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Week - 2

Assignment :-

Q1. Consider the vector:

$$u = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

$$v = \begin{bmatrix} 2 \\ 2 \\ -2 \\ 4 \end{bmatrix}$$

$$w = \begin{bmatrix} 1 \\ 2 \\ 0 \\ -2 \end{bmatrix}$$

i) Determine which two vectors are least similar to each other based on these norms:

a) l_2 norm:

$$dist(x, y) = \|x - y\|_2 = \|y - x\|_2 = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

$$\Rightarrow dist(\vec{u}, \vec{v}) = \sqrt{\sum_{i=1}^{n} (\vec{u} - \vec{v})^2}$$

$$= \sqrt{(1-2)^2 + (2-2)^2 + (0+2)^2 + (1-4)^2}$$

$$= \sqrt{(-1)^2 + 0^2 + 2^2 + (-3)^2}$$

$$= \sqrt{1 + 4 + 9} \quad = \sqrt{14}$$

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$$\Rightarrow dist(\vec{u},\vec{w}) = \sqrt{\sum_{i=1}^{n} (\vec{u}-\vec{w})^2}$$ (b)

$$= \sqrt{(1-1)^2+(2-2)^2+(0-0)^2+(1+2)^2}$$

$$= \sqrt{0^2+0^2+0^2+3^2}$$

$$= \sqrt{9} \quad = 3.$$

$$\Rightarrow dist(\vec{v},\vec{w}) = \sqrt{\sum_{i=1}^{n} (\vec{v}-\vec{w})^2}.$$

$$= \sqrt{(2-1)^2+(2-2)^2+(-2-0)^2+(4+2)^2}$$

$$= \sqrt{1^2+0^2+4+6^2}$$

$$= \sqrt{1+4+36} \quad = \sqrt{41}$$

Based on l_2 norm $\vec{v}$ & $\vec{w}$ are least similar to each other.

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b) l_1 norm:

$$dist(x, y) = \|x - y\|_1 = \|y - x\|_1 = \sum_{i=1}^{n} |x_i - y_i|$$

$$\Rightarrow dist(\vec{u}, \vec{v}) = \sum_{i=1}^{n} |\vec{u}_i - \vec{v}_i|$$

$$= |1-2| + |2-2| + |0+2| + |1-4|$$

$$= 1 + 0 + 2 + 3 = 6.$$

$$\Rightarrow dist(\vec{u}, \vec{w}) = \sum_{i=1}^{n} |\vec{u}_i - \vec{w}_i|$$

$$= |1-1| + |2-2| + |0-0| + |1+2|$$

$$= 0 + 0 + 0 + 3 = 3.$$

$$\Rightarrow dist(\vec{v}, \vec{w}) = \sum_{i=1}^{n} |v_i - w_i|$$

$$= |2-1| + |2-2| + |-2-0| + |4+2|$$

$$= 1 + 0 + 2 + 6 = 9$$

Based on l_1 norm $(\vec{v}, \vec{w})$ are least similar.

c) l_∞ norm

$$dist(x, y) = \|x-y\|_\infty = \|y-x\|_\infty = \max_{i=1,\dots n} |x_i - y_i|$$

$$\Rightarrow dist(\vec{u}, \vec{v}) = \max_{i=1,\dots n} |u_i - v_i|$$

$$= \max(|1-2|, |2-2|, |0+2|, |1-4|)$$

$$= \max(1, 0, 2, 3)$$

$$= 3$$

$$\Rightarrow dist(\vec{u}, \vec{w}) = \max_{i=1,\dots n} |u_i - w_i|$$

$$= \max(|1-1|, |2-2|, |0-0|, |1+2|)$$

$$= \max(0, 0, 0, 3)$$

$$= 3$$

$$\Rightarrow dist(\vec{v}, \vec{w}) = \max_{i=1,\dots n} |v_i - w_i|$$

$$= \max(|2-1|, |2-2|, |-2-0|, |4+2|)$$

$$= \max(1, 0, 2, 6)$$

$$= 6$$

Based on the l_∞ norm (v, w) are least similar to each other.

ii) Determine which two vectors are most similar to each other based on ~~those norms.~~ the cosine similarity measure:

$$\cos\theta = \frac{x \cdot y}{\|x\| \, \|y\|}$$

$$\Rightarrow \cos\theta(\vec{u}, \vec{v}) = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \, \|\vec{v}\|}$$

$$= \frac{\begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 2 \\ -2 \\ 4 \end{bmatrix}}{\sqrt{1^2+2^2+0^2+1^2} \, \sqrt{2^2+2^2+(-2)^2+4^2}}$$

$$= \frac{2+4+0+4}{\sqrt{1+4+0+1} \, \sqrt{4+4+4+16}}$$

$$= \frac{10}{\sqrt{6} \, \sqrt{28}}$$

$$= \frac{10}{\sqrt{168}}$$

iii)

Ans:-

$$\Rightarrow \cos\theta(\vec{u}, \vec{w}) = \frac{u \cdot w}{\|u\| \cdot \|w\|}$$

$$= \frac{\begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 0 \\ -2 \end{bmatrix}}{\sqrt{1^2+2^2+0^2+1^2}\sqrt{1^2+2^2+0^2+(-2)^2}}$$

$$= \frac{1+4+0-2}{\sqrt{1+4+1}\sqrt{1+4+4}}$$

$$= \frac{3}{\sqrt{6}\sqrt{9}} = \frac{3}{\sqrt{6} \times 3}$$

$$= \frac{1}{\sqrt{6}}$$

$$\Rightarrow \cos\theta(\vec{v}, \vec{w}) = \frac{v \cdot w}{\|v\| \cdot \|w\|}$$

$$= \frac{\begin{bmatrix} 2 \\ 2 \\ -2 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 0 \\ -2 \end{bmatrix}}{\sqrt{2^2+2^2+(-2)^2+4^2}\sqrt{1^2+2^2+0^2+(-2)^2}}$$

$$= \frac{2+4+0-8}{\sqrt{4+4+4+16}\sqrt{1+4+0+4}}$$

$$= \frac{-2}{\sqrt{28}\sqrt{9}} = \frac{-2}{3\sqrt{28}}$$

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Since $\cos\theta(\vec{u}, \vec{v}) = 10/\sqrt{168}$ is closest to 1
u & v are the most similar vectors.

iii) Explain the reason behind the difference in the results between (i) & (ii) if you observe any.

Ans:- The difference in the result of (i) & (ii) is because of the different measures of similarity

In (i) we used norm 1, 2-norm also called euclidean norm & ∞ norm or max norm to calculate the distance. but in (ii) we used $\cos\theta$ or the angle ~~at~~ that 2 vectors made at the origin point (The more the angle, the larger the distance).

So, we had different results in value form but conclusions were similar.

iv) Can the differences be resolved? Give details of your suggestion, if you have any and explain the outcome if your suggestions are applied.

Ans.- We can resolve the differences in (i) & (ii) by normalizing vectors & then calculating $\cos\theta$

For example;

$$l_2 \text{ norm } (\vec{u}) = \sqrt{1^2+2^2+0^2+1^2}$$

$$= \sqrt{1+4+0+1}$$

$$= \sqrt{6}$$

$$\|\vec{v}\| = \sqrt{2^2+2^2+(-2)^2+4^2}$$

$$= \sqrt{4+4+4+16}$$

$$= \sqrt{28}$$

$$\|\vec{w}\| = \sqrt{1^2+2^2+0^2+(-2)^2}$$

$$= \sqrt{1+4+0+4}$$

$$= \sqrt{9} = 3.$$

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To normalize, we divide them, each vector with their respective norms:

$$\vec{u}\_norm = \frac{u}{\|u\|} = \frac{u}{\sqrt{6}}$$

$$= \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 0 \\ 1/\sqrt{6} \end{bmatrix}$$

$$\vec{v}\_norm = \frac{v}{\|v\|} = \frac{v}{\sqrt{28}}$$

$$= \begin{bmatrix} 2/28 \\ 2/28 \\ -2/28 \\ 4/28 \end{bmatrix}$$

$$\vec{w}\_norm = \frac{w}{\|w\|} = \frac{w}{3}$$

$$= \begin{bmatrix} 1/3 \\ 2/3 \\ 0 \\ -2/3 \end{bmatrix}$$

Now we calculate $\cos\theta$ with normalized vectors

$$\cos\theta(\vec{u}\_norm, \vec{v}\_norm) = \frac{\vec{u}\_norm \cdot \vec{v}\_norm}{\|\vec{u}\_norm\| \cdot \|\vec{v}\_norm\|}$$

$$\Rightarrow \frac{\begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 0 \\ -1/\sqrt{6} \end{bmatrix} \cdot \begin{bmatrix} 2/\sqrt{28} \\ 2/\sqrt{28} \\ -2/\sqrt{28} \\ 4/\sqrt{28} \end{bmatrix}}{\sqrt{\left(\frac{1}{\sqrt{6}}\right)^2 + \left(\frac{2}{\sqrt{6}}\right)^2 + 0^2 + \left(-\frac{1}{\sqrt{6}}\right)^2}\ \sqrt{\left(\frac{2}{\sqrt{28}}\right)^2 + \left(\frac{2}{\sqrt{28}}\right)^2 + \left(-\frac{2}{\sqrt{28}}\right)^2 + \left(\frac{4}{\sqrt{28}}\right)^2}}$$

$$\Rightarrow \frac{\frac{2}{\sqrt{168}} + \frac{4}{\sqrt{168}} + \left(\frac{-2}{\sqrt{28}}\right) - \frac{4}{\sqrt{168}}}{\sqrt{\frac{1}{6} + \frac{2}{6} + 0 + \frac{1}{6}}\ \sqrt{\frac{4}{28} + \frac{4}{28} + \frac{4}{28} + \frac{16}{28}}}$$

$$\cancel{\Rightarrow \frac{7}{6}} \quad \frac{10\sqrt{168}}{16.8}.$$

Similarly, we can calculate,

$\cos\theta(\vec{u}\_norm, \vec{w}\_norm)$

&

$\cos\theta(\vec{v}\_norm, \vec{w}\_norm)$

This will give us absolute values of cosine with unit. Hence, we can compare to know which two are more similar.

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Q2 You are tasked with uncovering information about an incomplete matrix some of whose entities are unknown and denoted as a, b, c and d.

$$A = \begin{bmatrix} -1 & 0 & a \\ b & 4 & c \\ d & 0 & 0 \end{bmatrix}$$

i) Find the rank of A based on the values a, b, c and d.

$$A = \begin{bmatrix} -1 & 0 & a \\ b & 4 & c \\ d & 0 & 0 \end{bmatrix}$$

We reduce it to echlelon form.

$$A = \begin{bmatrix} -1 & 0 & a \\ 0 & 4 & c+ab \\ 0 & 0 & ad \end{bmatrix}$$

$R_2 \leftarrow R_2 + bR_1$
$R_3 \leftarrow R_3 + dR_1$

As shown above in rref (A), there is a possibility the R_3 is a zero row.
That means.

$a = 0$ or $d = 0$ or both $a \& d = 0$.

If $a = 0$,
Then,

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 4 & c \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & c/4 \\ 0 & 0 & 0 \end{bmatrix}$$ ⇒ We Unit Matrix.

There a 2 Pivots, ∴ Rank(A) = 2

If $d = 0$.

$$A = \begin{bmatrix} -1 & 0 & a \\ 0 & 4 & c+ab \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & -a \\ 0 & 1 & \frac{c+ab}{4} \\ 0 & 0 & 0 \end{bmatrix}$$ ⇒ Unit matrix.

We again have 2 pivots

∴ Rank(A) = 2

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For any other value of a, b, c & d matrix A; Rank of matrix A will be 3 as a or d will not be 0. Thus having no zero rows.

Q3. Consider the following matrix

$$A = \begin{bmatrix} 3 & -1 \\ 1 & 3 \\ 1 & 1 \end{bmatrix}$$

i) Construct $B = A^T A$, and find the eigenvalues and eigenvectors of B.

$$A = \begin{bmatrix} 3 & -1 \\ 1 & 3 \\ 1 & 1 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{bmatrix}$$

$$B = A^T . A$$

$$= \begin{bmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{bmatrix}_{2\times3} \cdot \begin{bmatrix} 3 & -1 \\ 1 & 3 \\ 1 & 1 \end{bmatrix}_{3\times2}$$

$$= \begin{bmatrix} (3\times3)+(1\times1)+(1\times1) & 3\times(-1)+(1\times3)+(1\times1) \\ (-1\times3)+(3\times1)+(1\times1) & -1\times(-1)+(3\times3)+(1\times1) \end{bmatrix}$$

$$= \begin{bmatrix} 9+1+1 & -3+3+1 \\ -3+3+1 & 1+9+1 \end{bmatrix}$$

$$B = \begin{bmatrix} 11 & 1 \\ 1 & 11 \end{bmatrix}$$

$$\det \left| \begin{bmatrix} 11 & 1 \\ 1 & 11 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0$$

$$\det \begin{vmatrix} 11-\lambda & 1 \\ 1 & 11-\lambda \end{vmatrix} = 0$$

$\Rightarrow (11-\lambda)^2 - 1 = 0$

$\Rightarrow (11-\lambda+1)(11-\lambda-1) = 0.$

$\Rightarrow (\lambda-12)(\lambda-10) = 0.$

$\therefore \lambda_1 = 12$ or $\lambda = 10$

If $\lambda = 12$

$$\begin{bmatrix} 11-12 & 1 \\ 1 & 11-12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$R_2 = R_2 + R_1$

$$\left[\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

This will mean it has infinite solutions.

$-x_1 + x_2 = 0$

$\Rightarrow x_1 = x_2 \qquad v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

If $\lambda = 10$

$$\begin{bmatrix} 11-10 & 1 \\ 1 & 11-10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{matrix} 0 \\ 0 \end{matrix}$$

$R_2 \leftarrow R_2 - R_1$.

$$\Rightarrow \begin{matrix} 1 & 1 & : & 0 \\ 0 & 0 & : & 0 \end{matrix}$$

$\Rightarrow x_1 + x_2 = 0$.

$\Rightarrow x_1 = -x_2$

if $x_1 = 1$, $x_2 = -1 \qquad v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

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ii) for positive eigenvalues, define $\sigma_i = \sqrt{\lambda_i}$ where $\lambda_1 \geq \lambda_2$. Construct matrix D as follows.

$$D = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \end{bmatrix}$$

Ans. $\lambda_1 \geq \lambda_2$

$\sigma = \sqrt{\lambda_i}$

We know.

$\lambda_1 = 12$.

$\lambda_2 = 10$

$\therefore \sigma_1 = \sqrt{12}$

$\sigma_2 = \sqrt{10}$

Hence,

$$D = \begin{bmatrix} \sqrt{12} & 0 \\ 0 & \sqrt{10} \\ 0 & 0 \end{bmatrix}$$

iii) If the eigenvectors of B are not orthonormal, orthonormalise them. Make a matrix V using the orthonormal vectors you obtained. The ordering of the columns of V should be the same as the ordering of the eigenvalues, that is $V = [v_1\ v_2]$. Show that V is an orthogonal matrix.

Ans. Check if v_1 perpendicular to v_2.

$$v_1 \cdot v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= 1 - 1$$

$$= 0$$

$$v_1 \cdot v_2 = 0$$

$\therefore$ v_1 is perpendicular to v_2

$$V = [v_1\ v_2]$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

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To check if V is an orthogonal matrix:

$$V^T = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$ This is interchangable.

$$\therefore V^T \cdot V = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+1 & 1-1 \\ 1-1 & 1+1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$= 2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

iv) **Find** eigenvectors of $C = AA^T$, and orthonormalize them.

$$C = AA^T$$

$$A = \begin{bmatrix} 3 & -1 \\ 1 & 3 \\ 1 & 1 \end{bmatrix} \cdot \quad A^T = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{bmatrix}$$

$$A \cdot A^T = \begin{bmatrix} (3\times3)+(-1)\times(-1) & (3\times1)+3\times(-1) & 3-1 \\ 3-3 & 1+9 & 1+3 \\ 3-1 & 1+3 & 1+1 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & 0 & 2 \\ 0 & 10 & 4 \\ 2 & 4 & 2 \end{bmatrix}$$

We will now solve by $(C - \lambda I)$

$$C - \lambda I = \begin{bmatrix} 10 & 0 & 2 \\ 0 & 10 & 4 \\ 2 & 4 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 10-\lambda & 0 & 2 \\ 0 & 10-\lambda & 4 \\ 2 & 4 & 2-\lambda \end{bmatrix}$$

Now;

$\det(C-\lambda I) = 0$.

We equate to find eigen values.

$$\det(C-\lambda I) = (10-\lambda)\left[(10-\lambda)\cdot(2-\lambda) - 4\times 4\right] + 2\left[-2(10-\lambda)\right] = 0.$$

$$\Rightarrow (10-\lambda)(\lambda^2 - 12\lambda + 20 - 16) + 4\lambda - 40 = 0$$

$$\Rightarrow (10-\lambda)(\lambda^2 - 12\lambda + 4) + 4\lambda - 40 = 0$$

$$\Rightarrow 10\lambda^2 - 120\lambda + 40 - \lambda^3 + 12\lambda^2 - 4\lambda^2 + 4\lambda^2 - 40 = 0$$

$$\Rightarrow \lambda^3 - 12\lambda^2 + 120\lambda = 0$$

$$\Rightarrow \lambda(\lambda-12)(\lambda-10) = 0.$$

We arrive at:

$$\lambda_2 = 12,\ \lambda_2 = 10\ \&\ \lambda_3 = 0$$

Eigenvector when $\lambda_1 = 12$.

$$\left[C - \lambda_1 I\right]\vec{x} = \vec{0}$$

$$\begin{bmatrix} 10-12 & 0 & 2 \\ 0 & 10-12 & 4 \\ 2 & 4 & 2-12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Augmented Matrix:

$$\left[\begin{array}{ccc|c} -2 & 0 & 2 & 0 \\ 0 & -2 & 4 & 0 \\ 2 & 4 & -10 & 0 \end{array}\right]$$

We divide by (-2)

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ -1 & -2 & 5 & 0 \end{array}\right]$$

$R_3 \leftarrow R_3 + R_1$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & -2 & 4 & 0 \end{array}\right]$$

$R_3 \leftarrow R_3 + 2R_2$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

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$\therefore$ x_1 & x_2 are dependent
& x_3 is independent.

$$x_1 = x_3$$
$$x_2 = 2x_3$$

If $x_3 = 5$, $5 \in R$.

$x_1 = 5$

$x_2 = 2 \times 5$

If name of eigenvector is $\vec{e}$, then.

$$\vec{e} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = 5 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

When $\lambda_2 = 10$, eigenvector $= \vec{e}_2$

then, $[C - \lambda_2 I]\vec{e}_2 = 0$

$$\begin{bmatrix} 10-10 & 0 & 2 \\ 0 & 10-10 & 4 \\ 2 & 4 & 2-10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 4 \\ 2 & 4 & -8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We interchange R_3 & R_1. & devide by 2 the matrix.

$$\left[\begin{array}{ccc|c} 1 & 2 & -4 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$R_3 \Leftarrow R_2 - 2R_3$.

$$\left[\begin{array}{ccc|c} 1 & 2 & -4 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

x is a dependent variable.
x_2 & x_3 are independent.

$$x_1 + 2x_2 - 4x_3 = 0 . \cancel{x}\cancel{2}$$

$$x_1 = -2x_2 + 4x_3$$

When,

$$x_3 = 0 \quad , \quad x_1 = -2x_2$$

$$\vec{e}_2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

Where x_2 is a independent variable

When $\lambda_3 = 0$, eigenvector = $\vec{e}_3$

$$[C - \lambda_3 I]\vec{e}_3 = \vec{0}$$

$$\begin{bmatrix} 10 & 0 & 2 \\ 0 & 10 & 4 \\ 2 & 4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \leftarrow 5R_3 - R_1$.

$$\left[\begin{array}{ccc|c} 10 & 0 & 2 & 0 \\ 0 & 10 & 4 & 0 \\ 0 & 20 & 8 & 0 \end{array}\right]$$

$R_3 \leftarrow R_3 - 2R_2$.

$$\left[\begin{array}{ccc|c} 5 & 0 & 1 & 0 \\ 0 & 5 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$5x_1 = -x_3$. $\quad$ $5x_2 = -2x_3$.

$x_1 = x_3/5$ $\quad$ $x_2 = -2x_3/5$.

x_1 & x_2 are dependent.
x_3 is independent.

$$\vec{e}_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1/5 \\ -2/5 \\ 1 \end{bmatrix} = x_3 = \begin{bmatrix} -1 \\ -2 \\ 5 \end{bmatrix}_{x_3}$$

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$\vec{e}_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

$$\vec{e}_3 = \begin{bmatrix} -1 \\ -2 \\ 5 \end{bmatrix}$$

Normalising

$$\vec{e}_1 = \frac{\vec{e}_1}{\|\vec{e}_1\|} = \frac{\vec{e}_1}{\sqrt{1^2+2^2+1^2}}$$

$$= \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

normalising $\vec{e}_2$

$$\frac{\vec{e}_2}{\|\vec{e}_2\|} = \frac{\vec{e}_2}{\sqrt{2^2+1^2}} = \frac{\vec{e}_2}{\sqrt{5}}$$

$$= \frac{1}{\sqrt{5}}\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

normalising $\vec{e}_3$

$$\frac{\vec{e}_3}{\sqrt{1^2+2^2+5^2}} = \frac{\vec{e}_3}{\sqrt{30}}$$

$$= \frac{1}{\sqrt{30}}\begin{bmatrix} -1 \\ -2 \\ 5 \end{bmatrix}$$