# Mathematics for Artificial Intelligence

Assignment 1

Saurabh Dharmadhikari
Saurabh.dharma01@gmail.com

Question 1:-

Consider the following logarithmic equation.

$$\log_x \left( \frac{8 - \log_5(x)}{\log_5(x)} \right)^{\log_3(x)} = 1$$

i) Find the value(s) of $x$ satisfying in the equation.

$$\log_x \left( \frac{8 - \log_5(x)}{\log_5(x)} \right)^{\log_3 x} = 1$$

$$\Rightarrow \log_x \left( \frac{8 - \frac{\log_3(x)}{\log_3 5}}{\frac{\log_3 x}{\log_3 5}} \right)^{\log_3 x} = 1$$

Change Of Base Rule -

$$\log_b a = \frac{\log_c a}{\log_c b}$$

$$\Rightarrow \log_x \left( \frac{\frac{8\log_3 5 - \log_3 x}{\log_3 5}}{\frac{\log_3 x}{\log_3 5}} \right)^{\log_3 x} = 1$$

$$\Rightarrow \log_x \left( \frac{8\log_3 5 - \log_3 x}{\log_3 x} \right)^{\log_3 x} = 1$$

Assume

$\log_3 x = y$

$\therefore \quad x = 3^y$

Thus, our previous equation will become :-

$$3^y = \left(\frac{8\log_3 5 - y}{y}\right)^y$$

$$\Rightarrow 3 = \frac{8\log_3 5 - y}{y}.$$

$$\Rightarrow 3y = 8\log_3 5 - y$$

$$\Rightarrow 4y = 8\log_3 5$$

$$\Rightarrow y = \frac{8\log_3 5}{4}$$

$$\Rightarrow y = 2\log_3 5$$

$\Rightarrow y = (\log_3(5^2))$ $\Longrightarrow$ Rule applied

$\log a^2 = 2\log a.$

$\Rightarrow \log_3 x = \log_3 25$ $\Longrightarrow$ Putting back the value of $y$.

$y = \log_3 x.$

$$\Rightarrow x = 25.$$

Hence, value for $x$ satisfying the equation is 25.

$x = 25.$

ii) Determine for what values of $x$ the logarithmic expression on the left side of the equation is defined.

$$\log_x \left( \frac{8 - \log_5(x)}{\log_5(x)} \right)^{\log_3 x} = 1.$$

$\Rightarrow 8 - \log_5(x) > 0 \Longrightarrow$ Rule:

argument $> 0$

Any given argument of a given logarithm must be a positive number greater than 0.

$\Rightarrow 8 > \log_5 x$

$\Rightarrow \log_5 x < 8$

$\Rightarrow x < 5^8$

$\Rightarrow x < 390625$

Question 2. Consider the function -

$$f(x) = |x^2 - 3x + 2| + |x - 3|$$

i) Find its derivatives using the concept of piecewise defined functions.

$$f(x) = |x^2 - 3x + 2| + |x - 3|$$
$$= |(x-2)(x-1)| + |(x-3)|$$

Let us Assume:-

When $x = 1$, $(x - 1) = 0$

$$\therefore f(x) = |(\cancel{x} - 2)(1-1)| + |(1-3)|$$
$$= 0 + |(-2)| = 2.$$

When $x = 2$,

$$f(x) = |(2-2)(2-1)| + |2-3|$$
$$= 0 + |-1| = 1$$

When $x = 3$,

$$f(x) = |(3-2)(3-1)| + |3-3|$$
$$= |(1)(2)| = 2.$$

When $x$ is at 1, 2 & 3, we know the value of $x$.

Lets see how function behaves in:

$x < 1$

$1 < x < 2$

$2 < x < 3$

$x > 3$

—

$x < 1$

$(x-1)$ is negative

$(x-2)$ is negative

$(x-3)$ is negative

$$f(x) = |(x-1)(x-2)| + |(x-3)|$$

$$= |(-ve \times -ve)| + |(-ve)|$$

$$= |+ve| + |-ve|$$

$$\Rightarrow (x-1)(x-2) - (x-3) \Longrightarrow \text{Modulus Rule.}$$

It implies that value will be converted to positive if itt is a negative value.

$$f(x) = x^2 - 3x + 2 - x + 3.$$

$$= x^2 - 4x + 5.$$

Derivative :-

$$f'(x) = 2x - 4.$$

If $1 < x < 2$

$(x-1)$ is positive

$(x-2)$ is negative

$(x-3)$ is negative

So,

$$f(x) = |(+ve) \times (-ve)| + |-ve|$$

$$= |-ve| + |-ve|$$

$$\Rightarrow -(x-1)(x-2) - (x-3).$$

$$\Rightarrow -(x^2 \cancel{+} -3x + 2) - (x-3)$$

$$\Rightarrow -x^2 + 3x - 2 - x + 3.$$

$$\Rightarrow -x^2 + 2x + 1.$$

Derivative :-

$$f'(x) = -2x + 2.$$

If $2 < x < 3$

$(x-1)$ is positive

$(x-2)$ is positive

$(x-3)$ is negative.

$$f(x) = x^2 - 3x + 2 - x + 3.$$

$$= x^2 - 4x + 5.$$

Derivative :-

$$f'(x) = 2x - 4.$$

If $x > 3$

$(x-1)$ is positive

$(x-2)$ is positive

$(x-3)$ is positive.

$$f(x) = (x^2 - 3x + 2) + (x-3)$$
$$= x^2 - 3x + 2 + x - 3.$$
$$= x^2 - 2x - 1.$$

Derivative -

$$f'(x) = 2x - 2.$$

Therefore Derivative for function is :

$$f'(x) = \begin{cases} 2x-4 & \text{when } x < 1 \\ -2x+2 & \text{when } 1 < x < 2 \\ 2x-4 & \text{when } 2 < x < 3 \\ 2x-2 & \text{when } x > 3 \end{cases}$$

ii) Also, determine the points when the function is not differentiable, if any exists.

At $x = 1$

$x = 2$

$x = 3$

function changes.

Question 3 :-

Consider the following function:

$$f(x) = \sqrt{x} + \sqrt{a-2x}$$

i) Suppose that the product of the maximum value of $f(x)$ & the minimum value of $f(x)$ is $\sqrt{12}$. Find the value of a.

$$f(x) = \sqrt{x} + \sqrt{(a-2x)}$$

$$\Rightarrow y = x^{1/2} + (a-2x)^{1/2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}x^{1/2-1} + \frac{1}{2}(a-2x)^{1/2-1} \times (-2)$$

$$= \frac{1}{2}x^{-1/2} - (a-2x)^{-1/2}$$

$$\Rightarrow = \frac{1}{2\sqrt{x}} - \frac{1}{\sqrt{a-2x}}$$

At stationary point $\frac{dy}{dx} = 0$

Hence,

$$\frac{1}{2\sqrt{x}} - \frac{1}{\sqrt{a-2x}} = 0$$

$$\sqrt{a-2x} - 2\sqrt{x} = 0$$

$$\sqrt{a-2x} = 2\sqrt{x}$$

Squaring both sides :-

$a - 2x = 4x$

$\Rightarrow a = 6x$

$\Rightarrow x = \frac{a}{6}$

Therefore $\frac{dy}{dx} = 0$, when $x = \frac{a}{6}$.

$f(x) = \sqrt{x} + \sqrt{a-2x}$

$x \geq 0$ $\Longrightarrow$ Square Root of expression under square root has to $\geq 0$ otherwise its irrational number.

&

$a - 2x \geq 0$

or.

$a \geq 2x$.

$x \leq \frac{a}{2}$

Domain of $x$

$x \geq 0$ & $x \leq \frac{a}{2}$

Let us see value of $f(x)$ when $x = 0$.

$f(0) = \sqrt{0} + \sqrt{a - 2\times 0}$.

$= \sqrt{a}$

Value of $f(x)$ when $x = \frac{a}{2}$.

$$f\left(\frac{a}{2}\right) = \sqrt{\frac{a}{2}} + \sqrt{a - 2\left(\frac{a}{2}\right)}$$

$$= \frac{1}{\sqrt{2}}\sqrt{a} = 0.707\sqrt{a}.$$

Value of $f(x)$ at $x = 0$ & $x = \frac{a}{2}$

$$f(0) = \sqrt{a}$$

$$f(a/2) = 0.707\sqrt{a}$$

$f(x)$ is stationary at $x = \frac{a}{6}$, $\frac{dy}{dx} = 0$

Value of $f(x)$ at $x = \frac{a}{6}$.

$$f\left(\frac{a}{6}\right) = \sqrt{\frac{a}{6}} + \sqrt{a - 2 \times \frac{a}{6}}$$

$$= \frac{\sqrt{a}}{\sqrt{6}} + \frac{2\sqrt{a}}{\sqrt{6}}.$$

$$= \frac{3\sqrt{a}}{6} = \frac{\sqrt{3} \times \sqrt{3}}{\sqrt{3} \times \sqrt{2}}\sqrt{a} = \sqrt{1.5} \times \sqrt{a} = 1.225\sqrt{a}.$$

Thus,

$$f\left(\frac{a}{6}\right) = 1.225\sqrt{a}$$

$$f\left(\frac{a}{6}\right) > f\left(\frac{a}{2}\right)$$

Hence,

$f\left(\frac{a}{6}\right)$ is maximum value.

$f\left(\frac{a}{2}\right)$ is minimum value.

It is given that: maximum value $f(x)$ × minimum value $f(x)$ $= \sqrt{12}$.

$\therefore f\left(\frac{a}{6}\right) \times f\left(\frac{a}{2}\right) = \sqrt{12}$

$\sqrt{\frac{3}{2}}\sqrt{a} \times \frac{1}{\sqrt{2}}\sqrt{a} = \sqrt{12}$.

$\Rightarrow \left(\frac{\sqrt{3}}{2}\right)a = \sqrt{12}$

$\Rightarrow a = 2\times\sqrt{\frac{12}{3}} \Rightarrow a = 2\times 2$

$\Rightarrow a = 4$

• Find $a = 4$ and plot the function with the following consideration

ii)

Find all $x$ - & $y$ - intercepts.

$a = 4$

$f(x) = \sqrt{x} + \sqrt{a-2x}$

$\Rightarrow f(x) = \sqrt{x} + \sqrt{4-2x}$

So, $x \geq 0$ & $(4-2x) \geq 0$

or.

$-2x \geq -4$ & $x \leq 2$.

$f(0) = \sqrt{0} + \sqrt{4-(2\times 0)}$

$= \sqrt{4} = 2$.

$f\left(\frac{a}{6}\right) = f\left(\frac{4}{6}\right) = f\left(\frac{2}{3}\right) = \sqrt{\frac{2}{3}} + \sqrt{4-\left(2\times\frac{2}{3}\right)}$

$= \sqrt{\frac{2}{3}} + \sqrt{\frac{8}{3}} = \frac{\sqrt{2}+\sqrt{8}}{\sqrt{3}}$

$\Rightarrow f\left(\frac{2}{3}\right) = \sqrt{6} = 2.45$.

When,

$x = 0$, $y = 2$.

$x = 2/3$, $y = 2.45$

$x = 2$, $y = \sqrt{2}$.

iii) Find all the stationary points & classify them. (you may convert the coordinates of the stationary point(s) into decimal for drawing purposes, however, follow the notes on page 1 of this assignment.

The stationary point is at:

$x = \frac{4}{6} = 0.667$

$y = \sqrt{6} = 2.45$.

iv) Determine the intervals for which the function is increasing, and the intervals for which the function is decreasing

$f(x)$ is increasing for the interval $0 \leq x \leq \frac{4}{6}$

after $x = \frac{4}{6}$, for the interval