y+z-3=0=2x+3y+4z-5$ .....(2)

Firstly, we changes the line in to symmetrical from .

Let l, m, n be the direction ratios of the line (1) then,

$l+m+n=0$

$2l+3m+4n=0$

By the cross multiplication, we get

$$\begin{vmatrix} 2(b-a) & b-a & b+c-a-d \\ 2a & \alpha & \alpha+\delta \\ 2\beta & \beta & \beta+\partial \end{vmatrix}$$

$$=2\begin{vmatrix} (b-a) & b-a & b+c-a-d \\ \alpha & \alpha & \alpha+\delta \\ \beta & \beta & \beta+\partial \end{vmatrix}$$

$=2\times 0=0$ $(\therefore C_1=C_3)$

Hence, the given two lines are co-planer Again, the equation of pl is which they lie is

$$\begin{vmatrix} x-x_1 & y-y & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0$$

$$=\begin{vmatrix} x-a+d & y-a & z-a-d \\ \alpha-\delta & \alpha & \alpha+\delta \\ \beta-\gamma & \beta & \beta+\gamma \end{vmatrix}=0$$

Now, $c_1 \to C_1-C_2$, $c_3 \to C_3-C_2$

$$\begin{vmatrix} x+d-y & y-a & z-d-y \\ -\delta & \alpha & \delta \\ -\gamma & \beta & \gamma \end{vmatrix}=0$$

or, $(x+d-y)(\alpha\gamma-\beta\delta)+(y-a)(-\delta y+\delta y)$

$+(z-d-y)(-\delta\beta+\gamma\alpha)=$

or, $(x+d-y)(\alpha\gamma-\beta\delta)+(z-d-y)(-\delta\beta+\gamma\alpha)=0$

or, $(\delta\gamma-\beta\delta)[x+d-y+z-d-y]=0$

or, $x-y+z-y=0$

or, $2y=x+z$ which is the required equation of the plane.

5. Show that $\frac{x+4}{3}=\frac{y+6}{5}=\frac{z-1}{-2}$ and $3x-2y+z+5=2x+3y+4z$ 4 are co-planer and Find the point of intersection.

Solution:

Let $\frac{x+4}{3}=\frac{y+6}{5}=\frac{z-1}{-2}=r$

Then the co-ordinates of any point on the first line are $(3r-4, 5r-6, -2r+1)$. The two given lines will be co-planer if the intersect at a point. But the second line is obtained by the intersectio of two plane

$3x-y+z+5=0$, $2x+3y+4z-4=0$

If the two given lines intersect, then there must be a common point between the first line and two planes.

The first line meets the plane

$3x-2y+z+5=0$ if

$3(3r-4)-2(5r-6)+(-2r+1)+5=0$

or, $9r-12-10r+12-2r+1+5=0$

or, $-3r=-6$

or, $r=2$

$\therefore$ The first line meets the plane $3x-2y+z+5=0$ at the point $(2, 4, -3)$

$\therefore$ The point $(2, 4, -3)$ clearly.

Satisfies the equation of the plane $2x+3y+4z-4=0$ Thus, two lines intersect at the point $(2, 4, -3)$ and hence they are coplanar.

6. Prove that the lines $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ and $4x-3y+1=0=5x-3z+2$ are coplaner. Also, Find their point of intersection.

Solution:

Let $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}=r$

Then the co-ordinates of any point on the first line are $(2r+1, 3r+2, 4r+3)$. The two given lines will be co-planer if they intersect at a point. But the second line is obtained by the intersection of two planes $4x-3y+1=0$ and $5x-3z+2=0$, If the two given lines are intersect, then there must be a common point between the first line and two planes. The first line meets the plane $4x-3y+1=0$ if $4(2r+1)-3(3r+2)+1=0$ or $r=-1$

The first line meets the plane $4x-3y+1=0$ at the point $(-1, -1, -1)$. The point $(-1, -1, -1)$ Clearly satisfies the equation of the plane $5x-3z+2=0$. Thus, two lines intersect at the point $(-1, -1, -1)$ and hence they are co-planer.

7. Prove that the line $x+y+z-3=0$, $2x+3y+4z-5$ is co-planer with the line $4x-y+5z-7=0$, $2x-5y-z-3=0$. Obtain the equation of plane in which they lie.

Solution:

One of the given line is $x+y+z-3=0=2x+3y+4z-5$ .....(2)

Firstly, we changes the line in to symmetrical from .

Let l, m, n be the direction ratios of the line (1) then,

$l+m+n=0$

$2l+3m+4n=0$

By the cross multiplication, we get

$\frac{\ell}{4-3}=\frac{m}{2-4}=\frac{n}{3-2}\Rightarrow\frac{\ell}{1}=\frac{m}{-2}=\frac{n}{1}$ The direction ratios of the line (1) are porportional to 1, –2, 1.

Let the line (1) meets the plane z = 0 at point $(x_1, y_1\ 0)$

Then , $x_1 + y_1 = 3$

$2x_1 + 3y_1 = 5$

Solving above equation the get $y_1 = -1$, x, = 4

Given line meets the plane z = 0 at (4, –1, 0)

Hence, the line is symmetrical form is

$$\frac{x-4}{1}=\frac{y+1}{-2}=\frac{z}{1}\text{.....(2)}$$

Also, $4x - y + 5z - 7 = 0$, $2x - 5y - z - 3$ .....(3)

The general equation of the plane through the second line is

$$4x - y + 5z - 7 + k\,(2x - 5y - z - 3) = 0$$

or, $(4 + 2k)\,x - (1 + 5k)\,y + (5 - k)\,z - 3k - 7 = 0$ where k is constant.

This will be parallel to line (2) if $(4+2k).\ 1 + (1+5k)$.

$$2 + (5 - k)\,.\,1 = 0$$

or, $11 + 11k = 0 \therefore k = -1$

Hence, the equation of the plane which contains the line (3) and parallel to the line (2) is

$$4x - y + 5z - 7 - 1\,(2x - 5y - z - 3) = 0$$

or, $2x + 4y + 6z - 4 = 0$

$\therefore$ This plane clearly passes trough the point (4, –1, 0) .

Since $2.4 + 4.(-1) + 6.0 - 4 = 0$

or, $8 - 4 - 4 = 0$

Hence, the plane also contains the line (2).

Hence two lines are co-planer. The equation of the plane in which they lie is $2x + 4y + 6z - 4 = 0$

or, $x + 2y + 3z - 2 = 0$

Exercise - 11

1. **Find the magnitude and the equation of the line of S.D between the lines $\frac{x-3}{1}=\frac{y-5}{2}=\frac{z-7}{-3}$ and $\frac{x+1}{3}=\frac{y+2}{-4}=\frac{z+3}{1}$**

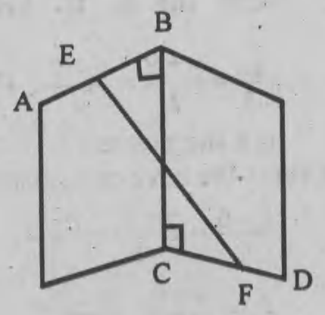
(BE 2068 Magh)

Solution:

Here, the equations of the lines are

$$\frac{x-3}{1}=\frac{y-5}{2}=\frac{z-7}{-3}\text{ and }\frac{x+1}{3}=\frac{y+2}{-4}=\frac{z+3}{1}$$

Let ℓ, m, n be the direction cosines of the line of shortest distance Bc. Since the line Bc is perpendicular to both AB and CD. Using the condition of perpendicular its,



$\ell + 2m - 3n = 0$

$3\ell - 4m + n = 0$

Solving by cross multiplication, we get

$$\frac{\ell}{2-12}=\frac{m}{-9-1}=\frac{n}{-4-6}$$

$$\text{or, }\frac{\ell}{-10}=\frac{m}{-10}=\frac{n}{-10}$$

$$\frac{\ell}{1}=\frac{m}{1}=\frac{n}{1}=\frac{\sqrt{\ell^2+m^2+n^2}}{\sqrt{1^2+1^2+1^2}}=\frac{1}{\sqrt{3}}$$

$$\therefore\ \ell=\frac{1}{\sqrt{3}},\ m=\frac{1}{\sqrt{3}},\ n=\frac{1}{\sqrt{3}}$$

Length of shortest distance Bc = Projection of EF on Bc.

$= (x_2 - x_1)\,\ell + (y_2 - y_1)^m + (z_2 - z_1)n$

$= (-1-3)\frac{1}{\sqrt{3}} + (-2-5)\frac{1}{\sqrt{3}} + (-3-7)\frac{1}{\sqrt{3}}$

$= \frac{-4}{\sqrt{3}} - \frac{7}{\sqrt{3}} - \frac{10}{\sqrt{3}}$

$= \left|\frac{-21}{\sqrt{3}}\right| = 7\sqrt{3}$

Length of shortest distance = $7\sqrt{3}$ and the equations of the line of shortest distance Bc are

$$\begin{vmatrix} x-3 & y-5 & z-7 \\ 1 & 2 & -3 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

or, $(x-3)(2+3)-(y-5)(1+3)+(z-7)(1-2)=0$

or, $5x-15-4y+20-z+7=0$

or, $5x-4y-z+12=0$

And $\begin{vmatrix} x+1 & y+2 & z+3 \\ 3 & -4 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 0$

or, $(x+1)(-4-1)-(y+2)(3-1)+(z+3)(3+4)=0$

or, $-5x-5-2y-4+7z+21=0$

or, $5x+2y-7z-12=0$

2. Find the S. D. between the lines $\frac{x-6}{3}=\frac{y-7}{-1}=\frac{z-4}{1}$ and $\frac{x}{-3}=\frac{y+9}{2}=\frac{z-2}{4}$. Find also the equation of shortest distance and the points. **(BE 2069 Bhadra)**

Solution: We have two equation of line is,

$$\frac{x-6}{3}=\frac{y-7}{-1}=\frac{z-4}{1} \quad \text{..... (i)}$$

$$\frac{x}{-3}=\frac{y+9}{2}=\frac{z-2}{4} \quad \text{..... (ii)}$$

Let l, m, n be the direction cosines of line of shortest distance.

$\therefore$ By the condition of perpendicularity,

$3l-m+n=0$

$-3l+2m+4n=0$

$$\therefore \frac{\ell}{-4-2}=\frac{m}{12+3}=\frac{n}{6-3}$$

$$=\frac{\ell}{-6}=\frac{m}{15}=\frac{n}{3}$$

$$\frac{\ell}{-2}=\frac{m}{5}=\frac{n}{1}=\frac{\sqrt{\ell^2+m^2+n^2}}{\sqrt{(-2)^2+(5)^2+(1)^2}}=\frac{1}{\sqrt{30}}$$

$$\therefore l=\frac{-2}{\sqrt{30}},\ m=\frac{5}{\sqrt{30}},\ n=\frac{1}{\sqrt{30}}$$

$\therefore$ The length of S. D. is

$=(x_2-x_1)l+(y_2-y_1)m+(z_2-z_1)n$

$$=(0-6)\frac{-2}{\sqrt{30}}+(-9-7)\frac{5}{\sqrt{30}}+(2-4).\frac{1}{\sqrt{30}}$$

$$=\frac{12}{\sqrt{30}}-\frac{80}{\sqrt{30}}-\frac{2}{\sqrt{30}}$$

$$=\frac{-70}{\sqrt{30}}=\frac{70}{\sqrt{30}}=3\sqrt{30}$$

Hence, equation of line of shertest distance is,

$$\begin{vmatrix} x-6 & y-7 & z-4 \\ 3 & -1 & 1 \\ \frac{-2}{\sqrt{30}} & \frac{5}{\sqrt{30}} & \frac{1}{\sqrt{30}} \end{vmatrix} = 0$$

and $\begin{vmatrix} x-0 & y-9 & z-2 \\ -3 & 2 & 4 \\ \frac{-2}{\sqrt{30}} & \frac{5}{\sqrt{30}} & \frac{1}{\sqrt{30}} \end{vmatrix} = 0$

If we change this in symmetrical form, we get the or, we get the points,

(3, 8, 3) and (– 3, – 7, 6) Ans.

Find the magnitude and equation of S.D between the lines

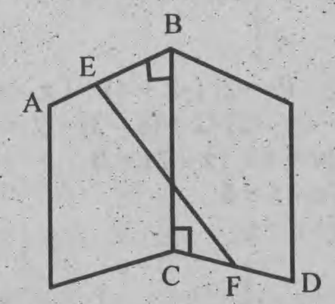
$$\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4} \text{ and } \frac{x-2}{3}=\frac{y-4}{4}=\frac{z-5}{5}$$

(BE 2069 Poush, 2068 Bhadra)

Solution:

Here, the equation of the line are

$$\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4} \text{ and } \frac{x-2}{3}=\frac{y-4}{4}=\frac{z-5}{5}$$



Let ℓ, m, n be the direction cosines of the line of shortest distance Bc. Since, the line BC is perpendicular to both AB and CD.

$\therefore \quad 2\ell+3m+4n=0$

and $\quad 3\ell+4m+5n=0$

Solving above these equation, we get

$$\frac{\ell}{15-16}=\frac{m}{12-10}=\frac{n}{8-9}$$

or, $$\frac{\ell}{-1}=\frac{m}{2}=\frac{n}{-1}=\frac{\sqrt{\ell^2+m^2+n^2}}{\sqrt{(1)^2+(2)^2+(1)^2}}=\frac{1}{\sqrt{6}}=\frac{1}{\sqrt{6}}$$

$\therefore$ $\ell=\frac{-1}{\sqrt{6}}, m=\frac{2}{\sqrt{6}}, n=\frac{-1}{\sqrt{6}}$

$\therefore$ Length of shortest distance Bc = Projection of EF on BC.

$= (x_2-x_1)\,l+(y_2-y_1)\,m+(z_2-z_1)\,n$

$$=(2-1)\left(\frac{-1}{\sqrt{6}}\right)+(4-2)\times\frac{2}{\sqrt{6}}+(5-3)\times\left(\frac{-1}{\sqrt{6}}\right)$$

$$=\frac{-1}{\sqrt{6}}+\frac{4}{\sqrt{6}}-\frac{2}{\sqrt{6}}=\frac{-3}{\sqrt{6}}+\frac{4}{\sqrt{6}}=\frac{1}{\sqrt{6}}$$

$\therefore$ Length of S.D $=\frac{1}{\sqrt{6}}$

Again, the equation of the line of shortest distance

Bc is $$\begin{vmatrix} x-1 & y-2 & z-3 \\ 2 & 3 & 4 \\ \frac{-1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} \end{vmatrix}=0$$

or, $$\frac{1}{\sqrt{6}}\begin{vmatrix} x-1 & y-2 & z-3 \\ 2 & 3 & 4 \\ -1 & 2 & -1 \end{vmatrix}=0$$

$\therefore$ $$\begin{vmatrix} x-1 & y-2 & z-3 \\ 2 & 3 & 4 \\ -1 & 2 & -1 \end{vmatrix}=0$$

or, $(x-1)(-3-8)-(y-2)(-2+4)+(z-3)(4+3)=0$

or, $(x-1)\times(-11)-(y-2)\times 2+(z-3)\times 7=0$

or, $-11x+11-2y+4+7z-21=0$

or, $-11x-2y+7z-6=0$

$\therefore$ $11x+2y-7z+6=0$

Again, $$\begin{vmatrix} x-2 & y-4 & z-5 \\ 3 & 4 & 5 \\ \frac{-1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} \end{vmatrix}=0$$

or, $$\begin{vmatrix} x-2 & y-4 & z-5 \\ 3 & 4 & 5 \\ -1 & 2 & -1 \end{vmatrix}=0$$

or, $(x-2)(-4-10)-(y-4)(-3+5)+(z-5)(6+4)=0$

or, $(x-2)(-14)-(y-4)\times 2+(z-5)\times 10=0$

or, $-14x+28-2y+8+10z-50=0$

or, $-14x-2y+10z-14=0$

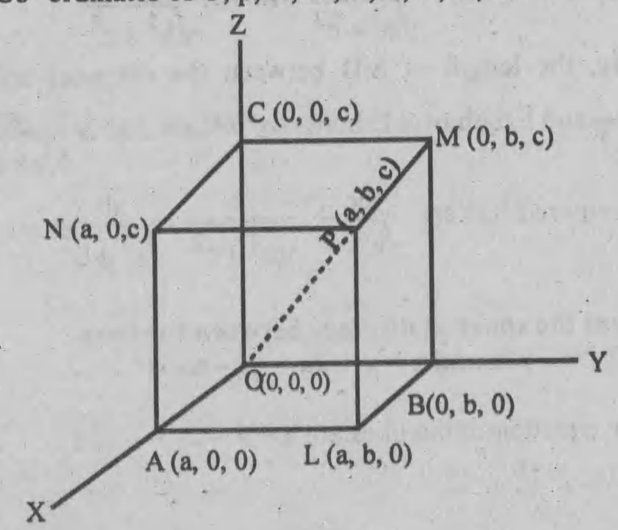
$\therefore$ $7x+y-5z+7=0$

Hence, eq^n of S. D is

$11x+2y-7z+6=0=7z+y-5z+7$

4. Show that the shortest distance between a diagonal of a rectangular parallepiped whose edges are a, b, c, and the edges not meeting it, are $\frac{bc}{\sqrt{b^2+c^2}}, \frac{ca}{\sqrt{c^2+a^2}}, \frac{ab}{\sqrt{a^2+b^2}}$

Here, Co- ordinates of O, p, A, M, C, L, B, N, are



(0 , 0, 0), (a, b, c), (a, 0, 0), (0, b, c)
(0, 0, c), (a, b, 0), (0, b, 0), (a, 0, c)

The equation of op is

$$\frac{x-0}{a-0}=\frac{y-0}{b-0}=\frac{z-0}{c-0}, \text{ i.e, } \frac{x-a}{a}=\frac{y-b}{0}=\frac{z-0}{0}$$

let ℓ, m, n be the direction cosines of the line of S.D between op and LB. Since, the line of S.D is perpendicular to both op and LB. So, using the condition of perpendicularly $a\ell + bm + cn = 0$ and $a\ell + 0m + 0n = 0$

Solving these we get

$$\frac{\ell}{0-0}=\frac{m}{ca-0}=\frac{n}{0-ab}$$

or, $$\frac{\ell}{0}=\frac{m}{ca}=\frac{n}{-ab}=\frac{\sqrt{\ell^2+m^2+b^2}}{\sqrt{0^2+(a)^2+(-ab)^2}}$$

$$=\frac{1}{\sqrt{0+c^2a^2+a^2b^2}}$$

$$=\frac{1}{\sqrt{a^2(b^2+c^2)}}=\frac{1}{a\sqrt{b^2+c^2}}$$

$\therefore$ $\ell=0, m=\frac{c}{\sqrt{b^2+c^2}}, n=\frac{-b}{\sqrt{b^2+c^2}}$

$\therefore$ Clearly, the line (1) and (2) passes through the point (0, 0, 0) and (0, b, 0) respectively.

Length of S.D = $(x_2-x_1)\ \ell+(y_2-y_1)\ m+(z_2-z_1)\ n$

$$=(0-0).0+(b-0).\frac{c}{\sqrt{b^2+c^2}}+(0-0)\frac{(-b)}{\sqrt{b^2+c^2}}$$

Similarly, the length of S.D between the diagonal op and AL is $\frac{ac}{\sqrt{a^2+c^2}}$ and length of S.D between OP and BM is $\frac{ab}{\sqrt{a^2+b^2}}$.

Hence, required S.D are $\frac{bc}{\sqrt{b^2+c^2}}, \frac{ac}{\sqrt{a^2+c^2}}, \frac{ab}{\sqrt{a^2+b^2}}$

5. Show that the shortest distance between the lines $x+a=2y=-12z$ and $x=y+2a=6z-6a$ is

Solution:

Here, the equation of the lines are $x+a=2y=-12z$.

i.e, $\frac{x+a}{12}=\frac{y-0}{6}=\frac{z-0}{-1}$

and $x=y+2a=6z-6a$

i.e, $\frac{x-a}{6}=\frac{y+2a}{6}=\frac{z-a}{1}$

Let ℓ, m, n be the direction cosines of the line s.D. Bc is between the line (1) and (2) So, It is Perpendicular to both (1) and (2), Using condition of Perpendicularity

$12\ell+6m-n=0$ and $6\ell+6m+n=0$

Solving, we get

$$\frac{\ell}{6+6}=\frac{m}{-6-12}=\frac{n}{72-36}$$

or, $$\frac{\ell}{12}=\frac{m}{-18}=\frac{n}{36}$$

or, $$\frac{\ell}{2}=\frac{m}{-3}=\frac{n}{6}=\frac{\sqrt{\ell^2+m^2+n^2}}{\sqrt{(2)^2+(-3)^2+(6)^2}}=\frac{1}{\sqrt{49}}=\frac{1}{7}$$

$\therefore$ Length of S.D. = $(x_2-x_1)\ \ell+(y_2-y_1)\ m+(z_2-z_1)_h$

$$=(0+a)\frac{2}{7}+(-2a-0)\times(\frac{-3}{7})+(a-0)\times\frac{6}{7}$$

$$=\frac{2a}{7}+\frac{6a}{7}+\frac{6a}{7}=\frac{14a}{7}=2a$$

$\therefore$ Length of line S.D = 2a

6. Find the megnitude of the line of shortest distance between the line $\frac{x}{4}=\frac{y+1}{3}=\frac{z-2}{2}$;

$5x-2y-3z+6=0$

$x-3y+2z-3=0$

Solution: Here, the equation of the lines are,

$\frac{x}{4}=\frac{y+1}{3}=\frac{z-2}{2}$ ..... (i)

$5x-2y-3z+6=0$

$x-3y+2z-3=0$ ..... (ii)

The equation of the plane through the line (ii) is

$5x-2y-3z+6+\lambda(x-3y+2z-3)=0$ ..... (iii)

$(5+\lambda)x+(-2-3\lambda)y+(-3+2\lambda)z+(6-3\lambda)=0$

It will be parallel to the line (i) if

$(5+\lambda)4+(-2-3\lambda)3+(-3+2\lambda)2=0$.

$20+4\lambda-6-9\lambda-6+4\lambda=0$

$-\lambda=-8$

$\lambda=8$

Substitute the value of λ in equation (iii)

$5x-2y-3z+6+8(x-3y+2z-3)=0$

$5x-2y-3z+6+8x-24y+16z-24=0$

$13x-26y+13z-18=0$

S.D. = Length of I^r from the point (0, – 1, 2) is

$$\frac{13\times0-26(-1)+13.2-18}{\sqrt{(13)^2+(26)^2+(13)^2}}=\frac{34}{13\sqrt{6}}$$

Exercise - 12

1. Find the equation of the sphere through the point (0, 0, 0), (0, 1, –1) (–1, 2, 0) and (1, 2, 3). Also, find its centre and radius.

Solution:

Let the equation of the sphere be

$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$.....(1)

Since (i) passes through (0, 0, 0) , (0, 1, – 1) (–1, 2, 0) and (1, 2, 3)

$0^2 + 0^2 + 0^2 + 2u.0 + 2v.0 + 2w.0 + d = 0$

i.e, $d = 0$ .....(ii)

or, i.e, $0^2 + 1^2 (-1)^2 + 2u.0 + 2v.0 (1) + 2w (-1) + d = 0$

or, $2v - 2w + 2 = 0$ $[\therefore d = 0]$

or, $v - w + 1 = 0$ .....(iii)

and or $(-1)^2 + 2^2 + 0^2 + 24 (-1) + 2v (2) + 2w.0 + d = 0$

or, $-2u + 4v + 5 = 0$ ..... (iv)

and $1^2 + 2^2 + 3^2 + 2v.1 + 2v.2 + 2w. 3 + d = 0$

or, $2v + 4v + 6w = 0$ .....(v) $[\because d = 0]$

Adding (iv) and (v), we get

$8v + 6w + 19 = 0$ .....(vi)

Multiplying (iii) by 6 and adding with (vi)

$6v - 6w + 6 = 0$

$\underline{8v + 6w + 19 = 0}$

$14v + 25 = 0 \therefore v = \frac{-25}{14}$

Putting the value of v in equation (iii), we get

$\frac{-25}{14} - w + 1 = 0$

or, $w = \frac{-11}{14}$

Putting the value of v in equation (iv)

$-2u + 4 \times \left(\frac{-25}{14}\right) + 5 = 0$

or, $2u = \frac{-15}{7}$

$\therefore$ $u = \frac{-15}{14}$

Putting the value of U, V, W, d in eq^n (i) , then the required equation of the sphere is

$x^2 + y^2 + z^2 + 2\left(\frac{-15}{14}\right)x + 2\left(\frac{-25}{14}\right)y + 2\left(\frac{-11}{14}\right)z + 0 = 0$

or, $x^2 + y^2 + z^2 - \frac{15}{7}x - \frac{25}{7}y \ \frac{-11}{7}z = 0$

or, $7(x^2 + y^2 + z^2) - 15x - 25y - 11z = 0$

2. Find the equation of the sphere passing through the points (3, 0, 2), (– 1, 1, 1) (2, – 5, 4) and having the centre on the plane 2x + 3y + 4z = 6

Solution: The general equation of sphere, is

$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ ..... (i)

Then sphere (i) passess through the point (3, 0, 2), (– 1, 1, 1) and (2, – 5, 4) we get

$3^2 + 0^2 + 2^2 + 2u.3 + 2v.0 + 2w.2 + d = 0$

$(-1)^3 + 1^2 + 2u. (-1) + 2v. 1 + 2w.1 + d = 0$

and $2^2 + (-5)^2 + 4^2 + 2u.2 + 2v (-5) + 2. 4 + d = 0$

i.e. $6u + 4w + 13 + d = 0$ ..... (ii)

$-2u + 2v + 2w + 3 + d = 0$ ..... (iii)

$4u - 10v + 8w + 45 + d = 0$ ..... (iv)

and centre of the sphere (–u, – v, – w) lies on the plane is

$-2u - 3v - 4w = 6$

$2u + 3v + 4w + 6 = 0$ ..... (v)

Solving (ii), (iii), (iv) and (v), we get

$u = 0, v = 2, w = -3, d = -1$

Then substitutes the, value of u, v, w and d in equation (i), we get

$x^2 + y^2 + z^2 + 4y - 6z - 1 = 0$

3. A spheere of constant radius 2k passess through the origin and meets the axes in A, B, C show that the locus of centroid of the tetrahedron ΔABC is the sphere $x^2 + y^2 + z^2 = k^2$.

Solution: The eqution of sphere is,

$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ ..... (i)

Which passess through the point

(a, 0, 0), (0, b, 0) (0, 0, c)

and through the origin (0, 0,0)

Then we get centre (– u, – v, – w) = (a, b, c)

$x^2 + y^2 + z^2 - ax - by - cz = 0$

Then by questions

radius = $\sqrt{u^2 + v^2 + w^2 - d}$

$4k^2 = \frac{a^2}{4} + \frac{b^2}{4} + \frac{c^2}{4}$

$16k^2 = a^2 + b^2 + c^2$ ..... (i)

Let p (x_1, y_1, z_1) be the centroid of the tetrahedron, ΔABC

$\therefore a = 4x_1, b = 4y_1$, and $c = 4z_1$

Then, from equation (i)

$16k^2 = 16x_1^2 + 16y_1^2 + 16z_1^2$

$\therefore k^2 = x_1^2 + y_1^2 + z_1^2$

$\therefore$ Locus of (x_1, y_1, z_1) is

$x^2 + y^2 + z^2 = k^2$

4. Find the equation of the sphere described on the joint of (0, 1, 0) (3 –5, 2) as diameter.

Solution:

The equation of the sphere described on the joint of (0, 1, 0) and (3, –5, 2) as diameter is

$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + (z - z_1)(z - z_2) = 0$

or, $(x - 0)(x - 3) + (y - 1)(y + 5) + (z - 0)(z - 2) = 0$

or, $x^2 - 3x + y^2 + 5y - y - 5 + z^2 - 2z + 2 + 3 = 0$

or, $x^2 + y^2 + z^2 - 3x + 4y - 2z - 5 + 5 = 0$

or, $x^2 + y^2 + z^2 - 3x + 4y - 2z = 0$

5. Find the equation of sphere through the circle $x^2 + y^2 + z^2 - 9 = 0$, $x + 3y + 4z - 2 = 0$ and the point (0, 0,0).

Solution:

The given equation of the circle is

$s = x^2 + y^2 + z^2 = 9$, $P = x + 3y + 4z - 2 = 0$.....(1)

The equation of the sphere through the circle (1) is

$S + \lambda p = 0$

$\Rightarrow \quad x^2 + y^2 + z^2 - 9 + \lambda(x + 3y + 4z - 2) = 0$ .....(2)

Since the surface passes through (0, 0, 0) then

$0 + 0 + 0 - 9 + \lambda(0 + 0 + 0 - 2) = 0$

$\Rightarrow \quad -9 - 2\lambda = 0$

$\Rightarrow \quad \lambda = \frac{-9}{2}$

$\therefore$ The equation of the sphere through the given circle (1) and (0, 0, 0) is $x^2 + y^2 + z^2 - 9 + \frac{-9}{2}(x + 3y + 4z - 2) = 0$

or, $2(x^2 + y^2 + z^2) - 18 - 9x - 27y - 36z + 18 = 0$

or, $2(x^2 + y^2 + z^2) - 9x - 27y - 36z = 0$

6. Find the centre and radius of the circle

(i) $x^2 + y^2 + z^2 - 8x + 4y + 8z - 45 = 0$, $x - 2y + 2z - 3 = 0$

(BE 2067 Mangsir, 2070 Magh)

Solution :

The given circle is

$x^2 + y^2 + z^2 - 8x + 4y + 8z - 45 = 0$, $x - 2y + 2z - 3 = 0$ .....(1)

The centre of the sphere of the given circle (1) is (4, –2, – 4) and radius of the sphere of the given circle

$R = \sqrt{U^2 + v^2 + w^2 - d}$

$= \sqrt{16 + 4 + 16 + 45}$

$= 9$

$\therefore$ The equation of any line through the centre of the sphere and perpendicular to the given plane $x - 2y + 2z = 3$ is

$\frac{x-4}{1} = \frac{y+2}{-2} = \frac{z+4}{2} = r$ (say) .....(2)

Any point on the line (2) is

$(r + 4, -2r - 2, 2r - 4)$

This point will lie on the given plane $x - 2y + 2z = 3$ if

$(4 + r) - 2(-2r - 2) + 2(2r - 4) = 3$

or, $r = \frac{1}{3}$

Hence the centre of the circle is $\left(\frac{13}{3}, \frac{-8}{3}, \frac{-10}{3}\right)$

Also, the length of the perpendicular from the centre of the sphere (4, – 2, –4) to the plane $x - 2y + 2z = 3$ is

$p = \left|\frac{4 - 2(-2) + 2(-4) - 3}{\sqrt{1 + 4 + 4}}\right| = \frac{3}{3} = 1$

$\therefore$ Therefore, the radius of circle $\sqrt{81 - 1}$

$= \sqrt{80}$

$= 4\sqrt{5}$

(ii) $x^2 + y^2 + z^2 + 12x - 12y - 16z + 11 = 0$, $2x + 2y + z = 17$

Solution:

The given circle is

$x^2 + y^2 + z^2 + 12x - 12y - 16z + 111 = 0$, $2x + 2y + z = 17$ ....

The centre of the sphere of the given circle (– 6, 6, 8) and radius of the sphere is

$R = \sqrt{u^2 + v^2 + w^2 - d} = \sqrt{(6)^2 + (-6)^2 + (-8)^2 - 111}$

$= \sqrt{36 + 36 + 64 - 111}$

$= \sqrt{136 - 111} = \sqrt{25} = 5$

$\therefore \quad R = 5$

The equation of any line through the centre of the sphere and perpendicular to the given plane $2x + 2y + z = 17$ is

$\frac{x - (-6)}{2} = \frac{y - 6}{2} = \frac{z - 8}{1}$

$\Rightarrow \quad \frac{x + 6}{2} = \frac{y - 6}{2} = \frac{z - 8}{1} = r$ (say) .....(2)

Any point of the line (2) has co-ordinates $(2r - 6, 2r + 6, r + 8)$.
This point will lie on the given plane.
$2x + 2y + z = 17$ if
$2(2r - 6) + 2(2r + 6) + r + 8 = 17$
$\Rightarrow \quad 9r + 8 = 17 \Rightarrow 9r = 9 \therefore r = 1$
$\therefore$ centre of the circle is
$(2.1 - 6, 2.1 + 6, 1+8)$
i.e, $(-4, 8, 9)$

Now, if p be the length of the perpendicular from the centre of the sphere $(-6, 6, 8)$ to the plane $2x + 2y + z = 17$ then,

$$P = \left|\frac{2(-6)+2.(6)+8-17}{\sqrt{2^2+2^2+1^2}}\right|$$

$$= \left|\frac{-12+12+8-17}{\sqrt{9}}\right| = \left|\frac{-9}{\sqrt{9}}\right| = |-3| = 3$$

Hence, radius of the circle $= \sqrt{R^2 - P^2}$
$= \sqrt{5^2 - 3^2}$
$= \sqrt{25-9}$
$= \sqrt{16}$
$= 4$

$\therefore$ Centre of the circle $= (-4, 8, 9)$ and radius $= 4$

7. Find the equation of the sphere having the circle

(i) $x^2 + y^2 + z^2 + 10y - 4z - 8 = 0, x + y + z = 3$ as a great circle

Solution

Here, the equation of the circle is
$x^2 + y^2 + z^2 + 10y - 4z - 8 = 0 \; x + y + z = 3$ .....(1)
The equation of the sphere through the circle (1) is
$x^2 + y^2 + z^2 + 10y - 4z - 8 + \lambda(x + y + z - 3) = 0$.....(2)
Comparing it with $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$
We find,

$$U = \frac{\lambda}{2}, v = \frac{\lambda+10}{2}, w = \frac{\lambda-4}{2} \; d = -8 - 3\lambda$$

Centre of the sphere is

$$(-u, -v, -w) = \left(-\frac{\lambda}{2}, -\frac{\lambda+10}{2}, -\frac{\lambda-4}{2}\right)$$

Since, the circle is great circle, so the centre of sphere must lies on the plane $x + y + z = 3$.

We have, $-\frac{\lambda}{2} - \frac{\lambda+10}{2} - \frac{\lambda-4}{2} = 3$

or, $\quad -\lambda - \lambda - 10 - \lambda + 4 = 0$
or, $\quad -3\lambda = 12$

$\therefore \lambda = -4$
Putting the value of λ in equation (2)
$x^2 + y^2 + z^2 + 10y - 4z - 8 + (-4)(x + y + z - 3) = 0$
or, $x^2 + y^2 + z^2 - 4x + 6y - 8z + 4 = 0$
is the required equation of the sphere.

(ii) $x^2 + y^2 + z^2 + 7y - 2z + 2 = 0, 2x + 3y + 4z - 8 = 0$ as great circle.

Solution:

Here, The eq^n of the circle is
$x^2 + y^2 + z^2 + 7y - 2z + 2 = 0, 2x + 3y + 4z - 8$, ...(1)
The eq^n of the sphere through the given circle (1) is
$x^2 + y^2 + z^2 + 7y - 2z + 2 + \lambda(2x + 3y + 4z - 8) = 0$ .....(2)
or, $\quad x^2 + y^2 + z^2 + 2\lambda x + (7 + 3\lambda)y - 2(1 - 2\lambda)z + (2 - 8\lambda) = 0$

and its centre is $\left(-\lambda, -\left(\frac{7+3\lambda}{2}\right), 1 - 2\lambda\right)$

If this circle is the great circle of this sphere, then its centre should lie on the plane of the circle. For this

or, $\quad 2(-\lambda) + 3\left(-\frac{7+3\lambda}{2}\right) + 4(1 - 2\lambda) = 8$

or, $\quad -2\lambda - \frac{21+9\lambda}{2} + 4 - 8\lambda = 8$

or, $\quad -4\lambda - 21 - 9\lambda + 8 - 16\lambda = 16$
or, $\quad 39\lambda = 39$
$\therefore \quad \lambda = 1$
Substituting the value of λ in eq^n (i) we have the required equation of the sphere as $x^2 + y^2 + z^2 - 2x + 4y - 6z + 10 = 0$

8. Find the equation of sphere which passes through the point $(0, -2, 3)$ and the circle $x^2 + y^2 = 4, z = 0$

Solution:

The given circle is
$x^2 + y^2 - 4 = 0, z = 0$ .....(1)
Any sphere through the circle (1) is $x^2 + y^2 + z^2 + \lambda z - 4 = 0$
.....(2) $[\therefore z = 0]$
The eq^n (2) passes through the points $(0, -2, 3)$
$\therefore \quad 0 + 4 + 9 + 3\lambda - 4 = 0$
$\therefore \quad 3\lambda = -9$
$\lambda = -3$
Then eq^n (2) becomes

$x^2 + y^2 + z^2 - 3z - 4 = 0$ which is the required equation of the sphere.

9.(i) $x^2 + y^2 + z^2 - 2x + 3y + 4z - 5 = 0$, $x^2 + y^2 + z^2 - 3x - 4y + 5z - 6 = 0$ and passing through the points (1, 1, 2)

Solution: The equation of the sphere through the given circle is

$x^2 + y^2 + z^2 - 2x + 3y + 4z - 5 + k(x^2 + y^2 + z^2 - 3x - 4y + 5z - 6) = 0$ ..... (i)

If it passess through the point (1, 1, 2)

$1^2 + 1^2 + 2^2 - 2 \times 1 + 3 \times 1 + 4 \times 2 - 5 + k(1^2 + 1^2 + 2^2 - 3 \times 1 - 4 \times 1 + 5 \times 2 - 6) = 0$

or, $10 + 3k = 0$

$\therefore k = -\frac{10}{3}$

Putting the value of k, in (i), the required equation of the sphere is

$x^2 + y^2 + z^2 - 2x + 3y + 4z - 5 - \frac{10}{3}(x^2 + y^2 + z^2 - 3x - 4y + 5z - 6) = 0$

or, $3(x^2 + y^2 + z^2 - 2x + 3y + 4z - 5) - 10(x^2 + y^2 + z^2 - 3x - 4y + 5z - 6) = 0$

or, $-7x^2 - 7y^2 - 7z^2 + 24x + 49y - 38z + 45 = 0$

or, $7x^2 + 7y^2 + 7z^2 - 24x - 49y - 38z - 45 = 0$

ii) $x^2 + y^2 + z^2 - 2x - 3y + 4z + 8 = 0$, $x^2 + y^2 + z^2 + 4x + 5y - 6z + 2 = 0$ and having its centre on the plane $4x - 5y - z = 3$

(BE 2068, Magh)

Solution:

Here, eq^n of the spheres are $s_1 = x^2 + y^2 + z^2 - 2x - 3y + 4z + 8 = 0$ and $s_2 = x^2 + y^2 + z^2 + 4x + 5y - 6z + 2 = 0$

The equation of a plane through the intersection spheres is $s_1 - s_2 = 0$

$\therefore$ $x^2 + y^2 + z^2 - 2x + 3y + 4z + 8 - (x^2 + y^2 + z^2 + 4x + 5y - 6z + 2) = 0$

or, $x^2 + y^2 + z^2 - 2x - 3y + 4z + 8 - x^2 - y^2 - z^2 - 4x - 5y + 6z - 2 = 0$

or, $-6x - 8y + 10z + 6 = 0$

or, $3x + 4y - 5z - 3 = 0$

The equation of the sphere through the circle

$x^2 + y^2 + z^2 - 2x - 3y + 4z + 8 = 0$, $3x + 4y - 5z - 3 = 0$ is $x^2 + y^2 + z^2 - 2x - 3y + 4z + 8 + \lambda(3x + 4y - 5z - 3) = 0$

or, $x^2 + y^2 + z^2 + (3\lambda - 2)x + (4\lambda - 3)y + (4 - 5\lambda)z + 8 - 3\lambda = 0$ .....(1)

Comparing it with

$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$

We find,

$u = \frac{3\lambda - 2}{2}$, $v = \frac{4\lambda - 3}{2}$, $w = \frac{4 - 5\lambda}{2}$ and $d = 8 - 3\lambda$

$\therefore$ Centre $(-u, -v, -w) = \left(\frac{3\lambda - 2}{2}, -\frac{4\lambda - 3}{2}, -\frac{4 - 5\lambda}{2}\right)$

Since the centre (–u, –v, –w) of the sphere lie on the plane $4x - 5y - z = 3$

or, $4\left(-\frac{3\lambda - 2}{2}\right) - 5\left(-\frac{4\lambda - 3}{2}\right) - \left(-\frac{4 - 5\lambda}{2}\right) = 3$

or, $-12\lambda + 8 + 20\lambda - 15 + 4 - 5\lambda = 6$

or, $3\lambda = 9$

$\therefore$ $\lambda = 3$

Putting the value of λ in (1)

$x^2 + y^2 + z^2 + (3 \times 3 - 2)x + (4 \times 3 - 3)y + (4 - 5 \times 3)z + 8 - 3 \times 3 = 0$

$\therefore$ $x^2 + y^2 + z^2 + 7x + 9y - 11z - 1 = 0$ is the required sphere.

Exercise - 13

(1) Find the equation of tangent plane to the spheres
(i) $X^2+y^2+z^2-x+3y+2z-3=0$ at the point (1, 1, –1)

Solution:

The given sphere is

$X^2+y^2+z^2-x+3y+2z-3=0$

or, $x^2+y^2+z^2+2\left(\frac{-1}{2}\right)x+\left(\frac{3}{2}\right)2y+2.1.z+1(-3)=0$

Comparing it with

$x^2+y^2+z^2+2ux+2vy+2wz+d=0$

We find,

$u=\frac{-1}{2}, v=\frac{3}{2}, w=1, d=-3$

We know that, the equation of tangent plane at (1, 1, –1) of the sphere is

$x\times1+y\times1+z\times(-1)+u(x+1)+v(y+1)+w(z-1)+d=0$

or, $x+y-z-\frac{1}{2}(x+1)+\frac{3}{2}(y+1)+1(z-1)-3=0$

or, $x+y-z\ \frac{-1}{2}x\ \frac{-1}{2}+\frac{3y}{2}+\frac{3}{2}+z-1-3=0$

or, $\frac{1}{2}x+\frac{5}{2}y-3=0$

$\therefore$ $x+5y-6=0$

(ii) $x^2+y^2+z^2-2x-4y+2z-3=0$ at the point (–1, 4, –2)

Solution,:

The given sphere is

$x^2+y^2+z^2-2x-4y+2z-3=0$

or, $x^2+y^2+z^2+2(-1)x+2(-2)y+2.1z+(-3)=0$.....(1)

Comparing (1) with $x^2+y^2+z^2+2ux+2vy+d=0$

We get

$u=1, v=-2, w=1, d=-3$

$\therefore$ The equation of tangent plane to the sphere (1) at (–1, 4, –2) is

$x\alpha+y\beta+z\gamma+u(x+\alpha)+v(y+\beta)+w(z+\gamma)+d=0$

or, $x(-1)+y.4+z(-2)+(-1)(x-1)+(-2)(y+4)+1(z-2)-3=0$

or, $-x+4y-2z-(x-1)-z(y+4)+(z-2)-3=0$

or, $-2x+2y-z-12=0$

or, $2x-2y+z+12=0$

Find the equation of tangent planes to the sphere $x^2+y^2+z^2=9$ Which are parallel to the plane $3x+4y+5z=7$

Solution:

The given sphere is $x^2+y^2+z^2=9$ .....(1)

Centre of sphere is (0, 0, 0) and the radius is 3.

$\therefore$ The given plane is $3x+4y+5z=7$ The general equation of the plane parallel to the given plane is

$3x+4y+5z+\lambda=0$ .....(2)

Also,

The length of the $\perp^r$ from the centre (0, 0, 0) the plane (2) is

$$P=\pm\left|\frac{3\times0+4\times0+5\times0+\lambda}{\sqrt{9+16+25}}\right|$$

$$\therefore\quad P=\pm\frac{\lambda}{\sqrt{50}}$$

$$p=\pm\frac{\lambda}{5\sqrt{2}}$$

By the condition of tangency,

$$3=\pm\frac{\lambda}{5\sqrt{2}}\quad\therefore\lambda=\pm15\sqrt{2}$$

$\therefore$ The required equation of the plane is $3x+4y+5z\pm15\sqrt{2}=0$

Find the two tangent planes to the sphere $x^2+y^2+z^2-4x+2y-6z+5=0$ which are parallel to the plane $2x+2y-z=0$

Solution:

The given sphere is

$x^2+y^2+z^2+2(-2)x+2.1y+2(-3)z+5=0$

The centre of the sphere is (–u, –v, –w) is (2, –1,3) and radius of the sphere is

$$=\sqrt{u^2+v^2+w^2-d}=\sqrt{(-2)^2+(1^2)+(-3)^2-5}$$

$$=\sqrt{4+1+9-5}=\sqrt{9}=3$$

The given plane is

$2x+2y-z=0$ .....(2)

The general equation of the plane parallel to the given plane (2) is $2x+2y-z+\lambda=0$ .....(3) where λ is a parameter

Also, the length of the perpendicular from the centre of the sphere, i.e,

$$\pm\frac{2\times2+2\times(-1)-3+\lambda}{\sqrt{2^2+2^2+(1)^2}}=\pm\frac{-1+\lambda}{\sqrt{9}}$$

$$=\pm\frac{-1+\lambda}{3}$$

The plane (3) will be tangent plane to the sphere if the length of the perpendicular from the centre of sphere i.e, (2, – 1, 3) to the plane (3) is equal to the radius of the sphere i.e, 3

$\therefore \pm\frac{\lambda-1}{3}=3$

$\therefore \pm(\lambda-1)=9$

$\therefore$ Taking positive and negative sign

$\therefore \lambda=8, 10$

$\therefore$ The required equations of the tangent planes are

$2x+2y-z+10=0, 2x+2y-z-8=0$

4. Prove that the plane $2x-y+2z-14=0$ touches the sphere $x^2+y^2+z^2-4x+2y-4=0$. Find the point of contact.

Solution:

We have $x^2+y^2+z^2-4x+2y-4=0$ is the equation of the sphere.

The centre and radius of the given sphere are (2, –1, 0) and $\sqrt{4+1+0+4}=\sqrt{9}=3$ respectively.

The length of perpendicular from the centre of the sphere to the given plane is $\left|\frac{2\times2-1(-1)+2\times0-14}{\sqrt{4+1+4}}\right|=3$

Which is equal to the radius of the sphere. Hence, the plane touches the sphere.

Again the equations of the line through the centre (2, –1, 0) and perpendicular or to the plane are $\frac{x-2}{2}=\frac{y+1}{-1}=\frac{z-0}{2}=r$ say

Any point on the line is

$(2r+2, -r-1, 2r)$. This will lie on the plane if

$2(2+2r)-(-1-r)+2.2r=14$

$\therefore r=1$

$\therefore$ Hence, point of contact is (4, –2, 2)

6. For what value of λ, the plane $x+y+z=\lambda\sqrt{3}$ touches the sphere

$x^2+y^2+z^2-2x-2y-2z-6=0$

Solution:

The given plane and the circle are $x+y+z=\lambda\sqrt{3}$ .....(1)

and $x^2+y^2+z^2-2x-2y-2z-6=0$ .....(2)

The centre of sphere (2) is (–u, –v, –w) - (1, 1, 1) and radius of sphere

$$=\sqrt{u^2+v^2+w^2-d}$$

$$=\sqrt{(-1)^2+(-1)^2+(-1)^2-(-6)}$$

$$=\sqrt{1+1+1+6}=3$$

New, the length of the perpendicular from the centre (1, 1, 1) to the plane (i) is

$$p=\pm\frac{1+1+1-\lambda\sqrt{3}}{\sqrt{1^2+1^2+1^2}}=\pm\frac{3-\lambda\sqrt{3}}{\sqrt{3}}=\pm(\sqrt{3}-\lambda)$$

The plane (1) touch the sphere (2) if the length of the perpendicular from the centre of sphere (1, 1, 1) to the plane (2) is equal to the radius of the plane, i. e. 3

i. e, $\pm(\sqrt{3}-\lambda)=3$

or, $\sqrt{3}-\lambda=\pm3$

or, $\lambda=\sqrt{3}\pm3$

If any tangent plane to the sphere $x^2+y^2+z^2=r^2$ makes intercepts a, b, c on the co-ordinates axes.

Prove that $a^{-2}+b^{-2}+c^{-2}=r^{-2}$

Solution:

Now, the equation of the sphere is $x^2+y^2+z^2=r^2$ .....(1)

We know that the equation of tangent plane at (x, y, z) with respect to sphere (1) is $xx_1+yy_1+zz_1=r^2$

Since, it meets the co-ordinates axes at (a, 0, 0) , (0, b, 0) and (0, 0, c)

Then

$ax_1+0+0=r^2 \quad \therefore x_1=\frac{r^2}{q}$

$0+by_1+0=r^2 \quad \therefore y_1=\frac{r^2}{b}$

$0+0+cz_1=r^2 \quad \therefore z_1=\frac{r^2}{c}$

Since, $(x_1, y_1, z_1,)$ lies on the sphere (1)

$x_1^2+y_1^2+z_1^2=r^2$

or, $\left(\frac{r^2}{a}\right)^2+\left(\frac{r^2}{b}\right)^2+\left(\frac{r^2}{c}\right)^2=r^2$

or, $\frac{r^4}{a^2}+\frac{r^4}{b^2}+\frac{r^4}{c^2}=r^2$

or, $r^4\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)=r^2$

$\therefore a^{-2}+b^{-2}+c^{-2}=\frac{1}{r^2}$

$\left|a^{-2}+b^{-2}+c^{-2}=r^{-2}\right|$

8. **Obtain the equation of the tangent planes to the sphere $x^2 + y^2 + z^2 + 6x - 2z + 1 = 0$ which passes through the line $3(16-x) = 3z = 2y + 30$**

Solution: Given sphere is $x^2 + y^2 + z^2 + 6x - 2z + 1 = 0$ .....(1)

The centre of sphere (1) is $(-u, -v, -w) = (-3, 0, 1)$ and radius of sphere is

$$R = \sqrt{u^2 + v^2 + w^2 - d}$$
$$= \sqrt{(-3)^2 + 0^2 + (-1)^2 - 1}$$
$$= \sqrt{9 + 1 - 1}$$
$$= 3$$

Also, The given line is $3(16-x) = 3z = 2y + 30$

or, $x + z - 16 = 0$ and $2y - 3z + 30 = 0$

Any plane through this line is

$x + z - 16 + \lambda(2y - 3z + 30) = 0$

or, $x + 2\lambda y + (1 - 3\lambda)z + (30\lambda - 16) = 0$ .....(2)

Again the length of perpendicular from the centre $(-3, 0, 1)$ to the plane (2) is

$$\left|\frac{(-3) + 2\lambda.0 + (1 - 3\lambda).1 + 30\lambda - 16}{\sqrt{(1)^2 + (2\lambda)^2 + (1 - 3\lambda)^2}}\right|$$

$$= \left|\frac{-3 + 1 - 3\lambda + 30\lambda - 16}{1 + 4\lambda^2 + 1 - 6\lambda + 9\lambda^2}\right| = \left|\frac{-18 + 27\lambda}{2 - 6\lambda + 13\lambda^2}\right|$$

The plane (2) touches the given sphere (1) if $\left|\frac{-18 + 27\lambda}{2 - 6\lambda + 13\lambda^2}\right| = 3$

Squaring gives,

$(-18 + 27\lambda)^2 = 9(2 - 6\lambda + 13\lambda^2)$

or, $9(-2 + 13\lambda)^2 = (2 - 6\lambda + 13\lambda^2)$

or, $9(4 - 12\lambda + 9\lambda^2) = 2 - 6\lambda + 13\lambda^2$

or, $36 - 108\lambda + 81\lambda^2 - 2 + 6\lambda - 13\lambda^2 = 0$

or, $68\lambda^2 - 3\lambda + 1 = 0$

or, $2\lambda^2 - \lambda - 2\lambda + 1 = 0$

or, $(2\lambda - 1)(\lambda - 1) == 0$

or, $\lambda = 1$, or, $\lambda = \frac{1}{2}$

Putting the value of λ in eq^n (2), we get the planes are

$x + z - 16 + 1(2y - 3z + 30) = 0$

and $x + z - 16 + \frac{1}{2}(2y - 3z + 30) = 0$

or, $x + 2y - 2z + 14 = 0$ and $2x + 2y - z - 2 = 0$

10. **Obtain the equation of the spheres through the circle $x^2 + y^2 + z^2 = 1$**
$2x + 4y + 5z = 6$ and touching the plane $z = 0$

Solution:

The equation of the sphere though the circle

$x^2 + y^2 + z^2 = 1$, $2x + 4y + 5z = 6$ is

$x^2 + y^2 + z^2 - 1 + \lambda(2x + 4y + 5z - 6) = 0$ .....(i)

$\therefore$ The centre of the sphere (1) is

$(-\lambda, -2\lambda, -\frac{5\lambda}{2})$

and

radius of the sphere is

$$= \sqrt{\lambda^2 + 4\lambda^2 + \frac{25\lambda^2}{4} + (6\lambda + 1)} = \frac{1}{2}\sqrt{45\lambda^2 + 24\lambda + 1}$$

The length of the perpendicular from the centre of the sphere $(-\lambda, -2\lambda, \frac{-5\lambda}{2})$ to the tangent plane $z = 0$ is

$$P = \pm\left|\frac{0 \times (-\lambda) + 0 \times (-2\lambda) - \frac{5\lambda}{2}}{\sqrt{6^2 + 0^2 + 1^2}}\right|$$

$\therefore \quad P = \pm \frac{5\lambda}{2}$

Using the condition of tangency,

$$\pm \frac{5\lambda}{2} = \frac{1}{2}\sqrt{45\lambda^2 + 24\lambda + 1}$$

We have, $5\lambda^2 + 6\lambda + 1 = 0$

$\therefore \lambda = -1, \frac{-1}{5}$

Substituting, the values of λ in (i), we get the required equations of spses as

$x^2 + y^2 + z^2 - 2x - 4y - 5z + 5 = 0$

and $5(x^2 + y^2 + z^2) - 2x - 4y - 5z + 1 = 0$

Exercise - 14

1. (i) Find the equation of the cone with vertex (1, 1, 0) and guiding curve is $y = 0, x^2 + z^2 = 4$.

Solution: Any line through (1, 1, 0) is

$$\frac{x-1}{\ell} = \frac{y-1}{m} = \frac{z-0}{n}$$ the line meet $y = 0$ at the point $(x_1, 0, z_1)$

then $\frac{x_1-1}{\ell} = \frac{-1}{m} = \frac{z_1}{n}$ .... (ii)

This gives, $x_1 = 1 - \frac{\ell}{m}, z_1 = -\frac{n}{m}$

If the point $(x_1, 0, z_1)$ also lies on the surface

$x^2 + z^2 = 4,\ x_1^2 + z_1^2 = 4$

$\therefore \left(1-\frac{\ell}{m}\right)^2 + \left(-\frac{n}{m}\right)^2 = 4$, eliminating l, m, n from equation (i)

From (i) $\frac{\ell}{m} = \frac{x-1}{y-1}$ and $\frac{n}{m} = \frac{z}{y-1}$

Putting the values in (iii), we get

$$\left(1-\frac{x-1}{y-1}\right)^2 + \left(-\frac{z}{y-1}\right)^2 = 4$$

or, $(y - 1 - x + 1)^2 + z^2 = 4(y-1)^2$

or, $(y - x)^2 + z^2 = 4(y^2 - 2y + 1)$

or, $y^2 - 2xy + x^2 + z^2 = 4y^2 - 8y + 4$

or, $x^2 - 3y^2 + z^2 - 2xy + 8y - 4 = 0$

This is the required equation cone.

(ii) Find the equation of the cone with vertex (α, β, γ) and base $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, z = 0$

Solution: Any line through (α, β, γ) is

$$\frac{x-\alpha}{\ell} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} \text{ ..... (i)}$$

If (i) meets $z = 0$ at $(x_1, y_1, 0)$

$\therefore \frac{x_1-\alpha}{\ell} = \frac{y_1-\beta}{m} = \frac{-\gamma}{n}$

or, $x_1 = \alpha - \frac{\ell\gamma}{n},\ y_1 = \beta - \frac{m\gamma}{n}$

If the points $(x_1, y_1, 0)$ lies on the surface $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\therefore \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1$

or, $b^2 x_1^2 + a^2 y_1^2 = a^2b^2$

or, $b^2\left(\alpha - \frac{\ell\gamma}{m}\right)^2 + a^2\left(\beta - \frac{m\gamma}{n}\right)^2 = a^2b^2$ ..... (ii)

From (i) $\frac{m}{n} = \frac{y-\beta}{z-\gamma}$ and $\frac{\ell}{n} = \frac{x-\alpha}{z-\gamma}$

Using the values in (iii), the required equation of the cone is

$$b^2\left(\alpha - \frac{x-\alpha}{z-\gamma}\right)^2 + a^2\left(\beta - \frac{y-\beta}{z-\gamma}\gamma\right)^2 = a^2b^2$$

$b^2(\alpha z - \gamma x)^2 + a^2(\beta z - \gamma y)^2 = a^2b^2(z-\gamma)^2$

(iii) Find the equation of the cone with vertex (3, 1, 2) and base $2x^2 + 3y^2 = 1, z = 1$ **(BE 2069 Bhadra)**

Solution: Any line through (3, 1, 2) is

$$\frac{x-3}{\ell} = \frac{y-1}{m} = \frac{z-2}{n} \text{ ..... (i)}$$

If (i) meets $z = 1$, we have

$$\frac{x-3}{\ell} = \frac{y-1}{m} = \frac{1-2}{n}$$

or, $\frac{x-3}{\ell} = \frac{y-1}{m} = \frac{-1}{n}$

or, $x = 3 - \frac{\ell}{n},\ y = 1 - \frac{m}{n}$

Putting the values of x and y in $2x^2 + 3y^2 = 1$

$$2\left(3-\frac{\ell}{n}\right)^2 + 3\left(1-\frac{m}{n}\right)^2 = 1 \text{ ..... (iii)}$$

From (I) $\frac{m}{n} = \frac{y-1}{z-2}$ and $\frac{\ell}{n} = \frac{x-3}{z-2}$

Using the value in (ii), the required equation of the cone is,

$$\left(3-\frac{x-3}{z-2}\right)^2 + 3\left(1-\frac{y-1}{z-2}\right)^2 = 1$$

or, $2(2z - 6 - x + 3)^2 + 3(z - 2 - y + 1)^2 = (z-2)^2$

or, $2(3z - x - 3)^2 + 3(z - y - 1)^2 = (z-2)^2$

or, $2(9z^2 + x^2 + 9 - 6zx + 6x - 18z) + 3(z^2 + y^2 + 1 - 2yz + 2y - 2z) = z^2 - 4z + 4$

or, $2x^2 + 3y^2 + 20z^2 - 6yz - 12zx + 12x + 6y - 38z - 17 = 0$

2. Find the equation of cone whose vertex is origin and guiding curve is $\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{1} = 1, x + y + z = 1$

Solution: The guiding curve is given by

$\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{1} = 1$ ..... (i)

$x + y + z = 1$ ..... (ii)

Making (i) homogeneous with the help of (ii) the equation of the required cone is,

$\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{1} = (x + y + z)^2$

or, $9x^2 + 4y^2 + 36z^2 = 36(x^2 + y^2 + z^2 + 2xy + 2yz + 2zx)$

or, $27x^2 + 32y^2 + 72(xy + yz + zx) = 0$

3. Find the equation of the cone whose vertex at origin and base is the circle

$y^2 + z^2 = b^2, x = a$

Solution: The base is composed of the equation

$y^2 + z^2 = b^2$ ..... (i)

$x = a$ ..... (ii)

From (ii) $\frac{x}{a} = 1$ ..... (iii)

Making (i) homogeneous with the help of (iii)

$y^2 + z^2 = b^2 . 1^2$

or, $y^2 + z^2 = b^2 . \left(\frac{x}{a}\right)^2$

or, $a^2(y^2 + z^2) = b^2x^2$ this is the required equation.

4. Find the equation of the cone whose vertex at the origin and direction cosines of the generators satisfy the relation $3l^2 + 4m^2 + 5n^2 = 0$

Solution: Any line through the origin (0, 0, 0)

With the direction cosines l, m, n is

$\frac{x}{\ell} = \frac{y}{m} = \frac{z}{n}$ ..... (i)

As given $3l^2 + 4m^2 + 5n^2 = 0$ ..... (ii)

Eliminating l, m, n from (i) and (ii)

The required equation of the cone is

$3x^2 + 4y^2 + 5z^2 = 0$

5. Show that the equation of the cone whose vertex is origin whose guiding curve is circle represented by $x^2 + y^2 + z^2 - ax - by - cz = 0$, $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

is $c(a^2 + b^2)xy + a(b^2 + c^2)yz + b(c^2 + a^2)zx = 0$

Solution: The guiding curve is represented by the equations

$x^2 + y^2 + z^2 - ax - by - cz = 0$ ..... (i)

$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ ..... (ii)

The equation of the cone with the vertex at the origin must be a homogeneous equation of degree 2 in x, y, z. So, making (i) homogeneous with the help of (ii), the required equation of the cone is,

$x^2 + y^2 + z^2 - ax . 1 - by . 1 - cz . 1 = 0$

or, $x^2 + y^2 + z^2 - ax\left(\frac{x}{a} + \frac{y}{b} + \frac{z}{c}\right) - by . \left(\frac{x}{a} + \frac{y}{b} + \frac{z}{c}\right) - cz . \left(\frac{x}{a} + \frac{y}{b} + \frac{z}{c}\right) = 0$

or, $-ax\left(\frac{y}{b} + \frac{z}{c}\right) - by\left(\frac{x}{a} + \frac{z}{c}\right) - cz\left(\frac{x}{a} + \frac{y}{b}\right) = 0$

or, $a^2x(cy + bz) + b^2y(cx + az) + c^2z(bx + ay) = 0$

or, $ca^2ay + a^2bzx + b^2cxy + ab^2yz + bc^2zx + c^2ayz = 0$

or, $c(a^2 + b^2)xy + a(b^2 + c^2)yz + b(c^2 + a^2)zx = 0$

6. Find the equation of right circular cone whose vertex at origin and axis is the line $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ with vertical angle 30°.

(BE 2070 Bhadra, Magh)

Ans. The vertex is 0 (0, 0, 0).Let P (x, y, z) be any point on the cone. Then the direction ratio of OP are x – 0, y – 0, z – 0 or x, y, z. As OP makes angle of 30° with the axis $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$

$$\cos 30^\circ = \frac{x.1 + y.2 + z.3}{\sqrt{x^2 + y^2 + z^2}\sqrt{1^2 + 2^2 + 3^2}}$$

$$\text{or, } \frac{\sqrt{3}}{2} = \frac{x + 2y + 3z}{\sqrt{x^2 + y^2 + z^2}\sqrt{14}}$$

or, $42(x^2 + y^2 + z^2) = 4(x + 2y + 3z)^2$

or, $21(x^2 + y^2 + z^2) = 2(x^2 + 4y^2 + 9z^2 + 4xy = 12yz + 6zx)$

or, $19x^2 + 13y^2 + 3z^2 - 8xy - 24yz - 12zx = 0$

This is the required equation of cone.

7. Find the equation of right circular cone whose vertex is at the origin and semi vertical angle α and having axis of z as its axis.

Solution: The vertex is 0 (0, 0, 0). Let P (x, y, z) be any point on the cone. Then direction ratios of OP are $x-0, y-0, z-0$ or x, y, z

The direction clsines of z-axis are 0, 0, 1.

As OP makes angle α with the axis of the cone (i.e., z-axis)

$$\cos\alpha = \frac{x.0+y.0+z.1}{\sqrt{x^2+y^2+z^2}}$$

$$\text{or, } \cos\alpha = \frac{z}{\sqrt{x^2+y^2+z^2}}$$

$\cos^2\alpha\,(x^2+y^2+z^2) = z^2$

or, $x^2+y^2+z^2 = z^2\sec^2\alpha$

or, $x^2+y^2 = z^2\sec^2\alpha - z^2$

or, $x^2+y^2 = z^2(\sec^2\alpha - 1)$

or, $x^2+y^2 = z^2\tan^2\alpha$

8. Find the equation of cone whose vertical angle is $\frac{\pi}{2}$ with vertex at the origin and its axis along the line $x = -2y = z$

Solution: Let P (x, y, z) be any point on the cone so that the direction ratio of the generator are x, y, z. The equation of the axis is $\frac{x}{1} = \frac{y}{-\frac{1}{2}} = \frac{z}{1}$

$$\text{or, } \frac{x}{2} = \frac{y}{-1} = \frac{z}{2}$$

As the vertical angle is $\frac{\pi}{2}$, the semi-vertical angle is $\frac{\pi}{4}$

$$\therefore \cos\frac{\pi}{4} = \frac{x.1+y.(-1)+z.2}{\sqrt{x^2+y^2+z^2}\sqrt{4+1+4}}$$

$$\text{or, } \frac{1}{\sqrt{2}} = \frac{x-y-2z}{3\sqrt{x^2+y^2+z^2}}$$

or, $9(x^2+y^2+z^2) = 2(x-y+2z)^2$

or, $9(x^2+y^2+z^2) = 2(x^2+y^2+4z^2-2xy-4yz+4zx)$

or, $7x^2+7y^2+z^2+4xy+8yz+8zx = 0$

Exercise - 15

1. Find the equation of the Cylinder whose guiding curve is conic. $ax^2+2hxy+by^2+2gx+2fy+c=0$, $z=0$ axis is the line $\frac{x}{\ell} = \frac{y}{m} = \frac{z}{n}$.

Solution: Let (α, β, γ) be any point on the cylinder. So, the equation of the generator through (α, β, γ) is

$$\frac{x-\alpha}{\ell} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} \quad \ldots\ldots \text{(i)}$$

Putting $z = 0$ in (i)

$$\frac{x-\alpha}{\ell} = \frac{y-\beta}{m} = \frac{0-\gamma}{n}$$

$$\text{or, } x = \alpha - \frac{\ell\gamma}{n} \text{ and } y = \beta - \frac{m\gamma}{n}$$

If $\left(\alpha - \frac{\ell\gamma}{n}, \beta - \frac{m\gamma}{n}, 0\right)$ is any point on the guiding curve.

$$a\left(\alpha-\frac{\ell\gamma}{n}\right)^2 + 2h\left(\alpha-\frac{\ell\gamma}{n}\right)\left(\beta-\frac{m\gamma}{n}\right) + b\left(\beta-\frac{m\gamma}{n}\right)^2$$
$$+ 2g\left(\alpha-\frac{\ell\gamma}{n}\right) + 2f\left(\beta-\frac{m\gamma}{n}\right) + c = 0$$

This shows that the loucus of (α, β, γ) or, the equation of cylinder is

$$a\left(x-\frac{\ell z}{n}\right)^2 + 2h\left(x-\frac{\ell z}{n}\right)\left(y-\frac{mz}{n}\right) + b\left(y-\frac{mz}{n}\right)^2$$
$$+ 2g\left(x-\frac{\ell z}{n}\right) + 2f\left(y-\frac{mz}{n}\right) + c = 0$$

or, $a(nx-lz)^2 + 2h(nx-lz)(xy-mz) + b(xy-mz)^2$
$+ 2gn(nx-lz) + 2fm(xy-mz) + cn^2 = 0$

2. Find the equation of the cylinder whose generaters are parallel to the line $\frac{x}{1} = \frac{y}{-2} = \frac{z}{3}$ and whose guiding curve is $x^2+2y^2=1$, $z=3$.

Solution: Let (x_1, y_1, z_1) be any point on the cylinder so that the equation of the generators through the point is

$$\frac{x-x_1}{1} = \frac{y-y_1}{-2} = \frac{z-z_1}{3} \quad \ldots\ldots\ldots \text{(i)}$$

Putting $z = 3$ in (i), we have

$$\frac{x-x_1}{1}=\frac{y-y_1}{-2}=\frac{3-z_1}{3}$$

or, $x = x_1 + \frac{3-z_1}{3}$, $y = y_1 + \frac{2(z_1-3)}{3}$

If $\left(x_1+\frac{3-z_1}{3},\ y_1+\frac{2(z_1-3)}{3},\ 3\right)$ is a point on the guiding curve.

$$\left(x_1+\frac{3-z_1}{3}\right)^2+2\left(y_1+\frac{2z_1-6}{3}\right)^2=1$$

Here, the equation of the cylinder is the locus of (x_1, y_1, z_1) is

$$\left(x_1+\frac{3-z}{3}\right)^2+2\left(y_1+\frac{2z-6}{3}\right)^2=1$$

or, $(3x - z + 3)^2 + 2(3y + 2z - 6)^2 = 9$

or, $9x^2 + z^2 + 9 - 6zx - 6z + 18x + 2(9y^2 + 4z^2 + 36 + 12yz - 24z - 36y) = 9$

or, $9x^2 + 18y^2 + 9z^2 + 24yz - 6zx + 18x - 72y - 54z + 7z\ 0$

or, $3x^2 + 6y^2 + 3z^2 + 8yz - 2zx + 6x - 24y - 18z + 24 = 0$

3. Find the equation of the cylinder which passess through $y^2 = 4ax$, $z = 0$ and whose generators are parallel to the line $\frac{x}{1}=\frac{y}{1}=\frac{z}{1}$.

Solution: Let (α, β, γ) be any point of the cylinder. So the equation of the generator through (α, β, γ) is

$$\frac{x-\alpha}{1}=\frac{y-\beta}{1}=\frac{z-\gamma}{1} \quad \text{..... (i)}$$

Putting $z = 0$ in (i)

$x - \alpha = y - \beta = -\gamma$

or, $x = \alpha - \gamma$ and $y = \beta - \gamma$

If $(\alpha - \gamma, \beta - \gamma, 0)$ is any point on the guiding curve.

$(\beta - \gamma)^2 = 4a(\alpha - \gamma)$

This shows that the locus of (α, β, γ)

or, the equation of the cylinder is

$(y - z)^2 = 4a(x - z)$

4. Find the equation of right circular cylinder of radius 2 whose axis is passess through (1, 2, 3) and has direction ratios 2, – 3, 6

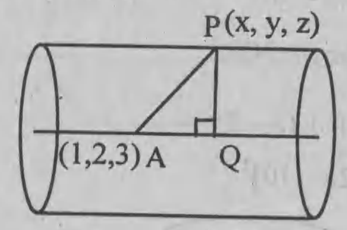
Solution: Let A be any point (1, 2, 3). the direction ratios of the axis are 2, – 3, 6. As $\sqrt{2^2+(-3)^2+6^2} = 7$, the direction cosines of AB are $\frac{2}{7}, \frac{-3}{7}, \frac{6}{7}$.

Let p (x, y, z) be any point on the cylinder. Join PA and draw PQ perpendicular to the axis.

Now,

$PA^2 = (x - 1)^2 + (y - 2)^2 + (z - 3)^2$

PQ = radius of the cylinder = 2



∴ AQ = Projection of PA on the axis

$$= (x-1)\frac{2}{7}+(y-2)\left(\frac{-3}{7}\right)+(z-3)\frac{6}{7}$$

$$= \frac{1}{7}(2x - 3y + 6z - 14)$$

From the right angle triangle PQA

$PA^2 = PQ^2 + QA^2$

or, $(x - 1)^2 + (y - 2)^2 + (z - 3)^2 = 4 + \frac{1}{49}(2x - 3y + 6z + 14)^2$

or, $49[(x - 1)^2 + (y - 2)^2 + 9z - 3)^2] = 196 + 4x^2 + 9y^2 + 36z^2 - 12xy - 36yz + 24zx - 56x\ 84y - 168z + 196$

or, $45x^2 + 40y^2 + 13z^2 + 26y^2 - 24zx + 12xy - 42x - 280y - 126z + 294 = 0$

Find the equation of right circular cylinder of radius 2 whose axis is the line $\frac{x-1}{2}=\frac{y-2}{1}=\frac{z-3}{2}$

Solution: The axis of the cylinder is

$$\frac{x-1}{2}=\frac{y-2}{1}=\frac{z-3}{2}$$

The direction ratios of the axis are, 2, 1, 2

The direction cosines are

$$\frac{2}{\sqrt{2^2+1^2+2^2}},\ \frac{2}{\sqrt{2^2+1^2+2^2}},\ \frac{2}{\sqrt{2^2+1^2+2^2}}$$

or, $\frac{2}{3}, \frac{1}{3}, \frac{2}{3}$

A point on (i) is R (1, 2, 3). take a point P (x, y, z) on the cylinder. Drawing PQ perpendicular to the axis and joining PR

$PR^2 = (x - 1)^2 + (y - 2)^2 + (z - 3)^2$

PQ = radius = 2

PQ = Projection of RP on the axis

$= (x-1)\,\frac{2}{3} + (y-2)\,\frac{1}{3} + \frac{2}{3}\,(z-3)$

$= \frac{1}{3}\,(2x - 2 + y - 2 + 2z - 6)$

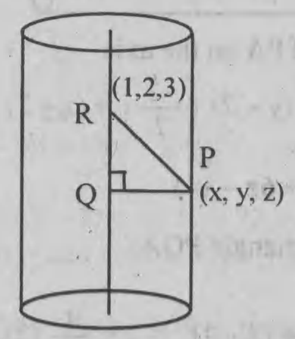
$= \frac{1}{3}\,(2x + y + 2z - 10)$

Now, from right triangle PQR,

$PR^2 = PQ^2 + QR^2$

or, $(x-1)^2 + (y-2)^2 + (z-3)^2$

$= 4 + \frac{1}{9}\,[2x + y + 2z - 10]^2$



or, $9\,(x^2 - 2x + 1 + y^2 - 4y + 4 + z^2 - 6z + 9)$

$= 36 + 4x^2 + y^2 + 4z^2 + 100 + 4xy + 4yz + 8zx - 40x - 20y - 40z$

or, $5x^2 + 8y^2 + 5z^2 - 4xy - 4yz - 8zx + 22x - 16y - 14z - 10 = 0$

This is the required equation of the cylinder.

6. Find the equation of the right circular cylinder having its base the circle $x^2 + y^2 + z^2 = 9$, $x - y + z = 3$ (BE 2068 Magh)

Solution: The given circle is the intersection of the sphere

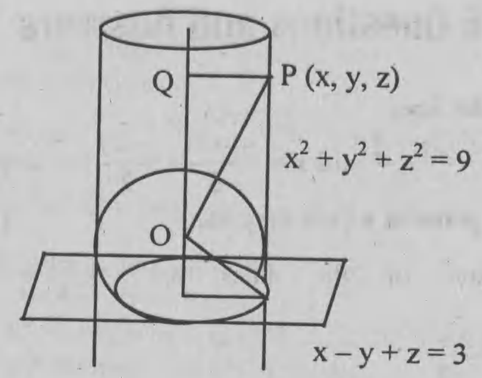
$x^2 + y^2 + z^2 = 9$ ..... (i)

and the plane $x - y + z = 3$ ..... (ii)

The centre of the sphere (i) is 0 (0, 0, 0) and its radius (R) = 3

The axis of the cylinder must be perpendicular to the plane and (ii) and pass through the centre of the sphere. Hence, the equation of the axis of the cylinder is $\frac{x-0}{1} = \frac{y-0}{-1} = \frac{z-0}{1}$

or, $\frac{x}{1} = \frac{y}{-1} = \frac{z}{1}$ ..... (iii)



$$\frac{1}{\sqrt{1^2 + (-1)^2 + 1^2}},\ \frac{-1}{\sqrt{1^2 + (-1)^2 + 1^2}},\ \frac{1}{\sqrt{1^2 + (-1)^2 + 1^2}}$$

or, $\frac{1}{\sqrt{3}},\ -\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}$

Length of the perpendicular from centre of the sphere to the plane (ii) is $p = \left|\frac{0 - 0 + 0 - 3}{\sqrt{3}}\right|$

or, $P = \sqrt{3}$

Now, the radius of the circle $= \sqrt{R^2 - P^2} = \sqrt{9-3} = \sqrt{6} = \sqrt{6}$

$\therefore$ The radius of the cylinder $= \sqrt{6}$

Let P (x, y, z) is any point on the cylinder. Draw PQ perpendicular to the axis.

Now, $P^2 = x^2 + y^2 + z^2$

PQ = radius of the cylinder $= \sqrt{6}$

OQ = Projection of OP on the axis

$= (x-0)\,\frac{1}{\sqrt{3}} + (y-0)\left(\frac{-1}{\sqrt{3}}\right) + (z-0)\,.\,\frac{1}{\sqrt{3}}$

$= \frac{1}{\sqrt{3}}\,(x - y + z)$

From the right triangle OPQ, $OP^2 = PQ^2 + OQ^2$

or, $x^2 + y^2 + z^2 = 6 + \frac{1}{3}\,(x - y + z)^2$

or, $3\,(x^2 + y^2 + z^2) = 18 + x^2 + y^2 + z^2 - 2xy - 2yz + 2zx$

or, $2x^2 + 2y^2 + 2z^2 + 2xy + 2yz - 2zx - 18 = 0$

or, $x^2 + y^2 + z^2 + xy + yz - zx - 9 = 0$

This is the required equation of the cylinder.

Important IOE Questions and Answers

1. Prove that the lines

$\frac{x+1}{-3}=\frac{y-3}{2}=\frac{z+2}{1}$ and $x=\frac{y-7}{-3}=\frac{z+7}{2}$ are coplanar and find the equation of plane in which they lie. **(BE 2069 Poush)**

Sol^n. The equation of the lines are $\frac{x+1}{-3}=\frac{y-3}{2}=\frac{z+2}{1}$

$\frac{x}{1}=\frac{y-7}{-3}=\frac{z+7}{2}$

Here, $l_1=-3, m_1=2, n_1=1, \quad x_2=-1, y_1=3, z_1=-2$

$l_2=1, m_2=-3, z_2=2$ and $x_2=0, y_2=7, z_2=-7$

If the lines (i) and (ii) are coplaner, then we have

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_2 \\ l_2 & m_2 & n_2 \end{vmatrix}=0$$

or, $\begin{vmatrix} -1-0 & 3-7 & -2+7 \\ -3 & 2 & 1 \\ 1 & -3 & 2 \end{vmatrix}=0$ or, $\begin{vmatrix} -1 & -4 & 5 \\ -3 & 2 & 1 \\ 1 & -3 & 2 \end{vmatrix}=0$

or, $-1(4+3)=4(-6-1)+5(9-2)=0$ or, $-35+35=0$

$0=0$

Hence the line (i) and (ii) are coplaner

The also, the equation of the plane containing the given lines (i) and (ii) is,

$$\begin{vmatrix} x+1 & y-3 & z+2 \\ -3 & 2 & 1 \\ 1 & -3 & 2 \end{vmatrix}=0$$

or, $(x+1)(4+3)-(y-3)(-6-1)+9z+2)(9-2)=0$

or, $7(x+1)+7(y-3)+7(z+2)=0$ or, $x+y+z=0$ Ans.

2. A variable plane is parallel to the plane $\frac{x}{a}+\frac{y}{b}+\frac{y}{c}=1$ and meets the axes in A, B, C. Prove that the circle ABC lies on the cone $\left(\frac{b}{c}+\frac{c}{b}\right)yz+\left(\frac{c}{a}+\frac{a}{c}\right)zx+\left(\frac{a}{b}+\frac{b}{a}\right)xy=0$ **(BE 2069 Poush)**

Sol^n. Here, the equation of any plane parallel to the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=0$ is

$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=\lambda$ or, $\frac{x}{a\lambda}+\frac{y}{b\lambda}+\frac{z}{c\lambda}=1$

This plane meet the axes at A $(a\lambda, 0, 0)$, B $(0, b\lambda, 0)$ and C $(0, 0, c\lambda)$ respectively.

The equation of any sphere through origin is

$x^2+y^2+z^2+2ux+2vy+2wz=0$

Since, it meets the axes at A, B and C,

or, $a^2\lambda^2+2ua\lambda=0$ or, $U=-\frac{\lambda}{2} \Rightarrow b^2\lambda^2+2vb\lambda=0$

$v=-\frac{b\lambda}{2}$ and $c^2\lambda^2+2wc\lambda=0$ or, $w=-\frac{c\lambda}{2}$

So, the equation of sphere (ii) gives,

$x^2+y^2+z^2-9\lambda x-b\lambda y-c\lambda z=0$

or, $x^2+y^2+z^2-\lambda(ax+by+cz)=0$ ..... (iii)

Eliminating and between (i) and (iii) we get,

$$x^2+y^2+z^2-(ax+by+cz)\left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}\right)=0$$

$$\text{or, } yz\left(\frac{b}{c}+\frac{c}{b}\right)+zx\left(\frac{c}{a}+\frac{a}{c}\right)+xy\left(\frac{a}{b}+\frac{b}{a}\right)=0 \text{ Ans.}$$

3. Find the equation of the right circular cylinder of radius 4 and axis the line $x=2y=-z$. **(BE 2069 Poush)**

Sol^n. Here, the equation of the axis of the cylinder is, $\frac{x}{2}=\frac{y}{1}=\frac{z}{-2}$

The direction ratio of the axis are, $2, 1, -2$

The direction cosines of the axis are $\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}$

We know that, the equation of the right circular cylinder with radius 4 and given axis is,

$$x^2+y^2+z^2-\left(\frac{2}{3}x+\frac{1}{3}y-\frac{2}{3}z\right)^2=4^2 \text{ or, } x^2+y^2+z^2-\frac{1}{9}(2x+y-2z)^2=16$$

or, $5x^2+8y^2+5z^2+4yz+8zx-4xy-144=0$

4. Obtain the equation of the sphere through the circle $x^2+y^2+z^2=9$, $x-2y+2z=5$ as a great circle. **(BE 2069 Bhadra, 2068 Bhadra)**

Sol^n. Here, the equation of the sphere through the circle is,

$x^2+y^2+z^2-9+\lambda(x-2y+2z-5)=0$

or, $x^2+y^2+z^2+\lambda x-2\lambda y+2\lambda z-5\lambda-9=0$ ..... (i)

Comparing equation (i) with the equation

$x^2+y^2+z^2+2ux+2vy+2wz+d=0$

We find, $U=\frac{\lambda}{2}, V=-\lambda, w=2\lambda$ and $d=-5\lambda-9$

Centre $=\left(-\frac{\lambda}{2}, \lambda, -\lambda\right)$. since, the circle is great circle of this sphere.

So, its centre lie on the plane $x-2y+2z=5$ if

$-\frac{\lambda}{2}-2\lambda+2(-\lambda)=5$ $\quad\therefore \lambda=-\frac{10}{9}$

Substituting the value of λ in (i), we get

$9(x^2+y^2+z^2)-10x+20y-20z-31=0$ Ans.

5. **Find the equation of right circular cylinder whose axis is the line $\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$ and whose radius 'r'.**

(BE 2069 Bhadra (Or))

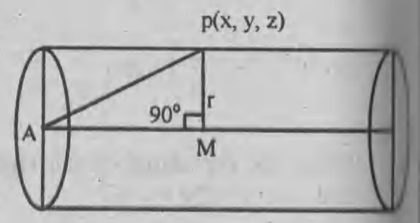
Sol^n. The axis of cylinder is,

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n} \text{ ..... (i)}$$

The direction cosines of line (i) are,

$$\frac{l}{\sqrt{l^2+m^2+n^2}}, \frac{m}{\sqrt{l^2+m^2+n^2}}, \frac{n}{\sqrt{l^2+m^2+n^2}}$$

A point on the line (i) is A (α, β, γ). Let p(x, y, z) be any point on the cylinder.



So, $Ap = \sqrt{(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2}$

Draw PM perpendicular to the axis AM.

Then PM = r and AM = Projection of AP on AM (axis)

$$= \frac{(x-\alpha)l+(y-\beta)m+(z-\gamma)n}{\sqrt{l^2+m^2+n^2}}$$

From the right angle triangle PAM

$(AP)^2 = (AM)^2 + (PM)^2 = [(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2]$

$$= \left[\frac{l(x-\alpha)+m(y-\beta)+n(z-\gamma)}{l^2+m^2+n^2}\right]+r^2$$

This is the required equation of right circular cylinder.

6. **Show that the lines $\frac{x+5}{3}=\frac{y+4}{1}=\frac{z-7}{-2}$ and $3x+2y+z-2=0$ $= x-3y+2z-13$ are coplanar and find the equation of the plane in which they lie.** **(BE 2068 Magh)**

Sol^n. Here, the equation of lines are $\frac{x+5}{3}=\frac{y+4}{1}=\frac{z-7}{-2}=r$(say)

and $3x+2y+z-2=0$, $x-3y+2z-13=0$

Any point on this line is P $(3r-5, r-4, -2r+7)$

This point which lie on the plane $3x+2y+z-2=0$

or (ii) if $3(3r-5)+2(r-4)+(-2r+7)-2=0$

or, $9r-15+2r-8-2r+7-2=0$

or, $9r=18$, $r=2$

The point P will also lie in the plane

$x-3y+2z-13=0$ of (ii) if $2r-5-2(r-4)+2(-2r+7)-13=0$

or, $3r-5-3r+12-4r+14-13=0$

or, $-4r=-8$, $r=2$

$\therefore$ Substitute r = 2 in the coordinates of P we get, $(1, -2, 3)$

Also, the equation of plane containing the line (ii) is,

$3x+2y+z-2+\lambda(x-3y+2z-13)=0$

If this plane contain the line (i), this point $(-5, -4, 7)$ must lie in this plane (iii),

So, $-15-8+7-2+\lambda(-5+12+14-13)=0$

or, $-18+8\lambda=0$, $\lambda=\frac{9}{4}$

Substitute the values of λ in equation (i) we get,

$21x-19y+22z-125=0$ Ans.

Prove that $ax^2+by^2+cz^2+2ux+2vy+2wz+d=0$ represents a cone if $\frac{u^2}{a}+\frac{v^2}{b}+\frac{\omega^2}{c}=d$ **(BE 2068, Magh)**

Sol^n. The general equation of second degree in x, y, z is $ax^2+by^2+cz^2=2ux+2vy+2wz+d=0$

Making this equation homogeneous by introducing a new veriable t, we get,

$F(x, y, z, t)=ax^2+by^2+cz^2+2uxt+2vyt+2wzt+dt^2=0$

Now, equating to zero the partial derivative of F.

We get, $\frac{\partial F}{\partial x}=2ax+2ut=0$ ..... (i)

$\frac{\partial F}{\partial y}=2by+2vt=0$ ..... (ii)

$\frac{\partial F}{\partial z}=2cz+2wt=0$ ..... (iii)

$\frac{\partial F}{\partial t}=2ux=2vy=2wz=2dt=0$ ..... (iv)

Solving the equation (i), (ii), (iii)

Taking t = 1, we get $x=-\frac{u}{a}$, $y=\frac{-v}{b}$, $z=\frac{-w}{c}$

These values must safety equation (iv), we get

$$-2u.\frac{u}{a}-2v.\frac{v}{b}-2w.\frac{w}{c}+2d=0 \quad \text{or, } \frac{u^2}{a}+\frac{v^2}{b}+\frac{w^2}{c}=d \text{ Ans.}$$

8. **Find the equation which touches the sphere $x^2+y^2+z^2+2x-6y+1=0$ at (1, 2, – 2) and passes through the point (1, –1, 0)**

(BE 2068 Bhadra (or))

Sol^n. Here, the equation of sphere is,

$x^2+y^2+z^2-2x-6y+1=0$ ..... (i)

Comparing it with $x^2+y^2+z^2+20x+2vyf\,2wztd=0$

We find U = 1, V = – 3, w = 0 and d = 1

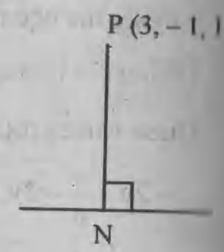
The equation of tangent plane at the point (1, 2, – 2) to the sphere (i) is
$xx_1 + yy_1 + zz_1 + U(x + x_1) + V(y + y_1) + w(z + z_1) + d = 0$
$x + 2y - 2z + 1(x + 1) - 3(y + z) + 0.(z - 2) + 1 = 0$
or, $x + 2y - 2z = x + 1 - 3y - 6 + 1 = 0$
or, $2x - y - 2z - 4 = 0$ ..... (ii)
∴ The equation of sphere through (i) and (ii) is,
$x^2 + y^2 + z^2 + 2x - 6y + 1 + \lambda(2x - y - 2z - 4) = 0$ ..... (iii)
Since, it passess through (1, – 1, 0)
So, $1 + 1 + 0 + 2 + 6 + 1 + \lambda(2 + 1 - 0 - 4)$ or, $11 - \lambda = 0$ or, $\lambda = 11$
∴ The equation of sphere (iii) gives,
$x^2 + y^2 + z^2 + 24x - 17y - 22z - 43 = 0$.

9. Find the equation of the cone with vertex (α, β, γ) and base $y^2 = 4ax$, $z = 0$
(BE 2068 Bhadra)

Sol^n. Let (α, β, γ) be the vertex of the cone then the equation of generator is
$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z\gamma}{n}$ which also lie on the parabola $y^2 = 4ax$ in the plane
$= 0$ in $\left(\beta - \frac{m\gamma}{n}\right)^2 = 4a\left(\alpha - \frac{l\gamma}{n}\right)$ ..... (ii)
Eliminating l, m,. n from (i) and (ii) we get,
$\left(\beta - \frac{y-\beta}{z-\gamma}.\gamma\right)^2 = 4a\left(\alpha a - \frac{x-\alpha}{z-\gamma}.\gamma\right)$ or, $(\beta z - \gamma y)^2 = 4a(\alpha z - \gamma x)(z - \gamma)$
This is the required equation of cone.

10. Find the length of perpendicular from the point (3, – 1, 11) to the line $\frac{x}{2} = \frac{y-2}{3} = \frac{z-3}{4}$. Also obtain the equation of the perpendicular.
(BE 2067 Mangsir, 2070 Bhadra)

Sol^n. The equation of the line is,
$\frac{x}{2} = \frac{y-2}{3} = \frac{z-3}{4} = r$(say)



Any point on this line is N (2r, 3r + 2, 4r + 3)
If PN is perpendicular to the given line so, the direction ratios of the line joining the point (3, – 1, 11) to N (2r, 3r + 2, 4r + 3) is
2r – 3, 3r + 2 + 1, 4r + 3 – 11
i.e. 2r – 3, 3r + 3, 4r – 8
Then using the condition of perpendicularity
$2(2r - 3) + 3(3r + 3) + 4(4r - 8) = 0$
or, $4r - 6 + 9r + 9 + 16r - 32 = 0$
or, $29r = 29$ ∴ $r = 1$
So, N (2r, 3r + 2, 4r + 3) = N (2, 3 + 2, 4 + 3) = N (2, 5, 7)

Thus, length of perpendicular,
$PN = \sqrt{(3-2)^2 + (-1-5)^2 + (11-7)^2} = \sqrt{1+36+16} = \sqrt{53}$

11. Find the length and equation of the shortest distance between the lines $\frac{x-3}{3} = \frac{y-8}{-1} = \frac{z-3}{1}$; $2x - 3y + 27 = 0 = 2y - z + 20$.
(BE 2067 Mangsir)

Sol^n. Here, the equation of the lines are
$\frac{x-3}{3} = \frac{y-8}{-1} = \frac{z-3}{1}$ ..... (i) and
$2x - 3y + 27 = 0, 2y - z + 20 = 0$ ..... (ii)
The equation of any plane through the line (ii) is,
$2x - 3y + 27 + \lambda(2y - z + 20) = 0$ ..... (iii)
or, $2x + (2\lambda - 3)y - \lambda z + 20\lambda + 27 = 0$
It will be parallel to the line (i) if
$2.3 + (2\lambda - 3)(-1) + (-\lambda).1 = 0$ ∵ $\lambda = 3$
Substituting the value of λ in equation (iii), we get, $2x + 3y - 3z + 87 = 0$
S.D. = length of perpendicular drawn from the point (3, 8, 3) of the line (i) on the plane (iv)
$= \frac{6 + 24 - 9 + 87}{\sqrt{4 + 9 + 9}} = \frac{108}{\sqrt{22}}$
∴ The equation of the plane through the line (i) and 1^r to (iv) is,
$$\begin{vmatrix} x-3 & y-8 & z-3 \\ 3 & -1 & 1 \\ 2 & 3 & -3 \end{vmatrix} = 0$$
or, $y + z - 11 = 0$
The equation of a plane through the line (ii) and perpendicular to (iv) is,
$2x - 3y + 27 + k(2y - z + 20) = 0$
or, $2x + (2k - 3)y - kz + 20k + 27 = 0$
It is perpendicular to (iv) if $4 + 6k - 9 + 3k = 0$ ∵ $k = \frac{5}{9}$
Substituting the value of k in (v), we get,
$18x - 17y - 5z + 343 = 0$
∴ The equations of the line of S. D. are
$y + z - 11 = 0, 18x - 17y - 5z = 343 = 0$ Ans.

12. Plane through OX and OY include an angle α. Show that their line of intersection lies on the cone $z^2(x^2 + y^2 + z^2) = x^2 y^2 \tan^2\alpha$.
(BE 2067 Mangsir)

Sol^n. The equation of a plane through OX is,
$by + cz = 0, y + \frac{c}{b}z = 0$ or, $y + n_1 z = 0$ ..... (i)
and the equation of plane through OY is,

$ax + cz = 0$ or, $x + \frac{c}{a}z = 0$ or, $x + n_2z = 0$ ..... (ii)

The direction ratios of the normal to the plane (i) and (ii) are 0, 1, n_1 and 1, 0, n_2 respectively.

If α be the angle between them.

$$\cos\alpha = \frac{0.1+1.0+n_1n_2}{\sqrt{\left(0+1+n_1^2\right)}\sqrt{\left(1+0+n_2^2\right)}} \text{ or, } \cos\alpha = \frac{n_1n_2}{\sqrt{1+n_1^2}\sqrt{1+n_2^2}} \text{ .... (iii)}$$

The intersection of the planes $y + n_1z = 0$ and $x + n_2z = 0$

Subject to the condition (iii), so, eliminating n_1, n_2 from (i) and (ii) and (iii), we get,

$$\cos\alpha = \frac{\left(-\frac{y}{z}\right)\left(-\frac{x}{z}\right)}{\sqrt{\left(1+\frac{y^2}{z^2}\right)}\sqrt{\left(1+\frac{x^2}{z^2}\right)}} = \frac{xy}{\sqrt{y^2+z^2}\sqrt{x^2+z^2}}$$

or, $\cos^2\alpha\,(y^2 + z^2)\,(x^2 + z^2) = x^2y^2$

or, $\cos^2\alpha\,(y^2x^2 + y^2z^2 + z^2x^2 + z^4) = x^2y^2$

or, $x^2y^2\cos^2\alpha + z^2\cos^2\alpha\,(x^2 + y^2 + z^2) = x^2y^2$

or, $z^2\cos^2\alpha\,(x^2 + y^2 + z^2) = x^2y^2\,(1 - \cos^2\alpha)$

or, $z^2\,(x^2 + y^2 + z^2) = x^2y^2\tan^2\alpha$ Ans.

13. Find the equation of the right circular cylinder whose guiding curve is the circle $x^2 + y^2 + z^2 - x - y - z = 0$, $x + y + z = 1$.

(BE 2067 Mangsir)

Sol^n. The equation of the circle is

$x^2 + y^2 + z^2 - x - y - z = 0$, $x + y + z = 1$ .... (i)

Comparing the sphere (i) with

$x^2 + y^2 + 2ux + 2vy + 2wz + d = 0$

we find,

$u = -\frac{1}{2}$, $v = -\frac{1}{2}$, $w = -\frac{1}{2}$

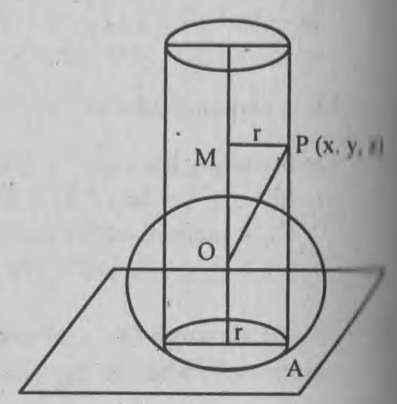
Centre of sphere is $(-u, -v, -w)$

$= \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$

Radius of sphere is

$R = \sqrt{u^2 + v^2 + w^2 - d}$

$= \sqrt{\frac{1}{4}+\frac{1}{4}+\frac{1}{4}-0} = \sqrt{\frac{3}{4}}$



The equation of a line through the point be $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ of the sphere and perpendicular to the plane of the circle (i) is,

$$\frac{x-\frac{1}{2}}{1} = \frac{y-\frac{1}{2}}{1} = \frac{z-\frac{1}{2}}{1}$$

So, the axis of the cylinder is $\frac{x-\frac{1}{2}}{1} = \frac{y-\frac{1}{2}}{1} = \frac{z-\frac{1}{2}}{1}$

$\therefore$ The length of 1^r from the centre $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ of the sphere to the plane $x + y + z = 1$ is, $p = \frac{\frac{1}{2}+\frac{1}{2}+\frac{1}{2}-1}{\sqrt{1^2+1^2+1^2}} = \frac{1}{2\sqrt{3}}$

We have the radius of the circle $\sqrt{R^2 - P^2} = \sqrt{\frac{3}{4} - \frac{1}{1^2}} = \sqrt{\frac{2}{3}}$

The radius of the cylinder $\sqrt{\frac{2}{3}}$

If p(x, y, z) be any point and on the cylinder, then, $OP^2 = OM^2 + PM^2$

or, $\left(x-\frac{1}{2}\right)^2 + \left(y-\frac{1}{2}\right)^2 = \left(z-\frac{1}{2}\right)^2$

$$= \left[\frac{1}{\sqrt{3}}\left(x-\frac{1}{2}\right) + \frac{1}{\sqrt{3}}\left(y-\frac{1}{2}\right) + \frac{1}{\sqrt{3}}\left(z-\frac{1}{2}\right)\right]^2 + \frac{2}{3}$$

or, $x^2 + y^2 + z^2 - yz - zx - xy - 1 = 0$ is the required equation of cylinder.

Solution of Differential Equation in Series and Special Functions

4

Initial value problems

Any differential equations together with an initial condition is called initial value problems.

- If $y = y_1(x)$ and $y = y_2(x)$ be the two solution of second order differential equation then the wronskin determinant is denoted and defined by

$$w(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

- If $y = y_1(x)$, $y = y_2(x)$ ..... $y = y_n(x)$ be the nth solution of nth order differential equation the wronskin determinant is denoted and defined by,

$$w(y_1, y_2, y_3, \ldots\ldots y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ y_1^{n-1} & y_2^{n-1} & & y_n^{n-1} \end{vmatrix}$$

Note

If the wronskin determinant $w(y_1, y_2, \ldots\ldots y_n) \neq 0$ then the solution $y_1, y_2, \ldots\ldots y_n$ are linearly independent otherwise linearly dependent.

Power series method

The second order differential equation is,

$$P_0 \frac{d^2y}{dx^2} + P_1 \frac{dy}{dx} \; P_2 y = 0$$

or, $P_o y'' + P_1 y' + P_2 y = 0$ ..... (i)

Where P_0, P_1, P_2 are function of x and $P_0 \neq 0$

Solving steps of (i) by using power series.

Let us assume that general solution of (i) is

$$y = C_0 + C_1x + C_2x^2 + C_3x^3 + \ldots\ldots + C_nx^n + \ldots\ldots = \sum_{m=0}^{\infty} C_m x^m \quad \ldots\ldots \text{(ii)}$$

Diff. (ii) w. r. t. 'x'

$$y' = C_1 + 2C_2x + 3C_3x^2 + \ldots\ldots + nCnx^{n-1} + \ldots\ldots = \sum_{m=1}^{\infty} C_m x^{m-1}$$

$$y'' = 2C_2 + 6C_3x + \ldots + n(n-1)C_nx^{n-2} + \ldots = \sum_{m=2}^{\infty} m(m-1)C_m x^{m-2}$$

Substituting the value of y, y' & y'' in (i)

and Equating the coefficient of various powers of x with zero we get

C_2, C_3, C_4 ..... interms of C_0 and C_1

Lastly, Substituting the values of C_2, C_3, C_4, ... in equation (ii) we get the required general solution in series form of the differential equation (i).

Legendre's equations

An equation of th form $(1 - x^2)y'' - 2xy' + n(n+1)y = 0$ is known as legendre's differential equation.

Note : $P_n(1) = 1$.

Rodrigve's formula

If $P_n(x)$ is a polynomial function of Legendre's equation of degree n then

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n.$$

Legendre's polynomials

$$P_n(x) = \sum_{r=0}^{N} \frac{(-1)^r (2n-r)!}{2^n r!(n-r)!(n-2r)!} x^{n-2r}$$

Note : $P_0(x) = 1$

$P_1(x) = x;$

$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$$

$$P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$$

$$P_6(x) = \frac{1}{6}(231x^6 - 351x^4 + 105x^2 - 5)$$

Bessel's equation

An equation of the form $x^2y'' + xy' + (x^2 - n^2)y = 0$ is known as Bessel's didffetrential equation.

1. $J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(n+m+1)} \left(\frac{x}{2}\right)^{n+2m}$ is called the Bessel's function of the first kind.

2. $J_{-n} = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m-n+1)} \left(\frac{x}{2}\right)^{-n+2m}$ is called the Bessel's function of the second kind.

Note:

1. $J_{n+1}(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(n+1+m+1)} \left(\frac{x}{2}\right)^{(n+1)+2m}$

2. $J_n(x) = \frac{x^n}{2^n \Gamma n+1} \left[1 - \frac{x^2}{4.1!(n+1)} + \frac{x^4}{4^2.2!(n+1)(n+2)} \cdots + \frac{(-1)^m x^{2m}}{4^m m!(n+1)(n+2)\ldots(n+m)}\right]$

3. $J_{-n}(x) = (-1)^n J_n(x)$

Recurrence relations for Bessel's functions

1. $x J_n'(x) = n J_n(x) - x J_{n+1}(x)$
2. $x J_n'(x) = -n J_n(x) + x J_{n-1}(x)$
3. $2J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$
4. $2n J_n(x) = x[J_{n-1}(x) + J_{n+1}(x)]$
5. $\frac{d}{dx}[x^{-n} J_n(x)] = -x^{-n} J_{n+1}(x)$
6. $\frac{d}{dx}[x^n J_n(x)] = x^n J_{n-1}(x)$

Exercise - 16

Solve the following initial value problems

1. $y'' + y' - 2y = 0,\ y(0) = 4,\ y'(0) = -5$ **[BE. 2065]**

Solution: Here the differential equation is,

$$y'' + y' - 2y = 0$$
$$\text{or, } (D^2 + D - 2)y = 0$$

This is linear differential equation of second order with constant coefficient. So its auxiliary equation is,

$$m^2 + m - 2 = 0$$
$$\text{or, } m^2 + 2m - m - 2 = 0$$
$$\text{or, } m(m+2) - 1(m+2) = 0$$
$$\text{or, } (m+2)(m-1) = 0$$

Either $(m+2) = 0$ or, $m - 1 = 0$

$\therefore m = -2$ $\therefore m = 1$

Thus $m = 1, -2$

So its general solution is

$$y = C_1 e^x + C_2 e^{-2x} \text{ ..... (i)}$$

Using the intial condition $y(0) = 4$, in (i)

$$4 = C_1 + C_2$$
$$\therefore C_1 + C_2 = 4 \text{ ..... (ii)}$$

Differentiating (i) with respect to x, we get

$$y' = C_1 e^x - 2C_2 e^{-2x} \text{ ..... (iii)}$$

Using the initial condition $y'(0) = -5$ in (iii) we get,

$$-5 = C_1 + 2C_2$$
$$\therefore C_1 + 2C_2 = -5 \text{ ..... (iv)}$$

Solving (ii) and (iv) we get,

$$C_1 = 1 \text{ and } C_2 = 3$$

Putting the value of C_1 and C_2 in (i) we get

$$y = e^x + 3e^{-2x} \text{ is the required solution.}$$

2. $y'' - y = 0,\ y(0) = 5,\ y'(0) = -3$

Solution: The differential equation is

Here, $y'' - y = 0$

$$\text{or, } (D^2 - 1)y = 0$$

This is linear differential equation of second order with constant coefficient. So its auxiliary equation is,

$$m^2 - 1 = 0$$
$$\therefore m = 1, -1$$

So its general solution is

$$y = C_1 e^x + C_2 e^{-x} \text{ ..... (i)}$$

Using the initial condition $y(0) = 5$ in (i) we get

$5 = C_1 + C_2$

$\therefore C_1 + C_2 = 5$ ..... (ii)

Differentiating (i) with respect to x we get,

$y' = C_1 e^x - C_2 e^{-x}$ ..... (iii)

Using the initial condition $y'(0) = -3$ in (iii) we get

$-3 = C_1 - C_2$

$\therefore C_1 - C_2 = -3$ ..... (iv)

Solving (ii) and (iv) we get,

$C_1 = 1$ and $C_2 = 4$

Putting the value of C_1 and C_2 in (i) we get

$y = e^x + 4e^{-x}$ is the required solution.

3. $y'' + 2y' + 5y = 0, y(0) = 1, y'(0) = 5$ [B.E. 2064. 2069 Poush]

Solution: Here, The differential equation is $y'' + 2y' + 5y = 0$

or, $(D^2 + 2D + 5)y = 0$

This is linear differential equation of second order with constant coefficient. So its auxiliary equation is,

$m^2 + 2m + 5 = 0$

$$\therefore m = \frac{-2 \pm \sqrt{(2)^2 - 4.1.5}}{2.1}$$

$$= \frac{-2 \pm \sqrt{-16}}{2}$$

$$= \frac{-2 \pm 4i}{2}$$

$= -1 \pm 2i$

So its general solution is

$y = e^{-x}(A \cos 2x + B \sin 2x)$ ..... (i)

Using the initial condition $y(0) = 1$ in (i)

$1 = A \quad \Rightarrow A = 1$ ..... (ii)

Differentiating (i) with respect to x we get

$y' = -2Ae^{-x} \sin 2x - Ae^{-x} \cos 2x + 2Be^{-x} \cos 2x - Be^{-x} \sin 2x$ ..... (iii)

Using the initial condition $y'(0) = 5$ in (iii) we get,

$5 = -A + 2B$

$\therefore -A + 2B = 5$ ..... (iv)

Solving (ii) and (iv) we get,

$A = 1$, and $B = 3$

Substituting the value of A and B in (i) we get

$y = e^{-x}(\cos 2x + 3 \sin 2x)$ is required solution.

4. $(D^2 + 4D + 5)y = 0, y(0) = 1, y'(0) = -3$

Solution: Here, The differential equation is $(D^2 + 4D + 5)y = 0$

This is linear differential equation of second order with constant coefficient. So its auxiliary equation is,

$m^2 + 4m + 5 = 0$

$$\therefore m = \frac{-4 \pm \sqrt{16 - 4.5}}{2.1}$$

$$= \frac{-4 \pm 2i}{2}$$

$= -2 \pm i$

So its general solution is

$y = e^{-2x}(A \cos x + B \sin x)$ ..... (i)

Using the initial condition $y(0) = 1$ in (i)

$1 = A \quad \Rightarrow A = 1$ ..... (ii)

Differentiating (i) with respect to x we get

$y' = e^{-2x}(-A \sin x + B \cos x) - 2(A \cos x + B \sin x)e^{-2x}$ .... (ii)

Using the initial condition $y'(0) = -3$ in (iii)

$-3 = B - 2A$

From (ii)

$-3 = B - 2.1$ or, $B = -1$

Substituting the value of A and B in (i)

$y = e^{-2x}(\cos x - \sin x)$ is the required solution.

$y'' - 4y' + 3y = 10e^{-2x}, y(0) = 1, y'(0) = 3$ (BE 2068 Bhadra)

Solution: Here, The differential equation is $y'' - 4y' + 3y = 10e^{-2x}$

or, $(D^2 - 4D + 3)y = 10e^{-2x}$

This is linear differential equation of second order with constant coefficient. So its auxiliary equation is,

$m^2 - 4m + 3 = 0$

$m^2 - 3m - m + 3 = 0$

or, $m(m - 3) - 1(m - 3) = 0$

or, $(m - 3)(m - 1) = 0$

Either $m - 3 = 0 \quad \Rightarrow m = 3$

or, $m - 1 = 0 \quad \Rightarrow m = 1$

$\therefore m = 1, 3$

$\therefore$ C. F. $= C_1 e^x + C_2 e^{3x}$

$$\text{and P. I.} = \frac{1}{(D^2 - 4D + 3)} . 10e^{-2x}$$

$$= \frac{1}{(-2)^2 - 4.(-2) + 3} 10e^{-2x}$$

$$= \frac{1}{4 + 8 + 3} 10e^{-2x}$$

$$= \frac{10}{15} e^{-2x}$$

$$= \frac{2}{3} e^{-2x}$$

$\therefore$ The general solution is

$y = C.F. + P.I$

$y = C_1e^x + C_2 e^{3x} + \frac{2}{3} e^{-2x}$ ..... (i)

Using initial condition y(0) = 1 in (i)

$1 = C_1 + C_2 + \frac{2}{3}$

or, $3C_1 + 3C_2 = 1$ ..... (ii)

Differentiating (i) with respect to x we get

$y' = C_1e^x + 3C_2 e^{3x} - \frac{2}{3} . 2 e^{-2x}$ ..... (iii)

Using the initial condition y' (0) = 3 in (iii)

$3 = C_1 + 3C_2 - \frac{4}{3}$

or, $3C_1 + 9C_2 = 13$ ..... (iv)

Solving (iii) and (iv) we get

$C_1 = \frac{5}{3}$ and $C_2 = 2$

Substituting the value of C_1 and C_2 in (i)

$y = \frac{5}{3} e^x + 2 e^{3x} + \frac{2}{3} e^{-2x}$ is the required solution.

6. $y'' - 3y' + 2y = 4x + e^{3x}$, $y(0) = 1$, $y'(0) = -1$

Solution: Here, the differential equation is $y'' - 3y' + 2y = 4x + e^{3x}$

or, $(D^2 - 3D + 2) y = 4x + e^{3x}$

This is linear differential equation of second order with constant coefficient. So its auxiliary equation is,

$m^2 - 3m + 2 = 0$

or, $m^2 - 2m - m + 2 = 0$

or, $m(m-2) - 1(m-2) = 0$

or, $(m-2)(m-1) = 0$

Either $m - 2 = 0 \Rightarrow m = 2$

or, $m - 1 = 0 \Rightarrow m = 1$

$\therefore m = 1, 2$

$\therefore C.F. = C_1 e^x + C_2 e^{2x}$

and $P.I. = \frac{1}{(D^2 - 3D + 2)} (4x + e^{3x})$

$= \frac{4}{2} \left(1 + \frac{D^2 - 3D}{2}\right)^{-1} x + \frac{1}{(D^2 - 3D + 2)} e^{3x}$

$= 2 \left\{1 - \left(\frac{D^2 - 3D}{2}\right) + \left(\frac{D^2 - 3D}{2}\right)^2 - .....\right\} x + \frac{1}{9 - 9 + 2} e^{3x}$

$= 2\left(x + \frac{3}{2}\right) + \frac{1}{2} e^{3x}$

$= 2x + 3 + \frac{1}{2} e^{3x}$

The general solution is

$y = C.F. + P.I.$

or, $y = C_1 e^x + C_2 e^{2x} + 2x + 3 + \frac{1}{2} e^{3x}$ ..... (i)

Using the initial condition y(0) = 1 in (i)

$1 = c_1 + c_2 + 3 + \frac{1}{2}$

$\therefore 2C_1 + 2C_2 + 6 + 1 = 0$

or, $2C_1 + 2C_2 = -5$ ..... (ii)

Differentiating (i) with respect to x we get

$y' = C_1e^x + 2C_2e^{2x} + 2 + \frac{3}{2} e^{3x}$ ..... (iii)

Using the initial condition y'(0) = – 1 in (iii)

or, $-1 = C_1 + 2C_2 + 2 + \frac{3}{2}$

or, $2C_1 + 4C_2 + 4 + 3 + 2 = 0$

or, $2C_1 + 4C_2 + 9 = 0$

or, $2C_1 + 4C_2 = -9$ ..... (iv)

Solving (ii) and (iii) we get,

$C_1 = -\frac{1}{2}$ and $C_2 = -2$

Substituting the value of C_1 and C_2 in (i)

$y = -\frac{1}{2} e^x - 2e^{2x} + 2x + 3 + \frac{1}{2} e^{3x}$

$y = 2x + 3 + \frac{1}{2} (e^{3x} - e^x) - 2e^{2x}$ is the required solution.

$\frac{d^2x}{dt^2} + \mu x = 0$, $\mu > 0$, given that $x = 1$ and $\frac{dx}{dt} = 0$ when $t = \frac{\pi}{2\sqrt{\mu}}$

Solution: Here, the differential equation is $\frac{d^2x}{dt^2} + \mu x = 0$

or, $(D^2 + \mu) x = 0$ where, $D = \frac{d}{dt}$

This is linear differential equation of second order with constant coefficient. So its auxiliary equation is,

$m^2 + \mu = 0$

$\Rightarrow \quad m = \pm i\sqrt{\mu}$

$\therefore$ The general solution is

$$x = C_1 \cos\sqrt{\mu} + C_2 \sin\sqrt{\mu}\, t \text{ ..... (i)}$$

Using the initial condition, $x = 1$ when $t = \dfrac{\pi}{2\sqrt{\mu}}$

$$1 = C_1 \cos\left\{\sqrt{\mu} \cdot \frac{\pi}{2\sqrt{\mu}}\right\} + C_2 \sin\left\{\sqrt{\mu} \cdot \frac{\pi}{2\sqrt{\mu}}\right\}$$

or, $1 = C_1 0 + C_2 . 1$

$\Rightarrow C_2 = 1$ ..... (ii)

Differentiating (i) with respect to x we get

$$\frac{dx}{dt} = -C_1\sqrt{\mu} \sin\sqrt{\mu}\, t + C_2\sqrt{\mu} \cos\sqrt{\mu}\, t \text{ ..... (iii)}$$

Using the initial condition $\dfrac{dx}{dt} = 0$ when $t = \dfrac{\pi}{2\sqrt{\mu}}$

$$0 = -C_1\sqrt{\mu} \sin\left\{\sqrt{\mu} \cdot \frac{\pi}{2\sqrt{\mu}}\right\} + C_2\sqrt{\mu} \cos\left\{\sqrt{\mu} \cdot \frac{\pi}{2\sqrt{\mu}}\right\}$$

$0 = -C_1\sqrt{\mu} + C_2\sqrt{\mu} . 0$

or, $-C_1\sqrt{\mu} = 0$

Since, $\mu > 0$

$\Rightarrow C_1 = 0$

Substituting the value of C_1 and C_2 in (i) we get,

$x = \sin\sqrt{\mu}\, t$ is the required solution.

8. $(D^2 - 4D + 5)\, y = 0$ given that $y = 1$, and $\dfrac{dy}{dx} = 2$ when $x = 0$

Solution: Here, the differential equation is $(D^2 - 4D + 5)y = 0$

This is linear differential equation of second order with constant coefficient. So its auxiliary equation is,

$$m^2 - 4m + 5 = 0$$

$$\therefore m = \frac{-(-4) \pm \sqrt{(-4)^2 - 4.1.5}}{2.1}$$

$$= \frac{4 \pm \sqrt{16 - 20}}{2}$$

$$= \frac{4 \pm 2i}{2}$$

$$= 2 \pm i$$

$\therefore$ The general solution is,

$$y = e^{2x}(C_1 \cos x + C_2 \sin x) \text{ ..... (i)}$$

Using the initial condition $y = 1$ when $x = 0$ in (i)

$1 = C_1 \quad \Rightarrow C_1 = 0$ ..... (ii)

Differentiating (i) with respect to x we get

$$\frac{dy}{dx} = e^{2x}(-C_1 \sin x + C_2 \cos x) + 2(C_1 \cos x + C_2 \sin x)\, e^{2x} \text{ .... (iii)}$$

Using the initial condition $\dfrac{dy}{dx} = 2$ when $x = 0$ in (iii) we get

$$3 = C_2 + 3C_1$$

From (i) $C_1 = 0$

$\therefore C_2 = 2$

Substituting the value of C_1 and C_2 in (i)

$y = 2e^{2x} \sin x$ is the required solution.

Show that the solution of $y''' + 3y'' + 3y' + y = 0$, $x > 0$ are linearly independent.

Solution: Here the differential equation is

$y''' + 3y'' + 3y' + y = 0, x > 0$

or, $(D^3 + 3D^2 + 3D + 1)\, y = 0$

This is linear differential equation of third order with constant coefficient. So its auxiliary equation is,

$$m^3 + 3m^2 + 3m + 1 = 0$$

or, $m^3 + m^2 + 2m^2 + 2m + m + 1 = 0$

or, $m^2(m + 1) + 2m(m + 1) + 1(m + 1) = 0$

or, $(m + 1)(m^2 + 2m + 1) = 0$

or, $(m + 1) . (m + 1)^2 = 0$

$\therefore m = -1, -1, -1$

$\therefore$ The general solution is

$$y = (C_1 + x C_2 + x^2 C_3)\, e^{-x}$$

$$= C_1 e^{x} + C_2 x e^{-x} + C_3 x^2 e^{-x}$$

so $y_1 = e^{-x}$, $y_2 = x e^{-x}$ and $y_3 = x^2 e^{-x}$

$\therefore f_1' = -e^{-x}, \quad f_2' - x e^{-x} + e^{-x}$

$y_1'' = e^{-x}, \quad y_2'' = x e^{-x} - e^{-x} - e^{-x}$

$= x e^{-x} - 2e^{-x}$

and $y_3' = -x^2 e^{-x} + 2x e^{-x}$

$y_3'' = x^2 e^{-x} - 2xe^{-x} - 2xe^{-x} + 2e^{-x}$

$= x^2 e^{-x} - 4xe^{-x} + 2e^{-x}$

Now,

$$W(y_1, y_2, y_3) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

$$= \begin{vmatrix} e^{-x} & xe^{-x} & x^2e^{-x} \\ -e^{-x} & -xe^{-x}+e^{-x} & -x^2e^{-x}+2xe^{-x} \\ e^{-x} & xe^{-x}-2e^{-x} & x^2e^{-x}-4xe^{-x}+2e^{-x} \end{vmatrix}$$

Taking common e^{-x}, from C_1, C_2 and C_3 resp.

$$= e^{-x}.e^{-x}.e^{-x} \begin{vmatrix} 1 & x & x^2 \\ -1 & -x+1 & -x^2+2x \\ 1 & x-2 & x^2-4x+2 \end{vmatrix}$$

Applying $R_1 \to R_1 + R_2$ and $R_3 \to R_3 + R_2$

$$= e^{-3x} \begin{vmatrix} 0 & 1 & 2x \\ -1 & -x+1 & -x^2+2x \\ 0 & -1 & -2x+2 \end{vmatrix}$$

Expanding along C_1

$$= e^{-3x}\{-(-1)\} \begin{vmatrix} 1 & 2x \\ -1 & -2x+2 \end{vmatrix}$$

$= e^{-3x}(-2x+2+2x)$

$= 2e^{-3x} \neq 0$

Hence, e^{-x}, xe^{-x}, x^2e^{-x} are linearly independent proved.

10. Show that the solution of $(D^4 + 4D^3 + 6D^2 + 4D + 1)\,y = 0$, $x > 0$ are lineraly independent.

Solution: Here, the differential equation is

$(D^4 + 4D^3 + 6D^2 + 4D + 1)\,y = 0$

This is linear differential equation of 4th order with constant coefficient. So its auxiliary equation is,

$m^4 + 4m^3 + 6m^2 + 4m + 1 = 0$

or, $m^4 + 1 + 4m^3 + 4m + 6m^2 = 0$

or, $(m^2)^2 + (1)^2 + 4m(m^2+1) + 6m^2 = 0$

or, $(m^2+1)^2 - 2m^2 + 4m(m^2+1) + 6m^2 = 0$

or, $(m^2+1)^2 + 4m(m^2+1) + 4m^2 = 0$

or, $(m^2+1)^2 + 2(m^2+1).2m + (2m)^2 = 0$

or, $(m^2+1+2m)^2 = 0$

or, $(m^2+2m+1)^2 = 0$

or, $\{(m+1)^2\}^2 = 0$

or, $(m+1)^4 = 0$

$\therefore m = -1, -1, -1, -1$

$\therefore$ The general solution is,

$y = (c_1 + xc_2 + x^2c_3 + x^3c_4)\,e^{-x}$

or, $y = c_1e^{-x} + c_2xe^{-x} + c_3x^2e^{-x} + c_4x^3e^{-x}$

so, $y_1 = e^{-x}$

$y_2 = xe^{-x}$

$y_3 = x^2e^{-x}$

& $y_4 = x^3e^{-x}$

$\therefore y_1' = -e^{-x}$

$y_1' = e^{-x}$

& $y_1'' = -e^{-x}$

$y_2' = -xe^{-x} + e^{-x}$

$y_2'' = xe^{-x} - e^{-x} - e^{-x}$

$= xe^{-x} - 2e^{-x}$

$\therefore y_3' = -x^2e^{-x} + 2xe^{-x}$

$\therefore y_3'' = x^3e^{-x} - 2xe^{-x} - 2xe^{-x} + 2e^{-x}$

$= x^2e^{-x} - 4xe^{-x} + 2e^{-x}$

$y_2''' = -xe^{-x} + e^{-x} + 2e^{-x}$

$y_3''' = -x^2e^{-x} + 2xe^{-x} + 4xe^{-x} - 4e^{-x} - 2e^{-x}$

and

$y_4' = -x^3e^{-x} + 3x^2e^{-x}$

$y_4'' = x^3e^{-x} - 3x^2e^{-x} - 3x^2e^{-x} + 6xe^{-x}$

$= x^3e^{-x} - 6x^3e^{-x} + 6xe^{-x}$

$y_4''' = -x^3e^{-x} + 3x^2e^{-x} + 6x^2e^{-x} - 12x\,e^{-x} - 6xe^{-x} + 6e^{-x}$

$y_4''' = -x^3e^{-x} + 9x^2e^{-x} - 18xe^{-x} + 6e^{-x}$

Now,

$$W(y_1, y_2, y_3, y_4) = \begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ y_1' & y_2' & y_3' & y_4' \\ y_1'' & y_2'' & y_3'' & y_4'' \\ y_1''' & y_2''' & y_3''' & y_4''' \end{vmatrix}$$

$$= \begin{vmatrix} e^{-x} & xe^{-x} & x^2e^{-x} & x^3e^{-x} \\ -e^{-x} & -xe^{-x}+e^{-x} & -x^2e^{-x}+2xe^{-x} & -x^3e^{-x}+3x^2e^{-x} \\ e^{-x} & xe^{-x}-2e^{-x} & x^2e^{-x}-4xe^{-x}+2e^{-x} & x^3e^{-x}-6x^2e^{-x}+6xe^{-x} \\ -e^{-x} & -xe^{-x}+3e^{-x} & -x^2e^{-x}+6xe^{-x}+6e^{-x} & -x^3e^{-x}+9x^2e^{-x}-18xe^{-x}+6e^{-x} \end{vmatrix}$$

Taking common from e^{-x} from R_1, R_2, R_3 & R_4 resp.

$$= e^{-x}.e^{-x}.e^{-x}.e^{-x} \begin{vmatrix} 1 & x & x^2 & x^3 \\ -1 & -x+1 & -x^2+2x & -x^3+3x^2 \\ 1 & x-2 & x^2-4x+2 & x^3-6x^2+6x \\ -1 & -x+3 & -x^2+6x+6 & -x^3+9x^2-18x+6 \end{vmatrix}$$

Applying $R_2 \to R_2 + R_1$, $R_3 \to R_3 - R_1$, $R_4 \to R_4 + R_1$.

$$= e^{-4x} \begin{vmatrix} 1 & x & x^2 & x^3 \\ 0 & 1 & 2x & 3x^2 \\ 0 & -2 & -4x+2 & -6x^2+6x \\ 0 & 3 & 6x+6 & 9x^2-18x+6 \end{vmatrix}$$

Expanding along C_1 :

$$= e^{-4x} \begin{vmatrix} 1 & 2x & 3x^2 \\ -2 & -4x+2 & -6x^2+6x \\ 3 & 6x+6 & 9x^2-18x+6 \end{vmatrix}$$

Applying $R_2 \to R_2 + 2R_1$ and $R_3 \to R_3 - 3R_1$

$$= e^{-4x}\begin{vmatrix} 1 & 2x & 3x^2 \\ 0 & 2 & 6x \\ 0 & 6 & -18x+6 \end{vmatrix}$$

Expanding along C_1

$$= e^{-4x}\begin{vmatrix} 2 & 6x \\ 6 & -18x+6 \end{vmatrix}$$

$= e^{-4x}(-36x + 12 - 36x)$

$= e^{-4x}(-72x + 12)$

$= -72xe^{-4x} + 12e^{-4x} \neq 0$

Hence e^{-x}, xe^{-x}, x^2e^{-x}, x^3e^{-x} are linearly independent.

11. Show that the solution of $(D^3 + D^2 + D + 1)\, y = 0$ $x > 0$ are linarly independent.

Solution: Here, the differential equation is $(D^3 + D^2 + D + 1)\, y = 0$

This is linear differential equation of 3th order with constant coefficient. So its auxiliary equation is,

$m^3 + m^2 + m + 1 = 0$

or, $m^3 + 1 + m^2 + m = 0$

or, $(m + 1)(m^2 - m + 1) + m(m + 1) = 0$

or, $(m + 1)(m^2 - m + 1 + m) = 0$

or, $(m + 1)(m^2 + 1) = 0$

$\therefore\ m = -1, m = \pm i$

$\therefore$ The general solution is,

$y = C_1e^{-x} + C_2 \cos x + C_3 \sin x$

So, $y_1 = e^{-x}$, $y_2 = \cos x$ $y_3 = \sin x$

Now,

$$W(y_1, y_2, y_3) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

$$= \begin{vmatrix} e^{-x} & \cos x & \sin x \\ -e^{-x} & -\sin x & \cos x \\ e^{-x} & -\cos x & -\sin x \end{vmatrix}$$

Taking common e^{-x} from C_1

$$= e^{-x}\begin{vmatrix} 1 & \cos x & \sin x \\ -1 & -\sin x & \cos x \\ 1 & -\cos x & -\sin x \end{vmatrix}$$

Applying $R_2 \to R_2 + R_1$ and $R_3 \to R_3 - R_1$

$$= e^{-x}\begin{vmatrix} 1 & \cos x & \sin x \\ -0 & -\sin x + \cos x & \cos x + \cos x \\ 0 & -\cos x - \cos x & -\sin x - \sin x \end{vmatrix}$$

Expanding along C_1

$$= e^{-x}\begin{vmatrix} -\sin x + \cos x & \sin x + \cos x \\ -2\cos x & -2\sin x \end{vmatrix}$$

$= e^{-x}(2\sin^2 x - 2\sin x \cos x + 2\sin x \cos x + 2\cos^2 x)$

$= 2e^{-x}(\sin^2 x + \cos^2 x)$

$= 2e^{-x} \neq 0$

Hence, e^{-x}, $\cos x$, $\sin x$ are linearly independent.

12. Test whether the solution of $(D^2 - 3D + 2)\, y = 0$ are linearly dependent or independent.

Solution: Here, $(D^2 - 3D + 2)y = 0$

This is linear differential equation of 2nd order with constant coefficient. So its auxiliary equation is,

$m^2 - 3m + 2 = 0$

or, $m^2 - 2m - m + 2 = 0$

or, $m(m - 2) - 1(m - 2) = 0$

or, $(m - 1)(m - 2) = 0$

$\therefore\ m = 1, 2$

$\therefore$ The general solution is

$y = C_1 e^x + C_2 e^{2x}$

So, $y_1 = e^x$, $y_2 = e^{2x}$

$$\text{Now, } w(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= \begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{vmatrix}$$

Taking common e^x & e^{2x} from C_1 and C_2 resp.

$$= e^x \cdot e^{2x}\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}$$

$= e^{3x}(2 - 1)$

$= e^{3x} \neq 0$

Hence, e^x & e^{2x} are linearly independent.

13. Verify that $D^4y = 0$, $x > 0$ have linearly independent solution.

Solution: Here, $D^4y = 0$

This is linear differential equation of 4th order with constant coefficient. So its auxiliary equation is,

$m^4 = 0$

$\therefore\ m = 0, 0, 0, 0$

$\therefore$ The general solution is, $y = c_1 + c_2 + c_3x^2 + c_4x^3$

So the solution of equations are,

$y_1 = 1, y_2 = x, y_3 = x^2, y_4 = x^3$

$$\text{Now, } \omega(y_1, y_2, y_3, y_4) = \begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ y_1' & y_2' & y_3' & y_4' \\ y_1'' & y_2'' & y_3'' & y_4'' \\ y_1''' & y_2''' & y_3''' & y_4''' \end{vmatrix}$$

$$= \begin{vmatrix} 1 & x & x^2 & x^3 \\ 0 & 1 & 2x & 3x^2 \\ 0 & 0 & 2 & 3x \\ 0 & 0 & 0 & 3 \end{vmatrix}$$

Expanding along C_1

$$= \begin{vmatrix} 1 & 2x & 3x^2 \\ 0 & 2 & 3x \\ 0 & 0 & 3 \end{vmatrix}$$

Expanding along C_1

$$= \begin{vmatrix} 2 & 3x \\ 0 & 3 \end{vmatrix}$$

$$= 6 \neq 0$$

Hence $1, x, x^2, x^3$ are linearly independent.

14. Verify that $D^3y = 0, x > 0$ have linearly independent solution.

Solution: Here, $D^3y = 0$

This is linear differential equation of 3th order with constant coefficient. So its auxiliary equation is,

$$m^3 = 0$$

$$\therefore m = 0, 0, 0$$

$\therefore$ The general solution is

$$y = C_1 + xC_2 + x^2C_3$$

So the solutions of equations are,

$$y_1 = 1, y_2 = x, y_3 = x^2$$

$$\text{Now, } \omega(y_1, y_2, y_3) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

$$= \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 0 & 2 \end{vmatrix}$$

Expanding along C_1

$$= \begin{vmatrix} 1 & 2x \\ 0 & 2 \end{vmatrix}$$

$$= 2 \neq 0$$

Hence, $1, x, x^2$ are linearly independent.

15. Show that the solutions of $(D^4 + 5D^2 + 4)\, y = 0, x > 0$ are linearly independent.

Solution: Here, $(D^4 + 5D^2 + 4)y = 0, x > 0$

This is linear differential equation of 4th order with constant coefficient. So its auxiliary equation is,

$$m^4 + 5m^2 + 4 = 0$$

$$\text{or, } m^2(m^2 + 4) + 1(m^2 + 4) = 0$$

$$\text{or, } (m^2 + 1)(m^2 + 4) = 0$$

Either $m^2 + 1 = 0 \quad \Rightarrow m = \pm i$

or, $m^2 + 4 = 0 \quad \Rightarrow m = \pm 2i$

$\therefore m = \pm i, 2i$

$\therefore$ The general solution is,

$$y = C_1 \cos x + C_2 \sin x + C_3 \cos 2x + C_4 \sin 2x$$

So the solution of the equation are,

$$y_1 = \cos x, \qquad y_2 = \sin x$$

$$y_3 = \cos 2x \qquad y_4 = \sin 2x$$

$$\text{Now, } \omega(y_1, y_2, y_3, y_4) = \begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ y_1' & y_2' & y_3' & y_4' \\ y_1'' & y_2'' & y_3'' & y_4'' \\ y_1''' & y_2''' & y_3''' & y_4''' \end{vmatrix}$$

$$= \begin{vmatrix} \cos x & \sin x & \cos 2x & \sin 2x \\ -\sin x & \cos x & -2\sin 2x & 2\cos 2x \\ -\cos x & -\sin x & -2\cos 2x & -4\sin 2x \\ \sin x & -\cos x & \sin 2x & -8\cos 2x \end{vmatrix}$$

Applying $R_3 \to R_3 + R_1$ and $R_4 \to R_4 + R_2$

$$= \begin{vmatrix} \cos x & \sin x & \cos 2x & \sin 2x \\ -\sin x & \cos x & -2\sin 2x & 2\cos 2x \\ 0 & 0 & -\cos 2x & -2\sin 2x \\ 0 & 0 & 6\sin 2x & -6\cos 2x \end{vmatrix}$$

Expanding along C_1

$$= \cos x \begin{vmatrix} \cos x & -2\sin 2x & 2\cos 2x \\ 0 & -2\cos 2x & -2\sin 2x \\ 0 & 6\sin 2x & -6\cos 2x \end{vmatrix}$$

$$+ \sin x \begin{vmatrix} \sin x & \cos 2x & \sin 2x \\ 0 & -2\cos 2x & -2\sin 2x \\ 0 & 6\sin 2x & -6\cos 2x \end{vmatrix}$$

Expanding along C_1 on both determinant,

$$= \cos^2 x \begin{vmatrix} -2\cos 2x & -2\sin 2x \\ 6\sin 2x & -6\cos 2x \end{vmatrix} + \sin^2 x \begin{vmatrix} -2\cos 2x & -2\sin x 2x \\ 6\sin 2x & -6\cos 2x \end{vmatrix}$$

Taking common – 2 and – 6 from R_1 and R_2 on both determinant

$$= 12\cos^2 x \begin{vmatrix} \cos 2x & \sin 2x \\ -\sin 2x & \cos 2x \end{vmatrix} + 12\sin^2 x \begin{vmatrix} \cos 2x & \sin 2x \\ -\sin 2x & \cos 2x \end{vmatrix}$$

$$= 12 \begin{vmatrix} \cos 2x & \sin 2x \\ -\sin 2x & \cos 2x \end{vmatrix} (\cos^2 x + \sin^2 x)$$

$= 12\ \{(\cos 2x)^2 + (\sin 2x)^2\}\ .\ 1$

$= 12\ .\ 1$

$= 12 \neq 0$

Hence, cos x, sin x, cos 2x, sin 2x are linearly independent.

16. Show that $y_1 = \cos x$, $y_2 = \sin x$ are linearly independent solution of $y'' + y = 0$

Solution: Here, $y'' + y = 0$

or, $(D^2 + 1)\, y = 0$

This is linear differential equation of 2th order with constant coefficient. So its auxiliary equation is,

$m^2 + 1 = 0$

or, $m = \pm i$

$\therefore$ The general solution is,

$y = C_1 \cos x + C_2 \sin x$

So the solution of the equations are,

$y_1 = \cos x,\ y_2 = \sin x$

Now, $\omega(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

$= \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}$

$= \cos^2 x + \sin^2 x$

$= 1 \neq 0$

Hence, the solution $y_1 = \cos x$ and $y_2 = \sin x$ are independent solution of the differential equation $y'' + y = 0$

Exercise - 17

Solve the following Differential equation by the power series method.

1. $y' - y = 0$

Solution: Here, the given differential equation is,

$y' - y = 0$ ..... (i)

Assume that the general solution of (i) is

$y = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + C_4 x^4 +$ ..... (ii)

Where, $c_0, c_1, c_2,$ ..... are constant

Differential (ii) w. r. t. x we get

$y' = e_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 +$ .....

Putting the value of y and y' to the equation (i) we get

$(c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + ...) - (c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^3 + ...) = 0$

or, $(c_1 - c_0) + (2c_2 - c_1)\, x + (3c_3 - c_2)\, x^2 + (4c_4 - c_3)\, x^3 + ... = 0$

Equating to zero the coefficient by various powers of x.

$c_1 - c_0 = 0 \quad \Rightarrow c_1 = c_0$

$2c_2 - c_1 = 0 \quad \Rightarrow c_2 = \frac{c_1}{2} = \frac{c_0}{2}$

$3c_3 - c_2 = 0 \quad \Rightarrow c_3 = \frac{c_2}{3} = \frac{c_0}{3.2} = \frac{c_0}{6}$

$4c_4 - c_3 = 0 \quad \Rightarrow c_4 = \frac{c_3}{4} = \frac{c_0}{6.4} = \frac{c_0}{24}$ and so on

Putting these in equation (ii) we get,

$$y = c_0 + cox + \frac{c_0}{2} x^2 + \frac{c_0}{6} x^3 + \frac{c_0}{24} x^4 .....$$

$$= c_0 \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + ...\right)$$

$\therefore y = c_0 e^x$ is the required solution.

2. $y' + y = 0$

Solution: Here, the given differential equation is

$y' + y = 0$ ..... (i)

Assume that the general solution of (i) is

$y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 +$ ..... (ii)

Where, $c_0, c_1, c_2,$ ..... are constants.

Differential (ii) w. r. t. x we get,

$y' = c_1 + 2c_2 + 3c_3 x^2 + 4c_4 x^3 +$ .....$= 0$

Putting the value of y and y' to the equation (i) we get

$(c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + ...) + (c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^3 + ...) = 0$

or, $(c_1 + c_0) + (2c_2 + c_1)x + (3c_3 + c_20\, x^2 + (4c_4 + c_3)x^3 + \ldots\ldots$

Equating to zero the various power of x

$$c_1 + c_0 = 0 \qquad \Rightarrow c_1 = -c_0$$

$$2c_2 + c_1 = 0 \qquad \Rightarrow c_2 = -\frac{c_1}{2} = +\frac{c_0}{2}$$

$$3c_3 + c_2 = 0 \qquad \Rightarrow c_3 = -\frac{c_2}{3} = -\frac{c_0}{2.3} = -\frac{c_0}{6}$$

$$4c_4 + c_3 = 0 \qquad \Rightarrow c_4 = -\frac{c_3}{4} = \frac{c_0}{6.4} = \frac{c_0}{24} \text{ and so on,}$$

Putting these value in equation (i) we get,

$$y = c_0 - c_0x + \frac{c_0}{2}x^2 - \frac{c_0}{6}x^3 + \frac{c_0x^4}{24} + \ldots\ldots$$

$$= c_0\left(1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots\ldots\right)$$

$\therefore\ y = c_0\, e^{-x}$ is the required solution.

3. $\quad (x + 1)\, y' + y = 0$

Solution: Here, the given differential equation is

$$(x + 1)\, y' + y = 0 \ \ldots\ldots\ldots \text{(i)}$$

Assume that the general solution of (i) is

$$y = \sum_{m=0}^{\infty} c_m x^m = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + \ldots\ldots\ c_nx^n + \ldots\ldots\text{(ii)}$$

Differential (ii) w. r. t. x. we get

$$y' = \sum_{m=1}^{\infty} mc_m x^{m-1} = c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + \ldots\ldots\ nc_nx^{n-1} + \ldots\ldots$$

Putting the value of y and y' in the given Differential equation (i), we get

$$(x + 1)\,(c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + \ldots\ldots + n\, c_nx^{n-1} + \ldots\ldots)$$
$$+ (c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + \ldots\ldots\ldots + c_nx^n + \ldots\ldots) = 0$$

or, $(c_1 + c_0) + (2c_2 + 2c_1)\, x + (3c_3 + 3c_2)\, x^2 + (4c_4 + 4c_3)x^3 + \ldots\ldots = 0$

Equating to zero the coefficient of various powers of x,

$$c_1 + c_0 = 0 \qquad \Rightarrow c_1 = -c_0$$
$$2c_2 + 2c_1 = 0 \qquad \Rightarrow c_2 = -c_1 = c_0$$
$$3c_3 + 3c_2 = 0 \qquad \Rightarrow c_3 = -c_2 = -c_0$$
$$4c_4 + 4c_3 = 0 \qquad \Rightarrow c_4 = -c_3 = c_0 \text{ and so on.}$$

Substituting these values in equation (ii) we get,

$$y = c_0 - c_0x + c_0x^2 - c_{0x}{}^3 + c_0x^4 + \ldots\ldots$$
$$= c_0\,(1 - x + x^2 - x^3 + x^4 + \ldots\ldots\ldots\ldots)$$
$$= c_0\,(1 + x)^{-1}$$

$$\therefore y = \frac{c_0}{(1+x)} \text{ is the required solution.}$$

4. $\quad y' - 3x^2y = 0$

Solution: Here, the given differential equation is

$$y' - 3x^2y = 0 \ \ldots\ldots\ldots \text{(i)}$$

Assume that the general solution of (i) is

$$y = \sum_{m=0}^{\infty} c_m x^m = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 \ldots\ldots + c_nx^n + \ldots\ldots \text{(ii)}$$

Differential (ii) w. r. t. x we get,

$$y' = \sum_{m=1}^{\infty} c_m x^{m-1} = c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + \ldots. + {}^{x}c_nx^{n-1} + \ldots$$

Putting the value of y and y' in the given differential equation (i) we get,

or, $(c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + \ldots\ldots\ldots\ldots)$
$$- (3x^2\,(c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 \ldots\ldots\,) = 0$$

or, $c_1 + 2c_2x + (3c_3 - 3c_0)\, x^2 + (4c_4 - 3c_1)\, x^3$
$$+ (5c_5 - 3c_2)\, x^5 + (6c_6 - 3c_3)\, x^6 + (7c_7 - 3c_4)\, x^7$$
$$+ (8c_7 - 3c_5)\, x^8 + (9c_8 - 3c_6)\, x^9 + \ldots\ldots\ldots = 0$$

Equating to zero the coefficient of various power of x.

$$c_1 = 0,\ 2c_2 = 0 \qquad \Rightarrow c_2 = 0$$
$$3c_3 - 3c_0 = 0 \qquad \Rightarrow c_3 = c_0$$
$$4c_4 - 3c_1 = 0 \qquad \Rightarrow c_4 = \frac{3c_1}{4} = 0$$
$$5c_5 - 3c_2 = 0 \qquad \Rightarrow c_5 = \frac{3c_2}{5} = 0$$
$$6c_6 - 3c_3 = 0 \qquad \Rightarrow c_6 = \frac{3}{6}\, c_3 = \frac{1}{2}\, c_0$$
$$7c_7 - 3c_4 = 0 \qquad \Rightarrow c_7 = \frac{3}{7}\, c_4 = 0$$
$$8c_8 - 3c_5 = 0 \qquad \Rightarrow c_8 = \frac{3}{8}\, c_5 = 0$$
$$9c_9 - 3c_6 = 0 \qquad \Rightarrow c_9 = \frac{3}{9}\, c_6 = \frac{1}{2} \cdot \frac{1}{3}\, c_0 = \frac{1}{6}\, c_o \text{ and so on.}$$

Substituting these values in equation (ii) we get,

$$y = c_0 + 0\,.\,x + 0\,.\,x^2 + c_0\,.\,x^3 + 0.\,x^4 + 0\,.\,x^5 + \frac{1}{2}x^6 + 0\,.\,x^7$$
$$+ 0\,.\,x^8 + \frac{1}{6}\,x^9 + \ldots\ldots\ldots$$

$$\therefore y = c_0\left(1+\frac{x^3}{1!}+\frac{x^6}{2!}+\frac{x^9}{3!}+.....\right)$$ is the required solution.

5. $(1-x^4)\,y' = 4x^3y$

Solution: Here, the given differential equation is,

$$(1-x^4)y' - 4x^3y = 0 \text{ ..... (i)}$$

Assume that the general solution of (i) is

$$y = \sum_{m=0}^{\infty} c_m x^m .$$

$$= c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + c_6x^6 + c_7x_7 + c_8x^8 + ... + c_nx^n + ..\text{(ii)}$$

Differential (ii) w. r. t. 'x' we get,

$$y' = \sum_{m=1}^{\infty} mc_m x^{m-1}$$

$$= c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + 5c_5x^4 + 6c_6x^5 + 7c_7x^6 + 8c_8x^7 + ... + nc_nx^{n-1} + ...$$

Putting the value of y and 'y' in the given Differential equation (i) we get,

$$(1-x^4)(c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + 5c_5x^4 + 6c_6x^5 + 7c_7x^6 + 8c_8x^7 + ...) - 4x^3(c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + c_6{}^x6 + c_7 + x^7 + + c^8x^8 + ...) = 0$$

or, $c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + 5c_5x^4 + 6c_6x^5 + 7c_7x^6 + 8c_8x^7 + ... - c_1x^4 - 2c_2x^5 - 3c_3x^6 - 4c_4x^7 - 5c_5x^8 - 6c_6x^9 + ... - 4c_0x^3 - 4c_1x^4 - 4c_2x^5 - 4c_3x^6 - 4c_4x^7 - 4c_5x^8 - .....$

or, $c_1 + 2c_2 + 3c_3\,x^2 + (4c_4 - 4c_0)x^3 + (5c_5 - c_1 - 4c_1)\,x^4 + (6c_6 - 2c_2 - 4c_2)\,x^5 + (7c_7 - 3c_3 - 4c_3)\,x^6 + (8\,c_8 - 4c_4 - 4c_4)\,x^7 + ...$

Equating to zero the coefficient of various power of x.

$c_1 = 0$

$2c_2 = 0 \quad \Rightarrow c_2 = 0$

$3c_3 = 0 \quad \Rightarrow c_3 = 0$

$4c_4 - 4c_0 = 0 \quad \Rightarrow c_4 = c_0$

$5c_5 - c_1 - 4c_1 = 0 \quad \Rightarrow c_5 = c_1 = 0$

$6c_6 - 2c_2 - 4c_2 = 0 \quad \Rightarrow c_6 = c_2 = 0$

$7c_7 - 3c_3 - 4c_3 = 0 \quad \Rightarrow c_7 = c_3 = 0$

$8c_8 - 4c_4 - 4c_4 = 0 \quad \Rightarrow c_8 = c_4 = c_0$ and sol on

Substituting these value in equation (ii), we get,

$$y = c_0 + 0\,.\,x + 0\,.\,x^2 + 0\,.\,x^3 + c_0\,.\,x^4 + 0.x^5 + 0.x^6 + 0\,.\,x^7 + c_0x^8 + ...)$$

$y = c_0\,(1 + x^4 + x^8 + ...)$ is the required.

6. $(x-3)\,y' = xy$

Solution: Here, the given differential equation is,

$$(x-3)\,y' = xy$$

or, $(x-3)\,y' - xy = 0$ ..... (i)

Assume that the general solution of (i) is,

$$y = \sum_{m=0}^{\infty} c_m x^m = c_0 + c_1x + c_2x^2 + c_3x^3 + ... + c_nx^n + ... \text{(ii)}$$

Differential equation (ii) w. r. t. 'x' we get,

$$y' = \sum_{m=1}^{\infty} mc_m x^{m-1}$$

$$= c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + ... nc_nx^{n-1} + ...$$

Putting the value of y and y' in the given Differential equation (i) we get,

$$(x-3)(c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + ...) - x(c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + ...) = 0$$

or, $c_1x + 2c_2x^2 + 3c_3x^3 + 4c_4x^4 + ... - 3c_1 - 6c_2x - 9c_{3x}{}^2 - 12c_4x^3 + ... - c_0x - c_1x^2 - c_2x^3 - c_3x^4 + ...$

or, $-3c_1 + (c_1 - 6c_2 - c_0)\,x + (2c_2 - 9c_3 - c_1)\,x^2 +)3c_3 - 12c_4 - c_2)x^3 + ...$

Equating to zero the coefficient of various power of x.

$-3c_1 = 0 \quad \Rightarrow c_1 = 0$

$c_1 - 6c_2 - c_0 = 0 \quad \Rightarrow c_2 = \frac{c_0}{6}$

$2c_2 - 9c_3 = 0$

$2.\frac{c_0}{6} - 9c_3 = 0 \quad \Rightarrow c_3 = \frac{c_0}{27}$ and so on.

Substituting these value in equation (ii) we gets,

$$y = c_0 + 0\,.\,x + \frac{c_0}{6}x^2 + \frac{c_0}{27}\,x^3 + ...$$

$$= c_0\left(1+\frac{x^2}{6}+\frac{x^3}{27}+.....\right)$$ is the required solution.

$y' = 2xy$ **(BE 2069 Poush)**

Solution: Given Differential equation is,

$$y' = 2xy$$

or, $y' - 2xy = 0$ ..... (i)

Assume that the general solution of (i) is,

$$y = \sum_{m=0}^{\infty} c_m x^m$$

$$= c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^5 + c_5x^5 + c_6x^6 + ... + c_nx^n + ... \text{(ii)}$$

Differential (ii) w. r. t. 'x' we get,

$$y' = \sum_{m=1}^{\infty} mc_m x^{m-1}$$

$$= c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + 5c_5x^4 + 6c_6x^5 + ... + nc_nx^{n-1} + ...$$

Putting the value of y and y' in the given Differential equation (i) we get,

$c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + 5c_5x^4 + 6c_6x^5 + ...$

$- 2x\,(c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^5 + c_5{}^x5 + c_6x^6 + ...$

or, $c_1 + (2c_2 - 2c_0)\,x + (3c_3 - 2c_1)\,x^2 + (4c_4 - 2c_2)x^3$

$+ (5c_5 - 2c_3)\,x^4 + (6c_6 - 2c_4)\,x^5 + ...$

Equating to zero the coefficient of various power of x.

$c_1 = 0 \qquad \Rightarrow c_1 = 0$

$2c_2 - 2c_0 = 0 \qquad \Rightarrow c_2 = c_0$

$3c_3 - 2c_1 = 0 \qquad \Rightarrow c_3 = \frac{2c_1}{3} = 0$

$4c_4 - 2c_2 = 0 \qquad \Rightarrow c_4 = \frac{c_2}{2} = \frac{c_0}{2}$

$5c_5 - 2c_3 = 0 \qquad \Rightarrow c_5 = \frac{2c_3}{5} = 0$

$6c_6 - 2c_4 = 0 \qquad \Rightarrow c_6 = \frac{c_4}{3} = \frac{c_0}{2.3} = \frac{c_0}{6}$ and so on.

Substituting these value in equation (ii) we get,

$$y = c_0 + 0\,.\,x + c_0x^2 + 0\,.\,x^3 + \frac{c_0}{2}\,x^4 + 0\,.\,x^5 + \frac{c_0}{6}\,x^6 + ...$$

$$= c_0\left(1 + x^2 + \frac{x^4}{2} + \frac{x^6}{6} + ...\right)$$ is the required solution.

8. $(x + 1)\,y' - (2x + 3)\,y = 0$

Solution: Given Differential equation is,

$$(x + 1)\,y' - (2x + 3y)\,y = 0 \;.....\; (i)$$

Assume that the general solution of (i) is,

$$y = \sum_{m=0}^{\infty} c_m x^m$$

$= c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + ... + c_nx^n + ...$ (ii)

Differential (ii) w. r. t. 'x' we get,

$$y' = \sum_{m=1}^{\infty} mc_m x^{m-1}$$

$= c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + ... + nx^{n-1} + ...$

Putting the value of y and y' in the given Differential equation (i) we get

$(x + 1)\,(c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + ...)$

$- (2x + 3)\,(c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + ...$

or, $c_1x + 2c_2x^2 + 3c_3x^3 + 4c_4x^4 + ...\; c_1 + 2c_2 + 3c_3x^2 + 4c_4x^3$

$+ ..... - 2c_0x - 2c_1x^2 - 2c_2x^3 - 2c_3x^4 - 2c_4x^5 .....- 3c_0 - 3c_1x - 3c_2x^2 -$

$3c_3x^3 - 3c_4x^4$

or, $(c_1 - 3c_0) + (c_1 + 2c_2 - 2c_0 - 3c_1)\,x + (2c_2 + 3c_3 - 2c_1 - 3c_2)\,x^2$

$+ (3c_3 + 4c_4 - 2c_2 - 3c_3)\,x^3 + ...$

Equating the zero the coefficient of various power of x,

$c_1 - 3c_0 = 0 \qquad \Rightarrow c_1 = 3c_0$

$c_1 + 2c_2 - 2c_0 - 3c_1 = 0$

or, $2c_2 - 2c_0 - 2c_1 = 0$

or, $2c_2 - 2c_0 - 6c_0 = 0 \qquad \Rightarrow c_2 = 4c_0$

$2c_2 + 3c_3 - 2c_1 - 3c_2 = 0$

or, $3c_3 - 2c_1 - c_2 = 0$

or, $3c_3 - 6c_0 - 4c_0 = 0 \qquad \Rightarrow c_3 = \frac{10c_0}{3}$

$3c_3 + 4c_4 - 2c_2 - 3c_3 = 0 \qquad \Rightarrow c_4 = 2c_0$ and so on

Substituting these values in equation (ii) we get,

$$y = c_0 + 3c_0x + 4c_0x^2 + \frac{10}{3}\,c_0x^3 + 2c_0x^4 + ...)$$

$$= c_0\left(1 + 3x + 4x^2 + \frac{10}{3}x^3 + 2x^4 + ...\right)$$ is the required solution.

$y'' + 4y = 0$

Solution: Given Differential equation is

$$y'' + 4y = 0 \;.....\; (i)$$

Assume that the general solution (i) is,

$$y = \sum_{m=0}^{\infty} c_m x^m$$

$= c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + c_6x^6 + ... + nx^n + ...$ (ii)

Differential equation (ii) w. r. t. 'x' we get,

$$y' = \sum_{m=2}^{\infty} m\,x^{m-1}$$

$= c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + 5c_5x^4 + 6c_6x^5 + ... + nx^{n-1}$

Again, Differential w.r.to x we get,

$$y'' = \sum_{m=1}^{\infty} m\,(m-1)\,x^{m-2}$$

$= 2c_2 + 6c_3x + 12c_4x^2 + 20c_5x^3 + 30c_6x^4 + ... + n\,(n-1)\,x^{n-2} + ...$

Putting the value of y and y" in the given Differential equation (i) we get,

$(2c_2 + 6c_3x + 12c_4x^2 + 20c_5x^3 + 30c_6x^4 + ...) + 4\,(c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + c^6x^6 + ...) = 0$

or, $(2c_2 + 4c_0) + (6c_3 + 4c_1)\,x + (12c_4 + 4c_2)x^2 + (20c_5 + 4c_3)\,x^3 + (30c_6 + 4c_4)\,x^4 + ... = 0$

Equating to zero the coefficient of various power of x, we get

$2c_2 + 4c_0 = 0 \qquad \Rightarrow c_2 = -2c_0$

$6c_3 + 4c_1 = 0 \qquad \Rightarrow c_3 = -4\,\frac{c_1}{6} = \frac{2c_1}{-3}$

$12c_4 + 4c_2 = 0 \qquad \Rightarrow c_5 = \frac{-4c^3}{20} = \frac{2}{3.5}c_1 = \frac{2}{15}c_1$

$30c_6 + 4c_4 = 0 \qquad \Rightarrow c_4 = -\frac{4c_3}{20} = \frac{2}{3.5}c_1 = \frac{2}{15}c_1$

$20c_5 + 4c_3 = 0 \qquad \Rightarrow c_6 = -\frac{4c_4}{30} = \frac{2\times 2.c_0}{3.15} = -\frac{4c_0}{45}$

and so on,

Substituting these values in equation (ii) we get,

$$y = c_0 + c_1x - 2c_0x^2 - \frac{2c_1}{3}x^3 + \frac{2c_0}{3}x^4 + \frac{2}{15}c_1x^5 - \frac{4}{45}c_0x^6 +$$

$$= c_0 - 2c_0x^2 + \frac{3}{2}c_0x^4 - \frac{4}{45}c_0x^6 - \ldots\ldots + c_1x - \frac{2}{3}c_1x^3 + \frac{2}{15}c_1x^5 - \ldots\ldots$$

$$\therefore y = c_0\left(1 - 2x^2 + \frac{2}{3}x^4 - \frac{4}{4}x^6 - \ldots\ldots\right) + c_1$$

$\left(1 - \frac{2}{3}x^3 + \frac{2}{15}x^5 - \ldots\ldots\right)$ is the required solution.

10. $y'' - 3y' + 2y = 0$

Solution: Given Differential equation is,

$y'' - 3y' + 2y = 0$ ..... (i)

Assume that the general solution of (i) is

$$y = \sum_{m=0}^{\infty} c_m x^m$$

$= c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + c_6x^6 + \ldots\ldots + c_nx^n + \ldots\ldots$ (2)

Differential (ii) w. r. t. x we get,

$$y' = \sum_{m=1}^{\infty} mc_m x^{m-1} = c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + 5c_5x^4 + 6c_6x^5 + \ldots\ldots + nc_n x^{n-1}$$

Again Differential w. r. to 'x' we get,

$$y'' = \sum_{m=2}^{\infty} m(m-1)c_m x^{m-1}$$

$= 2c_2 + 6c_3x + 12c_4x^2 + 20c_5x^3 + 30c_6x^4 + \ldots\ldots + n(n-1)c_nx^{n-2} +$

Putting the value of y, y' and y'' in the given Differential equation we get,

$2c_2 + 6c_3 + 12c_4x^2 + 20c_5x^3 + 30c_6x^4 + \ldots\ldots - 3(c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + 5c_5x^4 + \ldots\ldots\ldots\ldots + 2(c_0 + c_1x + c_2x^2 + c_3x^3 + (4x^4) \ldots\ldots\ldots\ldots$

or, $(2c_2 - 3c_1 + 2c_0) + (6c_3 - 6c_2 + 2c_1)x + (12c_4 - 9c_3 - 9c_3 + 2c_2)x^2$

$= (20c_5 - 12c_4 + 2c_3)x^3 + (30c_6 - 15c_5 + 24x^4 + \ldots\ldots\ldots\ldots$

Equating to zero the coefficient of various power of x, we get

$2c_2 - 3c_1 + 2c_0 = 0$

or, $2c_2 = 3c_1 - 2co$

or, $c_2 = \frac{3}{2}c_1 - c_0$

$6c_3 - 6c_2 + 2c_1 = 0$

or, $6c_3 = 6c_2 - 2c_1$

or, $c_3 = c_2 - \frac{2}{6}c_1$

$= \frac{3}{2}c_1 - c_0 - \frac{1}{3}c_1$

$= \frac{3}{2}c_1 - c_0 - \frac{3}{2}c_1$

$\left(\frac{9-2}{6}\right)c_1 - c_0$

$= \frac{7}{6}c_1 - c_0$

Putting these value in equation (ii) we get,

$y = c_0 + c_1x + \left(\frac{3}{2}c_1 - c_0\right)x^2 + \left(\frac{7}{6}c_1 - c_0\right)x^3 + \ldots\ldots$

is the required solution.

$(1 - x^2)y'' - 2xy' + 2y = 0$ **(BE 2067 Mangsir)**

Solution: Given diff. equation is,

$(1 - x^2)y'' - 2xy' + 2y = 0$ ..... (i)

Assume that the general solution of (i) is,

$$y = \sum_{m=0}^{\infty} c_m x^m$$

$= c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + c_6x^6 + \ldots\ldots + c_nx^n + \ldots\ldots$(2)

Differential (ii) w. r. t. x we get,

$$y' = \sum_{m=1}^{\infty} mc_m x^{m-1} = c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + 5c_5x^4 + 6c_6x^5 + x^5 + \ldots\ldots + nc_n x^{n-1}$$

Again Differential w. r. to 'x' we get,

$$y'' = \sum_{m=2}^{\infty} m(m-1)c_m x^{m-1}$$

$= 2c_2 + 6c_3x + 12c_4x^2 + 20c_5x^3 + 30c_6x^4 + \ldots\ldots + n(n-1)c_nx^{n-2} + \ldots$

Putting the value of y. y' and y'' in the given Differential equation (i) we get,

$(1 - x^2)(2c_2 + 6c_3x + 12c_4x^2 + 20c_5x^3 + 30c_6x^4 + \ldots\ldots)$

$- 2x(c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + 5c_5x^4 + \ldots\ldots\ldots\ldots)$

$+ 2(c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + \ldots\ldots\ldots\ldots)$

or, $2c_2 + 6c_3x + 12\,c_4x^2 + 20c_5x^3 + 30c_6x^4 + \ldots\ldots$

$$- 2c_2x^2 - 6c_3x^3 - 12c_4x^4 - 20c_5x^5 - 30c_6x^6 +$$

$$- 2c_1x - 4c_2x^2 - 6c_3x^3 - 8c_4x^4 - 10c_5x^5 + \ldots\ldots + 2c_0 + 2c_1x +$$

$$+ 2c_3x^3 + 2c_4x^4 +$$

or, $(2c_2 + 2c_0) + (6c_3 - 2c_1 + 2c_1)\,x + (12c_4 - 2c_2 - 4c_2 + 2c_2)\,x^2$

$+ (20c_5 - 6c_3 - 6c_3 + 2c_3)\,x^3 + (30c_6 - 12c_4 - 8c_4 + 2c_4)\,x^4 +$

Equating the zero, the coefficient of various power of 'x' we get,

$2c_2 + 2c_0 = 0 \qquad \Rightarrow c_2 = c_0$

$6c_3 - 2c_1 + 2c_1 = 0 \qquad \Rightarrow c_3 = 0$

$12c_4 - 2\,c_2 - 4c_2 + 2c_2 = 0$

or, $12c_4 - 4c_2 = 0$

or, $c_4 = \dfrac{4c^2}{12} = \dfrac{1}{3}c_0$

$20c_5 - 6c_3 - 6c_3 + 2c_3 = 0$

or, $20c_4 - 10c_3 = 0 \qquad \Rightarrow c_5 = \dfrac{10c_3}{20} = 0$

$30c_6 - 12c_4 - 8c_4 + 2c_4 = 0$

or, $30\,C_6 - 18_4 = 0$

$\Rightarrow c_6 = \dfrac{18}{30}\,c_4 = \dfrac{18}{30}\cdot\dfrac{1}{3}\,c_0 = \dfrac{1}{5}\,c_0$

Substituting these value in equation (ii) we get,

$y = c_0 + c_1x + c_0x^2 + 0\,.\,x^3 + \dfrac{1}{3}\,c_0x^4 + 0.x^5\,\dfrac{1}{5}c_0x^6 \ldots\ldots$

$= c_1x + c_0\left(1 + x^2 + \dfrac{1}{3}x^4 + \dfrac{1}{5}x^6 + \ldots\right)$ is the required solution

12. $y'' - 9y = 0$

Solution: Given Differential equation is,

$$y'' - 9y = 0$$

Assume that the general solution of (i) is

$$y = \sum_{m=0}^{\infty} c_m x^m$$

$= c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + c_6x^6 + \ldots\ldots + c_nx^n + \ldots\ldots$

Differential (ii) w. r. t. x we get,

$$y' = \sum_{m=1}^{\infty} mc_mx^{m-1} = c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + 5c_5x^4 + 6c_6$$

$$+ \ldots\ldots +$$

Again Differential w. r. to 'x' we get,

$$y'' = \sum_{m=2}^{\infty} m(m-1)c_mx^{m-1}$$

$= 2c_2 + 6c_3x + 12c_4x^2 + 20c_5x^3 + 30c_6x^4 + \ldots\ldots + n\,(n-1)\,c_nx^{n-2} + \ldots$

Putting the value of y. y' and y'' in the given Differential equation (i) we get,

$(2c_2 + 6c_3x + 12c_4x^2 + + 20c_5x^3 + 30c_6x^4 + \ldots\ldots)$

$$- 9\,(c_0 + c_1x + c_2x^2 + c_3x_3 + c_4x^4 + c_5x^5 + {}_6x^6 + \ldots\ldots\ldots\ldots)$$

or, $(2c_2 - 9c_0) + (6c_3 - 9c_1)\,x + (12c_4 - 9c_2)x^2 + (20c_5 - 9c_3)\,x^3$

$+ (30c_6 - 9c_4)x^4 + \ldots\ldots\ldots\ldots$

Equating to zero the coefficient of various power of x, we get

$2c_2 - 9c_0 = 0 \qquad \Rightarrow c_2 = \dfrac{9}{2}c_0$

$6c_3 - 9c_1 = 0 \qquad \Rightarrow c_3 = \dfrac{9}{6}c_1$

$12c_4 - 9c_2 = 0 \qquad \Rightarrow c_4 = \dfrac{9}{12}c_2 = \dfrac{9}{12}\times\dfrac{9}{2}c_0 = \dfrac{81}{24}c_0$

$20c_5 - 9c_3 = 0 \qquad \Rightarrow c_5 = \dfrac{9}{20}c_3 = \dfrac{9}{20}\times\dfrac{9}{6}c_1 = \dfrac{81}{120}c_1$

$30c_6 - 9c_4 = 0 \qquad \Rightarrow c_6 = \dfrac{9}{30}c_4 = \dfrac{9}{30}\times\dfrac{81}{24}\,c_0 = \dfrac{729}{720}c_0$

and so on,

Putting these value in equation (ii) we get,

$y = c_0 + c_1x + \dfrac{9}{2}\,c_0x^2 + \dfrac{9}{6}c_1x^3 + \dfrac{81}{24}\,c_0x^4 + \dfrac{81}{120}\,c_1x^5 + \dfrac{729}{720}\,c_0x^6 + \ldots\ldots$

$= \left(c_0 + \dfrac{9}{2}c_0x^2 + \dfrac{81}{24}c_0x^4 + \dfrac{729}{720}c_0x^6 + \ldots\right)$

$$+ \left(c_1x + \dfrac{9}{6}c_1x^3 + \dfrac{81}{120}c_1x^5 + \ldots\ldots\right)$$

$y = \left(1 + \dfrac{9}{2!}x^2 + \dfrac{81}{4!}x^3 + \dfrac{729}{61}x^6 + \ldots\ldots\right)c_0$

$+ c_1\left(x + \dfrac{9}{2!}x^2 + \dfrac{9}{3!}x^3 + \dfrac{81}{5!}x^5 + \ldots\ldots\right)$ is the required solution.

$y'' - y = x$

Solution: Given Differential equation is,

$$y'' - y = 0 \ldots\ldots\ldots\ldots (i)$$

Assume that the general solution of (i) is

$$y = \sum_{m=0}^{\infty} c_mx^m$$

$= c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + c_6x^6 + \ldots\ldots + c_nx^n + \ldots\ldots\ldots(2)$

Differential (ii) w. r. t. x we get,

$$y' = \sum_{m=1}^{\infty} mc_m x^{m-1} = c_1 - 2c_2x + 3c_3x^2 + 4c_4x^3 + 5c_5x^4 + 6c_6x^5 + \ldots\ldots + nc_n x^{n-1}$$

Putting the value of y. y' and y" in the given Differential equation we get,

$$(2c_2 + 6c_3x + 12c_4x^2 + 20c_5x^3 + 30c_6x^4 + \ldots\ldots) - (c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + \ldots\ldots)$$

or, $(2c_2 - c_0) + (6c_3 - c_1)\,x + (12c_4 - c_2)\,x^2 + (20c_5 - c_3)x^3 + (30c_6 - c_4)\,x^4 = 0$

Equating to zero the coefficient of various power of x we get,

$$2c_2 - c_0 = 0 \qquad \Rightarrow c_2 = \frac{c_0}{2}$$

$$6c_3 - c_1 = 0 \qquad \Rightarrow c_3 = \frac{c_1}{6}$$

$$12c_4 - c_2 = 0 \qquad \Rightarrow c_4 = \frac{c_2}{12} = \frac{1}{12} \times \frac{1}{2} c_0 = \frac{1}{24} c_0$$

$$20c_5 - c_3 = 0 \qquad \Rightarrow c_5 = \frac{16}{20} = \frac{1}{20} \times \frac{1}{6} c_1 = \frac{1}{120} c_1$$

Putting these value in equation (ii) we get,

$$y = c_0 + c_1x + \frac{c_0}{2}x^2 + \frac{c_1}{6}x^3 + \frac{1}{24}c_0x^4 + \frac{1}{120}c_1x^5 + \ldots\ldots\ldots$$

$$= c_0 + \frac{c_0}{2}x^2 + \frac{c_0}{24}x^4 + \ldots\ldots + c_1x + \frac{c_1}{6}x^3 + \frac{c_1}{120}x^5 + \ldots\ldots\ldots$$

$$= c_0\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots\ldots\right) + 4\left(x + \frac{x^3}{3!} + \frac{x^4}{5!} + \ldots\ldots\right) c_1$$

is the required solution.

14. $y'' + xy' + y = 0$ **(BE 2069 Poush, 2070 Magh)**

Solution: Here, $y'' + xy' + y = 0$ .....(i)

Assume that the general solution of (i) is

$$y = \sum_{m=0}^{\infty} c_m x^m$$

$$= c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + c_6x^6 + \ldots\ldots + c_nx^n + \ldots\ldots\ldots(2)$$

Differential (ii) w. r. t. x we get,

$$y' = \sum_{m=1}^{\infty} mc_m x^{m-1} = c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + 5c_5x^4 + 6c_6x^5 + \ldots\ldots + nc_n x^{n-1}$$

Putting the value of y. y' and y" in the given Differential equation we get,

$$(2c_2 + 6c_3x + 12c_4x^2 + 20c_5x^3 + 30c_6x^4 + \ldots\ldots\ldots) + x\,(c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + 5c_5x^4 + \ldots\ldots\ldots) + (c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + \ldots\ldots\ldots)$$

or, $(2c_2 + c_0) + (6c_3 + c_1 + c_1)\,x + (12c_4 + 2c_2 + c_2)\,x^2 + (20c_5 + 3c^3 + c_3)\,x^3 + (30c_6 + 4c_4 + c_4)\,x^4 + \ldots\ldots\ldots$

Equating to zero the coefficient of various power of x, we get

$$\text{or, } 2c_2 + c_0 = 0 \qquad \Rightarrow c_2 = -\frac{c_0}{2}$$

$$6c_3 + 2c_1 = 0 \qquad \Rightarrow c_3 = -\frac{2c_1}{6} = -\frac{c_1}{3}$$

$$12c_4 + 3c_2 = 0 \qquad \Rightarrow c_4 = -\frac{3c_2}{12} = -\frac{3}{12} \times \frac{-c_0}{2} = \frac{1}{8}c_0$$

$$20c_5 + 4c_3 = 0 \qquad \Rightarrow c_5 = -\frac{4}{20}c_3 = -\frac{4}{20} \times \frac{-c_1}{3} = \frac{1}{15}c_1$$

$$30c_6 + 5c_4 = 0 \qquad \Rightarrow c_6 = -\frac{5}{30}c_4 = -\frac{5}{30} \times \frac{1}{8}c_0x^6 = -\frac{1}{48}c_0$$

Putting these value in equation (ii) we get,

$$y = c_0 + c_1x - \frac{c_0}{2}x^2 - \frac{c_1}{3}x^3 + \frac{c_0}{8}x^4 + \frac{c_1}{15}x^5 - \frac{1}{48}c_0 + x^6 + \ldots\ldots\ldots$$

$$= \left(c_0 - \frac{c_0}{2}x^2 + \frac{c_0}{8}x^4 - \frac{c_0}{48}x^6 + \ldots\ldots\right) + \left(c_1x - \frac{c_1}{3}x^3 + \frac{c_1}{15}x^5 + \ldots\ldots\right)$$

$$= c_0\left(1 - \frac{x^2}{2} + \frac{x^4}{2.4} - \frac{x^6}{2.4.6} + \ldots\ldots\right) + c_1\left(x - \frac{x^3}{3} + \frac{x^5}{3.5}\ldots\ldots\ldots\right)$$

is the required solution.

15. $y'' + x^2y = 0$ **(BE 2068 Bhadra)**

Solution: Given Differential equation is,

$$y'' + x^2y = 0 \ldots\ldots\ldots\ldots (i)$$

Assume that the general solution of (i) is

$$y = \sum_{m=0}^{\infty} c_m x^m$$

$$= c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + c_6x^6 + \ldots\ldots + c_nx^n + \ldots\ldots\ldots(2)$$

Differential (ii) w. r. t. x we get,

$$y' = \sum_{m=1}^{\infty} mc_m x^{m-1} = c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + 5c_5x^4 + 6c_6x^5 + \ldots\ldots + nc_n x^{n-1}$$

Putting the value of y, y' and y" in the given Differential equation (i) we get,

$$(2c_2 + 6c_3x + 12c_4x^2 + 20c_5x^3 + 30c_6x^4 + \ldots\ldots\ldots\ldots) + x^2(c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + \ldots\ldots\ldots\ldots)$$

or, $2c_2 + 6c_3x + (12c_4 + c_0)x^2 + (20c_5 + c_1)x^3 + (30c_6 + c_2)x^4 + \ldots\ldots$

Equating to zero the coefficient of various power of x, we get,

$2c_2 = 0 \qquad \Rightarrow c_2 = 0$

$6c_3 = 0 \qquad \Rightarrow c_3 = 0$

$12c_4 + c_0 = 0 \qquad \Rightarrow c_4 = \dfrac{-c_0}{12}$

$20c_5 + c_1 = 0 \qquad \Rightarrow c_5 = \dfrac{c_1}{20}$

$30c_6 + c_2 = 0 \qquad \Rightarrow c_6 = \dfrac{c_2}{30} = 0$ and so on

Putting these value in equation (ii) we get,

$$y = c_0 + c_1x + 0 \,.\, x^2 + 0 \,.\, x^3 - \frac{c_0}{12}x^4 - \frac{c_1}{20}x^5 + 0 \,.\, x^6 + \ldots.$$

$$= \left(c_0 - \frac{c_0}{12}x^4 + \ldots..\right) + \left(c_1x - \frac{c_1}{20}x^5 + \ldots..\right)$$

$$= c_0\left(1 - \frac{x^4}{12} + \ldots\ldots\right) + c_1\left(x - \frac{x_5}{20} + \ldots..\right)$$ is the required solution.

Exercise - 18

Prove that the following Legendre's functions.

i. $\dfrac{2}{5}p_3(x) + \dfrac{3}{5}p_1(x) = x^3$

Solution: We know that Legendre's polynomial are,

$p_0(x) = 1, \qquad p_1(x) = x$

$p_2(x) = \dfrac{1}{2}(3x^2 - 1)$ and $p_3(x) = \dfrac{1}{2}(5x^3 - 3x)$

Now, LHS $= \dfrac{2}{5}p_3(x) + \dfrac{3}{5}p_1(x)$

$= \dfrac{2}{5} \cdot \dfrac{1}{2}(5x^3 - 3x) + \dfrac{3}{5}x = \dfrac{1}{5}(5x^3 - 3x) + \dfrac{3}{5}x$

$= \dfrac{1}{5} \cdot 5x^3 - \dfrac{3}{5}x + \dfrac{3}{5}x$

$= x^3$ R.H.S. Proved.

ii. $x^5 = \dfrac{8}{63}\left[p_5(x) + \dfrac{7}{2}p_3(x) + \dfrac{27}{8}p_1(x)\right]$

Solution: We know that Legendre's polynomial are,

$p_0(x) = 1, \qquad p_1(x) = x$

$p_2(x) = \dfrac{1}{2}(3x^2 - 1) \qquad p_3(x) = \dfrac{1}{2}(5x^3 - 3x)$

$p_4(x) = \dfrac{1}{8}(35x^4 - 30x^2 + 3)$ and $p_5(x) = \dfrac{1}{8}(63x^5 - 70x^3 + 15x)$

Now, RHS

$= \dfrac{8}{63}\left[p_5(x) + \dfrac{7}{2}p_3(x) + \dfrac{27}{8}p_1(x)\right]$

$= \dfrac{8}{63}\left[\left\{\dfrac{1}{8}(63x^5 - 70x^3 + 15x\right\} + \dfrac{7}{2}\left\{\dfrac{1}{2}\left(5x^3 - 3x\right)\right\} + \dfrac{27}{8}x\right]$

$= \dfrac{8}{63} \times \dfrac{1}{8} \times 63x^5 - \dfrac{8}{63} \times \dfrac{1}{8} \times 70x^3 + \dfrac{8}{63} \times \dfrac{1}{8} \times 15x$

$+ \dfrac{8}{63} \times \dfrac{7}{2} \times \dfrac{1}{2} \times 5x^3 - \dfrac{8}{63} \times \dfrac{7}{2} \times \dfrac{1}{2} \times 3x + \dfrac{8}{63} \times \dfrac{27}{8}x$

$= x^5 - \dfrac{10}{9}x^3 + \dfrac{5}{21}x + \dfrac{10}{9}x^3 - \dfrac{2}{3}x + \dfrac{3x}{7}$

$= x^5 + \dfrac{5x - 14x + 9x}{21} = x^5 + \dfrac{14x - 14x}{21} = x^5$ L.H.S. Proved.

Express the following in terms of Legendre's polynomials.

3. $f(x) = 1 + x - x^2$

Solution: Here the given function is,

$f(x) = 1 + x - x^2$ ..... (i)

We know that Legendre's polynomials are,

$p_0(x) = 1, \Rightarrow 1 = p_0(x), \quad p_1(x) = x \Rightarrow x = p_1(x)$

and, $p_2(x) = \frac{1}{2}(3x^2 - 1)$

or, $3x^2 - 1 = 2\,p_2(x)$

$\therefore x^2 = \frac{1}{3} + \frac{2}{3}p_2(x)$

$x^2 = \frac{1}{3}p_0(x) + \frac{2}{3}p_2(x)$

So (i) can we expressed as,

$f(x) = p_0(x) + p_1(x) - \frac{1}{3}p_0(x) - \frac{2}{3}p_2(x)$

$= \frac{2}{3}p_0(x) + p_1(x) - \frac{2}{3}p_2(x)$

$\therefore f(x) = \frac{2}{3}p_0(x) + p_1(x) - \frac{2}{3}p_2(x)$ Ans.

4. $f(x) = 5x^3 + x$

Solution: Here the given function is,

$f(x) = 5x^3 + x$ ..... (i)

We know that Legendre's polynomials are,

$p_0(x) = 1, \Rightarrow 1 = p_0(x); \quad p_1(x) = x \Rightarrow x = p_1(x)$

$p_2(x) = \frac{1}{2}(3x^2 - 1)$

or, $3x^2 - 1 = 2p_2(x)$

$\therefore x^2 = \frac{1}{3} + \frac{2}{3}p_2(x)$

$x^2 = \frac{1}{3}p_0(x) + \frac{2}{3}p_2(x)$

and $p_3(x) = \frac{1}{2}(5x^3 - 3x)$

$5x^3 - 3x = 2\,p_3(x)$

$\therefore x^3 = \frac{3}{5}x + \frac{2}{5}p_3(x)$

$\frac{3}{5}p_1(x) + \frac{2}{5}p_3(x)$

So, (i) can be expressed as

$f(x) = 5\left[\frac{3}{5}p_1(x) + \frac{2}{5}p_3(x)\right] + p_1(x)$

$= 5 \times \frac{3}{5}p_1(x) + 5 \times \frac{2}{5}p_3(x) + p_1(x)$

$= 3p_1(x) + 2p_3(x) + p_1(x)$

$= 2\,p_3(x) + 4p_1(x)$

$\therefore f(x) = 2p_3(x) + 4p_1(x)$

5. $f(x) = x^3 - 5x^2 + 6x + 1$ **(BE 2068 Bhadra)**

Solution: Here the given function is,

$f(x) = x^3 - 5x^2 + 6x + 1$

We know that legendre's polynomials are

$p_0(x) = 1 \Rightarrow 1 = p_0(x); \quad p_1(x) = x \Rightarrow x = p_1(x)$

$\therefore x = p_1(x)$

$p_2(x) = \frac{1}{2}(3x^2 - 1)$

or, $3x^2 - 1 = 2p_2(x)$

$\therefore x^2 = \frac{1}{3}p_0(x) + \frac{2}{3}p_2(x)$

$x^2 = \frac{1}{3}p_0(x) + \frac{2}{3}p_2(x)$

and $p_3(x) = \frac{1}{2}(5x^3 - 3x)$

or, $5x^3 - 3x = 2\,P_3(x)$

$x^3 = \frac{3}{5}p_1(x) + \frac{2}{5}p_3(x)$

So, (i) can be expressed as,

$f(x) = \frac{3}{5}p_1(x) + \frac{2}{5}p_3(x) - \frac{5}{3}p_0(x)$

$- \frac{10}{3}p_2(x) + 6p_1(x) + p_0(x)$

$= \frac{2}{5}p_3(x) - \frac{10}{3}p_2(x) + \frac{33}{5}p_1(x) - \frac{2}{3}p_0(x)$

$\therefore f(x) = \frac{2}{5}p_3(x) - \frac{10}{3}p_2(x) + \frac{33}{5}p_1(x) - \frac{2}{3}p_0(x)$

Prove the following Bessel's functions.

6. $J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\sin x$.

Solution: we know that

$$J_n(x) = \frac{x^n}{2^n\sqrt{n+1}} \sum_{m=0}^{\infty} \frac{(-1)^m}{4^m m!(n+1)(n+2).....(n+m)} x^{2m}$$

$$= \frac{x^n}{2^n\sqrt{n+1}}\left[1 - \frac{1}{4(n+1)}x^2 + \frac{1}{4^2 2!(n+1)(n+2)}x^4 - .....\right]$$

Putting $n = \frac{1}{2}$ we get,

$$J_{\frac{1}{2}}(x) = \frac{x^{\frac{1}{2}}}{2^{\frac{1}{2}}\sqrt{\frac{3}{2}}}\left[1 - \frac{1}{4.\frac{3}{2}}x^2 + \frac{1}{16.2\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)}x^4 - ....\right]$$

$$= \sqrt{\frac{x}{2}}\frac{1}{\frac{1}{2}\sqrt{\frac{1}{2}}}\left[1 - \frac{1}{6}x^2 + \frac{1}{120}x^4 - .....\right]$$

$$= \sqrt{\frac{x}{2}}.\frac{2}{\sqrt{\pi}}\left[1 - \frac{1}{3!}x^2 + \frac{1}{5!}x^4 - .....\right]$$

$$= \sqrt{\frac{x}{2}}.\frac{2}{\pi}.\frac{1}{x}\left[x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - .....\right] \text{ since } \Gamma(n+1) = n\Gamma(n)$$

$$= \sqrt{\frac{2}{\pi x}}\left[x - \frac{1}{3!}x^3 + \frac{1}{5}x^5 - .....\right]$$

$$\therefore J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sin x \quad \text{Proved.}$$

7. $J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cos x.$

Solution: We know that

$$J_n(x) = \frac{x^n}{2^n\sqrt{n+1}} \sum_{m=0}^{\infty} \frac{(-1)^m}{4^m m!(n+1)(n+2).....(n+m)} x^2 m$$

$$= \frac{x^n}{2^n\sqrt{n+1}}\left[1 - \frac{1}{(n+1)}x^2 + \frac{1}{4^2.2!(n+1)(n+2)}x^4 - .....\right]$$

Putting $n = -\frac{1}{2}$ we get.

$$J_{-\frac{1}{2}}(x) = \frac{x^{-\frac{1}{2}}}{2^{-\frac{1}{2}}\sqrt{\frac{-1}{2}+1}}\left[1 - \frac{1}{4.\frac{1}{2}}x^2 + \frac{1}{16.2.\frac{1}{2}.\frac{3}{2}}x^4 - .....\right]$$

$$= \left(\frac{2}{x}\right)^{\frac{1}{2}}\frac{1}{\sqrt{\frac{1}{2}}}\left[1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - .....\right]$$

$$= \sqrt{\frac{2}{x}}.\frac{1}{\sqrt{\pi}}\left[1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - .....\right] = \sqrt{\frac{2}{\pi x}} \cos x.$$

$$J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cos x. \text{ Proved.}$$

8. $J_{\frac{3}{2}}(x) = \sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x} - \cos x\right)$ **[B.E. 2064]**

Solution:

We know that recurrence relation of Bessel's function.

$2nJ_n(x) = x\,[J_{n-1}(x) + J_{n+1}(x)]$

Putting $n = \frac{1}{2}$, we get,

$$2\,\frac{1}{2}\,J_{\frac{1}{2}}(x) = x\left[J_{-\frac{1}{2}}(x) + J_{\frac{3}{2}}(x).\right]$$

$$J_{\frac{1}{2}}(x) - x\,J_{-\frac{1}{2}}(x) = x\,J_{\frac{3}{2}}(x).$$

$$x\,J_{\frac{3}{2}}(x) = J_{\frac{1}{2}}(x) - x\,J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\sin x - x\sqrt{\frac{2}{\pi x}}\cos x.$$

Since $J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sin x$ and

$$J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cos x.$$

$$\text{or, } x\,J_{\frac{3}{2}}(x) = \sqrt{\frac{2}{\pi x}}\,[\sin x - \cos x]$$

$$\therefore \quad J_{\frac{3}{2}}(x) = \sqrt{\frac{2}{\pi x}}\left[\frac{\sin x}{x} - \cos x\right] \text{ Proved.}$$

9. $J_{-\frac{3}{2}}(x) = \sqrt{\frac{2}{\pi x}}\left(\frac{-\cos x}{x} - \sin x\right)$

Solution:

We know that recurrence relation of the Bessel's function

$2n\,J_n(x) = x[J_{n-1}(x) + J_{n+1}(x)]$

Putting $n = -\frac{1}{2}$ we get,

$$2.\left(-\frac{1}{2}\right)J_{-\frac{1}{2}}(x)=x\left[J_{-\frac{1}{2}}(x)+J_{\frac{1}{2}}(x)\right]$$

or, $-J_{-\frac{1}{2}}(x)-x\,J_{\frac{1}{2}}(x)=x\,J_{-\frac{3}{2}}(x)$

or, $x\,J_{-\frac{3}{2}}(x)=-J_{-\frac{1}{2}}(x)-x\,J_{\frac{1}{2}}(x)=-\sqrt{\frac{2}{\pi x}}\cos x-x\sqrt{\frac{2}{\pi x}}\sin x$

Since, $J_{-\frac{1}{2}}(x)=x\sqrt{\frac{2}{\pi x}}\cos x$

$$J_{\frac{1}{2}}(x)=\sqrt{\frac{2}{\pi x}}\cos x$$

or, $x\,J_{-\frac{3}{2}}(x)=\sqrt{\frac{2}{\pi x}}(-\cos x-x\sin x)$

$\therefore$ $J_{-\frac{3}{2}}(x)=\sqrt{\frac{2}{\pi x}}\left(-\frac{\cos x}{x}-\sin x\right)$ Proved.

10. $J_{\frac{5}{2}}(x)=\sqrt{\frac{2}{\pi x}}\left\{\frac{3-x^2}{x^2}\sin x-\frac{3}{2}\cos x\right\}$

[B.E. 2065, 2067 Mangsir]

Solution: We know that recurrence relation of Bessel's function.

$2n\,J_n(x)=x[J_{n-1}(x)+J_{n+1}(x)]$

Putting $n=\frac{3}{2}$ we get,

$$2.\frac{3}{2}\,J_{\frac{3}{2}}(x)=x\left[J_{\frac{1}{2}}(x)+J_{\frac{5}{2}}(x)\right]$$

or, $3\,J_{\frac{3}{2}}(x)=x\,J_{\frac{1}{2}}(x)+x\,J_{\frac{5}{2}}(x)$

or, $3\,J_{\frac{3}{2}}(x)-x\,J_{\frac{1}{2}}(x)=x\,J_{\frac{5}{2}}(x)$

$$x\,J_{\frac{5}{2}}(x)=3\,J_{\frac{3}{2}}(x)-x\,J_{\frac{1}{2}}(x)=3\sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x}-\cos x\right)-x\sqrt{\frac{2}{\pi x}}\sin x.$$

Since $J_{\frac{3}{2}}(x)=\sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x}-\cos x\right)$ and $J_{\frac{1}{2}}(x)=\sqrt{\frac{2}{\pi x}}\sin x.$

or, $x\,J_{\frac{5}{2}}(x)=\sqrt{\frac{2}{\pi x}}\left[\frac{3\sin x}{x}-3\cos x-x\sin x\right]$

or, $x\,J_{\frac{5}{2}}(x)=\sqrt{\frac{2}{\pi x}}\left(\frac{3-x^2}{x}\sin x-3\cos x\right)$

$$J_{\frac{5}{2}}(x)=\sqrt{\frac{2}{\pi x}}\left[\frac{3-x^2}{x^2}\sin x-\frac{3\cos x}{x}\right]$$ Proved.

$$J_{-\frac{5}{2}}(x)=\sqrt{\frac{2}{\pi x}}\left\{\frac{3}{x}\sin x+\frac{3-x^2}{x^2}\cos x\right\}$$

Solution: we know that the recurrence relation of the Bessel's functions

$2nJ_n(x)=x[J_{n-1}(x)+J_{n+1}(x)]$

Putting $n=-\frac{3}{2}$

$$2.\left(-\frac{3}{2}\right)J_{-\frac{3}{2}}(x)=x\left[J_{-\frac{5}{2}}(x)+J_{-\frac{1}{2}}(x)\right]$$

$$-3\,J_{-\frac{3}{2}}(x)=x\,J_{-\frac{5}{2}}(x)+x\,J_{-\frac{1}{2}}(x)$$

$$-3\,J_{-\frac{3}{2}}(x)-x\,J_{-\frac{1}{2}}(x)=x\,J_{-\frac{5}{2}}(x)$$

$$x\,J_{-\frac{5}{2}}(x)=-3\,J_{-\frac{3}{2}}(x)-x\,J_{-\frac{1}{2}}(x)$$

$$=3\sqrt{\frac{2}{\pi x}}\left(\frac{\cos x}{x}+\sin x\right)-x\sqrt{\frac{2}{\pi x}}\cos x$$

Since, $J_{-\frac{3}{2}}(x)=\sqrt{\frac{2}{\pi x}}\left(\frac{-\cos x}{x}-\sin x\right)$ and $J_{-\frac{1}{2}}(x)=\sqrt{\frac{2}{\pi x}}\cos x.$

$$=\sqrt{\frac{2}{\pi x}}\left(3\frac{\cos x}{x}+3\sin x-x\cos x\right)$$

$$=\sqrt{\frac{2}{\pi x}}\left[\frac{3-x^2}{x}\cos x+3\sin x\right]$$

$$J_{-\frac{5}{2}}(x)=\sqrt{\frac{2}{\pi x}}\left[\frac{3-x^2}{x^2}\cos x+\frac{3}{x}\sin x\right]$$

$$J_{-\frac{5}{2}}(x)=\sqrt{\frac{2}{\pi x}}\left[\frac{3}{x}\sin x+\frac{3-x^2}{x^2}\cos x\right]$$ Proved.

$J_4(x)=\left(\frac{48}{x^3}-\frac{8}{x}\right)J_1(x)+\left(1-\frac{24}{x^2}\right)J_0(x)$ **[B.E.2067 Mangsir]**

Solution: We know that the recurrence relation of the Bessel's function,

$2n\,J_n(x)=x\,[J_{n-1}(x)+J_{n+1}(x)]$

$J_{n+1}(x)=\frac{2n}{x}\,J_n(x)-J_{n-1}(x)$ .....(i)

Putting n = 3 in (i) we get

or, $J_4(x) = \frac{2.3}{x} J_3(x) - J_2(x)$

or, $J_4(x) = \frac{6}{x} J_3(x) - J_2(x)$ ..... (ii)

Putting n = 2 in (i)

or, $J_3(x) = \frac{2.2}{x} J_2(x) - J_1(x)$

or, $J_3(x) = \frac{4}{x} J_2(x) - J_1(x)$ ..... (iii)

from (ii) & (iii)

$$J_4(x) = \frac{6}{x}\left[\frac{4}{x}J_2(x) - J_1(x)\right] - J_2(x)$$

$$= \frac{24}{x^2}J_2(x) - \frac{6}{x}J_1(x) - J_2(x)$$

$$= \left(\frac{24}{x^2} - 1\right)J_2(x) - \frac{6}{x}J_1(x) \text{ .....(iv)}$$

Put n = 1 in (i)

$$J_2(x) = \frac{2}{x} J_1(x) - J_0(x) \text{ ..... (v)}$$

from (iv) and (v)

$$J_4(x) = \left(\frac{24}{x^2} - 1\right)\left[\frac{2}{x}J_1(x) - J_0(x)\right] - \frac{6}{x}J_1(x)$$

$$= \left(\frac{48}{x^3} - \frac{2}{x}\right)J_1(x) - \left(\frac{24}{x^2} - 1\right)J_0(x) - \frac{6}{x}J_1(x)$$

$$= \left(\frac{48}{x^3} - \frac{2}{x} - \frac{6}{x}\right)J_1(x) - \left(\frac{24}{x^2} - 1\right)J_0(x)$$

$$= \left(\frac{48}{x^3} - \frac{8}{x}\right)J_1(x) + \left(1 - \frac{24}{x}\right)J_0(x)$$

$\therefore$ $J_4(x) = \left(\frac{48}{x^3} - \frac{8}{x}\right)J_1(x) + \left(1 - \frac{24}{x}\right)J_0(x)$ Proved.

13. $J_{n+3}(x) + J_{n+5}(x) = \frac{2}{x}(x+4) J_{n+4}(x)$. (BE 2068 Bhadra

Solution: we know that the recurrence relation of the Bessel function.

$2n J_n(x) = x[J_{n-1}(x) + J_{n+1}(x)]$ .....(i)

Putting n = n + 4 in (i) we get,

$2(n + 4) J_{n+4}(x) = x [J_{4-1}(x) + J_{n+4+1}(x)]$

or. $2(n + 4) J_{n+4}(x) = x [J_{n+3}(x) + J_{n+5}(x)]$

$x[J_{n+3}(x) + J_{n+5}(x)] = 2(n + 4) J_{n+4}(x)$

$J_{n+3}(x) + J_{n+5}(x) = \frac{2}{x}(n + 4) J_{n+4}(x)$ Proved.

14. $4J''_n(x) = J_{n-2}(x) - 2J_n(x) + J_{n+2}(x)$

[B.E. 2066, 2069 Poush, 2070 Bhadra]

Solution: We know that the recurrence relation of the Bessel's function.

$2J_n(x) = J_{n-1}(x) - J_{n+1}(x)$ .....(i)

Diff. eq^n (i) w.r.t 'x' we get,

$2J'_n(x) = J'_{n-1}(x) - J'_{n+1}(x)$

Multiplying both sides by 2

$4J'_n(x) = 2J'_{n-1}(x) - 2J'_{n+1}(x)$ .....(ii)

Putting n = n – 1 and n + 1 in (i) resp. we get

$2J_{n-1}(x) = J_{n-2}(x) - J_n(x)$.....(iii)

$2J_{n+1}(x) = J_n(x) - J_{n+2}(x)$ .....(iv)

Putting the value of

$2J_{n-1}(x)$ and $2J_{n+1}(x)$ in (ii)

$4J'_n(x) = J_{n-2}(x) - J_n(x) - J_n(x) + J_{n+2}(x)$

$= J_{n-2}(x) - 2J_n(x) + J_{n+2}(x)$.

$4J'_n(x) - 2J_n(x) + J_{n+2}(x)$ Proved.

15. $4J''_0(x) + 3J'_0(x) + J_3(x) = 0$

Solution: we know that the recurrence relation of the Bessel's function

$2J_n(x) = J_{n-1}(x) - J_{n+1}(x)$ .....(i)

Diff. (i) w.r.t 'x' we get,

$2J'_n(x) = J'_{n-1}(x) - J'_{n+1}(x)$

Multiplying both sides by 2

$4J'_n(x) = 2J'_{n-1}(x) - 2J'_{n+1}(x)$ .....(ii)

Putting n = n – 1 and n + 1 in (i) resp. we get

$2J_{n-1}(x) = J_{n-2}(x) - J_n(x)$.....(iii)

$2J_{n+1}(x) = J_n(x) - J_{n+2}(x)$ .....(iv)

Putting the value of

$2J_{n-1}(x)$ and $2J_{n+1}(x)$ in (ii)

$4J'_n(x) = J_{n-2}(x) - J_n(x) - J_n(x) + J_{n+2}(x)$

$= J_{n-2}(x) - 2J_n(x) + J_{n+2}(x)$

$4J'_n(x) = J_{n-2}(x) - 2J_n(x) + J_{n+2}(x)$ .....(v)

diff. (v) w.r.t 'x'

$4J''_n(x) = J'_{n-2}(x) - 2J'_n(x) + J'_{n+2}(x)$

Multiplying on bothe sides by 2

$8J''_n(x) = 2J'_{n-2}(x) - 4J'_n(x) + J'_{n+2}(x)$ .....(vi)

Put n = n – 2 and n + 2 in (i) resp. we get

$2J'_{n-2}(x) = J_{n-3}(x) - J_{n-1}(x)$ ..... (vii)

and $2J'_{n+2}(x) = J_{n+1}(x) - J_{n+3}(x)$ .....(viii)

Substituting the value of $2J'_{n-2}(x)$, $2J'_n(x)$ and $2J'_{n+2}(x)$ in (vi) we get

$8J'''_n(x) = J_{n-3}(x) - J_{n-1}(x) - 2J_{n-1}(x) + 2J_{n+1}(x) + J_{n+1}(x) - J_{n+3}(x)$ .....

or, $8J_n''' = J_{n-3}(x) - 3J_{n-1}(x) + 3J_{n+1}(x) - J_{n+3}$ ..... (ix)

Putting n = 0 in (ix)

$8J'''_0(x) = J_{-3}(x) - 3J_{-1}(x) + 3J_1(x) - J_3(x)$

$= -J_3(x) + 3J_1(x) + 3J_1(x) - J_3(x)$

Since, $J_{-n}(x) = (-1)^n J_n(x)$.

or, $8J'''_0(x) = -2J_3(x) + 6J_1(x)$

$= -2J_3(x) - J'_0(x)$

Since, $J'_0(x) = -J_1(x)$.

or, $8J'''_0(x) + 2J_3(x) + 6J'_0(x) = 0$

or, $4J'''_0(x) + J_3(x) + 2J'_0(x) = 0$

or, $4J'''_0(x) + 2J'_0(x) + J_3(x) = 0$ Proved.

16. $J'_0(x) = -J_1(x)$

Solution: We know that the recurrence relation of the Bessel function.

$J'_n(x) = nJ_n(x) - xJ_{n+1}(x)$ .....(i)

Putting n = 0 in (i)

or, $xJ'_0(x) = 0.\ J_0(x) - xJ_{0+1}(x)$

or, $xJ'_0(x) = -xJ_1(x)$

or, $\therefore J'_0(x) = -J_1(x)$ Proved.

17. $J_2(x) - J_0(x) = 2J''_0(x)$.

Solution: we know that the recurrence relation of the Bessel function.

$2J'_n(x) = J_{n-1}(x) - J_{n+1}(x)$ .....(i)

$2J''_n(x) = J'_{n-1}(x) - J'_{n+1}(x)$

Multiplying both sides by 2

$4J''_n(x) = 2J'_{n-1}(x) - 2J'_{n+1}(x)$ .....(ii)

Putting n = n – 1 and n + 1 in (i) resp. we get

$2J'_{n-1}(x) = J_{n-2}(x) - J_n(x)$.....(iii)

$2J'_{n+1}(x) = J_n(x) - J_{n+2}(x)$ .....(iv)

Putting the value of

$2J'_{n-1}(x)$ and $2J'_{n+1}(x)$ in (ii)

$4J''_n(x) = J_{n-2}(x) - J_n(x) - J_n(x) + J_{n+2}(x)$

$= J_{n-2}(x) - 2J_n(x) + J_{n+2}(x)$.

$4J''_n(x) = J_{n-2}(x) - 2J_n(x) + J_{n+2}(x)$ .....(v)

Put n = 0 in (v)

$4J''_0(x) = J_{-2}(x) - 2J_0(x) + J_2(x)$

or, $4J''_0(x) = (-1)^2 J_2(x) - 2J_0(x) + J_2(x)$

Since $J_{-n}(x) = (-1)^n J_n(x)$

or, $4J''_0(x) = 2J_2(x) - 2J_0(x)$

or, $2J''_0(x) = J_2(x) - J_0(x)$

or, $J_2(x) - J_0(x) = 2J''_0(x)$ Proved.

$J_3(x) + 3J'_0(x) + 4J'''_0(x) = 0$

Solution: We know that the recurrence relation of the Bessel's function.

Solution: *Same as Q.N (15)*

$J_{-\frac{1}{2}}(x) = J_{\frac{1}{2}}(x)\cot x$.

Solution: we know that

$$J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\cos x \text{ .....(i)}$$

$$\text{and } J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\cos x \text{.....(ii)}$$

dividing (i) by (ii)

$$\frac{J_{-\frac{1}{2}}(x)}{J_{\frac{1}{2}}(x)} = \frac{\sqrt{\frac{2}{\pi x}}\cos x}{\sqrt{\frac{2}{\pi x}}\sin x} \quad \text{or,} \quad \frac{J_{-\frac{1}{2}}(x)}{J_{\frac{1}{2}}(x)} = \frac{\cos x}{\sin x}$$

$J_{-\frac{1}{2}}(x) = J_{\frac{1}{2}}(x)\cot x$ Proved.

$\left[J_{\frac{1}{2}}(x)\right]^2 + \left[J_{-\frac{1}{2}}(x)\right]^2 = \frac{2}{\pi x}$

Solution: We know that,

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\sin x \text{ .....(i) and } J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\cos x \text{ ..... (ii)}$$

Squaring and adding (i) and (ii)

$$\left[J_{\frac{1}{2}}(x)\right]^2 + \left[J_{-\frac{1}{2}}(x)\right]^2 = \left(\sqrt{\frac{2}{\pi x}}\sin x\right)^2 + \left(\frac{\sqrt{2}}{\pi x}\cos x\right)^2$$

$$= \frac{2}{\pi x}[\sin^2 x + \cos^2 x] = \frac{2}{\pi x}$$

$$\left[J_{\frac{1}{2}}(x)\right]^2 + \left[J_{-\frac{1}{2}}(x)\right]^2 = \frac{2}{\pi x} \text{ Proved.}$$

Important IOE Questions and Answers

1. Show that the solutions of $x^2y''' - 3xy'' + 3y' = 0$, $(x > 0)$ are linearly independent. (BE 2069 Fall)

Sol^n. Given differential equation is,

$x^2y''' - 3xy'' + 3y' = 0$ ..... (i) $\qquad x > 0$

Multiplying (i) by x

$x^3y''' - 3x^2y'' + 3xy' = 0$

or, $(x^3D^3 - 3x^2D^2 + 3xD)y = 0$ ..... (ii)

Which is homogeneous differential equation.

So, put $x = ez \Rightarrow z = \log x$

Then, $xD = \delta$ where, $\delta = \dfrac{d}{dz}$

$x^2D^2 = \delta(\delta - 1)$ and $x^3D^3 = \delta(\delta - 1)(\delta - 2)$

Now (ii) becomes,

$(\delta(\delta-1)(\delta - 2) - 3\delta(\delta - 1) + 2\delta)\, y = 0$

$(\delta^3 - 6\delta^2 + 8\delta)\, y = 0$

Its auxiliary equation is, $m^3 - 6m^2 + 8m = 0 \Rightarrow m = 0, 2, 4$

Its general solution in, $y = C_1 + C_2e^{2z} + C_3e^{4y}$

$y = C_1 + C_2x^2 + C_3x^4$

Here, $y_1(x) = 1$, $y_2(x) = x^2$, $y_3(x) = x^4$

$\therefore y_1'(x) = 0$, $y_2'(x) = 2x$, $y_3'(x) = 12x^2$

$$\text{Now, } W(y_1, y_2, y_3) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} 1 & x^2 & x^4 \\ 0 & 2x & 4x^3 \\ 0 & 2 & 12x^2 \end{vmatrix} = 16x^3 \neq 0 \text{ since}$$

$\therefore$ Which shows that $1, x^2, x^4$ are linearly inependent.

2. Find the general solution of Legender's differential equation. (BE 2069 Poush (or), 2068 Bhadra (or), 2070

Sol^n. The Legendre's differential equation is

$(1 - x^2)\, y'' - 2xy' + n(n + 1)\, y = 0$ ..... (i)

Here $P_0(x) = 1 - x^2 \neq 0$, when $x = 0$ so that,

$x = 0$ is an ordinary point.

Let the general solution of (i) is,

$$y = C_0 + C_1x + C_2x^2 + C_3x^3 + \ldots\ldots = \sum_{m=0}^{\infty} C_m x^m \text{ ..... (ii)}$$

Differentiating (i) with respect to 'x'

$$y' = \sum_{m=0}^{\infty} mC_m x^{m-1}, \qquad y'' = \sum_{m=0}^{\infty} m(m-1)\, C_m x^{m-2}$$

Substituting these to (i) we get,

$$(1 - x^2) \sum_{m=0}^{\infty} m(m-1)\, C_m x^{m-2} - 2x \sum_{m=0}^{\infty} mC_m x^{m-1} + n(n+1) \sum_{m=0}^{\infty} C_m x^m = 0$$

$$\text{or, } \sum_{m=2}^{\infty} m(m-1)\, C_m x^{m-2} - \sum_{m=2}^{\infty} m(m-1)\, C_m x^m - 2\sum_{m=1}^{\infty} m\, C_m x^m + n(n+1) \sum_{m=0}^{\infty} C_m x^m = 0$$

or, $2.1C_2 + 3.2C_3x + 4.3C_4x^2 + \ldots (5 + 2)(3 + 1)\, C_{5+2}\, x^5 + \ldots\ldots$

$- 2.1C_2x^2 - \ldots\ldots 0150\ 5(5 - 1)\, C_5\, x^5 - \ldots\ldots - 2.1C_1x - 2.2C_2x^2$

$- \ldots\ldots 2.5\,.\, C_5x^5 - \ldots\ldots + (n + 1)\, C_0 + n(n + 1)\, C_1x + n(n + 1)\, C_2x^2$

$+ \ldots\ldots + n(n + 1)\, C_5x^5 + \ldots\ldots\ldots = 0$

Equating to zero the coefficient of various power of x we get,

$6C_3 + n(n + 1)\, C_1 = 0 \Rightarrow 6C_3 - 2C_1 + n(n + 1)\, C_1 = 0$

In general, $(5 + 2)(5 + 1)\, C_{5+2} - 5(5 - 1)\, C_5 - 25C_5 + n(n + 1)\, C_5 = 0$

or, $(5 + 2)(5 + 1)\, C_{5+2} + (n - 5)(n + 5 + 1)\, C_5 = 0$

$$\Rightarrow C_{5+2} = -\frac{(n-5)(n+5-1)}{(5+2)(5+1)} C_5$$

This is called recurrevee relation, it gives each coefficient in terms of arbitrary constant C_0 and C_1

where s = 0, 1, 2, 3, ...

$$\therefore C_2 = -\frac{n(n+1)}{2!} C_0,\ C_3 = -\frac{(n-1)(n+2)}{3!} C_1$$

$$C_4 = \frac{-(n-2)(n+3)}{4.3} C_2 = n\frac{(n-2)(n+3)(n+1)}{4!} C_0$$

$$C_5 = \frac{-(n-3)(n+4)}{5.4} C_3 = \frac{(n-3)(n-1)(n+2)(n+4)}{5!} C_1$$

and so on putting the values of C_2, C_3, C_4, C_5 ..... etc in (ii) we get,

$$y = C_0\left(1 - \frac{(n+1)}{2!}x^2 + \frac{n(n-2)(n+1)(n+3)}{4!}x^4 - \ldots\ldots\right)$$

$$+ C_1\left(x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!}x^5 + \ldots\ldots\right)$$

$\therefore y = C_0y_1(x) + C_1y_2(x)$ is called the required general solution of the Legendre's equation.

$$\text{Where, } y_1(x) = 1 - \frac{(n+1)}{2!}x^2 + \frac{n(n-2)(n+1)(n+3)}{4!}x^4 + \ldots\ldots$$

$$y_2(x) = x - \frac{n+(n+2)}{3!}x^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!}x^5 + \ldots\ldots$$

3. Prove the Bessell's function $\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)$:

(BE 2069 Bha

Sol^n. We have the Bessell's function is,

$$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(n+m+1)}\left(\frac{x}{2}\right)^{n+2m} \quad \text{..... (i)}$$

$\therefore$ Multiplying both sides by x^n

$$x^n J_n(x) = x^n \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(n+m+1)}\left(\frac{x}{2}\right)^{n+2m}$$

$$x^n J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(n+m+1)}\left(\frac{x}{2}\right)^{2n+2m} \quad \text{..... (ii)}$$

Differentiating (ii) w.r. t. 'x'

$$\frac{d}{dx}[x^n{}_n(x)] = \frac{d}{dx}\sum_{m=0}^{\infty} \frac{(-1)^m 2^n}{m!\,\Gamma(n+m+1)}\left(\frac{x}{2}\right)^{2n+2m}$$

$$= 2^n \sum_{m=0}^{\infty} \frac{(-1)^m (2n+2m)}{m!\,\Gamma(n+m+1)} \cdot \left(\frac{x}{2}\right)^{2n+2m-1} \cdot \frac{1}{2}$$

$$= 2^n \sum_{m=0}^{\infty} \frac{(-1)^m (n+m)}{m!\,(n+m)\,\Gamma(n+m)}\left(\frac{x}{2}\right)^{2n-2m-1}$$

$$= 2^n \cdot \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(n+m)}\left(\frac{x}{2}\right)^{n} \cdot \left(\frac{x}{2}\right)^{n-2m-1}$$

$$= x^n \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(n-1+m+1)}\left(\frac{x}{2}\right)^{n-1+2m} = x^n J_{n-1}(x)$$

$\therefore \frac{d}{dx}[x^n J_n(x)] = x^n J_{n-1}(x)$ Proved.

4. Solve by power series method the differential equation $y'' - 4xy' + (4x^2 - 2)\,y = 0$ **(BE 2068 Bhadra, 2070 Bhadra)**

Sol^n. Given differential equation is, $y'' - 4xy' + (4x^2 - 2)\,y = 0$ ..... (i)

Let us assome that the general solution of (i) is,

$y = c_1 + c_1x + c_2x^2 + c_3x^3 + \ldots + e_nx^n + \ldots$ ..... (ii)

Differentiating (ii) w. r. t. x e get,

$y = c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + \ldots + nc_nx^{4-1} + \ldots$

and $y'' = 2(2x + 3.2.c_3x + 4.3.c_4x^2 + \ldots + n(n-1)\,c_nx^{n-2} + \ldots$

Substituting these in (i) we get,

$2c_2 + 3.2c_3x + 4.3.c_4x^2 + 5.4c_5x^3 + \ldots - 4xc_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + \ldots + (4x^2 - 2)(c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^3 + \ldots) = 0$

or, $(2c_2 - 2c_0) + (6c_3 - 6c_1)\,x + (12c_4 - 10c_2 + 4c_0) + (20c_5 - 14c_3 + 4c_1)\,x^3 + (30c_6 - 18c_4 + 4c_2)\,x^4 + \ldots = 0$

Equating to zero the coefficient of various powers of x we get,

$c_2 = 6_0$

$6c_3 = 6c_1$, i.e. $c_3 = c_1$

$12c_4 - 10c_2 + 4c_0 = 0$ i.e. $c_4 = \frac{c_0}{2}$, $20c_5 - 14c_3 + 4c_1 = 0$ i.e. $c_5 = \frac{c_1}{2}$

$30c_6 - 18c_4 + 4c_2 = 0$ i.e. $c_6 = \frac{5c_0}{30} = \frac{c_0}{6}$ and so on.

Putting these to (ii) we get,

$$y = c_0 + c_1x + c_0x^2 + c_1x^3 + \frac{c_0}{2}x^4 + \frac{c_1}{2}x^5 + \frac{c_0}{6}x^6 + \ldots$$

$$= c_0\left(1 + x^2 + \frac{x^4}{2} + \frac{x^6}{6} + \ldots\right) + c_1\left(x + x^3 + \frac{x^5}{2} + \ldots\right)$$ is the required general solution.

5. Prove that $\frac{d}{dx}[x^{-n}J_n(x)] = -x^{-n}J_{n+1}(x)$.

(BE 2068 Bhadra (or), 2070 Magh)

Sol^n. We have, $J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(n+m+1)}\left(\frac{x}{2}\right)^{n+2m}$

Multiplying both sides by x^{-n}

$$x^{-n} J_n(x) = x^{-n} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(n+m+1)}\left(\frac{x}{2}\right)^{n+2m} = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(n+m+1)} 2^{-n}\left(\frac{x}{2}\right)^{2m}$$

Differentiating w. r. t. 'x'

$$\frac{d}{dx}(x^{-n}J_nx) = \frac{d}{dx}\left[\sum_{m=0}^{\infty} \frac{(-1)^m 2^{-n}}{m!\,\Gamma(n+m+1)}\left(\frac{x}{2}\right)^{2m}\right]$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m 2^{-n}\, 2m 2^{-n}}{m!\,\Gamma(n+m+1)}\left(\frac{x}{2}\right)^{2m-1} = x^{-n} \sum_{m=1}^{\infty} \frac{(-1)^m}{(m-1)!\,\Gamma(n+m+1)}\left(\frac{x}{2}\right)^{n+2m-1}$$

Taking $m - 1 = s$, we get,

$$\frac{d}{dx}[x^{-n}J_n(x)] = x^{-n} \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,\Gamma(n+s+1)}\left(\frac{x}{2}\right)^{n+1+2s}$$

$$= -x^{-n} \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,\Gamma(n+s+1)}\left(\frac{x}{2}\right)^{n+1+2s} = -x^{-n} J_{n+1}(x)$$

$\therefore \frac{d}{dx}[x^{-n}J_n(x)] = -x^{-n} J_{n+1}(x)$ Proved.

6. **Solve in series:** $(1+x^2)\dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}-y=0$ **(BE 2067 Mangsir)**

Sol^n. Given differential equation is, $(1+x^2)\dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}-y=0$

or, $(1+x^2)\,y''+xy'-y=0$ ..... (i)

Let us, assume that the general solution of (i) is

$y=c_0+c_1x+c_2x^2+c_3x^3+c_4x^4+c_5x^5+c_6x^6+$ ..... (v)

Diff. w. r. t. 'x'

$y'=c_1+2c_2x+3c_3x^2+4c_4x^3+5c_5x^4+6c_6x^5+$ .....

$y''=2c_2+3.2c_3x+4.3c_4x^2+5.4.c_5x^3+6.5c_6x^4+$ ....

Substituting these values in (i)

$(1+x^2)(2c_2+3.2c_3x+4.3c_4x^2+5.4c_5x^3+6.5c_6x^4+\ldots)$

$+x(c_1+2c_2x+3c_3x^2+4c_4x^2+5c_5x^4+6c_6x^5+\ldots)$

$-(c_0+c_1x+c_2x^2+c_3x^3+c_4x^4+c_5x^5+c_6x^6+\ldots)=0$

or, $2c_2+3.2c_3x+4.3c_4x^2+5.4c_5x^3+6.5c_6x^4+\ldots+2c_2x^2+3.2c_3x^3$

$+4.3c_4x^4+\ldots+c_1x+2c_2x^2+3c_3x^3+4c_4x^4+\ldots$

$-c_0-c_1x-c_2x^2-c_3x^3-c_4x^4$ .....

Equating the various coefficient of x with zero we get,

$2c_2-c_0\Rightarrow c_2=\dfrac{c_0}{2}$

$6c_3+c-c_1=0\Rightarrow c_3=0$

$12c_4=2c_2+2c_2-c_2\Rightarrow c_4=-3\dfrac{c_2}{12}=-\dfrac{c_0}{8}$

$20c_5+6c_3+3c_3-c_3=0\Rightarrow c_3=0$

$30c_6+12c_4+4c_4-c_4=0\Rightarrow c_6=\dfrac{15c_4}{-30}=\dfrac{c_0}{16}$ and so on.

Substituting the value of c_2, c_3, c_4, c_5, c_6 ..... in (ii)

$y=c_0\left(1+\dfrac{1}{2}x^2-\dfrac{1}{8}x^4+\dfrac{1}{16}x^6-\ldots\right)+c_1x$ is the required general solution.

7. **Show that:** $P_n(x)=\dfrac{1}{2^n n dx^n}(x^2-1)^n$ **(BE 2067 Mangsir)**

Sol^n. Let $v-(x^2-1)^n$

Then $\dfrac{dv}{dx}=v_1=n(x^2-1)^{n-1}.2x$

or, $(x^2-1)\,v_1=2nx\,(x^2-1)^n$ or, $(x^2-1)\,v_1=2nxv$

or, $(1-x^2)\,v_1+2nxv=0$

Differentiating it (n + 1) times by Leibnitz's theorem, we get,

$(1-x^2)\,v_{n+2}+(n+1)(-2n)\,v_{n+1}+\dfrac{(n+1)n}{2}(-2)\,v_n$

$+2n\,[xv_{n+1}+(n+1)\,v_n]=0$

or, $(1-x^2)\dfrac{d^2v_n}{dx^2}-2x\dfrac{d(v_n)}{dx}+n(n+1)\,v_n=0$

This is Legendre's differential equation and its solution is $P_n(x)$.

So, $P_n(x)=Cv_n=C\dfrac{d^n}{dx^n}(x^2-1)^n=C\dfrac{d^n}{dx^n}[(x-1)^n(x+1)^n]$

$$=c\left[(x-1)^n\frac{d^n}{dx^n}(x+1)^n+{}^nC_1n(x-1)^{n-1}\frac{d^{n-1}}{dx^{n-1}}(x+1)^n+\ldots+(x+1)^2\frac{d^n}{dx^n}(x-1)^n\right]$$

$$=c\left[(x-1)^n n!+{}^nC_1n(x-1)^{n-1}\frac{d^{n-1}}{dx^{n-1}}(x+1)^n+\ldots+(x+1)^n n!\right]$$

To determine the constiatns, C.

Put x = 1 we get,

$P_n(1)=C\,[0+0+0\ldots+2^n n!]$ or, $1=c2^n n!$ $\therefore c=\dfrac{1}{2^n n!}$

Substituting the values of c in (i) we get,

$p_n(x)=\dfrac{1}{2^n n!}\dfrac{d^n}{dx^n}\left(x^2-1\right)^n$ Proved.

Vector Algebra and Calculus

5

Product of three vectors

The scalar triple product is the scalar product of two vectors in which one of them must be vector produce of two vectors.

If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are three vectors. Then the scalar or dot product of $\vec{a} \times \vec{b}$ with $\vec{c}$ is called the scalar product of three vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ and is denoted by $\vec{a}.\vec{b} \times \vec{c} = [\vec{a}\ \vec{b}\ \vec{c}]$

Note : $(\vec{a} \times \vec{b}).\vec{c} = (\vec{b} \times \vec{c}).\vec{a} = (\vec{c} \times \vec{a}).\vec{b} = V$

represent the volume of prarallelepiped forming by the sides $\vec{a}$, $\vec{b}$ and $\vec{c}$.

Proporties of scalar triple product

- The scalar triple product of unit vector, $\vec{i}$, $\vec{j}$ and $\vec{k}$ is unity.

  i.e. $[\vec{i}, \vec{j}, \vec{k}] = 1$

- Scalar triple product of vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ can be expressed in determinant from.

  If $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$

  $\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$

  $\vec{c} = c_1\vec{i} + c_2\vec{j} + c_3\vec{k}$

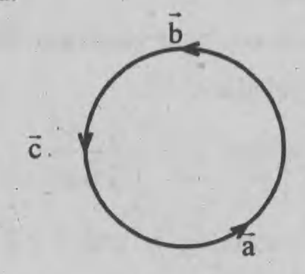
  Then,

  $$[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

- In a scalar triple product, the position of dot and cross can be interchanged without changing its value.

  i.e. $(\vec{a} \times \vec{b}).\vec{c} = \vec{a}.(\vec{b} \times \vec{c})$

- The value of scalar triple product of three vectors remains unchanged if the vectors are in cyclic order otherwise the value of scalar triple product be changed.



i.e. $\vec{a}.(\vec{b} \times \vec{c}) = \vec{b}.(\vec{c} \times \vec{a}) = \vec{c}.(\vec{a} \times \vec{b})\ \vec{b}$

i.e. $[\vec{a}\,\vec{b}\,\vec{c}] = [\vec{b}\,\vec{c}\,\vec{a}] = [\vec{c}\,\vec{a}\,\vec{b}]$

- Three vectors are coplanar if their scalier triple product be zero.
- The value of scalar triple product be zero when two of the vector are equal.

  i.e. $[\vec{a}\,\vec{a}\,\vec{c}] = 0$

- The value of scalar triple product be zero when two of the vector are parallel.

Vector tipple product

Vector triple product is also cross product of two vectors in which one of them must be cross product of two vectors.

Note : $\vec{a} \times (\vec{b} \times \vec{c}) \neq (\vec{a} \times \vec{b}) \times \vec{c}$

Expression for vector triple product

If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are any three vectors then,

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a}.\vec{c})\,\vec{b} - (\vec{a}.\vec{b})\,\vec{c}$$

Product of four vectors

Scalar product of four vectors is also scalar product of two vectors in which both of then must be cross product of two vectors.

If $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$ are four vectors, then vectors is

$(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d})$ where $(\vec{a} \times \vec{b})$ and $(\vec{c} \times \vec{d})$ are two vectors quantities.

Determination of $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d})$

$$(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = [\vec{a}\,\vec{c}\,\vec{d}]\,\vec{b} - [\vec{b}\,\vec{c}\,\vec{d}]\,\vec{a}$$

and

$$\left(\vec{a}\times\vec{b}\right)\times\left(\vec{c}\times\vec{d}\right) = \left[\vec{a}\,\vec{b}\,\vec{d}\right]\vec{c} - \left[\vec{a}\,\vec{b}\,\vec{c}\right]\vec{d}$$

Reciprocal system of vectors

Let $\vec{a}'$, $\vec{b}'$, $\vec{c}'$, be a set of non-coplanar vectors so that $\left[\vec{a}\,\vec{b}\,\vec{c}\right] \neq 0$ then the three vectors defined by the relations.

$$\vec{a}' = \frac{\vec{b}\times\vec{c}}{\left[\vec{a}\,\vec{b}\,\vec{c}\right]},\ \vec{b}' = \frac{\vec{c}\times\vec{a}}{\left[\vec{a}\,\vec{b}\,\vec{c}\right]} \text{ and } \vec{c}' = \frac{\vec{a}\times\vec{b}}{\left[\vec{a}\,\vec{b}\,\vec{c}\right]}$$

respectively to the planes containing $\vec{b}$ and $\vec{c}$, $\vec{c}$ and $\vec{a}$, $\vec{a}$ and $\vec{b}$ are called reciprocal system to $\vec{a}$, $\vec{b}$, $\vec{c}$.

Properties of reciprocal system of vectors

- If $\vec{a}$, $\vec{b}$, $\vec{c}$ are $\vec{a}'$, $\vec{b}'$, $\vec{c}'$ are reciprocal systems of vectors. Then,

  $\vec{a}\,.\,\vec{a}' = \vec{b}\,.\,\vec{b}' = \vec{c}\,.\,\vec{c}' = 1$
- The dot product of any vector of one system with a vector of the other system which does not correspond to it, is zero.

  i.e. $\vec{a}\,.\,\vec{b}' = \vec{b}'\,.\,\vec{c}' = \vec{c}\,.\,\vec{a} = 0$
- The vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{a}'$, $\vec{b}'$, $\vec{c}'$ are mutually reciprocal

  i.e. $\vec{a}' = \dfrac{\vec{b}\times\vec{c}}{\left[\vec{a}\,\vec{b}\,\vec{c}\right]}$, $\vec{b}' = \dfrac{\vec{c}\times\vec{a}}{\left[\vec{a}\,\vec{b}\,\vec{c}\right]}$ and $\vec{c}' = \dfrac{\vec{a}\times\vec{b}}{\left[\vec{a}\,\vec{b}\,\vec{c}\right]}$ then

  $$\vec{a} = \frac{\vec{b}'\times\vec{c}'}{\left[\vec{a}'\,\vec{b}'\,\vec{c}'\right]},\ \vec{b} = \frac{\vec{c}'\times\vec{a}'}{\left[\vec{a}'\,\vec{b}'\,\vec{c}'\right]} \text{ and } \vec{c} = \frac{\vec{a}'\times\vec{b}'}{\left[\vec{a}'\,\vec{b}'\,\vec{c}'\right]}$$
- The scalar triple product $\left[\vec{a}\,\vec{b}\,\vec{c}\right]$ and $\left[\vec{a}'\,\vec{b}'\,\vec{c}'\right]$ are reciprocally connected.

  i.e. $\left[\vec{a}'\,\vec{b}'\,\vec{c}'\right] = \dfrac{1}{\left[\vec{a}'\,\vec{b}'\,\vec{c}'\right]}$
- The orthonormal vector triad $\vec{i}$, $\vec{j}$, $\vec{k}$ is itself reciprocal.

  i.e. $\vec{i}' = \vec{i}$, $\vec{j}' = \vec{j}$ and $\vec{k}' = \vec{k}$
- Any vector $\vec{r}$ may be expressed as a linear combination of the non coplanar vectors $\vec{a}$, $\vec{b}$, $\vec{c}$

  $$\vec{r} = \left(\vec{r}\,.\,\vec{a}\right)\vec{a} + \left(\vec{r}\,.\,\vec{b}\right)\vec{b} + \left(\vec{r}\,.\,\vec{c}\right)\vec{c}$$
- Any vector $\vec{r}$ may be expressed as linear combination of $\vec{a}'$, $\vec{b}'$, $\vec{c}'$

  i.e. $\vec{r} = \left(\vec{r}\,.\,\vec{a}\right)\vec{a}' + \left(\vec{r}\,.\,\vec{b}\right)\vec{b}' + \left(\vec{r}\,.\,\vec{c}\right)\vec{c}'$

Vector differential operater's

The vector differential operator del is denoted by ∇. It is defined as $\nabla = \left(\vec{i}\dfrac{\partial}{\partial x} + \vec{j}\dfrac{\partial}{\partial y} + \vec{k}\dfrac{\partial}{\partial z}\right)$

$$= \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$\text{div } \vec{v} = \nabla\,.\,\vec{v}$$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right).(v)$$

$$= \left(\vec{i}\frac{\partial}{\partial x}\vec{v} + \vec{j}\frac{\partial}{\partial y}\vec{v} + \vec{k}\frac{\partial}{\partial z}\vec{v}\right)$$

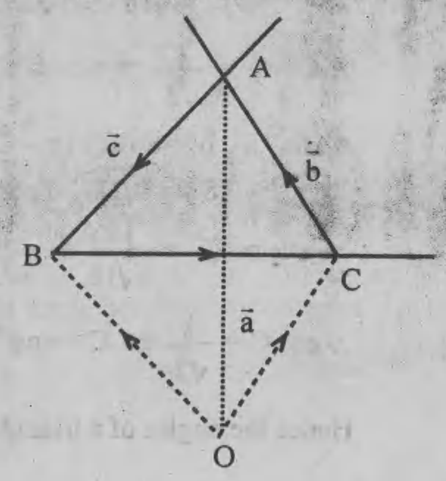
$$\text{curl } \vec{v} = \nabla\,.\,\vec{v}$$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\times\vec{v}$$

Exercise - 19

1. **Find the angles of a triangle whose vertices are $7\vec{j}+10\vec{k}$, $-\vec{i}+6\vec{j}+6\vec{k}$ and $-4\vec{i}+9\vec{j}+6\vec{k}$**

Solution: Let the given vertifces of triangle are A, B and C. Let O be the origion and A. B and C be the position vectors, then,



$\overrightarrow{OA} = 7\vec{j} + 10\vec{k}$

$\overrightarrow{OB} = -\vec{i} + 6\vec{j} + 6\vec{k}$

and $\overrightarrow{OC} = -4\vec{i} + 9\vec{j} + 6\vec{k}$

Now, $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= (-\vec{i} + 6\vec{j} + 6\vec{k}) - (7\vec{j} + 10\vec{k})$

$\therefore\ \vec{c} = -\vec{i} - \vec{j} - 4\vec{k}$

Also, $|\overrightarrow{AB}| = |-\vec{i} - \vec{j} - 4\vec{k}|$

or, $c = \sqrt{1+1+16}$

$= \sqrt{18}$

Again, $\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$

$$= (-4\vec{i} + 9\vec{j} + 6\vec{k}) - (-\vec{i} + 6\vec{j} + 6\vec{k})$$

$\therefore\ \vec{a} = -3\vec{i} + 3\vec{j}$

Also, $|\overline{BC}| = |-3\vec{i} + 3\vec{j}|$

$$a = \sqrt{9+9} = \sqrt{18}$$

Again, $\overline{CA} = \overline{OA} - \overline{OC}$

$$= (7\vec{j} + 10\vec{k}) - (-4\vec{i} + 9\vec{j} + 6\vec{k})$$

$\vec{b} = 4\vec{i} - 2\vec{j} + 4\vec{k}$

Also, $|\overline{CA}| = \left|4\vec{i} - 2\vec{j} + 4\vec{k}\right|$

$$b = \sqrt{16+4+16} = \sqrt{36} = 6$$

Again, $\vec{a}\,.\,\vec{c} = (-3\vec{i} + 3\vec{j})\,.\,(-\vec{i} - \vec{j} - 4\vec{k})$

$$= 3 - 3$$
$$= 0$$

$\therefore\ \vec{a}$ is perpendicular to $\vec{c}$.***

Again, $\vec{a}\,.\,\vec{b} = (-3\vec{i} + 3\vec{j})\,.\,(4\vec{i} - 2\vec{j} + 4\vec{k})$

$$= -12 - 6 = -18$$

Again, $\vec{b}\,.\,\vec{C} = (4\vec{i} - 2\vec{j} + 4\vec{k})\,.\,(-\vec{i} - \vec{j} - 4\vec{k})$

$$= -4 + 2 - 16$$
$$= -18$$

We have, $\vec{b}\,.\,\vec{C} = bc\cos(\pi - A)$

$$-18 = -6\,.\,\sqrt{18}\ \cos A$$

or, $\text{Cos } A = \dfrac{18}{6\sqrt{18}} = \dfrac{3}{3\sqrt{2}} = \dfrac{1}{\sqrt{2}}$

$\therefore\ \text{Cos } A = \dfrac{1}{\sqrt{2}} \Rightarrow A = \cos^{-1}\left(\dfrac{1}{\sqrt{2}}\right) = \dfrac{\pi}{4}$

Again, $\vec{a}\,.\,\vec{b} = ab\cos(\pi - C)$

$$-18 = -\sqrt{18}\ .\,6\cos C$$

or, $\cos C = \dfrac{18}{6\sqrt{18}} = \dfrac{3}{3\sqrt{2}} = \dfrac{1}{\sqrt{2}}$

$\therefore\ \cos C = \dfrac{1}{\sqrt{2}} \Rightarrow C = \cos^{-1}\left(\dfrac{1}{\sqrt{2}}\right) = \dfrac{\pi}{4}$

Hence the angles of a triangles are $\dfrac{\pi}{2}, \dfrac{\pi}{4}, \dfrac{\pi}{4}$ Ans.

Find the unit vector perpendicular to each of the vector $2\vec{i} - \vec{j} + \vec{k}$ and $3\vec{i} + 4\vec{j} - \vec{k}$ and calculate the since of the angle between the two vectors.

Solution: Here, $\vec{a} = 2\vec{i} - \vec{j} + \vec{k}$

and $\vec{b} = 3\vec{i} + 4\vec{j} - \vec{k}$

We know that the vector $\vec{a} \times \vec{b}$ is

Perpendicular to both $\vec{a}$ and $\vec{b}$

So, $\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & 1 \\ 3 & 4 & -1 \end{vmatrix}$

$$= (1-4)\,\vec{i} + (3+2)\,\vec{j} + (8+3)\,\vec{k}$$

$\vec{a} \times \vec{b} = -3\vec{i} + 5\vec{j} + 11\,\vec{k}$

$|\vec{a} \times \vec{b}| = \left|-3\vec{i} + 5\vec{j} + 11\,\vec{k}\right|$

$$= \sqrt{9+25+121}$$
$$= \sqrt{155}$$

The unit vector along $\vec{a} \times \vec{b}$ is

$$= \frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$$

$= \dfrac{1}{\sqrt{155}}\,(-3\vec{i} + 5\vec{j} + 11\,\vec{k})$ Ans.

Also, $|\vec{a}| = \left|2\vec{i} - \vec{j} + \vec{k}\right|$

$$= \sqrt{4+1+1} = \sqrt{6}$$

and, $|\vec{b}| = \left|3\vec{i} + 4\vec{j} - \vec{k}\right|$

$$= \sqrt{9+16+1}$$
$$= \sqrt{26}$$

We have,

$\vec{a} \times \vec{b} = |\vec{a}|\,|\vec{b}|\sin\theta$ where θ is the angle between two vectors $\vec{a}$ and $\vec{b}$.

$\dfrac{1}{\sqrt{155}} = \sqrt{6}\ \sqrt{26}\ \sin\theta$

$\sin\theta = \dfrac{\sqrt{155}}{\sqrt{6 \times 26}} = \dfrac{155}{\sqrt{156}}$

$\therefore \theta = \sin^{-1}\left(\frac{155}{\sqrt{156}}\right)$ Ans.

3. If $|\vec{a}+\vec{b}| = |\vec{a}-\vec{b}|$ then show that $\vec{a}$ and $\vec{b}$ are perpendicular.

Solution: Given $|\vec{a}+\vec{b}| = |\vec{a}-\vec{b}|$

or, $|\vec{a}+\vec{b}|^2 = |\vec{a}-\vec{b}|^2$

or, $(\vec{a}+\vec{b})^2 = (\vec{a}-\vec{b})^2$

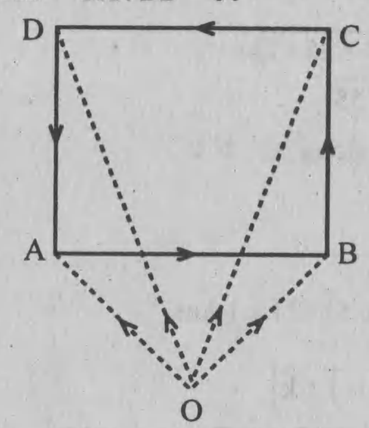
or, $\vec{a}^2 + 2\vec{a}.\vec{b} + \vec{b}^2 = \vec{a}^2 - 2\vec{a}.\vec{b} + \vec{b}^2$

or, $4\vec{a}.\vec{b} = 0$

$\Rightarrow \vec{a}\times\vec{b} = 0$

$\therefore$ $\vec{a}$ is perpendicular to $\vec{b}$ proved.

4. If A, B, C and D are any four points, show that $\overrightarrow{AB}.\overrightarrow{CD} + \overrightarrow{BC}.\overrightarrow{AD} + \overrightarrow{CA}.\overrightarrow{BD} = 0$.



Solution: Given A, B, C and D are any four points. and let O be the origin and $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$, $\overrightarrow{OC} = \vec{c}$, $\overrightarrow{OD} = \vec{d}$

Then, $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \vec{b} - \vec{a}$

$\overrightarrow{CD} = \overrightarrow{OD} - \overrightarrow{OC} = \vec{d} - \vec{c}$ 54

$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = \vec{c} - \vec{b}$

$\overrightarrow{BD} = \overrightarrow{OD} - \overrightarrow{OB} = \vec{d} - \vec{b}$

$\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA} = \vec{d} - \vec{a}$

$\overrightarrow{CA} = \overrightarrow{OA} - \overrightarrow{OC} = \vec{a} - \vec{c}$

Now, L. H. S.

$\overrightarrow{AB}.\overrightarrow{CD} + \overrightarrow{BC}.\overrightarrow{AD} + \overrightarrow{CA}.\overrightarrow{BD}$

$= (\vec{b}-\vec{a}).(\vec{d}-\vec{c}) + (\vec{c}-\vec{b}).(\vec{d}-\vec{a}) + (\vec{a}-\vec{c}).(\vec{d}-\vec{b})$

$= \vec{b}.\vec{d} - \vec{b}.\vec{c} - \vec{a}.\vec{d} + \vec{a}.\vec{c} + \vec{c}.\vec{d} - \vec{c}.\vec{a} - \vec{b}.\vec{d} + \vec{b}.\vec{a}$
$+ \vec{a}.\vec{d} - \vec{a}.\vec{b} - \vec{c}.\vec{d} + \vec{c}.\vec{b} = 0$

using $\vec{a}.\vec{b} = \vec{b}.\vec{a}$

$= 0$ R. H. S.

$\therefore \overrightarrow{AB}.\overrightarrow{CD} + \overrightarrow{BC}.\overrightarrow{AD} + \overrightarrow{CA}.\overrightarrow{BD} = 0$ Proved.

If $\hat{a}$ and $\hat{b}$ are unit vectors and θ is the angle between them, show that

$\sin\frac{\theta}{2} = \frac{1}{2}|\hat{a}-\hat{b}|$

Solution: Since $\vec{a}$ and $\vec{b}$ are unit vectors

$\therefore |\vec{a}| = 1,\ |\vec{b}| = 1$ ..... (i)

So, $\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

$\therefore \vec{a}.\vec{b} = \cos\theta$ ..... (ii)

Now, R. H. S.

$\frac{1}{2}|\vec{a}-\vec{b}| = \sqrt{\frac{1}{4}(\vec{a}-\vec{b})^2}$

$= \frac{1}{2}\sqrt{\vec{a}^2 - 2\vec{a}.\vec{b} + \vec{b}^2}$

$= \frac{1}{2}\sqrt{|\vec{a}|^2 - 2\vec{a}.\vec{b} + |\vec{b}|^2}$

$= \frac{1}{2}\sqrt{1 - 2\cos\theta + 1}$ By using (i) and (ii)

$= \frac{1}{2}\sqrt{2 - 2\cos\theta} = \frac{1}{2}\sqrt{2(1-\cos\theta)} = \frac{1}{2}\sqrt{2\times 2\sin^2\frac{\theta}{2}}$

$= \sin\frac{\theta}{2}$ R. H. S. Hence proved.

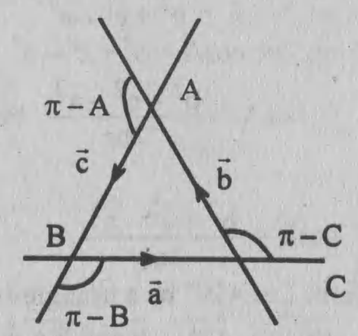
Prove vectorially.

$a = b\cos C + c\cos B$

Solution: Let ABC be triangle such that

$\overrightarrow{AB} = \vec{c}$, $\overrightarrow{BC} = \vec{a}$ and $\overrightarrow{CA} = \vec{b}$

From figure



$\overrightarrow{BC} = \overrightarrow{BA} + \overrightarrow{AC}$

or, $\vec{a} = -\vec{c} - \vec{b}$

Taking dot product on both sides by $\vec{a}$ we get

$\vec{a} \cdot \vec{a} = -(\vec{a} \cdot \vec{c}) - (\vec{a} \cdot \vec{b})$

or, $a^2 = -[ac \cos(\pi - B) + ab \cos(\pi - C)]$

or, $a^2 = a\,c \cos B + ab \cos C$

or, $a^2 = a\,[c\cos B + b\cos C]$

$\therefore a = b\cos C + c \cos B$ Proved.

(ii) $C = a\cos B + b \cos A$

Solution: From same figure of (i)

$\overrightarrow{AB} = \overrightarrow{AC} + \overrightarrow{CB}$

or, $\vec{c} = -\vec{b} - \vec{a}$

Taking dot producat both sides by $\vec{c}$ we get

$\vec{c} \cdot \vec{c} = -(\vec{b} \cdot \vec{c}) - (\vec{a} \cdot \vec{c})$

or, $c^2 = -bc \cos(\pi - A) - ac \cos(\pi - B)$

or, $c^2 = bc \cos A + ac \cos B$

or, $c^2 = c\,[b\cos A + a \cos B]$

$\therefore c = a\cos B + b\cos A$ Proved.

(iii) $\text{Cos}A = \dfrac{b^2 + c^2 - a^2}{2bc}$

Solution: Let ABC be a triangle such that,

$\overrightarrow{AB} = \vec{c}$, $\overrightarrow{BC} = \vec{a}$ and $\overrightarrow{AC} = \vec{b}$

Then, from figure

$\overrightarrow{BC} = \overrightarrow{BA} + \overrightarrow{AC}$

$\vec{a} = -\vec{c} + \vec{b}$

$\therefore \vec{a} = \vec{b} - \vec{c}$

Now squaring on both sides

$(\vec{a})^2 = (\vec{b} - \vec{c})^2$

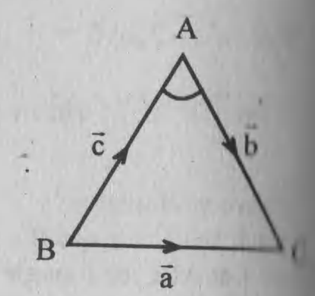
or, $a^2 = \vec{b}^2 - 2\vec{b} \cdot \vec{c} + \vec{c}^2$

or, $a^2 = \vec{b}^2 + \vec{c}^2 - 2\vec{b} \cdot \vec{c}$

or, $2\vec{b} \cdot \vec{c} = b^2 + c^2 - a^2$

or, $2bc \cos A = b^2 + c^2 - a^2$

$\therefore \cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$ Proved.



(iv) $\cos B = \dfrac{b^2 + c^2 - a^2}{2ac}$

Solution: Let ABC be a triangle such that

$\overrightarrow{BC} = \vec{a}$, $\overrightarrow{CA} = \vec{b}$ and $\overrightarrow{BA} = \vec{c}$

Then from figure

$\overrightarrow{CA} = \overrightarrow{CB} + \overrightarrow{BA}$

$\vec{b} = -\vec{a} + \vec{c}$

$\vec{b} = \vec{c} - \vec{a}$

Now squaring on both sides,

$(\vec{b})^2 = (\vec{c} - \vec{a})^2$

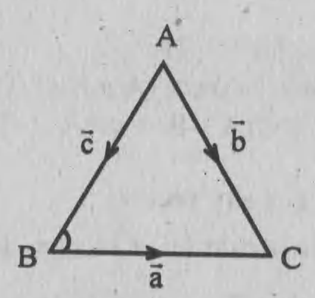
$b^2 = \vec{c}^2 - 2\vec{c} \cdot \vec{a} + \vec{a}^2$

or, $b^2 = c^2 - 2\vec{c} \cdot \vec{a} + a^2$

or, $2\vec{c} \cdot \vec{a} = c^2 + a^2 - b^2$

or, $2ca \cos B = c^2 + a^2 - b^2$

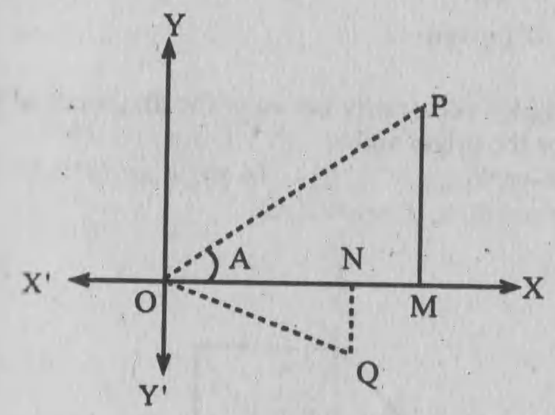
$\cos B = \dfrac{c^2 + a^2 - a^2}{2ac}$ Proved.



$\cos(A + B) = \cos A \cos B - \sin A \sin B$

Solution: Let OX and OY be the two co-ordinate axes. Let P and Q be two points such that $\angle XOP = A$, $\angle XOQ = B$

$\therefore \angle OPQ = A + B$



$\therefore \overrightarrow{OP} = (OM, MP)$

$= (OP \cos A, OP \sin A)$

$\overrightarrow{OQ} = (ON, -NQ)$

$= (OQ \cos B, -OQ \sin B)$

Since $\angle POQ = A + B$

$\therefore \overrightarrow{OP} \cdot \overrightarrow{OQ} = |\overrightarrow{OP}|\,|\overrightarrow{OQ}| \cos(A + B)$

$= (OP\,(OQ) \cos(A + B)$ ..... (i)

Again,

$\overrightarrow{OP} \cdot \overrightarrow{OQ} = (OP \cos A, OP \sin A) \cdot (OQ \cos B, -OQ\sin B)$

$= OP\,OQ \cos A \cos B - OP\,OQ \sin A \sin B$

$\overrightarrow{OP} \cdot \overrightarrow{OQ} = (OP)\,(OQ)\,(\cos A \cos B - \sin A \sin B)$ ..... (ii)

From (i) & (ii)

(OP) (OQ) cos (A + B) = (OP) (OQ) [cosA cosB – sinA sinB]

$\therefore$ cos (A + B) = cosA cosB – sinA sinB Proved.

7. If $\vec{a}$ is any vector,

Prove that $(\vec{a}.\vec{i})\vec{i}+(\vec{a}.\vec{j})\vec{j}+(\vec{a}.\vec{k})\vec{k}=\vec{a}$.

Solution: Let $\vec{a} = a_1\vec{i}+a_2\vec{j}+a_3\vec{k}$

Where $\vec{i}$, $\vec{j}$ and $\vec{k}$ are multually perpendicular vector.

Now, L. H. S.

$$= (\vec{a}.\vec{i})\vec{i}+(\vec{a}.\vec{j})\vec{j}+(\vec{a}.\vec{k})\vec{k}$$

$$= \left\{(a_1\vec{i}+a_2\vec{j}+a_3\vec{k}).\vec{i}\right\}\vec{i}+\left\{(a_1\vec{i}+a_2\vec{j}+a_3\vec{k}).\vec{j}\right\}\vec{j}$$

$$+\left\{(a_1\vec{i}+a_2\vec{j}+a_3\vec{k})\right\}\vec{k}$$

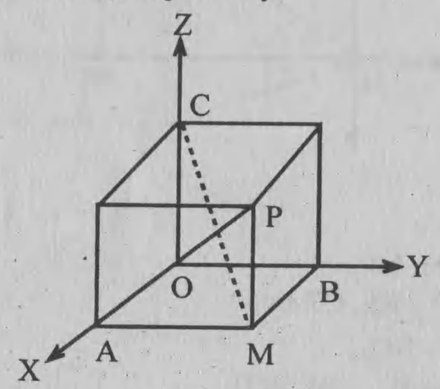
$$= a_1\vec{i}+a_2\vec{j}+a_3\vec{k}$$

$= \vec{a}$ R. H. S. Proved.

8. Find the angles vectorially between the diagonals of a cube.

Solution: Let O be the origin and let OA = OB = OC = a

Then the co-ordinates of A, B, C, M and P are (a, 0, 0), (0, a, 0), (0, 0, a), (a, a, 0) and (a, a, a) respectively.



Now, $\overrightarrow{OP}$ = (a, a, a)

and $\overrightarrow{CM}$ = (a – 0, a – 0, 0 – a)

= (a, a, – a)

Again $\overrightarrow{OP}$. $\overrightarrow{CM}$ = (a, a, a) . (a, a, – a)

$= a^2 + a^2 - a^2$

$= a^2$

$|\overrightarrow{OP}| = |(a, a, a)| = \sqrt{a^2+a^2+a^2} = \sqrt{3a^2} = a\sqrt{3}$

and $|\overrightarrow{CM}| = |(a, a, -a)| = \sqrt{a^2+a^2+a^2} = \sqrt{3a^2} = a\sqrt{3}$

Let θ be the angle between the two diayonals, Then,

$$\cos\theta = \frac{\overrightarrow{OP}\,.\,\overrightarrow{CM}}{|\overrightarrow{OP}|\,|\overrightarrow{CM}|}$$

$$= \frac{a^2}{(a\sqrt{3})(a\sqrt{3})} = \frac{1}{3}$$

$\therefore \theta = \cos^{-1}\left(\frac{1}{3}\right)$ Ans.

9. If a line makes angle α, β, γ, δ with four diagonais of a cube prove that

$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = \frac{4}{3}$.

Solution: Let O be the origin and OA, OB, OC be the edge of the cube AQBRCSA each of length 1 unit lies an x, y, z axis respectively. The co-ordinates of A, B, C are (1, 0, 0), (0, 1, 0) and (0, 0, 1) respectively, co-ordinates of Q, R, S are (1, 1, 0), (0, 1, 1) and (1, 0, 1) respectively.

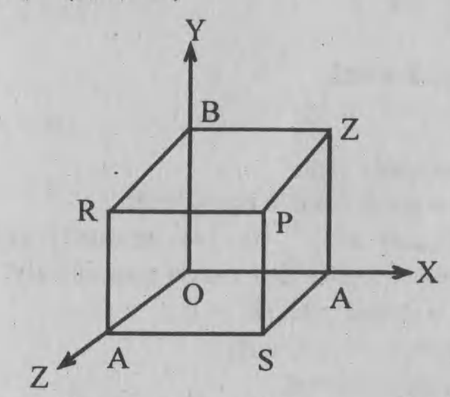
Similarly, $\overrightarrow{AR} = \overrightarrow{OR} - \overrightarrow{OA}$

= (0, 1, 1) – (1, 0, 0)

= (– 1, 1, 1)

$\overrightarrow{BS}$ = (1, – 1, 1)

$\overrightarrow{CQ}$ = (1, 1, – 1)



Let OT by any line whose position vector is xyz and it makes an angle α, β, γ, δ with the vectors.

$\overrightarrow{OP}$, $\overrightarrow{AR}$, $\overrightarrow{CQ}$, $\overrightarrow{BS}$ respectively.

Now, $\cos\alpha = \overrightarrow{OT}\,.\,\overrightarrow{OP}$

$$= \frac{(x, y, z) \cdot (1, 1, 1)}{\sqrt{x^2+y^2+z^2}\sqrt{1+1+1}}$$

$$= \frac{x+y+z}{\sqrt{3}\sqrt{x^2+y^2+z^2}}$$

Similarly, $\cos\beta = \overrightarrow{OT} \cdot \overrightarrow{AR}$

$$\frac{-x+y+z}{\sqrt{3}\sqrt{x^2+y^2+z^2}}$$

$$\cos\gamma = \frac{\overrightarrow{OT} \cdot \overrightarrow{CQ}}{|\overrightarrow{OT}| \times |\overrightarrow{CQ}|} = \frac{x+y-z}{\sqrt{3}\sqrt{x^2+y^2+z^2}}$$

and $\cos\delta = \dfrac{\overrightarrow{OT} \cdot \overrightarrow{BS}}{|\overrightarrow{OT}| \times |\overrightarrow{BS}|}$

$$= \frac{x-y+z}{\sqrt{3}\sqrt{x^2+y^2+z^2}}$$

Now, L. H. S.

$= \cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta$

$$= \frac{(x+y+z)^2+(-x+y+z)^2+(x+y-z)^2+(x-y+z)^2}{3(x^2+y^2+z^2)}$$

$$= \frac{4(x^2+y^2+z^2)}{3(x^2+y^2+z^2)}$$

$= \frac{4}{3}$ R. H. S. Proved.

10. Prove vectorically that
$\sin(A + B) = \sin A \cos B + \cos A \sin B$

Solution: Let XOX^1 and YOY^1, the two mutually perpendicular straight lines represented x-axis and y-axis respectively.

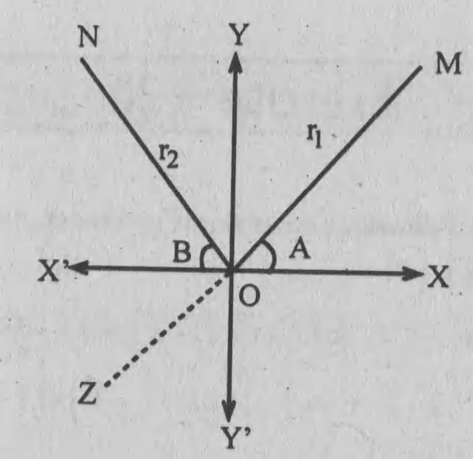
Let $\angle XOM = A$ and $\angle NOX^1 = B$

So that $\angle MON = \pi - (A + B)$

Let $OM = r_1$ and $ON = r_2$

Then the co-ordinates of M and N are

$(r_1\cos A, r_1\sin A)$ and $(r_2\cos(\pi - B), r_2 \sin(\pi - B))$



So,

$\overrightarrow{OM} = (r_1\cos A, r_1\sin A)$

$= (r_1 \cos A, r_1 \sin A, 0)$

and $\overrightarrow{ON} = (r_2 \cos(\pi - B), r_2 \sin(\pi - B)$

$= (-r_2 \cos B, r_2 \sin B)$

$= (-r_2\cos B, r_2 \sin B \,.\, 0)$

$\overrightarrow{OM} \times \overrightarrow{ON} = (r_1 \cos A, r_1 \sin A, 0) \times (-r_2 \cos B, r_2\sin B, 0)$

$= (0, 0, r_1\cos A r_2 \sin B + r_1 \sin A\, r_2\cos B)$

$= (0, 0, r_1r_2 \cos A \sin B + r_1r_2 \sin A \cos B)$

$\therefore |\overrightarrow{OM} \times \overrightarrow{ON}| = r_1r_2 (\cos A \sin B + \sin A \cos B)$

Since, $\pi - (A + B)$ is the angle between OM and ON so,

$$\sin[\pi - (A + b)] = \frac{|\overrightarrow{OM} \times \overrightarrow{ON}|}{|\overrightarrow{OM}|\,|\overrightarrow{ON}|}$$

$$\text{or, } \sin(A + B) = \frac{r_1r_2(\sin B \cos A + \sin A \cos B)}{r_1r_2}$$

$\therefore \sin(A + B) = \sin A \cos B + \cos A \sin B$ Proved.

Exercise - 20

1. Compute the following scalar triple producet.

(i) $\left(3\vec{i}-\vec{j}+2\vec{k}\right)\left(2\vec{i}+\vec{j}-\vec{k}\right)\times\left(\vec{i}-2\vec{j}+2\vec{k}\right)$

Solution: Given $\left(3\vec{i}-\vec{j}+2\vec{k}\right)\left(2\vec{i}+\vec{j}-\vec{k}\right)\times\left(\vec{i}-2\vec{j}+2\vec{k}\right)$

$$=\left(3\vec{i}-\vec{j}+2\vec{k}\right).\left[\left(2\vec{i}+\vec{j}-\vec{k}\right)\times\left(\vec{i}-2\vec{j}+2\vec{k}\right)\right]$$

$$=\begin{vmatrix}3 & -1 & 2\\ 2 & 1 & -1\\ 1 & -2 & 2\end{vmatrix}$$

$$=3(2-2)+1(4+1)+2(-4-1)$$
$$=0+5-10$$
$$=-5 \text{ Ans.}$$

(ii) $\left(2\vec{i}-3\vec{j}+\vec{k}\right).\left(\vec{i}-\vec{j}+2\vec{k}\right)\times\left(2\vec{i}+\vec{j}-\vec{k}\right)$

Solution: Given $\left(2\vec{i}-3\vec{j}+\vec{k}\right).\left(\vec{i}-\vec{j}+2\vec{k}\right)\times\left(2\vec{i}+\vec{j}-\vec{k}\right)$

$$=\left(2\vec{i}-3\vec{j}+\vec{k}\right).\left[\left(\vec{i}-\vec{j}+2\vec{k}\right)\times\left(2\vec{i}+\vec{j}-\vec{k}\right)\right]$$

$$=\begin{vmatrix}2 & -3 & 1\\ 1 & -1 & 2\\ 2 & 1 & -1\end{vmatrix}$$

$$=2(1-2)+3(-1-4)+1(1+2)$$
$$=-2-15+3$$
$$=-14 \text{ Ans.}$$

(iii) $\left(\vec{i}-2\vec{j}+3\vec{k}\right).\left(2\vec{i}+\vec{j}-\vec{k}\right)\times\left(\vec{j}+\vec{k}\right)$

Solution: Given $\left(\vec{i}-2\vec{j}+3\vec{k}\right).\left(2\vec{i}+\vec{j}-\vec{k}\right)\times\left(\vec{j}+\vec{k}\right)$

$$=\left(\vec{i}-2\vec{j}+3\vec{k}\right).\left[\left(2\vec{i}+\vec{j}-\vec{k}\right)\times\left(0.\vec{i}+\vec{j}+\vec{k}\right)\right]$$

$$=\begin{vmatrix}1 & -2 & 3\\ 2 & 1 & -1\\ 0 & 1 & 1\end{vmatrix}$$

$$=1(1+1)+2(2-0)+3(2-0)$$
$$=2+4+6$$
$$=12 \text{ Ans.}$$

2. Find the volumes of the parallepiped whose edges are represented by the vectors.

(i) $\vec{i}+2\vec{j}-\vec{k}, \vec{i}-\vec{j}+\vec{k}$ **and** $\vec{i}+\vec{j}+\vec{k}$

Solution: Let, $\vec{A}=\vec{i}+2\vec{j}-\vec{k}$, $\vec{B}=\vec{i}-\vec{j}+\vec{k}$ and $\vec{C}=\vec{i}+\vec{j}+\vec{k}$

Since the volume of the parallelopiped with edges $\vec{A}$, $\vec{B}$, $\vec{C}$ is given by $[\vec{A}\ \vec{B}\ \vec{C}]$

$\therefore$ Required volume (v) $=[\vec{A}\ \vec{B}\ \vec{C}]$

$$=\vec{A}\,.\,\vec{B}\times\vec{C}$$

$$=\begin{vmatrix}1 & 2 & -1\\ 1 & -1 & 1\\ 1 & 1 & 1\end{vmatrix}$$

$$=1(-1-1)-2(1-1)-1(1+1)$$
$$=-2-0-2$$
$$=-4$$

$\therefore$ v = 4 cubic units Ans.

(ii) $\vec{i}+2\vec{j}+3\vec{k}, \vec{i}-2\vec{j}+3\vec{k}$ **and** $3\vec{i}+4\vec{j}-5\vec{k}$

Solution: Let, $\vec{A}=\vec{i}+2\vec{j}+3\vec{k}$, $\vec{B}=\vec{i}-2\vec{j}+3\vec{k}$ and $\vec{C}=3\vec{i}+4\vec{j}-5\vec{k}$

Since the volume of the parallelopiped with edges $\vec{A}$, $\vec{B}$, $\vec{C}$ is given by $[\vec{A}\ \vec{B}\ \vec{C}]$

$\therefore$ Required volume (v) $=[\vec{A}\ \vec{B}\ \vec{C}]$

$$=\vec{A}\,.\,\vec{B}\times\vec{C}$$

$$=\begin{vmatrix}1 & 2 & 3\\ 1 & -2 & 3\\ 3 & 4 & -5\end{vmatrix}$$

$$=1(10-12)-2(-5-9)+3(4+6)$$
$$=-2+28+30$$
$$=56$$

$\therefore$ v = 56 cubic units Ans.

(iii) $2\vec{i}+3\vec{j}-4\vec{k}$, $\vec{i}+\vec{j}$, **and** $\vec{i}-\vec{j}+\vec{k}$

Solution: Let, $\vec{A}=2\vec{i}+3\vec{j}-4\vec{k}$, $\vec{B}=\vec{i}+\vec{j}+0\vec{k}$. and $\vec{C}=\vec{i}-\vec{j}+\vec{k}$

Since the volume of the parallelopiped with edges $\vec{A}$, $\vec{B}$, $\vec{C}$ is given by $[\vec{A}\ \vec{B}\ \vec{C}]$

$\therefore$ Required volume (v) $=[\vec{A}\ \vec{B}\ \vec{C}]$

$$=\vec{A}\,.\,\vec{B}\times\vec{C}$$

$$= \begin{vmatrix} 2 & 3 & -4 \\ 1 & 1 & 0 \\ 1 & -1 & 1 \end{vmatrix}$$

$= 2(1-0) - 3(1-0) - 4(-1-1)$

$= 2 - 3 + 8$

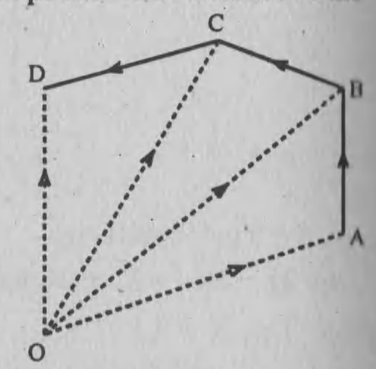
$= 7$

$\therefore$ v = 7 cubic units Ans.

3. Prove that the following four points are coplanar.

(i) $2\vec{i}+3\vec{j}-\vec{k},\ \vec{i}-2\vec{j}+3\vec{k},\ 3\vec{i}+4\vec{j}-2\vec{k},\ \vec{i}-6\vec{j}+6\vec{k},$

Solution: Let A, B, C, D be the given four points with reference to the origin O then we have,



$\overrightarrow{OA} = 2\vec{i}+3\vec{j}-\vec{k}$

$\overrightarrow{OB} = \vec{i}-2\vec{j}+3\vec{k}$

$\overrightarrow{OC} = 3\vec{i}+4\vec{j}-2\vec{k}$

$\overrightarrow{OD} = \vec{i}-6\vec{j}+6\vec{k}$

Now, $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= \left(\vec{i}-2\vec{j}+3\vec{k}\right) - \left(2\vec{i}+3\vec{j}-\vec{k}\right)$

$= -\vec{i}-5\vec{j}+4\vec{k}$

$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$

$= \left(3\vec{i}+4\vec{j}-2\vec{k}\right) - \left(\vec{i}-2\vec{j}+3\vec{k}\right)$

$= 2\vec{i}+6\vec{j}-5\vec{k}$

$\overrightarrow{CD} = \overrightarrow{OD} - \overrightarrow{OC}$

$= \left(\vec{i}-6\vec{j}+6\vec{k}\right) - \left(3\vec{i}+4\vec{j}-2\vec{k}\right)$

$= -2\vec{i}-10\vec{j}+8\vec{k}$

The four points A, B, C, D will be copolanar if

$\left[\overrightarrow{AB}\ \overrightarrow{BC}\ \overrightarrow{CD}\right] = 0$

i.e. $\overrightarrow{AB} \cdot \overrightarrow{BC} \times \overrightarrow{CD} = 0$

Now, $\overrightarrow{AB} \cdot \overrightarrow{BC} \times \overrightarrow{CD}$

$$= \begin{vmatrix} -1 & -5 & 4 \\ 2 & 6 & -5 \\ -2 & -10 & 8 \end{vmatrix}$$

$= -1(48-50) + 5(16-10) + 4(-20+12)$

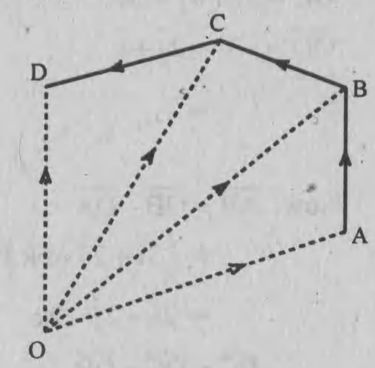
$= 2 + 30 - 32$

$= 0$

Which shows that the three vectors $\overrightarrow{AB}, \overrightarrow{BC}, \overrightarrow{CD}$ and hence the given four points are coplanar. Proved.

(ii) $4\vec{i}-2\vec{j}+\vec{k},\ 5\vec{i}+\vec{j}+6\vec{k},\ 2\vec{i}+2\vec{j}-5\vec{k}$ **and** $3\vec{i}+5\vec{j}$

Solution: Let A, B, C, D, be the given four points with reference to the origin O. Then we have



$\overrightarrow{OA} = 4\vec{i}-2\vec{j}+\vec{k}$

$\overrightarrow{OB} = 5\vec{i}+\vec{j}+6\vec{k}$

$\overrightarrow{OC} = 2\vec{i}+2\vec{j}-5\vec{k}$

$\overrightarrow{OD} = 3\vec{i}+5\vec{j}+0.\vec{k}$

Now, $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= \left(5\vec{i}+\vec{j}+6\vec{k}\right) - \left(4\vec{i}-2\vec{j}+\vec{k}\right)$

$= \vec{i}+3\vec{j}+5\vec{k}$

$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$

$= \left(2\vec{i}+2\vec{j}-5\vec{k}\right) - \left(5\vec{i}+\vec{j}+6\vec{k}\right)$

$= -3\vec{i}+\vec{j}-11\vec{k}$

$\overrightarrow{CD} = \overrightarrow{OD} - \overrightarrow{OC}$

$= \left(3\vec{i}+5\vec{j}+0.\vec{k}\right) - \left(2\vec{i}+2\vec{j}-5\vec{k}\right)$

$= \vec{i}+3\vec{j}+5\vec{k}$

The four points A, B, C, D will be coplanar if,

$\left[\overrightarrow{AB}\ \overrightarrow{BC}\ \overrightarrow{CD}\right] = 0$

i.e. $\overrightarrow{AB} \cdot \overrightarrow{BC} \times \overrightarrow{CD} = 0$

Now, $\overrightarrow{AB} \cdot \overrightarrow{BC} \times \overrightarrow{CD}$

$$= \begin{vmatrix} 1 & 3 & 5 \\ -3 & 1 & -11 \\ 1 & 3 & 5 \end{vmatrix}$$

$= 0$

$\therefore R_1 = R_3$

Which shows that the three vectors $\overrightarrow{AB}, \overrightarrow{BC}, \overrightarrow{CD}$ and hence the given four poitns are coplanar. Proved.

(iii) $\vec{i}+4\vec{j}-3\vec{k},\ 3\vec{i}+2\vec{j}-5\vec{k},\ 3\vec{i}+8\vec{j}-5\vec{k}$ **and** $-3\vec{i}+2\vec{j}+\vec{k}$

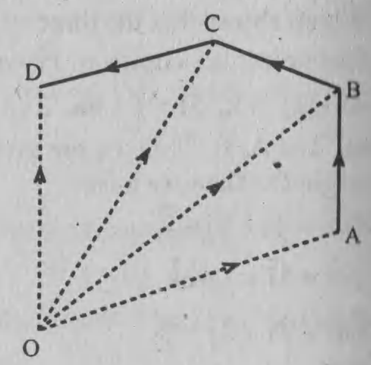
Solution: Let A, B, C, D, be the given four points with reference to the origin O. Then we have

$\overrightarrow{OA}=\vec{i}+4\vec{j}-3\vec{k}$

$\overrightarrow{OB}=3\vec{i}+2\vec{j}-5\vec{k}$

$\overrightarrow{OC}=3\vec{i}+8\vec{j}-5\vec{k}$

$\overrightarrow{OD}=-3\vec{i}+2\vec{j}+\vec{k}$



Now, $\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}$

$=\left(3\vec{i}+2\vec{j}-5\vec{k}\right)-\left(\vec{i}+4\vec{j}-3\vec{k}\right)$

$=2\vec{i}-2\vec{j}-2\vec{k}$

$\overrightarrow{BC}=\overrightarrow{OC}-\overrightarrow{OB}$

$=\left(3\vec{i}+8\vec{j}-5\vec{k}\right)-\left(3\vec{i}+2\vec{j}-5\vec{k}\right)$

$=0\,.\,\vec{i}-6\vec{j}+0\,.\,\vec{k}$

$\overrightarrow{CD}=\overrightarrow{OD}-\overrightarrow{OC}$

$=\left(-3\vec{i}+2\vec{j}+\vec{k}\right)-\left(3\vec{i}+8\vec{j}-5\vec{k}\right)$

$=-6\vec{i}-6\vec{j}+6\vec{k}$

The four points A, B, C, D will be coplanar if,

$$\left[\overrightarrow{AB}\ \overrightarrow{BC}\ \overrightarrow{CD}\right]=0$$

i.e. $\overrightarrow{AB}\,.\,\overrightarrow{BC}\times\overrightarrow{CD}=0$

Now, $\overrightarrow{AB}\,.\,\overrightarrow{BC}\times\overrightarrow{CD}$

$$=\begin{vmatrix} 2 & -2 & -2 \\ 0 & -6 & 0 \\ -6 & -6 & 6 \end{vmatrix}$$

Expanding Along R_2

$$=-6\begin{vmatrix} 2 & -2 \\ -6 & 6 \end{vmatrix}$$

$=-6\,(12-12)$

$=0$

Which shows that the three vectors $\overrightarrow{AB}, \overrightarrow{BC}, \overrightarrow{CD}$ and hence the given four points are coplanar. Proved.

(iv) $-\vec{i}+2\vec{j}-4\vec{k},\ 2\vec{i}-\vec{j}+3\vec{k},\ 6\vec{i}+2\vec{j}-\vec{k}$ **and** $-12\vec{i}-\vec{j}-3\vec{k}$

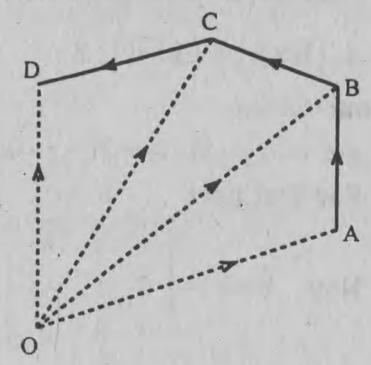
Solution: Let A, B, C, D, be the given four points with reference to the origin O. Then we have

$\overrightarrow{OA}=-\vec{i}+2\vec{j}-4\vec{k}$

$\overrightarrow{OB}=2\vec{i}-\vec{j}+3\vec{k}$

$\overrightarrow{OC}=6\vec{i}+2\vec{j}-\vec{k}$

$\overrightarrow{OD}=-12\vec{i}-\vec{j}+3\vec{k}$



Now, $\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}$

$=\left(2\vec{i}-\vec{j}+3\vec{k}\right)-\left(-\vec{i}+2\vec{j}-4\vec{k}\right)$

$=3\vec{i}-3\vec{j}+7\vec{k}$

$\overrightarrow{BC}=\overrightarrow{OC}-\overrightarrow{OB}$

$=\left(6\vec{i}+2\vec{j}-\vec{k}\right)-\left(2\vec{i}-\vec{j}+3\vec{k}\right)$

$=4\vec{i}+3\vec{j}-4\,\vec{k}$

$\overrightarrow{CD}=\overrightarrow{OD}-\overrightarrow{OC}$

$=\left(-12\vec{i}-\vec{j}-3\vec{k}\right)-\left(6\vec{i}+2\vec{j}-\vec{k}\right)$

$=-18\vec{i}-3\vec{j}-2\vec{k}$

The four points A, B, C, D will be coplanar if,

$$\left[\overrightarrow{AB}\ \overrightarrow{BC}\ \overrightarrow{CD}\right]=0$$

i.e. $\overrightarrow{AB}\,.\,\overrightarrow{BC}\times\overrightarrow{CD}=0$

Now, $\overrightarrow{AB}\,.\,\overrightarrow{BC}\times\overrightarrow{CD}$

$$=\begin{vmatrix} 3 & -3 & 7 \\ 4 & 3 & -4 \\ -18 & -3 & -2 \end{vmatrix}$$

$=3\,(-6-12)+3\,(-8-72)+7\,(-12+54)$

$=-54-240+294$

$=-294+294$

$=0$

Which shows that the three vectors $\overrightarrow{AB}, \overrightarrow{BC}, \overrightarrow{CD}$ and hence the given four points are coplanar. Proved.

4. If $\vec{a}=\vec{i}-2\vec{j}-3\vec{k}$, $\vec{b}=2\vec{i}+\vec{j}-\vec{k}$, $\vec{c}=\vec{i}+3\vec{j}-2\vec{k}$

Find $\vec{a}.(\vec{b}\times\vec{c})$ and $(\vec{a}\times\vec{b}).\vec{c}$ and hence verify that,

$\vec{a}.(\vec{b}\times\vec{c})=(\vec{a}\times\vec{b}).\vec{c}$

Solution: Given,

$\vec{a}=\vec{i}-2\vec{j}-3\vec{k}$, $\vec{b}=2\vec{i}+\vec{j}-\vec{k}$ and $\vec{c}=\vec{i}+3\vec{j}-2\vec{k}$

For first part

Now, $\vec{b}\times\vec{c}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 2 & 1 & -1\\ 1 & 3 & 2\end{vmatrix}$

$=\vec{i}(-2+3)-\vec{j}(-4+1)+\vec{k}(6-1)$

$=\vec{i}+3\vec{j}+5\vec{k}$

Also, $\vec{a}\times\vec{b}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 1 & -2 & -3\\ 2 & 1 & -1\end{vmatrix}$

$=\vec{i}(2+3)-\vec{j}(-1+6)+\vec{k}(1+4)$

$=5\vec{i}-5\vec{j}+5\vec{k}$

$\therefore\ \vec{a}.(\vec{b}\times\vec{c})=(\vec{i}-2\vec{j}-3\vec{k}).(\vec{i}+3\vec{j}+5\vec{k})$

$=1-6-15$

$=-20$ ..... (i)

$\therefore\ \vec{a}.(\vec{b}\times\vec{c})=-20$ Ans.

and $(\vec{a}\times\vec{b}).\vec{c}=(5\vec{i}-5\vec{j}+5\vec{k}).(\vec{i}+3\vec{j}-2\vec{k})$

$=5-15-10$

$=-20$ ..... (ii)

$(\vec{a}\times\vec{b}).\vec{c}=-20$ Ans.

For second part

From (i) and (ii) we get,

$\vec{a}.(\vec{b}\times\vec{c})=(\vec{a}\times\vec{b}).\vec{c}$ proved.

5. Find 'p' such that the vectors

$2\vec{i}-\vec{j}+\vec{k}$, $\vec{i}+2\vec{j}+3\vec{k}$, $3\vec{i}+p\vec{j}+5\vec{k}$ are coplanar. [B. E. 2064]

Solution: Let, $\vec{a}=2\vec{i}-\vec{j}+\vec{k}$

$\vec{b}=\vec{i}+2\vec{j}+3\vec{k}$

and $\vec{c}=3\vec{i}+p\vec{j}+5\vec{k}$

If the three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are

Coplanar, then the scalar triple product of those vector $\vec{a}$, $\vec{b}$, $\vec{c}$ must be zero.

i.e. $[\vec{a}\,\vec{b}\,\vec{c}]=0$

or, $\vec{a}.\vec{b}\times\vec{c}=0$

or, $\begin{vmatrix}2 & -1 & 1\\ 1 & 2 & 3\\ 3 & p & 5\end{vmatrix}=0$

or, $2(10-3p)+1(5-9)+1(p-6)=0$

or, $20-6p-4+p-6=0$

or, $-5p+20-10=0$

or, $-5p=-10$

or, $p=\dfrac{-10}{-5}$

$\therefore\ p=2$ Ans.

If $\vec{a},\vec{b},\vec{c}$ be three non-copeanar vectors such that,

$\vec{a}=a_1\vec{\ell}+a_2\vec{m}+a_3\vec{n}$

$\vec{b}=b_1\vec{\ell}+b_2\vec{m}+b_3\vec{n}$

and $\vec{c}=c_1\vec{\ell}+c_2\vec{m}+c_3\vec{n}$

Prove that $[\vec{a}\,\vec{b}\,\vec{c}]=\begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}[\vec{\ell}\,\vec{m}\,\vec{n}]$

Solution: Given, $\vec{a}=a_1\vec{\ell}+a_2\vec{m}+a_3\vec{n}$

$\vec{b}=b_1\vec{\ell}+b_2\vec{m}+b_3\vec{n}$

and $\vec{c}=c_1\vec{\ell}+c_2\vec{m}+c_3\vec{n}$

Now,

$[\vec{a}\,\vec{b}\,\vec{c}]=\vec{a}.\vec{b}\times\vec{c}$

$=(a_1\vec{\ell}+a_2\vec{m}+a_3\vec{n}).[(b_1\vec{\ell}+b_2\vec{m}+b_3\vec{n})\times(c_1\vec{\ell}+c_2\vec{m}+c_3\vec{n})]$

$=(a_1\vec{\ell}+a_2\vec{m}+a_3\vec{n}).[b_1c_1(\vec{\ell}\times\vec{\ell})+b_1c_2(\vec{\ell}\times\vec{m})$

$+b_1c_3(\vec{\ell}\times\vec{n})+b_2c_1(\vec{m}\times\vec{\ell})+b_2c_2(\vec{m}\times\vec{m})+b_2c_3(\vec{m}\times\vec{n})$

$+b_3c_1(\vec{n}\times\vec{\ell})+b_3c_2(\vec{n}\times\vec{m})+b_3c_3(\vec{n}\times\vec{n})]$

$$= \left(a_1\vec{\ell}+a_2\vec{m}+a_3\vec{n}\right).\left[b_1c_2\left(\vec{\ell}\times\vec{m}\right)+b_1c_3\left(\vec{\ell}\times\vec{n}\right)\right.$$

$$\left.+b_2c_1\left(\vec{m}\times\vec{\ell}\right)+b_2c_3\left(\vec{m}\times\vec{n}\right)+b_3c_1\left(\vec{n}\times\vec{\ell}\right)+b_3c_2\left(\vec{n}\times\vec{m}\right)\right]$$

Since, $\vec{\ell}\times\vec{\ell}=\vec{m}\times\vec{m}=\vec{n}\times\vec{n}=0$

$$= \left(a_1\vec{\ell}+a_2\vec{m}+a_3\vec{n}\right).\left[(b_1c_2-b_2c_1)\left(\vec{\ell}\times\vec{m}\right)\right.$$

$$\left.+(b_2c_3-b_3c_2)\left(\vec{m}\times\vec{n}\right)+(b_3c_1-b_1c_3)\left(\vec{n}\times\vec{\ell}\right)\right]$$

$\left(\vec{n}\times\vec{m}\right)=-\left(\vec{m}\times\vec{n}\right)$ and $\left(\vec{\ell}\times\vec{n}\right)=-\left(\vec{n}\times\vec{\ell}\right)$

Since $\left(\vec{m}\times\vec{\ell}\right)=\left(\vec{\ell}\times\vec{m}\right)$, $\left(\vec{n}\times\vec{m}\right)=-\left(\vec{m}\times\vec{n}\right)$ and $\left(\vec{\ell}\times\vec{n}\right)=-\left(\vec{n}\times\vec{\ell}\right)$

$$= a_1(b_1c_2-b_2c_1)\left[\vec{\ell}\,\vec{\ell}\,\vec{m}\right]+a_1(b_2c_3-b_3c_2)\left[\vec{\ell}\,\vec{m}\,\vec{n}\right]$$

$$+a_1(b_3c_1-b_1c_3)\left[\vec{\ell}\,\vec{n}\,\vec{\ell}\right]+a_2(b_1c_2-b_2c_1)\left[\vec{m}\,\vec{\ell}\,\vec{m}\right]$$

$$+a_2(b_2c_3-b_3c_2)\left[\vec{m}\,\vec{m}\,\vec{n}\right]+a_2(b_3c_1-b_1c_3)\left[\vec{m}\,\vec{n}\,\vec{\ell}\right]$$

$$+a_3(b_1c_2-b_2c_1)\left[\vec{n}\,\vec{\ell}\,\vec{m}\right]+a_3(b_2c_3-b_3c_2)\left[\vec{n}\,\vec{m}\,\vec{n}\right]$$

$$+a_3(b_3c_1-b_1c_3)\left[\vec{n}\,\vec{n}\,\vec{\ell}\right]$$

$$= 0+a_1(b_2c_3-b_3c_2)\left[\vec{\ell}\,\vec{m}\,\vec{n}\right]+0+0+0+a_2(b_3c_1-b_1c_3)$$

$$\left[\vec{m}\,\vec{n}\,\vec{\ell}\right]+a_3(b_1c_2-b_2c_1)\left[\vec{n}\,\vec{\ell}\,\vec{m}\right]+0+0$$

But $\left[\vec{\ell}\,\vec{\ell}\,\vec{m}\right]=0$ e.t.c., when two vector are equal, value of scalar triple produced is zero.

$$= [a_1(b_2c_3-b_3c_2)+a_2(b_3c_1-b_1c_3)+a_3(b_1c_2-b_2c_1)]\left[\vec{\ell}\,\vec{m}\,\vec{n}\right]$$

$$\therefore \left[\vec{a}\,\vec{b}\,\vec{c}\right]=\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}\left[\vec{\ell}\,\vec{m}\,\vec{n}\right]$$

Hence proved.

7. **Prove that if $\vec{\ell},\vec{m},\vec{n}$ be three non-coplanar vectors then $\left[\vec{\ell}\,\vec{m}\,\vec{n}\right]$**

$$\left[\vec{a}\times\vec{b}\right]=\begin{vmatrix} \vec{\ell}.\vec{a} & \vec{\ell}.\vec{b} & \vec{\ell} \\ \vec{m}.\vec{a} & \vec{m}.\vec{b} & \vec{m} \\ \vec{n}.\vec{a} & \vec{n}.\vec{b} & \vec{n} \end{vmatrix}$$ **(BE 2069 Bhadra)**

Solution: Since every vector can be expessed in terms of three matually perpendicular unit vectors $\vec{i}$, $\vec{j}$ & $\vec{k}$ So, $\vec{\ell}=\ell_1\vec{i}+\ell_2\vec{j}+\ell_3\vec{k}$

$\vec{m}=m_1\vec{i}+m_2\vec{j}+m_3\vec{k}$

and $\vec{n}=n_1\vec{i}+n_2\vec{j}+n_3\vec{k}$

Let $\vec{a}=a_1\vec{i}+a_2\vec{j}+a_3\vec{k}$

and $\vec{b}=b_1\vec{i}+b_2\vec{j}+b_3\vec{k}$

Then,

$$\left[\vec{\ell}\,\vec{m}\,\vec{n}\right]=\vec{\ell}.\vec{m}\times\vec{n}$$

$$=\begin{vmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{vmatrix}$$

and $\vec{a}\times\vec{b}=\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

Now, $\left[\vec{\ell}\,\vec{m}\,\vec{n}\right]\left(\vec{a}\times\vec{b}\right)$

$$=\begin{vmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{vmatrix}\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$=\begin{vmatrix} \Sigma\ell_1\vec{i} & \Sigma\ell_1a_1 & \Sigma\ell_1b_1 \\ \Sigma m_1\vec{i} & \Sigma m_1a_1 & \Sigma m_1b_1 \\ \Sigma n_1\vec{i} & \Sigma n_1a_1 & \Sigma n_1b_1 \end{vmatrix}$$

$\therefore$ By applying the rwo by row multiplication of two determinants.

$$=\begin{vmatrix} \vec{\ell} & \vec{\ell}.\vec{a} & \vec{\ell}.\vec{b} \\ \vec{m} & \vec{m}.\vec{a} & \vec{m}.\vec{b} \\ \vec{n} & \vec{n}.\vec{a} & \vec{n}.\vec{b} \end{vmatrix}$$

Since, $\vec{\ell}.\vec{a}=l_1a_1+l_2a_2+l_3a_3$

$\vec{m}.\vec{a}=m_1a_1+m_2a_2+m_3a_3$

$\vec{n}.\vec{a}=n_1a_1+n_2a_2+n_3a_3$

$\vec{\ell}.\vec{b}=l_1b_1+l_2b_2+l_3b_3$

$\vec{\ell}.\vec{b}=m_1b_1+m_2b_2+m_3b_3$

$\vec{\ell}.\vec{b}=n_1b_1+n_2b_2+n_3b_3$

$$= -\begin{vmatrix} \vec{\ell}.\vec{a} & \vec{\ell} & \vec{\ell}.\vec{b} \\ \vec{m}.\vec{a} & \vec{m} & \vec{m}.\vec{b} \\ \vec{n}.\vec{a} & \vec{n} & \vec{n}.\vec{b} \end{vmatrix}$$

Hence,

$$1\left(\vec{a}\times\vec{b}\right) = \begin{vmatrix} \vec{\ell}.\vec{a} & \vec{\ell}.\vec{b} & \vec{\ell} \\ \vec{m}.\vec{a} & \vec{m}.\vec{b} & \vec{m} \\ \vec{n}.\vec{a} & \vec{n}.\vec{b} & \vec{n} \end{vmatrix} \text{ Proved.}$$

9. If $\vec{a}=\vec{i}-2\vec{j}+\vec{k}$, $\vec{b}=2\vec{i}+\vec{j}+\vec{k}$ and $\vec{c}=\vec{i}+2\vec{j}-\vec{k}$ then verify the $\vec{a}\times\left(\vec{b}\times\vec{c}\right)=\left(\vec{a}.\vec{c}\right)\vec{b}-\left(\vec{a}.\vec{b}\right)\vec{c}$ [B.E. 2063]

Solution: Given, $\vec{a}=\vec{i}-2\vec{j}+\vec{k}$, $\vec{b}=2\vec{i}+\vec{j}+\vec{k}$ and $\vec{c}=\vec{i}+2\vec{j}-\vec{k}$

Now, $\vec{b}\times\vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & 1 \\ 1 & 2 & -1 \end{vmatrix}$

$$= (-1-2)\vec{i}+(1+2)\vec{j}+(4-1)\vec{k}$$

$$= -3\vec{i}+3\vec{j}+3\vec{k}$$

Also,

$$\vec{a}\times\left(\vec{b}\times\vec{c}\right) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -2 & 1 \\ -3 & 3 & 3 \end{vmatrix}$$

$$= 3\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -2 & 1 \\ -1 & 1 & 1 \end{vmatrix}$$

$$= 3\left\{(-2-1)\vec{i}+(-1-1)\vec{j}+(1-2)\vec{k}\right\}$$

$$= 3\left\{-3\vec{i}-2\vec{j}-\vec{k}\right\}$$

$\therefore\ \vec{a}\times\left(\vec{b}\times\vec{c}\right) = -9\vec{i}-6\vec{j}-3\vec{k}$ ..... (i)

and $\left(\vec{a}.\vec{c}\right)\vec{b}-\left(\vec{a}.\vec{b}\right)\vec{c}$

$$= \left\{\left(\vec{i}-2\vec{j}+\vec{k}\right).\left(\vec{i}-2\vec{j}-\vec{k}\right)\right\}\left(2\vec{i}+\vec{j}+\vec{k}\right)$$

$$-\left\{\left(\vec{i}-2\vec{j}+\vec{k}\right).\left(2\vec{i}+\vec{j}+\vec{k}\right)\right\}\left(\vec{i}+2\vec{j}-\vec{k}\right)$$

$$= (1-4-1)\left(2\vec{i}+\vec{j}+\vec{k}\right)-\left\{(2-2+1)\left(\vec{i}+2\vec{j}-\vec{k}\right)\right\}$$

$$= -4\left(2\vec{i}+\vec{j}+\vec{k}\right)-\left(\vec{i}+2\vec{j}-\vec{k}\right)$$

$$= \left(-8\vec{i}-4\vec{j}-4\vec{k}\right)-\left(\vec{i}+2\vec{j}-\vec{k}\right)$$

$$= -9\vec{i}-6\vec{j}-3\vec{k}$$

$\therefore\ \left(\vec{a}.\vec{c}\right)\vec{b}-\left(\vec{a}.\vec{b}\right)\vec{c} = 9\vec{i}-6\vec{j}-3\vec{k}$ ..... (ii)

From (i) and (ii)

$\vec{a}\times\left(\vec{b}\times\vec{c}\right) = \left(\vec{a}.\vec{c}\right)\vec{b}-\left(\vec{a}.\vec{b}\right)\vec{c}$ is verified

10. Prove that, $\vec{a}\times\left(\vec{b}\times\vec{a}\right)=\left(\vec{a}\times\vec{b}\right)\times\vec{a}$

Solution: L. H. S. $= \vec{a}\times\left(\vec{b}\times\vec{a}\right)$

$$= \left(\vec{a}.\vec{a}\right)\vec{b}-\left(\vec{a}.\vec{b}\right)\vec{a}$$

$$= \left(\vec{a}^2\right)\vec{b}-\left(\vec{a}.\vec{b}\right)\vec{a}$$

$$= a^2b-\left(\vec{a}.\vec{b}\right)\vec{a}$$

R. H. S. $= \left(\vec{a}\times\vec{b}\right)\times\vec{a}$

$$= -\left\{\vec{a}\times\left(\vec{a}\times\vec{b}\right)\right\}$$

$$= -\left\{\left(\vec{a}.\vec{b}\right)\vec{a}-\left(\vec{a}.\vec{a}\right)\vec{b}\right\}$$

$$= -\left(\vec{a}.\vec{b}\right)\vec{a}+\left(\vec{a}^2\right)\vec{b}$$

$$= a^2\vec{b}-\left(\vec{a}.\vec{b}\right)\vec{a}$$

$\therefore$ L. H. S. = R. H. S. proved.

11. Show that

$\vec{a}\times\left(\vec{b}\times\vec{c}\right), \vec{b}\times\left(\vec{c}\times\vec{a}\right), \vec{c}\times\left(\vec{a}\times\vec{b}\right)$ are coplanar. [B. E. 2065]

Solution: Given three vectors

$\vec{a}\times\left(\vec{b}\times\vec{c}\right), \vec{b}\times\left(\vec{c}\times\vec{a}\right), \vec{c}\times\left(\vec{a}\times\vec{b}\right)$ are copanar if their scalar triple product is zero.

i.e. $\left\{\vec{a}\times\left(\vec{b}\times\vec{c}\right)\right\}.\left[\left\{\vec{b}\times\left(\vec{c}\times\vec{a}\right)\right\}\times\left\{\vec{c}\times\left(\vec{a}\times\vec{b}\right)\right\}\right]=0$ ..... (i)

Now, $\vec{a}\times\left(\vec{b}\times\vec{c}\right) = \left(\vec{a}.\vec{c}\right)\vec{b}-\left(\vec{a}.\vec{b}\right)\vec{c}$

and $\left\{\vec{b}\times(\vec{c}\times\vec{a})\right\}\times\left\{\vec{c}\times(\vec{a}\times\vec{b})\right\}$

$$=\left\{(\vec{b}.\vec{a})\,\vec{c}-(\vec{b}.\vec{c})\,\vec{a}\right\}\times\left\{(\vec{c}.\vec{b})\,\vec{a}-(\vec{c}.\vec{a})\,\vec{b}\right\}$$

$$=(\vec{b}.\vec{a})\,(\vec{c}.\vec{b})\,(\vec{c}\times\vec{a})-(\vec{b}.\vec{a})\,(\vec{c}.\vec{a})\,(\vec{c}\times\vec{b})$$

$$-(\vec{b}.\vec{c})\,(\vec{c}.\vec{b})\,(\vec{a}\times\vec{a})+(\vec{b}.\vec{c})\,(\vec{c}.\vec{a})\,(\vec{a}\times\vec{b})$$

$$=(\vec{b}.\vec{a})\,(\vec{c}.\vec{b})\,(\vec{c}\times\vec{a})-(\vec{b}.\vec{a})\,(\vec{c}.\vec{a})\,(\vec{c}\times\vec{b})$$

$$+(\vec{b}.\vec{c})\,(\vec{c}.\vec{a})\,(\vec{a}\times\vec{b})\text{ since }\vec{a}\times\vec{a}=0$$

Now, L. H. S. of ..... (i)

$$=\left\{\vec{a}\times(\vec{b}\times\vec{c})\right\}.\left[\left\{\vec{b}\times(\vec{c}\times\vec{a})\right\}\times\left\{\vec{c}\times(\vec{a}\times\vec{b})\right\}\right]$$

$$=\left\{(\vec{a}.\vec{c})\,\vec{b}-(\vec{a}.\vec{b})\,\vec{c}\right\}.\left[(\vec{b}.\vec{a})\,(\vec{c}.\vec{b})\,(\vec{c}\times\vec{a})\right.$$

$$\left.-(\vec{b}.\vec{a})\,(\vec{c}.\vec{a})\,(\vec{c}\times\vec{b})+(\vec{b}.\vec{c})\,(\vec{c}.\vec{a})\,(\vec{a}\times\vec{b})\right]$$

$$=(\vec{a}.\vec{c})\,(\vec{b}.\vec{a})\,(\vec{c}.\vec{b})\,(\vec{b}.\vec{c}\times\vec{a})$$

$$-(\vec{a}.\vec{c})\,(\vec{b}.\vec{a})\,(\vec{c}.\vec{a})\,(\vec{b}.\vec{c}\times\vec{b})+(\vec{a}.\vec{c})\,(\vec{b}.\vec{c})\,(\vec{c}.\vec{a})\,(\vec{b}.\vec{a}\times\vec{b})$$

$$-(\vec{a}.\vec{b})\,(\vec{b}.\vec{a})\,(\vec{c}.\vec{b})\,(\vec{c}.\vec{c}\times\vec{a})+(\vec{a}.\vec{b})\,(\vec{b}.\vec{a})\,(\vec{c}.\vec{a})\,(\vec{c}.\vec{c}\times\vec{b})$$

$$-(\vec{a}.\vec{b})\,(\vec{b}.\vec{c})\,(\vec{c}.\vec{a})\,(\vec{c}.\vec{a}\times\vec{b})$$

$$=(\vec{a}.\vec{c})\,(\vec{b}.\vec{a})\,(\vec{c}.\vec{b})\left[\vec{b}\ \vec{c}\ \vec{a}\right]-0+0-0+0$$

$$-(\vec{a}.\vec{b})\,(\vec{b}.\vec{c})\,(\vec{c}.\vec{a})\left[\vec{c}\ \vec{a}\ \vec{b}\right]$$

Since, $\left[\vec{b}\ \vec{c}\ \vec{b}\right]=\left[\vec{b}\ \vec{a}\ \vec{b}\right]=\left[\vec{c}\ \vec{c}\ \vec{a}\right]=\left[\vec{c}\ \vec{c}\ \vec{b}\right]=0$

$$=\left[(\vec{a}.\vec{c})\,(\vec{b}.\vec{a})\,(\vec{c}.\vec{b})-(\vec{a}.\vec{b})\,(\vec{b}.\vec{c})\,(\vec{c}.\vec{a})\right]\left[\vec{a}\ \vec{b}\ \vec{c}\right]=0$$

Since, $\left[\vec{b}\ \vec{c}\ \vec{a}\right]=\left[\vec{c}\ \vec{a}\ \vec{b}\right]=\left[\vec{a}\ \vec{b}\ \vec{c}\right]$

$=0$

$\therefore$ L. H. S. $=0$

Thus the given three vector are coplanar.

12. $\left[\vec{b}\times\vec{c}\quad\vec{c}\times\vec{a}\quad\vec{a}\times\vec{b}\right]=\left[\vec{a}\ \vec{b}\ \vec{c}\right]^2$ **[B. E. 2063]**

Solution. L. H. S. $=\left[\vec{b}\times\vec{c}\quad\vec{c}\times\vec{a}\quad\vec{a}\times\vec{b}\right]$

$$=(\vec{b}\times\vec{c})\,.\left\{(\vec{c}\times\vec{a})\times(\vec{a}\times\vec{b})\right\}$$

$$=(\vec{b}\times\vec{c})\,.\left\{\vec{m}\times(\vec{a}\times\vec{b})\right\}\quad\text{where }\vec{m}=\vec{c}\times\vec{a}$$

$$=(\vec{b}\times\vec{c})\,.\left\{(\vec{m}.\vec{b})\,\vec{a}-(\vec{m}.\vec{a})\,\vec{b}\right\}$$

$$=(\vec{b}\times\vec{c})\,.\left\{(\vec{a}\times\vec{c}).\vec{b}\right\}\vec{a}-\left\{(\vec{c}\times\vec{a}).\vec{a}\right\}\vec{b}$$

$$=(\vec{b}\times\vec{c})\,.\left\{\left[\vec{a}\ \vec{b}\ \vec{c}\right]\vec{a}-0\right\}$$

$$=\left[\vec{a}\ \vec{b}\ \vec{c}\right](\vec{b}\times\vec{c})\,.\,\vec{a}$$

$$=\left[\vec{a}\ \vec{b}\ \vec{c}\right]^2\text{ R. H. S. Proved.}$$

13. If $\vec{a},\vec{b},\vec{c}$ be three unit vectors such that $\vec{a}\times(\vec{b}\times\vec{c})=\frac{\vec{b}}{2}$. Find the angles which $\vec{a}$ makes with $\vec{b}$ and $\vec{c}$. Where $\vec{b}$ and $\vec{c}$ being non-parallel vectors.

Solution: Since $\vec{a},\vec{b},\vec{c}$ are unit vectors

$\therefore\ |\vec{a}|=1,\ |\vec{b}|=1,\ |\vec{c}|=1$

By given condition, we have,

$$\vec{a}\times(\vec{b}\times\vec{c})=\frac{1}{2}\vec{b}$$

or, $(\vec{a}.\vec{c})\,\vec{b}-(\vec{a}.\vec{b})\,\vec{c}=\frac{1}{2}\vec{b}$

or, $(\vec{a}.\vec{c})\,\vec{b}-\frac{1}{2}\vec{b}-(\vec{a}.\vec{b})\,\vec{c}=0$

or, $\left(\vec{a}.\vec{c}-\frac{1}{2}\right)\vec{b}+\left[-(\vec{a}.\vec{b})\,\vec{c}\right]=0$

Since $\vec{b}$ and $\vec{c}$ being non parallel.

So either,

$\vec{a}.\vec{c}-\frac{1}{2}=0$ or, $\vec{a}.\vec{b}=0$

Case : 1

If $\vec{a}.\vec{c}-\frac{1}{2}=0$

$\Rightarrow\ \vec{a}.\vec{c}=\frac{1}{2}$

Now, If ϕ be the angle between $\vec{a}$ and $\vec{c}$ then,

$\vec{a}.\vec{c}=|\vec{a}|\,|\vec{c}|\cos\phi$

or, $\frac{1}{2}=1\,.\,1\cos\phi$

or, $\cos\phi = \frac{1}{2}$

or, $\cos\phi = \cos\frac{\pi}{3}$

$\therefore\ \phi = \frac{\pi}{3}$

Hence the angle between $\vec{a}$ and $\vec{c}$ is $\frac{\pi}{3}$ Ans.

Case : II $\vec{a}\,.\,\vec{b} = 0$

Now, if θ be the angle between $\vec{a}$ and $\vec{b}$ then,

$\vec{a}\,.\,\vec{b} = |\vec{a}|\ |\vec{b}|\ \cos\theta$

or, $0 = 1\,.\,1\cos\theta$

or, $\cos\theta = 0$

or, $\cos\theta = \cos\frac{\pi}{2}$

$\therefore\ \theta = \frac{\pi}{2}$

Hence the angle between $\vec{a}$ and $\vec{b}$ is $\frac{\pi}{2}$ Ans.

14. Prove that $b^2\vec{a} = (\vec{a}\,.\,\vec{b})\vec{b} + \vec{b}\times(\vec{a}\times\vec{b})$ where $\vec{a}, \vec{b}, \vec{c}$ are any three vectors.

Proof: R. H. S. $= (\vec{a}\,.\,\vec{b})\vec{b} + \vec{b}\times(\vec{a}\times\vec{b})$

$$= (\vec{a}\,.\,\vec{b})\vec{b} + (\vec{b}\,.\,\vec{b})\,\vec{a} - (\vec{b}\,.\,\vec{a})\,\vec{b}$$

$$= (\vec{a}\,.\,\vec{b})\vec{b} + (\vec{b}^2)\,\vec{a} - (\vec{a}\,.\,\vec{b})\vec{b}$$

$$= \vec{b}^2\,\vec{a}$$ L. H. S. proved.

15. If $\vec{a}, \vec{b}$ and $\vec{c}$ are any three vectors then prove that $\vec{a}\times(\vec{b}\times\vec{c}) + \vec{b}\times(\vec{c}\times\vec{a}) + \vec{c}\times(\vec{a}\times\vec{b}) = 0$

Solution: L. H. S. $= \vec{a}\times(\vec{b}\times\vec{c}) + \vec{b}\times(\vec{c}\times\vec{a}) + \vec{c}\times(\vec{a}\times\vec{b})$

$$= (\vec{a}\,.\,\vec{c})\,\vec{b} - (\vec{a}\,.\,\vec{b})\,\vec{c} + (\vec{b}\,.\,\vec{a})\,\vec{c} - (\vec{b}\,.\,\vec{c})\,\vec{a} + (\vec{c}\,.\,\vec{b})\,\vec{a} - (\vec{c}\,.\,\vec{a})\,\vec{b}$$

$$= (\vec{a}\,.\,\vec{c})\,\vec{b} - (\vec{a}\,.\,\vec{b})\,\vec{c} + (\vec{a}\,.\,\vec{b})\,\vec{c} - (\vec{b}\,.\,\vec{c})\,\vec{a} + (\vec{b}\,.\,\vec{c})\,\vec{a} - (\vec{a}\,.\,\vec{c})\,\vec{b}$$

Since $\vec{a}\,.\,\vec{b} = \vec{b}\,.\,\vec{a}$ etc.

16. If $\vec{u}, \vec{v}, \vec{w}$ are three vectors defined by the relations.

$$\vec{u} = \frac{\vec{b}\times\vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]},\ \vec{v} = \frac{\vec{c}\times\vec{a}}{[\vec{a}\,\vec{b}\,\vec{c}]},\ \vec{w} = \frac{\vec{a}\times\vec{b}}{[\vec{a}\,\vec{b}\,\vec{c}]}$$

where $[\vec{a}\ \vec{b}\ \vec{c}] \neq 0$

Prove that, $\vec{a}\times\vec{u} + \vec{b}\times\vec{v} + \vec{c}\times\vec{w} = 0$

Solution: Given, $\vec{u} = \frac{\vec{b}\times\vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]}$, $\vec{v} = \frac{\vec{c}\times\vec{a}}{[\vec{a}\,\vec{b}\,\vec{c}]}$ and $\vec{w} = \frac{\vec{a}\times\vec{b}}{[\vec{a}\,\vec{b}\,\vec{c}]}$

Where $[\vec{a}\ \vec{b}\ \vec{c}] \neq 0$

L. H. S. $= \vec{a}\times\vec{u} + \vec{b}\times\vec{v} + \vec{c}\times\vec{w}$

$$= \vec{a}\times\frac{\vec{b}\times\vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]} + \vec{b}\times\frac{\vec{c}\times\vec{a}}{[\vec{a}\,\vec{b}\,\vec{c}]} + \vec{c}\times\frac{\vec{a}\times\vec{b}}{[\vec{a}\,\vec{b}\,\vec{c}]}$$

$$= \frac{1}{[\vec{a}\,\vec{b}\,\vec{c}]}\left[\vec{a}\times(\vec{b}\times\vec{c}) + \vec{b}\times(\vec{c}\times\vec{a}) + \vec{c}\times(\vec{a}\times\vec{b})\right]$$

$$= \frac{1}{[\vec{a}\,\vec{b}\,\vec{c}]}\left[(\vec{a}\,.\,\vec{c})\,\vec{b} - (\vec{a}\,.\,\vec{b})\,\vec{c} + (\vec{b}\,.\,\vec{a})\,\vec{c} - (\vec{b}\,.\,\vec{c})\,\vec{a} + (\vec{c}\,.\,\vec{b})\,\vec{a} - (\vec{c}\,.\,\vec{a})\,\vec{b}\right]$$

$$= \frac{1}{[\vec{a}\,\vec{b}\,\vec{c}]}\left[(\vec{a}\,.\,\vec{c})\,\vec{b} - (\vec{a}\,.\,\vec{b})\,\vec{c} + (\vec{a}\,.\,\vec{b})\,\vec{c} - (\vec{b}\,.\,\vec{c})\,\vec{a} + (\vec{b}\,.\,\vec{c})\,\vec{a} - (\vec{a}\,.\,\vec{c})\,\vec{b}\right]$$

Since, $\vec{a}\,.\,\vec{b} = \vec{b}\,.\,\vec{a}$ e. t. c.

$$= \frac{0}{[\vec{a}\,\vec{b}\,\vec{c}]}$$

$= 0$ R. H. S. proved.

17. If $\vec{a}, \vec{b}, \vec{c}$ are three non-copeanar vectors then express $\vec{b}\times\vec{c}$, $\vec{c}\times\vec{a}$, $\vec{a}\times\vec{b}$ in terms of $\vec{a}, \vec{b}, \vec{c}$.

Solution: Given, $\vec{a}, \vec{b}, \vec{c}$ are three non-coplanar vectors.

Lat, $\vec{b}\times\vec{c} = \ell\vec{a} + m\vec{b} + n\vec{c}$ ..... (i)

Where, ℓ, m, n are scalars.

Taking dot product on both sides by $(\vec{b}\times\vec{c})$

$$\left(\vec{b}\times\vec{c}\right).\left(\vec{b}\times\vec{c}\right)=\left(\ell\,\vec{a}+m\vec{b}+n\vec{c}\right).\left(\vec{b}\times\vec{c}\right)$$

$$=\ell\,\vec{a}.(\vec{b}\times\vec{c})+m\,\vec{b}.\left(\vec{b}\times\vec{c}\right)+n\,\vec{c}.\left(\vec{b}\times\vec{c}\right)$$

$$=\ell\left[\vec{a}\,\vec{b}\,\vec{c}\right]+m\left[\vec{b}\;\vec{b}\;\vec{c}\right]+n\left[\vec{c}\;\vec{b}\;\vec{c}\right]$$

$$=\ell\left[\vec{a}\,\vec{b}\,\vec{c}\right]+m.0+n.0$$

$$\therefore\ \ell=\frac{\left(\vec{b}\times\vec{c}\right).\left(\vec{b}\times\vec{c}\right)}{\left[\vec{a}\,\vec{b}\,\vec{c}\right]}$$

Similarly, taking dot product on both sides (i) by $\vec{c}\times\vec{a}$ and $\vec{a}\times\vec{b}$ we get

$$m=\frac{\left(\vec{b}\times\vec{c}\right).\left(\vec{c}\times\vec{a}\right)}{\left[\vec{a}\,\vec{b}\,\vec{c}\right]}\ \text{and}\ n=\frac{\left(\vec{b}\times\vec{c}\right).\left(\vec{a}\times\vec{b}\right)}{\left[\vec{a}\,\vec{b}\,\vec{c}\right]}$$

Substituting the value of ℓ, m, n in equation (i) we get,

$$\vec{b}\times\vec{c}=\frac{\left(\vec{b}\times\vec{c}\right).\left(\vec{c}\times\vec{a}\right)}{\left[\vec{a}\,\vec{b}\,\vec{c}\right]}\vec{a}+\frac{\left(\vec{b}\times\vec{c}\right).\left(\vec{c}\times\vec{a}\right)}{\left[\vec{a}\,\vec{b}\,\vec{c}\right]}\vec{b}+\frac{\left(\vec{b}\times\vec{c}\right).\left(\vec{a}\times\vec{b}\right)}{\left[\vec{a}\,\vec{b}\,\vec{c}\right]}\vec{c}\quad\text{Ans.}$$

Similarly, we can express $\vec{c}\times\vec{a}$ and $\vec{a}\times\vec{b}$ interms of $\vec{a},\vec{b},\vec{c}$ Ans.

18. Show that four points $\vec{a}$, $\vec{b}$, $\vec{c}$, $\vec{d}$ are coplanar if

$$\left[\vec{b}\;\vec{c}\;\vec{d}\right]+\left[\vec{c}\;\vec{a}\;\vec{d}\right]+\left[\vec{a}\;\vec{b}\;\vec{d}\right]=\left[\vec{a}\;\vec{b}\;\vec{c}\right]$$

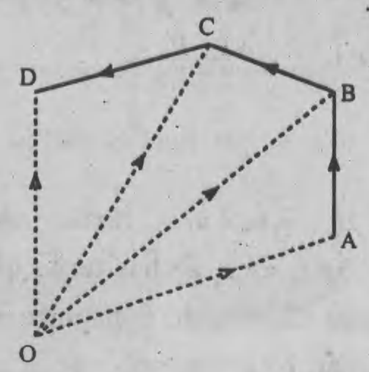
Solution: Let A, B, C and D be the given four points with reference to the origin O. Then we have

$\overrightarrow{OA}=\vec{a}$

$\overrightarrow{OB}=\vec{b}$

$\overrightarrow{OC}=\vec{c}$ and

$\overrightarrow{OD}=\vec{d}$



Now, $\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=\vec{b}-\vec{a}$

$\overrightarrow{BC}=\overrightarrow{OC}-\overrightarrow{OB}$

$=\vec{c}-\vec{b}$

$\overrightarrow{CD}=\overrightarrow{OD}-\overrightarrow{OC}$

$=\vec{d}-\vec{c}$

The four points $\vec{A},\vec{B},\vec{C},\vec{D}$ will be coplanar if,

$$\left[\overrightarrow{AB}\ \overrightarrow{BC}\ \overrightarrow{CD}\right]=0$$

i.e. $\overrightarrow{AB}.\overrightarrow{BC}\times\overrightarrow{CD}=0$

Now, $f\left(\vec{b}-\vec{a}\right).\left\{\left(\vec{c}-\vec{b}\right)\times\left(\vec{d}-\vec{c}\right)\right\}=0$

or, $f\left(\vec{b}-\vec{a}\right).\left\{\vec{c}\times\vec{d}-\vec{c}\times\vec{c}-\vec{b}\times\vec{d}+\vec{b}\times\vec{c}\right\}=0$

or, $f\left(\vec{b}-\vec{a}\right).\left\{\left(\vec{c}\times\vec{d}\right)-\left(\vec{b}\times\vec{d}\right)+\left(\vec{b}\times\vec{c}\right)\right\}=0$

Since $\vec{c}\times\vec{c}=0$

or, $f\,\vec{b}.\left(\vec{c}\times\vec{d}\right)-\vec{b}.\left(\vec{b}\times\vec{d}\right)+\vec{b}.\left(\vec{b}\times\vec{c}\right)-\vec{a}.\left(\vec{c}\times\vec{d}\right)+\vec{a}.\left(\vec{b}\times\vec{d}\right)-\vec{a}.\left(\vec{b}\times\vec{c}\right)=0$

or, $f\left[\vec{b}\;\vec{c}\;\vec{d}\right]-\left[\vec{b}\;\vec{b}\;\vec{d}\right]+\left[\vec{b}\;\vec{b}\;\vec{c}\right]+\left[\vec{a}\;\vec{c}\;\vec{d}\right]+\left[\vec{a}\;\vec{b}\;\vec{d}\right]-\left[\vec{a}\,\vec{b}\,\vec{c}\right]=0$

Since $\left[\vec{b}\;\vec{b}\;\vec{d}\right]=\left[\vec{b}\;\vec{b}\;\vec{c}\right]=0$

or, $f\left[\vec{b}\;\vec{c}\;\vec{d}\right]-0+0-\left[\vec{a}\;\vec{c}\;\vec{d}\right]+\left[\vec{a}\;\vec{b}\;\vec{d}\right]-\left[\vec{a}\,\vec{b}\,\vec{c}\right]=0$

or, $f\left[\vec{b}\;\vec{c}\;\vec{d}\right]-\left[\vec{a}\;\vec{c}\;\vec{d}\right]+\left[\vec{a}\;\vec{b}\;\vec{d}\right]=\left[\vec{a}\,\vec{b}\,\vec{c}\right]$

or, $f\left[\vec{b}\;\vec{c}\;\vec{d}\right]+\left[\vec{c}\;\vec{a}\;\vec{d}\right]+\left[\vec{a}\;\vec{b}\;\vec{d}\right]=\left[\vec{a}\,\vec{b}\,\vec{c}\right]$

Since $\left[\vec{a}\;\vec{c}\;\vec{d}\right]=-\left[\vec{c}\;\vec{a}\;\vec{d}\right]$

19. If $\vec{a}$, $\vec{b}$, $\vec{c}$ be three vectors such that $\vec{a}\times\vec{b}=\vec{c}$ and $\vec{b}\times\vec{c}=\vec{a}$, then prove that the three vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ are orthogonal in pairs and $|\vec{b}|=1$, $|\vec{c}|=|\vec{a}|$.

Solution: Given $\vec{a}$, $\vec{b}$, $\vec{c}$ are three vectors.

and we have,

$\vec{a}\times\vec{b}=\vec{c}$ ..... (i)

$\vec{b}\times\vec{c}=\vec{a}$ ..... (ii)

Now, $\vec{a}.\vec{b}=\left(\vec{b}\times\vec{c}\right).\vec{b}$ from (ii)

$\vec{a}.\vec{b}=\left[\vec{b}\;\vec{c}\;\vec{b}\right]=0$ ..... (iii)

Hence $\vec{a}$ and $\vec{b}$ are orthogonal.

Also, $\vec{b}.\vec{c} = \vec{b}.\left(\vec{a}\times\vec{b}\right)$ from (i)

$$= \left[\vec{b}\ \vec{a}\ \vec{b}\right] = 0 \text{ ..... (iv)}$$

Hence $\vec{b}$ and $\vec{c}$ are orthogonal.

and $\vec{a}.\vec{c} = \vec{a}.\left(\vec{a}.\vec{c}\right)$ from (i)

$$= \left[\vec{a}\ \vec{a}\ \vec{c}\right] = 0 \text{ ..... (v)}$$

Hence $\vec{a}$ and $\vec{c}$ are orthogonal.

Thus the three vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ are orthogonel in pairs. Proved.

For second part

From (i) and (ii) putting the value of $\vec{c}$ in (ii) we get

$\vec{b}\times\left(\vec{a}\times\vec{b}\right) = \vec{a}$

or, $\left(\vec{b}.\vec{b}\right)\vec{a} - \left(\vec{b}.\vec{a}\right)\vec{b} = \vec{a}$

or, $\left(\vec{b}^2\right)\vec{a} - 0\left(\vec{b}\right) = \vec{a}$

from (iii) $\vec{a}.\vec{b} = 0$

or, $\vec{b}^2\vec{a} - \vec{a} = 0$

or, $(\vec{b}^2 - 1)\vec{a} = 0$

$\therefore\ \vec{a} \neq 0 \Rightarrow \vec{b}^2 - 1 = 0$

or, $\vec{b}^2 = 1$

$\therefore\ |\vec{b}| = 1$ ..... (vi) Proved.

Also, $|c|^2 = \vec{c}.\vec{c}$

$$= \left(\vec{a}\times\vec{b}\right).\left(\vec{a}\times\vec{b}\right) = \vec{a}\times\vec{b}.\left(\vec{a}\times\vec{b}\right)$$

$$= \vec{a}.\left[\vec{b}\times\left(\vec{a}\times\vec{b}\right)\right]$$

$$= \vec{a}.\left\{\left(\vec{b}.\vec{b}\right)\vec{a} - \left(\vec{b}.\vec{a}\right)\vec{b}\right\}$$

$$= \left(\vec{b}.\vec{b}\right)\left(\vec{a}\times\vec{a}\right) - \left(\vec{a}.\vec{b}\right)\left(\vec{a}.\vec{b}\right)$$

$= \vec{a}^2\vec{b}^2 - 0$ from (iii) $\vec{a}.\vec{b} = 0$

$$= |\vec{a}|^2\ |\vec{b}|^2$$

$\therefore\ |c|^2 = |\vec{a}|^2\ 1^2$ from (vi) $|\vec{b}| = 1$

$\therefore\ |\vec{c}| = |\vec{a}|$ Proved.

20. If three vectors $a\vec{i}+\vec{j}+\vec{k}$, $\vec{i}+b\vec{j}+\vec{k}$, $\vec{i}+\vec{j}+c\vec{k}$ are coplanar then prove that ($a \neq 1, b \neq 1, c \neq 1$)

$$\frac{1}{1-a}+\frac{1}{1-b}+\frac{1}{1-c} = 1$$

Solution: Let $= \vec{A} = a\vec{i}+\vec{j}+\vec{k}$, $\vec{B} = \vec{i}+b\vec{j}+\vec{k}$ and $\vec{C} = \vec{i}+\vec{j}+c\vec{k}$

If three vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ are coplanar then, $\left[\vec{A}\ \vec{B}\ \vec{C}\right] = 0$

i.e. $\vec{A}.\left(\vec{B}\times\vec{C}\right) = 0$

or, $\begin{vmatrix} a & 1 & 1 \\ 1 & b & 1 \\ 1 & 1 & c \end{vmatrix} = 0$

Applying $R_1 \to R_1 - R_2$, $R_3 \to R_3 - R_2$

or, $\begin{vmatrix} a-1 & 1-b & 0 \\ 1 & b & 1 \\ 0 & 1-b & c-1 \end{vmatrix} = 0$

Expanding along R_1;

or, $(a-1)\begin{vmatrix} b & 1 \\ 1-b & c-1 \end{vmatrix} - (1-b)\begin{vmatrix} 1 & 1 \\ 0 & c-1 \end{vmatrix} = 0$

or, $(a-1)\{b(c-1)-(1-b)\} - (1-b)(c-1) = 0$

or, $b(a-1((c-1)-(a-1)(1-b)-(1-b)(c-1) = 0$

or, $b(1-a)(1-c)+(1-a)(1-b)+(1-b)(1-c) = 0$

Dividing each term by $(1-a)(1-b)(1-C)$

or, $\frac{1}{(1-b)}+\frac{1}{(1-c)}+\frac{1}{(1-a)} = 0$

or, $\frac{1-(1-b)}{(1-b)}+\frac{1}{(1-c)}+\frac{1}{(1-a)} = 0$

or, $\frac{1}{(1-b)}-1+\frac{1}{1-c}+\frac{1}{1-a}$

or, $\frac{1}{1-a}+\frac{1}{1-b}+\frac{1}{1-c} = 1$ Proved.

Exercise - 21

1. If $\vec{a} = 7\vec{i} + 6\vec{j} + 5\vec{k}$, $\vec{b} = 4\vec{i} + 3\vec{j} + 2\vec{k}$ $\vec{c} = 2\vec{i} - 3\vec{j} + 4\vec{k}$ and $\vec{d} = 5\vec{i} + 6\vec{j} - 7\vec{k}$ Then find the value of $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d})$ and $(\vec{a} \times \vec{b}).(\vec{c} \times \vec{d})$

Solution: Given,

$\vec{a} = 7\vec{i} + 6\vec{j} + 5\vec{k}$

$\vec{b} = 4\vec{i} + 3\vec{j} + 2\vec{k}$

$\vec{c} = 2\vec{i} - 3\vec{j} + 4\vec{k}$

and $\vec{d} = 5\vec{i} + 6\vec{j} - 7\vec{k}$

Now,

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 7 & 6 & 5 \\ 4 & 3 & 2 \end{vmatrix}$$

$= \vec{i}(12-15) - \vec{j}(14-20) + \vec{k}(21-24)$

$= -3\vec{i} + 6\vec{j} - 3\vec{k}$

$$\text{And } \vec{c} \times \vec{d} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -3 & 4 \\ 5 & 6 & -7 \end{vmatrix}$$

$= \vec{i}(21-24) - \vec{j}(-14-20) + \vec{k}(12+15)$

$= -3\vec{i} + 34\vec{j} + 27\vec{k}$

For, $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = (-3\vec{i} - 6\vec{j} - 3\vec{k}) \times (-3\vec{i} + 34\vec{j} + 27\vec{k})$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -3 & -6 & -3 \\ -3 & 34 & 27 \end{vmatrix}$$

$= \vec{i}(162+102) - \vec{j}(-81-9) + \vec{k}(-102+18)$

$\therefore (\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = 264\vec{i} + 90\vec{j} - 84\vec{k}$ Ans.

For,

$(\vec{a} \times \vec{b}).(\vec{c} \times \vec{d}) = (-3\vec{i} + 6\vec{j} - 3\vec{k}).(-3\vec{i} + 34\vec{j} + 27\vec{k})$

$= 9 + 204 - 81$

$= 132$ Ans.

2. Prove that

$[\vec{a} \times (\vec{b} + \vec{c})] \times [\vec{b} \times (\vec{c} + \vec{a})] = [\vec{a}\ \vec{b}\ \vec{c}]\,[\vec{a} + \vec{b} + \vec{c}]$

Solution. LHS $= [\vec{a} \times (\vec{b} + \vec{c})] \times [\vec{b} \times (\vec{c} + \vec{a})]$

$= [(\vec{a} \times \vec{b}) + (\vec{a} \times \vec{c})] \times [(\vec{b} \times \vec{c}) + (\vec{b} \times \vec{a})]$

$= (\vec{a} \times \vec{b}) \times (\vec{b} \times \vec{c}) + (\vec{a} \times \vec{b}) \times (\vec{b} \times \vec{a}) + (\vec{a} \times \vec{c}) \times (\vec{b} \times \vec{c}) + (\vec{a} \times \vec{c}) \times (\vec{b} \times \vec{a})$

$= [\vec{a}\ \vec{b}\ \vec{c}]\vec{b} - [\vec{a}\ \vec{b}\ \vec{b}]\vec{c} + [\vec{a}\ \vec{b}\ \vec{a}]\vec{b} - [\vec{a}\ \vec{b}\ \vec{b}]\vec{a} + [\vec{a}\ \vec{c}\ \vec{c}]\vec{b} - [\vec{a}\ \vec{c}\ \vec{b}]\vec{c} - [\vec{a}\ \vec{c}\ \vec{a}]\vec{b} - [\vec{a}\ \vec{c}\ \vec{b}]\vec{a}$

$= [\vec{a}\ \vec{b}\ \vec{c}]\vec{b} - 0 + 0 - 0 + 0 - [\vec{a}\ \vec{c}\ \vec{b}]\vec{c} + 0 + [\vec{a}\ \vec{c}\ \vec{b}]\vec{a}$

Since, $[\vec{a}\ \vec{b}\ \vec{b}.] = 0$ etc.

$= [\vec{a}\ \vec{b}\ \vec{c}]\vec{b} + [\vec{a}\ \vec{b}\ \vec{c}]\vec{c} + [\vec{a}\ \vec{b}\ \vec{c}]\vec{a}$ Since $[\vec{a}\ \vec{c}\ \vec{b}] = -[\vec{a}\ \vec{b}\ \vec{c}]$

$= [\vec{a}\ \vec{b}\ \vec{c}](\vec{b} + \vec{c} + \vec{a})$

$= [\vec{a}\ \vec{b}\ \vec{c}](\vec{a} + \vec{b} + \vec{c})$ R. H. S proved.

3. Prove that

$\vec{a} \times \{\vec{a} \times (\vec{a} \times \vec{b})\} = (\vec{a}.\vec{a})(\vec{b} \times \vec{a})$

Solution. L.H.S $= \vec{a} \times \{\vec{a} \times (\vec{a} \times \vec{b})\}$

$= \vec{a} \times \{(\vec{a}.\vec{b})\vec{a} - (\vec{a}.\vec{a})\vec{b}\}$

$= (\vec{a}.\vec{b})(\vec{a} \times \vec{a}) - (\vec{a}.\vec{a})(\vec{a} \times \vec{b})$

$= 0 - (\vec{a}.\vec{a})(\vec{a} \times \vec{b})$

$= (\vec{a}.\vec{a})(\vec{b} \times \vec{a})$ R.H. S proved.

4. Prove that,

$$2(\vec{c} \times \vec{d}) \times (\vec{a} \times \vec{b}) = \begin{vmatrix} \vec{a} & a_1 & a_2 & a_3 \\ \vec{b} & b_1 & b_2 & b_3 \\ -\vec{c} & c_1 & c_2 & c_3 \\ -\vec{d} & d_1 & d_2 & d_3 \end{vmatrix}$$

Proof:

Since every vector can be expressed in terms of three mutually perpendicular vectors $\vec{i}$, $\vec{j}$, $\vec{k}$ so that

$\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$

$\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$

$\vec{c} = c_1\vec{i} + c_2\vec{j} + c_3\vec{k}$

and $\vec{d} = d_1\vec{i} + d_2\vec{j} + d_3\vec{k}$.

We know that

$$[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix},$$

$$[\vec{b}\ \vec{c}\ \vec{d}] = \begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{vmatrix} \text{ etc.}$$

We know,

$(\vec{c}\times\vec{d})\times(\vec{a}\times\vec{b}) = [\vec{c}\ \vec{d}\ \vec{b}]\vec{a} - [\vec{c}\ \vec{d}\ \vec{a}]\vec{b}$ ..... (i)

Also, $(\vec{c}\times\vec{d})\times(\vec{a}\times\vec{b}) = [\vec{c}\ \vec{a}\ \vec{b})\vec{d} - [\vec{d}\ \vec{a}\ \vec{b})\vec{c}$ .....(ii)

Adding (1) and (ii), we have,

$2(\vec{c}\times\vec{d})\times(\vec{a}\times\vec{b}) = [\vec{c}\ \vec{d}\ \vec{b}]\vec{a} - [\vec{c}\ \vec{d}\ \vec{a}]\vec{b} + [\vec{c}\ \vec{a}\ \vec{b}]\vec{d} - [\vec{d}\ \vec{a}\ \vec{b}]\vec{c}$

$= [\vec{b}\ \vec{c}\ \vec{d}]\vec{a} - [\vec{a}\ \vec{c}\ \vec{d}]\vec{b} + [\vec{a}\ \vec{b}\ \vec{d}](-\vec{c}) - [\vec{a}\ \vec{b}\ \vec{c}](-\vec{d})$

$$= \begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{vmatrix}\vec{a} - \begin{vmatrix} a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{vmatrix}\vec{b}$$

$$+ \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix}(-\vec{c}) - \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}(-\vec{d})$$

$$= \begin{vmatrix} \vec{a} & a_1 & a_2 & a_3 \\ \vec{b} & b_1 & b_2 & b_3 \\ -\vec{c} & c_1 & c_2 & c_3 \\ -\vec{d} & d_1 & d_2 & d_3 \end{vmatrix}$$

Hence, $2(\vec{c}\times\vec{d})\times(\vec{a}\times\vec{b}) = \begin{vmatrix} \vec{a} & a_1 & a_2 & a_3 \\ \vec{b} & b_1 & b_2 & b_3 \\ -\vec{c} & c_1 & c_2 & c_3 \\ -\vec{d} & d_1 & d_2 & d_3 \end{vmatrix}$ Proved.

5. **Prove that,** $\vec{a}\times\{\vec{b}\times(\vec{c}\times\vec{d})\}$

$= (\vec{b}.\vec{d})(\vec{a}\times\vec{c}) - (\vec{b}.\vec{c})(\vec{a}\times\vec{d})$

$= [\vec{a}\ \vec{c}\ \vec{d}]\vec{b} - (\vec{b}.\vec{c})(\vec{a}\times\vec{d})$

Proof: L.H. S = $\vec{a}\times\{\vec{b}\times(\vec{c}\times\vec{d})\}$

$\vec{a}\times\{(\vec{b}.\vec{d})\vec{c} - (\vec{b}.\vec{c})\vec{d}\}$

$= (\vec{b}.\vec{d})(\vec{a}\times\vec{c}) - (\vec{b}.\vec{c})(\vec{a}\times\vec{d})$

Which proves first part.

For second part.

L.H.S = $\vec{a}\times[\vec{b}\times(\vec{c}\times\vec{d})]$

$= \vec{a}\times(\vec{b}\times\vec{m})$ where $\vec{m} = \vec{c}\times\vec{d}$

$= (\vec{a}.\vec{m})\vec{b} - (\vec{a}.\vec{b})\vec{m}$

$= [\{\vec{a}.(\vec{c}\times\vec{d})\}]\vec{b} - (\vec{a}.\vec{b})(\vec{c}\times\vec{d})$

$= [\vec{a}\ \vec{c}\ \vec{d}]\vec{b} - (\vec{a}.\vec{b})(\vec{c}\times\vec{d})$ Proved.

Show that

$$\vec{c}\begin{vmatrix} \vec{a}.\vec{a} & \vec{a}.\vec{b} \\ \vec{a}.\vec{b} & \vec{b}.\vec{b} \end{vmatrix} = \begin{vmatrix} \vec{c}.\vec{a} & \vec{a}.\vec{b} \\ \vec{c}.\vec{b} & \vec{b}.\vec{b} \end{vmatrix}\vec{a} + \begin{vmatrix} \vec{a}.\vec{a} & \vec{c}.\vec{a} \\ \vec{a}.\vec{b} & \vec{c}.\vec{b} \end{vmatrix}\vec{b}$$

Where, $\vec{a}$, $\vec{b}$, $\vec{c}$ are coplanar and $\vec{a}$ is not parallel to $\vec{b}$.

Solution:

Since $\vec{a}, \vec{b}, \vec{c}$ are coplanar and $\vec{a}$ is not parallel to $\vec{b}$

So let,

$\vec{c} = \ell\vec{a} + m\vec{b}$ where, ℓ, m are scalars. .....(1)

Taking cross product on both sides (1) by $\vec{b}$ we get,

$\vec{c}\times\vec{b} = (\ell\vec{a} + m\vec{b})\times\vec{b}$

or, $(\vec{c}\times\vec{b}) = \ell(\vec{a}\times\vec{b}) + m(\vec{b}\times\vec{b})$

or, $\vec{c}\times\vec{b} = \ell(\vec{a}\times\vec{b}) + m.0$ since $\vec{b}\times\vec{b} = 0$

$\therefore (\vec{c}\times\vec{b}) = \ell(\vec{a}\times\vec{b})$..... (ii)

Again taking dot produce on both sides (ii) by $\vec{a}\times\vec{b}$ we get

$(\vec{c}\times\vec{b}).(\vec{a}\times\vec{b}) = \ell(\vec{a}\times\vec{b}).(\vec{a}\times\vec{b})$

$$\therefore \ell = \frac{(\vec{c}\times\vec{b}).(\vec{a}\times\vec{b})}{(\vec{a}\times\vec{b}).(\vec{a}\times\vec{b})}$$

Also, taking cross product on both sides (i) by $\vec{a}$ we get,

$\vec{c}\times\vec{a} = (\ell\vec{a}\times m\vec{b})\times\vec{a}$

or, $\vec{c}\times\vec{a} = m(\vec{b}\times\vec{a})$ .....(iii)

Again taking dot product on both sides (iii) by $(\vec{b}\times\vec{a})$ we get,

$(\vec{c}\times\vec{a}).(\vec{b}\times\vec{a}) = m(\vec{b}\times\vec{a}).(\vec{b}\times\vec{a})$

$$m = \frac{(\vec{c}\times\vec{a}).(\vec{b}\times\vec{a})}{(\vec{b}\times\vec{a}).(\vec{b}\times\vec{a})}$$

Substituting the value of l and m in eq^n (i) we get

$$\vec{c} = \frac{(\vec{c}\times\vec{b}).(\vec{a}\times\vec{b})}{(\vec{a}\times\vec{b}).(\vec{a}\times\vec{b})}\vec{a} + \frac{(\vec{c}\times\vec{a}).(\vec{b}\times\vec{a})}{(\vec{b}\times\vec{a}).(\vec{b}\times\vec{a})}\vec{b}$$

or, $[(\vec{a}\times\vec{b}).(\vec{a}\times\vec{b})]\vec{c} = [(\vec{c}\times\vec{b}).(\vec{a}\times\vec{b})]\vec{a} + [(\vec{c}\times\vec{a}).(\vec{b}\times\vec{a})]\vec{b}$

or, $\begin{vmatrix} \vec{a}.\vec{a} & \vec{a}.\vec{b} \\ \vec{b}.\vec{a} & \vec{b}.\vec{b} \end{vmatrix} \vec{c} = \begin{vmatrix} \vec{c}.\vec{a} & \vec{c}.\vec{b} \\ \vec{b}.\vec{a} & \vec{b}.\vec{b} \end{vmatrix} \vec{a} + \begin{vmatrix} \vec{c}.\vec{b} & \vec{c}.\vec{a} \\ \vec{a}.\vec{b} & \vec{a}.\vec{a} \end{vmatrix} \vec{b}$

$\therefore$ $\begin{vmatrix} \vec{a}.\vec{a} & \vec{a}.\vec{b} \\ \vec{a}.\vec{b} & \vec{b}.\vec{b} \end{vmatrix} \vec{c} = \begin{vmatrix} \vec{c}.\vec{a} & \vec{a}.\vec{b} \\ \vec{c}.\vec{b} & \vec{b}.\vec{b} \end{vmatrix} \vec{a} + \begin{vmatrix} \vec{a}.\vec{a} & \vec{c}.\vec{a} \\ \vec{a}.\vec{b} & \vec{c}.\vec{b} \end{vmatrix} \vec{b}$ Proved.

7. Proved that

$[\vec{a}\times\vec{b} \;\; \vec{c}\times\vec{d} \;\; \vec{e}\times\vec{f}]$

$= [\vec{a}\;\vec{b}\;\vec{d}]\,[\vec{c}\;\vec{e}\;\vec{f}] - [\vec{a}\;\vec{b}\;\vec{c}]\,[\vec{d}\;\vec{e}\;\vec{f}]$

Proof: L.H.S.,

$[\vec{a}\times\vec{b} \;\; \vec{c}\times\vec{d} \;\; \vec{e}\times\vec{f}]$

$= (\vec{a}\times\vec{b}).\{(\vec{c}\times\vec{d})\times(\vec{e}\times\vec{f})\}$

$= (\vec{a}\times\vec{b}).\{[\vec{c}\;\vec{e}\;\vec{f}]\vec{d} - [\vec{d}\;\vec{e}\;\vec{f}]\vec{c}\}$

$= \{\vec{a}\times\vec{b}).\vec{d}\,\{[\vec{c}\;\vec{e}\;\vec{f}]\} - \{(\vec{a}\times\vec{b}).\vec{c}\}\,[\vec{d}\;\vec{e}\;\vec{f}]$

$= [\vec{a}\;\vec{b}\;\vec{d}]\,[\vec{c}\;\vec{e}\;\vec{f}] - [\vec{a}\;\vec{b}\;\vec{c}]\;[\vec{d}\;\vec{e}\;\vec{f}]$ R.H. S Proved

8. Prove that,

$\vec{a}\times[\vec{b}\times\{\vec{c}\times(\vec{d}\times\vec{e})\}]$

$= \{\vec{a}.\vec{d})(\vec{c}.\vec{e}) - (\vec{c}.\vec{d})(\vec{a}.\vec{e})\}\vec{b} + (\vec{a}.\vec{b})\{(\vec{c}.\vec{d})\vec{e} - (\vec{c}.\vec{e})\vec{d}\}$

Proof : L.H.S.

$\vec{a}\times[\vec{b}\times\{\vec{c}\times(\vec{d}\times\vec{e})\}]$

$= \vec{a}\times\vec{b}\times\{(\vec{c}.\vec{e})\vec{d} - (\vec{c}.\vec{d})\vec{e}\}]$

$= \{(\vec{c}.\vec{e})(\vec{a}.\vec{d}) - (\vec{c}.\vec{d})(\vec{a}.\vec{e})\vec{b}\}$

$-(\vec{a}\;.\;\vec{b})\{(\vec{c}.\vec{e})\vec{d} - (\vec{c}.\vec{d})\vec{e}\}$

R. H. S. proved.

9. Find a set of vectors reciprocal to the following vectors.

$\vec{a} = 2\vec{i} - \vec{j} + 3\vec{k}$, $\vec{b} = -\vec{i} + 3\vec{j} + 3\vec{k}$

$\vec{c} = \vec{i} + \vec{j} - 2\vec{k}$ **and verify that** $[\vec{a}\;\vec{b}\;\vec{c}]\,[\vec{a}'\vec{b}'\vec{c}'] = 1$.

Solution: Given

$\vec{a} = 2\vec{i} - \vec{j} + 3\vec{k}$,

$\vec{b} = -\vec{i} + 3\vec{j} + 3\vec{k}$,

and $\vec{c} = \vec{i} + \vec{j} - 2\vec{k}$

Let the vectors reciprocal system to $\vec{a}, \vec{b}, \vec{c}$ are $\vec{a}', \vec{b}', \vec{c}'$ respectively

Now, $[\vec{a}\;\vec{b}\;\vec{c}] = \begin{vmatrix} 2 & -1 & 3 \\ -1 & 3 & 3 \\ 1 & 1 & -2 \end{vmatrix}$

$= 2(-6-3) + 1(2-3) + 3(-1-3)$

$= -18 - 1 - 12$

$[\vec{a}\;\vec{b}\;\vec{c}] = -31$ .....(1)

$\therefore [\vec{a}\;\vec{b}\;\vec{c}] = -31 \neq 0$

Hence $\vec{a}', \vec{b}', \vec{c}'$ exist.

Again, $\vec{b}\times\vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 3 & 3 \\ 1 & 1 & -2 \end{vmatrix}$

$= \vec{i}(-6-3) - \vec{j}(2-3) + \vec{k}(-1-3)$

$= -9\vec{i} + \vec{j} - 4\vec{k}$

Also,

$\vec{c}\times\vec{a} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & -2 \\ 2 & -1 & 3 \end{vmatrix}$

$= \vec{i}(3-2) - \vec{j}(3+4) + \vec{k}(-1-2)$

$= \vec{i} - 7\vec{j} - 3\vec{k}$

and

$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & 3 \\ -1 & 3 & 3 \end{vmatrix}$

$= \vec{i}(-3-9) - \vec{j}(6+3) + \vec{k}(6-1)$

$= -12\vec{i} - 9\vec{j} + 5\vec{k}$.

Now, $\vec{a}' = \dfrac{\vec{b}\times\vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]} = \dfrac{-9\vec{i} + \vec{j} - 4\vec{k}}{-31} = \dfrac{9}{31}\vec{i} - \dfrac{1}{31}\vec{j} + \dfrac{4}{31}\vec{k}$

Similarly $\vec{b}' = \dfrac{\vec{c}\times\vec{a}}{[\vec{a}\,\vec{b}\,\vec{c}]} = \dfrac{-1}{31}(\vec{i} - 7\vec{j} - 3\vec{k}) = \dfrac{-1}{31}\vec{i} + \dfrac{7}{31}\vec{j} + \dfrac{3}{31}\vec{k}$

& $\vec{c}' = \dfrac{\vec{a}\times\vec{b}}{[\vec{a}\,\vec{b}\,\vec{c}]} = \dfrac{-1}{31}(-12\vec{i} - 9\vec{j} + 5\vec{k}) = \dfrac{12}{31}\vec{i} - \dfrac{9}{31}\vec{j} - \dfrac{5}{31}\vec{k}$

Now,

$[\vec{a}'\vec{b}'\vec{c}'] = \vec{a}'.(\vec{b}'\times\vec{c}')$

$$= \begin{vmatrix} \frac{9}{31} & \frac{1}{31} & \frac{4}{31} \\ \frac{-1}{31} & \frac{7}{31} & \frac{3}{31} \\ \frac{12}{31} & \frac{9}{31} & \frac{-5}{31} \end{vmatrix}$$

Taking $\frac{1}{31}$ from R_1, R_2 and R_3 resp.

$$= \left(\frac{1}{31}\right)\left(\frac{1}{31}\right)\cdot\left(\frac{1}{31}\right)\begin{vmatrix} 9 & -1 & 4 \\ 1 & 7 & 3 \\ 12 & 9 & -5 \end{vmatrix}$$

$$= \left(\frac{1}{31}\right)\cdot\left(\frac{1}{961}\right)[9(-35-27)+1(5-36)+4(-9-84)]$$

$$= \frac{1}{29791}[-558-31-372]$$

$$= \frac{1}{29791}\times(-961)$$

$$\left[\vec{a}'\vec{b}'\vec{c}'\right] = -\frac{1}{31} \text{.....(2)}$$

Now, $[\vec{a}\,\vec{b}\,\vec{c}]\left[\vec{a}'\vec{b}'\vec{c}'\right] = -31\times\left(-\frac{1}{31}\right) = 1$

$\therefore [\vec{a}\,\vec{b}\,\vec{c}]\,[\vec{a}'\vec{b}'\vec{c}'] = 1$ Hence verity.

10. Find a set of vectors reciprocal to the following vectors.

(i) $2\vec{i}-3\vec{j}+4\vec{k}$, $\vec{i}+2\vec{j}-\vec{k}$ and $3\vec{i}-\vec{j}+2\vec{k}$

Solution

Let $\vec{a} = 2\vec{i}-3\vec{j}+4\vec{k}$

$\vec{b} = \vec{i}+2\vec{j}-\vec{k}$

and $\vec{c} = 3\vec{i}-\vec{j}+2\vec{k}$.

Let the vectors reciprocal system to $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{a}'\vec{b}'\vec{c}'$ respectively

Now, $[\vec{a}\,\vec{b}\,\vec{c}] = \vec{a}\,.\,(\vec{b}\times\vec{c})$

$$= \begin{vmatrix} 2 & -3 & 4 \\ 1 & 2 & -1 \\ 3 & -1 & 2 \end{vmatrix}$$

$= 2(4-1)+3(2+3)+4(-1-6)$

$= 6+15-28$

$= -7$

$\therefore [\vec{a}\,\vec{b}\,\vec{c}] = -7 \neq 0$.

Hence $\vec{a}'\vec{b}'\vec{c}'$ exists.

Again, $\vec{b}\times\vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & -1 \\ 3 & -1 & 2 \end{vmatrix}$

$= \vec{i}(4-1)-\vec{j}(2+3)+\vec{k}(-1-6)$

$\vec{b}\times\vec{c} = 3\vec{i}-5\vec{j}-7\vec{k}$

Also, $\vec{c}\times\vec{a} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -1 & 2 \\ 2 & -3 & 4 \end{vmatrix}$

$= \vec{i}(-4+6)-\vec{j}(12-4)+\vec{k}(-9+2)$

$= 2\vec{i}-8\vec{j}-7\vec{k}$

and $\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -3 & 4 \\ 1 & 2 & -1 \end{vmatrix}$

$= \vec{i}(3-8)-\vec{j}(-2-4)+\vec{k}(4+3)$

$= -5\vec{i}+6\vec{j}+7\vec{k}$

Now,

$$\vec{a}' = \frac{\vec{b}\times\vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]} = \frac{3\vec{i}-5\vec{j}-7\vec{k}}{-7} = \frac{-3}{7}\vec{i}+\frac{5}{7}\vec{j}+\vec{k}$$

$$\vec{b}' = \frac{\vec{c}\times\vec{a}}{[\vec{a}\,\vec{b}\,\vec{c}]} = \frac{2\vec{i}-8\vec{j}-7\vec{k}}{-7} = \frac{-2}{7}\vec{i}+\frac{8}{7}\vec{j}+\vec{k}$$

$$\vec{c}' = \frac{\vec{a}\times\vec{b}}{[\vec{a}\,\vec{b}\,\vec{c}]} = \frac{-5\vec{i}+6\vec{j}+7\vec{k}}{-7} = \frac{5}{7}\vec{i}-\frac{6}{7}\vec{j}-\vec{k}$$

(ii) $2\vec{i}-3\vec{j}$, $\vec{i}+\vec{j}-\vec{k}$ and $3\vec{i}-\vec{k}$.

Solution

let $\vec{a} = 2\vec{i}-3\vec{j}+0\vec{k}$

$\vec{b} = \vec{i}+\vec{j}-\vec{k}$

and $\vec{c} = 3\vec{i}-0\vec{j}-\vec{k}$.

Let the vectors reciprocal system to $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{a}'\vec{b}'\vec{c}'$ respectively.

Now,

$$[\vec{a}\ \vec{b}\ \vec{c}\,] = \vec{a}\,.\,\vec{b}\times\vec{c}$$

$$= \begin{vmatrix} 2 & -3 & 0 \\ 1 & 1 & -1 \\ 3 & 0 & -1 \end{vmatrix}$$

$$= 2(-1) + 3\,(-1+3) + 0$$

$$= -2 + 6$$

$$= -4$$

$$\therefore\ [\vec{a}\ \vec{b}\ \vec{c}\,] = 4 \neq 0$$

Here $\vec{a}'\vec{b}'\vec{c}'$ exists.

Again, $$\vec{b}\times\vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & -1 \\ 3 & 0 & -1 \end{vmatrix}$$

$$= \vec{i}\,(-1) - \vec{j}\,(-1+3) + \vec{k}\,(-3)$$

$$= -\vec{i} - 2\vec{j} - 3\vec{k}$$

Also, $$\vec{c}\times\vec{a} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 0 & -1 \\ 2 & -3 & 0 \end{vmatrix}$$

$$= \vec{i}\,(-3) - \vec{j}\,.(2) + \vec{k}\,(-9)$$

$$= -3\vec{i} - 2\vec{j} - 9\vec{k}$$

and $$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -3 & 0 \\ 1 & 1 & -1 \end{vmatrix}$$

$$= \vec{i}(3-0) - \vec{j}\,(-2-0) + \vec{k}\,(2+3)$$

$$= 3\vec{i} + 2\vec{j} + 5\vec{k}$$

Now, $$\vec{a}' = \frac{\vec{b}\times\vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]} = -\frac{1}{4}\vec{i} - \frac{1}{4}\vec{j} - \frac{3}{4}\vec{k}$$

$$\vec{b}' = \frac{\vec{c}\times\vec{a}}{[\vec{a}\ \vec{b}\ \vec{c}]} = -\frac{3}{4}\vec{i} - \frac{1}{4}\vec{j} - \frac{9}{4}\vec{k}$$

and $$\vec{c}' = \frac{\vec{a}\times\vec{b}}{[\vec{a}\ \vec{b}\ \vec{c}]} = \frac{3}{4}\vec{i} + \frac{1}{2}\vec{j} + \frac{5}{4}\vec{k}$$

(iii) $\mathbf{2\vec{i} + 3\vec{j} - \vec{k},\ \vec{i} - \vec{j} - 2\vec{k},\ -\vec{i} + 2\vec{j} + 2\vec{k}}$ **(BE 2070 Bhadra)**

Solution: Let, $\vec{a} = 2\vec{i} + 3\vec{j} - \vec{k}$

$\vec{b} = \vec{i} - \vec{j} - 2\vec{k}$

and $\vec{c} = -\vec{i} + 2\vec{j} + 2\vec{k}$

Let the vectors reciprocal system to $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{a}', \vec{b}', \vec{c}'$ respectively.

Now, $$\left[\vec{a}\ \vec{b}\ \vec{c}\right] = \begin{vmatrix} 2 & 3 & -1 \\ 1 & -1 & -2 \\ -1 & 2 & 2 \end{vmatrix}$$

$$= 2\,(-2+4) - 3\,(2-2) - 1\,(2-1)$$

$$= 4 - 0 - 1$$

$$= 3$$

$$\therefore\ \left[\vec{a}\ \vec{b}\ \vec{c}\right] = 3 \neq 0$$

Hence, $\vec{a}', \vec{b}', \vec{c}'$ exists.

Again, $$\vec{b}\times\vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & -2 \\ -1 & 2 & 2 \end{vmatrix}$$

$$= \vec{i}(-2+4) - \vec{j}\,(2-2) + \vec{k}\,(2-1)$$

$$= 2\vec{i} + \vec{k}$$

Also, $$\vec{c}\times\vec{a} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 2 & 2 \\ 2 & 3 & -1 \end{vmatrix}$$

$$= \vec{i}(-2-6) - \vec{j}\,(1-4) + \vec{k}\,(-3-4)$$

$$= -8\vec{i} + 3\vec{j} - 7\vec{k}$$

and $$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & -1 \\ 1 & -1 & -2 \end{vmatrix}$$

$$= \vec{i}(-6-1) - \vec{j}\,(-4+1) + \vec{k}\,(-2-3)$$

$$= -7\vec{i} + 3\vec{j} - 5\vec{k}$$

Now, $$\vec{a}' = \frac{\vec{b}\times\vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]} = \frac{2}{3}\vec{i} + \frac{1}{3}\vec{k}$$

$$\vec{b}' = \frac{\vec{c}\times\vec{a}}{[\vec{a}\ \vec{b}\ \vec{c}]} = -\frac{8}{3}\vec{i} + \vec{j} - \frac{7}{3}\vec{k}$$

and $\vec{c}' = \dfrac{\vec{a}\times\vec{b}}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]} = -\dfrac{7}{3}\vec{i}+\vec{j}-\dfrac{5}{3}\vec{k}$

(iv) $-\vec{i}+\vec{j}+\vec{k},\ \vec{i}-\vec{j}+\vec{k},\ \vec{i}+\vec{j}-\vec{k}$ **(BE 2068 Bhadra)**

Solution: Let, $\vec{a} = -\vec{i}+\vec{j}+\vec{k}$

$\vec{b} = \vec{i}-\vec{j}+\vec{k}$

and $\vec{c} = \vec{i}+\vec{j}-\vec{k}$

Let the vectors reciprocal system to $\vec{a}, \vec{b}, \vec{c}$ and $\vec{a}', \vec{b}', \vec{c}'$ respectively

Now, $\left[\vec{a}\ \vec{b}\ \vec{c}\right] = \vec{a}\,.\,\vec{b}\times\vec{c}$

$$= \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$= -1\,(1-1) - 1\,(-1-1) + 1\,(1+1)$

$= 0+2+2$

$= 4$

$\therefore \left[\vec{a}\ \vec{b}\ \vec{c}\right] = 4 \neq 0$

Hence, $\vec{a}', \vec{b}', \vec{c}'$ exists.

Again, $\vec{b}\times\vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$

$= \vec{i}(1-1) - \vec{j}\,(-1-1) + \vec{k}\,(1+1)$

$= 2\vec{j}+2\vec{k}$

$= 2\left(\vec{j}+\vec{k}\right)$

Also, $\vec{c}\times\vec{a} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{vmatrix}$

$= \vec{i}(1+1) - \vec{j}\,(1-1) + \vec{k}\,(1+1)$

$= 2\vec{i}+2\vec{k}$

$= 2\left(\vec{i}+\vec{k}\right)$

and $\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix}$

$= \vec{i}(1+1) - \vec{j}\,(-1-1) + \vec{k}\,(1-1)$

$= 2\vec{i}+2\vec{j}$

$= 2\left(\vec{i}+\vec{j}\right)$

Now, $\vec{a}' = \dfrac{\vec{b}\times\vec{c}}{\left[\vec{a}\,\vec{b}\,\vec{c}\right]} = \dfrac{2\left(\vec{j}+\vec{k}\right)}{4} = \dfrac{\vec{i}+\vec{k}}{2} = \dfrac{1}{2}\vec{i}+\dfrac{1}{2}\vec{k}$

$\vec{b}' = \dfrac{\vec{c}\times\vec{a}}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]} = \dfrac{2\left(\vec{i}+\vec{k}\right)}{4} = \dfrac{\vec{i}+\vec{k}}{2} = \dfrac{1}{2}\vec{i}+\dfrac{1}{2}\vec{k}$

and $\vec{c}' = \dfrac{\vec{a}\times\vec{b}}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]} = \dfrac{2\left(\vec{i}+\vec{j}\right)}{4} = \dfrac{\vec{i}+\vec{j}}{2} = \dfrac{1}{2}\vec{i}+\dfrac{1}{2}\vec{j}$

11. If $\vec{a}, \vec{b}, \vec{c}$ and $\vec{a}', \vec{b}', \vec{c}'$ are reciprocal system of vectors then. Show that

(i) $\vec{a}.\vec{a}'+\vec{b}.\vec{b}'+\vec{c}.\vec{c}' = 3$ **[2062]**

Solution: Given, $\vec{a}, \vec{b}, \vec{c}$ and $\vec{a}', \vec{b}', \vec{c}'$ are reciprocal system of vectors then by definition.

$\vec{a}' = \dfrac{\vec{b}\times\vec{c}}{\left[\vec{a}\,\vec{b}\,\vec{c}\right]}$, $\vec{b}' = \dfrac{\vec{c}\times\vec{a}}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]}$ and $\vec{c}' = \dfrac{\vec{a}\times\vec{b}}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]}$

Now, L. H. S. $= \vec{a}.\vec{a}'+\vec{b}.\vec{b}'+\vec{c}.\vec{c}'$

$= \vec{a}\,.\,\dfrac{\vec{b}\times\vec{c}}{\left[\vec{a}\,\vec{b}\,\vec{c}\right]} + \vec{b}\,.\,\dfrac{\vec{c}\times\vec{a}}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]} + \vec{c}\,.\,\dfrac{\vec{a}\times\vec{b}}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]}$

$= \dfrac{1}{\left[\vec{a}\,\vec{b}\,\vec{c}\right]}\left[\vec{a}\,.\left(\vec{b}\times\vec{c}\right) + \vec{b}.\left(\vec{c}\times\vec{a}\right) + \vec{c}.\left(\vec{a}\times\vec{b}\right)\right]$

$= \dfrac{1}{\left[\vec{a}\,\vec{b}\,\vec{c}\right]}\left\{\left[\vec{a}\,\vec{b}\,\vec{c}\right] + \left[\vec{b}\ \vec{c}\ \vec{a}\right] + \left[\vec{c}\ \vec{a}\,\vec{b}\right]\right\}$

$= \dfrac{1}{\left[\vec{a}\,\vec{b}\,\vec{c}\right]}\,3\left[\vec{a}\,\vec{b}\,\vec{c}\right]$

Since, $\left[\vec{b}\ \vec{c}\ \vec{a}\right] = \left[\vec{a}\ \vec{b}\ \vec{c}\right]$ and $\left[\vec{c}\ \vec{a}\,\vec{b}\right] = \left[\vec{a}\ \vec{b}\ \vec{c}\right]$

$= 3$

Hence, $\vec{a}.\vec{a}'+\vec{b}.\vec{b}'+\vec{c}.\vec{c}' = 3$ Proved.

(ii) $\left(\vec{a}+\vec{b}\right).\vec{a}'+\left(\vec{b}+\vec{c}\right).\vec{b}'+\left(\vec{c}+\vec{a}\right).\vec{c}' = 3$

Solution: Given, $\vec{a}, \vec{b}, \vec{c}$ and $\vec{a}', \vec{b}', \vec{c}'$ are the reciprocal system of vectors then by definition.

$$\vec{a}' = \frac{\vec{b}\times\vec{c}}{\left[\vec{a}\,\vec{b}\,\vec{c}\right]}, \qquad \vec{b}' = \frac{\vec{c}\times\vec{a}}{\left[\vec{a}\,\vec{b}\,\vec{c}\right]} \quad \text{and } \vec{c}' = \frac{\vec{a}\times\vec{b}}{\left[\vec{a}\,\vec{b}\,\vec{c}\right]}$$

Now, L. H. S. $= \left(\vec{a}+\vec{b}\right).\vec{a}' + \left(\vec{b}+\vec{c}\right).\vec{b}' + \left(\vec{c}+\vec{a}\right).\vec{c}'$

$$= \left(\vec{a}+\vec{b}\right).\frac{\vec{b}\times\vec{c}}{\left[\vec{a}\,\vec{b}\,\vec{c}\right]} + \left(\vec{b}+\vec{c}\right).\frac{\vec{c}\times\vec{a}}{\left[\vec{a}\,\vec{b}\,\vec{c}\right]} + \left(\vec{c}+\vec{a}\right).\frac{\vec{a}\times\vec{b}}{\left[\vec{a}\,\vec{b}\,\vec{c}\right]}$$

$$= \frac{1}{\left[\vec{a}\,\vec{b}\,\vec{c}\right]}\Big[\left(\vec{a}+\vec{b}\right).\left(\vec{b}\times\vec{c}\right) + \left(\vec{b}+\vec{c}\right).\left(\vec{c}\times\vec{a}\right) + \left(\vec{c}+\vec{a}\right).\left(\vec{a}\times\vec{b}\right)\Big]$$

$$= \frac{1}{\left[\vec{a}\,\vec{b}\,\vec{c}\right]}\Big[\vec{a}.\left(\vec{b}\times\vec{c}\right) + \vec{b}.\left(\vec{b}\times\vec{c}\right) + \vec{b}.\left(\vec{c}\times\vec{a}\right) + \vec{c}.\left(\vec{c}\times\vec{a}\right) + \vec{c}.\left(\vec{a}\times\vec{b}\right) + \vec{a}.\left(\vec{a}\times\vec{b}\right)\Big]$$

$$= \frac{1}{\left[\vec{a}\,\vec{b}\,\vec{c}\right]}\Big\{\left[\vec{a}\,\vec{b}\,\vec{c}\right] + \left[\vec{b}\,\vec{b}\,\vec{c}\right] + \left[\vec{b}\;\vec{c}\;\vec{a}\right] + \left[\vec{c}\;\vec{c}\;\vec{a}\right] + \left[\vec{c}\;\vec{a}\;\vec{b}\right] + \left[\vec{a}\;\vec{a}\;\vec{b}\right]\Big\}$$

$$= \frac{1}{\left[\vec{a}\,\vec{b}\,\vec{c}\right]}\Big\{\left[\vec{a}\,\vec{b}\,\vec{c}\right] + 0 + \left[\vec{a}\,\vec{b}\,\vec{c}\right] + 0\left[\vec{a}\,\vec{b}\,\vec{c}\right] + 0\Big\}$$

Since $\left[\vec{b}\;\vec{b}\;\vec{c}\right] = 0$ etc and $\left[\vec{c}\;\vec{a}\;\vec{b}\right] = \left[\vec{a}\;\vec{b}\;\vec{c}\right]$

$$= \frac{1}{\left[\vec{a}\,\vec{b}\,\vec{c}\right]}.3\left[\vec{a}\;\vec{b}\;\vec{c}\right]$$

$= 3$

Hence, $\left(\vec{a}+\vec{b}\right).\vec{a}' + \left(\vec{b}+\vec{c}\right).\vec{b}' + \left(\vec{c}+\vec{a}\right).\vec{c}' = 3$ Proved.

12. If $\vec{a}' = \vec{u} = \dfrac{\vec{b}\times\vec{c}}{[\vec{a}\;\vec{b}\;\vec{c}]}$, $\vec{b}' = \vec{v} = \dfrac{\vec{c}\times\vec{a}}{[\vec{a}\;\vec{b}\;\vec{c}]}$ and $\vec{c}' = \vec{w} = \dfrac{\vec{a}\times\vec{b}}{[\vec{a}\;\vec{b}\;\vec{c}]}$

(i) Find $\left[\vec{u}\;\vec{v}\;\vec{w}\right]$

Solution: Given $\vec{a}' = \vec{u} = \dfrac{\vec{b}\times\vec{c}}{[\vec{a}\;\vec{b}\;\vec{c}]}$

$$\vec{b}' = \vec{v} = \frac{\vec{c}\times\vec{a}}{[\vec{a}\;\vec{b}\;\vec{c}]}$$

$$\text{and } \vec{c}' = \vec{w} = \frac{\vec{a}\times\vec{b}}{[\vec{a}\;\vec{b}\;\vec{c}]}$$

Now, $\left[\vec{u}\;\vec{v}\;\vec{w}\right] = \vec{u}.\vec{v}\times\vec{w}$

Here, $\vec{v}\times\vec{w} = \dfrac{\vec{c}\times\vec{a}}{[\vec{a}\;\vec{b}\;\vec{c}]} \times \dfrac{\vec{a}\times\vec{b}}{[\vec{a}\;\vec{b}\;\vec{c}]}$

$$= \frac{1}{[\vec{a}\;\vec{b}\;\vec{c}]^2}\left(\vec{c}\times\vec{a}\right)\times\left(\vec{a}\times\vec{b}\right)$$

$$= \frac{1}{[\vec{a}\;\vec{b}\;\vec{c}]^2}\,[\vec{c}\;\vec{a}\;\vec{b}]\,\vec{a} - [\vec{c}\;\vec{a}\;\vec{a}]\,\vec{b}$$

$$= \frac{[\vec{a}\;\vec{b}\;\vec{c}]\,\vec{a}}{[\vec{a}\;\vec{b}\;\vec{c}]^2} - 0$$

Since $[\vec{c}\;\vec{a}\;\vec{a}] = 0$ and $[\vec{c}\;\vec{a}\;\vec{b}] = [\vec{a}\;\vec{b}\;\vec{c}]$

$$\therefore\ \vec{v}\times\vec{w} = \frac{\vec{a}}{[\vec{a}\;\vec{b}\;\vec{c}]}$$

Again, $\vec{u}.\vec{v}\times\vec{w} = \dfrac{\vec{b}\times\vec{c}}{[\vec{a}\;\vec{b}\;\vec{c}]}\cdot\dfrac{\vec{a}}{[\vec{a}\;\vec{b}\;\vec{c}]}$

$$= \frac{\left[\vec{b}\;\vec{c}\;\vec{a}\right]}{[\vec{a}\;\vec{b}\;\vec{c}]^2}$$

$$= \frac{\left[\vec{a}\;\vec{b}\;\vec{c}\right]}{[\vec{a}\;\vec{b}\;\vec{c}]^2}$$

$$\therefore\ \vec{u}.\vec{v}\times\vec{w} = \frac{1}{[\vec{a}\;\vec{b}\;\vec{c}]}$$

Thus $[\vec{u}.\vec{v}\times\vec{w}] = \dfrac{1}{[\vec{a}\;\vec{b}\;\vec{c}]}$ Ans.

(ii) Show that $\vec{a} = \dfrac{\vec{b}'\times\vec{c}'}{[\vec{a}'\;\vec{b}'\;\vec{c}']}$, $\vec{b} = \dfrac{\vec{c}'\times\vec{a}'}{[\vec{a}'\;\vec{b}'\;\vec{c}']}$ and $\vec{c} = \dfrac{\vec{a}'\times\vec{b}'}{[\vec{a}'\;\vec{b}'\;\vec{c}']}$

Solution: We have, $\vec{a}' = \dfrac{\vec{b}\times\vec{c}}{[\vec{a}\;\vec{b}\;\vec{c}]}$, $\vec{b}' = \dfrac{\vec{c}\times\vec{a}}{[\vec{a}\;\vec{b}\;\vec{c}]}$ and $\vec{c}' = \dfrac{\vec{a}\times\vec{b}}{[\vec{a}\;\vec{b}\;\vec{c}]}$

Now, $\left[\vec{a}'\;\vec{b}'\;\vec{c}'\right] = \vec{a}'.\vec{b}'\times\vec{c}'$

$$= \frac{\vec{b}\times\vec{c}}{[\vec{a}\;\vec{b}\;\vec{c}]}\cdot\frac{\vec{c}\times\vec{a}}{[\vec{a}\;\vec{b}\;\vec{c}]}\times\frac{\vec{a}\times\vec{b}}{[\vec{a}\;\vec{b}\;\vec{c}]}$$

$$= \frac{1}{[\vec{a}\;\vec{b}\;\vec{c}]^3}\left\{\left(\vec{b}\times\vec{c}\right).\left(\vec{c}\times\vec{a}\right)\times\left(\vec{a}\times\vec{b}\right)\right\}$$

$$= \frac{1}{[\vec{a}\ \vec{b}\ \vec{c}]^3}\left\{\left(\vec{b}\times\vec{c}\right).\left[\vec{c}\ \vec{a}\ \vec{b}\right]\vec{a}-\left[\vec{c}\ \vec{a}\ \vec{a}\right]\vec{b}\right\}$$

$$= \frac{1}{[\vec{a}\ \vec{b}\ \vec{c}]^3}\left\{\left(\vec{b}\times\vec{c}\right).\left[\vec{a}\ \vec{b}\ \vec{c}\right]\vec{a}-0.\vec{b}\right\}$$

$$= \frac{[\vec{a}\ \vec{b}\ \vec{c}]}{[\vec{a}\ \vec{b}\ \vec{c}]^3}\left\{\left(\vec{b}\times\vec{c}\right).\vec{a}\right\}$$

$$= \frac{[\vec{a}\ \vec{b}\ \vec{c}]}{[\vec{a}\ \vec{b}\ \vec{c}]^3}\left[\vec{b}\ \vec{c}\ \vec{a}\right]$$

$$= \frac{[\vec{a}\ \vec{b}\ \vec{c}]^2}{[\vec{a}\ \vec{b}\ \vec{c}]^3}$$

$$= \frac{1}{[\vec{a}\ \vec{b}\ \vec{c}]}$$

$$\therefore \left[\vec{a}'\ \vec{b}'\ \vec{c}'\right] = \frac{1}{[\vec{a}\ \vec{b}\ \vec{c}]}$$

Now $\dfrac{\vec{b}'\times\vec{c}'}{[\vec{a}'\ \vec{b}'\ \vec{c}']}$

$$= \frac{1}{[\vec{a}'\ \vec{b}'\ \vec{c}']}\left[\vec{b}'\times\vec{c}'\right]$$

$$= [\vec{a}\ \vec{b}\ \vec{c}]\left\{\frac{\vec{c}\times\vec{a}}{[\vec{a}\ \vec{b}\ \vec{c}]}\times\frac{\vec{a}\times\vec{b}}{[\vec{a}\ \vec{b}\ \vec{c}]}\right\}$$

$$= \frac{[\vec{a}\ \vec{b}\ \vec{c}]}{[\vec{a}\ \vec{b}\ \vec{c}]^2}\left\{\left(\vec{c}\times\vec{a}\right)\times\left(\vec{a}\times\vec{b}\right)\right\}$$

$$= \frac{1}{[\vec{a}\ \vec{b}\ \vec{c}]}\left\{\left[\vec{c}\ \vec{a}\ \vec{b}\right]\vec{a}-\left[\vec{c}\ \vec{a}\ \vec{a}\right]\vec{b}\right\}$$

$$= \frac{1}{[\vec{a}\ \vec{b}\ \vec{c}]}\left[\vec{a}\ \vec{b}\ \vec{c}\right]\vec{a}$$

Since $\left[\vec{c}\ \vec{a}\ \vec{b}\right]=\left[\vec{a}\ \vec{b}\ \vec{c}\right]$

$$= \vec{a}$$

Hence, $\vec{a} = \dfrac{\vec{b}'\times\vec{c}'}{[\vec{a}'\ \vec{b}'\ \vec{c}']}$ Proved.

Also, $\dfrac{\vec{c}'\times\vec{a}'}{[\vec{a}'\ \vec{b}'\ \vec{c}']}$

$$= \frac{1}{[\vec{a}'\ \vec{b}'\ \vec{c}']}\left(\vec{c}'\times\vec{a}'\right)$$

$$= \left[\vec{a}\ \vec{b}\ \vec{c}\right]\left\{\frac{\vec{a}\times\vec{b}}{[\vec{a}\ \vec{b}\ \vec{c}]}\times\frac{\vec{b}'\times\vec{c}'}{[\vec{a}\ \vec{b}\ \vec{c}]}\right\}$$

$$= \frac{[\vec{a}\ \vec{b}\ \vec{c}]}{[\vec{a}\ \vec{b}\ \vec{c}]^2}\left\{\left(\vec{a}\times\vec{b}\right)\times\left(\vec{b}\times\vec{c}\right)\right\}$$

$$= \frac{1}{[\vec{a}\ \vec{b}\ \vec{c}]}\left\{\left[\vec{a}\ \vec{b}\ \vec{c}\right]\vec{b}-\left[\vec{a}\ \vec{b}\ \vec{b}\right]\vec{b}\right\}$$

$$= \frac{1}{[\vec{a}\ \vec{b}\ \vec{c}]}.\left[\vec{a}\ \vec{b}\ \vec{c}\right]\vec{b} = \vec{b}$$

Hence $\vec{b} = \dfrac{\vec{c}'\times\vec{a}'}{[\vec{a}'\ \vec{b}'\ \vec{c}']}$ Proved.

and $\dfrac{\vec{a}'\times\vec{b}'}{[\vec{a}'\ \vec{b}'\ \vec{c}']}$

$$= \frac{1}{[\vec{a}'\ \vec{b}'\ \vec{c}']}\left(\vec{a}'\times\vec{b}'\right)$$

$$= \left[\vec{a}\ \vec{b}\ \vec{c}\right]\left\{\frac{\vec{b}\times\vec{c}}{[\vec{a}\ \vec{b}\ \vec{c}]}\times\frac{\vec{c}\times\vec{a}}{[\vec{a}\ \vec{b}\ \vec{c}]}\right\}$$

$$= \frac{[\vec{a}\ \vec{b}\ \vec{c}]}{[\vec{a}\ \vec{b}\ \vec{c}]^2}\left\{\left(\vec{b}\times\vec{c}\right)\times\left(\vec{c}\times\vec{a}\right)\right\}$$

$$= \frac{1}{[\vec{a}\ \vec{b}\ \vec{c}]}\left\{\left[\vec{b}\ \vec{c}\ \vec{a}\right]\vec{c}-\left[\vec{b}\ \vec{c}\ \vec{c}\right]\vec{a}\right\}$$

$$= \frac{1}{[\vec{a}\ \vec{b}\ \vec{c}]}\left\{\left[\vec{a}\ \vec{b}\ \vec{c}\right]\vec{c}-0\right\}$$

$$= \frac{[\vec{a}\ \vec{b}\ \vec{c}]\ \vec{c}}{[\vec{a}\ \vec{b}\ \vec{c}]}$$

$$= \vec{c}$$

Hence, $\vec{c} = \dfrac{\vec{a}'\times\vec{b}'}{[\vec{a}'\ \vec{b}'\ \vec{c}']}$ Proved.

Exercise - 22

1. Find the parametric equations for the line through P and parallel to $\vec{a}$.

(i) $P(4, 2, -3)$, $\vec{a} = \left(\frac{1}{3}, 2, \frac{1}{2}\right)$

Solution: We know parametric equation of a line through P (4, 2, – 3) and parallel to $\vec{a} = \left(\frac{1}{3}, 2, \frac{1}{2}\right)$ are,

$x = 4 + \frac{1}{3}t,\ y = 2 + 2t$

and $z = -3 + \frac{1}{2}t$. where $f \in \mathbb{R}$

(ii) $P(5, 0, -2)$, $\vec{a} = (-1, -4, 1)$

Solution: We know parametric equation of a line through P (5, 0, – 2) and parallel to

$\vec{a} = (-1, -4, 1)$ are

$x = 5 - t,\ y = 0 - 4t$ and $z = -2 + t$

Where $t \in \mathbb{R}$

or, $x = 5 - t,\ y = -4t$ and $z = -2 + t$

(iii) $P(0, 0, 0)$, $\vec{a} = (0, 1, 0)$

Solution: We know parametric equation of a line through P (0, 0, 0) and parallel to $\vec{a} = (0, 1, 0)$ are

$x = 0 + 0.t,\ y = 0 + 1.t,\ z = 0 + 0.t$

where $t \in \mathbb{R}$

or, $x = 0,\ y = t,\ z = 0$ Ans.

(iv) $P(1, 2, 3)$, $\vec{a} = (1, 2, 3)$

Solution: We know parametric equation of a line through P (1, 2, 3) and Parallel to $\vec{a} = (1, 2, 3)$ are

$x = 1 + 1t,\ y = 2 + 2t,\ z = 3 + 3t$

Where $t \in \mathbb{R}$

or, $x = 1 + t,\ y = 2(1 + t),\ z = 3(1 + t)$ Ans.

2. Find the parametric equations for the line through P_1 (5, – 2, 4), P_2 (2, 6, 1). Also find the points at which the line intersects each of the co-ordinate planes.

Solution: Given two points are,

$P_1(5, -2, 4)$ and $P_2(2, 6, 1)$

Let 0 be the origion then

$\overrightarrow{OP_1} = (5, -2, 4)$ and $\overrightarrow{OP_2} = (2, 6, 1)$

The vector $\vec{a}$ corresponding to $\overrightarrow{P_1P_2}$ is

$$\begin{aligned}\vec{a} = \overrightarrow{P_1P_2} &= \overrightarrow{OP_2} - \overrightarrow{OP_1}\\ &= (2, 6, 1) - (5, -2, 4)\\ &= (2-5, 6+2, 1-4)\\ &= (-3, 8, -3)\end{aligned}$$

So the equation of the line through $P_1(5, -2, 4)$ and parallel to $\vec{a} = \overrightarrow{P_1P_2} = (-3, 8, -3)$ is,

$x = 5 - 3t,\ y = -2 = 8t$ and $z = 4 - 3t$

Where, $t \in \mathbb{R}$

If the line intersects z-axis at R (x, y, z)

Then $z = 0$

$\therefore\ 4 - 3t = 0$

$\Rightarrow t = \frac{4}{3}$

So, $x = 5 - 3 . \frac{4}{3}$ and $y = -2 + 8 . \frac{4}{3}$

or, $x = 5 - 4$ and $y = -\frac{6+32}{3} = \frac{26}{3}$

or, $x = 1$

Hence R is the point with coordinates $\left(1, \frac{26}{3}, 0\right)$. Ans.

If the line intersects y-axis at R (x, y, z)

Then $y = 0$

$\therefore\ -2 + 8t = 0$

$\Rightarrow t = \frac{1}{4}$

So, $x = 5 - 3 . \frac{1}{4}$ and $z = 4 - 3 . \frac{1}{4}$

or, $x = \frac{20-3}{4}$ and $y = \frac{16-3}{4}$

or, $x = \frac{17}{3}$ and $y = \frac{13}{4}$

Hence R is the point with coordinates

$\left(\frac{17}{3}, 0, \frac{13}{4}\right)$ Ans.

If the line intersects x-axis at R (x, y, z)

Then $x = 0$

$\therefore\ 5 - 3t = 0$

$\Rightarrow t = \frac{5}{3}$

So, $y = -2 + 8 \cdot \frac{5}{3}$ and $z = 4 - 3 \cdot \frac{5}{3}$

or, $y = \frac{-6+40}{3}$ and $z = \frac{12-15}{3}$

or, $y = \frac{34}{3}$ and $= -1$

Hence, R is the point with co-ordinates $\left(0, \frac{34}{3}. -1\right)$ Ans.

3. Find the equation of the plane passing through (– 11, 4, – 2) with normal vector $6\vec{i} - 5\vec{j} - \vec{k}$.

Solution: We know the equation of the plane through the point (– 11, 4, – 2) with normal vector.

$\vec{a} = \left(6\vec{i} - 5\vec{j} - \vec{k}\right) = (6 - 5, -1)$ is

$a_1(x - x_1) + a_2(y - y_1) + a_3(z - z_1) = 0$

$= 6(x + 11) - (y - 4) - 1(z + 2) = 0$

or, $6x - 5y - z + 66 + 20 - 2 = 0$

or, $6x - 5y - z + 84 = 0$ Ans.

4. Find the equation of plane through P (– 2, 5, – 8) with normal vector (a) $\vec{i}$ (b) $\vec{j}$ (c) $\vec{k}$.

Solution: We know the equation of the plane through the point P (– 2, 5, – 8) with normal vector.

$\vec{a} = \vec{i} = (1, 0, 0)$ is

$a_1(x - x_1) + a_2(y - y_1) + a_3(z - z_1) = 0$

$= 1(x + 2) + 0(y - 5) + 0(z + 8) = 0$

$= x + 2$ Ans.

(b) We know the equation of the plane through the point P (– 2, 5, – 8) with normal vectors

$\vec{a} = \vec{j} = (0, 1, 0)$ is

$a_1(x - x_1) + a_2(y - y_1) + a_3(z - z_1) = 0$

$= 0(x + 2) + 1(y - 5) + 0(z + 8) = 0$

$= y - 5$ Ans.

(c) We know the equation of the plane through the point P (– 2, 5, – 8) with normal vectors

$\vec{a} = \vec{k} = (0, 0, 1)$ is

$a_1(x - x_1) + a_2(y - y_1) + a_3(z - z_1) = 0$

$= 0(x + 2) + 0(y - 5) + 1(z + 8) = 0$

$= z + 8$ Ans.

5. Find the equation of a plane through (2, 5, – 6) parallel to the palen 3x – 2z = 10 by the vector method.

Solution: The given plane is,

$3x - y + 2z = 0$ ..... (i)

The normal vector to this plane is

$\vec{N} = 3\vec{i} - \vec{j} + 2\vec{k} = (3, -1, 2)$

The required plane is parallel to the plane

(i) and $\vec{N}$ is normal to the required plane

Let $\vec{r}$ (x, y, z) be any point on the required plane. If $\vec{a} = (2, 5, -6)$

Then, $\left(\vec{r} - \vec{a}\right)\vec{N} = 0$

or, $(x - 2, y - 5, z + 6)(3, -1, 2)$

or, $3(x - 2) - 1(y - 5) + 2(z + 6) = 0$

or, $3x - y + 2z - 6 + 5 + 12 = 0$

or, $3x - y + 2z + 11 = 0$ Ans.

6. Find the equation of a plane through origin and the points P (0, 2, 5) and Q (1, 4, 0) by the vector method.

Solution: Let the points P (0, 2, 5), Q (1, 4, 0) and the points 0 (0, 0, 0) determine a plane that contains the triangle.

The vectors $\vec{a}$ and $\vec{b}$ corresponding to $\overrightarrow{PQ}$ and $\overrightarrow{PO}$ are,

$\vec{a} = \overrightarrow{PQ} = \overrightarrow{O'Q} - \overrightarrow{O'P} = (1, 4, 0) - (0, 2, 5)$

$= (1 - 0, 4 - 2, 0 - 5)$

$= (1, 2, -5)$

and $\vec{b} = \overrightarrow{PQ} = \overrightarrow{O'Q} - \overrightarrow{O'P} = (0, 0, 0) - (0, 2, 5)$

$= (0, -2, -5)$

The vector $\vec{a} \times \vec{b}$ is normal to the plane determined by P, Q and O so that

$\vec{a} \times \vec{b} = (1, 2, -5) \times (0, -2, -5)$

$= \left(\vec{i} + 2\vec{j} - 5\vec{k}\right) \times \left(0.\vec{i} - 2\vec{j} - 5\vec{k}\right)$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & -5 \\ 0 & -2 & -5 \end{vmatrix}$$

$$= \vec{i}(-10-10) - \vec{j}(-5-0) + \vec{k}(-2-0)$$

$$= -20\vec{i} + 5\vec{j} - 2\vec{k}$$

We know that equation of the plane through P (0, 2, 5) and normal to the vector (– 20, 5, – 2) is

$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$

or, $-20(x - 0) + 5(y - 2) - 2(z - 5) = 0$

or, $-20x + 5y - 10 - 2z + 10 = 0$

or, $-20x + 5y - 2z = 0$

$\therefore 20x - 5y + 2z = 0$

7. Find the equation of a line is symmetric from through P_1 (5, – 2, 4), P_2 (2, 6, 1) by the vector method.

Solution: Given points are,

P_1 (5, – 2, 4) and P_2 (2, 6, 1)

Let 0 be the origion then,

$\overrightarrow{OP_1} = (5, -2, 4)$ and $\overrightarrow{OP_2} + (2, 6, 11)$

Let the vector $\vec{a}$ corresponding to $\overrightarrow{P_1P_2} = \overrightarrow{OP_2} - \overrightarrow{OP_1}$

$= (2, 6, 1) - (5, -2, 4)$

$= (2 - 5, 6 + 2, 1 - 4)$

$= (-3, 8, -3)$

The parametric equation for the line through P_1 (5, – 2, 4) and parallel to the vectors $\vec{a} = (-3, 8, -3)$ is

$x = 5 - 3t$, $y = -2 + 8t$ and $z = 4 - 3t$

Where $t \in \mathbb{R}$

Solving each equation for 't' we get

$t = \frac{x-5}{-3}$, $t = \frac{y+2}{8}$ and $t = \frac{z-4}{-3}$

$\therefore \frac{x-5}{-3} = \frac{y+2}{8} = \frac{z-4}{-3}$

$\Rightarrow \frac{x-5}{-3} = \frac{y+2}{8} = \frac{z-4}{-3}$ Ans.

8. Find the equation of a line in symmetric form through P_1 (4, 2, – 3), P_2 (– 3, 2, 5) by the vector method.

Solution: Given points are,

P_1 (4, 2, – 3) and P_2 (– 3, 2, 5)

Let 0 be the origin then,

$\overrightarrow{OP_1} = (4, 2, -3)$ and $\overrightarrow{OP_2}$ (– 3, 2, 5)

Let the vector $\vec{a}$ corresponding to $\overrightarrow{P_1P_2}$ is

$\vec{a} = \overrightarrow{P_1P_2} = \overrightarrow{OP_2} - \overrightarrow{OP_1}$

$= (-3, 2, 5) - (4, 2, -3)$

$= (-7, 0, 8)$

The parametric equation for the line through P_1 (4, 2, – 3) and parallel to the vector $\vec{a} = (-7, 0, 8)$ is,

$x = 4 - 7t$, $y = 2 + 0t$ and $z = -2 + 2t$

Where $t \in \mathbb{R}$

$\therefore x = 4 - 7t$, $y = 2$, and $z = -3 + 8t$

Solving each equation for t we get,

$t = \frac{x-4}{7}$, $y - 2 = 0$, and $t = \frac{z+3}{2}$

$\therefore \frac{x-4}{7} = \frac{z+3}{2}$, $y - 2 = 0$ Ans.

Exercise - 23

1. If $\vec{r} = t^2\vec{i} - t\vec{j} + (2t+1)\vec{k}$

Find the value of (i) $\frac{d\vec{r}}{dt} \cdot \frac{d^2\vec{r}}{dt^2}$ (ii) $\left|\frac{d\vec{r}}{dt}\right|$ at $t = 0$

(iii) $\left|\frac{d^2\vec{r}}{dt^2}\right|$ at $t = 0$ (iv) $\vec{r} \cdot \frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}$

Solution:

(i) Given, $\vec{r} = t^2\vec{i} - t\vec{j} + (2t+1)\vec{k}$

$$\frac{d\vec{r}}{dt} = 2t\vec{i} - \vec{j} + 2\vec{k} \text{ and } \frac{d^2\vec{r}}{dt^2} + 2\vec{i}$$

$$\because \frac{d\vec{r}}{dt} \cdot \frac{d^2\vec{r}}{dt^2} = (2t\vec{i} - \vec{j} + 2\vec{k}) \cdot (2\vec{i})$$

$$= 4t \text{ Ans.}$$

(ii) Given, $\vec{r} = t^2\vec{i} - t\vec{j} + (2t+1)\vec{k}$

or, $\left|\frac{d\vec{r}}{dt^2}\right| = \sqrt{4t^2 + 1^2 + 4} = \sqrt{4t^2 + 5}$

At $t = 0$, $\left|\frac{d\vec{r}}{dt^2}\right| = \sqrt{4 \times 0 + 5} = \sqrt{5}$

(iii) Given, $\frac{d^2\vec{r}}{dt^2} = 2\vec{i}$

At $t = 0$, $\left|\frac{d^2\vec{r}}{dt^2}\right| = \sqrt{4} = 2$

(iv) Given, $\vec{r} \cdot \frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}$

$$= [t^2\vec{i} - t\vec{j} + (2t+1)\vec{k}] \cdot \{[2t\vec{i} - \vec{j} + 2\vec{k}) \times 2\vec{i}\}$$
$$= [t^2\vec{i} - t\vec{j} + (2t+1)\vec{k}] \cdot [2\vec{k} + 4\vec{j}]$$
$$= -4t + 4t + 2$$
$$= 2 \text{ Ans.}$$

If $\vec{r}_1 = t^3\vec{i} + t^2\vec{j} + t\vec{k}$

and $\vec{r}_2 = (t+1)\vec{i} + (t+2)\vec{j} - 3t\vec{k}$ find.

$\frac{d}{dt}\left(\vec{r}_1 \cdot \vec{r}_2\right)$ at $t = 2$

Solution: Given, $\vec{r}_1 = t^3\vec{i} - t^2\vec{j} + t\vec{k}$

$$\vec{r}_2 = (t+1)\vec{i} + (t+2)\vec{j} - 3t\vec{k}$$

Now, $\vec{r}_1 \cdot \vec{r}_2 = (t^3\vec{i} + t^2\vec{j} + t\vec{k}) \cdot (t+1)\vec{i} + (t+2)\vec{j} - 3t\vec{k}$

$$= t^3(t+1) + t^2(t+2) + t(-3t)$$
$$= t^4 + t^3 + t^3 + 2t^2 - 3t^2$$
$$= t^4 + 2t^3 - t^2$$

$$\therefore \frac{d}{dt}\left(\vec{r}_1 \cdot \vec{r}_2\right) = 4t^3 + 6t^2 - 2t$$

At $t = 2$; $\frac{d}{dt}\left(\vec{r}_1 \cdot \vec{r}_2\right)$

$$= 4 \times 2^3 + 6 \times 2^2 - 2 \times 2$$
$$= 32 + 24 - 4$$
$$= 52 \text{ Ans.}$$

$\frac{d}{dt}\left(\vec{r}_1 \times \vec{r}_2\right)$ at $t = 2$

Solution: Given, $\vec{r}_1 = t^3\vec{i} - t^2\vec{j} + t\vec{k}$

$$\vec{r}_2 = (t+1)\vec{i} + (t+2)\vec{j} - 3t\vec{k}$$

$\vec{r}_1 \times \vec{r}_2 = (t^3\vec{i} + t^2\vec{j} + t\vec{k}) \times \{(t+1)\vec{i}\ (t+2)\vec{j} - 3t\vec{k}\}$

$$= (t^4 + 2t^3)\vec{k} + 3t^4\vec{j} - t^3\vec{k} - t^2\vec{k} - 3t^3\vec{i} + t^2\vec{j} + t\vec{j} - t^2\vec{i} - 2t\vec{i}$$
$$= (t^4 + 2t^3 - t^2 - t^3)\vec{k} + (3t^4 + t^2 + t)\vec{j} - 3t^3 + (t^2 + 2t)\vec{i}$$

$$\frac{d}{dt}\left(\vec{r}_1 \times \vec{r}_2\right) = (4t^3 + 6t^2 - 2t - 3t^2)\vec{k} + (12t^3 + 2t + 1)\vec{j} - (9t^2 + 2t + 2)\vec{i}$$

At $t = 2$; $\frac{d}{dt}(\vec{r}_1 \times \vec{r}_2)$

$$= 40\vec{k} + 101\vec{j} - 42\vec{i}$$
$$= 42\vec{i} + 101\vec{j} + 40\vec{k}$$

$\frac{d}{dt}\left(\vec{r}_1 \cdot \vec{r}_2\right) = \vec{r}_1 \cdot \frac{d\vec{r}_2}{dt} + \frac{d\vec{r}_1}{dt} \cdot \vec{r}_2$

Solution: From (i), we have,

$$\frac{d}{dt}\left(\vec{r}_1 \cdot \vec{r}_2\right) = 4t^3 + 6t^2 - 2t \text{ ..... (i)}$$

Also, we have

$$\vec{r}_1 \cdot \frac{d\vec{r}_2}{dt} + \frac{d\vec{r}_1}{dt} \cdot \vec{r}_2 = (t^3\vec{i} + t^2\vec{j} + t\vec{k}) \cdot \frac{d}{dt}[(t+1)\times\vec{i} + (t+2)\vec{j} - 3t\vec{k}]$$

$$+ \frac{d}{dt}[t^3\vec{i} + t^2\vec{j} + t\vec{k}] \cdot (t+1)\times\vec{i} + (t+2)\vec{j} - 3t$$

$$= (t^3\vec{i} + t^2\vec{j} + t\vec{k}) \cdot (\vec{i} + \vec{j} - 3\vec{k}) + (3t^2\vec{i} + 2t\vec{j} + \vec{k})$$

$$\cdot (t+1)\vec{i} + (t+2)\vec{k} - 3t$$

$$= t^3 + t^2 - 3t + 3t^3 + 3t^2 + 2t^2 + 4t - 3t$$

$$= 4t^3 + 6t^2 - 2t \text{ ..... (ii)}$$

From (i) and (ii) we conclude that

$$\frac{d}{dt}\left(\vec{r}_1 \cdot \vec{r}_2\right) = \vec{r}_1 \cdot \frac{d\vec{r}_2}{dt} + \frac{d\vec{r}_1}{dt} \cdot \vec{r}_2 \text{ Ans.}$$

(iv) $\frac{d}{dt}\left(\vec{r}_1 \times \vec{r}_2\right) = \vec{r}_1 \times \frac{d\vec{r}_2}{dt} + \frac{d\vec{r}_1}{dt} \times \vec{r}_2$

Solution: From (ii) we have,

$$\frac{d}{dt}\left(\vec{r}_1 \times \vec{r}_2\right) =$$

Also, we have

$$(4t^3 + 6t^2 - 2t - 3t^2)\vec{k} + (12t^3 + 2t + 1)\vec{j} - (9t^2 + 2t + 2)\vec{i})$$

$$\vec{r}_1 \times \frac{d\vec{r}_2}{dt} + \frac{d\vec{r}_1}{dt} \times \vec{r}_2 = (t^3\vec{i} + t^2\vec{j} + t\vec{k}) \times \frac{d}{dt}[(t+1)^2\vec{i} + (t+2)\vec{j} - 3t\vec{k}]$$

$$+ \frac{d}{dt}[t^3\vec{i} + t^2\vec{j} + t\vec{k}] \times [(t+1)^2\vec{i} + (t+2)\vec{j} - 3t$$

$$= (t^3\vec{i} + t^2\vec{j} + t\vec{k}) \times [\vec{i} + \vec{j} + \vec{k}] + [3t^2\vec{i} + 2t\vec{j} + \vec{k}]$$

$$\times [(t+1)^2\vec{i} + (t+2)\vec{j} - 3t$$

$$= t^3\vec{k} - t^3\vec{j} - t^2\vec{k} + t^2\vec{i} + t\vec{j} - t\vec{i} + (3t^3 + 6t^2)\vec{k} + 9t^2\vec{j}$$

$$- (2t^2 + 2t)\vec{k} - 6t^2\vec{i} + (t+1)\vec{j} - (t+2)$$

$$= (4t^3 + 6t^2 - 2t - 3t^2)\vec{k} + (12t^3 + 2t + 1)\vec{j}$$

$$- (9t^2 + 2t + 2)\vec{i} \text{ .....}$$

$\therefore$ From (i) and (ii) we culcude that

$$\frac{d}{dt}\left(\vec{r}_1 \times \vec{r}_2\right) = \vec{r}_1 \times \frac{d\vec{r}_2}{dt} + \frac{d\vec{r}_1}{dt} \times \vec{r}_2 \text{ Ans.}$$

3. (i) If $\vec{r} = \vec{a}e^{mt} + \vec{b}e^{nt}$ where $\vec{a}$ and $\vec{b}$ are constant vectors, show that $\frac{d^2\vec{r}}{dt^2} - (m+n)\frac{d\vec{r}}{dt} mn\vec{r} = 0$

Solution: We have, $\vec{r} = \vec{a}e^{mt} + \vec{b}e^{nt}$ ..... (i)

$$\therefore \frac{d\vec{r}}{dt} = m\vec{a}e^{mt} + n\vec{b}e^{nt}$$

$$\therefore \frac{d^2\vec{r}}{dt^2} = m^2\vec{a}e^{mt} + n^2\vec{b}e^{nt} \text{ ..... (ii)}$$

Then from question,

$$\frac{d^2\vec{r}}{dt^2} - (m+n)\frac{d\vec{r}}{dt} + mn\vec{r} = 0$$

$$\therefore m^2\vec{a}e^{mt} + n^2\vec{b}e^{nt} - (m+n)[m\vec{a}e^{mt} + n\vec{b}e^{nt}] + mn\vec{r} = 0$$

$$\therefore m^2\vec{a}e^{mt} + n^2\vec{b}e^{nt} - m^2\vec{a}e^{mt} - mn\vec{a}e^{mt} - n^2\vec{b}e^{nt} - mn\vec{b}e^{nt}$$

$$+ mn\vec{a}e^{mt} + mn\vec{b}e^{nt} = 0$$

$\therefore 0 = 0$ Hence, proved.

(ii) If $\vec{r} = \vec{a}e^{ct} + \vec{b}e^{-ct}$ where $\vec{a}$ and $\vec{b}$ are constant vectors, show that $\frac{d^2\vec{r}}{dt^2} - c^2\vec{r} = 0$

Solution: We have, $\vec{r} = \vec{a}e^{ct} + \vec{b}e^{-ct}$ ..... (i)

$$\frac{d\vec{r}}{dt} = \vec{a}ce^{ct} - \vec{b}ce^{-ct}$$

$$\frac{d^2\vec{r}}{dt^2} = \vec{a}c^2e^{ct} + \vec{b}c^2e^{-ct}$$

$$\frac{d^2\vec{r}}{dt^2} = c^2[\vec{a}e^{ct} + \vec{b}e^{-ct}]$$

$$\frac{d^2\vec{r}}{dt^2} = c^2\vec{r} \quad \text{using (i)}$$

$$\therefore \frac{d^2\vec{r}}{dt^2} - c^2\vec{r} = 0 \text{ Proved.}$$

(iii) For the curve $x = 3t, y = 3t^2, z = 2t^3$
Prove that $\left[\dot{\vec{r}}\ \ddot{\vec{r}}\ \dddot{\vec{r}}\right] = 216$

Solution: Given, equation of the curve is,

$x = 3t, y = 3t^2, z = 2t^3$

We have,

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

or, $\vec{r} = 3t\vec{i} + 3t^2\vec{j} + 2t^3\vec{k}$

Differentiating w. r. to t,

$$\frac{d\vec{r}}{dt} = \dot{\vec{r}} = 3\vec{i} + 6t\vec{j} + 6t^2\vec{k}$$

$$\frac{d^2\vec{r}}{dt^2} = \ddot{\vec{r}} = 0 + 6\vec{j} + 12t\vec{k}$$

$$\frac{d^3\vec{r}}{dt^3} = \dddot{\vec{r}} = 0 + 12 = 12$$

we have, $\left[\dot{\vec{r}}\ \ddot{\vec{r}},\ \dddot{\vec{r}}\right] = \begin{vmatrix} 3 & 6t & 6t^2 \\ 0 & 6 & 12t \\ 0 & 0 & 12 \end{vmatrix}$

$= 3 \times 6 \times 12$

$= 216$

$\therefore \left[\dot{\vec{r}}\ \ddot{\vec{r}},\ \dddot{\vec{r}}\right] = 216$ Ans.

4. (i) Particle moves along the cuve x = 4cost, y = 4 sint, z = 6t. Find the velocity an acceleration at time t = 0 and t = $\frac{\pi}{2}$. Also find their magnitudes. **(BE 2070 Bhadra)**

Solution: Here, the equation of the curve is x = 4cost, y = 4 sint, z = 6t

We have, $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$= 4\cos t\,\vec{i} + 4\sin t\,\vec{j} + 6t\vec{k}$

Differentiating it, with respect to t

$$\frac{d\vec{r}}{dt} = \dot{\vec{r}} = -4\sin t\,\vec{i} + 4\cos t\,\vec{j} + 6\vec{k}$$

$$\frac{d^2\vec{r}}{dt^2} = \ddot{\vec{r}} = -4\sin t\,\vec{i} - 4\sin t\,\vec{j} + 0$$

At, t = 0

Velocity, $\frac{d\vec{r}}{dt} = 4\vec{j} + 6\vec{k}$

Magnitudes = $\left|\frac{d\vec{r}}{dt}\right| = |4\vec{j} + 6\vec{k}| = \sqrt{16+36} = \sqrt{52} = 2\sqrt{17}$

At, t = $\frac{\pi}{2}$

Velocity, $\frac{d\vec{r}}{dt} = -4\vec{i} + 6\vec{k}$

Magnitudes = $\left|\frac{d\vec{r}}{dt}\right| = |-4\vec{i} + 6\vec{k}| = \sqrt{16+36} = \sqrt{52} = 2\sqrt{17}$

At, t = 0

Acceleration, $\frac{d^2\vec{r}}{dt^2} = -4\vec{i} - 0 = -4\vec{i}$

At t = $\frac{\pi}{2}$

Magnitudes $\left|\frac{d^2\vec{r}}{dt^2}\right| = |-4\vec{i}| = \sqrt{16} = 4$

Acceleration, $\frac{d^2\vec{r}}{dt^2} = 0 - 4\vec{j} = -4\vec{j}$

Magnitudes; $\left|\frac{d^2\vec{r}}{dt^2}\right| = |-4\vec{j}| = \sqrt{16} = 4$

(ii) A particle moves along the curve $x = t^3 + 1$, $y = t^2$, $z = 2t$. Find the velocity and accelaration at time t = 1. Also find their magnitudes.

Solution: Here, the equation of the curve is $x = t^3 + 1$, $y = t^2$, $z = 2t$

We have, $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$= (t^3 + 1)\,\vec{i} + t^2\vec{j} + 2t\vec{k}$

Differentiating it, with respect to t

$$\frac{d\vec{r}}{dt} = 3t^2\vec{i} + 2t\vec{j} + 2\vec{k}$$

$$\frac{d^2\vec{r}}{dt^2} = 6t\vec{i} + 2\vec{j}$$

At t = 1

Velocity $\frac{d\vec{r}}{dt} = 3\vec{i} + 2\vec{j} + 2\vec{k}$

Magnitudes = $\left|\frac{d\vec{r}}{dt}\right| = |3\vec{i} + 2\vec{j} + 2\vec{k}| = \sqrt{9+4+4} = \sqrt{25} = 5$

At t = 1

Acceleration, $\frac{d^2\vec{r}}{dt^2} = 6\vec{i} + 2\vec{j}$

Magnitudes = $\left|\frac{d\vec{r}}{dt}\right| = |6\vec{i} + 2\vec{j}| = \sqrt{36+4} = \sqrt{40} = 2\sqrt{10}$

5. (a) If $\vec{r}_1 = xyz\,\vec{i} + xz^2\vec{j} - y^3\vec{k}$ and $\vec{r}_2 = x^3\,\vec{i} - xyz\vec{j} + x^2z\vec{k}$, then find the value of

(i) $\dfrac{\partial^2}{\partial x\,\partial y}(\vec{r}_1 \times \vec{r}_2)$ **and** $\dfrac{\partial^2}{\partial x\,\partial y}(\vec{r}_2 \times \vec{r}_1)$

Solution: Here, $\vec{r}_1 = xyz\,\vec{i} + xz^2\,\vec{j} - y^3\,\vec{k}$ and

$\vec{r}_2 = x^3\,\vec{i} - xyz\,\vec{j} + x^2z\,\vec{k}$

$$\vec{r}_1 \times \vec{r}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ xyz & xz^2 & -y^3 \\ x^3 & -xyz & x^2z \end{vmatrix}$$

$$\therefore\ \vec{r}_1 \times \vec{r}_2 = (x^3z^3 - xy^4z)\ \vec{i} + \vec{j}\,(-x^3y^3 - x^3yz^2) + \vec{k}\,(-x^2y^2z^2 - x^4z^2)$$

Particatty Differentiating it w.r.to y.

$$\therefore\ \frac{\partial}{\partial y}(\vec{r}_1 \times \vec{r}_2) = -4xy^3z\,\vec{i} - (3x^3y^2 + x^3z^2)\ \vec{j} - (2x^2yz^2)\ \vec{k}$$

Partially, Differentiating it w.r.to x.

$$\frac{\partial^2}{\partial x\,\partial y}(\vec{r}_1 \times \vec{r}_2) = -4y^3z\,\vec{i} - (9x^2y^2 + 3x^2z^2)\ \vec{j} - 4xyz^2\,\vec{k}$$

and similarly we get $\dfrac{\partial^2}{\partial x\,\partial y}(\vec{r}_1 \times \vec{r}_2)$

(ii) $\dfrac{\partial^2\vec{r}_1}{\partial y^2} \times \dfrac{\partial^2\vec{r}_2}{\partial x^2}$ **at the point (1, 1, 0)**

Solution: Have, $\vec{r}_1 = xyz\,\vec{i} + xz^2\,\vec{j} - y^3\,\vec{k}$

$$\therefore\ \frac{\partial\vec{r}_1}{\partial y} = xz\,\vec{i} - 3y^2\,\vec{k}$$

$$\therefore\ \frac{\partial^2\vec{r}_1}{\partial y^2} = -6y\,\vec{k}$$

and $\vec{r}_2 = x^3\,\vec{i} - xyz\,\vec{j} - x^2z\,\vec{k}$

$$\therefore\ \frac{\partial\vec{r}_2}{\partial x} = 3x^2\,\vec{i} - yz\,\vec{j} - 2xz\,\vec{k}$$

$$\therefore\ \frac{\partial^2\vec{r}_2}{\partial x^2} = 6x\,\vec{i} + 2z\,\vec{k}$$

From the question,

$$\therefore\ \frac{\partial^2\vec{r}_1}{\partial y^2} \times \frac{\partial^2\vec{r}_2}{\partial x^2} = -6y\,\vec{k} \times [6x\,\vec{i} + 2z\,\vec{k}]$$

$$= -36xy\,\vec{j}$$

at the point (1, 1, 0)

$$\frac{\partial^2\vec{r}_1}{\partial y^2} \times \frac{\partial^2\vec{r}_2}{\partial x^2} = -36\,\vec{j}$$

(b) If $\vec{r}_1 = x^2yz\,\vec{i} - 2xz^2\vec{j} + xz^2\vec{k}$ and $\vec{r}_2 = 2z\vec{i} + y\vec{j} - x^2\vec{k}$, then find the values of $\dfrac{\partial^2}{\partial x\,\partial y}(\vec{r}_1 \times \vec{r}_2)$ and $\dfrac{\partial^2}{\partial x\,\partial y}(\vec{r}_2 \times \vec{r}_1)$. Verify that

$$\frac{\partial^2}{\partial x\,\partial y}(\vec{r}_1 \times \vec{r}_2) = \frac{-\partial^2}{\partial x\,\partial y}(\vec{r}_2 \times \vec{r}_1)$$

Solution: Here, $\vec{r}_1 = x^2yz\,\vec{i} - 2xz^2\,\vec{j} + xz^2\,\vec{k}$

and $\vec{r}_2 = 2z\,\vec{i} + y\,\vec{j} - x^2\,\vec{k}$

$$\therefore\ \vec{r}_1 \times \vec{r}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x^2yz & -2xz^2 & xz^2 \\ 2z & y & -x^2 \end{vmatrix}$$

$$\therefore\ \vec{r}_1 \times \vec{r}_2 = (2x^3z^2 - xyz^2)\ \vec{i} - (-x^4yz - 2xz^3)\ \vec{j} + (x^2y^2z + 4xz^3)\ \vec{k}$$

Partially differentiation it w. r. t. y

$$\frac{\partial(\vec{r}_1 \times \vec{r}_2)}{\partial y} = (-xz^2)\ \vec{i} + x^4z\,\vec{k} + 2x^2yz\,\vec{k}$$

Differentiating partialy w. r. t. x

$$\frac{\partial^2}{\partial x\,\partial y}(\vec{r}_1 \times \vec{r}_2) = -z^2\,\vec{i} + 4x^3z\,\vec{j} + 4xyz\,\vec{k}$$

At (1, 0, 2), $\dfrac{\partial^2}{\partial x\,\partial y}(\vec{r}_1 \times \vec{r}_2) = -4\,\vec{i} + 8\,\vec{j}$ ..... (i)

Also, $\vec{r}_2 \times \vec{r}_1 = (2z\,\vec{i} + y\,\vec{j} - x^2\,\vec{k}) \times (x^2yz\,\vec{i} - 2xz^2\,\vec{j} + xz^2\,\vec{k})$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2z & y & -x^2 \\ x^2yz & -2xz^2 & xz^2 \end{vmatrix}$$

$$\therefore\ \vec{r}_2 \times \vec{r}_1 = (xyz^2 - 2x^3z^2)\ \vec{i} - (2xz^3 + x^4yz)\ \vec{j} + (-4xz^3 - x^2y^2z)\ \vec{k}$$

Partially differentiating it partially w. r. t. y

$$\frac{\partial}{\partial y}(\vec{r}_2 \times \vec{r}_1) = xz^2\vec{i} - x^4z\vec{j} - 2x^2yz\vec{k}$$

Partially differentiating it partially w. r. t. x

$$\frac{\partial^2}{\partial x\,\partial y}(\vec{r}_2 \times \vec{r}_1) = z^2\vec{i} - 4x^3z\vec{j} - 4xyz\vec{k}$$

At (1, 0, 2);

$$\frac{\partial^2}{\partial x\,\partial y}(\vec{r}_2 \times \vec{r}_1) = 4\vec{i} - 8\vec{j} = -(4\vec{i} - 8\vec{j}) \text{ ..... (ii)}$$

From (i) and (ii) we conclude that

$$\frac{\partial^2}{\partial x\,\partial y}(\vec{r}_1 \times \vec{r}_2) = -\frac{\partial^2}{\partial x\,\partial y}(\vec{r}_2 \times \vec{r}_1) \text{ is verified.}$$

6. A particle moves so that its position vector is given by $\vec{r} = \cos wt\,\vec{i} + \sin wt\,\vec{j}$, where ω is a constant, show that the velocity $\vec{v}$ is perpendicular to $\vec{r}$ and the acceleration $\vec{f}$ is directed forwards the origin and has magnitude proportion to the distance from the origin.

Solution: Given, $\vec{r} = \cos\omega t\,\vec{i} + \sin\omega t\,\vec{j}$ ..... (i)

Differentiating w. r. t. 't'

$$\frac{d}{dt}\vec{r} = \vec{v} = (-\sin\omega t\,\vec{i} + \cos\omega t\,\vec{j})\,\omega \text{ ..... (ii)}$$

Now, $\vec{v}.\vec{r} = \vec{r}\,.\,\vec{r}$

$= \omega(-\sin\omega t\,\vec{i} + \cos\omega t\,\vec{j}).(\cos\omega t\,\vec{i} + \sin\omega t\,\vec{j}($

$= t\sin\omega t\cos\omega t + \sin\omega t\cos\omega t = 0$

$\therefore$ The velocity $\vec{v}$ is perpendicular to $\vec{r}$.

Differentiating (ii) w. r. t. 't'

(ii) $\vec{a} = \frac{d^2\vec{r}}{d^2t} = \frac{d}{dt}[(-\sin\omega t\,\vec{i} + \cos\omega t\,\vec{j})\,w]$

$= -(\cos\omega t\,\vec{i} + \sin\omega t\,\vec{j})\,w^2$

$= -\omega^2\vec{r}$

Since, ω^2 is constant, so the acceleration $\vec{a}$ has magnitude porportional to $\vec{r}$, the distance of the point from the origin and the negative sign shows that acceleration is directed towards the centre.

7. Evaluate the desivatives of the following

(i) $\hat{r} = \frac{\vec{r}}{r}$

Solution: Given, $\hat{r} = \frac{\vec{r}}{r}$

Differentiating with respect to t

$$\frac{d\hat{r}}{dt} = \frac{r\frac{d\vec{r}}{dt} - \vec{r}\frac{dr}{dt}}{r^2}$$

$$\text{or, } \frac{d\hat{r}}{dt} = \frac{r}{r^2}\frac{d\vec{r}}{dt} - \frac{\vec{r}}{r^2}\frac{dr}{dt}$$

$$\therefore \frac{d\hat{r}}{dt} = \frac{1}{r}\frac{d\vec{r}}{dt} - \frac{\vec{r}}{r^2}\frac{dr}{dt} \text{ Ans.}$$

(ii) $\vec{r}^2 + \frac{1}{\vec{r}^2}$

Solution: Given, $\vec{r}^2 + \frac{1}{\vec{r}^2}$

$= \vec{r}^2 + \vec{r}^{-2}$

Let $U = \vec{r}^2 + \vec{r}^2$

Differentiating with respect to 't'

$$\frac{du}{dt} = \frac{d}{dt}(\vec{r}^2 + \vec{r}^2)$$

$$= \frac{dr^2}{dr}\cdot\frac{dr}{dt} + \frac{dr^2}{dr}\cdot\frac{dr}{dt}$$

$$= 2r\frac{dr}{dt} - 2\vec{r}^3\frac{dr}{dt}$$

$$= 2\left(r - \frac{1}{r^3}\right)\frac{dr}{dt}$$

(iii) $\vec{r} \times \left(\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}\right)$

Sol^n. Given, $\vec{r} \times \left(\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}\right)$

Differentiating w. r. t. 't'

$$= \frac{d}{dr}\left[\vec{r} \times \left(\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}\right)\right]$$

$$= \frac{d\vec{r}}{dt} \times \left(\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}\right) + \vec{r} \times \left(\frac{d^2\vec{r}}{dt^2} \times \frac{d^2\vec{r}}{dt^2}\right) + \vec{r} \times \left(\frac{d\vec{r}}{dt} \times \frac{d^3\vec{r}}{dt^3}\right)$$

$$= \frac{d\vec{r}}{dt}\times\left(\frac{d\vec{r}}{dt}\times\frac{d^2\vec{r}}{dt^2}\right)+\vec{r}\times\left(\frac{d\vec{r}}{dt}\times\frac{d^3\vec{r}}{dt^3}\right) \qquad \left[\because \frac{d^2\vec{r}}{dt^2}\times\frac{d^2\vec{r}}{dt^2}=0\right]$$

(iv) $\dfrac{\vec{r}\times\vec{a}}{\vec{r}.\vec{a}}$

Solution: Given, $\dfrac{\vec{r}\times\vec{a}}{\vec{r}.\vec{a}}$

Differentiate it w. r. t. t

$$\frac{d}{dt}\left[\frac{\vec{r}\times\vec{a}}{\vec{r}.\vec{a}}\right]=\frac{(\vec{r}.\vec{a})\left\{\frac{d\vec{r}}{dt}\times\vec{a}\right\}-(\vec{r}\times\vec{a})\frac{d\vec{r}}{dt}.\vec{a}}{(\vec{r}.\vec{a})^2}$$

$$=\frac{1}{\vec{r}.\vec{a}}\frac{d\vec{r}}{dt}\times\vec{a}-\frac{(\vec{r}\times\vec{a})}{(\vec{r}.\vec{a})^2}\left(\frac{d\vec{r}}{dt}.\vec{a}\right)$$

$$\therefore \frac{d}{dt}\left[\frac{\vec{r}\times\vec{a}}{\vec{r}.\vec{a}}\right]=\frac{1}{\vec{r}.\vec{a}}\frac{d\vec{r}}{dt}\times\vec{a}-\frac{(\vec{r}\times\vec{a})}{(\vec{r}.\vec{a})^2}\left(\frac{d\vec{r}}{dt}.\vec{a}\right) \text{ Ans.}$$

(v) $\vec{r}.\dfrac{d\vec{r}}{dt}\times\dfrac{d^2\vec{r}}{dt^2}$

Solution: Given $\vec{r}.\dfrac{d\vec{r}}{dt}\times\dfrac{d^2\vec{r}}{dt^2}$

Differentiate with respect ot t

$$\frac{d}{dt}\left[\vec{r}.\frac{d\vec{r}}{dt}\times\frac{d^2\vec{r}}{dt^2}\right]=\vec{r}.\frac{d}{dt}\left[\frac{d\vec{r}}{dt}\times\frac{d^2\vec{r}}{dt^2}\right]+\frac{d\vec{r}}{dt}.\left(\frac{d\vec{r}}{dt}\times\frac{d^2\vec{r}}{dt^2}\right)$$

$$=\vec{r}.\left\{\frac{d\vec{r}}{dt}\times\frac{d^3\vec{r}}{dt^3}+\frac{d^2\vec{r}}{dt^2}\times\frac{d^2\vec{r}}{dt^2}\right\}+0$$

$$=\vec{r}.\frac{d\vec{r}}{dt}\times\frac{d^3\vec{r}}{dt^3}+\left[\vec{r}.\frac{d^2\vec{r}}{dt^2}\times\frac{d^2\vec{r}}{dt^2}\right]$$

$$=\vec{r}.\frac{d\vec{r}}{dt}\times\frac{d^3\vec{r}}{dt^3}+0$$

$$\therefore \frac{d}{dt}\left[\vec{r}.\frac{d\vec{r}}{dt}\times\frac{d^2\vec{r}}{dt^2}\right]=\vec{r}.\frac{d\vec{r}}{dt}\times\frac{d^3\vec{r}}{dt^3} \text{ Ans.}$$

(vi) $\dfrac{\vec{r}+\vec{a}}{\vec{r}^2+\vec{a}^2}$ **where $\vec{a}$ is a constant vector.**

Solution: Given, $\dfrac{\vec{r}+\vec{a}}{\vec{r}^2+\vec{a}^2}$ where $\vec{a}$ is constant vector.

Differentiate it w. r. to 't'

$$\frac{d}{dt}\left[\left|\frac{\vec{r}+\vec{a}}{\vec{r}^2+\vec{a}^2}\right|\right]=\frac{\left(\vec{r}^2+\vec{a}^2\right)\left(\frac{d\vec{r}}{dt}+0\right)-(\vec{r}+\vec{a})2\vec{r}.\frac{d\vec{r}}{dt}}{\left(\vec{r}^2+\vec{a}^2\right)^2}$$

$$=\frac{1}{\left(\vec{r}^2+\vec{a}^2\right)}\frac{d\vec{r}}{dt}-\frac{2\vec{r}\ 2\vec{r}(\vec{r}+\vec{a})}{\left(\vec{r}^2+\vec{a}^2\right)^2}.\frac{d\vec{r}}{dt}$$

$$=\frac{1}{\vec{r}^2+\vec{a}^2}\frac{d\vec{r}}{dt}-\frac{2\vec{r}.\frac{d\vec{r}}{dt}}{\left(\vec{r}^2+\vec{a}^2\right)^2}(\vec{r}+\vec{a})$$

$$\therefore \frac{d}{dt}\left[\left|\frac{\vec{r}+\vec{a}}{\vec{r}^2+\vec{a}^2}\right|\right]=\frac{1}{\vec{r}^2+\vec{a}^2}\frac{d\vec{r}}{dt}-\frac{2\vec{r}.\frac{d\vec{r}}{dt}}{\left(\vec{r}^2+\vec{a}^2\right)^2}(\vec{r}+\vec{a}) \text{ Ans.}$$

8. **If $\dfrac{d\vec{a}}{dt}=\vec{c}\times\vec{a}$, $\dfrac{d\vec{b}}{dt}=\vec{c}\times\vec{b}$**

Show that $\dfrac{d}{dt}\left(\vec{a}\times\vec{b}\right)=\vec{c}\times\left(\vec{a}\times\vec{b}\right)$

Solution: Given that $\dfrac{d\vec{a}}{dt}=\vec{c}\times\vec{a}$,

$$\frac{d\vec{b}}{dt}=\vec{c}\times\vec{b}$$

We have, $\dfrac{d}{dt}\left(\vec{a}\times\vec{b}\right)=\vec{a}\times\dfrac{d\vec{b}}{dt}+\dfrac{d\vec{a}}{dt}\times\vec{b}$

$$=\vec{a}\times(\vec{c}\times\vec{b})+(\vec{c}\times\vec{a})\times\vec{b}$$

$$=\left(\vec{a}.\vec{b}\right)\vec{c}-\left(\vec{a}.\vec{c}\right)\vec{b}+\left(\vec{b}.\vec{c}\right)\vec{a}-\left(\vec{b}.\vec{a}\right)\vec{c}$$

$$=\left(\vec{b}.\vec{c}\right)\vec{a}-\left(\vec{a}.\vec{c}\right)\vec{b}$$

$$\frac{d}{dt}\left(\vec{a}\times\vec{b}\right)=\vec{c}\times\left(\vec{a}\times\vec{b}\right) \text{ Proved.}$$

9. **If $\vec{r}$ is a unit vector, prove that**

$$\left|\vec{r}\times\frac{d\vec{r}}{dt}\right|=\left|\frac{d\vec{r}}{dt}\right|$$

Solution: $r^2=1$, So, $\vec{r}.\dfrac{d\vec{r}}{dt}=0$

Since $\bar{r} = 1$ squaring on both sides

$\because$ The angle between $\bar{r}$ and $\frac{d\bar{r}}{dt}$ is $90°$

Now, $\left|\bar{r}\times\frac{d\bar{r}}{dt}\right| = |\bar{r}|\left|\frac{d\bar{r}}{dt}\right| \sin 90°$

$$= 1 \cdot \left|\frac{d\bar{r}}{dt}\right| \cdot 1 = \left|\frac{d\bar{r}}{dt}\right|$$

10. Evaluate

(i) $\frac{d}{dt}\left\{\left(\bar{r}\times\frac{d\bar{r}}{dt}\right)\times\left(\frac{d^2\bar{r}}{dt^2}\right)\right\}$

Solution: Given, $\frac{d}{dt}\left\{\left(\bar{r}\times\frac{d\bar{r}}{dt}\right)\times\left(\frac{d^2\bar{r}}{dt^2}\right)\right\}$

$$= \left[\frac{d\bar{r}}{dt}\times\frac{d\bar{r}}{dt}+\bar{r}\times\frac{d^2\bar{r}}{dt^2}\right]\times\frac{d^2\bar{r}}{dt^2}+\left(\bar{r}\times\frac{d\bar{r}}{dt}\right)\times\frac{d^3\bar{r}}{dt^2}$$

$$\therefore \left(\bar{r}\times\frac{d^2\bar{r}}{dt^2}\right)\times\frac{d^2\bar{r}}{dt^2}+\left(\bar{r}\times\frac{d\bar{r}}{dt}\right)\times\frac{d^3\bar{r}}{dt^2}$$

(ii) $\frac{d}{dt}\left(\bar{r}_1\times\frac{d\bar{r}_2}{dt}-\frac{d\bar{r}_1}{dt}\times\bar{r}_2\right)$

Solution: Given, $\frac{d}{dt}\left(\bar{r}_1\times\frac{d\bar{r}_2}{dt}-\frac{d\bar{r}_1}{dt}\times\bar{r}_2\right)$

$$= \bar{r}_1\times\frac{d^2\bar{r}_2}{dt^2}+\frac{d\bar{r}_1}{dt}\times\frac{d\bar{r}_2}{dt}-\left[\frac{d\bar{r}_1}{dt}\times\frac{d\bar{r}_2}{dt}+\frac{d^2\bar{r}_1}{dt^2}\times\bar{r}_2\right]$$

$$= \bar{r}_1\times\frac{d^2\bar{r}_2}{dt^2}-\frac{d^2\bar{r}_1}{dt^2}\times\bar{r}_2 \text{ Ans.}$$

(iii) $\frac{d}{dt}\left(\bar{r}_1\cdot\frac{d\bar{r}_2}{dt}-\frac{d\bar{r}_1}{dt}\cdot\bar{r}_2\right)$

Solution: We have, $\frac{d}{dt}\left(\bar{r}_1\cdot\frac{d\bar{r}_2}{dt}-\frac{d\bar{r}_1}{dt}\cdot\bar{r}_2\right)$

$$= \frac{d\bar{r}_1}{dt}\cdot\frac{d\bar{r}_2}{dt}+\bar{r}_1\cdot\frac{d^2\bar{r}_2}{dt^2}-\frac{d\bar{r}_1}{dt}\cdot\frac{d\bar{r}_2}{dt}-\frac{d^2\bar{r}_1}{dt^2}\cdot\bar{r}_2$$

$$= \bar{r}_1\cdot\frac{d^2\bar{r}_2}{dt^2}-\frac{d^2\bar{r}_1}{dt^2}\cdot\bar{r}_2 \text{ Ans.}$$

(iv) $\frac{d^2}{dt^2}\left\{\left(\bar{r}\times\frac{d\bar{r}}{dt}\right)\times\frac{d^2\bar{r}}{dt^2}\right\}$

Solution: We have,

$$\frac{d^2}{dt^2}\left\{\left(\bar{r}\times\frac{d\bar{r}}{dt}\right)\times\frac{d^2\bar{r}}{dt^2}\right\}$$

$$= \left[\frac{d}{dt}\left\{\left(\bar{r}\times\frac{d\bar{r}}{dt}\right)\times\frac{d^2\bar{r}}{dt^2}\right\}\right]$$

$$= \frac{d}{dt}\left\{\left[\frac{d\bar{r}}{dt}\times\frac{d\bar{r}}{dt}+\bar{r}\times\frac{d^2\bar{r}}{dt^2}\right]\times\frac{d^2\bar{r}}{dt^2}+\left(\bar{r}\times\frac{d\bar{r}}{dt}\right)\times\frac{d^3\bar{r}}{dt^3}\right\}$$

$$\frac{d}{dt} = \left\{\left(\frac{d\bar{r}}{dt}\times\frac{d\bar{r}}{dt}\right)\times\frac{d^2\bar{r}}{dt^2}+\left(\bar{r}\times\frac{d^2\bar{r}}{dt^2}\right)\times\frac{d^2\bar{r}}{dt^2}+\left(\bar{r}\times\frac{d\bar{r}}{dt}\right)\times\frac{d^3\bar{r}}{dt^3}\right\}$$

$$= \left[\frac{d^2\bar{r}}{dt^2}\times\frac{d\bar{r}}{dt}+\frac{d\bar{r}}{dt}\times\frac{d^2\bar{r}}{dt^2}\right]\times\frac{d^2\bar{r}}{dt^2}\left[\left(\frac{d\bar{r}}{dt}\times\frac{d\bar{r}}{dt}\right)\times\frac{d^3\bar{r}}{dt^3}\right]$$

$$+\left[\frac{d\bar{r}}{dt}\times\frac{d^2\bar{r}}{dt^2}+\bar{r}\times\frac{d^3\bar{r}}{dt^3}\right]\times\frac{d^2\bar{r}}{dt^2}+\left(\bar{r}\times\frac{d^2\bar{r}}{dt^2}\right)\times\frac{d^3\bar{r}}{dt^3}$$

$$+\left[\frac{d\bar{r}}{dt}\times\frac{d^2\bar{r}}{dt^2}+\bar{r}\times\frac{d^2\bar{r}}{dt^2}\right]\times\frac{d^3\bar{r}}{dt^3}+\left(\bar{r}\times\frac{d\bar{r}}{dt}\right)\times\frac{d^4\bar{r}}{dt^4}$$

$$= \left(\frac{d\bar{r}}{dt}\times\frac{d^2\bar{r}}{dt^2}\right)\times\frac{d^2\bar{r}}{dt^2}+\left(\bar{r}\times\frac{d^3\bar{r}}{dt^3}\right)\times\frac{d^2\bar{r}}{dt^2}$$

$$+2\left(\bar{r}\times\frac{d^2\bar{r}}{dt^2}\right)\times\frac{d^3\bar{r}}{dt^3}+\left(\bar{r}\times\frac{d^2\bar{r}}{dt^2}\right)\times\frac{d^4\bar{r}}{dt^4}$$

Exercise - 24

1. Evaluate

(i) $\int_1^2 (2t^2\vec{i} + t\vec{j} - 3t^2\vec{k})\,dt$

Solution: Given, $\int_1^2 (2t^2\vec{i} + t\vec{j} - 3t^2\vec{k})\,dt$

$$= \left[\frac{2t^3}{3}\vec{i} + \frac{t^2}{2}\vec{j} - \frac{3t^3}{3}\vec{k}\right]_1^2$$

$$= \frac{2}{3}[2^3 - 1^3]\vec{i} + \frac{1}{2}[2^2 - 1]\vec{j} - [2^3 - 1)\vec{k}$$

$$= \frac{2}{3}\times 7\vec{i} + \frac{1}{2}.3\vec{j} - 7\vec{k}$$

$$= \frac{14}{3}\vec{i} + \frac{3}{2}\vec{j} - 7\vec{k} \text{ Ans.}$$

(ii) $\int_1^2 \left(\vec{r}\times\frac{d^2\vec{r}}{dt^2}\right)dt$, **where** $\vec{r} = 2t^2\vec{i} + t\vec{j} - 3t^2\vec{k}$

Solution: Given, $\vec{r} = 2t^2\vec{i} + t\vec{j} - 3t^2\vec{k}$

$$\frac{d\vec{r}}{dt} = 4t\vec{i} + \vec{j} - 6t\vec{k}$$

$$\therefore \frac{d^2\vec{r}}{dt^2} = 4\vec{i} - 6\vec{k}$$

$$\vec{r}\times\frac{d^2\vec{r}}{dt^2} = (2t^2\vec{i} + t\vec{j} - 3t^2\vec{k})\times(4\vec{i} - 6\vec{k}) = -6t\vec{i} - 4t\vec{k}$$

Now, $\int_1^2 \left(\vec{r}\times\frac{d^2\vec{r}}{dt^2}\right)dt = \int_1^2 (-6t\vec{i} - 4t\vec{k})\,dt$

$$= \left[-3t^2\vec{i} - 2t^2\vec{k}\right]_1^2 = -9\vec{i} - 6\vec{k} \text{ Ans.}$$

(iii) $\int_2^3 \left(\vec{r}.\frac{d\vec{r}}{dt}\right)dt$ **where**

$\vec{r} = 2\vec{i} - \vec{j} + 2\vec{k}$ **when** $t = 2$

$4\vec{i} - 2\vec{j} + 3\vec{k}$ **when** $t = 3$

Solution: $\int_2^3 \left(\vec{r}.\frac{d\vec{r}}{dt}\right)dt = \left[\frac{r^2}{2}\right]_2^3$

$$= \frac{1}{2}[(4\vec{i} - 2\vec{j} + 3\vec{k})^2 - (2\vec{i} - \vec{j} + 2\vec{k})^2]$$

$$= \frac{1}{2}[(4^2 + (-2)^2 + 3^2) - (2)^2 + (-1)^2 + 2)^2]$$

$$= \frac{1}{2}(16 + 4 + 9 - (4 + 1 + 4)]$$

$$= \frac{1}{2}[29 - 9]$$

$$= \frac{1}{2}\times 20$$

$$= 10$$

$$\therefore \int_2^3 \left(\vec{r}.\frac{d\vec{r}}{dt}\right)dt = 10 \text{ Ans.}$$

(iv) $\int_0^1 (t\vec{i} + e^t\vec{j} + e^{-2t}\vec{k})\,dt$

Solution : $\left[\frac{t^2}{2}\vec{i} + \frac{e^t}{1}\vec{j} + \frac{e^{-2t}}{-2}\vec{k}\right]_0^1$

$$= \frac{1}{2}\vec{i} + (e - 1)\vec{j} + \frac{1}{2}(1 - e^2)\vec{k} \text{ Ans.}$$

(v) $\int_0^2 (\vec{r}_1\times\vec{r}_2)\,dt =$ **where** $\vec{r}_1 = 2\vec{i} + t\vec{j} - \vec{k}$

$\vec{r}_2 = t\vec{i} + 2\vec{j} + 3\vec{k}$

Solution: Given that,

$\vec{r}_1 = 2\vec{i} + t\vec{j} - \vec{k}$

$\vec{r}_2 = t\vec{i} + 2\vec{j} + 3\vec{k}$

We have,

$\vec{r}_1\times\vec{r}_2 = (2\vec{i} + t\vec{j} - \vec{k})\times(t\vec{i} + 2\vec{j} + 3\vec{k})$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & t & -1 \\ t & 2 & 3 \end{vmatrix}$$

$$= \vec{i}\,(3t + 2) - \vec{j}\,(6 + t) + \vec{k}\,(4 - t^2)$$

$$= (3t + 2)\vec{i} - (6 + t)\vec{j} + (4 - t^2)\vec{k}$$

So, $\int_0^2 (\vec{r}_1\times\vec{r}_2)\,dt = \int_0^2 \left\{(3t + 2)\vec{i} - (6 + t)\vec{j} + (4 - t^2)\vec{k}\right\}dt$

$$= \left[\left(\frac{3t^2}{2} + 2t\right)\vec{i} - \left(6t + \frac{t^2}{2}\right)\vec{j} + \left(4t - \frac{t^3}{3}\right)\vec{k}\right]_0^2$$

$$= 10\,\vec{i} - 14\,\vec{j} + \frac{16}{3}\,\vec{k}$$

$$\int_0^2 \left(\vec{r}_1 \times \vec{r}_2\right) dt = 10\,\vec{i} - 14\,\vec{j} + \frac{16}{3}\,\vec{k} \text{ Proved.}$$

(vi) $\int_0^1 \vec{r}_1 \cdot \left(\vec{r}_1 \times \vec{r}_2\right) dt$ **where** $\vec{r}_1 = 2t^2\vec{i} + t\vec{j} - 2t^2\vec{k}$

$$\vec{r}_2 = t^2\vec{i} + 2t\vec{j} - t^2\vec{k}$$

$$\vec{r}_3 = t^2\vec{i} - t\vec{j} - t^2\vec{k}$$

Solution: Here, $\vec{r}_2 \times \vec{r}_3 = (t^2\,\vec{i} + 2t\,\vec{j} - t^2\,\vec{k}) \times (t^2\,\vec{i} - t\,\vec{j} - t^2\,\vec{k})$

$$= t^3\,\vec{k} + t^4\,\vec{j} - 2t^3\,\vec{k} - 2t^3\,\vec{i} - t^4\,\vec{j} - t^3\,\vec{i}$$

$$= -3t^3\,\vec{i} - 3t^3\,\vec{k}$$

$$\therefore\ r_1 \cdot \left(\vec{r}_2 \times \vec{r}_3\right) = (2t^2\,\vec{i} - t\,\vec{j} + 2t^2\,\vec{k}) \cdot (-3t^3\,\vec{i} - 3t^3\,\vec{k})$$

$$= -6t^5 - 6t^5$$

$$= -12t^5$$

$$\therefore\ \int_0^1 \vec{r}_1 \cdot \left(\vec{r}_1 \times \vec{r}_2\right) dt = \int_0^1 -12\,t^5\,dt$$

$$= -12\left[\frac{t^6}{6}\right]_0^1$$

$$= \frac{-12}{6}$$

$$= -2$$

2. **Given that** $\vec{r}_1 = 2t^2\vec{i} + 3(t-1)\vec{j} + 4t^2\vec{k}$ **and**

$$\vec{r}_2 = (t-1)\vec{i} + t^2\vec{j} + (t-2)\vec{k}\text{, find}$$

(i) $\int_1^2 \left(\vec{r}_1 \times \vec{r}_2\right) dt$

$$= \int_1^2 \left[\left(2t^2\vec{i} + 3(t-1)\vec{j} + 4t^2\vec{k}\right) \cdot \left((t-1)\vec{i} + t^2\vec{j} + (t-2)\vec{k}\right)\right]$$

Solution: $\int_1^2 [\,2t^3 - 2t^2 + 3t^2 - 3t^2 + 4t^3 - 8t^2]$

$$= \int_1^2 (9t^3 - 13t^2)\,dt$$

$$= \left[\frac{9t^4}{4}\right]_1^2 - \left[\frac{13t^3}{3}\right]_1^2$$

$$= \frac{9}{4}\,[16-2) - \frac{13}{3}\,[8-1]$$

$$= \frac{9\times 15}{4} - \frac{13}{3}\times 7$$

$$= \frac{135}{4} - \frac{91}{3}$$

$$= \frac{41}{12} \text{ Ans.}$$

(ii) $\int_1^2 \left(\vec{r}_1 \cdot \frac{d\vec{r}_2}{dt}\right) dt$

Solution: Now, $\vec{r}_2 = (t-1)\,\vec{i} + t^2\,\vec{j} + (t-2)\,\vec{k}$

$$\because\ \frac{d\vec{r}_2}{dt} = \vec{i} + 2t\,\vec{j} + \vec{k}$$

$$\therefore\ \int_1^2 \left(\vec{r}_1 \cdot \frac{d\vec{r}_2}{dt}\right) dt = [(2t^2\,\vec{i} + 3(t-1)\,\vec{j} + 4t^2\,\vec{k}) \cdot [\,\vec{i} + 2t\,\vec{j} + \vec{k}\,]$$

$$= \int_1^2 (2t^2 + 6t^2 - 6t + 4t^2)\,dt$$

$$= \int_1^2 (12t^2 - 6t)\,dt$$

$$= \left[\frac{12t^3}{3} - \frac{6t^2}{2}\right]_1^2$$

$$= \frac{13}{3}\,[8-1] - \frac{6}{2}\,[4-1]$$

$$= \frac{12}{3}\times 7 - \frac{6}{2}\times 3$$

$$= \frac{84}{3} - \frac{18}{2}$$

$$= 19 \text{ Ans.}$$

(iii) $\int_1^2 \left(\vec{r}_1 \times \frac{d^2\vec{r}_2}{dt^2}\right) dt$

Solution: We know from (ii)

$$\because\ \frac{d\vec{r}_2}{dt} = \vec{i} + 2t\,\vec{j} + \vec{k}$$

$$\therefore\ \frac{d^2\vec{r}_2}{dt^2} = 2\,\vec{j}$$

$$\therefore\ \int_1^2 \left(\vec{r}_1 \times \frac{d^2\vec{r}_2}{dt^2}\right) dt = \int_1^2 \left[\,2t^2\vec{i} + 3\,(t-1)\vec{j} + 4t^2\vec{k}) \times 2\vec{j}\,\right] dt$$

$$= \int_1^2 (4t^2\,\vec{k} - 8t^2\,\vec{i}\,]\,dt$$

$$= \left[\frac{4t^3}{3}\right]_1^2 \vec{k} - \left[\frac{8t^3}{3}\right]_1^2 \vec{i}$$

$$= \frac{4}{3}[8-1]\,\vec{k} - \frac{8}{3}[8-1]\,\vec{i}$$

$$= \frac{4}{3}\times 7\vec{k} - \frac{8}{3}\times 7\vec{i}$$

$$= \frac{28}{3}(\vec{k} - 2\vec{i}) \text{ Ans.}$$

3. Solve for $\vec{r}$ to the equation.

(i) $\vec{a}\times\frac{d^2\vec{r}}{dt^2} = \vec{b}$ where $\vec{a}$ and $\vec{b}$ are constant vectors. Given that $\vec{r} = 0$, $\frac{d\vec{r}}{dt^2} = 0$ at $t = 0$

Solution: Here, $\vec{a}\times\frac{d^2\vec{r}}{dt^2} = \vec{b}$ ..... (i)

$$\vec{a}\times\frac{d^2\vec{r}}{dt^2} = \vec{b}\,t + c \text{ ..... (ii)}$$

Since, $\frac{d\vec{r}}{dt} = 0$ at $t = 0$

$\because$ Then equation (ii) becomes $\vec{a}\times\vec{dt} = \vec{bt}$

$\therefore c = 0$

Again integrating (iii) we get

$$\therefore\ \vec{a}\times\vec{r} = \frac{\vec{b}t^2}{2} + d \text{ ..... (iv)}$$

Since, $\vec{r} = 0$ at $t = 0$

$\Rightarrow d = 0$

Then from equation (iv)

$$\therefore\ \vec{a}\times\vec{r} = \frac{\vec{b}t^2}{2} \text{ Ans.}$$

(ii) $\frac{d^2\vec{r}}{dt^2} = e^t\vec{i} + e^{2t}\vec{j} + \vec{k}$

Given that $\frac{d\vec{r}}{dt} = \vec{i} + \vec{j}$ at $\vec{t} = 0$ and $\vec{r} = 0$ at $t = 0$

Solution: We have,

$$\frac{d^2\vec{r}}{dt^2} = e^t\vec{i} + e^{2t}\vec{j} + \vec{k} \text{ ..... (i)}$$

Integrating (i), we get

$$\frac{d^2\vec{r}}{dt^2} = e^t\vec{i} + \frac{e^{2t}}{2}\vec{j} + t\vec{k} + c \text{ ..... (ii)}$$

Since, $\frac{d\vec{r}}{dt} = \vec{i} + \vec{j}$ at $t = 0$

$$\therefore\ \vec{i} + \vec{j} = \vec{i} + \frac{1}{2}\vec{j} + c$$

$$\therefore\ c = \frac{1}{2}\vec{j}$$

Then from equation (ii)

$$\frac{d\vec{r}}{dt} = e^t\vec{i} + \frac{e^{2t}}{2}\vec{j} + t\vec{k} + \frac{1}{2}\vec{j} \text{ ..... (iii)}$$

Again integrating, we get,

$$\vec{r} = e^t + \vec{i} + \frac{e^{2t}}{4}\vec{j} + \frac{t^2}{2}\vec{k} + \frac{t}{2}\vec{j} + d \text{ ..... (iv)}$$

$\vec{r} = 0$ at $t = 0$

$$\therefore\ 0 = \vec{i} + \frac{1}{4}\vec{j} + d$$

$$\therefore\ d = -\vec{i} + \frac{1}{4}\vec{j}$$

Then from equation (iv) we get

$$\vec{r} = e^t\vec{i} + \frac{e^{2t}}{2}\vec{j} + \frac{t^2}{2}\vec{k} + \frac{t}{2}\vec{j} - \vec{i}\,.\,\frac{-1}{4}\vec{j}$$

$$\therefore\ \left(e^t - 1\right)\vec{i} + \left(\frac{e^{2t}}{4} + \frac{t}{2} - \frac{1}{4}\right)\vec{j} + \left(\frac{t^2}{2} + 1\right)\vec{k} \text{ Ans.}$$

If $\frac{d\vec{r}}{dt} = t^2\vec{i} + (6t+1)\vec{j} + 8t^3\vec{k}$,

and $\vec{r}(0) = 2\vec{i} - 3\vec{j} + \vec{k}$, find $\vec{r}$

Solution: Given, $\frac{d\vec{r}}{dt} = t^2\vec{i} + (6t+1)\vec{j} + 8t^3\vec{k}$ ..... (i)

Integrating (i) we get

$$\vec{r} = \frac{t^3}{3}\vec{i} + \left(\frac{6t^2}{2} + t\right)\vec{j} + \frac{8t^4}{4}\vec{k} + c \text{ ..... (ii)}$$

Since, $\vec{r} = 2\vec{i} - 3\vec{j} + \vec{k}$ at $t = 0$

$$\therefore\ 2\vec{i} - 3\vec{j} + \vec{k} = 0 + 0 + 0 + c$$

$$\therefore\ c = 2\vec{i} - 3\vec{j} + \vec{k}$$

Then from equation (ii),

$$\therefore\ \vec{r} = \frac{t^3}{3}\vec{i} + \left(\frac{6t^2}{2} + t\right)\vec{j} + \frac{8t^4}{4}\vec{k} + 2\vec{i} - 3\vec{j} + \vec{k}$$

$$= \left(\frac{t^3}{3} + 2\right)\vec{i} + \left(\frac{6t^2}{2} + t - 3\right)\vec{j} + \left(\frac{8t^4}{4} + 1\right)\vec{k} \text{ Ans.}$$

5. **If** $\frac{d^2\vec{r}}{dt^2} = 6t\vec{i} - 12t^2\vec{j} + \vec{k}$

and $\vec{r}'(0) = \vec{i} + 2\vec{j} - 3\vec{k}$

$\vec{r}(0) = 7\vec{i} + \vec{j}$ **find** $\vec{r}$

Solution: We have,

$$\frac{d^2\vec{r}}{dt^2} = 6t\vec{i} - 12t^2\vec{j} + \vec{k} \text{ ..... (i)}$$

Integrating (i), we get

$$\frac{6t^2}{2}\vec{i} - \frac{12t^3}{3}\vec{j} + t\vec{k} + c \text{ ..... (ii)}$$

Since, $\vec{r}'(0) = \vec{i} + 2\vec{j} - 3\vec{k}$ at $t = 0$

$\therefore\ \vec{i} + 2\vec{j} - 3\vec{k} = c + 0$

$\therefore\ c = \vec{i} + 2\vec{j} - 3\vec{k}$

Then from (ii)

$$\frac{d\vec{r}}{dt} = \frac{6t^2}{2}\vec{i} - \frac{12t^3}{3}\vec{j} + t\vec{k} + \vec{i} + 2\vec{j} + \vec{k} \text{ ..... (iii)}$$

Again, integrating, we get

$$\vec{r} = \frac{6}{2}\ \frac{t^3}{3}\vec{i} - \frac{12t^4}{3r^4}\vec{j} + \frac{t^2}{2}\vec{k} + t\vec{i} + 2t\vec{j} + t\vec{k} + d \text{ ..... (iv)}$$

Since $\vec{r} = 7\vec{i} + \vec{k}$ at $t = 0$

$d = 7\vec{i} + \vec{k}$

Then from equation (iv)

$$\vec{r} = \frac{6}{6}t^3\vec{i} - \frac{12}{12}t^4\vec{j} + \frac{t^2}{2}\vec{k} + t\vec{i} + 2t\vec{j} + t\vec{k} + 7\vec{i} + \vec{k}$$

$$\therefore\ \vec{r} = (t^3 + t + 7)\vec{i} + (2t - t^4)\vec{j} + \left(\frac{t^2}{2} - 3t + 1\right)\vec{k} \text{ Ans.}$$

Exercise - 25

1. **Find the gradient, divergence and curl (whichever possible) of the following scalar and vector point functions.**

(i) $\vec{v} = 3x^2\vec{i} + 5xy^2\vec{j} + xyz^3\vec{k}$ **at the point (1, 2, 3)**

Solution: Curl $\vec{v}$ and dir $\vec{v}$ is possible.

Now, curl $\vec{v} = \nabla \times \vec{v}$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \times \left(3x^2\vec{i} + 5xy^2\vec{j} + xyz^3\vec{k}\right)$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2 & 5xy^2 & 3xyz \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}\left(xyz^2\right) - \frac{\partial}{\partial z}\left(5xy^2\right)\right] - \vec{j}\left[\frac{\partial}{\partial x}\left(xyz^3\right) - \frac{\partial}{\partial z}\left(3x^2\right)\right]$$

$$+\vec{k}\left[\frac{\partial}{\partial x}\left(5xy^2\right) - \frac{\partial}{\partial y}\left(3x^2\right)\right]$$

$$\Rightarrow\ xz^3\vec{i} - \vec{j}yz^3 + \vec{k}5y^2$$

$\therefore$ At the point (1, 2, 3), curl $\vec{v} = 1\,(3)^3\,\vec{i} - 2(3)^3\,\vec{j} + 5(2)^2\,\vec{k}$

$= 27\,\vec{i} - 54\,\vec{j} + 20\,\vec{k}$

$\therefore$ div $\vec{v} = \nabla\,.\,\vec{v}$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right).\left(3x^2\vec{i} + 5xy^2\vec{j} + xyz^3\vec{k}\right)$$

$= 6x + 10xy + 3xyz^2$

$\therefore$ At the point (1, 2, 3)

$\therefore$ div $\vec{v} = 6 + 10 \times 2 \times 1 + 3.1.2.9$

$= 6 + 20 + 54$

$= 80$ Ans.

(ii) $\vec{v} = xyz\vec{i} + 3x^2y\vec{j} + (xz^2 - y^2z)\,\vec{k}$

Solution: Curl $\vec{v}$ and div $\vec{v}$ is possible.

$\therefore$ curl $\vec{v} = \nabla \times \vec{v}$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \times \left(xyz\vec{i} + 3x^2y\vec{j} + (xz^2 - y^2z)\,\vec{k}\right)$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & 3x^2y & xz^2 - y^2z \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}\left(xz^2 - y^2z\right) - \frac{\partial}{\partial z}\left(3x^2y\right)\right] - \vec{j}\left[\frac{\partial}{\partial x}\left(xz^2 - y^2z\right) - \frac{\partial}{\partial z}(xyz)\right]$$

$$+\vec{k}\left[\frac{\partial}{\partial x}\left(3x^2y\right) - \frac{\partial}{\partial y}(xyz)\right]$$

$$= \vec{i}(-2yz) - \vec{j}\left[\left(z^2\right) - xy\right] + \vec{k}(6xy - xz)$$

$$= - -2yz\vec{i} - (2 - xy)\vec{j} + \vec{k}(6xy - xz)$$

and div $\vec{v} = \nabla \cdot \vec{v}$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \cdot \left(xyz\vec{i} + 3x^2y\vec{j} + (xz^2 - y^2z)\vec{k}\right)$$

$\Rightarrow yz + 3x^2 + 2xz - y^2$ Ans.

(iv) $\phi = \log(x^2 + y^2 + z^2)$

Solution: Grad $\phi = \nabla\phi$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)[\log(x^2 + y^2 + z^2)]$$

$$= \vec{i}\frac{\partial}{\partial x}\left[\log\left(x^2 + y^2 + z^2\right)\right] + \vec{j}\frac{\partial}{\partial y}\left[\log\left(x^2 + y^2 + z^2\right)\right]$$

$$+\vec{k}\frac{\partial}{\partial z}\left[\log\left(x^2 + y^2 + z^2\right)\right]$$

$$= \frac{2x\vec{i}}{x^2 + y^2 + z^2} + \frac{2y\vec{j}}{x^2 + y^2 + z^2} + \frac{2z\vec{k}}{x^2 + y^2 + z^2} \text{ Ans.}$$

(v) $\phi = \log \sin^{-1}(r)$ where $r = |\vec{r}|$, $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

Solution: Since ϕ is a scalar point function. So, it's gradient exists or possible but its divergent and curl do not exist.

Now, grade $\phi = \nabla\phi = \Sigma\vec{i}\,\frac{\partial\phi}{\partial x}$

$$= \Sigma\vec{i}\,\frac{\partial\phi}{\partial x}\left[\log \sin^{-1} r\right]$$

$$= \Sigma\vec{i}\,\frac{1}{\sin^{-1} r}\,\frac{1}{\sqrt{i - r^2}}\,\frac{\partial\phi}{\partial x}\left[\frac{\partial r}{\partial x} = \frac{x}{r}\right]$$

$$= \frac{1}{\sin^{-1} r\sqrt{1 - r^2}}\,\Sigma\vec{i}\,\frac{x}{r}$$

$$= \frac{1}{\sin^{-1} r\sqrt{1 - r^2}} \cdot \frac{1}{r}\Sigma\vec{i}\,x$$

$$= \frac{1}{\sin^{-1} r\sqrt{1 - r^2}}\left(\frac{\vec{r}}{r}\right) \text{ Ans.}$$

(vi) $\vec{v} = x^2yz\vec{i} + xy^2z\vec{j} + xyz^2\vec{k}$

Solution: Here, curl $\vec{v}$ and div $\vec{v}$ is possible.

$\therefore$ curl $\vec{v} = \nabla \times \vec{v}$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \times \left(x^2yz\vec{i} + xy^2z\vec{j} + xyz^2\vec{k}\right)$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2yz & xy^2z & xyz^2 \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}\left(xyz^2\right) - \frac{\partial}{\partial z}\left(xy^2z\right)\right] - \vec{j}\left[\frac{\partial}{\partial x}\left(xyz^2\right) - \frac{\partial}{\partial z}\left(x^2yz\right)\right]$$

$$+\vec{k}\left[\frac{\partial}{\partial x}\left(xy^2z\right) - \frac{\partial}{\partial y}\left(x^2yz\right)\right]$$

$$= \left[xz^2 - xy^2\right]\vec{i} - \vec{j}\left[yz^2 - x^2z\right] + \vec{k}\left[y^2z - x^2z\right]$$

$$= -x\left(z^2 - y^2\right)\vec{i} + y\left(x^2 - z^2\right)\vec{j} + z\left(y^2 - x^2\right)\vec{k}$$

And div $\vec{v} = \nabla \times \vec{v}$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \cdot \left(x^2yz\vec{i} + xy^2z\vec{j} + xyz^2\vec{k}\right)$$

$= 2xyz + 2xyz + 2xyz$

$= 6xyz$ Ans.

(vii) $\vec{v} = x^2z\vec{i} - 2y^3z^2\vec{j} + xy^2z\vec{k}$ at the point (1, – 1, 1)

Solution: Now, curl $\vec{v}$ and dir $\vec{v}$ exist.

$\therefore$ curl $\vec{v} = \nabla \times \vec{v}$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \times \left(x^2z\vec{i} - 2y^3z^2\vec{j} + xy^2z\vec{k}\right)$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2z & -2y^3z^2 & xy^2z \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}\left(xy^2z\right)-\frac{\partial}{\partial z}\left(-2y^3z^2\right)\right]-\vec{j}\left[\frac{\partial}{\partial x}\left(xy^2z\right)-\frac{\partial}{\partial z}\left(x^2z\right)\right]$$

$$+\vec{k}\left[\frac{\partial}{\partial x}\left(-2y^3z^2\right)-\frac{\partial}{\partial y}\left(x^2z\right)\right]$$

$$= \vec{i}\left[2xyz-4y^3z\right]-\vec{j}\left[y^2z-x^2\right]+\vec{k}[0+0]$$

$$\Rightarrow \left[2xyz+4y^3z\right]\vec{i}-\left[y^2z-x^2\right]\vec{j}$$

At the point (1, – 1, 1)

$$\therefore \text{Curl } \vec{v} = \left[2\times(1)\cdot(-1).1+4(-1)^3.1\right]\vec{i}-\left[(-1)^2.1-1^2\right]\vec{j}$$

$$= -6\vec{i} \text{ Ans.}$$

$$\text{div}\,\vec{v} = \nabla\times\vec{v} = \left(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}\right).\left(x^2z\vec{i}-2y^3z^2\vec{j}+xy^2z\vec{k}\right)$$

$$= 2xy-6y^2z^2+xy^2$$

At the point (1, – 1, 1); div $\vec{v}$

$= 2 . 1 . 1 - 6 (-1) . 1 + 1 . (-1)^2 = 2 + 6 + 1 = 9$

(viii) $\phi = 3x^2y - y^3z^3$ at the point (1, – 2, – 1)

Solution: Now, grad $\phi = \nabla\phi = \left(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}\right)(3x^2y-y^3z^3)$

$$= \vec{i}(6xy)+\vec{j}\left(3x^2-3y^2z^2\right)+\vec{k}\left(-y^3.2z\right)$$

Grad ϕ at the point (1, – 2, – 1)

$$\Rightarrow \vec{i}(-12)+\vec{j}\left(3.1-3.4(1)\right)+\vec{k}(-16)$$

$$\Rightarrow -12\vec{i}-9\vec{j}-16\vec{k} \text{ Ans.}$$

2. (i) Find the directional derivative of $\phi = x^2yz + 4xz^2$ at the point (1, – 2, – 1) in the direction $2\vec{i}-\vec{j}-2\vec{k}$.

Solution: Let $\vec{v} = 2\vec{i}-\vec{j}-2\vec{k}$

$$\therefore \frac{\vec{v}}{|v|} = \frac{2\vec{i}-\vec{j}-2\vec{k}}{\sqrt{4+1+4}} = \frac{2\vec{i}-\vec{j}-2\vec{k}}{3} \quad \text{..... (i)}$$

Again, gradient $\phi = \nabla\phi$

$$= \left(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}\right)(x^2yz+4xz^2)$$

$$= \vec{i}\left(2xyz+4z^2\right)-\vec{j}\left(x^2z\right)+\vec{k}\left(x^2y+8zx\right)$$

$$= \left(2xyz+4z^2\right)\vec{i}+x^2z\vec{j}+\left(x^2y+8xz\right)\vec{k}$$

At (1, – 2, – 1);

Grad $\phi = 8\vec{i}-\vec{j}-10\vec{k}$ ..... (ii)

We know that the directional derivative of the function ϕ in the direction of the vector $\vec{v}$ at the point (1, – 2, – 1)

$$\text{grad } \phi \frac{\vec{v}}{|v|}$$

$\therefore$ Required directional derivative

$$= (8\vec{i}-\vec{j}-10\vec{k}).\left(\frac{2\vec{i}-\vec{j}-2\vec{k}}{3}\right)$$

From (i) and (ii)

$$= \frac{1}{3}(16+1+20) = \frac{37}{3} \text{ Ans.}$$

(ii) In what direction from the point (3, 1, – 2) is the directional derivative of $\phi = x^2y^2z^4$ maximum ? Find also the megnitutive of this maximum.

Solution: Given function is $\phi = x^2y^2z^4$

grad $\phi = \nabla\phi$

$$= \left(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}\right)(x^2y^2z^4)$$

$$= 2xy^2z^4\vec{i}+2x^2yz^4\vec{j}+4x^2y^2z^3\vec{k}$$

At the point (3, 1, – 2) is

grade $\phi = 2.3.1.16\vec{i} + 2.3^2.1.16\vec{j} + 4.9.1(-8)\vec{k}$

$$= 96\vec{i}+188\vec{j}-188\vec{k} = 96(\vec{i}+3\vec{j}-3\vec{k})$$

Then magnitude of grad $\phi = 16\vec{i}+188\vec{j}-188\vec{k}$

$$= \sqrt{(96)^2+(188)^2+(188)^2}$$

$$= 96$$

(iii) Find the unit vector normal to the surface $z = x^2 + y^2$ at the point (– 1, – 2, 5)

Solution: Given surface $z = x^2 + y^2$

$$\phi = z - x^2 - y^2$$

$\therefore$ grad $\phi = \nabla\phi$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(z - x^2 - y^2)$$

$$= -2x\,\vec{i} - 2y\,\vec{j} + \vec{k}$$

$\therefore$ At the point $(-1, -2, 5)$ is

grad $\phi = 2\,\vec{i} + 4\,\vec{j} + \vec{k}$

$$\therefore \text{ unit normal vector} = \frac{\text{grad } \phi}{|\text{grad } \phi|}$$

$$= \frac{2\vec{i} + 4\vec{j} + \vec{k}}{\sqrt{4+16+1}}$$

$$= \frac{2\vec{i} + 4\vec{j} + \vec{k}}{\sqrt{21}} \text{ Ans.}$$

(iv) Find the angle between the normal to the surface $x \log z = y^2 - 1$ and $x^2y + z$ at the point (1, 1, 1) **(BE 2069 Bhadra)**

Solution: We have, to given surfaces are,

$\phi_1 = x \log z - y^2 + 1$

and $\phi_2 = x^2y + z$

Then, grad $\phi_1 = \nabla\phi_1$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(x \log z - y^2 + 1)$$

$$= \log z\,\vec{i} - 2y\,\vec{j} + \vec{k}\,\frac{x}{z}$$

At the point (1, 1, 1) is grade $\phi_1 = \log 1\,\vec{i} - 2y\,\vec{j} + \vec{k} = -2y\,\vec{j} + \vec{k}$

Unit normal vector to surface ϕ_1 at the point (1, 1, 1)

$$= \frac{-2\vec{j} + \vec{k}}{\sqrt{4+1}} = \frac{-2\vec{j} + \vec{k}}{\sqrt{5}}$$

and grad $\phi_2 = \nabla\phi_2$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(x^2y + z)$$

$$= 2xy\,\vec{i} + x^2\,\vec{j} + \vec{k}$$

$\therefore$ Unit normal vector to the surface ϕ_2 at the point (1, 1, 1) is,

$2\,\vec{i} + \vec{j} + \vec{k}$

$$\therefore \hat{n}_2 = \frac{2\vec{i} + \vec{j} + \vec{k}}{\sqrt{4+1+1}} = \frac{2\vec{i} + \vec{j} + \vec{k}}{\sqrt{6}}$$

Then θ be the angle between their normal then $\cos\theta = \hat{n}_1 . \hat{n}_2$

$$\cos\theta = \frac{\left(-2\vec{j} + \vec{k}\right)}{\sqrt{5}} \cdot \left(\frac{2\vec{i} + \vec{j} + \vec{k}}{\sqrt{6}}\right)$$

$$= \frac{-2+1}{\sqrt{30}} = \frac{-1}{\sqrt{30}}$$

$$\therefore \cos\theta = \frac{-1}{\sqrt{30}}$$

$$\therefore \theta = \cos^{-1}\frac{1}{\sqrt{30}}$$

(v) Find a unit normal to the surface $x^3 + y^3 + 3xy = 3$ and the point (1, 2, – 1)

Solution: Given surface is $x^3 + y^3 + 3xyz - 3 = 0$

$\therefore$ let $\phi = x^3 + y^3 + 3xyz - 3$

$\therefore$ grad$\phi = \nabla\phi$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(x^3 + y^3 + 3xyz - 3$$

$$= 3x^2\,\vec{i} + 3y^2\,\vec{j} + 2yz\,\vec{i} + 3xz\,\vec{j} + 3xy\,\vec{k}$$

grad$\phi = (3x^2 + 3yz)\,\vec{i} + (3y^2 + 3xz)\,\vec{j} + 3xy\,\vec{k}$

$\because$ unit normal vector at the point (1, 2, – 1)

is grad $\phi = [3 + 3.2\,(-1)]\,\vec{i} + [(3.4 + 3.1.(-1)]\,\vec{j} + 6\,\vec{k}$

$$= -3\,\vec{i} + 9\,\vec{j} + 6\,\vec{k}$$

$$\hat{n}_1 = \frac{-3\vec{i} + 9\vec{j} + 6\vec{k}}{\sqrt{9+81+36}} = \frac{-3\vec{i} + 9\vec{j} + 6\vec{k}}{\sqrt{126}}$$

$$= \frac{-\vec{i} + 3\vec{j} + 2\vec{k}}{\sqrt{14}}$$

3. (i) If $\phi = \log(x^2 + y^2 + z^2)$, find div (grad ϕ) and curl (grad ϕ)

Solution: Given $\phi = \log(x^2 + y^2 + z^2)$

$\therefore$ grad $\phi = \nabla\phi$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)[\log(x^2 + y^2 + z^2)]$$

$$= \frac{2x}{x^2+y^2+z^2}\,\vec{i} + \frac{2xy}{x^2+y^2+z^2}\,\vec{j} + \frac{2z}{x^2+y^2+z^2}\,\vec{k}$$

$$= \frac{2x\,\vec{i} + 2y\vec{j} + 2z\vec{k}}{x^2+y^2+z^2}$$

$\therefore$ div (grad ϕ) = ∇ . grad ϕ

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \cdot \left(\frac{2x\,\vec{i} + 2y\vec{j} + 2z\vec{k}}{x^2+y^2+z^2}\right)$$

$$= \frac{2}{x^2+y^2+z^2}$$

Again, curl (grad ϕ)

$\therefore \nabla \times \text{grad}\,\phi$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \times \left(\frac{2x\,\vec{i} + 2y\vec{j} + 2z\vec{k}}{x^2+y^2+z^2}\right)$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{2x}{x^2+y^2+z^2} & \frac{2y}{x^2+y^2+z^2} & \frac{2z}{x^2+y^2+z^2} \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}\left(\frac{2z}{x^2+y^2+z^2}\right) - \frac{\partial}{\partial z}\left(\frac{2y}{x^2+y^2+z^2}\right)\right]$$

$$- \vec{j}\left[\frac{\partial}{\partial x}\left(\frac{2x}{x^2+y^2+z^2}\right) - \frac{\partial}{\partial z}\left(\frac{2z}{x^2+y^2+z^2}\right)\right]$$

$$+ \vec{k}\left[\frac{\partial}{\partial x}\left(\frac{2y}{x^2+y^2+z^2}\right) - \frac{\partial}{\partial y}\left(\frac{2x}{x^2+y^2+z^2}\right)\right]$$

$= 0.$

(ii) If $\vec{v} = x^2y\,\vec{i} - 2xz\,\vec{j} + 2yz\,\vec{k}$

Find div $\vec{v}$, curl $\vec{v}$ and curl curl $\vec{v}$

Solution: Given, $\vec{v} = x^2y\,\vec{i} - 2xz\,\vec{j} + 2yz\,\vec{k}$

$\therefore$ div $\vec{v} = \nabla \cdot \vec{v}$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \cdot \left(x^2y\,\vec{i} - 2xz\,\vec{j} + 2yz\,\vec{k}\right)$$

$= 2xy + 0 + 2y$

$= 2xy + 2y$

$= 2y\,(x + 1)$

curl $\vec{v} = \nabla \times \vec{v}$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \times \left(x^2y\,\vec{i} - 2xz\,\vec{j} + 2yz\,\vec{k}\right)$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & -2xz & 2yz \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(2yz) - \frac{\partial}{\partial z}(-2xz)\right] - \vec{j}\left[\frac{\partial}{\partial x}(2yz) - \frac{\partial}{\partial z}\left(x^2y\right)\right]$$

$$+ \vec{k}\left[\frac{\partial}{\partial x}(-2xz) - \frac{\partial}{\partial y}\left(x^2y\right)\right]$$

$= 2z\,\vec{i} + 2x\,\vec{i} - 2z\,\vec{k} - x^2\vec{k}$

$= (2z + 2x)\,\vec{i} - (2z + x^2)\,\vec{k}$

$= 2\,(x + z)\,\vec{i} - (x^2 + 2z)\,\vec{k}$

Now, curl curl $\vec{v}$

From above, we know that

curl $\vec{v} = (2z+2x)\,\vec{i} - 2z + x^2)\,\vec{k}$

$\therefore$ curl (curl $\vec{v}$) $= \nabla \times$ curl $\vec{v}$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \times \left((2z+2x)\,\vec{i} - 2z + x^2)\,\vec{k}\right)$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2z+2x & 0 & -(2z+x^2) \end{vmatrix}$$

$$= \vec{i}\left[\frac{-\partial}{\partial y}\left(2z+x^2\right) - \frac{\partial}{\partial z}\times 0\right] - \vec{j}\left[\frac{\partial}{\partial x}\left(2z+x^2\right) - \frac{\partial}{\partial z}(2z+2x)\right]$$

$$+ \vec{k}\left[0 + \frac{\partial}{\partial y}(2z+2x)\right]$$

$= -2x\,\vec{j} + 2\,\vec{j} = 2\,(x+1)\vec{j}$

(iii) If $f = x + y + z$, $g = x^2 + y^2 + z^2$ and $h = xy + yz + zx$ show that [grad f grad g, grad h] = 0 **(BE 2068 Bhadra (or))**

Solution: Given, $f = x + y + z$, $g = x^2 + y^2 + z^2)$ and $h = xy + yz + zx$

Now, grad f = $\left(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}\right)(x+y+z)$

$= \vec{i}+\vec{j}+\vec{k}$ ..... (i)

grad g = $\left(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}\right)(x^2+y^2+z^2)$

$= 2x\vec{i}+2y\vec{j}+2z\vec{k}$ ..... (ii)

and again, grad w = $\left(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}\right)(xy+yz+zx)$

$= (y+z)\vec{i}+(x+z)\vec{j}+(y+x)\vec{k}$ ..... (iii)

From (i), (ii), and (iii) we get

[grad f grad g grad w]

= gra f . (grad g × grad h)

$$= \begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ y+z & x+z & y+x \end{vmatrix}$$

$= 2y(y+x) - 2z(x+z) - 2x(y+x) + 2z(y+z) + 2x(x+z) - 2y(x+z)$

$= 2y^2 + 2xy - 2xz - 2z^2 - 2xy - 2x^2 + 2yz + 2z^2 + 2x^2 + 2zx - 2yz = 0$

Thus, [grad f grad g grad h] = 0 Hence proved.

(iv) Find the constant a so that the vector

$(ax^2y+yz)\vec{i}+(xy^2-xz^2)\vec{j}+(2xyz-2x^2y^2)\vec{k}$

is solenoidal.

Solution: The given vector is solenodial if

div $\vec{v} = 0$

where $\vec{v} = \left(ax^2y+yz\right)\vec{i}+\left(xy^2-xz^2\right)\vec{j}+\left(2xyz-2x^2y^2\right)\vec{k}$

$\therefore \nabla . \vec{v} = 0$

$\Rightarrow \left(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}\right).$

$\left[\left(ax^2+yz\right)\vec{i}+\left(xy^2-xz^2\right)\vec{j}+\left(2xyz-2x^2y^2\right)\right]$

$= 2axy + 2xy + 2xy = 0$

$= 2axy + 4xy = 0$

$= xy(2a+4) = 0$

$xy \neq 0,$

$2a + 4 = 0 \Rightarrow 2a = -4$

or, $\frac{-4}{2} = -2$

$\therefore a = -2$

Find the constants a, b, c. So that the vector,

$\vec{v} = (x+2y+az)\vec{i}+(bx-3y-z)\vec{j}+(4x+cy+2z)\vec{k}$ is

irrotational.

Solution: Given vector is,

$\vec{v} = (x+2y+az)\vec{i}+(bx-3y-z)\vec{j}+(4x+cy+2z)\vec{k}$

Now, curl $\vec{F} = \nabla\times\vec{F}$

$\left(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}\right)\times\left[(x+2y+az)\vec{i}+(bx-3y-z)\vec{j}+(4x+cy+2z)\vec{k}\right]$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix}$$

$\therefore$ curl $\vec{F} = (c+1)\vec{i}+(a-4)\vec{j}+(b-2)\vec{k}$

Now, the vector $\vec{F}$ is 0 imotational if curl $\vec{F} = 0$

i.e. $(c+1)\vec{i}+(a-4)\vec{j}+(b-2)\vec{k} = 0\vec{i}+0\vec{j}+0\vec{k}$

$c+1 = 0, a-4 = 0, b-2 = 0$

$c = -1, a = 4, b = 2$

$\therefore a = 4, b = 2, c = 1$ Ans.

Find n so that $r^n\vec{r}$ is solenodial. **(BE 2070 Magh)**

Solution: Now, $\vec{a}$ is solenodial if $\nabla.\vec{a} = 0$

Now, $\nabla.\vec{a} = 0 \Rightarrow \Sigma\vec{i}\frac{\partial}{\partial x}.(r^n\vec{r}) = 0$

$\Rightarrow \Sigma\vec{i}.\frac{\partial}{\partial x}.(r^n\vec{r}) = 0$

$\Rightarrow \Sigma\vec{i}\left[nr^{n-1}\frac{\partial r}{\partial x}\vec{r}+r^n\frac{\partial\vec{r}}{\partial x}\right] = 0$ ..... (i)

But, we know that $\frac{\partial r}{\partial x} = \frac{x}{r}$ and $\frac{\partial\vec{r}}{\partial x} = \vec{i}$

Hence, equation (i) becomes

$\Sigma\vec{i}.\left(nr^{n-1}\frac{x}{r}\vec{r}+r^n\vec{i}\right) = 0$

or, $nr^{n-2}\Sigma\left(\vec{i}.\vec{r}\right)x + r^n\Sigma\left(\vec{i}.\vec{i}\right) = 0$

or, $nr^{n-2}\Sigma x^2 + r^n \Sigma 1 = 0$
or, $nr^{n-2} . r^2 + r^n . 3 = 0$
or, $nr^n + 3r^n = 0$
$r^n (n+3) = 0$ So, $n + 3 = 0$
Since $r^n \neq 0$ i.e. $n = -3$ Ans.

(vii) Show that the vector $3y^4z^2\vec{i} + 4x^3z^2\vec{j} + 3x^2y^2\vec{k}$ is sdenodial.

Solution: Let, $\vec{v} = 3y^4z^2\vec{i} + 4x^3z^2\vec{j} + 3x^2y^2\vec{k}$

To show $\vec{v}$ in solenodial

If div $\vec{v} = 0$

$\therefore$ div $\vec{v} = \nabla . \vec{v}$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) . \left(3y^4z^2\vec{i} + 4x^3z^2\vec{j} + 3x^2y^2\vec{k}\right)$$

$$= \frac{\partial}{\partial x}\left(3y^4z^2\right) + \frac{\partial}{\partial y}\left(4x^3z^2\right) + \frac{\partial}{\partial z}\left(3x^2y^2\right)$$

$= 0 + 0 + 0$
$= 0$
$\therefore$ div $\vec{v} = 0$

Hence, $\vec{v}$ is solenodial proved.

(viii) If $\phi = \log (xy + yz + zx)$ and $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$, find $\vec{r}\,\nabla\phi$ $\vec{r} \times \nabla\phi$

Solution: Here, $\phi = \log (xy + yz + zx)$

and $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$$\therefore \nabla\phi = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) [\log (xy + yz + zx)]$$

$$\left[\frac{1}{xy+yz+z} . (y+z)\right]\vec{i} + \left[\frac{1}{xy+yz+z} . (x+z)\right]\vec{j}$$

$$+ \left[\frac{1}{xy+yz+z} . (x+y)\right]$$

$$\therefore \vec{r} . \nabla\phi = (x\vec{i} + y\vec{j} + z\vec{k})$$

$$. \left[\left(\frac{y+z}{xy+yz+z}\right)\vec{i} + \left(\frac{x+z}{xy+yz+z}\right)\vec{j} + \left(\frac{x+y}{xy+yz+z}\right)\right]$$

$$= \frac{xy+xz}{xy+yz+zx} + \frac{xy+yz}{xy+yz+zx} + \frac{xz+yz}{xy+yz+zx}$$

$$= \frac{2xy+2yz+2zx}{xy+yz+zx}$$

$$= \frac{2(xy+yz+zx)}{(xy+yz+zx)}$$

$= 2$

If be the position vector of a point in space and $\vec{a}$ is constant vector then prove that

div $\{\vec{r}\times(\vec{a}\times\vec{r})\} = -2\vec{r}.\vec{a}$

Solution: L. H. S = div $\{\vec{r}\times(\vec{a}\times\vec{r})\}$

$= \nabla . \{\vec{r}\times(\vec{a}\times\vec{r})\}$

$= \Sigma\vec{i}\frac{\partial}{\partial x} . \{\vec{r}\times(\vec{a}\times\vec{r})\}$

$= \Sigma\vec{i} . \frac{\partial}{\partial x}\{\vec{r}\times(\vec{a}\times\vec{r})\}$

By definition of divergance of vector function.

$$= \Sigma\vec{i} . \left[\frac{\partial\vec{r}}{\partial x}\times(\vec{a}\times\vec{r}) + \vec{r}\times\frac{\partial}{\partial x}(\vec{a}\times\vec{r})\right]$$

$$= \Sigma\vec{i} . \left[\vec{i}\times(\vec{a}\times\vec{r}) + \vec{r}\times\left(\frac{\partial\vec{a}}{\partial x}\times\vec{r} + \vec{a}\times\frac{d\vec{r}}{dx}\right)\right]$$

Since $\vec{a}$ is constant vector

$$\frac{\partial\vec{a}}{\partial x} = 0$$

Also, $\vec{r}$ is postition vector

$$\frac{d\vec{r}}{dx} = \vec{i}$$

$$= \Sigma\vec{i} . \left[\vec{i}\times(\vec{a}\times\vec{r}) + \vec{r}\times\left(\vec{0}\times\vec{r} + \vec{a}\times\vec{i}\right)\right]$$

$$= \Sigma\vec{i} . \left[\vec{i}\times(\vec{a}\times\vec{r}) + \vec{r}\times(\vec{a}\times\vec{r})\right]$$

$$= \Sigma\vec{i} . \left\{(\vec{i}.\vec{r})\vec{a} - (\vec{i}.\vec{a})\vec{r} + (\vec{r}.\vec{r})\vec{a} - (\vec{r}.\vec{a})\vec{i}\right\}$$

Since, $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ and $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$

Then $\vec{i} . \vec{r} = x$ and $\vec{i} . \vec{a} = a_1$ etc.

$$= \Sigma\vec{i} . \left\{x\vec{a} - a_1\vec{r} + x\vec{a} - (\vec{r}.\vec{a})\vec{i}\right\}$$

$$= \Sigma\vec{i}\,x . \vec{a} - \Sigma\vec{i}a_1.\vec{r} + \Sigma\vec{i}x . \vec{a} - (\vec{r}.\vec{a})\Sigma(\vec{i}.\vec{i})$$

$$= \vec{r}.\vec{a} - \vec{a}.\vec{r} + \vec{r}.\vec{a} - 3(\vec{a}.\vec{r})$$

$$= -2(\vec{r}.\vec{a})$$

Thus div $\{\vec{r}\times(\vec{a}\times\vec{r})\} = -2\,\vec{r}.\vec{a}$ Hence proved.

(ii) **Curl $\{\vec{r}\times(\vec{a}\times\vec{r})\} = 3\vec{r}\times\vec{a}$**

Solution: L. H. S. = Curl $\{\vec{r}\times(\vec{a}\times\vec{r})\}$

$$= \nabla\times\{\vec{r}\times(\vec{a}\times\vec{r})\}$$

$$= \Sigma\vec{i}\frac{\partial}{\partial x}\times\{\vec{r}\times(\vec{a}\times\vec{r})\}$$

$$= \Sigma\vec{i}\times\frac{\partial}{\partial x}\{\vec{r}\times(\vec{a}\times\vec{r})\}$$

$$= \Sigma\vec{i}\times\left\{\frac{\partial\vec{r}}{\partial x}\times(\vec{a}\times\vec{r})+\vec{r}\times\frac{\partial}{\partial x}(\vec{a}\times\vec{r})\right\}$$

$$= \Sigma\vec{i}\times\left\{\frac{\partial\vec{r}}{\partial x}\times(\vec{a}\times\vec{r})+\vec{r}\times\left(\frac{\partial\vec{a}}{\partial x}\times\vec{r}+\vec{a}\times\frac{\partial\vec{r}}{\partial x}\right)\right\}$$

Since $\frac{\partial\vec{a}}{\partial x}=0$ and $\frac{d\vec{r}}{dx}=\vec{i}$

$$= \Sigma\vec{i}\times\left\{\vec{i}\times(\vec{a}\times\vec{r})+\vec{r}\times(\vec{a}\times\vec{i})\right\}$$

$$= \Sigma\vec{i}\times\left\{(\vec{i}.\vec{r})\vec{a}-(\vec{i}.\vec{a})\vec{r}+(\vec{r}.\vec{i})\vec{a}-(\vec{r}.\vec{a})\vec{i}\right\}$$

$$= \Sigma\vec{i}\times\{x\vec{a}-a_1.\vec{r}+x\vec{a}-(\vec{r}.\vec{a})\vec{i}\}$$

Since, $\vec{i}.\vec{r}=x$ and $\vec{i}.\vec{a}=a_1$

$$= \Sigma\vec{i}\,x\times\vec{a}-\Sigma\vec{i}a_1\times\vec{r}+\Sigma\vec{i}x\times\vec{a}-(\vec{r}.\vec{a})\Sigma(\vec{i}\times\vec{i})$$

$$= \vec{r}\times\vec{a}-\vec{a}\times\vec{r}+\vec{r}\times\vec{a}-(\vec{r}.\vec{a})\,\Sigma\vec{0}$$

$$= \vec{r}\times\vec{a}+\vec{r}\times\vec{a}+\vec{r}\times\vec{a}-\vec{0}$$

$$= 3\vec{r}\times\vec{a}$$

Thus, curl $\{\vec{r}\times(\vec{a}\times\vec{r})\} = 3\vec{r}\times\vec{a}$

(iii) $\nabla\times\left(\frac{\vec{a}\times\vec{r}}{r}\right)=\frac{\vec{a}}{r}+\frac{\vec{a}.\vec{r}}{r^3}\vec{r}$

Solution: L. H.S. = $\nabla\times\left(\frac{\vec{a}\times\vec{r}}{r}\right)$

$$= \vec{i}\frac{\partial}{\partial x}\times\left\{(\vec{a}\times\vec{r})r^{-1}\right\}$$

$$= \Sigma\vec{i}\times\frac{\partial}{\partial x}\left\{(\vec{a}\times\vec{r})r^{-1}\right\}$$

$$= \Sigma\vec{i}\times\left\{(\vec{a}\times\vec{r})\frac{\partial}{\partial x}(r^{-1})+r^{-1}\frac{\partial}{\partial x}(\vec{a}\times\vec{r})\right\}$$

$$= \Sigma\vec{i}\times\left\{(\vec{a}\times\vec{r})(-1)+r^{-2}\frac{\partial r}{\partial x}+r^{-1}\left(\frac{\partial\vec{a}}{\partial x}\times\vec{r}+\vec{a}\times\frac{\partial\vec{r}}{\partial x}\right)\right\}$$

$$= \Sigma\vec{i}\times\left\{(\vec{a}\times\vec{r})\left(-\frac{x}{r^3}\right)+r^{-1}(\vec{0}+\vec{a}\times\vec{i})\right\}$$

Since, $\frac{d\vec{r}}{dx}=\vec{i}$, $\frac{dr}{dx}=\frac{x}{r}$ and $\frac{\partial\vec{a}}{\partial x}=\vec{0}$

$$= -\frac{1}{r^3}\left[\Sigma\vec{i}x\times(\vec{a}\times\vec{r})\right]+r^{-1}\Sigma\vec{i}\times(\vec{a}\times\vec{i})$$

$$= -\frac{1}{r^3}\{\vec{r}\times(\vec{a}\times\vec{r})\}+\frac{1}{r}\left\{\Sigma\vec{i}\times(\vec{a}\times\vec{i})\right\}$$

$$= -\frac{1}{r^3}\{(\vec{r}.\vec{r})\vec{a}-(\vec{r}.\vec{a})\vec{r}\}+\frac{1}{r}\Sigma\left\{(\vec{i}.\vec{i})\vec{a}-(\vec{i}.\vec{a})\vec{i}\right\}$$

$$= -\frac{1}{r^3}\left\{r^2\vec{a}-(\vec{r}.\vec{a})\vec{r}\right\}+\frac{1}{r}\{3\vec{a}-\vec{a}\}$$

$$= -\frac{1}{r^3}\left\{r^2\vec{a}-(\vec{r}.\vec{a})\vec{r}\right\}+\frac{2\vec{a}}{r}$$

$$= -\frac{\vec{a}}{r}+\left(\frac{\vec{a}.\vec{r}}{r^3}\right)\vec{r}+\frac{2\vec{a}}{r}$$

$$= \frac{\vec{a}}{r}+\left(\frac{\vec{a}.\vec{r}}{r^3}\right)\vec{r}$$

Thus, curl $\left(\frac{\vec{a}\times\vec{r}}{r}\right)=\frac{\vec{a}}{r}+\left(\frac{\vec{a}.\vec{r}}{r^3}\right)\vec{r}$ Hence proved.

(iv) $\nabla\times\{(\vec{a}.\vec{r})\vec{a}\}=0$

Solution: L.H.S. = $\nabla\times\{(\vec{a}.\vec{r})\vec{a}\}$

$$= \Sigma\vec{i}\frac{\partial}{\partial x}\times\{(\vec{a}.\vec{r})\vec{a}\}$$

$$= \Sigma\vec{i}\times\frac{\partial}{\partial x}\{(\vec{a}.\vec{r})\vec{a}\}$$

$$= \Sigma\vec{i}\times\left[\left\{\frac{\partial}{\partial x}(\vec{a}.\vec{r})\right\}\vec{a}+(\vec{a}.\vec{r})\frac{\partial\vec{a}}{\partial x}\right]$$

$$= \Sigma\vec{i}\times\left\{\left(\frac{\partial\vec{a}}{\partial x}.\vec{r}+\vec{a}.\frac{\partial\vec{r}}{\partial x}\right)\vec{a}+(\vec{r}.\vec{a})\frac{\partial\vec{a}}{\partial x}\right\}$$

Since, $\frac{\partial\vec{a}}{\partial x}=\vec{0}$ and $\frac{\partial\vec{r}}{\partial x}=\vec{i}$

$$= \Sigma\vec{i}\times\left\{(\vec{0}.\vec{r}+\vec{a}.i)\vec{a}+(\vec{a}.\vec{r})\vec{0}\right\}=\Sigma\vec{i}\times\left\{(\vec{a}.\vec{i})\vec{a}\right\}$$

$$= \Sigma\vec{i}\times a_1\vec{a} \qquad = \Sigma\vec{i}\,a_1\times\vec{a} \qquad = \vec{a}\times\vec{a}=0$$

Thus $\nabla\times\{(\vec{a}.\vec{r})\vec{a}\}=0$ Hence proved.

(v) $\nabla.\left(\frac{\vec{a}\times\vec{r}}{r^n}\right)=0$

Solution: $\nabla.\left[\left(\frac{\vec{a}\times\vec{r}}{r^n}\right)\right]=\nabla\left[r^{-n}(\vec{a}\times\vec{r})\right]$

We know that $\nabla(f\vec{a})=f(\vec{\nabla}.\vec{a})+(\nabla f).\vec{a}$

$$\nabla.\left[r^{-n}(\vec{a}\times\vec{r})\right]=r^{-n}\left[\nabla.(\vec{a}\times\vec{r})\right]+\left(\nabla r^{-n}\right).(\vec{a}\times\vec{r})$$

$$=r^{-n}\left[\nabla.(\vec{a}\times\vec{r})\right]+\left(\vec{i}\frac{\partial r^{-n}}{\partial x}+\vec{j}\frac{\partial r^{-n}}{\partial y}+\vec{k}\frac{\partial r^{-n}}{\partial z}\right).(\vec{a}\times\vec{r})$$

Since $\vec{a}$ is constant, $\nabla\times\vec{a}=0$

Also, $\nabla.(\vec{a}\times\vec{r})=0$

Hence, $\nabla.\left[r^{-n}(\vec{a}\times\vec{r})\right]$

$$=0+\left(-r^{-n-1}\frac{\partial r}{\partial x}\vec{i}-r^{-n-1}\frac{\partial r}{\partial y}\vec{j}-r^{-n-1}\frac{\partial r}{\partial z}\vec{k}\right).(\vec{a}\times\vec{r})$$

$$=0-r^{-n}\left(\frac{x\vec{i}+y\vec{j}+z\vec{k}}{r}\right).(\vec{a}\times\vec{r})$$

$$=\frac{-r^{-n}}{r}\left[\vec{r}.(\vec{a}\times\vec{r})\right]=0=\text{R.H.S. proved.}$$

(vii) $\nabla\times\left\{(\vec{a}.\vec{r})\vec{r}\right\}=\vec{a}\times\vec{r}$

Solution: L. H. S. $=\nabla\times(\vec{a}.\vec{r})\,\vec{r}$

$$=\Sigma\vec{i}\frac{\partial}{\partial x}\times\left\{(\vec{a}.\vec{r})\,\vec{r}\right\}$$

$$=\Sigma\vec{i}\times\left\{\frac{\partial}{\partial x}(\vec{a}.\vec{r})\vec{r}\right\}$$

$$=\Sigma\vec{i}\times\left\{\frac{\partial r}{\partial x}(\vec{a}.\vec{r})+(\vec{a}.\vec{r})\frac{\partial\vec{r}}{\partial x}\right\}$$

$$=\Sigma\vec{i}\times\left[\vec{r}\left\{\vec{a}.\frac{\partial\vec{r}}{\partial x}+\vec{r}.\frac{\partial\vec{a}}{\partial x}\right\}+(\vec{a}.\vec{r})\frac{\partial\vec{r}}{\partial x}\right]$$

$$=\Sigma\vec{i}\times\left[\vec{r}\left\{(\vec{a}.\vec{i})+\vec{r}.\vec{0}\right\}+(\vec{a}.\vec{r})\vec{i}\right]$$

$$=\Sigma\vec{i}\times(\vec{a}.\vec{i})\vec{r}+\Sigma\vec{i}\times(\vec{a}.\vec{r})\vec{i}$$

$$=\Sigma(\vec{i}.\vec{i})\vec{a}\times\vec{r}+0=\Sigma\vec{a}\times\vec{r}=\vec{a}\times\vec{r}\ \text{R. H. S.}$$

Hence, $\nabla\times(\vec{a}.\vec{r})\,\vec{r}=\vec{a}\times\vec{r}$ Proved.

(viii) $\nabla\times\left(\frac{\vec{a}\times\vec{r}}{r^n}\right)=\frac{-(n-2)}{r^n}\vec{a}+n\frac{(\vec{a}.\vec{r})}{r^{n+2}}\vec{r}$

Solution: $\nabla\times\left[\frac{\vec{a}\times\vec{r}}{r^n}\right]=\nabla\times\left[\vec{a}\times\left(\frac{\vec{r}}{r^n}\right)\right]$

$$=\left(\Sigma\vec{i}\frac{\partial}{\partial x}\right)\times\left(\vec{a}\times\frac{\vec{r}}{r^n}\right)=\Sigma\vec{i}\times\frac{\partial}{\partial x}\left(\vec{a}\times\frac{\vec{r}}{r^n}\right)$$

$$=\Sigma\vec{i}\times\frac{\partial}{\partial x}\left\{\frac{\partial\vec{a}}{\partial x}\times\frac{\vec{r}}{r^n}+\vec{a}\times\frac{\partial}{\partial x}\left(\frac{\vec{r}}{r^n}\right)\right\}$$

$$=\Sigma\vec{i}\times\left\{\vec{a}\times\frac{\partial}{\partial x}\left(\frac{\vec{r}}{r^n}\right)\right\}$$

Using the formula

$$\frac{d}{dt}(\vec{a}\times\vec{b})=\frac{d\vec{a}}{dt}\times\vec{b}+\vec{a}\times\frac{a\vec{b}}{dt}$$

Since $\frac{\partial\vec{a}}{\partial x}=0$ because $\vec{a}$ is a costant vector

$$=\Sigma\left\{\vec{i}.\frac{\partial}{\partial x}\left(\frac{\vec{r}}{r^n}\right)\right\}\vec{a}-\left\{(\vec{i}.\vec{a})\frac{\partial}{\partial x}\left(\frac{\vec{r}}{r^n}\right)\right\}$$

$$=\Sigma\left\{\vec{i}.\frac{\partial}{\partial x}\left(\frac{\vec{r}}{r^n}\right)\right\}\vec{a}-\Sigma(\vec{i}.\vec{a})\frac{\partial}{\partial x}\left(\frac{\vec{r}}{r^n}\right)$$

Now, $\Sigma\vec{i}.\frac{\partial}{\partial x}\left(\frac{\vec{r}}{r^n}\right)\vec{a}=\Sigma\left\{\vec{i}.(a_1\vec{i}+a_2\vec{j}+a_3\vec{k})\right\}\frac{\partial}{\partial x}\left(\frac{\vec{r}}{r^n}\right)$

$$=\Sigma a_1\frac{\partial}{\partial x}\left(\frac{\vec{r}}{r^n}\right)$$

$$=a_1\frac{\partial}{\partial x}\left(\frac{\vec{r}}{r^n}\right)\times a_2.\frac{\partial}{\partial y}\left(\frac{\vec{r}}{r^n}\right)+a_3\frac{\partial}{\partial z}\left(\frac{\vec{r}}{r^n}\right)$$

$$=(\vec{a}.\nabla)\frac{\vec{r}}{r^n}$$

$$\therefore\nabla\times\left(\frac{\vec{a}\times\vec{r}}{r^n}\right)=\nabla.\left(\frac{\vec{r}}{r^n}\right)\vec{a}-(\vec{a}.\nabla)\left(\frac{\vec{r}}{r^n}\right)\ \text{..... (i)}$$

Now, $\left[\nabla.\left(\frac{\vec{r}}{r^n}\right)\right]\vec{a}=\vec{a}\left[\nabla.\left(r^{-n}\vec{r}\right)\right]$

$$=\vec{a}\left[r^{-n}(\nabla.\vec{r})+\nabla(r^{-n}).\vec{r}\right]$$

$$= \vec{a}\ [r^{-n}(3) + (-n)\, r^{-n-2}\ \vec{r}\ .\ r]$$
$$= \vec{a}\ (3r^{-n} - nr^{-n-2}\, r^2]$$
$$= \vec{a}\ [3r^{-n} - nr^{-n}]$$
$$= (3-n)\, r^{-n}\ \vec{a}\ \text{..... (ii)}$$

Also, $(\vec{a}\,.\,\nabla)\left(\dfrac{\vec{r}}{r^n}\right) = (\vec{a}\ .\ \nabla)(r^{-n}\ \vec{r}\,)$

$$= r^{-n}(\vec{a}\,.\,\nabla)\vec{r} + (\vec{r}\,(\vec{a}\,.\,\nabla)\, r^{-n}$$

$$= r^{-n}\ \Sigma\left\{\vec{a}\,.\,\vec{i}\,.\ \frac{\partial\vec{r}}{\partial x} + \vec{a}\,.\vec{j}\ \frac{\partial\vec{r}}{\partial y} + \vec{a}\,.\vec{k}\ \frac{\partial\vec{r}}{\partial z}\right\} + \vec{r}\ (\vec{a}\ .\ \nabla r^{-n})$$

$$= r^{-n}\left\{\vec{a}\,.\,\vec{i}\ \vec{i} + \vec{a}\,.\vec{j}\,\vec{j} + \vec{a}\,.\,\vec{k}\,\vec{k}\right\} + \vec{r}\,.\vec{a}\ (nr^{-n-2}\vec{r}\,)$$

$$= r^{-n}\ \vec{a} - \frac{n}{r^{n+2}}\ (\vec{a}\,.\,\vec{r})\ \text{..... (iii)}$$

Hence substitute the (ii) and (iii) in (i), we get

$$\nabla\times\left[\frac{\vec{a}\times\vec{r}}{r^n}\right] = (3-n)\, r^{-n}\ \vec{a} - \left\{-r^{-n}\vec{a} - \frac{n}{r^{n+2}}(\vec{a}\,.\,\vec{r})\vec{r}\right\}$$

$$= \left(\frac{3^{-n}}{r^n}\right)\vec{a}\ \frac{\vec{a}}{r^n} + \frac{n}{r^{n+2}}(\vec{a}\,.\,\vec{r})\ \vec{r}$$

$$= \left(\frac{2^{-n}}{r^n}\right)\vec{a} + \frac{n}{r^{n+2}}(\vec{a}\,.\,\vec{r})\ \vec{r}$$

(ix) $\nabla\,.\,\hat{r} = \dfrac{2}{r}$

Solution: Since, $\hat{r} = \dfrac{\vec{r}}{|\vec{r}|}$

$$\hat{r} = \frac{\vec{r}}{r}$$

Now, L. H. S. $= \nabla\,.\,\hat{r} = \Sigma\vec{i}\ \dfrac{\partial}{\partial x}\ .\ \hat{r}$

$$= \Sigma\vec{i}\,.\left\{\frac{\partial}{\partial x}\left(\frac{\vec{r}}{r}\right)\right\} = \Sigma\vec{i}\ .\left\{\frac{r.\dfrac{d\vec{r}}{dx} - \vec{r}\,\dfrac{dr}{dx}}{r^2}\right\} = \Sigma\vec{i}\,.\left\{\frac{r\,\vec{i} - \vec{r}\,\dfrac{x}{r}}{r^2}\right\}$$

$$= \Sigma\vec{i}\,.\left\{\frac{r^2\,\vec{i} - \vec{r}\,x}{r^3}\right\} = \Sigma\vec{i}\,.\left\{\frac{\vec{i}}{r} - \frac{\vec{r}x}{r^3}\right\} = \Sigma\ \frac{\vec{i}\,.\,\vec{i}}{r} - \Sigma\frac{\vec{i}\,x.\,\vec{r}}{r^3}$$

$$= \frac{3}{r} - \frac{\vec{r}\,.\,\vec{r}}{r^3} = \frac{3}{r} - \frac{r^2}{r^3} = \frac{3}{r} - \frac{1}{r} = \frac{2}{r}\ \text{R. H. S.}$$

Hence, $\nabla\,.\,(\hat{r}) = \dfrac{2}{r}$ Proved.

(x) Prove that $\nabla\left(\nabla\,.\,\dfrac{\vec{r}}{r}\right) = -\dfrac{2\vec{r}}{r^3}$

Solution: $\nabla\,.\ \dfrac{\vec{r}}{r} = \nabla.\ \left(r^{-1}\,\vec{r}\right)$

$$= r^{-1}\ (\nabla\,.\,\vec{r}) + (\nabla r^{-1})\,.\ \vec{r}$$

$$= r^{-1}\left(\vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z}\right).\left(x\vec{i} + y\vec{j} + z\vec{k}\right)$$

$$+ \left(\vec{i}\,\frac{\partial r^{-1}}{\partial x} + \vec{j}\,\frac{\partial r^{-1}}{\partial y} + \vec{k}\,\frac{\partial r^{-1}}{\partial z}\right).\ \vec{r}$$

$$= 3r^{-1} + \left(-r^{-2}\,\frac{\partial r}{\partial x}\,\vec{i} - r^{-2}\,\frac{\partial r}{\partial y}\,\vec{j} - r^{-2}\,\frac{\partial r}{\partial z}\,\vec{k}\right).\ \vec{r}$$

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$r^2 = x^2 + y^2 + z^2,\ \frac{\partial r}{\partial x} = \frac{x}{r},\ \frac{\partial r}{\partial y} = \frac{y}{z},\ \frac{\partial r}{\partial z} = \frac{z}{r}$$

Hence, $\nabla\,.\ \dfrac{\vec{r}}{r} = 3r^{-1} - r^{-2}\left(\dfrac{x}{r}\,\vec{i} + \dfrac{y}{r}\,\vec{j} + \dfrac{z}{r}\,\vec{k}\right).\ \vec{r}$

$$= 3r^{-1} - r^{-2}\left(\frac{\vec{r}}{r}\right).\ \vec{r} = 3r^{-1} - r^{-2}\left(\frac{\vec{r}\,.\,\vec{r}}{r}\right)$$

$$= 3r^{-1} - r^{-2}\left(\frac{r^2}{r}\right) = 2r^{-1}$$

$$\nabla\left(\nabla\,.\,\frac{\vec{r}}{r}\right) = \vec{i}\,\frac{\partial}{\partial x}\left(2r^{-1}\right) + \vec{j}\,\frac{\partial}{\partial y}\left(2r^{-1}\right) + \vec{k}\,\frac{\partial}{\partial z}\left(2r^{-1}\right)$$

$$= -2r^{-2}\,\frac{\partial r}{\partial x}\ \vec{i} - 2r^{-2}\,\frac{\partial r}{\partial y}\ \vec{j} - 2r^{-2}\,\frac{\partial r}{\partial z}\ \vec{k}$$

$$= -2r^{-2}\left(\frac{x}{r}\,\vec{i} + \frac{y}{r}\,\vec{j} + \frac{z}{r}\,\vec{k}\right)$$

$$= -2r^{-2}\ \frac{\vec{r}}{r}$$

$$= -\frac{2}{r^3}\ \vec{r}\ \text{R. H. S. proved.}$$

5. If $\vec{a}$ and $\vec{b}$ are constant vectors, prove that,

(i) $\nabla\,.\left\{\left(\vec{b}\,.\,\vec{r}\right)\vec{a}\right\} = \vec{a}\,.\,\vec{b}$

Solution: Let $\vec{a} = a_1\vec{i} + a_2\vec{j} + b_3\vec{k}$

$$\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$$

and $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

Now, $\vec{b}\,.\,\vec{r} = \left(b_1\vec{i} + b_2\vec{j} + b_3\vec{k}\right).\left(x_1\vec{i} + y_2\vec{j} + z_3\vec{k}\right)$

$= b_1x + b_2y + b_3z$ ..... (i)

Now, L. H. S. $= \nabla\,.\left\{\left(\vec{b}.\vec{r}\right)\vec{a}\right\}$

$= \Sigma\vec{i}.\frac{\partial}{\partial x}\left\{\left(\vec{b}.\vec{r}\right)\vec{a}\right\}$

$= \Sigma\vec{i}.\left[\left(\vec{b}.\vec{r}\right)\frac{\partial\vec{a}}{\partial x} + \left\{\frac{\partial}{\partial x}\left(\vec{b}.\vec{r}\right)\vec{a}\right\}\right]$ ..... (ii)

since, $\vec{a}$ is constant vector

$\therefore\ \frac{\partial\vec{a}}{\partial x} = \vec{0}$

Then (ii) becomes,

$\nabla\,.\left\{\left(\vec{b}.\vec{r}\right)\vec{a}\right\} = \Sigma\vec{i}.\left[\left\{\frac{\partial}{\partial x}\left(\vec{b}.\vec{r}\right)\right\}\vec{a}\right]$

$= \Sigma\vec{i}.\left[\vec{a}\,\frac{\partial}{\partial r}\left(b_1x + b_2y + b_3z\right)\right]$

Since, $\vec{b}\,.\,\vec{r} = b_1x + b_2y + b_3z$

$= \Sigma\vec{i}.\,(\vec{a}\,b_1) = \vec{a}\,.\,\Sigma\,\vec{i}b_1 = \vec{a}\,.\,\vec{b}$ R. H. S.

Hence, $\nabla\,.\,\{(\vec{b}\,.\,\vec{r}\,)\,\vec{a}\,\} = \vec{a}\,.\,\vec{b}$ Proved.

(ii) $\nabla\times\left\{\vec{a}\times\left(\vec{b}\times\vec{r}\right)\right\} = \vec{a}\times\vec{b}$

Solution: Let $\vec{a} = a_1\vec{i} + a_2\vec{j} + b_3\vec{k}$

$\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$

and $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

Now, L. H. S.

$\nabla\times\left\{\vec{a}\times\left(\vec{b}\times\vec{r}\right)\right\} = \Sigma i\frac{\partial}{\partial x}\times\left\{\vec{a}\times\left(\vec{b}\times\vec{r}\right)\right\}$

$= \Sigma\vec{i}\,\frac{\partial}{\partial x}\left\{\vec{a}\times\left(\vec{b}\times\vec{r}\right)\right\}$

$= \Sigma\vec{i}\times\left\{\vec{a}\times\left\{\frac{\partial}{\partial x}\times\left(\vec{b}\times\vec{r}\right) + \frac{\partial\vec{a}}{\partial x}\times\left(\vec{b}\times\vec{r}\right)\right\}\right\}$

$= \Sigma\vec{i}\times\left[\vec{a}\times\left\{\frac{\partial b}{\partial x}\times\vec{r} + \vec{b}\times\frac{\partial\vec{r}}{\partial x}\right\} + \frac{\partial\vec{a}}{\partial x}\times\left(\vec{b}\times\vec{r}\right)\right]$ ..... (i)

Since $\vec{a}$ and $\vec{b}$ are constant vectors

$\therefore\ \frac{\partial\vec{a}}{\partial x} = \frac{\partial\vec{b}}{\partial x} = \vec{0}$

Also, $\frac{\partial\vec{r}}{\partial x} = \vec{i}$

Hence the above expression (i) becomes

$\nabla\times\left\{\vec{a}\times\left(\vec{b}\times\vec{r}\right)\right\} = \Sigma\vec{i}\times\left[a\times\left\{\vec{0} + \vec{b}\times\vec{i}\right\} + \vec{0}\right]$

$= \Sigma\vec{i}\times\left[\vec{a} + \left(\vec{b}\times\vec{i}\right)\right] = \Sigma\vec{i}\times\left[\left(\vec{a}\,.\,\vec{i}\right)\vec{b} - \left(\vec{a}\,.\,\vec{b}\right)\vec{i}\right]$

$= \Sigma\left(\vec{a}\,.\,\vec{i}\right)\left(\vec{i}\times\vec{b}\right) - \Sigma\left(\vec{a}\,.\,\vec{b}\right)\left(\vec{i}\times\vec{i}\right)$

Since$\Sigma\ \vec{i}\times\vec{i} = \vec{0}$

$\therefore\ \nabla\times\left\{\vec{a}\times\left(\vec{b}\times\vec{r}\right)\right\} = \Sigma\left(\vec{a}\,.\,\vec{i}\right)\left(\vec{i}\times\vec{b}\right) - 0$

$\Sigma\left(\vec{a}\,.\,\vec{i}\right)\left(\vec{i}\times\vec{b}\right)$

since $\vec{a}\,.\,\vec{i} = a_1$

$= \Sigma a_1\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 0 \\ b_1 & b_2 & b3 \end{vmatrix}$

$= \Sigma a_1\left[\vec{i}(0-0) + \vec{j}(0-b_3) + \vec{k}(b_2-0)\right]$

$= \Sigma a_1\left[\vec{k}b_2 - \vec{j}b_3\right]$

$= a_1\left[\vec{k}b_2 - \vec{j}b_3\right] + a_2\left[\vec{i}b_3 - \vec{k}b_1\right] + a_3\left[\vec{j}b_1 - \vec{i}b_2\right]$

$= \vec{i}\left(a_2b_3 - b_2a_3\right) + \vec{j}\left(a_3b_1 - a_1b_3\right) + \vec{k}\left(a_1b_2 - a_2b_1\right)$

$= \vec{a}\times\vec{b}$

Thus, $\nabla\times\left\{\vec{a}\times\left(\vec{b}\times\vec{r}\right)\right\} = \vec{a}\times\vec{b}$ Hence proved.

(iii) $\nabla.\left\{\vec{r}\times\left(\vec{a}\times\vec{b}\right)\right\} = 0$

Solution: Since $r = x\vec{i} + y\vec{j} + z\vec{k}$

Now, L. H. S.

$\nabla.\left\{\vec{r}\times\left(\vec{a}\times\vec{b}\right)\right\}$

$= \Sigma\vec{i}\,\frac{\partial}{\partial x}.\left\{\vec{r}\times\left(\vec{a}\times\vec{b}\right)\right\} = \Sigma\vec{i}\,.\frac{\partial}{\partial x}\left\{\vec{r}\times\left(\vec{a}\times\vec{b}\right)\right\}$

$= \Sigma\vec{i}.\left\{\frac{\partial\vec{r}}{\partial x}\times\left(\vec{a}\times\vec{b}\right) + \vec{r}\times\frac{\partial}{\partial x}\left(\vec{a}\times\vec{b}\right)\right\}$ ..... (i)

Since $\vec{a}\times\vec{b}$ is constant vector

$\therefore\ \frac{\partial}{\partial x}(\vec{a}\times\vec{b}) = 0$ so the equation (i) becomes

$$\nabla.\left\{\vec{r}\times\left(\vec{a}.\vec{b}\right)\right\} = \Sigma\vec{i}.\left\{\frac{\partial\vec{r}}{\partial x}\times\left(\vec{a}\times\vec{b}\right)\right\}$$

$$= \Sigma\vec{i}.\left\{\vec{i}\times\left(\vec{a}\times\vec{b}\right)\right\} \text{ since } \frac{\partial\vec{r}}{\partial x} = \vec{i}$$

$= \Sigma\ 0$ [In scalartriple product is zero when two vector are equal)

$$\therefore \nabla.\left\{\vec{r}\times\left(\vec{a}\times\vec{b}\right)\right\} = 0 \text{ Proved.}$$

(v) $\vec{a}.\nabla\left\{\vec{b}.\nabla\left(\frac{1}{r}\right)\right\} = \frac{3\left(\vec{a}.\vec{r}\right)\left(\vec{b}.\vec{r}\right)}{r^5} - \frac{\vec{a}.\vec{b}}{r^3}$

Solution: Let $\vec{a} = a_1\vec{i}+a_2\vec{j}+a_3\vec{k}$, $\vec{b} = b_1\vec{i}+b_2\vec{j}+b_3\vec{k}$

$$\nabla\left(\frac{1}{r}\right) = \vec{i}\frac{\partial}{\partial x}\left(\frac{1}{r}\right)+\vec{j}\frac{\partial}{\partial y}\left(\frac{1}{r}\right)+\vec{k}\frac{\partial}{\partial z}\left(\frac{1}{r}\right)$$

$$= \vec{i}\left(\frac{-1}{r^2}\frac{\partial r}{\partial x}\right)+\vec{j}\left(\frac{-1}{r^2}\frac{\partial r}{\partial y}\right)+\vec{k}\left(\frac{-1}{r^2}\frac{\partial r}{\partial z}\right)$$

$$\vec{r} = x\vec{i}+y\vec{j}+z\vec{k},\ \vec{r}^{\,2} = x^2+y^2+z^2$$

But $\frac{\partial r}{\partial x} = \frac{x}{r}, \frac{\partial r}{\partial y} = \frac{y}{z}, \frac{\partial r}{\partial z} = \frac{z}{r}$

$$\nabla\left(\frac{1}{r}\right) = \vec{i}\left(\frac{-1}{r^2}\frac{x}{r}\right)+\vec{j}\left(\frac{-1}{r^2}.\frac{y}{r}\right)+\vec{k}\left(\frac{-1}{r^2}\frac{z}{r}\right)$$

$$= \frac{-1}{r^3}\left(x\vec{i}+y\vec{j}+z\vec{k}\right)$$

$$= \frac{-r}{r^3}$$

$$\vec{b}.\nabla\left(\frac{1}{r}\right) = \left(b_1\vec{i}+b_2\vec{j}+b_3\vec{k}\right).\frac{-\left(x\vec{i}+y\vec{j}+z\vec{k}\right)}{r^3}$$

$$= -\left(\frac{b_1x+b_2y+b_3z}{r^3}\right)$$

$= \phi$ (say)

$$\nabla\left(\vec{b}.\nabla\frac{1}{r}\right) = \nabla\phi = \vec{i}\frac{\partial\phi}{\partial x}+\vec{j}\frac{\partial\phi}{\partial y}+\vec{k}\frac{\partial\phi}{\partial z}$$

$$\Rightarrow \frac{\partial\phi}{\partial x} = \frac{-\partial}{\partial x}\left(\frac{b_1x+b_2y+b_3z}{r^3}\right)$$

$$= -\left[\frac{b_1r^3-\left(b_1x+b_2y+b_3z\right)\frac{\partial}{\partial x}r^3}{r^6}\right]$$

$$= -\left[\frac{b_1r^3-\left(b_1x+b_2y+b_3z\right)3r^2\frac{\partial r}{\partial x}}{r^6}\right]$$

$$= -\left[\frac{b_1r^3-\left(\vec{b}.\vec{r}\right)3r^2\frac{x}{r}}{r^6}\right] = \left(-\frac{b_1r^2+3\left(\vec{b}.\vec{r}\right)x}{r^5}\right)$$

Similarly, $\frac{\partial\phi}{\partial y} = -\frac{b_2r^2+3\left(\vec{b}.\vec{r}\right)y}{r^5}$

and $\frac{\partial\phi}{\partial z} = -\frac{\left(b_3r^2+3\left(\vec{b}.\vec{r}\right)y\right)}{r^5}$

$$\nabla\phi = \left(\vec{i}\frac{\partial\phi}{\partial x}+\vec{j}\frac{\partial\phi}{\partial y}+\vec{k}\frac{\partial\phi}{\partial z}\right)$$

$$= \frac{-\left(b_1\vec{i}+b_2\vec{j}+b_3\vec{k}\right)}{r^3}+\frac{3\left(\vec{b}.\vec{r}\right)\left(x\vec{i}+y\vec{j}+z\vec{k}\right)}{r^5}$$

$$= \frac{-\vec{b}}{r^3}+\frac{3\left(\vec{b}.\vec{r}\right)\vec{r}}{r^5}$$

$$\therefore \vec{a}.\nabla\phi = \vec{a}.\nabla\left(\vec{b}.\nabla\frac{1}{r}\right) = -\frac{\vec{a}.\vec{b}}{r^3}+\frac{3\left(\vec{a}.\vec{r}\right)\left(\vec{b}.\vec{r}\right)}{r^5}$$

Hence, $\vec{a}.\nabla\left\{\vec{b}.\nabla\left(\frac{1}{r}\right)\right\} = \frac{3\left(\vec{a}.\vec{r}\right)\left(\vec{b}.\vec{r}\right)}{r^5}-\frac{\vec{a}.\vec{b}}{r^3}$

Important IOE Questions and Answers

1. Show that $(\vec{b}\times\vec{c})\times(\vec{c}\times\vec{a})=[\vec{a}\,\vec{b}\,\vec{c}]\vec{c}$ and deduce $[\vec{b}\times\vec{c}\ \ \vec{c}\times\vec{a}\ \ \vec{a}\times\vec{b}]=[\vec{a}\,\vec{b}\,\vec{c}]^2$

(BE 2069 Poush, 2068, 2070 Magh)

Sol^n. For the first part

Let $\vec{b}\times\vec{c}=\vec{u}$ then, $(\vec{b}\times\vec{c})\times(\vec{c}\times\vec{a})=\vec{u}\times(\vec{c}\times\vec{a})$

$=(\vec{u}\times\vec{a})\vec{c}-(\vec{u}\times\vec{c})\vec{a}=(\vec{b}\times\vec{c}.\vec{a})\vec{c}-(\vec{b}\times\vec{c}.\vec{c})\vec{a}=[\vec{b}\,\vec{c}\,\vec{a}]\vec{c}-0$

$=[\vec{a}\,\vec{b}\,\vec{c}]\vec{c}$ Proved.

Second part

$[\vec{b}\times\vec{c}\ \ \vec{c}\times\vec{a}\ \ \vec{a}\times\vec{b}]$

$=(\vec{b}\times\vec{c})\times(\vec{c}\times\vec{a}).(\vec{a}\times\vec{b})=[\vec{a}\,\vec{b}\,\vec{c}]\vec{c}.(\vec{a}\times\vec{b})=[\vec{a}\,\vec{b}\,\vec{c}][\vec{c}\,\vec{a}\,\vec{b}]$

$\therefore [\vec{b}\times\vec{c}\ \ \vec{c}\times\vec{a}\ \ \vec{a}\times\vec{b}]=[\vec{a}\,\vec{b}\,\vec{c}][\vec{a}\,\vec{b}\,\vec{c}]=[\vec{a}\,\vec{b}\,\vec{c}]^2$

2. Prove that the necessary and sufficient condition for the function $\vec{a}$ of scalar variable to have a constant direction is $\vec{a}\times\frac{d\vec{a}}{dt}=0$

(BE 2069 Poush, 2068 Magh, 2070 Magh)

Sol^n. Necessary condition

Let $\vec{a}=a\,.\,\tilde{a}$

Where a denotes the magnitude of $\vec{a}$ and $\tilde{a}$ is a unit vector whose magnitude is unity for all values of t along the direction of $\vec{a}$.

Differentiating (i) w. r. t. 't'

$\frac{d\vec{a}}{dt}+\frac{da}{dt}\tilde{a}$

Pre-multiplying both sides vectorically by,

$\vec{a}\times\frac{d\vec{a}}{dt}=\vec{a}\times\left(a\frac{d\tilde{a}}{dt}+\frac{da}{dt}\tilde{a}\right)=a\tilde{a}\times\left(a\frac{d\tilde{a}}{dt}+\frac{da}{dt}\tilde{a}\right)$ $\quad(\because \vec{a}=a\tilde{a})$

$=a^2\tilde{a}\times\frac{d\tilde{a}}{dt}+a\frac{da}{dt}(\tilde{a}\times\tilde{a})$ or, $\vec{a}\times\frac{d\vec{a}}{dt}=a^2\tilde{a}\times\frac{d\tilde{a}}{dt}$

Since $\tilde{a}$ is constant vector $\frac{d\tilde{a}}{dt}=0$

$\therefore \vec{a}\times\frac{d\vec{a}}{dt}=0$ Hence the necessary condition.

Sufficient condition

Let $\vec{a}\times\frac{d\vec{a}}{dt}=0$ we have to prove that $\vec{a}$ has constant direction.

Now $\vec{a}\times\frac{d\tilde{a}}{dt}=0$ or, $a\tilde{a}\times\frac{d(a\tilde{a})}{dt}=0$ or, $a^2\tilde{a}\times\frac{d\tilde{a}}{dt}=0$

$\therefore \tilde{a}\times\frac{d\tilde{a}}{dt}=0$ ..... (i)

Again $\tilde{a}$ is a vector function of constant magnitude and hence

$\tilde{a}\,.\,\frac{d\tilde{a}}{dt}=0$ ..... (ii)

From (i) and (ii) we have,

$\frac{d\tilde{a}}{dt}=0$. Thus $\tilde{a}$ to have constant direction.

Find the angle between the surface $x^2+y^2+z^2=9$ and $z=x^2+y^2-3$ at the point (2, – 1, 2)

(BE 2069 Poush)

Sol^n. We know that angle between two surface $x^2+y^2+z^2=9$ and $z=x^2+y^2-3$ is the angle between their normals at (2, – 1, 2)

Let $\phi_1=x^2+y^2+z^2-9=0$

$\phi_2(x)=z-x^2-y^2+3=0$

Normal to the surface ϕ_1 is,

$$\text{grad}\,\phi_1=\nabla\phi_1=\left(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}\right)\phi_1=\vec{i}\frac{\partial\phi_1}{\partial x}+\vec{j}\frac{\partial\phi_1}{\partial y}+\vec{k}\frac{\partial\phi_1}{\partial z}$$

$$=\frac{\vec{i}\partial(x^2+y^2+z^2-9)}{\partial x}+\frac{\vec{j}\partial(x^2+y^2+z^2-9)}{\partial y}+\frac{\vec{k}\partial(x^2+y^2+z^2-9)}{\partial z}$$

$=2x\vec{i}+2y\vec{j}+2z\vec{k}$

Let $\tilde{n}_1$ = unit normal to the surface ϕ, at (2, – 1, 2)

$$=\frac{4\vec{i}-2\vec{j}+4\vec{k}}{\sqrt{16+4+16}}=\frac{1}{3}\left(2\vec{i}-2\vec{j}+2\vec{k}\right)$$

Normal to the surface ϕ_2 is,

$$\text{Grad}\,\phi_2=\nabla\phi_2=\left(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}\right)\phi_2=\vec{i}\frac{\partial\phi_2}{\partial x}+\vec{j}\frac{\partial\phi_2}{\partial y}+\vec{k}\frac{\partial\phi_2}{\partial z}$$

$$=\frac{\vec{i}\partial(z-x^2-y^2+3)}{\partial x}+\frac{\vec{j}\partial(z-x^2-y^2+3)}{\partial y}+\frac{\vec{k}\partial(z-x^2-y^2+3)}{\partial z}=-2x\vec{i}-2y\vec{j}+\vec{k}$$

$\tilde{n}_2$ = Unit normal to the surface ϕ_2 at (2, – 1, 2)

$$=\frac{-4\vec{i}+2\vec{j}+\vec{k}}{\sqrt{16+4+1}}=\frac{1}{\sqrt{21}}\left(-4\vec{i}+2\vec{j}+\vec{k}\right)$$

If θ be angle between their normals, then $\cos\theta=\tilde{n}_1.\tilde{n}_2$

$$=\frac{1}{3}\left(2\vec{i}-2\vec{j}+2\vec{k}\right).\frac{1}{\sqrt{21}}\left(-4\vec{i}+2\vec{j}+\vec{k}\right)=-\frac{8}{3\sqrt{21}}$$

$\therefore \cos\theta = -\dfrac{8}{3\sqrt{21}}$ $\quad \therefore \theta = \cos^{-1}\left(\dfrac{8}{3\sqrt{21}}\right)$ Ans.

4. Prove that the necessary and sufficient condition for the vector function of a scalar variable t have a constant magnitude is $\vec{a}\cdot\dfrac{d\vec{a}}{dt} = 0$.

(BE 2069 Bhadra, 2067 Mangsir, 2070 Bhadra)

Sol^n. Necessary condition

Let a vector function $\vec{a}$ of scalar variable be of constant magnitude

i.e. $|\vec{a}| = a$ (constant)

$|\vec{a}|^2 = a^2 \Rightarrow \vec{a}\cdot\vec{a} = a^2$

Differentiating w. r. t 't'

$\dfrac{d}{dt}(\vec{a}\cdot\vec{a}) = 0$

$\vec{a}\cdot\dfrac{d\vec{a}}{dt} + \dfrac{d\vec{a}}{dt}\cdot\vec{a} = 0$ or, $2\vec{a}\cdot\dfrac{d\vec{a}}{dt} = 0$ $\quad \therefore \vec{a}\cdot\dfrac{d\vec{a}}{dt} = 0$ Hence the necessary condition.

It is also shows that of $\vec{a}$ is vector function of constant magnitude, then $\vec{a}$ and $\dfrac{d\vec{a}}{dt}$ are perpendicular.

Sufficient condition

Let $\vec{a}\cdot\dfrac{d\vec{a}}{dt} = 0$, we have to show that $\vec{a}$ is of constant magnitude.

We have $\vec{a}\cdot\vec{a} = |\vec{a}|^2$

Differentiating w.r.t. 't'

$\dfrac{d}{dt}(\vec{a}\cdot\vec{a}) = \dfrac{d}{dt}|\vec{a}|^2$

or, $\vec{a}\cdot\dfrac{d\vec{a}}{dt} + \dfrac{d\vec{a}}{dt}\cdot\vec{a} = 2|\vec{a}|\dfrac{d|\vec{a}|}{dt} \Rightarrow \vec{a}\cdot\dfrac{d\vec{a}}{dt} = |\vec{a}|\dfrac{d|\vec{a}|}{dt}$

or, $|\vec{a}|\dfrac{d|\vec{a}|}{dt} = 0$, But $|\vec{a}| \neq 0$ $\therefore \dfrac{d|\vec{a}|}{dt} = 0$

Hence, $|\vec{a}|$ = constant. i.e. $\vec{a}$ is of constant magnitude. Proved.

5. Find the divergence and curl of $\vec{F}$ where $\vec{F}$ = grad $(x^3 + y^3 + z^3 - 3xyz)$

(BE 2068 Magh)

Sol^n. Given, F = grad $(x^3 + y^3 + z^3 - 3xyz)$

$\nabla(x^3 + y^3 + z^3 - 3xyz)$

$= \left(\vec{i}\dfrac{\partial}{\partial x} + \vec{j}\dfrac{\partial}{\partial y} + \vec{k}\dfrac{\partial}{\partial z}\right)\cdot\{(3x^2 - 3yz)\vec{i} + (3y^2 - 3xz)\vec{j} + (3z^2 - 3xy)\vec{k}\}$

$= \dfrac{\partial(3x^2 - 3yz)}{\partial x} + \dfrac{\partial}{\partial y}(3y^2 - 3xz) + \dfrac{\partial(3z^2 - 3xy)}{\partial z}$

$= 6x + 6y + 6z = 6(x + y + z)$ Ans.

Curl $\vec{F} = \nabla\times\vec{F}$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 3x^2 - 3yz & 3y^2 - 3xy & 3z^2 - 3xy \end{vmatrix}$$

$= \vec{i}(-3x + 3x) + \vec{j}(-3y + 3y) + \vec{k}(-3z + 3z) = 0$

$\therefore$ Cur $\vec{F} = 0$ Ans.

Prove that $\vec{b}\times\vec{c}, \vec{c}\times\vec{a}$ and $\vec{a}\times\vec{b}$ are coplanar or non-coplanar according as $\vec{a}, \vec{b}, \vec{c}$ are coplanar or non-coplanar.

(BE 2068 Bhadra)

Sol^n. We have, $\left[\vec{b}\times\vec{c}\ \ \vec{c}\times\vec{a}\ \ \vec{a}\times\vec{b}\right] = \vec{b}\times\vec{c}\cdot\{(\vec{c}\times\vec{a})\times(\vec{a}\times\vec{b})\}$

Let $\vec{c}\times\vec{a} = \vec{u}$

$= (\vec{b}\times\vec{c})\cdot\{\vec{u}\times(\vec{a}\times\vec{b})\} = (\vec{b}\times\vec{c})\cdot\{(\vec{u}\times\vec{b})\vec{a} - (\vec{c}\times\vec{a}\cdot\vec{a})\vec{b}\}$

$= (\vec{b}\times\vec{c})\cdot\{(\vec{c}\times\vec{a}\cdot\vec{b})\vec{a} - (\vec{c}\times\vec{a}\cdot\vec{a})\vec{b}\} = (\vec{b}\times\vec{c})\cdot\{(\vec{a}\ \vec{b}\ \vec{c})\vec{a} - 0\}$

$= [\vec{a}\ \vec{b}\ \vec{c}]\,(\vec{b}\times\vec{c}\cdot\vec{a}) = [\vec{a}\ \vec{b}\ \vec{c}]\,(\vec{b}\ \vec{c}\ \vec{a}) = [\vec{a}\ \vec{b}\ \vec{c}][\vec{a}\ \vec{b}\ \vec{c}]$

$= [\vec{a}\ \vec{b}\ \vec{c}]^2$ $\quad \therefore [\vec{b}\times\vec{c}\ \ \vec{a}\times\vec{a}\ \ \vec{a}\times\vec{b}] = [\vec{a}\ \vec{b}\ \vec{c}]^2$

If $\vec{a}\ \vec{b}\ \vec{c}$ are coplaner, these $(\vec{a}\ \vec{b}\ \vec{c}) = 0$ and hence $[\vec{b}\times\vec{c}\ \ \vec{a}\times\vec{a}\ \ \vec{a}\times\vec{b}] = 0$

$\therefore$ The vector $\vec{b}\times\vec{c}\ \ \vec{c}\times\vec{a}\ \ \vec{a}\times\vec{b}$ are complaner.

If $\vec{a}\ \vec{b}\ \vec{c}$ are non coplaner

Then $(\vec{a}\ \vec{b}\ \vec{c}) \neq 0$

$\therefore$ The vectors $\vec{b}\times\vec{c}\ \ \vec{c}\times\vec{a}\ \ \vec{a}\times\vec{b}$ are non complanar.

Prove that curl $(\vec{a}\times\vec{b}) = \vec{a}$ div $\vec{b} - (\vec{a}\cdot\nabla)\vec{b}$

(BE 2068 Bhadra, Poush)

Sol^n. We have $(\vec{a}\times\vec{b}) = \nabla\times(\vec{a}\times\vec{b}) = \sum\vec{i}\times\dfrac{\partial}{\partial x}(\vec{a}\times\vec{b})$

$= \vec{i}\times\left(\vec{a}\times\dfrac{\partial\vec{b}}{\partial x} + \dfrac{\partial\vec{a}}{\partial x}\times\vec{b}\right) = \sum\vec{i}\times\left(\vec{a}\times\dfrac{\partial\vec{b}}{\partial x}\right) + \sum\vec{i}\times\left(\dfrac{\partial\vec{a}}{\partial x}\times\vec{b}\right)$

$= \sum\left\{\left(\vec{i}\cdot\dfrac{\partial\vec{b}}{\partial x}\right)\vec{a} - (\vec{i}\cdot\vec{a})\dfrac{\partial\vec{b}}{\partial x}\right\} + \sum\left\{(\vec{i}\cdot\vec{b})\dfrac{\partial\vec{a}}{\partial x} - \left(\vec{i}\cdot\dfrac{\partial\vec{a}}{\partial x}\right)\vec{b}\right\}$

Infinite Series

6

Infinite sequence

The sum of the terms of an infinite sequence {an} is an expression of the from $a_1 + a_2 + a_3 + \ldots\ldots a_n + \ldots\ldots$

Which is called an infinite series and it is denoted by

$$\sum_{n=1}^{\infty} a_n \text{ or } \Sigma a_n.$$

Convergent and divergent and infinite series

An infinite series $\sum_{n=1}^{\infty} u_n$ is said to be convergent, divergent or oscillatory according as its sequence $\{s_n\}$ of partial sum $s_1, s_2, s_3 \ldots\ldots s_n$ converges, diverges or oscillates.

1. If s_n tends to finite limit as $n \to \infty$, then the series Σu_n is convergent.
2. If s_n tends to ∞ as $n \to \infty$, then the series Σu_n is divergent.
3. If s_n doesn't tends to a unique limit as $n \to \infty$, then the series Σu_n is oscillatory.

Exercise - 26

Determine the following series and convergents or divergent if it converges find its sum.

1. $$3+\frac{3}{(-4)}+\frac{3}{(-4)^2}+\ldots\ldots+\frac{3}{(-4)^{n-1}}+\ldots\ldots$$

Solution: Let $S_n = 3+\frac{3}{(-4)}+\frac{3}{(-4)^2}+\ldots\ldots+\frac{3}{(-4)^{n-1}}+\ldots\ldots$

$$= \frac{3\left[1-\left(-\frac{1}{4}\right)^n\right]}{1+\frac{1}{4}} \quad \left[\text{In a G. S., } S_n = \frac{a\left(1-r^n\right)}{1-r}\right]$$

$$= 3 \times \frac{4}{5}\left[1-(-1)^n 4^{-n}\right]$$

$$= \frac{12}{5}\left[1-4^{-n}\right] \text{ if n is even}$$

$$\text{and } \frac{12}{5}\left[1+4^{-n}\right] \text{ if n is odd}$$

In both cases, $\lim_{n\to\infty} S_n = \frac{12}{5}$ which is finite

Hence given series is convergent. [As $n \to \infty$, $4^{-n} \to 0$]

2. $$\frac{1}{4.5}+\frac{1}{5.6}+\frac{1}{6.7}+\ldots\ldots+\frac{1}{(n+3)(n+4)}+\ldots\ldots$$

Solution: The nth partial sum of the given series is,

$$S_n = \frac{1}{4}-\frac{1}{5}+\frac{1}{5}-\frac{1}{6}+\ldots\ldots\frac{1}{n+3}-\frac{1}{n+4}$$

$$= \frac{1}{4}-\frac{1}{n+4}$$

$$\therefore \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left[\frac{1}{4}-\frac{1}{n+4}\right]$$

$$= \frac{1}{4}-\lim_{n\to\infty}\frac{1}{n+4} = \frac{1}{4}-\frac{1}{\infty+4} = \frac{1}{4}-\frac{1}{\infty} = \frac{1}{4}-0$$

$$\lim_{n\to\infty} S_n = \frac{1}{4} \text{ which is finite.}$$

Hence, the given series is convergent and its sum is $\frac{1}{4}$.

3. $$\frac{5}{1.2}+\frac{5}{2.3}+\frac{5}{3.4}+\ldots\ldots+\frac{5}{n(n+1)}+\ldots\ldots$$

Solution: The nth partial sum of given series is,

$$S_n = 5\left[\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\ldots\ldots+\frac{1}{n(n+1)}\right]$$

$$= 5\left[\frac{1}{1}-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\frac{1}{3}-\frac{1}{4}+\ldots\ldots\frac{1}{n}-\frac{1}{n+1}\right]$$

$$= 5\left[1-\frac{1}{n+1}\right] = \frac{5(n+1-1)}{n+1} = \frac{5n}{n+1}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\frac{5n}{n+1} = \lim_{n\to\infty}\frac{5n}{n\left(1+\frac{1}{n}\right)}$$

$$= \lim_{n\to\infty}\left(\frac{5}{1+\frac{1}{n}}\right) = \frac{5}{1+\frac{1}{\infty}} = 5$$

Hence the given series is convergent and its sum is 5.

4. $3+\frac{3}{2}+\frac{3}{3}+\ldots\ldots\ldots\ldots\frac{3}{n}+\ldots\ldots\ldots\ldots$

Solution: The given series may be written as

$$3\left[1+\frac{1}{2}+\frac{1}{3}+\ldots\ldots\ldots+\frac{1}{n}+\ldots\ldots\ldots\ldots\right]$$

Now, consider the series

$$= 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\ldots\ldots\ldots\ldots$$

$$= 1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\left(\frac{1}{9}+\frac{1}{10}+\ldots\ldots\frac{1}{16}\right)+\ldots\ldots$$

$$> 1+\frac{1}{2}+\left(\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right)+\left(\frac{1}{16}+\frac{1}{16}+\ldots\ldots\frac{1}{16}\right)+\ldots\ldots$$

$$= 1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\ldots\ldots\ldots\ldots$$

n^{th} term of the series $\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\ldots\ldots\ldots\ldots$ is $\frac{n}{2}$

Which tends to ∞ as n tends to ∞. So, the series $1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\ldots\ldots$ is divergent and hence the given series is divergent.

5. $\frac{3.1}{4}+\frac{3.2}{9}+\frac{3.3}{14}+\ldots\ldots\ldots\ldots+\frac{3n}{5n-1}+\ldots\ldots\ldots\ldots$

Solution: Here, n^{th} term

$$U_n = \frac{3n}{5n-1}$$

$$\therefore \lim_{n\to\infty} U_n = \lim_{n\to\infty}\frac{3n}{5n-1}$$

$$= \lim_{n\to\infty}\frac{3n}{n\left(5-\frac{1}{n}\right)}$$

$$= \lim_{n\to\infty}\frac{3}{\left(5-\frac{1}{n}\right)}$$

$$= \frac{3}{5} \neq 0$$

If a series is convergent $\lim_{n\to\infty} U_n = 0$

Hence, the given series is divergent.

6. $\Sigma\left(\frac{1}{8^n}+\frac{1}{n(n+1)}\right)$

Solution: For the given series

$$S_n = \frac{1}{8}+\frac{1}{1.2}+\frac{1}{8^2}+\frac{1}{2.3}+\frac{1}{8^3}+\frac{1}{3.4}+\ldots\ldots+\frac{1}{8^n}+\frac{1}{n(n-1)}$$

$$= \frac{1}{8}+\frac{1}{8^2}+\frac{1}{8^3}+\ldots\ldots+\frac{1}{8^n}+\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\ldots\ldots+\frac{1}{n(n-1)}$$

$$= \frac{\frac{1}{8}\left(1-\frac{1}{8^n}\right)}{1-\frac{1}{8}}+1-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\frac{1}{3}-\frac{1}{4}+\ldots\ldots+\frac{1}{n-1}-\frac{1}{n}$$

$$S_n = \frac{\frac{1}{8}\left(1-\frac{1}{8^n}\right)}{\frac{7}{8}}+1-\frac{1}{n}$$

$$\therefore \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left[\frac{\frac{1}{8}\left(1-\frac{1}{8^n}\right)}{\frac{7}{8}}+1-\frac{1}{n}\right]$$

$$\therefore \lim_{n\to\infty} S_n = \frac{\frac{1}{8}(1-0)}{\frac{7}{8}}+1-0 = \frac{1}{7}+1 = \frac{8}{7}$$

Hence, the given series is convergent and its sum is $\frac{8}{7}$

7. $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots\ldots\ldots\ldots+\frac{1}{n}+\ldots\ldots\ldots$

Solution: $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots\ldots\ldots\ldots+\frac{1}{n}$

$$1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\left(\frac{1}{9}+\frac{1}{10}+\ldots\ldots+\frac{1}{16}+\ldots\ldots\right)$$

$$> 1+\frac{1}{2}+\left(\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right)+\left(\frac{1}{16}+\frac{1}{16}+\ldots\ldots\frac{1}{16}\right)+\ldots\ldots$$

$$= 1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\ldots\ldots$$

n^{th} term of the series $\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\ldots\ldots$ is $\frac{n}{2}$

Which tends to ∞ as n tends to ∞. So, the series $1 + \frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+.....$ is divergent and hence the given series is divergent.

8. $\frac{1}{3}+\frac{2}{5}+\frac{3}{7}+.....$

Solution: Hence, n^{th} term

$$U_n = \frac{n}{3+(n-1)2} = \frac{n}{2n+1}$$

$$\therefore \lim_{n\to\infty} U_n = \lim_{n\to\infty} \frac{n}{2n+1}$$

$$= \lim_{n\to\infty} \frac{n}{n\left(2+\frac{1}{n}\right)} = \lim_{n\to\infty} \frac{1}{2+1} = \frac{1}{2} \neq 0$$

If a series is convergent $\lim_{n\to\infty} U_n = 0$

Hence, the given series is divergent.

10. Show that $\Sigma\left(1+\frac{1}{2n}\right)^n$ is divergent.

Solution: Here, the nth term

$$U_n = \left(1+\frac{1}{2n}\right)^n$$

$$\therefore \lim_{n\to\infty} U_n = \lim_{n\to\infty} \left(1+\frac{1}{2n}\right)^n$$

Let $2n = z$ then as $n \to \infty$, $z \to \infty$ and

$$\lim_{n\to\infty} U_n = \lim_{z\to\infty} \left(1+\frac{1}{z}\right)^{\frac{z}{2}} = \lim_{z\to\infty} \left[\left(1+\frac{1}{z}\right)^{z}\right]^{\frac{1}{2}}$$

$= e^{\frac{1}{2}} \neq 0$ If a series in convergent $\lim_{n\to\infty} n = 0$

Hence, the given series is divergent.

Exercise - 27

Determine whether the following series are convergent or divergent.

1. $1+\frac{1}{2}+\frac{1}{5}+\frac{1}{10}+.....\frac{1}{(n^2+1)}+.....$

Solution: The given series is,

$$\Sigma U_n = \Sigma \frac{1}{n^2+1}$$

Let $\Sigma V_n = \Sigma \frac{1}{n^2}$

$$\therefore \lim_{n\to\infty} \frac{U_n}{V_n} = \lim_{n\to\infty} \frac{\frac{1}{n^2+1}}{\frac{1}{n^2}}$$

$$= \lim_{n\to\infty} \frac{n^2}{n^2+1}$$

$$= \lim_{n\to\infty} \frac{n^2}{n^2\left(1+\frac{1}{n^2}\right)}$$

$$= \lim_{n\to\infty} \left(\frac{1}{1+\frac{1}{n^2}}\right)$$

= 1 which is finite and non-zero.

Hence, ΣU_n and ΣV_n are both divergent or both convergent.

As $\Sigma \frac{1}{n^p}$ is convergent for $P > 1$, ΣV_n is convergent

Hence, the given series convergent.

2. $\frac{1}{1.2.3}+\frac{3}{2.3.4}+\frac{5}{3.4.5}+.....$ **[2062]**

Solution: The given series is,

$$\Sigma U_n = \Sigma \frac{2n-1}{n(n+1)(n+2)}$$

Let, $\Sigma V_n = \Sigma \frac{1}{n^2}$

Now, $\lim_{n\to\infty} \frac{U_n}{V_n} = \lim_{n\to\infty} \frac{\frac{2n-1}{n(n+1)(n+2)}}{\frac{1}{n^2}}$

$$= \lim_{n\to\infty} \frac{n^2(2n-1)}{n(n+1)(n+2)}$$

$$= \lim_{n\to\infty} \frac{n^3\left(2-\frac{1}{n}\right)}{n^3\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}$$

$$= \lim_{n\to\infty} \frac{\left(2-\frac{1}{n}\right)}{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}$$

$= 2$ which is finite and non-zero.

Hence, ΣU_n and ΣV_n are both divergent or both convergent.

As $\Sigma \frac{1}{n^p}$ is convergent for $P > 1$, ΣV_n is convergent

Hence, the given series convergent.

3. $\frac{2}{1^p}+\frac{3}{2^p}+\frac{4}{3^p}+\frac{5}{4^p}+ \ldots\ldots\ldots$ [2062]

Solution: The given series is,

$$\Sigma U_n = \Sigma \frac{n+1}{n^p}$$

Let, $\Sigma V_n = \Sigma \frac{1}{n^{p-1}}$

So, $\lim_{n\to\infty} \frac{U_n}{V_n} = \lim_{n\to\infty} \frac{\frac{n+1}{n^p}}{\frac{1}{n^{p-1}}}$

$$= \lim_{n\to\infty} \frac{n^{p-1}(n+1)}{n^p}$$

$$= \lim_{n\to\infty} \frac{n^p\left(1+\frac{1}{n}\right)}{n^p}$$

$$= \lim_{n\to\infty} \left(1+\frac{1}{n}\right)$$

$= 1$ which is finite and non-zero.

Hence, ΣU_n and ΣV_n are both divergent or both convergent.

As $\Sigma \frac{1}{n^p}$ is convergent for $P > 1$, ΣV_n and divergent for $p \le 1$, ΣV_n is convergent for $p - 1 > 1$ or $p > 2$ and divergent for $p - 1 \le 1$ or $p \le 2$

Hence, the series is convergent for $p > 2$ and divergent for $p \le 2$.

4. $\frac{1}{1.2}+\frac{2}{3.4}+\frac{3}{5.6}+ \ldots\ldots\ldots$

Solution: The given series is $\Sigma U_n = \Sigma \frac{n}{(2n-1).2n} = \Sigma \frac{1}{4n-2}$

Let, $\Sigma V_n = \Sigma \frac{1}{n}$

So, $\lim_{n\to\infty} \frac{U_n}{V_n} = \lim_{n\to\infty} \frac{\frac{1}{4n-2}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{4n-2} = \lim_{n\to\infty} \frac{n}{n\left(4-\frac{2}{n}\right)}$

$$= \lim_{n\to\infty} \left(\frac{1}{4-\frac{2}{n}}\right)$$

$= \frac{1}{4}$ which is finite and non-zero.

Hence, ΣU_n and ΣV_n are both divergent or both convergent.

As $\Sigma \frac{1}{n} = \Sigma V_n$ is divergent, the given series is divergent.

5. $\Sigma \frac{\sqrt{n}}{n^2+1}$

Solution: The given series is $\Sigma U_n = \Sigma \frac{\sqrt{n}}{n^2+1}$

Let, $\Sigma V_n = \Sigma \frac{n^{\frac{1}{2}}}{n^2} = \Sigma \frac{1}{n^{\frac{3}{2}}}$

$\therefore \lim_{n\to\infty} \frac{U_n}{V_n} = \lim_{n\to\infty} \frac{\frac{\sqrt{n}}{n^2+1}}{\frac{1}{n^{\frac{3}{2}}}}$

$$= \lim_{n\to\infty} \frac{n^{\frac{1}{2}}}{n^2+1} \cdot \frac{n^{\frac{3}{2}}}{1} = \lim_{n\to\infty} \frac{n^2}{n^2+1} = \lim_{n\to\infty} \frac{n^2}{n^2\left(1+\frac{1}{n^2}\right)}$$

$$= \lim_{n\to\infty}\left(\frac{1}{1+\frac{1}{n^2}}\right)$$

$= 1$ which is finite and non-zero.

Hence, $\sum U_n$ and $\sum V_n$ are both divergent and both convergent.

As, $\sum \frac{1}{n^p}$ is convergent for $p > 1$.

$\sum V_n$ is convergent.

Hence, the given series is convergent.

6. $\sum\left(\sqrt[3]{n^3+1-n}\right)$ **[2064]**

Solution: $\sum U_n = \sum\left(\sqrt[3]{n^3+1}-n\right)$

$$\text{But, } \sqrt[3]{n^3+1}-n = \frac{\left(\sqrt[3]{n^3+1}-n\right)\left[\left(n^3+1\right)^{\frac{2}{3}}+\left(n^3+1\right)^{\frac{1}{3}}.n+n^2\right]}{\left(n^3+1\right)^{\frac{2}{3}}+\left(n^3+1\right)^{\frac{1}{3}}.n+n^2}$$

$$= \frac{n^3+1-n^3}{\left(n^3+1\right)^{\frac{2}{3}}+\left(n^3+1\right)^{\frac{1}{3}}.n+n^2}$$

$$\therefore \sum U_n = \frac{1}{\left(n^3+1\right)^{\frac{2}{3}}+\left(n^3+1\right)^{\frac{1}{3}}.n+n^2}$$

Let, $\sum V_n = \sum \frac{1}{n^2}$

$$\text{So, } \lim_{n\to\infty}\frac{U_n}{V_n} = \lim_{n\to\infty}\frac{1}{\left(n^3+1\right)^{\frac{2}{3}}+\left(n^3+1\right)^{\frac{1}{3}}.n+n^2}.n^2$$

$$= \lim_{n\to\infty}\frac{n^2}{\left(n^3\right)^{\frac{2}{3}}\left(1+\frac{1}{n^3}\right)^{\frac{2}{3}}+n^2\left(1+\frac{1}{n^3}\right)^{\frac{1}{3}}+n^2}$$

$$= \lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n^3}\right)^{\frac{2}{3}}+\left(1+\frac{1}{n^3}\right)^{\frac{1}{3}}+1}$$

$= \frac{1}{3}$ which is finite and non-zero.

Hence, $\sum U_n$ and $\sum V_n$ are both divergent and covergent.

As, $\sum \frac{1}{n^p}$ is convergent for $p > 1$, $\sum V_n$ is convergent and hence the given series is convergent.

7. $\sum \frac{n}{1+n\sqrt{n+1}}$

Solution: The given series is,

$$\sum U_n = \sum \frac{n}{1+n\sqrt{n^2+1}}$$

$$\text{Let, } \sum V_n = \sum \frac{1}{n^{\frac{3}{2}}-1} = \sum \frac{1}{n^{\frac{1}{2}}}$$

$$\text{So, } \lim_{n\to\infty}\frac{U_n}{V_n} = \lim_{n\to\infty}\frac{\frac{n}{1+n\sqrt{n+1}}}{\frac{1}{n^{\frac{1}{2}}}}$$

$$= \lim_{n\to\infty}\frac{n}{1+n\sqrt{n+1}}.\frac{n^{\frac{1}{2}}}{1}$$

$$= \lim_{n\to\infty}\frac{n^{\frac{3}{2}}}{n^{\frac{3}{2}}\left(\frac{1}{n^{\frac{3}{2}}}+\sqrt{1+\frac{1}{n}}\right)} = \lim_{n\to\infty}\frac{1}{\frac{1}{n^{\frac{3}{2}}}+\sqrt{1+\frac{1}{n}}}$$

$$= \frac{1}{0+\sqrt{1+0}}$$

$= 1$ which is finite and non-zero

Hence $\sum U_n$ and $\sum V_n$ are both divergent and convergent,

As, $\sum \frac{1}{n^p}$ is divergent for $p < 1$. $\sum V_n$ is divergent, the given series is divergent.

8. $\sum \sqrt{n^4-1}-n^2$ **[2058]**

Solution: The given series is,

$$\sum U_n = \sum\left(\sqrt{n^4-1}-n^2\right)$$

But, $\sqrt{n^4-1}-n^2 = \dfrac{\left(\sqrt{n^4-1}-n^2\right)\left(\sqrt{n^4-1}+n^2\right)}{\sqrt{n^4-1}+n^2}$

$$= \frac{n^4-1-n^4}{\sqrt{n^4-1}+n^2}$$

$$\therefore\ \Sigma U_n = \frac{-1}{\sqrt{n^4-1}+n^2}$$

Let, $\Sigma U_n = \dfrac{1}{n^2}$

So, $\lim\limits_{n\to\infty} \dfrac{U_n}{V_n} = \lim\limits_{n\to\infty} \dfrac{-1}{\sqrt{n^4-1}+n^2} \cdot n^2$

$$= \lim_{n\to\infty} \frac{-n^2}{n^2\left(\sqrt{1-\frac{1}{n^4}}+1\right)}$$

$= -\dfrac{1}{2}$ which is finite and non-zero.

Hence, ΣU_n and ΣV_n are both divergent and both convergent.

As, $\Sigma \dfrac{1}{n^p}$ is convergent for $n > 1$. ΣV_n is convergent and hence, the given series is convergent.

9. $\displaystyle\sum_{n=0}^{\infty} \frac{2n^3+5}{4n^5+1}$

Solution: The given series is,

$$\Sigma U_n = \Sigma \frac{2n^3+5}{4n^5+1}$$

Let, $\Sigma V_n = \Sigma \dfrac{1}{n^2}$

$$\therefore\ \lim_{n\to\infty} \frac{U_n}{V_n} = \lim_{n\to\infty} \frac{\frac{2n^3+5}{4n^5+1}}{\frac{1}{n^2}}$$

$$= \lim_{n\to\infty} \frac{n^2\left(2n^3+5\right)}{4n^5+1} = \lim_{n\to\infty} \frac{n^5\left(2+\frac{5}{n^3}\right)}{n^5\left(4+\frac{1}{n^5}\right)}$$

$= \dfrac{2}{4}$ which is finite and non-zero.

Hence, ΣU_n and ΣV_n are both divergent or both convergent.

As $\Sigma \dfrac{1}{n^p}$ is convergent for $p > 1$

ΣV_n is convergent.

Hence, the given series is convergents.

10. $\Sigma\left\{\sqrt{n+1}-\sqrt{n}\right\}$

Solution: The given series is,

$$\Sigma U_n = \Sigma\left(\sqrt{n+1}-\sqrt{n}\right)$$

But, $V_n = \sqrt{n+1}-\sqrt{n}$

$$= \left(\sqrt{n+1}-\sqrt{n}\right) \times \frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+n}$$

$$= \frac{n+1-n}{\sqrt{n+1}+\sqrt{n}} = \frac{1}{\sqrt{n+1}+\sqrt{n}}$$

Let, $\Sigma V_n = \Sigma \dfrac{1}{n^{\frac{1}{2}}}$

So, $\lim\limits_{n\to\infty} \dfrac{U_n}{V_n} = \lim\limits_{n\to\infty} \dfrac{1}{\sqrt{n+1}+\sqrt{n}} \times n^{\frac{1}{2}}$

$$= \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n}\left(\sqrt{1+\frac{1}{n}}+1\right)}$$

$= \dfrac{1}{2}$ which is finite and non-zero.

Hence, ΣU_n and ΣV_n are both divergent or both convergent

As $\Sigma \dfrac{1}{n^p}$ is divergent for $p < 1$, ΣV_n is divergent.

Hence, the given series divergent.

Exercise - 28

Test the following series for convergence by applying ratio test.

1. $x + 2x^2 + 3x^3 + 4x^4 + \ldots\ldots\ldots\ldots, x>0$

Solution: Here, $U_n = n\,x^n$

$$\therefore \lim_{n\to\infty} \frac{U_{n+1}}{U_n}$$

$$= \lim_{n\to\infty} \frac{(n+1)\,x^{n+1}}{nx^n}$$

$$= \lim_{n\to\infty} \frac{n+1}{n}\,x$$

$$= \lim_{n\to\infty} \left(1+\frac{1}{n}\right)x = \left(1+\frac{1}{\infty}\right)x = (1+0)\,x = x$$

Hence, the series is convergent for $x < 1$ and divergent for $x > 1$.

If $x = 1$, the series

$= 1 + 2 + 3 + 4 + \ldots\ldots\ldots\ldots$

Which is divergent.

Hence, the series is conergent of $x < 1$ and divergent of $x \geq 1$.

2. $1 + 3x + 5x^2 + 7x^3 + \ldots\ldots\ldots\ldots, x > 0$

Solution: Here, $U_n = (n^{th}$ term of $1, 3, 5, 7, \ldots\ldots\ldots)\,x^{n-1}$

$= [1 + (n-1)\,2]\,x^{n-1}$

$= (2n-1)\,x^{n-1}$

$$\therefore \lim_{n\to\infty} \frac{U_{n+1}}{U_n}$$

$$= \lim_{n\to\infty} \frac{[2(n+1)-1]\,x^{n+1-1}}{(2n-1)\,x^{n-1}}$$

$$= \lim_{n\to\infty} \frac{(2n+2-1)\,x^n}{(2n-1)\,x^{n-1}}$$

$$= \lim_{n\to\infty} \frac{2n+1}{2n-1}\,x$$

$$= \lim_{n\to\infty} \frac{n\left(2+\frac{1}{n}\right)}{n\left(2-\frac{1}{n}\right)}\,x$$

$$= \frac{2+0}{2+0}\,x = x$$

Hence, the series is convergent for $x < 1$ and divergent for $x > 1$

If $x = 1$ the series

$1 + 3 + 5 + 7 + \ldots\ldots\ldots\ldots$

For which n^{th} particle sum

$S_n = 1 + 3 + 5 + 7 + \ldots\ldots\ldots\ldots + (2n-1)$

$$= \frac{n}{2}[2\,.\,1 + (n-1)\,2]$$

$$= \frac{n}{2}\,[2 + 2n - 2]$$

$$= \frac{n}{2}\,.\,2n$$

$$= n^2$$

$$\therefore \lim_{n\to\infty} S_n = \lim_{n\to\infty} n^2 = \infty$$

i.e. the series $1 + 3 + 5 + 7 + 1 \ldots\ldots\ldots$ is divergent

Hence, the series is convergent $x < 1\ \infty$

and divergent of $x \geq 1$.

3. $1 + \frac{2^p}{2!} + \frac{3^p}{3!} + \frac{4^p}{4!} + \ldots\ldots\ldots\ldots, x > 0$

Solution: Here, $U_n = \frac{n^p}{n!}$

$$\therefore \lim_{n\to\infty} \frac{U_{n+1}}{U_n} = \lim_{n\to\infty} \frac{\frac{(n+1)^p}{(n+1)!}}{\frac{n^p}{n!}}$$

$$= \lim_{n\to\infty} \frac{(n+1)^p}{(n+1)!} \times \frac{n!}{n^p} = \lim_{n\to\infty} \frac{(n+1)^p n!}{(n+1)n!\,n^p} = \lim_{n\to\infty} \frac{(n+1)^p}{(n+1)n^p}$$

$$= \lim_{n\to\infty} \frac{n^p\left(1+\frac{1}{n}\right)^p}{(n+1)n^p} = \lim_{n\to\infty} \frac{\left(1+\frac{1}{n}\right)^p}{(n+1)}$$

$$= \frac{\left(1+\frac{1}{\infty}\right)^p}{(\infty+1)} = \frac{(1+0)}{\infty} = \frac{1}{\infty} = 0 < 1$$

Hence the series is convergent for all values of P.

4. $2x + \frac{3x^2}{8} + \frac{4x^2}{27} + \ldots\ldots\ldots\ldots\ x > 0$

Solution: Here, $U_n = \frac{(n+1)}{n^3}\,x^n$

and $U_{n+1} = \frac{n+2}{(n+1)^3} \cdot x^{n+1}$

Now, $\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = \lim_{n\to\infty} \frac{(n+2)}{(n+1)^3} x^{n+1} \cdot \frac{n^3}{(n+1)x^n}$

$$= x \lim_{n\to\infty} \frac{(n+2)\, n^3}{(n+1)^3 (n+1)}$$

$$= x \lim_{n\to\infty} \frac{\left(1+\frac{2}{n}\right)}{\left(n+\frac{1}{n}\right)^3 \left(n+\frac{1}{n}\right)}$$

$$= x$$

$\therefore$ ΣU_n converges for $x < 1$ and diverges for $x > 1$.

When $x = 1$

$\therefore 2 + \frac{3}{8} + \frac{4}{27} + \ldots\ldots + \frac{n+1}{n^3} + \ldots\ldots\ldots\ldots$

The general term of the series is

$$U_n = \frac{n+1}{n^3}$$

Now, $\lim_{n\to\infty} \frac{U_n}{V_n} = \lim_{n\to\infty} \frac{(n+1)}{n^3} \times n^2$

$$= \lim_{n\to\infty} \left(1+\frac{1}{n}\right)$$

$= 1$, a finite quantity.

But we know that $\Sigma V_n = \sum \frac{1}{n^2}$, with $p > 1$, is convergent.

Hence, the series ΣU_n is convergent for $x \le 1$ and divergent for $x > 1$.

5. $\mathbf{x + \frac{3}{5}x^2 + \frac{8}{10}x^3 + \frac{15}{17}x^4 + \ldots\ldots + \frac{n^2-1}{n^2+1}x^2 + \ldots\ldots x > 0}$

[BE 2059, 2070 Magh]

Solution: Here, $U_n = \frac{n^2-1}{n^2+1} x^n$

$$= \lim_{n\to\infty} \frac{U_{n+1}}{U_n} = \lim_{n\to\infty} \frac{\frac{(n+1)^2-1}{(n+1)^2+1} x^{n+1}}{\frac{n^2-1}{(n^2+1)} x^n}$$

$$= \lim_{n\to\infty} \frac{(n+1)^2-1}{(n+1)^2+2} \cdot \frac{n^2+1}{n^2-1} x$$

$$= \lim_{n\to\infty} \frac{n^2\left[\left(1+\frac{1}{n}\right)^2 - \frac{1}{n^2}\right] . n^2\left(1+\frac{1}{n^2}\right)}{n^2\left[\left(1+\frac{1}{n^2}\right)^2 + \frac{1}{n^2}\right] . n^2\left(1-\frac{1}{n^2}\right)} x$$

$= x$

Hence, the series is convergent for $x < 1$ and divergent for $x > 1$.

If $x = 1$, the series

$$= 1 + \frac{3}{5} + \frac{8}{10} + \frac{15}{17} + \ldots\ldots + \frac{n^2-1}{n^2+1}$$

Its nth term $t_n = \frac{n^2-1}{n^2+1}$

$$\therefore \lim_{n\to\infty} \frac{n^2-1}{n^2+1} = \lim_{n\to\infty} \frac{n^2\left(1+\frac{1}{n^2}\right)}{n^2\left(1+\frac{1}{n^2}\right)}$$

$$= \lim_{n\to\infty} \frac{\left(1-\frac{1}{n^2}\right)}{\left(1+\frac{1}{n^2}\right)} = 1$$

Which is not zero. Hence the series is divergent for $x = 1$.

Hence, the series is covergent for $x < 1$ and divergent for $x \ge 1$.

6. $\mathbf{\frac{1^2}{3} + \frac{2^2}{3^2} + \frac{3^2}{3^3} + \ldots\ldots\ldots\ldots\ldots\ldots}$

Solution: Here, $U_n = \frac{n^2}{3^n}$

$$\therefore \lim_{n\to\infty} \frac{U_{n+1}}{U_n} = \lim_{n\to\infty} \frac{\frac{(n+1)^2}{3^{n+1}}}{\frac{n^2}{3^n}}$$

$$= \lim_{n\to\infty} \frac{(n+1)^2}{3^{n+1}} \cdot \frac{3^n}{n^2}$$

$$= \lim_{n\to\infty} \frac{n^2\left(1+\frac{1}{n}\right)^2}{3n^2}$$

$$= \frac{1}{3} < 1$$

Hence, the series is convergent.

7. $\frac{x}{1.2}+\frac{x^2}{2.3}+\frac{x^3}{3.4}+\ldots\ldots\ldots\ldots, x>0$ **(BE 2069 Bhadra)**

Solution: Here, $U_n = \frac{x^n}{n(n+1)}$

$$\therefore \lim_{n\to\infty} \frac{U_{n+1}}{U_n} = \lim_{n\to\infty} \frac{\frac{x^{n+1}}{(n+1)(n+2)}}{\frac{x^n}{n(n+1)}}$$

$$= \lim_{n\to\infty} \frac{x^{n+1}}{(n+1)(n+2)} \cdot \frac{n(n+1)}{x^n}$$

$$= \lim_{n\to\infty} \frac{n(n+1)}{(n+1)(n+2)} x$$

$$= \lim_{n\to\infty} \frac{n^2\left(1+\frac{1}{n}\right)}{n^2\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)} x$$

$$= x$$

Hence, the series is convergent for $x < 1$ and divergent for $x > 1$

If $x = 1$, the series,

$$\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\ldots\ldots\ldots\ldots$$

The nth partial sum of the given series is,

$$S_n = \frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\ldots\ldots\ldots+\frac{1}{n(n+1)}$$

$$= \frac{1}{1}-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\ldots\ldots+\frac{1}{n}-\frac{1}{n+1}$$

$$= 1-\frac{1}{n+1}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1-\frac{1}{n+1}\right)$$

$= 1 - 0 = 1$ which is finite.

Hence, the series is convergent for $x = 1$

Hence, the series is convergent for $x \le 1$ and divergent for $x > 1$.

8. $1+\frac{2!}{2^2}+\frac{3!}{3^3}+\frac{4!}{4^4}+\ldots\ldots\ldots\ldots$

Solution: Here, $U_n = \frac{n!}{n^n}$

$$\therefore \lim_{n\to\infty} \frac{U_{n+1}}{U_n} = \lim_{n\to\infty} \frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}}$$

$$= \lim_{n\to\infty} \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!}$$

$$= \lim_{n\to\infty} \frac{(n+1)\,n!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!}$$

$$= \lim_{n\to\infty} \frac{(n+1)}{(n+1)^{n+1}} \cdot n^n$$

$$= \lim_{n\to\infty} \frac{n^n}{(n+1)^n} = \lim_{n\to\infty} \frac{n^n}{n^n\left[1+\frac{1}{n}\right]^n}$$

$$= \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n}$$

$$= \frac{1}{e}$$

$$= \frac{1}{2.71828} < 1$$

Hence the series is convergent.

9. $3+\frac{3^2}{2!}+\frac{3^3}{3!}+\frac{3^4}{4!}+\ldots\ldots\ldots\ldots$

Solution: Here, $U_n = \frac{3^n}{n!}$

$$\therefore \lim_{n\to\infty} \frac{U_{n+1}}{U_n} = \lim_{n\to\infty} \frac{\frac{3^{n+1}}{(n+1)!}}{\frac{3^n}{n!}}$$

$$= \lim_{n\to\infty} \frac{3^{n+1}}{(n+1)!} \cdot \frac{n!}{3^n}$$

$$= \lim_{n\to\infty} \frac{3^{n+1}.\,n!}{(n+1)\,n!\,3^n}$$

$$= \lim_{n\to\infty} \frac{3}{n+1} = \frac{3}{\infty+1} = \frac{3}{\infty} = 0 < 1$$

Hence, the series is convergent.

10. $1 + \frac{2^2}{2!} + \frac{3^3}{3!} + \frac{4^4}{4!} + \ldots\ldots + \frac{n^n}{n!} + \ldots\ldots\ldots$

Solution: Here, $U_n = \frac{n^n}{n!}$

$$\lim_{n\to\infty} \frac{\frac{(n+1)^{n+1}}{(n+1)\,!}}{\frac{n^n}{n!}}$$

$$= \lim_{n\to\infty} \frac{(n+1)^{n+1}}{(n+1)\,!} \cdot \frac{n!}{n^n}$$

$$= \lim_{n\to\infty} \frac{(n+1)^{n+1}}{(n+1)\,n!} \frac{n!}{n^n}$$

$$= \lim_{n\to\infty} \frac{(n+1)^{n+1}}{(n+1)n^n}$$

$$= \lim_{n\to\infty} \frac{(n+1)^n}{n^n}$$

$$= \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n$$

$$= \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n$$

$= e = 2.71808 \ldots\ldots > 1$

Hence, the series is divergent.

Test the following series for Convergence by applying root test.

12, $x + \frac{3}{5}x^2 + \frac{8}{10}x^3 + \frac{15}{17}x^4 + \ldots\ldots + \frac{n^2-1}{n^2+1}x^n + \ldots\ldots\ldots\ldots$

Solution: Here, $U_n = \frac{n^2-1}{n^2+1}x^n$

$$\therefore \lim_{n\to\infty} \sqrt[n]{U_n} = \lim_{n\to\infty} \sqrt[n]{\frac{n^2-1}{n^2+1}\,x^n}$$

$$= \lim_{n\to\infty} \sqrt[n]{\frac{n^2-1}{n^2+1}}\,x$$

$$= \lim_{n\to\infty} \left(\frac{n^2-1}{n^2+1}\right)^{\frac{1}{n}} x$$

$$= \frac{(1-0)^0}{(1+0)}\,x = x$$

So, the given series is convergent is $x < 1$ and divergent if $x > 1$.

If, $x = 1$, the series

$$= 1 + \frac{3}{5} + \frac{8}{10} + \frac{15}{17} + \ldots\ldots\ldots\ldots + \frac{n^2+1}{n^2+1}$$

It's n^{th} term $t_n = \frac{n^2-1}{n^2+1}$

$$\lim_{n\to\infty} \frac{n^2-1}{n^2+1} = \lim_{n\to\infty} \frac{n^2\left(1-\frac{1}{n^2}\right)}{n^2\left(1+\frac{1}{n^2}\right)}$$

$$= \lim_{n\to\infty} \frac{1-\frac{1}{n^2}}{1+\frac{1}{n^2}}$$

$$= 1$$

Which is not zero. Hence, the series is divergent for $x = 1$.

Hence, the series is convergent for $x < 1$ and divergent for $x \geq 1$

13. $\sum\left(1+\frac{1}{n}\right)^{-n^2}$

Solution: $U_n = \left(1+\frac{1}{n}\right)^{-n^2}$

$$\therefore \lim_{n\to\infty} \sqrt[n]{U_n} = \lim_{n\to\infty} \left[\left(1+\frac{1}{n}\right)^{-n^2}\right]^{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{-n}$$

$$= \lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)^n\right]^{-1} = \lim_{n\to\infty}\left[\frac{n^2\left(1-\frac{1}{n^2}\right)}{n^2\left(1+\frac{1}{n^2}\right)}\right]^{\frac{1}{n}} x$$

$$= e^{-1} = \frac{1}{2.71828} < 1$$

Hence, the series is convergent.

Thus the series is convergent for $x < 1$ and divergent for $x \geq 1$

14. $\sum\left(\frac{nx}{1+n}\right)^n$

Solution: Here, $U_n = \left(\frac{nx}{1+n}\right)^n$

$$\therefore \lim_{n\to\infty}\sqrt[n]{U_n} = \lim_{n\to\infty}\left[\left(\frac{nx}{1+n}\right)^n\right]^{\frac{1}{n}}$$

$$= \lim_{n\to\infty}\frac{nx}{1+n} = \lim_{n\to\infty}\frac{nx}{n\left(1+\frac{1}{n}\right)} = \lim_{n\to\infty}\frac{x}{\frac{1}{n}+1}$$

$$= x$$

Hence, the series is convergent if $x < 1$ and divergent if $x > 1$.

If $x = 1$. The series

$$\sum\left(\frac{n}{1+n}\right)^n$$

Here the nth terms

$$U_n = \left(\frac{n}{1+n}\right)^n$$

$$\therefore \lim_{n\to\infty} U_n = \lim_{n\to\infty}\left(\frac{n}{1+n}\right)^n$$

$$= \lim_{n\to\infty}\frac{n^n}{(1+n)^n} = \lim_{n\to\infty}\frac{n^n}{n^n\left[\frac{1}{n}+1\right]^n} = \lim_{n\to\infty}\frac{1}{\left(\frac{1}{n}+1\right)^n}$$

$$= \left(\frac{1}{\infty}+1\right)^n = (0+1)^n = 1^n = 1 \neq 0$$

If the series is couvergent $\lim_{n\to\infty} U_n = 0$ hence the series is divergent.

thus the given series is convergent for $x < 1$ and divergent for $x \geq 1$.

15. $\sum\frac{1}{n^n}$

Solution: Here, $U_n = \frac{1}{n^n} = n^{-n}$

$$\therefore \lim_{n\to\infty}\sqrt[n]{U_n} = \lim_{n\to\infty}\left(n^{-n}\right)^{\frac{1}{n}}$$

$$= \lim_{n\to\infty} n^{-1}$$

$$= \lim_{n\to\infty}\frac{1}{n} = \frac{1}{\infty}\,0 < 1$$

Hence, the series is convergent.

16. $\sum\frac{1}{(\log n)^n}$

Solution: Here, $U_n = \frac{1}{(\log n)^n} = (\log n)^{-n}$

$$\therefore \lim_{n\to\infty}\sqrt[n]{U_n} = \lim_{n\to\infty}\left[(\log n)^{-n}\right]^{\frac{1}{n}}$$

$$= \lim_{n\to\infty}(\log n)^{-1}$$

$$= \lim_{n\to\infty}\frac{1}{\log n} = \frac{1}{\infty} = 0 < 1$$

Hence the series is convergent.

Exercise - 30

Apply integral test to test the convergence of the series.

1. $\sum_{n=1}^{\infty} \frac{1}{n^2+n}$

Solution: Here, $f(n) = \frac{1}{n^2+n}$

$\therefore \quad f(x) = \frac{1}{x^2+x}$

Now, $\int^{\infty} f(x)\,dx = \int^{\infty} \frac{1}{x^2+x}\,dx$

$$= \lim_{m\to\infty} \int^{m} \frac{1}{x(x+1)}\,dx$$

$$= \lim_{m\to\infty} \int^{m} \left(\frac{1}{x} - \frac{1}{x+1}\right) dx$$

$$= \lim_{m\to\infty} \left[\log x - \log(x+1)\right]_1^m$$

$$= \lim_{m\to\infty} \left[\log \frac{x}{x+1}\right]_1^m$$

$$= \lim_{m\to\infty} \left[\log \frac{m}{m+1} - \log\frac{1}{2}\right]$$

$$= \lim_{m\to\infty} \left[\log \frac{m}{m\left(1+\frac{1}{m}\right)} - \log\frac{1}{2}\right]$$

$$= \lim_{m\to\infty} \left[\log \frac{1}{1+\frac{1}{m}} - \log\frac{1}{2}\right]$$

$= \log 1 - \log(2^{-1})$

$= \log 2$ which is finite.

Hence the given series is convergent.

2. $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

Solution: $f(n) = \frac{1}{n^2+1}$

$\therefore \quad f(x) = \frac{1}{x^2+1}$

Now, $\int^{\infty} f(x)\,dx = \int^{\infty} \frac{1}{x^2+1}\,dx$

$$= \lim_{m\to\infty} \int^{m} \frac{1}{x^2+1}\,dx$$

$$= \lim_{m\to\infty} \left[\tan^{-1} x\right]_1^m$$

$$= \lim_{m\to\infty} \left[\tan^{-1} m - \tan^{-1} 1\right]$$

$$= \frac{\pi}{2} - \frac{\pi}{4}$$

$= \frac{\pi}{4}$ which is finite.

Hence the given series is convergent.

3. $\sum_{n=1}^{\infty} \frac{1}{(3+2n)^2}$

Solution: Here, $f(n) = \frac{1}{(3+3n)^2}$

$\therefore f(x) = \frac{1}{(3+3x)^2}$

Now, $\int^{\infty} f(x)\,dx = \int^{\infty} \frac{1}{(3+2x)^2}\,dx$

$$= \lim_{m\to\infty} \int^{m} \frac{1}{(3+2x)^2}\,dx$$

$$= \lim_{m\to\infty} \left[\frac{-1}{2(3+2x)}\right]_1^m$$

$$= \lim_{m\to\infty} \left[\frac{-1}{2(3+2m)} + \frac{1}{2(3+2)}\right]$$

$= \frac{1}{10}$ which is finite.

Hence, the given series is convergent.

4. $\sum_{n=1}^{\infty} \frac{1}{4n+7}$

Solution: Here, $f(n) = \frac{1}{4n+7}$

$\therefore f(x) = \dfrac{1}{4x+7}$

Now, $\int^{\infty} f(x)\,dx$

$$= \lim_{m\to\infty} \int^{m} \frac{1}{4x+7}\,dx$$

$$= \lim_{m\to\infty} \left[\frac{1}{4}\log\,(4x+7)\right]_1^m$$

$$= \lim_{m\to\infty} \left[\frac{1}{4}\log\,(4m+7) - \frac{1}{4}\log\,11\right]$$

$$= \frac{1}{4}\log\infty - \frac{1}{4}\log\,11$$

$$= \infty - \frac{1}{4}\log 11$$

$$= \infty$$

Hence, the given series is divergent.

5. $\displaystyle\sum_{n=1}^{\infty} \frac{\log n}{n}$

Solution: Here, $f(n) = \dfrac{\log n}{n}$

$\therefore f(x) = \dfrac{\log x}{x}$

Now, $\int^{\infty} f(x)\,dx = \int^{\infty} \dfrac{\log x}{x}\,dx$

$$= \lim_{m\to\infty} \int^{m} \frac{\log x}{x}\,dx$$

Putting $\log x = t$, $\dfrac{1}{x}\,dx = dt$

When $x = 1$, $t = \log 1 = 0$

When $x = m$, $t = \log m$

$$\therefore \int^{\infty} f(x)\,dx = \lim_{m\to\infty} \int_0^{\log m} t\,dt$$

$$= \lim_{m\to\infty} \left[\frac{1}{2}t^2\right]_0^{\log m}$$

$$= \lim_{m\to\infty} \left[\frac{1}{2}(\log m)^2 - 0\right]$$

$$= \frac{1}{2}(\log\infty)^2 - 0 = \infty$$

Hence, the given series is divergent.

6. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{2n+1}}$

Solution: Here, $f(n) = \dfrac{1}{\sqrt[3]{2n+1}}$

$\therefore f(x) = \dfrac{1}{\sqrt[3]{2x+1}}$

Now, $\int^{\infty} f(x)\,dx = \int^{\infty} \dfrac{1}{\sqrt[3]{2x+1}}\,dx$

$$= \lim_{m\to\infty} \int^{m} (2x+1)^{-\frac{1}{3}}\,dx$$

$$= \lim_{m\to\infty} \left[\frac{(2x+1)^{\frac{2}{3}}}{2\times\frac{2}{3}}\right]_1^m$$

$$= \lim_{m\to\infty} \left[\frac{3}{4}(2m+1)^{\frac{2}{3}} - \frac{3}{4}\,3^{\frac{2}{3}}\right]$$

$$= \infty$$

Hence, the given series is divergent.

7. $\sum n\,e^{-n^2}$

Solution: Here, $f(n) = n\,e^{-n^2}$

$\therefore f(x) = x\,e^{-x^2}$

Now, $\int^{\infty} f(x)\,dx = \int^{\infty} x\,e^{-x^2}\,dx$

$$= \lim_{m\to\infty} \int^{m} x\,e^{-x^2}\,dx$$

Let $x^2 = t$, so that $2xdx = dt$

When $x = 1$, $t = 1$, and when $x = m$, $t = m^2$

$$\therefore \int^{\infty} f(x)\,dx = \lim_{m\to\infty} \int^{m^2} \frac{1}{2}\,e^{-t}\,dt$$

$$= \lim_{m\to\infty} \left[-\frac{e^{-t}}{2}\right]_1^{m^2}$$

$$= \lim_{m\to\infty} \left[-\frac{1}{2}\,e^{-m^2} + \frac{1}{2}\,e^{-1}\right]$$

$$= -\frac{1}{2}e^{-\infty} + \frac{1}{2e}$$

$$= 0 + \frac{1}{2e}$$

$$= \frac{1}{2e} \text{ which is finite.}$$

Hence, the given series is convergent.

8. $\displaystyle\sum_{n=1}^{\infty} \frac{\tan^{-1} n}{n+n^2}$

Solution: Here $f(n) = \dfrac{\tan^{-1} n}{1+n^2}$

$$\therefore f(x) = \frac{\tan^{-1} x}{1+x^2}$$

Now, $\displaystyle\int^{\infty} f(x)\,dx = \int^{\infty} \frac{\tan^{-1} x}{1+x^2}\,dx$

$$= \lim_{m\to\infty} \int^{m} \frac{\tan^{-1} x}{1+x^2}\,dx$$

Put $\tan^{-1} x = t$, $\dfrac{1}{1+x^2}\,dx = dt$

When $x = 1$, $t = \dfrac{\pi}{4}$

When $x = m$, $t = \tan^{-1} m$

$$\therefore \int^{\infty} f(x)\,dx = \lim_{m\to\infty} \int_{\frac{\pi}{4}}^{\tan^{-1} m} t\,dt$$

$$= \lim_{m\to\infty} \left[\frac{t^2}{2}\right]_{\frac{\pi}{4}}^{\tan^{-1} m}$$

$$= \lim_{m\to\infty} \left[\frac{1}{2}\left(\tan^{-1} m\right)^2 - \frac{1}{2}\cdot\frac{\pi^2}{16}\right]$$

$$= \frac{1}{2}\left(\frac{\pi}{2}\right)^2 - \left(\frac{\pi^2}{32}\right)$$

$$= \frac{\pi^2}{8} - \frac{\pi^2}{32} \text{ which is finite.}$$

Hence, the given series is convergent.

9. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\sqrt[3]{\log n}}$

Solution: Here, $f(n) = \dfrac{1}{n\sqrt[3]{\log n}}$

$$\therefore f(x) = \frac{1}{x\sqrt[3]{\log x}}$$

Now, $\displaystyle\int^{\infty} f(x)\,dx = \int^{\infty} \frac{1}{x\sqrt[3]{\log x}}\,dx$

Putting $\log x = t$, $\dfrac{1}{x}\,dx = dt$

When $x = 1$, $t = 0$, when $x = m$, $t = \log m$

$$\therefore \int^{\infty} f(x)\,dx = \lim_{m\to\infty} \int_0^{\log m} \frac{1}{\sqrt[3]{t}}\,dt$$

$$= \lim_{m\to\infty} \int_0^{\log m} t^{-\frac{1}{3}}\,dt = \lim_{m\to\infty} \left[\frac{3}{2}t^{\frac{2}{3}}\right]_0^{\log m}$$

$$= \lim_{m\to\infty} \left[\frac{3}{2}(\log m)^{\frac{2}{3}} - 0\right]$$

$$= \frac{3}{2}(\log \infty)^{\frac{2}{3}} - 0$$

$$= \infty$$

Hence, the given series is divergent.

10. $\displaystyle\sum_{n=1}^{\infty} n\,e^{-n}$

Solution: Hence, $f(n) = n\,e^{-n}$

$$\therefore f(x) = x\,e^{-x}$$

Now, $\displaystyle\int^{\infty} f(x)\,dx = \int^{\infty} x\,e^{-x}\,dx = \lim_{m\to\infty} \int^{m} x\,e^{-x}\,dx$

$$= \lim_{m\to\infty} \left\{\left[x\left(-e^{-x}\right)_1^m - \int^m \left(1-e^{-x}\right)dx\right]\right\}$$

[integrating by parts]

$$= \lim_{m\to\infty} \left\{-m\,e^{-m} + e^{-1} - \left[e^{-x}\right]_1^m\right\}$$

$$= \lim_{m\to\infty} \{-me^{-m} + e^{-1} - e^{-m} + e^{-1}\}$$

$$= \frac{2}{e} \text{ which is finite.}$$

As, $\lim_{m\to\infty} me^{-m} = \lim_{m\to\infty} \frac{m}{e^m} = \lim_{m\to\infty} \frac{1}{e^m} = 0$

[Using < Hospital rue]

and $\lim_{m\to\infty} e^{-m} = 0$

Hence, the given series is convergent.

Determine each alternating series is convergent conditionally convergent or divergent.

11. $\sum_{n=1}^{\infty}(-1)^n \frac{1}{n^2}$

Solution: Here, $\lim_{n\to\infty} |U_n| = \lim_{n\to\infty} \frac{1}{n^2}$ which is convergent because

$\lim_{n\to\infty} \frac{1}{n^p}$ is convergent for $p > 1$.

Thus the given series is absolutely convergent and hence, it is convergent.

12. $\sum_{n=1}^{\infty}(-1)^n \frac{n^2+4}{2^n}$

Solution: Here, $\sum_{n=1}^{\infty}|U_n| = \sum_{n=1}^{\infty}\frac{n^2+4}{2^n}$

$$\text{Now, } \lim_{n\to\infty}\frac{|U_{n+1}|}{|U_n|} = \lim_{n\to\infty}\left|\frac{(n+1)^2+4}{2^{n+1}}\cdot\frac{2^n}{n^2+4}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(n+1)^2+4}{2(n^2+4)}\right|$$

$$= \lim_{n\to\infty}\left|\frac{n^2\left[\left(1+\frac{1}{n}\right)^2+\frac{4}{n^2}\right]}{2n^2\left(1+\frac{4}{n^2}\right)}\right|$$

$$= \left|\frac{1}{2}\right| = \frac{1}{2} < 1$$

$\therefore$ By De alembert's ratio test, the given series is absolutely convergent and hence it is convergent.

13. $\sum_{n=1}^{\infty}(-1)^n \frac{1}{\sqrt{2n+1}}$

Solution: Here, $\sum_{n=1}^{\infty}|U_n| = \sum_{n=1}^{\infty}\frac{1}{\sqrt{2n+1}}$

Now, Let $\sum_{n=1}^{\infty}V_n = \frac{1}{\sqrt{n}}$

$$\therefore \lim_{n\to\infty}\frac{|U_n|}{V_n} = \lim_{n\to\infty}\frac{\frac{1}{\sqrt{2n+1}}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty}\sqrt{\frac{n}{2n+1}}$$

$$= \lim_{n\to\infty}\sqrt{\frac{n}{n\left(2+\frac{1}{n}\right)}}$$

$$= \frac{1}{\sqrt{2}} \text{ which is finite and non-zero}$$

$\therefore \sum_{n=1}^{\infty}V_n = \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ is divergent because $\sum_{n=1}^{\infty}\frac{1}{n^p}$ is divergent for $p <$ 1.

By comparison test, the series is not absolutely convergent, further, the series is an alternative series and each term is numerically less than the preceding term.

$\therefore \lim_{n\to\infty}\frac{1}{\sqrt{2n+1}} = 0$

Hence, by Libinitz's rule, the given series is convergent. As the series is not absolutely convergent, the series is conditionally convergent.

14. $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{\log(n+1)}$

Solution: Here, $\sum_{n=1}^{\infty}|U_n| = \sum_{n=1}^{\infty}\frac{1}{\log(n+1)}$

$$\text{Now, } \lim_{n\to\infty}\frac{|U_{n+1}|}{|U_n|} = \lim_{n\to\infty}\left|\frac{\frac{1}{\log(n+2)}}{\frac{1}{\log(n+1)}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{\frac{1}{n+1}}{\frac{1}{n+2}}\right| \quad \text{[Apply L' Hospitals' rule]}$$

$$= \lim_{n\to\infty} \left|\frac{n+2}{n+1}\right| = \lim_{n\to\infty} \left|\frac{n\left(1+\frac{2}{n}\right)}{n\left(1+\frac{1}{n}\right)}\right| = |1|$$

$$= 1$$

The De Alemberis ratio test fails.

$$\lim_{n\to\infty} n\left[\frac{|U_n|}{|U_{n+1}|}-1\right] = \lim_{n\to\infty} n\left[\frac{\log(n+2)}{\log(n+1)}-1\right]$$

$$= \lim_{n\to\infty} n\left[\frac{\log(n+2)-\log(n+1)}{\log(n+1)}\right]$$

$$= \lim_{n\to\infty} \frac{n}{\log(n+1)} \cdot \log\frac{n+2}{n+1}$$

$$= \lim_{n\to\infty} \frac{n}{\log(n+1)} \cdot \log\left(1+\frac{1}{n+1}\right)$$

$$= \lim_{n\to\infty} \frac{n}{(n+1)\log(n+1)} \cdot \log\left(1+\frac{1}{n+1}\right)^{n+1}$$

$$= \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)\log(n+1)} \log\left(1+\frac{1}{n+1}\right)^{n+1}$$

$$= \frac{1}{(1+0)\log\infty}$$

$$= 0 < 1$$

So by Raabe's test

$\sum_{n=1}^{\infty}|U_n|$ is divergent, i.e., the given series is not absolutely convergent.

Further, each term is numerically less than the preceeding term and

$$\lim_{n\to\infty} U_n = \lim_{n\to\infty} \frac{1}{\log(n+1)} = 0$$

So, by leibinitz's test, teh series is convergent. Hence, the series is conditionally convergent.

15. $\sum_{n=1}^{\infty}(-1)^n \frac{n}{\log n}$

Solution: $\lim_{n\to\infty} \frac{n}{\log n}$ [Indeterminant form $\frac{\infty}{\infty}$]

$$= \lim_{n\to\infty} \frac{1}{\frac{1}{n}} \qquad \text{[L'Hospital rule]}$$

$$= \lim_{n\to\infty} n = \infty$$

Hence, the given series is divergent.

16. $\sum_{n=1}^{\infty}(-1)^n \frac{5}{n^3+1}$

Solution: Here, $\sum_{n=1}^{\infty}|U_n| = \sum_{n=1}^{\infty}\frac{5}{n^3+1}$

Now, $\lim_{n\to\infty} \frac{|U_{n+1}|}{|U_n|} = \lim_{n\to\infty} \left|\frac{5}{(n+1)^3+1} \cdot \frac{n^3+1}{5}\right|$

$$= \lim_{n\to\infty} \left|\frac{n^3+1}{(n+1)^3+1}\right| = \lim_{n\to\infty} \left|\frac{n^3\left(1+\frac{1}{n^3}\right)}{n^3\left[1+\left(\frac{1}{n}\right)^3+\frac{1}{n^3}\right]}\right| = 1$$

$\therefore$ The De Hember'ts ratio test fails.

To test the convergent, we use the Raabe's test.

$$\therefore \lim_{n\to\infty} n\left[\frac{|U_n|}{|U_{n+1}|}-1\right] = \lim_{n\to\infty} n\left[\frac{(n+1)^3+1}{n^3+1}-1\right]$$

$$= \lim_{n\to\infty} n\left[\frac{(n+1)^3+1-n^3-1}{n^3+1}\right] = \lim_{n\to\infty} n\left[\frac{3n^2+3n+1}{n^3+1}\right]$$

$$= \lim_{n\to\infty} \frac{3n^3+3n^2+n}{n^3+1} = \lim_{n\to\infty} \frac{x^3\left[3+\frac{3}{n}+\frac{1}{n^2}\right]}{x^3\left[1+\frac{1}{n^3}\right]}$$

$$= \lim_{n\to\infty} \left(\frac{3+\frac{3}{n}+\frac{1}{n^2}}{1+\frac{1}{n^3}}\right) = \frac{3+0+0}{1+0} = 3 > 1$$

Hence the series is convergent.

Exercise - 31

Find the interval of convergence of the power series:

1. $\sum_{n=1}^{\infty} \frac{2^n (x-3)^n}{n+3}$ **(BE 2068 Magh)**

Solution: Here, $u_n = \frac{2^n (x-3)^n}{n+3}$ and $u_{n+1} = \frac{2^{n+1}(x-3)^{n+1}}{n+4}$

$$\therefore \lim_{x\to\infty} \left|\frac{u_{n+1}}{u_n}\right|$$

$$= \left|\frac{2^{n+1}(x-3)^{n+1}}{n+4} \times \frac{n+3}{2^n (x-3)^n}\right| = |2(x-3|$$

So the series is convergent for

$|2(x-3)| < 1$

$|x-3| < \frac{1}{2}$ $\Rightarrow -\frac{1}{2} < x-3 < \frac{1}{2}$

$\Rightarrow \frac{5}{2} < x < \frac{7}{2}$ and divergent for $|2(x-3)| > 1$

At $x = \frac{5}{2}$

$$u_n = \frac{2^n\left(\frac{5}{2}-3\right)^n}{n+3} = \frac{2^n\left(\frac{-1}{2}\right)^n}{n+3} = \frac{(-1)^n}{n+3}$$

Which is alternative series with

$U_n = \frac{1}{n+3}$

$$\lim_{n\to\infty} U_n = \lim_{n\to\infty} \frac{1}{n+3} = \frac{1}{\infty+3} = \frac{1}{\infty} = 0$$

By Leibnit z's test

The series is convergent for $x = \frac{5}{2}$ at $x = \frac{7}{2}$;

$$U_n = \sum \frac{2^n\left(\frac{7}{2}-3\right)^n}{n+1} = \sum \frac{2^n\left(\frac{1}{2}\right)^n}{(n+3)} = \sum \frac{1}{(n+3)}$$

Taking $U_n = \frac{1}{n+3}$ and $u_n = \frac{1}{n}$

$$\therefore \lim_{n\to\infty} \frac{U_n}{V_n} = \lim_{n\to\infty} \frac{n}{n+3} = 1 \text{ finite}$$

By comparison test as $\sum v_n$ diverges for $n = \frac{7}{2}$

Which shows that the given series is convergent for

$\frac{5}{2} \le x < \frac{7}{2}$

Hence the interval of convergence is $\left[\frac{5}{2}, \frac{7}{2}\right]$

$$\sum_{n=1}^{\infty} \frac{nx^n}{5^n}$$

Solution: We have, $U_n = \frac{nx^n}{5^n}$, $U_{n+1} = \frac{(n+1)x^{n+1}}{5^{n+1}}$

$$\therefore \frac{U_{n+1}}{U_n} = \frac{(n+1)x^{n+1}}{5^{n+1}} \cdot \frac{5^n}{nx^n} = \frac{n+1}{n} \cdot \frac{x}{5}$$

$$\therefore \lim_{n\to\infty} \left|\frac{U_{n+1}}{U_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)x}{5n}\right| = \left|\frac{x}{5}\right|$$

So, the series is convergent for $\left|\frac{x}{5}\right| < 1$

i.e. $-1 < \frac{x}{5} < 1$

or, $-5 < x < 5$ and divergent for $\left|\frac{x}{5}\right| < 1$

For $x = 5$, the series is,

$\sum_{n=1}^{\infty} \frac{n5^n}{5^n} = \sum_{n=1}^{\infty} n = 1 + 2 - 3 + \ldots\ldots\ldots$ which is divergent.

For, $x = -5$, the series is

$\sum_{n=1}^{\infty} \frac{n(-5)^n}{5^n} = \sum_{n=1}^{\infty} (-1)^n\, n = -1 + 2 - 3 + 4 - \ldots\ldots\ldots$

Which is alternating series with $U_n = n$

So, $\lim_{n\to\infty} U_n \neq 0$. Hence, the series is not convergent for $x = -5$

Hence, the interval of convergence is $(-5, 5)$

$$\sum_{n=1}^{\infty} \frac{x^n}{n+4}$$

Solution: We have, $U_n = \frac{x^n}{n+4}$, $U_{n+1} = \frac{x^{n+1}}{n+5}$

So, $\frac{U_{n+1}}{U_n} = \frac{x^{n+1}}{n+5} \cdot \frac{n+4}{x^n} = \frac{n+4}{n+5} \cdot x$

$$\therefore \lim_{n\to\infty}\left|\frac{U_{n+1}}{U_n}\right| = \lim_{n\to\infty}\left|\frac{n+4}{n+5}\cdot x\right| = \lim_{n\to\infty}\left|\frac{1+\frac{4}{n}}{1+\frac{5}{n}}\cdot x\right| = |x|$$

So, the series is convergent for $|x| < 1$. i.e.,
$-1 < x < 1$ and divergent for $|x| > 1$

For $x = 1$, the series is $\sum_{n=1}^{\infty}\frac{1}{n+4}$

Taking, $U_n = \frac{1}{n+4}$ and $V_n = \frac{1}{n}$

$$\therefore \lim_{n\to\infty}\frac{U_n}{V_n} = \lim_{n\to\infty}\frac{n}{n+4} = 1 \text{ (finite)}$$

By comparison test as $\sum V_n$ diverges, the series diverges for $x = 1$.
For, $x = -1$, the series is

$\sum_{n=1}^{\infty}\frac{(-1)^n}{n+4}$ which is an alternative series.

With $U_n = \frac{1}{n+4}$, $\lim_{n\to\infty} U_n = 0$. By leibnitz's test the series is convergent for $x = -1$
Which shows that the given series is convergent for $-1 \le x < 1$.
Hence the interval of convergence is $[-1, 1)$.

4. $\sum_{n=1}^{\infty}\frac{n^2x^n}{2^n}$

Solution: We have, $U_n = \frac{n^2x^n}{2^n}$, $U_{n+1} = \frac{(n+1)^2\, x^{n+1}}{2^{n+1}}$

$$\text{So, } \frac{U_{n+1}}{U_n} = \frac{(n+1)^2\, x^{n+1}}{2^{n+1}}\cdot\frac{2^n}{n^2x^n} = \left(\frac{(n+1)^2}{n}\right)\cdot\frac{x}{2}$$

$$\therefore \lim_{n\to\infty}\left|\frac{U_{n+1}}{U_n}\right| = \lim_{n\to\infty}\left|\left(\frac{n+1}{n}\right)^2\cdot\frac{x}{2}\right| = \left|\frac{x}{2}\right|$$

So, the series is convergent for $\left|\frac{x}{2}\right| < 1$.

i.e., $-1 < \frac{x}{2} < 1$

or, $-2 < x < 2$ and divergent for $\left|\frac{x}{2}\right| > 1$.

For $x = 2$, the series is

$\sum_{n=1}^{\infty} n^2$ which is divergent as $n^2 \to \infty$ as $n \to \infty$

For $x = -2$, the series is

$\sum_{n=1}^{\infty}(-1)^n n^2$ which is alternative.

Series with $U_n = n^2$, $\lim_{n\to\infty} U_n \ne 0$

So, by Leibinitz's test, the series is not
Convergent for $x = -2$
Which shows that the given series convergent for $-2 < x < 2$.
Hence, the interval of convergence is $(-2, 2)$

5. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}x^n}{\sqrt{n}}$

Solution: We have, $U_n = \frac{(-1)^{n+1}x^n}{\sqrt{n}}$, $U_{n+1} = \frac{(-1)^{n+2}.x^{n+1}}{\sqrt{n+1}}$

$$\text{So, } \frac{U_{n+1}}{U_n} = \frac{(-1)^{n+2}.x^{n+1}}{\sqrt{n+1}}\cdot\frac{\sqrt{n}}{(-1)^{n+1}.x^n} = -\sqrt{\frac{n}{n+1}}.x$$

$$\therefore \lim_{n\to\infty}\left|\frac{U_{n+1}}{U_n}\right| = \lim_{n\to\infty}\left|-\sqrt{\frac{n}{n+1}}.x\right| = \lim_{n\to\infty}\left|\frac{1}{\sqrt{1+\frac{1}{n}}}.x\right|$$

$$= |x|$$

So, the series is convergent for $|x| < 1$
i.e., $-1 < x < 1$ and divergent for $|x| > 1$
For $x = 1$, the series is,

$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{\sqrt{n}}$ which is alternative series with $U_n = \frac{1}{\sqrt{n}}$

$\therefore \lim_{n\to\infty} U_n = 0$.

So, by Leibnitz's test, the series is convergent for $x = -1$
For $x = -1$

The series $U_n = \frac{(-1)^{n+1}.(1)^n}{\sqrt{n}} = \frac{(-1)^{2n+1}}{\sqrt{n}}$

BP test, U_n in divrgent.
Thus the interval of convergence is $(-1, 1)$

6. $x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}-\ldots\ldots\ldots\ldots$

Solution: We have, $U_n=\frac{(-1)^{n+1}.x^n}{n}$, $U_{n+1}=\frac{(-1)^{n+2}.x^{n+1}}{n+1}$

So, $\frac{U_{n+1}}{U_n}=\frac{(-1)^{n+2}.x^{n+1}}{n+1}\cdot\frac{n}{(-1)^{n+1}.x^n}=-\frac{n}{n+1}.x$

$\therefore \lim_{n\to\infty}\left|\frac{U_{n+1}}{U_n}\right|=\lim_{n\to\infty}\left|-\frac{n}{n+1}.x\right|=|x|$

So, the series is convergent for $|x|<1$,
i.e. $-1<x<1$ and divergent for $|x|>1$
For $x=1$, the series is

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots\ldots\ldots\ldots$$

This is an alternating series with $U_n=\frac{1}{n}$

$\lim_{n\to\infty} U_n=0$, **So, by leibnity's test**

The series is convergent
For $x=-1$, the series is

$$-1-\frac{(-1)^2}{2}+\frac{(-1)^3}{3}\ldots\ldots\ldots\ldots$$

or, $-1-\frac{1}{2}-\frac{1}{3}\ldots\ldots\ldots\ldots$

or, $-\left(1+\frac{1}{2}+\frac{1}{3}+\ldots\ldots\right)$ which is divergent because a p-series with p = 1 is divergent.
Which snows that the given series is convergent for $-1<x\le 1$.
Hence, the interval of the convergence is $(-1, 1]$

7. $x-\frac{x^2}{2^2}+\frac{x^3}{3^2}-\frac{x^4}{4^2}+\ldots\ldots\ldots\ldots$

Solution: We have, $U_n=\frac{(-1)^{n+1}.x^n}{n^2}$, $U_{n+1}=\frac{(-1)^{n+2}x^{n+1}}{(n+1)^2}$

So, $\frac{U_{n+1}}{U_n}=\frac{(-1)^{n+1}.x^n}{n^2}\cdot\frac{n^2.}{(-1)^{n+1}.x^n}=\left(\frac{n}{n+1}\right)^2.x$

$\therefore \lim_{n\to\infty}\left|\frac{U_{n+1}}{U_n}\right|=\lim_{n\to\infty}\left|-\left(\frac{n}{n+1}\right)^2.x\right|=|x|$

So, the series is convergent for $|x|<1$

i.e. $-1<x<1$ and divergent for $|x|>1$
For $x=1$, the series is

$$1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\ldots\ldots\ldots\ldots$$

This is alternating series with

$U_n=\frac{1}{n^2}$. $\lim_{n\to\infty} U_n=0$

So, by leibinitz test the series is convergent.
For $x=-1$, the series is

$$-1-\frac{(-1)^2}{2^2}+\frac{(-1)^3}{3^2}\ldots\ldots\ldots\ldots=-1-\frac{1}{2^2}-\frac{1}{3^2}\ldots\ldots\ldots\ldots$$

$$=-\left(1+\frac{1}{2^2}+\frac{1}{3^2}+\ldots\ldots\right)$$

Which is convergent as a p-series with $p=2>1$ is convergent.
Which shows that the given series is convergent for $-1\le x\le 1$.
Hence the interval of convergence is $[-1, 1]$

8. $\sum_{n=1}^{\infty}\frac{n(x+2)^n}{3^{n+1}}$

Solution: Here, the general term of the series is,

$U_n=\lim_{x\to\infty}\frac{n(x+2)^n}{3^{n+1}}$ and $U_{n+1}=\lim_{x\to\infty}\frac{(n+1)(x+2)^{n+1}}{3^{n+2}}$

$\therefore \lim_{x\to\infty}\left|\frac{u_{n+1}}{u_n}\right|$

$=\lim_{x\to\infty}\left|\frac{(n+1)(x+2)^{n+1}}{3^{n+2}}\cdot\frac{3^{n+1}}{n(n+2)^n}\right|=\left|\frac{x+2}{3}\right|$

The series converges for $\left|\frac{x+2}{3}\right|<1$ i.e. $|x+2|<3$

For the interval of the series
$|x+2|<3$
$\Rightarrow -3<x+2<3 \Rightarrow -5<x<1$

and divergent for $\left|\frac{x+2}{3}\right|>1$

At $x=-5$

$u_n=\frac{n(-5+2)^n}{3^{n+1}}=\frac{n(-3)^n}{3^n.3}=\frac{n(-1)^n}{3}$

Which is divergent.

At $x = 1$, $u_n = \dfrac{x(-1)^n}{3^{n+1}} = \dfrac{3}{n} = \dfrac{1}{3}\ \dfrac{1}{n^{-1}}$

by p-test, which is divergent

$\therefore$ Which shows that the given series is convergent for $-5 < x < 1$ thus the interval of convergence is, $(-5, 1)$

9. $\displaystyle\sum_{n=1}^{\infty} \frac{(x+2)^n}{3^n\, n}$

Solution: We have, $U_n = \dfrac{(x+2)^n}{3^n n}$ and $U_{n+1} = \dfrac{(x+2)^{n+1}}{3^{n+1}(n+1)}$

So, $\dfrac{U_{n+1}}{U_n} = \dfrac{(x+2)^{n+1}}{3^{n+1}(n+1)} \cdot \dfrac{3^n n}{(x+2)^n} = \dfrac{n}{n+1}\ \dfrac{x+2}{3}$

$\therefore \displaystyle\lim_{n\to\infty} \left|\frac{U_{n+1}}{U_n}\right| = \lim_{n\to\infty} \left|\frac{n}{n+1} \cdot \frac{x+2}{3}\right| = \left|\frac{x+2}{3}\right|$

So, the series is convergent for $\left|\dfrac{x+2}{3}\right| < 1$.

i.e. $-1 < \dfrac{x+2}{3} < 1$

or, $-3 < x+2 < 3$

or, $-5 < x < 1$

and divergent for $\left|\dfrac{x+2}{3}\right| > 1$.

For $x = -5$, the series is

$\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^n}{3^n n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ which is an alternative series with $U_n = \dfrac{1}{n}$

and $\displaystyle\lim_{n\to\infty} U_n = 0$

So, by leibinitz's test, the series is convergent for $x = -5$

For $x = 1$, the series is,

$= \displaystyle\sum_{n=1}^{\infty} \frac{3^n}{3^n . n} = \sum_{n=1}^{\infty} \frac{1}{n}$ which is a p-series with $p = 1$ is divergent.

Which shows that the given series is convergent for $-5 \le x < 1$.

Hence, the interval of convergence is $[-5, 1)$

10. $\displaystyle\sum_{n=1}^{\infty} \frac{nx^n}{(n+1)(n+2)}$ $(x > 0)$

Solution: We have, $U_n = \dfrac{nx^n}{(n+1)(n+2)}$ and $U_{n+1} = \dfrac{(n+1)\,x^{n+1}}{(n+2)(n+3)}$

So, $\dfrac{U_{n+1}}{U_n} = \dfrac{(n+1)\,x^{n+1}}{(n+2)(n+3)} \cdot \dfrac{(x+1)(x+2)}{nx^n}$

$\therefore \displaystyle\lim_{n\to\infty} \left|\frac{U_{n+1}}{U_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)\,x^{n+1}}{(n+2)(n+3)} \cdot \frac{(n+1)(n+2)}{nx^n}\right|$

$= \displaystyle\lim_{n\to\infty} \left|\frac{(n+1)^2(n+2)\,x}{n(n+2)(n+3)}\right|$

$= \displaystyle\lim_{n\to\infty} \left|\frac{n^3\left(1+\frac{1}{n}\right)^2\left(1+\frac{2}{n}\right)x}{n^3\left(1+\frac{2}{n}\right)\left(1+\frac{3}{n}\right)}\right| = |x|$

So, the series is covergent for $|x| < 1$.

i.e. $-< x < 1$ and divergent for $|x| > 1$

For $x = 1$, the series is,

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)}$ with $U_n = \dfrac{n}{(n+1)(n+2)}$

Take $U_n = \dfrac{1}{n}$ so that $\dfrac{U_n}{V_n} = \dfrac{n^2}{(n+1)(n+2)}$

$\therefore \displaystyle\lim_{n\to\infty} \frac{U_n}{V_n} = \lim_{n\to\infty} \frac{n^2}{(n+1)(n+2)}$

$= \displaystyle\lim_{n\to\infty} \frac{n^2}{n^2\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)} = 1$ (finite)

So, By comparison test,

$\displaystyle\sum_{n=1}^{\infty} \frac{n}{(n+1)(n+2)}$ is divergent as $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent. As $x > 0$, the series is convergent for the interval $(0, 1)$

11. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n (x-3)^n}{n+1}$ **(BE 2070 Bhadra)**

Solution: We have, $U_n = \dfrac{(-1)^n (x-3)^n}{n+1}$, $U_{n+1} = \dfrac{(-1)^{n+1}(x-3)^{n+1}}{n+2}$

So, $\dfrac{U_{n+1}}{U_n} = \dfrac{(-1)^{n+1}(x-3)^{n+1}}{n+2} \cdot \dfrac{n+1}{(-1)^n (x-3)^n} = -\dfrac{n+1\,(x-3)}{n+2}$

$$\therefore \lim_{n\to\infty}\left|\frac{U_{n+1}}{U_n}\right| = \lim_{n\to\infty}\left|-\frac{n+1}{n+2}(x-3)\right| = |x-3|$$

So, the series is convergent for $|x-3|<1$.

i.e., $-1 < x-3 < 1$

or, $2 < x < 4$ and divergent for $|x-3|>4$

For $x = 2$, the series is,

$$\sum_{n=1}^{\infty}\frac{(-1)^n(2-3)^n}{n+1} = \sum_{n=1}^{\infty}\frac{1}{n+1} \text{ which is divergent,}$$

Because if we take $U_n = \frac{1}{n+1}$ and $V_n = \frac{1}{n}$

We see that $\frac{U_n}{V_n} = \frac{n}{n+1}$ and thus $\lim_{n\to\infty}\frac{U_n}{V_n} = 1$

So, by comparision test,

as ΣV_n is divergent, ΣU_n is also divergent

For $x = 4$, the series is

$$\sum_{n=1}^{\infty}\frac{(-1)^n(4-3)^n}{n+1} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n+1} \text{ which is an alternative series with}$$

$U_n = \frac{1}{n+1}$

$\lim_{n\to\infty} U_n = 0$. So by Leibinitz test, the series is convergent for $x = 4$.

Which shows that the given series is convergent for $2 < x \le 4$. Hence interval of convergence is $(2, 4]$.

12. $$\sum_{n=1}^{\infty}\frac{(n+1)(x-4)^n}{10^n}$$

Solution: We have, $U_n = \frac{(n+1)(x-4)^n}{10^n}$, $U_{n+1} = \frac{(n+2)(x-4)^{n+1}}{10^{n+1}}$

$$\text{So, } \frac{U_{n+1}}{U_n} = \frac{(n+2)(x-4)^{n+1}}{10^{n+1}}\cdot\frac{10^n}{(n+1)(x-4)^n}$$

$$= \frac{n+2}{n+1}\cdot\frac{x-4}{10}$$

$$\therefore \lim_{n\to\infty}\left|\frac{U_{n+1}}{U_n}\right| = \lim_{n\to\infty}\left|\frac{n+2}{n+1}\cdot\frac{x-4}{10}\right| = \left|\frac{x-4}{10}\right|$$

So, the series is convergent for $\left|\frac{x-4}{10}\right| < 1$

i.e. $-1 < \left|\frac{x-4}{10}\right| < 1$

or, $-10 < x-4 < 10$

or, $-6 < x < 14$ and divergent for $\left|\frac{x-4}{10}\right| > 1$.

For $x = -6$, the series is,

$$\sum_{n=1}^{\infty}\frac{(n+1)(-6-4)^n}{10^n} = \sum_{n=1}^{\infty}(-1)^n(n+1) \text{ which is an alternative}$$

series with $U_n = n+1$

and thus $\lim_{n\to\infty} U_n \ne 0$. So by leibinitz test it is divergent.

For $x = 14$, the series is

$$\sum_{n=1}^{\infty}\frac{(n+1)(14-4)^n}{10^n} = \sum_{n=1}^{\infty}(n+1) \text{ which is also divergent. so, the}$$

given series is convergent for $-6 < x < 14$

Hence the interval of convergence is $(-6, 14)$

13. $$\sum_{n=1}^{\infty}\frac{3^{2n}(x-2)^n}{n+1}$$

Solution: We have, $U_n = \frac{3^{2n}(x-2)^n}{n+1}$

$$\text{and } U_{n+1} = \frac{3^{2n+2}(x-2)^{n+1}}{n+2}\cdot\frac{n+1}{3^{2n}(x-2)^n} = 9\,\frac{n+1}{n+2}(x-2)$$

$$\text{So, } \lim_{n\to\infty}\left|\frac{U_{n+1}}{U_n}\right| = \lim_{n\to\infty}\left|9\frac{n+1}{n+2}\cdot(x-2)\right| = |9(x-2)|$$

So, the series is convergent for $|9(x-2)| < 1$

or, $-1 < 9x-18 < 1$

or, $17 < 9x < 19$

or, $\frac{17}{9} < x < \frac{19}{9}$ and divergent for $|9(x-2)| > 1$.

For $x = \frac{17}{9}$, the series is

$$\sum_{n=1}^{\infty}\frac{3^{2n}\left(\frac{17}{9}-2\right)^n}{n+1} = \sum_{n=1}^{\infty}\frac{3^{2n}\left(-\frac{1}{9}\right)^n}{n+1} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n+1}$$

Which is an alternative series with $U_n = \frac{1}{n+1}$

and thus $\lim_{n\to\infty} U_n = 0$.

So by Leibnitz's test it is convergent.

$$\sum_{n=1}^{\infty} \frac{3^{2n}\left(\frac{19}{9}-2\right)^n}{n+1}$$

For $x = \frac{19}{9}$, the series is

$$= \sum_{n=1}^{\infty} \frac{1}{(n+1)} \text{ which is divergent}$$

So, which shows that the given series is convergent for $\frac{17}{9} \le x < \frac{19}{9}$

Hence the interval of convergence is $\left[\frac{17}{9}, \frac{19}{9}\right)$

14. $\sum_{n=1}^{\infty} \frac{n^2(x+4)^n}{2^{3n}}$

Solution: We have, $U_n = \frac{n^2(x+4)^n}{2^{3n}}$

and $U_{n+1} = \frac{(n+1)^2\ (x+4)^{n+1}}{2^{3n+3}}$

So, $\frac{U_{n+1}}{U_n} = \frac{(n+1)^2(x+4)^{n+1}}{2^{3n+3}} \cdot \frac{2^{3n}}{n^2(x+4)^n}$

$= \left(\frac{n+1}{n}\right)^2 \cdot \frac{x+4}{8}$

So, $\lim_{n\to\infty} \left|\frac{U_{n+1}}{U_n}\right| = \lim_{n\to\infty} \left|\left(\frac{n+1}{n}\right)^2 \cdot \frac{x+4}{8}\right| = \left|\frac{x+4}{8}\right|$

So, the series is convergent for $\left|\frac{x+4}{8}\right| < 1$

i.e. $-1 < \frac{x+4}{8} < 1$

or, $-8 < x+4 < 8$

or, $-12 < x < 4$

and divergent for $\left|\frac{x+4}{8}\right| > 1$

For, $x = -12$, the series is $\sum_{n=1}^{\infty} \frac{n^2(-8)^n}{8^n} = \sum_{n=1}^{\infty} n^2(-1)^n$

Which is an alternating series

Which $U_n = n^2$ and thus $\lim_{n\to\infty} U_n \ne 0$

So, by Leibnitz's test, it is not convergent.

For $x = 4$, the series is,

$\sum_{n=1}^{\infty} \frac{n^2(8)^n}{8^n} = \sum_{n=1}^{\infty} n^2$ which is also divergent.

Which shows that the given series is covergent for $-12 < x < 4$

Hence the interval of convergence is $(-12, 4)$

15. $\sum_{n=1}^{\infty} \frac{(-1)^n(x+1)^n}{2^n}$

Solution: Let $U_n = \frac{(-1)^n(x+1)^n}{2n}$ and $U_{n+1} = \frac{(-1)^{n+1}(x+1)^{n+1}}{2^{n+1}}$

$$\therefore \lim_{n\to\infty} \left|\frac{U_{n+1}}{U_n}\right| = \lim_{n\to\infty} \left|\frac{(-1)^{n+1}(x+1)^{n+1}}{2^{n+1}} \cdot \frac{2^n}{(-1)^n(x+1)^n}\right|$$

$$= \lim_{n\to\infty} \left|(-1) \cdot \frac{2^n(x+1)}{2^{n+1}}\right|$$

$$= \lim_{n\to\infty} \left|\frac{x+1}{2}\right| = \left|\frac{x+1}{2}\right|$$

Hence, the series is convergent for $\left|\frac{x+1}{2}\right| < 1$.

$|x+1| < 2$.

Hence, the radius of convergene is $R = 2$

Since, the seris is centered at $x = -1$, it with converge in the interval $(-3, 1)$

$[|x+1| < 2 \Rightarrow -2 < x+1 < 2 \Rightarrow -3 < x < 1]$

Furthermore, at the endpoints,

We have, when $x = -3$

$\therefore \sum_{n=0}^{\infty} \frac{(-1)^n(-2)^n}{2^n} = \sum_{n=1}^{\infty} \frac{2^n}{2^n} = 1 \ne 0$ diverges for $n = -3$

when $x = 1$

$$\sum_{n=0}^{\infty}\frac{(-1)^n(-2)^n}{2^n} = \sum_{n=1}^{\infty}(-1)^n \text{ diverges.}$$

Thus, the interval of convergence is (– 3, 1)

16. $1+\frac{1}{5}x+\frac{2}{5^2}x^2+\frac{3}{5^2}x^3+.....$

Solution: Here, the general term of the series is

$$u_n = \frac{nx^n}{5^n}$$

$$\text{and } u_{n+1} = \frac{(n+1)x^{n+1}}{5^{n+1}}$$

$$\therefore \lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|$$

$$= \lim_{x\to\infty}\frac{(n+1)x^{n+1}}{5^{n+1}}\times\frac{5^n}{nx^n}$$

$$= \left|\frac{n}{5}\right| < 1$$

i.e., $|x| < 5$

$\therefore R = 5$

For the interval of the series,

$|x| < 5,\ -5 < x < 5$

At $x = -5$

$$u_n = \frac{n(-5)^n}{5^n}$$

$u_n = n(-1)^n$

Which is divergent

and at $x = 5$

$$u_n = \frac{n5^n}{5^n} = n$$

Which is shows that the given series is convergent for $-5 < x < 5$

Hence the interval of convergece is (– 5, 5)

17. $\sum_{n=1}^{\infty}\frac{3^n x^n}{(n+1)^2}$

Solution: Here, The general term of the series is,

$$u_n = \frac{3^n x^n}{(n+1)^2} \text{ and } u_{n+1} = \frac{3^{n+1}x^{n+1}}{(n+2)^2}$$

Now, $\lim_{x\to\infty}\left|\frac{u_{n+1}}{u_n}\right|$

$$= \lim_{x\to\infty}\left|\frac{3^{n+1}x^{n+1}}{(n+2)^2}\times\frac{(n+1)^2}{3^n x^n}\right|$$

$= |3x|$

The ratio test, the series is convergent for $|3x| < 1$ or, $|x| < \frac{1}{3}$

i.e., $-\frac{1}{3} < x < \frac{1}{3}$ and divergenct for $|3x|$. 1

When $x = -\frac{1}{3}$

$$u_n = \frac{3^n\left(-\frac{1}{3}\right)^n}{(n+1)^2} = \frac{(-1)^n}{(n+1)^2}$$

Here, the series is converegent.

At $x = \frac{1}{3}$

$$u_n = \frac{3^n\left(\frac{1}{3}\right)^n}{(n+1)^2} = \frac{(1)^n}{(n+1)^2}$$

By p-test,

The series is convergent

$\therefore$ Interval is $\left[-\frac{1}{3}, \frac{1}{3}\right]$

18. $\sum_{n=0}^{\infty}\frac{x^n}{2^n}$

Solution: Let $u_n = \lim_{x\to 0}\frac{x^n}{2^n}$

$$\text{and } u_{n+1} = \lim_{x\to 0}\frac{x^{n+1}}{2^{n+1}}$$

$$\therefore \lim_{x\to 0}\left|\frac{x^{n+1}}{2^{n+1}}\times\frac{2^n}{x^n}\right| = \lim_{x\to 0}\left|\frac{x}{2}\right|$$

So the series is convergent for

$\left|\frac{x}{2}\right| < 1$

$|x| < 2$

$\therefore -2 < x < 2$ and divergence of $|x| > 1$
When $x = -2$, the series becomes,

$$u_n = \lim_{x\to 0} \frac{(-2)^n}{2^n} = (-1)^n$$

Which is divergent by alternating series,
When, $x = 2$

$$u_n = \lim_{n\to 0} \frac{2^n}{2^n} = 1$$

Which is also, divergent,
$\therefore$ Internal of the given series is
$(-2, 2)$ Ans.

Find the radius of convergence of the power series:

19. $1 + 2x + 3x^2 + 4x^3 + \ldots\ldots\ldots\ldots$

Solution: We have, $U_n = nx^{n-1}$, $U_{n+1} = (n+1)\,x^n$

$$\text{So, } \frac{U_{n+1}}{U_n} = \frac{(n+1)\,x^n}{nx^{n-1}} = \frac{n+1}{n}\,.\,x$$

$$\therefore \lim_{n\to\infty}\left|\frac{U_{n+1}}{U_n}\right| = \lim_{n\to\infty}\left|\frac{n+1}{n}\,.\,x\right|$$

$$= \lim_{n\to\infty}\left|\left(1+\frac{1}{n}\right).\,x\right| = |x|$$

So, the series is con vergents for $|x| < 1$
i.e. $-1 < x < 1$ and divergent for $|x| > 1$

$$\text{The radius of covergence} = \frac{1-(-1)}{2} = 1$$

20. $1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^4}{4} + \ldots\ldots\ldots\ldots$

Solution: We have, $U_n = \dfrac{(-1)^{n+1}x^{n-1}}{n}$, $U_{n+1} = \dfrac{(-1)^{n+2}x^n}{n+1}$

$$\therefore \lim_{n\to\infty}\left|\frac{U_{n+1}}{U_n}\right| = \lim_{n\to\infty}\left|-\frac{n}{n+1}\,.\,x\right|$$

$$= \lim_{n\to\infty}\left|\left(\frac{1}{1+\frac{1}{n}}\right).\,x\right| = |x|$$

So, the series is convergent for $|x| < 1$.
i.e. $-1 < x < 1$ and divergent for $|x| > 1$.

$$\text{The radius of convergence} = \frac{1-(-1)}{1} = 1.$$

21. $\sum \dfrac{nx^n}{(n+1)^2}$

Solution: We have, $U_n = \dfrac{nx^n}{(n+1)^2}$ and $U_{n+1} = \dfrac{(n+1)\,x^{n+1}}{(n+1)^2}$

$$\text{So, } \frac{U_{n+1}}{U_n} = \frac{(n+1)\,x^{n+1}}{(x+2)^2}\,.\,\frac{(n+1)^2}{nx^n} = \frac{(n+1)^3}{n(n+2)^2}\,x$$

$$\therefore \lim_{n\to\infty}\left|\frac{U_{n+1}}{U_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^3}{n(n+2)^2}\,.\,x\right|$$

$$= \lim_{n\to\infty}\left|\frac{n^3\left(1+\frac{1}{n}\right)^3}{n^3\left(1+\frac{1}{n}\right)^2}\,.\,x\right| = |x|$$

So, the series is convergent for $|x| < 1$.
i.e., $-1 < x < 1$ and divergent for $|x| > 1$

$$\therefore \text{The radius of convergence} = \frac{1-(-1)}{2} = 1$$

22. $\sum \dfrac{2^n x^n}{n!}$ **(BE 2069 Bhadra)**

Solution: We have, $U_n = \dfrac{2^n x^n}{n!}$, and $U_{n+1} = \dfrac{2^{n+1}x^{n+1}}{(n+1)\,!}$

$$\text{So, } \frac{U_{n+1}}{U_n} = \frac{2^{n+1}\,x^{n+1}}{(n+1)\,!}\,.\,\frac{n!}{2^n x^n} = \frac{2x\,n!}{(n+1)\,n!} = \frac{2x}{n+1}$$

$$\therefore \lim_{n\to\infty}\left|\frac{U_{n+1}}{U_n}\right| = \lim_{n\to\infty}\left|\frac{2x}{n+1}\right| = 0 < 1$$

So, the series is convergent for all values of x.
The radius of convergence $= \infty$

23. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n-1)^n}{n}$

Solution: We have, $U_n = \dfrac{(-1)^{n+1}(x-1)^n}{n}$ and $U_{n+1} = \dfrac{(-1)^{n+2}(x-1)^{n+1}}{n+1}$

So, $\frac{U_{n+1}}{U_n} = \frac{(-1)^{n+2}(x-1)^{n+1}}{n+1} \cdot \frac{n}{(-1)^{n+1}(x-1)^n}$

$$= -\frac{n}{n+1}(x-1)$$

$$\therefore \lim_{n\to\infty}\left|\frac{U_{n+1}}{U_n}\right| = \lim_{n\to\infty}\left|-\frac{n}{n+1}\cdot(x-1)\right|$$

$$= \lim_{n\to\infty}\left|\frac{1}{1+\frac{1}{n}}\cdot(x-1)\right| = |x-1|$$

So, the series is convergent for $|x-1|<1$
i.e., $-1<x-1<1$ and divergent $|x-1|>1$
$-1<x-1<1 \Rightarrow 0<x<2.$

The radius of covergence $= \frac{2-0}{2} = 1$

24. $\sum_{n=1}^{\infty}\frac{n^n x^n}{n!}$

Solution: We have, $U_n = \frac{n^n x^n}{n!}$ and $U_{n+1} = \frac{(n+1)^{n+1}x^{n+1}}{(n+1)!}$

So, $\frac{U_{n+1}}{U_n} = \frac{(n+1)^n x^{n+1}}{(n+1)!}\cdot\frac{n!}{n^n x^n}$

$$= \frac{(n+1)^n(n+1)\,x}{(n+1)n!}\cdot\frac{n!}{n^n}$$

$$= \left(\frac{n+1}{n}\right)^n x$$

$$\therefore \lim_{n\to\infty}\left|\frac{U_{n+1}}{U_n}\right| = \lim_{n\to\infty}\left|\left(\frac{n+1}{n}\right)^n . x\right|$$

$$= \lim_{n\to\infty}\left|\left(1+\frac{1}{n}\right)^n . x\right| = |ex| = e|x|$$

So, the series is convergenct for $e|x|<1$ and divergent for $e|x|>1$

$e|x|<1 \Rightarrow |x|<\frac{1}{e}$ or, $-\frac{1}{e}<x<\frac{1}{e}$

Radius of convergence $= \frac{\frac{1}{e}-\left(-\frac{1}{e}\right)}{2} = \frac{1}{e}$

25. $\sum_{n=1}^{\infty}\frac{(-3)^n x^n}{\sqrt{n+1}}$

Solution: We have, $U_n = \frac{(-3)^n x^n}{\sqrt{n+1}}$ and $U_{n+1} = \frac{(-3)^{n+1}x^{n+1}}{\sqrt{n+2}}$

So, $\frac{U_{n+1}}{U_n} = \frac{(-3)^{n+1}.x^{n+1}}{\sqrt{n+2}}\cdot\frac{\sqrt{n+1}}{(-3)^n x^n}$

$$= -3\sqrt{\frac{n+1}{n+2}}\,x$$

$$\therefore \lim_{n\to\infty}\left|\frac{U_{n+1}}{U_n}\right| = \lim_{n\to\infty}\left|-3\sqrt{\frac{n+1}{n+2}}.x\right|$$

$$= \lim_{n\to\infty}\left|-3\sqrt{\frac{1+\frac{1}{n}}{1+\frac{2}{n}}}.x\right| = 3|x|$$

So, the series is convergent for $3|x|<1$
and divergent for $3|x|>1$

$3|x|<1 \Rightarrow |x|<\frac{1}{3}$

or, $-\frac{-1}{3}<x<\frac{1}{3}$

$\therefore$ Radius of convergence $= \frac{\frac{1}{3}-\left(-\frac{1}{3}\right)}{2} = \frac{1}{3}$

Important IOE Questions and Answers

1. Test the convergence of the series $1+\frac{1}{2^2}+\frac{2^2}{3^3}+\frac{3^3}{4^4}+\ldots\ldots\ldots$

(BE 2068 Magh)

Sol^n. Here, the given series is $1+\frac{1}{2^2}+\frac{2^2}{3^3}+\frac{3^3}{4^4}+\ldots\ldots\ldots$

Ominiting the first term, its general term is,

$$U_n=\frac{n^4}{(n+1)^{n+1}}$$

We, have by the root-test,

$$\lim_{n\to\infty}\sqrt[n]{U_n}=\lim_{n\to\infty}\sqrt[n]{\frac{n^n}{(n+1)^{n+1}}}=\lim_{n\to\infty}\sqrt[n]{\frac{n^n}{(n+1)^n(n+1)}}$$

$$=\lim_{n\to\infty}\frac{n}{n+1}\frac{1}{(n+1)^{\frac{1}{n}}}=\lim_{n\to\infty}\frac{n}{n\left(1+\frac{1}{n}\right)(n+1)^{\frac{1}{n}}}$$

$$=\lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)}\cdot\frac{1}{(n+1)^{\frac{1}{n}}}=\lim_{n\to\infty}\frac{1}{e^{\frac{\log(n+1)}{n}}}$$

$$\therefore\ \lim_{n\to\infty}\sqrt[n]{U_n}=1$$

Hence, by the root test, the series is convergent.

2. Test the convergence of the series:

$$2x+\frac{3x^2}{8}+\frac{4x^3}{27}+\ldots\ldots+\frac{(n+1)}{n^3}x^n+\ldots\ldots\ldots\ldots$$ **(BE 2068 Bhadra)**

Sol^n. Given series is, $2x+\frac{3x^2}{8}+\frac{4x^3}{27}+\ldots\ldots+\frac{(n+1)}{n^3}x^n+\ldots\ldots$

Here, $U_n=\frac{(n+1)}{n^3}x^n$ and $U_{n+1}=\frac{n+2}{(n+1)^3}x^{n+1}$

Now, $\lim_{n\to\infty}\frac{U_{n+1}}{U_n}=\lim_{n\to\infty}\frac{(n+2)}{(n+1)^3}x^{n+1}\frac{n^3}{(n+1)x^n}$

$$=x\lim_{n\to\infty}\frac{(n+2)n^3}{(n+1)^3.(n+1)}=x\lim_{n\to\infty}\frac{\left(1+\frac{2}{n}\right)}{\left(1+\frac{1}{n}\right)^3\left(1+\frac{1}{n}\right)}=x$$

By the ratio test $\sum U_n$ converges for $x<1$ and diverges for $x>1$.

When $x=1$

$$U_n=\frac{n+1}{n^3}\Rightarrow V_n=\frac{1}{n^2}$$

Now, $\lim_{n\to\infty}\frac{U_n}{V_n}=\lim_{n\to\infty}\frac{(n+1)}{n^3}\times n^2=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)=1$, a finite quantity

But we know that $\sum V_n=\sum\frac{1}{n^2}$, being p-series with $p>1$, is convgent

Hence, the series $\sum U_n$ is convergent for $x=1$

So, The series $\sum U_n$ is convsgent for $x\le 1$ and divergent for $x>1$.

3. Find the radius of convergence and the interval of convergence of the power series:

$$\sum_{n=0}^{\infty}\frac{(-1)^n(x-3)^n}{n+1}$$ **(BE 2068 Bhadra)**

Sol^n. Given series is, $\sum_{n=0}^{\infty}\frac{(-1)^n(x-3)^n}{n+1}$

Its general term is, $U_n=\frac{(-1)^n(x-3)^n}{n+1}$

and $U_{n+1}=\frac{(-1)^{n+1}(x-3)^{n+1}}{n+2}$

By the ratio test,

$$\lim_{n\to\infty}\left|\frac{U_{n+1}}{U_n}\right|=\lim_{n\to\infty}\left|\frac{(-1)^{n+1}(x-3)^{n+1}}{n+2}\times\frac{n+1}{(-1)^n(x-3)^n}\right|$$

$$=|x-3|\lim_{n\to\infty}\left|\frac{n+1}{n+2}\right|=|x-3|\lim_{n\to\infty}\left|\frac{n\left(1+\frac{1}{n}\right)}{n\left(1+\frac{2}{n}\right)}\right|=|x-3|\frac{1+0}{1+0}$$

$$\therefore\ \lim_{n\to\infty}\left|\frac{U_{n+1}}{U_n}\right|=|x-3|$$

If $|x-3|<1$, then $\lim_{n\to\infty}\left|\frac{U_{n+1}}{U_n}\right|<1$ and by the ratio test, the series is comvergent.

Thus, the series is convergent for $|x-3|<1$

i.e. $-1<x-3<1$

i.e. $2<x<4$

when $x=2$, the given series becomes,

$$\sum_{n=0}^{\infty}\frac{(-1)^n(x-3)^n}{n+1}=\sum_{n=0}^{\infty}\frac{1}{n+1}=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots\ldots\ldots$$

Clearly, this is P-series with p = 1 divergent.
When, x = 4, the series becomes,

$$\sum_{n=0}^{\infty}\frac{(-1)^n(x-3)^n}{n+1}=\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots\ldots\ldots\ldots$$

This is alternating series with $V_{n+1} > U_n$

and $\lim_{n\to\infty} U_n = \lim_{n\to\infty}\frac{1}{n+1}=0$

Thus, the series is convergent for x = 4
Hence, the given series is convergent in
So, the interval of convergent is (2, 4)
Radius = 1 with centre = 3

4. Test the convergence of the series $\frac{1}{2}+\frac{2}{3}x+\left(\frac{3}{4}\right)^2x^2+\left(\frac{4}{5}\right)^3x^3+\ldots\ldots\ldots\ldots$ (BE 2067 Mangsir)

Sol^n. Given, $\frac{1}{2}+\frac{2}{3}x+\left(\frac{3}{4}\right)^2+\left(\frac{4}{5}\right)^3x^3+\ldots\ldots$ for positive velues of x.

Omiting the first term, general term is

$$U_n=\left(\frac{n+1}{n+2}\right)^n x^n$$

By applying root test

$$\lim_{n\to\infty}\sqrt[n]{U_n}=\lim_{n\to\infty}\sqrt[n]{\left(\frac{n+1}{n+2}\right)^n x^n}$$

$$=x\lim_{n\to\infty}\frac{n+1}{n+2}=x\lim_{n\to\infty}\frac{1+\frac{1}{n}}{1+\frac{2}{n}}=x$$

Therefore the series is convergent if x<1 and divergent if x>1

If x = 1, then $\lim_{n\to\infty}\sqrt[n]{U_n}=1$, the root test fails.

Therefore, $\lim_{n\to\infty}U_n=\lim_{n\to\infty}\left(\frac{n+1}{n+2}\right)^n$

$$=\lim_{n\to\infty}\frac{\left(1+\frac{1}{n}\right)^n}{\left(1+\frac{2}{n}\right)^n}=\lim_{n\to\infty}\frac{\left(1+\frac{1}{n}\right)^n}{\left[\left(1+\frac{2}{n}\right)^{\frac{n}{2}}\right]^2}=\frac{e}{e^2}=\frac{1}{2}\neq 0$$

Thus, the series is divergent for x = 1 Ans.

IOE Exam 2071 (Bhadra)

(Regular)

1. State Euler's theorem for a homogeneous function of two independent variable and verify it for the function $u = x^n . \sin\left(\frac{y}{x}\right)$.

Sol^n. *See exericse-2, Q. No. 1(iv) in this book.*

2. Find the extreme value $x^2 + y^2 + z^2$ subject to the condition $x + y + z = 1$ and $xyz + 1 = 0$

Sol^n. *See exercise- 3, Q. No. 9 in this book.*

3. Evaluate $\iint xy\,(x + y)\,dxdy$ over the area between $y = x^2$ and $y = x$.

Sol^n. *See exercise- 4, Q. No. 3 in this book.*

4. Evaluate the integral by changing to polar coordinates $\int_0^1\int_x^{\sqrt{2x-x^2}}(x^2+y^2)\,dy\,dx$.

Sol^n. *See exercise- 5, Q. No. 2(iv) in this book.*

OR

Find the triple integration the volume of sphere $x^2 + y^2 + z^2 = a^2$.

Sol^n. *See exercise- 7, Q. No. 14 in this book.*

5. Show that the lines $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ and $4x - 3y + 1 = 0 = 5x - 3z + 2$ are coplanar. Also find their point of intersection.

Sol^n. *See exercise- 10, Q. No. 6*

6. Find the length and equation of the shortest distance between the lines $\frac{x-3}{3}=\frac{y-8}{-1}=\frac{z-3}{1}$ and $2x - 3y + 27 = 0, 2y - z + 20 = 0$

Sol^n. *See important IOE question, Page no. 197, Q. No. 11*

7. Find the centre and radius of the circle $x^2 + y^2 + z^2 - 8x + 4y + 8z - 45 = 0, x - 2y + 2z - 3 = 0$

Sol^n. *See exercise-12, Q. No. 6(i)*

8. Find the equation of right circular cone whose vertex at origin and axis the line $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$ with the vertical angle 30°.

Sol^n. *See exercise- 14, Q. No. 6*

OR

Find the equation of the right circular cylinder having for its base the circle $x^2 + y^2 + z^2 = 9, x - y + z = 3$.

Sol^n. *See exercise-15, Q. No. 6*

9. Solve by the power series method the differential equation $y'' - 4xy' + (4x^2 - 2)y = 0$

Sol^n. *See important IOE question, page 244, Q. No. 4,*

10. Test whether the solutions of $y''' - 2y'' - y' + 2y = 0$ are linearly independent or dependent.

Sol^n. Given, differential equation is,

$y''' - 2y'' - y' + 2y = 0$ ..... (i)

or, $D^3y - 2D^2y - Dy + 2y = 0$ where $D = \frac{d}{dx}$

or, $(D^3 - 2D^2 - D + 2)\, y = 0$

It's Auxilary equation is,

$m^3 - 2m^2 - m + 2 = 0$

or, $m^2 (m - 2) - 1 (m - 2) = 0$

or, $(m^2 - 1) (m - 2) = 0$

Either $m^2 - 1 = 0 \Rightarrow m = \pm 1$

or, $m - 2 = 0 \Rightarrow m = 2$

$\therefore$ The solution of (i) is,

$y = C_1e^x + C_2e^{-x} + C_3e^{-2x}$

Here, $y_1(x) = e^x,\ y_2(x) = e^{-x},\ y_3(x) = e^{2x}$

$\therefore y'_1(x) = e^x, y'_2(x) = -e^{-x}, y'_3(x) = 2e^{2x}$

$\therefore y''_1(x) = e^x, y''_2(x) = e^{-x}, y''_3(x) = 4e^{2x}$

Now the Wronskin determinant

$$W[y_1(x), y_2(x), y_3(x)] = \begin{vmatrix} y_1(x) & y_2(x) & y_3(x) \\ y'_1(x) & y'_2(x) & y'_3(x) \\ y''_1(x) & y''_2(x) & y''_3(x) \end{vmatrix}$$

$$= \begin{vmatrix} e^x & e^{-x} & e^{2x} \\ e^x & -e^{-x} & 2e^{2x} \\ e^x & e^{-x} & 4e^{2x} \end{vmatrix}$$

Taking common e^x, e^{-x} and e^{2x} from C_1, C_2 and C_3 respectively,

$$= e^x . e^{-x} . e^{2x} \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 1 & 1 & 4 \end{vmatrix}$$

Applying $C_1 \to C_1 - C_2$

$$= e^{2x} \begin{vmatrix} 0 & 1 & 1 \\ 2 & -1 & 2 \\ 0 & 1 & 4 \end{vmatrix}$$

Expanding along C_1,

$$= e^{2x} (-2)(4 - 1) = -6e^{2x} \neq 0$$

Hence the e^x, e^{-x} and e^{2x} are linearly independent.

11. Show that : $J_{-\left(\frac{5}{2}\right)}(x) = \sqrt{\frac{2}{\pi x}}\left(\frac{3}{x}\sin x + \frac{3 - x^2}{x^2}\cos x\right)$.

Sol^n. *See exercise- 18, Q. No. 11 in this book.*

12. If $\vec{a}, \vec{b}, \vec{c}$ and $\vec{a}', \vec{b}', \vec{c}'$ are the reciprocal system of vectors, then prove that

$$\vec{a}' \times \vec{b}' + \vec{b}' \times \vec{c}' + \vec{c}' \times \vec{a}' = \frac{\vec{a} + \vec{b} + \vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}']},\ \left[\vec{a}\,\vec{b}\,\vec{c}\right] \neq 0.$$

Sol^n. Since $\vec{a}, \vec{b}, \vec{c}$ and $\vec{a}', \vec{b}', \vec{c}'$ are reciprocal system of vectors.

So, $\vec{a}' = \frac{\vec{b} \times \vec{c}}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]}$, $\vec{b} = \frac{\vec{c} \times \vec{a}}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]}$ and $\vec{c}\,\frac{\vec{a} \times \vec{b}}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]}$

Now, L.H.S. $= \vec{a}' \times \vec{b}' + \vec{b}' \times \vec{c}' + \vec{c}' \times \vec{a}'$

$$= \frac{\vec{b} \times \vec{c}}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]} \times \frac{\vec{c} \times \vec{a}}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]} + \frac{\vec{c} \times \vec{a}}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]} \times \frac{\vec{a} \times \vec{b}}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]} + \frac{\vec{a} \times \vec{b}}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]} \times \frac{\vec{b} \times \vec{c}}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]}$$

$$= \frac{1}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]^2}\left\{(\vec{b} \times \vec{c}) \times (\vec{c} \times \vec{a})\right\} + \left\{(\vec{c} \times \vec{a}) \times (\vec{a} \times \vec{b})\right\} + \left\{(\vec{a} \times \vec{b}) \times (\vec{b} \times \vec{c})\right\}$$

$$= \frac{1}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]^2}\left\{\left[\vec{b}\ \vec{c}\ \vec{a}\right]\vec{c} - \left[\vec{b}\ \vec{c}\ \vec{c}\right]\vec{a}\right\} + \left\{\left[\vec{c}\ \vec{a}\ \vec{b}\right]\vec{a} - \left[\vec{c}\ \vec{a}\ \vec{a}\right]\vec{b}\right\}$$

$$+ \left\{\left[\vec{a}\ \vec{b}\ \vec{c}\right]\vec{b} - \left[\vec{a}\ \vec{b}\ \vec{b}\right]\vec{c}\right\}$$

$$= \frac{1}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]^2}\left\{\left[\vec{a}\ \vec{b}\ \vec{c}\right]\vec{c} - 0 + \left[\vec{a}\ \vec{b}\ \vec{c}\right]\vec{a} - 0 + \left[\vec{a}\ \vec{b}\ \vec{c}\right]\vec{b} - 0\right\}$$

Since $\left[\vec{b}\ \vec{c}\ \vec{a}\right] = \left[\vec{c}\ \vec{a}\ \vec{b}\right] = \left[\vec{a}\ \vec{b}\ \vec{c}\right]$ and

$= \left[\vec{b}\ \vec{c}\ \vec{c}\right] = \left[\vec{c}\ \vec{a}\ \vec{a}\right] = \left[\vec{a}\ \vec{b}\ \vec{c}\right] = 0$

$$= \frac{1}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]^2}\left[\vec{a}\ \vec{b}\ \vec{c}\right]\left[\vec{c} + \vec{a} + \vec{b}\right]$$

$$= \frac{1}{\left[\vec{a}\ \vec{b}\ \vec{c}\right]}\left(\vec{a} + \vec{b} + \vec{c}\right)$$ R.H.S. Proved.

13. The necessary and sufficient condition for the function $\vec{a}$ of scalar variable t to have a constant direction is $\vec{a} \times \frac{d\vec{a}}{dt} = 0$.

Sol^n. *See important IOE question, Page no. 348, Q. No. 2*

14. Find the directional derivative of $\phi = x^2yz + 4xz^2$ at the point $(1, -2, -1)$ in the direction of vector $2\vec{i} - \vec{j} - 2\vec{k}$.

Sol^n. *See exercise- 25, Q. No. 2(i) in this book.*

OR

If $\vec{a}$ is a constant vector and $\vec{r}$ be the position vector, then, prove that $\nabla \times (\vec{a} \times \vec{r}) = 2\vec{a}$.

Sol^n. Since $\vec{a}$ is a constant vector and $\vec{r}$ be the position vector.

So, Let $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ and $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$

Now, L. H. S.

$\nabla \times (\vec{a} \times \vec{r})$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \times (\vec{a} \times \vec{r})$$

$$= \vec{i} \times \frac{\partial(\vec{a} \times \vec{r})}{\partial x} + \vec{j} \times \frac{\partial(\vec{a} \times \vec{r})}{\partial y} + \vec{k} \times \frac{\partial(\vec{a} \times \vec{r})}{\partial z}$$

$$= \vec{i} \times \left(\vec{a} \times \frac{\partial \vec{r}}{\partial x}\right) + \vec{j} \times \left(\vec{a} \times \frac{\partial \vec{r}}{\partial y}\right) + \vec{k} \times \left(\vec{a} \times \frac{\partial \vec{r}}{\partial z}\right)$$

$= \vec{i} \times (\vec{a} \times \vec{i}) + \vec{j} \times (\vec{a} \times \vec{j}) + \vec{k} \times (\vec{a} \times \vec{k})$ Since, $\frac{\partial r}{\partial x} = \vec{i}, \frac{\partial r}{\partial r} = \vec{j}$ and $\frac{\partial r}{\partial z} = \vec{k}$

$$= (\vec{i} \cdot \vec{i})\vec{a} - (\vec{i} \cdot \vec{a})\vec{i} + (\vec{j} \cdot \vec{j})\vec{a} - (\vec{j} \cdot \vec{a})\vec{j} + (\vec{k} \cdot \vec{k})\vec{a} - (\vec{k} \cdot \vec{a})\vec{k}$$

$$= \vec{a} - a_1\vec{i} + \vec{a} - a_2\vec{j} + \vec{a} - a_3\vec{k} = 3\vec{a} - (a_1\vec{i} + a_2\vec{j} + a_3\vec{k})$$

$= 3\vec{a} - \vec{a} = 2\vec{a}$ R.H.S. Proved.

15. Determine whether the series is convergent or divergent $\sum_{n=1}^{\infty}\left(\sqrt[3]{n^3+1} - 1\right)$.

Sol^n. *See exercise- 27, Q. No. 8 in this book.*

16. Find the interval and radius of convergence of the power series: $\sum_{n=1}^{\infty} \frac{2^n \cdot (x-3)^n}{n+3}$.

Sol^n. *See exercise- 31, Q. No. 1 in this book.*

IOE Exam 2071 (Bhadra)

(Old Back- 2065 & Earlier Batch)

1. Describe and sketch the graph of the equation $r = \frac{12}{3 + 2\cos\theta}$.

Sol^n. *Does not contain the new syllabus.*

2. If $z(x + y) = x^2 + y^2$, then show that $\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)$.

Sol^n. Given $z(x + y) = x^2 + y^2$ ..... (i)

Differentiating (i) partially with respect to x and y respectively.

$$\frac{\partial}{\partial x}\{z.(x+y)\} = \frac{\partial}{\partial x}(x^2 + y^2) \Rightarrow z\frac{\partial(x+y)}{\partial x} + (x+y)\frac{\partial z}{\partial x} = 2x$$

$$z + (x+y)\frac{\partial z}{\partial x} = 2x \qquad \text{or, } (x+y)\frac{\partial z}{\partial x} = 2x - z$$

$$\text{or, } \frac{\partial z}{\partial x} = \frac{2x - z}{(x+y)} \text{ and } \frac{\partial}{\partial y}\{z.(x+y)\} = \frac{\partial}{\partial y}(x^2 + y^2)$$

$$z\frac{\partial}{\partial y}(x+y) + (x+y)\frac{\partial z}{\partial y} = 2y$$

$$\text{or, } z + (x+y)\frac{\partial z}{\partial y} = 2y \quad \text{or, } (x+y)\frac{\partial z}{\partial y} = 2y - z \quad \text{or, } \frac{\partial z}{\partial y} = \frac{2y - z}{x+y}$$

$$\text{Now L. H. S.} = \left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2$$

$$= \left(\frac{2x-z}{x+y} - \frac{2y-z}{x+y}\right)^2 = \left(\frac{2x-z-2y+z}{x+y}\right)^2 = \left(\frac{2x-2y}{x+y}\right)^2 = 4\left(\frac{x-y}{x+y}\right)^2 \text{ ... (i)}$$

Also,

$$\text{R. H. S.} = 4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right) = 4\left(1 - \frac{2x-z}{x+y} - \frac{2y-z}{x+y}\right) = 4\left(\frac{x+y-2x+z-2y+z}{x+y}\right)$$

$$= 4\left(\frac{2z - x - y}{x+y}\right) \text{ Since } z(x+y) = x^2 + y^2 \Rightarrow z = \frac{x^2 + y^2}{x+y}$$

$$= \frac{4}{(x+y)}\left[2\frac{(x^2+y^2)}{x+y} - (x+y)\right]$$

$$= \frac{4}{(x+y)}\left[2\frac{(x^2+y^2)-(x-y)^2}{x+y}\right] = \frac{4}{(x+y)}\left(\frac{2x^2+2y^2-x^2-2xy+y^2}{x+y}\right)$$

$$= \frac{4}{(x+y)}\left(\frac{x^2-2xy+y^2}{x+y}\right) = \frac{4(x-y)^2}{(x+y)(x+y)} = 4\left(\frac{x-y}{x+y}\right)^2 \text{ ..... (iii)}$$

From (ii) and (iii)

$$\left(\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}\right)^2 = 4\left(1-\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}\right) \text{ Proved.}$$

3. Find the maximum & minimum values of the function $20 - x^2 - y^2 - z^2$.

Sol^n. *See exercise- 3, Q. No. 2(v) in this book.*

4. Evaluate the double integral $\int_0^a \int_0^x \frac{\cos y}{\sqrt{(a-x)(a-y)}}\,dxdy$, by changing the order of integration.

Sol^n. *See exercise-5, Q. No. 1 (viii) in this book.*

5. Find by double integration, the area of the region bounded by $y^2 = x^3$ and $x = y$.

Sol^n. *See exercise- 7, Q. No. 4 in this book.*

6. Test the convergence of the series $\sum_{n=1}^{\infty} \frac{5^n}{n3^{n+1}} x^n$.

Sol^n. Given series $\sum_{n=1}^{\infty} \frac{5^n}{n^{3n+1}} \cdot x^n$

Its general term of the given series is

$$u_n = \frac{5^n}{n3^{n+1}} x^n \qquad \therefore u_{n+1} = \frac{5^{n+1}}{(n+1)\,3^{n+2}} \cdot x^{n+1}$$

By ratio test, $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \left\{\frac{5^{n+1}}{(n+1)\,3^{n+2}} \cdot x^{n+1} \times \frac{n3^{n+1}}{5^n . x^n}\right\}$

$$= \lim_{n\to\infty}\left(\frac{5x}{3}\cdot\frac{n}{n+1}\right) = \lim_{n\to\infty}\left(\frac{5x}{3}\,\frac{n}{n\left(1+\frac{1}{n}\right)}\right) = \frac{5x}{3}\cdot\frac{1}{\left(1+\frac{1}{\infty}\right)} = \frac{5x}{3}$$

By ratio test given series is convergent for $x < 1$ and divergent for $x > 1$.
When $x = 1$ given series becomes,

$$\sum_{n=1}^{\infty} \frac{5^n}{n3^{n+1}}$$

Here, $u_n = \frac{5^n}{n3^{n+1}} \qquad \therefore u_{n+1} = \frac{5^{n+1}}{(n+1)\,3^{n+2}}$

By ratio test, $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \left\{\frac{5^{n+1}}{(n+1)\,3^{n+2}} \times \frac{n3^{n+1}}{5^n}\right\}$

$$= \lim_{n\to\infty}\left(\frac{5}{3}\,\frac{n}{n+1}\right) = \lim_{n\to\infty}\left\{\frac{5}{3}\,\frac{n}{n\left(1+\frac{1}{n}\right)}\right\} = \frac{5}{3}\,\frac{1}{\left(1+\frac{1}{\infty}\right)} = \frac{5}{3} > 1$$

Hence the given series is divergent for $x = 1$
Thus the given series is convergent for $x < 1$ and divergent for $x \geq 1$

7. Find the interval and radius of convergence of the power series $\sum_{n=1}^{\infty} \frac{n(x+2)^n}{3^{n+1}}$

Sol^n. *See exercise-31, Q. No. 8 in this book.*

8. Show that $[\vec{a}+\vec{b}\ \ \vec{b}+\vec{c}\ \ \vec{c}+\vec{a}] = 2\left[\vec{a}\,\vec{b}\,\vec{c}\right]$.

Sol^n. L. H. S. $= [\vec{a}+\vec{b}\ \ \vec{b}+\vec{c}\ \ \vec{c}+\vec{a}]$

$= (\vec{a}+\vec{b}).\ (\vec{b}+\vec{c}) \times (\vec{c}+\vec{a})\}$

$= (\vec{a}+\vec{b}).\ \{(\vec{b}\times\vec{c}) + (\vec{b}\times\vec{a}) + (\vec{c}\times\vec{c}) + (\vec{c}\times\vec{a})$

$= (\vec{a}+\vec{b}).\{(\vec{b}\times\vec{c}) + (\vec{b}\times\vec{a}) + 0 + (\vec{c}\times\vec{a})\}$ Since $\vec{c}\times\vec{c} = 0$

$= \vec{a}.(\vec{b}\times\vec{c}) + \vec{a}.(\vec{b}\times\vec{a}) + \vec{a}\,(\vec{c}\times\vec{a}) + \vec{b}.(\vec{b}\times\vec{c}) + \vec{b}.(\vec{b}\times\vec{a}) + \vec{b}.(\vec{c}\times\vec{a})$

$= \left[\vec{a}\ \vec{b}\ \vec{c}\right] + \left[\vec{a}\ \vec{b}\ \vec{a}\right] + \left[\vec{a}\ \vec{c}\ \vec{a}\right] + \left[\vec{b}\ \vec{b}\ \vec{c}\right] + \left[\vec{b}\ \vec{b}\ \vec{a}\right] + \left[\vec{b}\ \vec{c}\ \vec{a}\right]$

$= \left[\vec{a}\ \vec{b}\ \vec{c}\right] + 0 + 0 + 0 + 0 + \left[\vec{a}\ \vec{b}\ \vec{c}\right]$

Since $\left[\vec{a}\ \vec{b}\ \vec{a}\right] = \left[\vec{a}\ \vec{c}\ \vec{a}\right] = \left[\vec{b}\ \vec{b}\ \vec{c}\right] = \left[\vec{b}\ \vec{b}\ \vec{a}\right] = 0$

$= 2\left[\vec{a}\ \vec{b}\ \vec{c}\right]$ = R. H. S. Proved.

9. Show that the vector function $\vec{a}$ of scolar variable t has constant direction if and only if $\vec{a} \times \frac{d\vec{a}}{dt} = 0$.

Sol^n. *See important IOE question, Page no. 348, Q. No. 2*

10. Show that the lines $\frac{x-5}{4} = \frac{y-7}{4} = \frac{z+3}{-5}$; $\frac{x-8}{7} = \frac{y-4}{1} = \frac{z-5}{3}$ are coplanar. Also find the plane containing them.

Sol^n. *See exercise-10, Q. No. 1*

11. Find the magnitude and the equation of the shortest distance between the lines $\frac{x}{2} = \frac{y}{-3} = \frac{z}{1}$ and $\frac{x-2}{3} = \frac{y-1}{-5} = \frac{z+2}{2}$.

Sol^n. Given equation are : $\frac{x}{2} = \frac{y}{-3} - \frac{z}{1}$ and $\frac{x-2}{3} = \frac{y-1}{-5} = \frac{z+2}{2}$

Let l, m, n be the directions cosines of S.D.
So, $2l - 3m + n = 0$, $3l - 5m + 2n = 0$

Solving, we get, $\frac{\ell}{1} = \frac{m}{1} = \frac{n}{1} = \frac{1}{\sqrt{3}} \qquad \therefore l = \frac{1}{\sqrt{3}}, m = \frac{1}{\sqrt{3}}, n = \frac{1}{\sqrt{3}}$

Length of S. D. $= (x_2 - x_1)\, l + (y_2 - y_1)\, m + (z_2 - z_1) n$

$$= (2-0)\frac{1}{\sqrt{3}} + (1-0)\frac{1}{\sqrt{3}} + (-2-0)\frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}}$$

$\therefore$ Equation of S.D.

$$\begin{vmatrix} x & y & z \\ 2 & -3 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 0 \text{ and } \begin{vmatrix} x-2 & y-1 & z+2 \\ 3 & -5 & 2 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

$\therefore 4x + y - 5z = 0$ and $7x + y - 8z = 31$

12. Find the equation of the sphere passing through the circle $x^2 + y^2 = 4$, $z = 0$ and is intersected by the plane $x + 2y + 2z = 0$ is a circle of radius 3.

Sol^n. Here the equation of sphere through the given circle is,

$x^2 + y^2 + z^2 - 4 + \lambda z = 0$ ..... (i)

and its centre is $(-u, -v, -w) = \left(0, 0, -\frac{\lambda}{2}\right)$

radius $= \sqrt{\frac{\lambda^2}{4} + 4}$

Also, length of perpendicular from centre $\left(0, 0, -\frac{\lambda}{2}\right)$ of the sphere to the plane $x + 2y + 2z = 0$ is

$$\left|\frac{1.0 + 2.0 + 2\left(-\frac{\lambda}{2}\right)}{\sqrt{1+4+4}}\right| = \frac{\lambda}{3}$$

We have, $r^2 = R^2 - p^2$

or, $\frac{\lambda^2}{4} + 4 - \frac{\lambda^2}{9} = 9$ or, $\frac{5\lambda^2}{36} = 5 \Rightarrow \lambda^2 = \frac{36 \times 5}{5} = 36 \Rightarrow \lambda = \pm 6$

Then from equation (i) we get,

$x^2 + y^2 + z^2 - 4 \pm 6z = 0$

13. Prove that the equation $ax^2 + by^2 + cz^2 + 2ux + 2vy + 2wz = d = 0$ represents a cone if $\frac{u^2}{a} + \frac{v^2}{b} + \frac{w^2}{c} = d$.

Sol^n. *See important IOE question, Page no. 195, Q. No. 7*

14. Solve the differential equation $x^2 \frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 4x^3$.

Sol^n. *Does not contain the new syllabus.*

15. Solve the differential equation $y'' - 9y = 0$ by the power series method.

Sol^n. *See exercise- 17, Q. No. 12 in this book.*

16. Express the function $f(x) = 1 + x - x^2$ in terms Legendre's polynomial.

Sol^n. *See exercise-18, Q. no. 3 in this book.*