An Exploration of Random Rotation-Symmetric Bosonic Quantum Error Correcting Codes Under Noise from Loss

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A thesis presented for the degree of
Bachelors of Science in Computer Science

Department of Computer Science
University of Colorado at Boulder
March 14, 2023

1 Abstract

At present it is believed that it is not possible to build a quantum computer without error correction. The traditional approach to quantum error correction is to sacrifice several physical qubits to encode one logical qubit. An alternate approach is to encode a logical qubit into a single physical system with a large Hilbert space, such as the quantum harmonic oscillator. Because the latter requires less physical systems, it is deemed to be "hardware efficient". Bosonic codes are codes that encode one or more qubits into the Hilbert space of a quantum harmonic oscillator. In this thesis, bosonic codes with a phase space rotation symmetry are studied. The performance of known codes, such as the cat code and the binomial code, are compared to randomly generated codes that also have rotation symmetry. Through a numerical study, the performance of the proposed random codes are investigated when subject to the dominant noise channel for bosons, loss noise. Notably, these random codes can perform better than the best known rotation code, the binomial code, and they can perform significantly better than the cat code. Our findings suggest that evaluating the performance of new bosonic codes against random codes is a valuable benchmark for identifying good quantum error correcting codes.

2 Acknowledgements

This work was done under the guidance and supervision of Dr. Josh Combes, and the helpful assistance of Aiko Kyle, Noah Lordi, and Steven Liu. Thank you all.

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3 Introduction

Quantum computers are theoretically able to perform time-consuming tasks such as prime-factorizations or chemical simulations in a fraction of the time that classical computers would require. However, quantum information is extremely susceptible to all sorts of different noise. Due the fragility and transience of quantum information, an important part of every quantum computation process is the correction of errant information. It is currently believed that, not only will error correction be required for the construction of a viable quantum computer, but that most of what a quantum computer will do is correct errors.

Error correction codes are a subspace of a larger state space. Bosonic rotation codes are specifically a subspace of the Hilbert space afforded by a quantum harmonic oscillator. There are many recent experimental results showing the efficacy of bosonic codes. Three bosonic codes have reached the break even point, which is the point at which error correction isn't harming information fidelity. In 2016, Ofek et al. achieved break-even with cat codes, a specific type of bosonic rotation code [1]. In 2023, Ni et al. achieved the same benchmark with binomial codes, another type of bosonic rotation code [2]. Also in 2023, Sivak et al. managed to experimentally realize a code that beat break-even by a factor of 2.27 using a GKP code, which is a bosonic code [3]. Consequently, theoretical progress on bosonic codes and their performances have direct impacts on what is experimentally realized and what is possible for future quantum computers.

As development into bosonic error correction codes continues, it would be nice to have a suitable benchmark to compare with. Randomly generated codes may potentially provide that benchmarking standard.

4 Background

4.1 The Classical Repetition Code

A discussion of classical errors and classical error correction is instructive for the quantum analog. Classical information can always be encoded in binary digits (bits) so this discussion of classical error correction will only look at bits. Information can be corrupted by errors in many different ways, but the simplest error that classical information can experience is the flipping of bits from 0 to 1 or 1 to 0. When managed by a computer, information can experience errors in many different processes, such as when in storage or during transfer. Any process where information can be corrupted can be modelled as that information passing through a noisy channel. For the sake of modelling, it is assumed that information only has a risk of corruption when passing through a noisy channel, and otherwise the information is perfectly fine. Passing through a noisy channel doesn't necessarily mean that the information is moving per se; even information sitting on a hard drive may be subject to errors, and so the act of time passing could be modelled as passing through a noisy channel.

Example of Classical Bit-Flip Errors

Original Message	Noisy Channel	Received Message
1101	→	1001

Figure 1: Example of information passing through a noisy channel. The channel has a chance to flip bits of information that passes through. In this example, only the second bit was flipped, and the rest of the information remained intact.

The process of error correction is therefore to encode information before sending it through a noisy channel in a manner that ensures that the original information is recoverable despite errors experienced in the noisy channel.

The simplest form of error correction is the classical repetition code. The classical repetition code is an information encoding that protects against errors by duplicating information, so when errors happen, there is enough of the original information in the encoded message to still recover what the original un-encoded message was. A key requirement of this repetition code is that errors are rare enough that it can be assumed that the original information is more prevalent than errant information after errors have occured.

The properties of this repetition code are best demonstrated when bits are repeated until there are three copies of every bit in the encoded message. The encoded message, after being subjected to errors, can then be corrected by looking at the most common bit in each triplet of bits (a majority vote for each triplet). If a triplet of bits isn't all the same, it must be the case that an error occurred. As long as errors are sufficiently uncommon, each triplet of bits should have more of the original information than errant information. The process of taking the most common bit in each triplet is the recovery procedure to attempt to recover the original information.

Example of Classical Repetition				
Original Message	Repeated Message	Noisy Channel	Received Message	Corrected Message
1101	111 111 000 111	$\rightarrow$	101 111 001 011	1101

Figure 2: Example of a classical message passing through a noisy channel. Before passing through, the original message has each bit duplicated three times. After going through the noisy channel, the received message has some errors, but there is enough of the original message that the original message can be recovered by taking the most common bit in each triplet of bits.

The mapping of $0 \rightarrow 000$ and $1 \rightarrow 111$ is not uniquely the best. There exist other 1-bit to 3-bit encodings that work exactly as well as just repetition. Those are all different versions of this repetition code.

The classical repetition code is far from the best classical error correction scheme. To encode information of n bits, $3n$ bits are required, which is a large inflation of needed bits. There are far better classical error correction schemes, but the repetition code is an instructive starting point for the quantum analog.

4.2 Quantum States

Quantum information is more complicated than classical information. Whereas a classical bit is either 0 or 1, a quantum bit (qubit) can be in any superposition of 0 and 1. Unsurprisingly, quantum information is therefore notated with the language of quantum states. A discussion of pure states is used to inform the background of the more generalized mixed states.

In the same way that classical information has many different physical encodings, quantum information also has many different physical realizations. The specific implementations of quantum computing aren't particularly relevant for this thesis besides the fact that the systems must be made of bosons in a quantum harmonic oscillator.

4.2.1 Pure States

The notation

$$|\psi\rangle$$

is used to denote a quantum state with the name ψ. The symbols surrounding ψ in $|\psi\rangle$ are called a ket. All kets are complex column-vectors of unit length in a Hilbert space. For some $|\psi\rangle \neq |\phi\rangle$, the notation

$$\alpha |\psi\rangle + \beta |\phi\rangle$$

is used to denote a quantum state which is the superposition of $|\psi\rangle$ and $|\phi\rangle$. The α and β terms denote a complex-valued amplitude of each state's presence in the superposition; they essentially correspond to how much their associated state is present within the superposition. Superpositions can consist of more than two states too.

The notation

$$\langle\psi|$$

denotes a bra of ψ, as opposed to a ket. A bra is the dual of a ket. In terms of matrix operations, a bra is the conjugate transpose of a ket. For example,

$$|\psi\rangle = \alpha |\phi\rangle + \beta |\varphi\rangle \rightarrow \langle\psi| = \alpha^* \langle\phi| + \beta^* \langle\varphi| .$$

With the notion of states and their conjugate transpose, the natural next step is to consider inner products. Inner products are an invaluable operation when working with quantum states. The notation

$$\langle\psi|\phi\rangle$$

is used to represent the inner product of the state $|\psi\rangle$ with $|\phi\rangle$. An inner product is a bra-ket. The inner product $\langle\psi|\phi\rangle$ can be thought of as the amplitude of $|\psi\rangle$ in $|\phi\rangle$. Accordingly, inner products are complex scalar numbers, unlike quantum states. Importantly, order can matter for inner products because

$$\langle\psi|\phi\rangle = \langle\phi|\psi\rangle^* ,$$

which means that only real-valued inner products are order-independent. For quantum states, the inner product of a quantum state with itself is always 1. Two quantum states are considered orthogonal if their inner product is 0.

Quantum states are descriptions of the state of the system, but operators are another important part of the picture that describe how states change. Operators are typically denoted with a hat, so

$$\hat{O}$$

describes an operator called O. The action of an operator, $\hat{O}$, upon a state, $|\psi\rangle$, is notated as

$$\hat{O}\,|\psi\rangle$$

and the output is another quantum state. Importantly, operators are matrices, so for operators $\hat{O}$ and $\hat{P}$ acting on a state $|\psi\rangle$,

$$\hat{P}\hat{O}\,|\psi\rangle$$

means that $\hat{O}$ acts on $|\psi\rangle$ first, and then $\hat{P}$ acts on the output of that. The operations are associative, so that isn't the only unique interpretation. However, the actions don't necessarily commute, so

$$\hat{P}\hat{O}\,|\psi\rangle \neq \hat{O}\hat{P}\,|\psi\rangle$$

in general. To determine whether two operators commute, a value called the commutator is calculated. The commutator of operators $\hat{O}$ and $\hat{P}$ is given as

$$\left[\hat{O}, \hat{P}\right] = \hat{O}\hat{P} - \hat{P}\hat{O}.$$

If the commutator of two operators is 0, then those operators can commute.

An important constraint on allowed operators is that they be unitary, so for any valid operator $\hat{O}$, it must be true that

$$\hat{O}^{\dagger}\hat{O} = \hat{O}\hat{O}^{\dagger} = \hat{I}\,,$$

where $\hat{I}$ is the identity operator which doesn't change or affect states it acts upon. This constraint makes it so that operators are reversible: $\hat{O}$ is undone by $\hat{O}^{\dagger}$.

One may notice that there are many different ways to represent quantum states. For example, for different states $|\phi\rangle$, $|\varphi\rangle$, $|\gamma\rangle$, and $|\eta\rangle$, one could represent a state in a 2-dimensional Hilbert space as either a linear combination of $|\phi\rangle$ and $|\varphi\rangle$, or as a linear combination of $|\gamma\rangle$ and $|\eta\rangle$. The states that are being used to represent all other states are called the basis states. Basis states are often orthogonal, but need not be as long as they are linearly independent: basis states must be able to represent states not representable by all the other basis states. The dimension of a Hilbert space is the number of basis states required to represent any possible state within that state space.

Changing the basis representation of a state requires a discussion of projectors. Let us suppose there is a state $|\psi\rangle$ in some arbitrary basis, and we want to convert it to its representation in the basis given by the following set of states

$$\{|b_0\rangle\,, |b_1\rangle\,, |b_2\rangle\,, \ldots\}.$$

For an arbitrary basis state, $|b_i\rangle$, we know that $\langle b_i|\psi\rangle$ is the "amount" of $|b_i\rangle$ present in $|\psi\rangle$, or, more specifically, the amplitude of $|b_i\rangle$ in $|\psi\rangle$. Therefore, we can represent $|\psi\rangle$ in the new basis with at least one term that we know

$$|\psi\rangle = \ldots + \langle b_i|\psi\rangle\,|b_i\rangle + \ldots.$$

Since this procedure was for an arbitrary basis element b_i, we can apply it for every basis element and get that

$$|\psi\rangle = \langle b_0|\psi\rangle\,|b_0\rangle + \langle b_1|\psi\rangle\,|b_1\rangle + \ldots.$$

Since $\langle b_i|\psi\rangle$ is a scalar, we can write the above equation more suggestively as

$$|\psi\rangle = |b_0\rangle\!\langle b_0|\,|\psi\rangle + |b_1\rangle\!\langle b_1|\,|\psi\rangle + \ldots.$$

This form may suggest that some sort of operator is being applied to $|\psi\rangle$. We can write this operator as

$$\sum_i |b_i\rangle\!\langle b_i| = \hat{I}.$$

Each term of this new basis-changing operator, $|b_i\rangle\!\langle b_i|$, is called a projector. Projectors generally take the form

$$|\psi\rangle\!\langle\psi|$$

for some state $|\psi\rangle$. The projector $|\psi\rangle\langle\psi|$ is an operator that acts on a state and gives back the state $|\psi\rangle$ multiplied by the amplitude of $|\psi\rangle$ in the original state.

With all these tools, we are now able to tackle the topic of measuring quantum states. Observables are specified by hermitian operators. An operator, $\hat{O}$ is hermitian when

$$\hat{O} = \hat{O}^{\dagger}.$$

All measurable values of $\hat{O}$ are encoded as its eigenvalues. Importantly, measuring a state will also change what state it is. For example, if λ is an eigenvalue of $\hat{O}$ with eigenstate $|\lambda\rangle$, then if λ is measured after measuring with $\hat{O}$, the remaining state will be $|\lambda\rangle$. In order to get any meaningful measurement, observable operators must have multiple eigenvalues. If the operator $\hat{M}$ has eigenvalues

$$\{\lambda_0, \lambda_1, \lambda_2, \ldots\},$$

then the probability of measuring a state $|\psi\rangle$ to $|\lambda_i\rangle$ is given by

$$\Pr(\lambda_i \mid |\psi\rangle) = |\langle\lambda_i|\psi\rangle|^2. \tag{1}$$

This measurement probability is known as the Born rule. When working with a specific observable, it is often useful to express states in the basis of the observable's eigenstates. When represented as such, finding the probabilities is much simpler. If, for example, $|\psi\rangle$ is represented as

$$|\psi\rangle = c_0 |\lambda_0\rangle + c_1 |\lambda_1\rangle + \ldots + c_i |\lambda_i\rangle + \ldots$$

then the probability of measuring λ_i with $\hat{M}$ is given by

$$|c_i|^2.$$

Importantly, only the magnitude of c_i matters – not its phase. This effectively means that $e^{i\theta} |\psi\rangle$ and $|\psi\rangle$ are basically the same because there are no measurable differences between the states. The magnitudes don't change when a global phase is applied. Additionally, the probability for all measurement outcomes must sum to 1; this requirement is called the normalization condition.

In order to represent multiple quantum states in different systems, we combine them with the Kronecker product. The system of states $|\psi\rangle$ and $|\phi\rangle$ is represented as

$$|\psi\rangle \otimes |\phi\rangle = |\psi\rangle |\phi\rangle = |\psi\phi\rangle.$$

Operators also combine in the same way, so two operators, $\hat{O}$ and $\hat{M}$ combine as

$$\hat{O} \otimes \hat{M}.$$

The trace of an operator in this notation is given as

$$\mathrm{Tr}\big[\hat{O}\big] = \sum_i o_{ii} = \sum_i \langle i|\hat{O}|i\rangle,$$

where o_{ij} are the matrix elements of $\hat{O}$ and all $|i\rangle$ form some basis for the entire Hilbert space. The trace of an operator is scalar valued. However the partial trace is operator valued. Consider an operator $\hat{\Upsilon}$ defined on the tensor product of two Hilbert spaces $H_A \otimes H_B$. The partial trace over H_A is

$$\mathrm{Tr}_A\left[\hat{\Upsilon}\right] = \sum_i \big(\langle i| \otimes \hat{I}\big)\hat{\Upsilon}\big(|i\rangle \otimes \hat{I}\big)$$

and the partial trace over H_B is

$$\mathrm{Tr}_B\left[\hat{\Upsilon}\right] = \sum_i \big(\hat{I} \otimes \langle i|\big)\hat{\Upsilon}\big(\hat{I} \otimes |i\rangle\big).$$

In the special case where $\hat{\Upsilon} = \hat{A} \otimes \hat{B}$, for some operators $\hat{A}$ in H_A and $\hat{B}$ in H_B, we get that

$$\mathrm{Tr}_A\left[\hat{\Upsilon}\right] = \sum_i \langle i|\hat{A}|i\rangle \otimes \hat{I}\hat{B}\hat{I} = \mathrm{Tr}\big[\hat{A}\big]\hat{B}$$

and also that $\mathrm{Tr}_B\left[\hat{\Upsilon}\right] = \mathrm{Tr}\big[\hat{B}\big]\hat{A}$.

There is now finally enough background to describe qubits. Qubits are in a two-dimensional state space, and are often described in the basis

$$\{|0\rangle, |1\rangle\}$$

which is called the Z-basis. This is because these basis states are taken to be the eigenstates of the Pauli-Z operator, and have corresponding eigenvalues of 1 and −1 respectively. However, the X-basis is also commonly used. The X-basis is given by $\{|+\rangle, |-\rangle\}$, where

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

and

$$|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle).$$

4.2.2 Mixed States

States represented by kets are called pure states because there is always some basis in which there exists some measurement that will have a definite result with probability 1. A generalization of pure states comes in the form of mixed states. Whereas pure states can only express quantum superpositions, mixed states can also express classical mixtures of states. Mixed states can only be expressed through density matrices.

Density matrices are not denoted with bras or kets or any other fancy symbols besides their labelling symbol. The projector of a pure state is how one can represent it as a density matrix. For example, the density matrix of a pure state, $|\psi\rangle$, which is called ρ, is notated as

$$\rho = |\psi\rangle\langle\psi| .$$

This can be generalized to multiple states at different probabilities. For a set of pure states $\{|\psi_0\rangle, |\psi_1\rangle, \ldots\}$ and their corresponding probabilities $\{p_0, p_1, \ldots\}$, the mixed state σ is defined as

$$\sigma = \sum_i p_i |\psi_i\rangle\langle\psi_i| ,$$

where the sum of the probabilities must equal 1. This requirement manifests in σ as the requirement that its trace be 1.

Everything about bases and Kronecker products that was true for pure states works similarly for density matrices.

Operators for density matrices work much the same way that they do for pure states. However, an important difference is that they must act on both sides of the density matrix. For example, for an operator $\hat{O}$, it acts on the density matrix ρ as

$$\hat{O}\rho\hat{O}^\dagger$$

This relation becomes apparent when considering when ρ encodes only a pure state $|\psi\rangle$, so the previous relation simplifies to

$$\hat{O}|\psi\rangle\langle\psi|\hat{O}^\dagger$$

which is sensible because $\hat{O}$ acts on both the bra and the ket in valid manners.

Quantum measurement theory deals with mapping the mathematical formalism of quantum states and gates to probabilities that are observable in an experiment. In particular, for a mixed state ρ, and a collection of generalized measurement operators $\{\hat{M}_k\}$, the state after the measurement is

$$\rho_k = \frac{\hat{M}_k\rho\hat{M}_k^\dagger}{\mathrm{Tr}\left[\hat{M}_k\rho\hat{M}_k^\dagger\right]} ,$$

where the measurement operators must obey $\hat{M}_k^\dagger\hat{M}_k \geq 0$ and $\sum_k \hat{M}_k^\dagger\hat{M}_k = \hat{I}$ in order to obey the laws of probability. The probability for outcome k is

$$\Pr(k \mid \rho) = \mathrm{Tr}\left[\hat{M}_k^\dagger\hat{M}_k\rho\right]. \tag{2}$$

The measurement update rule is like Bayes' rule for quantum states where ρ is the prior state and ρ_k is the posterior state conditioned on outcome k.

4.2.3 Channels and the Kraus Representation

Channels for quantum information are similar to channels for classical information. They act upon the information that passes through them. Unlike operators, channels need not be reversible. Mathematically, quantum channels are represented as functions acting on mixed states. The following equation shows an example channel

$$B_q(\rho) = q\hat{X}\rho\hat{X} + (1-q)\hat{I}\rho\hat{I}\,, \tag{3}$$

where q is the probability that the channel applies the Pauli-X operator to the input state, and $1-q$ is the remaining probability in which the identity operator is instead applied.

An oft-used manner of representing channels is the Kraus representation. All channels have a Kraus representation. A channel in the Kraus representation is instead given as a set of operators. The operators that make up the set of the Kraus representation are called that channel's Kraus operators. Importantly, Kraus operators are not really operators in the sense that they need not be unitary. If supposing a channel, C, has the set of Kraus operators $\{K_0, K_1, \ldots\}$, then the channel as a function is given by

$$C(\rho) = \sum_i K_i \rho K_i^\dagger. \tag{4}$$

The set of Kraus operators for B_q is accordingly

$$\left\{\sqrt{q}\hat{X}, \sqrt{1-q}\hat{I}\right\}. \tag{5}$$

One can verify that the set given above in (5) is accurate by plugging it into equation (4) and recovering equation (3). Additionally, since $\hat{X}$ and $\hat{I}$ are unitary, $\sqrt{q}\hat{X}$ and $\sqrt{1-q}\hat{I}$ cannot be unitary, so the Kraus operators for B_q are indeed not unitary.

4.3 Code Performance

A quantum error correcting code is an encoding of quantum information in such a manner as to try and protect that information against errors. The types of errors that quantum information can experience vary wildly depending on the physical realization of the quantum system. Additionally, specific quantum error correcting codes may be extremely resilient against certain types of errors but be relatively useless against other types of errors.

There are many different possible metrics to evaluate how "good" an error correcting code is. In general, any metric must suss out whether an error correcting code can allow information to pass through a noisy channel and allow the original information to be decoded well. The subsequent sections describe the way in which codes are evaluated in this thesis.

4.3.1 Average Gate Fidelity

Gates are operators that are supposed to be applied to states. However, since gates in reality are imperfect, they are often modelled as channels. Since all the subsequent math works for gates as channels, even channels that are not typically thought of as gates will be put through the subsequent formulas. Specifically, these formulas for gates will be used to evaluate noise channels.

The concept of average gate fidelity plays a central role in the evaluation of a code's fidelity, but to understand either, an understanding of fidelity is required. Fidelity is a measure of indistinguishability between two objects. A fidelity of 1 means that two objects are completely indistinguishable, whereas a fidelity of 0 means that two objects are as distinct as possible. For quantum states, distinguishability is determined by whether there exists some sort of measurement that can pick apart the states. [4]

The fidelity of two density matrices ρ and σ is given as

$$F(\rho,\sigma) = \left(\mathrm{Tr}\left(\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}\right)\right)^2, \tag{6}$$

where the equation returns how close ρ is to σ on a scale from 0 to 1. This equation can then be used to get a gate fidelity between two gates, $O(\rho)$ and $M(\rho)$, that act on a density matrix, ρ, and give back a density matrix. The gate fidelity is therefore

$$F(O,M,\rho) = \left(\mathrm{Tr}\left(\sqrt{\sqrt{O(\rho)}M(\rho)\sqrt{O(\rho)}}\right)\right)^2, \tag{7}$$

and it gives how similar O and M are when acting on a specific state ρ. Importantly, the fidelity for the same two gates can vary depending on the state. [4]

To get a more general idea of how gates O and M compare, it is useful to not need to compare them for specific states. Average gate fidelity achieves exactly this purpose. Average gate fidelity is calculated by taking the specific gate fidelity for a general state, and integrating that and averaging that out across all states. If $\{K_0, K_1, \ldots\}$ is the set of Kraus operators for gate O, then the average gate fidelity ends up being

$$\overline{F}(O, M) = \frac{d + \sum_i \left|\mathrm{Tr}\left(K_i M^\dagger\right)\right|^2}{d^2 + d}, \tag{8}$$

where d is the dimension of the Hilbert space that the operators act on. [5]

4.3.2 Noise

Noise is taken to be harmful but unavoidable aspect of managing quantum states. Generally less impactful noise is desired. Noise is generally modelled as a channel, especially because noise tends not to be completely reversible.

There are many different types of noise that quantum states can undergo, but generally, the less destructive the noise is, the higher the noise channel's fidelity is when compared against the identity channel. The identity channel is defined as

$$I(\rho) = \rho$$

which takes in a state and gives back the same state unchanged.

Because it is desired that noise is as un-impactful as possible, the ideal noise channel is the identity channel. A noise channel that is the identity channel means that there is no change to the information passing through the channel. This makes the identity channel an ideal target of comparison for fidelity for any noise channel: the higher a noise channel's fidelity is with the identity channel, the "better" that noise is. As such, plugging in I for the second argument of the function given by equation (8) gives a noise channel's average fidelity

$$\overline{F}(O) = \frac{d + \sum_i \left|\mathrm{Tr}(K_i)\right|^2}{d^2 + d}, \tag{9}$$

which is essentially a measure of how "good" or unimpactful the noise is. [5]

4.3.3 Recoveries

In the repetition code example, after passing through the noisy channel, a recovery procedure was applied to the errant information to attempt to get the original information. Recovery procedures in the quantum case tend to be more difficult to construct because one cannot just see the states and correct them. Measurements can disrupt state. Recovery procedures tend to require more clever tricks to indirectly measure whether and where errors have occurred. Additionally, quantum information can undergo many different types of errors when passing through a noisy channel. Most classical error correction schemes only deal with bit flips.

In this thesis, only a theoretical optimal recovery channel is considered. For a noise channel, $\mathcal{E}$, the optimal recovery channel is given by $\mathcal{R}^\star$ where

$$\mathcal{R}^\star(\mathcal{E}) = \arg\max_{\mathcal{R}} \overline{F}(\mathcal{R} \circ \mathcal{E}) \tag{10}$$

The obtained recovery is only realizable with perfect knowledge of the error channel and is therefore not physically realistic in actual quantum computing systems. There is a semi-definite programming procedure for determining $\mathcal{R}^\star$, but the specific details of that are outside the scope of this thesis. [6]

There exist other recovery procedures that aren't optimal but are more realistic. For example, other recovery operations may be used that don't presuppose a perfect mathematical model of the noise. Those recoveries are not considered in this thesis because the optimal recovery still gives a good measure of how possible it is to correct errors after a noise channel. The optimal recovery provides a sense for the bound.

4.3.4 Code Fidelity

A code's fidelity is the fidelity of encoding information, sending it through a channel, and decoding/recovering the original information. Those processes are composed into a single channel, and the average gate fidelity of that channel is the code's fidelity. If E is the channel that encodes information into a code, $\mathcal{E}$ is the noise channel, and $\mathcal{R}$ is the channel that recovers/decodes the received information, then the code's fidelity is given by

$$\overline{F}(\mathcal{R} \circ \mathcal{E} \circ E). \tag{11}$$

This formula gives how close the actions of encoding, passing through a noisy channel, and recovering/decoding are to the identity channel. The better the code is, the closer the combined channel is to the identity channel, where information can pass through unchanged. Consequently, one can think of a code as being a modification to the noise channel, and a good code is one that modifies a noise channel to have a good recovery.

Within this thesis, a code is evaluated based on its optimal fidelity and its fidelity under no recovery. The optimal fidelity is a code's fidelity under an optimal recovery. A code's fidelity under no recovery is how resilient the information encoded by the code is when no recovery procedure other than decoding is applied after passing through a noisy channel. The optimal recovery and no recovery fidelities can be explicitly written as

$$\overline{F}(\mathcal{R}^\star(\mathcal{E} \circ E) \circ \mathcal{E} \circ E)$$

and

$$\overline{F}(E^\dagger \circ \mathcal{E} \circ E)$$

respectively where $\mathcal{R}^\star$ is given by equation (10).

4.4 Quantum Harmonic Oscillator

In classical mechanics, a harmonic oscillator is a spring-like system. The quantum harmonic oscillator is much like a classical harmonic oscillator. The harmonic oscillator potential is quadratic. The quantum harmonic oscillator is a general structure that can arise when some observable and its time-derivative have a specific relationship to each other. The canonical example is with the observable of position, $\hat{x}$, and its time derivative of momentum, $\hat{p}$, when they conspire together to give a Hamiltonian (total energy for the system) of

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega\hat{x}^2\,, \tag{12}$$

where m is the mass of the oscillator, ω is the angular frequency of oscillaton, and $[\hat{x}, \hat{p}] = i$. Notice that the form of this total energy is similar to a classical example of a mass on a spring. Importantly, a quantum harmonic oscillator has discrete, evenly-spaced allowed energy levels. This is unlike other quantum systems, like infinite or finite square wells or the hydrogen atom.

A quantum harmonic oscillator has infinitely many energy levels. The Fock basis is the set of states given by

$$\{|0\rangle, |1\rangle, |2\rangle, \dots\}\,,$$

which in this scenario, refers to the energy levels of a harmonic oscillator. Consequently, a state in an ideal quantum harmonic oscillator lives in an infinite-dimensional Hilbert space. In practice, usually there is a cutoff of energy levels because one does not have access to infinite energy.

Three important operators for the quantum harmonic oscillator are the two ladder operators and the number operator. The ladder operators are so-named because the energy levels of the harmonic oscillator can be thought of as rungs of a ladder, and the ladder operators change the energy level of the Fock basis states. The ladder operator $\hat{a}$ acts on a state by lowering its energy level. The conjugate-transpose of $\hat{a}$ raises the energy level of the acted upon state. The number operator, $\hat{n} = \hat{a}^\dagger\hat{a}$, is an operator such that all Fock basis states are eigenstates of it. The operators work like

$$\hat{a}\,|n\rangle = \sqrt{n}\,|n-1\rangle \qquad \hat{a}^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle \qquad \hat{n}\,|n\rangle = \hat{a}^\dagger\hat{a}\,|n\rangle = n\,|n\rangle\,, \tag{13}$$

where $|n\rangle$ is one of the Fock basis states.

Coherent states are eigenstates of the lowering operator, and can be considered to be the states that represent classical motion in a harmonic oscillator. Coherent states are denoted as $|\alpha\rangle$ where α is a complex value and the eignevalue of the lowering operator. Coherent states are expressed in the Fock basis as

$$|\alpha\rangle = \exp\left(-\frac{|\alpha|^2}{2}\right)\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\,|n\rangle\,. \tag{14}$$

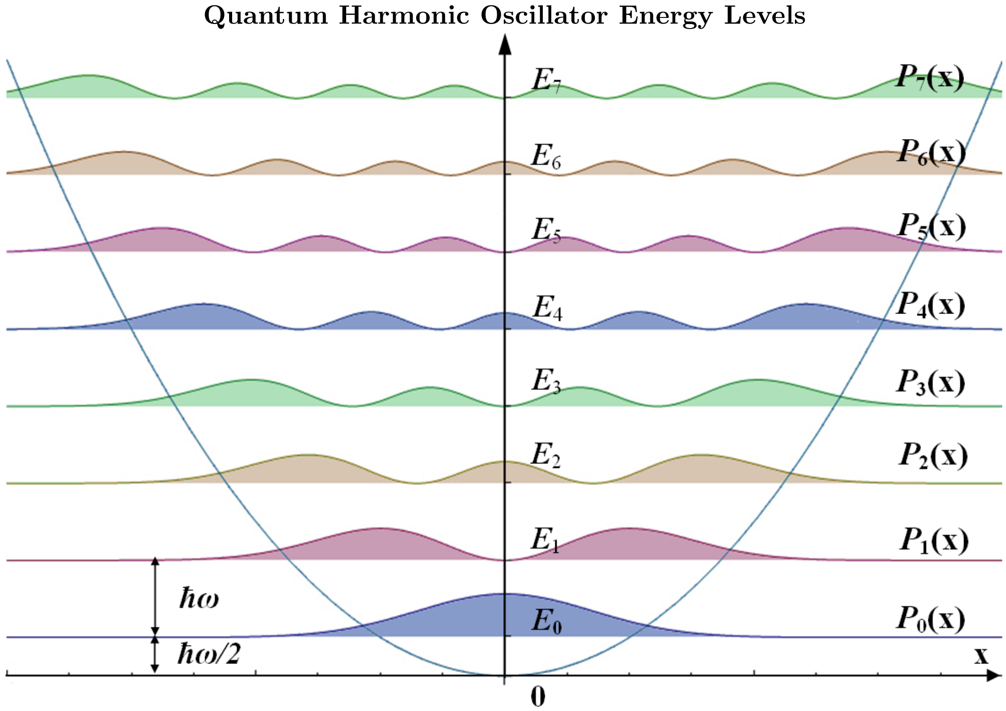

Figure 3: This plot is the visualization of the energy of a quantum harmonic oscillator. The background parabola in blue represents the harmonic oscillator potential. Each energy is denoted as E_i, starting from E_0 and going to E_7. The curves with colored areas labelled with $P_i(x)$ represent the probability distributions of finding a particle at at any point along the x axis when at the corresponding energy level. The ground state is at an energy of $\hbar\omega/2$, and every subsequent energy level is $\hbar\omega$ more than the previous. This image was taken from [7].

A natural number in a ket will refer to a Fock basis state unless explicitly stated otherwise to be a coherent state. [7]

One can encode information as a bosons' energy levels in a quantum harmonic oscillator. A state of $|2\rangle$, for example, can be realized by trapping two bosons, photons for example, in a quantum harmonic oscillator. The difference between using a boson and a fermion is that multiple bosons can occupy the same state whereas fermions cannot. Only bosonic codes are considered in this thesis.

A quantum harmonic oscillator can be an advantageous realization of qubits compared to other physical systems. For instance, it allows one to encode multiple qubits with a single harmonic oscillator since it is infinite-dimensional. A two-level atom, on the other hand, can encode only qubit. However, these considerations are outside the scope of this thesis.

A quantum harmonic oscillator can undergo many different forms of noise. The most prevalent form is loss noise, with dephasing noise also being not too insignificant [8]. While dephasing noise is important, it is not considered within this thesis. Loss noise represents a loss of energy, which can also be caused by the loss of bosons from the oscillator. Loss noise as a channel has the Kraus operators of

$$\left\{ \frac{(1-\exp(-\kappa t))^{\frac{k}{2}}}{\sqrt{k!}} \exp\left(-\frac{\kappa \hat{n} t}{2}\right) \hat{a}^k \mid k \in \mathbb{N} \right\}, \tag{15}$$

where κ can be considered as the degree or strength of loss noise and t represents time. κ must have units of s^{-1}, so it can accordingly be thought of as a rate. The interpretation of loss noise as a decrease of total energy can be inferred from the operators present within each Kraus operator: there are $\hat{n}$ and $\hat{a}$ operators, but no $\hat{a}^\dagger$ operators, so these Kraus operators can only reduce the number of bosons. [9]

4.5 Wigner Plots

Wigner plots are a useful visualization tool for viewing quantum states. The Wigner plot of a state shows a quasi-probability distribution of that state in phase space. Phase spaces are the possible combinations of a relevant observable and its derivative. Again, canonically, the used observable is position and its derivative is taken to be momentum. A Wigner plot is a quasi-probability distribution because it can have negative values, but it is similar to a joint distribution in that one can marginalize over either position or momentum to get a probability distribution. The exact formula or method of generating Wigner plots is unimportant for the work in this thesis.

Example Wigner Plot of the Trivial Code

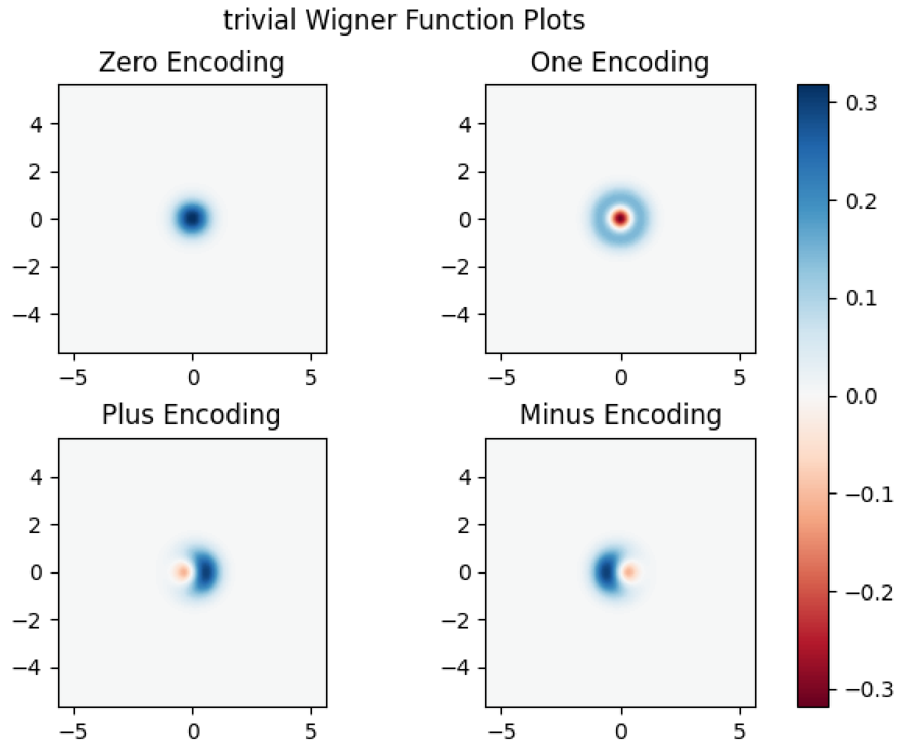

Figure 4: These are the Wigner plots for the simplest quantum code. In this code, the logical 0 state is encoded as the ground state of the quantum harmonic oscillator, and the logical 1 state is encoded as the first excited state of the quantum harmonic oscillator. If the quantum harmonic oscillator was in position and momentum, then position would be the horizontal axis of the plots and momentum would be the vertical. The axes are normalized with respect to coherent states of the quantum harmonic oscillator, but this scaling is done with natural units and is generally unimportant.

4.6 Bosonic Rotation Codes

Bosonic rotation codes are encodings of a logical 0 and a logical 1 state into a quantum harmonic oscillator with bosons such that there is a discrete rotational symmetry in the phase space. This rotational symmetry is visible on the Wigner plots of the encodings. [10]

Importantly, the logical zero and one states are encoded with different sets of Fock basis states. More specifically, the logical zero state is encoded with the following set of states

$$\{|2kN\rangle \mid k \in \mathbb{N}\} \tag{16}$$

and logical one is encoded with the next set of states

$$\{|(2k+1)N\rangle \mid k \in \mathbb{N}\} \tag{17}$$

where N is the degree of rotational symmetry of the code. Though all states could be used, there is often some cutoff after which higher-numbered Fock states are not used. [10]

4.6.1 Binomial and Cat Codes

Binomial and cat codes are specific types of rotation codes. They both tend to very robust against loss and dephasing errors. Cat codes are a commonly used form of error correction, but binomial codes are increasingly

Example Wigner Plot of a Cat Code

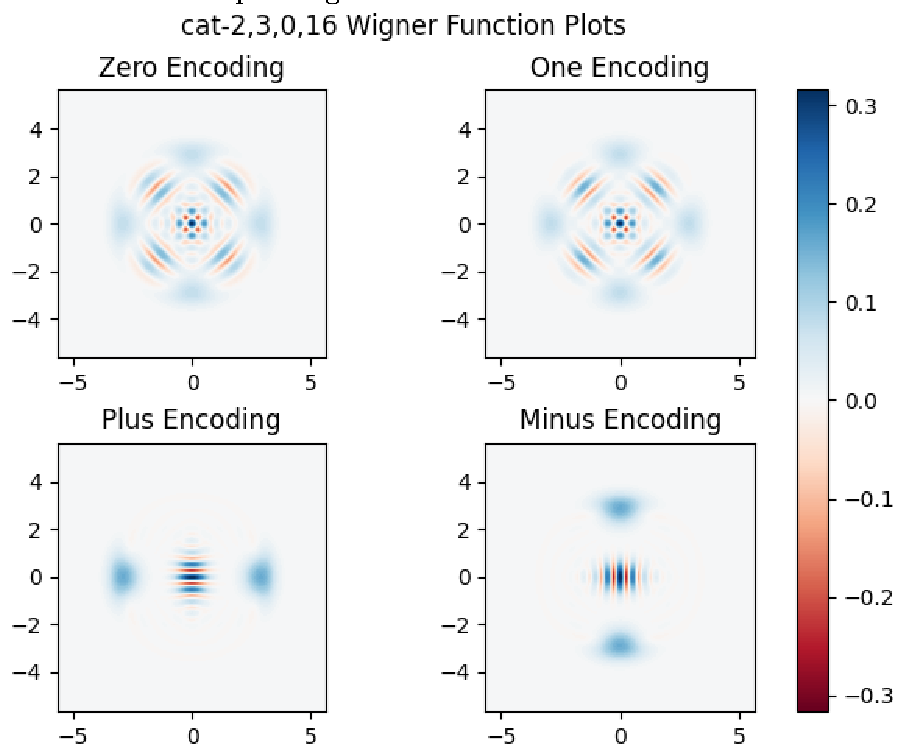

Figure 5: These are the Wigner plots for a cat code. The parameters for this cat code are that it has a twofold rotation symmetry, a primitive coherent state of $|3\rangle$, 0 squeezing (squeezing is unimportant for this thesis), and a Fock dimension of 16. This cat code is made by superposing the coherent state $|3\rangle$ 4 times at different displacements and rotations for both the logical 0 and 1 states.

being used as the benchmark code due to their resilience. Binomial codes are now amongst the most successful forms of quantum error correction. Both binomial and cat codes have been implemented in physical systems that have reached the break-even point where quantum error correction is not harming the information fidelity. [10][1][2]

Cat codes are generated by taking a coherent state and repeating it with different displacements and rotations such that the resulting code is a rotation code. The logical 0 and 1 states, denoted as $|0_L\rangle$ and $|1_L\rangle$ respectively, are equivalently given by

$$|0_L\rangle = \frac{1}{\mathcal{N}} \sum_{j=0}^{2N} \hat{D}\left(\alpha \exp\left(\frac{i\pi}{N} j\right)\right) |0\rangle \tag{18}$$

and

$$|1_L\rangle = \frac{1}{\mathcal{N}} \sum_{j=0}^{2N} (-1)^j \hat{D}\left(\alpha \exp\left(\frac{i\pi}{N} j\right)\right) |0\rangle , \tag{19}$$

where $\hat{D}$ is the displacement operator given by $\hat{D}(s) = \exp\left(s\hat{a}^\dagger - s^*\hat{a}\right)$, α is the coherent state eigenvalue, and $\mathcal{N}$ is the scalar that ensures that the states are of unit length. Cat codes are considered to be simple and were used as the "default" rotation code before binomial codes were created. [10][5]

Binomial codes get their name from their use of binomial coefficients as amplitudes for the number states. The logical plus and minus states, denoted as $|+_L\rangle$ and $|-_L\rangle$ respectively, are given as

$$|+_L\rangle = \frac{1}{\mathcal{N}} \sum_{j=0}^{n+2} \sqrt{\binom{n+1}{j}} |jN\rangle \tag{20}$$

and

$$|-_L\rangle = \frac{1}{\mathcal{N}} \sum_{j=0}^{n+2} (-1)^j \sqrt{\binom{n+1}{j}} |jN\rangle , \tag{21}$$

Structure of an $N = 2$ Rotation Code

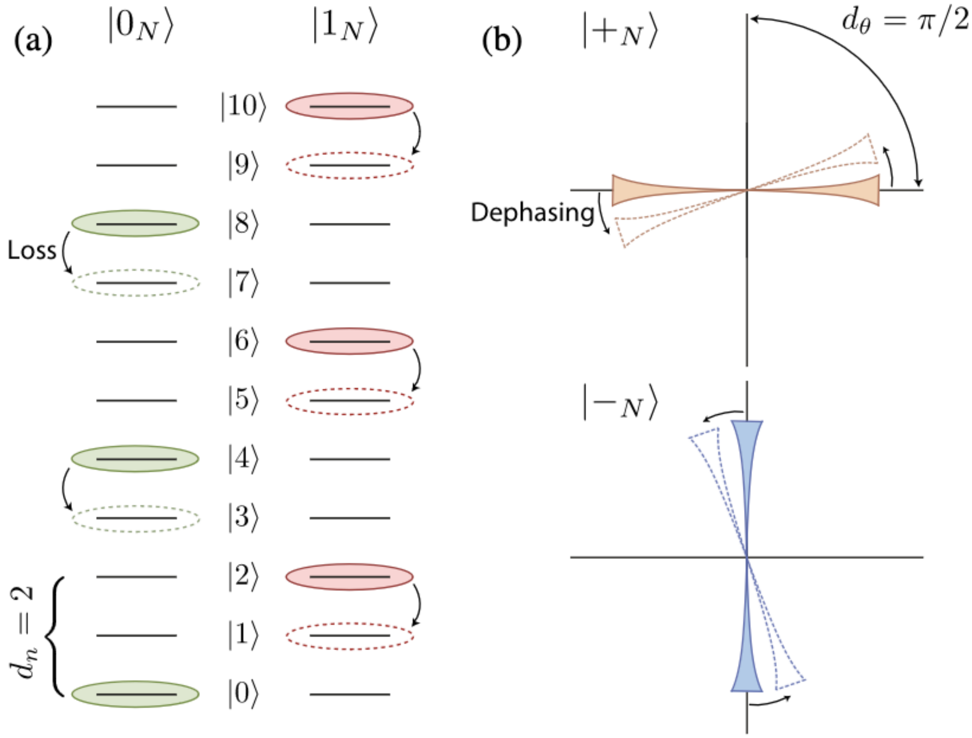

Figure 6: This figure demonstrates the structure of an $N = 2$ rotation code. Section (a) of the figure indicates which Fock basis states are used for which logical state. Section (b) shows a cartoon phase space plot for potential plus and minus states. The logical 0 state has support on the $|2Ni\rangle$ states and the logical 1 state has support on the $|(2N+1)i\rangle$ states for all $i \in \mathbb{N}$ up to a cutoff. Because there is a twofold rotation symmetry, the plus and minus states are the same when rotating by π radians. Loss noise is the loss of bosons, which lowers the energies in a state, as demonstrated in section (a). Dephasing noise is demonstrated in section (b) but is unimportant for this thesis. This figure was taken from [10].

where $\mathcal{N}$ is a scalar such that the resulting states are normalized. Binomial codes are among the best rotation-symmetric bosonic codes. [11][10]

4.7 Motivation for Random Codes

The motivation to analyze random codes in this area comes from their performance in other cases. Randomly generated covariant codes that are generated by operators that commute with $\mathrm{U}(1)$ and $\mathrm{SU}(d)$ elements reach the limits of what is possible with error correction against erasure noise. The operators that generate random codes are called Haar random unitaries[1]; they can be applied to a state and essentially a random state is given back. [12]

In this project however, the codes are not covariant and there is no known relevant selection criteria for the used Haar random unitaries. Additionally, the relevant type of noise for the quantum harmonic oscillator is loss noise as opposed to erasure noise. The point of this project was to determine whether random codes could succeed in this area.

5 Methodology

5.1 General Methodology

In order to determine whether random codes could provide viable encodings, their correcting capabilities were determined numerically and compared to the error-correction capabilities of binomial codes in different regimes.

[1]This is a simple tutorial on the concept `https://pennylane.ai/qml/demos/tutorial_haar_measure.html`.

Example Wigner Plot of a Binomial Code

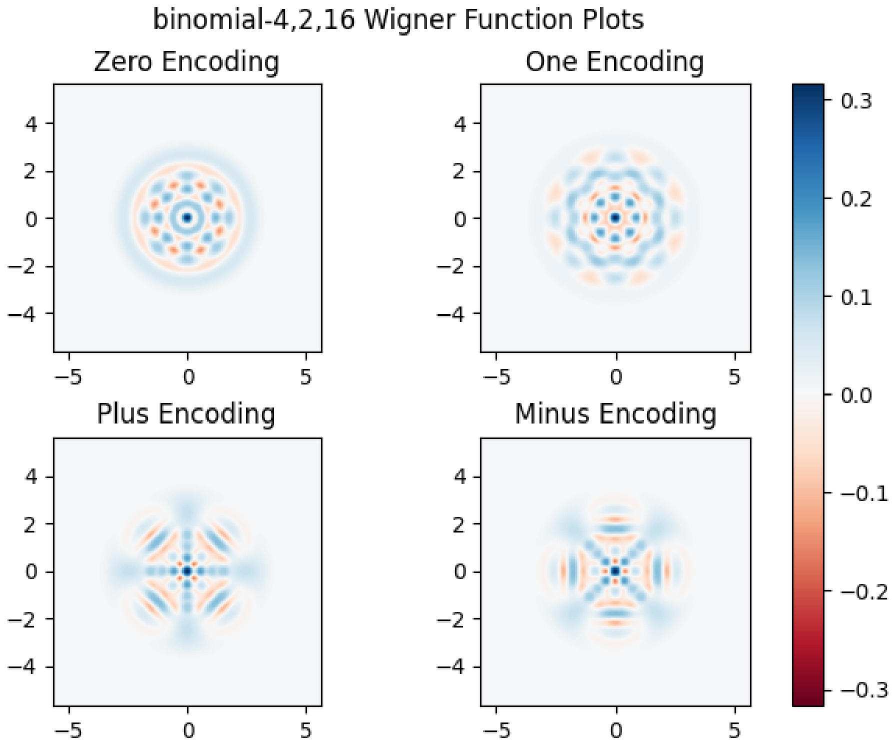

Figure 7: These are the Wigner plots for a specific binomial code. This code has a fourfold rotation symmetry, an average boson number of 2, and a dimension cutoff of 16.

Each regime consisted of a unique loss noise amount and degree of rotation symmetry of the code.

In a regime of a specific rotation symmetry degree and specific loss noise amount, multiple binomial codes and random codes were generated and compared. The generated codes varied in their average photon number. Low average photon numbers were used because those tended to generate the most performant codes in terms of correcting capabilities. The exact average photon numbers weren't directly controllable because the random codes were generated in such a manner that the photon number couldn't be specified exactly.

The separation of regimes by loss noise amount, κ was important because the correcting capability of a code depends on the error channel, and different κs lead to different error channels. The additional separation of regimes by rotation symmetry degree was done somewhat arbitrarily.

Then, for each generated code, its optimal recovery and its no recovery fidelities were calculated. The best fidelities between random codes and binomial codes were then compared within each regime. If a random code had a better fidelity than a binomial code in a regime, it was compared to the best binomial code in the regime. The similarities of the codes were compared in order to determine whether the random codes were "accidentally recreating" binomial codes or whether the random codes were making different codes that were better.

5.1.1 Generation of Random Codes

The generation of random codes involved projecting Haar random states onto the code-word projectors of rotation codes. For a Haar random state in the Fock basis, $|\psi\rangle$, the generated encoding scheme for logical 0 (hence labelled with the state $|0_L\rangle$) and logical 1 (hence labelled with the state $|1_L\rangle$) was

$$|0_L\rangle = \frac{1}{\mathcal{N}_0}\left(\sum_{i=0}^{\infty} |2iN\rangle\langle 2iN|\right)|\psi\rangle \tag{22}$$

and

$$|1_L\rangle = \frac{1}{\mathcal{N}_1}\left(\sum_{i=0}^{\infty} |(2i+1)N\rangle\langle (2i+1)N|\right)|\psi\rangle\ , \tag{23}$$

where $\mathcal{N}_0$ and $\mathcal{N}_1$ are scalars such that the resulting state is normalized, and N is the degree of rotation symmetry for the generated code.

Wigner Plots of an Arbitrary Random Code

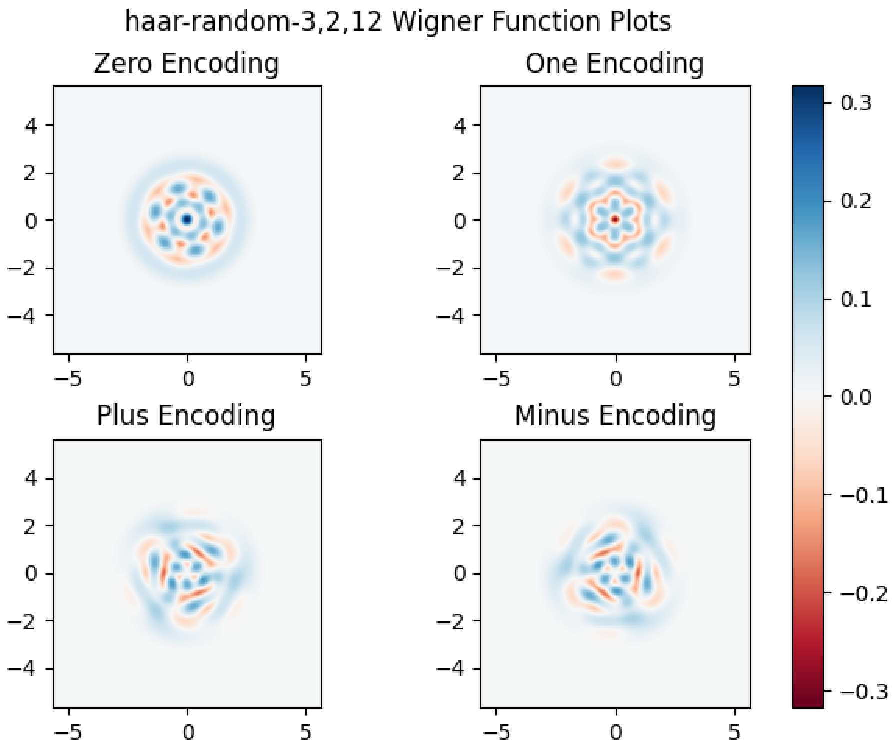

Figure 8: These are the Wigner plots for a set of encodings given by an arbitrary random code generated by the previously described procedure. The Wigner plots for random codes vary wildly, and this plot is just one specific example. In this example, the code has a threefold symmetry, an attempted average boson number of 2, and dimension cutoff of 12.

5.1.2 Comparison of Winning Random Codes to Binomial Codes

After calculating the fidelities of random codes and binomial codes in a regime, the best codes for each regime were taken and their fidelities were compared. Random codes that were "winners" (i.e. those that had the highest fidelity of all codes in a regime) were taken and compared against the best binomial code in their regime. This comparison was carried out in order to figure out whether winning random codes generated codes with similar structure to binomial codes, or whether better codes could be found with completely different structures.

The comparison of winning random codes to binomial codes involved comparing their fidelities, comparing their Wigner plots, and inspecting the magnitude squared of the inner product of their logical states. Using a b subscript for logical states of the best binomial code in a regime and the r subscript for logical states of the best random code in a regime, the following quantities were calculated

$$|\langle 0_b | 0_r \rangle|^2 \qquad |\langle 1_b | 1_r \rangle|^2 \qquad |\langle +_b | +_r \rangle|^2. \tag{24}$$

The closer any of these quantities were to 1, the more similar those specific encodings were. The calculation of $|\langle -_b | -_r \rangle|^2$ is redundant because

$$|\langle +_b | +_r \rangle|^2 = |\langle -_b | -_r \rangle|^2.$$

These specific calculations give meaningful values because both the binomial and random codes use the same Fock basis states for their 0 and 1 encodings in a given regime, so all that differs between their encodings is the distribution of amplitudes. The calculations given by equation (24) then capture the similarity of the coefficients used in the encodings.

5.2 Methodology Constraints

Only rotation symmetry degrees of $N = 2$ through $N = 4$ were examined because they seemed to produce the best binomial and random codes. This seems to be consistent with prior work which examined rotation codes up to $N = 5$. [10]

In order to keep the project size manageable, the codes were only subjected to loss noise. However, dephasing noise is another important type of noise that is ignored in this work. Moreover, based on preliminary results (not presented in this thesis), the random codes were especially bad at correcting dephasing errors. This underperformance seemed to vary based on the method of generating random codes, but no systematic study was carried out due to time constraints. However, loss noise is the more prevalent and therefore more important noise type to account for in most bosonic qubit implementations. This was an interesting insight and we did not have time to understand and then explain the bad performance. [8]

The number of trials for how many random codes were generated for each regime was arbitrarily determined so that some winners would get generated, but also so that the code wouldn't take too long to run and iterate upon. A more thorough study may run with more trials and allow the code to run for a while. Ultimately, understanding what portion of random codes outperform binomial codes in each regime is of interest, but due to time and computational constraints, that analysis is not presented in this thesis. Instead, this work merely demonstrates that random codes are capable of performing as good, or better than, state-of-the-art encodings.

6 Results

After performing the numerical experiments, many "winning" random codes were found. However, unsurprisingly, most random codes were not winners. There weren't winning random codes in each regime, but there were winning random codes for every rotation symmetry degree, N. In regimes with smaller amounts of loss noise, κ, the binomial codes tended to have higher fidelities than random codes. It is unclear if this indicates a lack of performance of random codes in a low-noise environment, or if binomial codes are dominant in this regime.

It was possible that the high-fidelity random codes that were generated were nearly the same codes as binomial codes. While occasionally a random code was extremely similar to a binomial code, such random codes were the exception rather than the rule. The winning random codes tended to be unlike binomial codes, even if their optimal average fidelities were close. Subjectively, the random codes tended to have relatively different Wigner plots, and the dot products from equation (24) tended to vary. Some would be close to 1 and others closer to 0; there didn't seem to be any sort of pattern. The differences in fidelities between the winning random codes and the best binomial codes tended to be extremely small: differences were on the order of 10^{-4} to 10^{-5} typically. Such differences are small, but since the high-performing fidelities are already high, the fractional differences are relatively large.

7 Conclusion

In this work we showed that random codes can achieve comparable or better performance than binomial codes in a variety of loss noise channels. Furthermore, we were able to generate codes that were significantly different than the binomial codes we compared against. While we were not able to complete a full study of the prevalence of the highly performant random codes, early research indicates that random codes were more likely to perform better than binomial codes when more loss was present. This was true regardless of the degree of rotational symmetry. Although most random codes do not outperform the best bosonic encodings, we have shown that it is possible to randomly generate codes that are better than binomial codes in certain regimes. Such winning random codes are generally distinct from binomial codes and can occur at all different values of rotation degree symmetry, N.

We emphasize that binomial codes are very high-performing codes in the noise regimes that were studied. Even with this high bar, we were able to demonstrate a proof-of-concept by generating random codes that performed better in some cases. Whether binomial codes always perform better at lower κ remains to be seen.

8 Further Research

There is much room for improvement on the work performed in this thesis. The foremost improvement is to more thoroughly check lower κ regimes and run trials with more samples. The most useful continuation to this project would be to analyze random codes with dephasing noise in addition to loss noise. Another useful change would be to check the fidelities of other recoveries instead of just the optimal recovery. A novel way to continue the

Numerically Determined Fidelities for Binomial and Random Codes

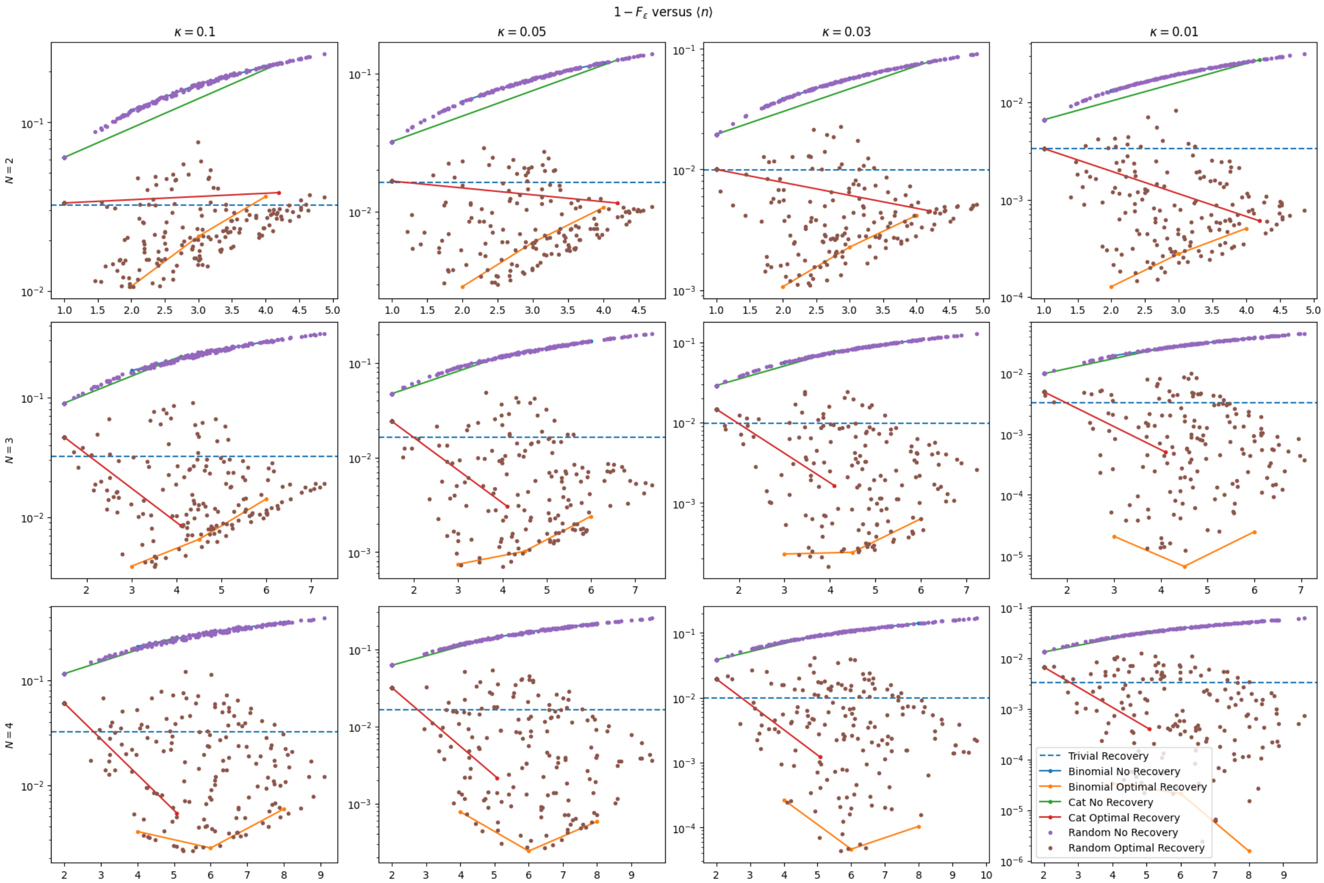

Figure 9: There are twelve plots, one for each regime. Each regime is specified by a rotation symmetry, N, and a loss noise amount, κ. Each plot has error rate ($1 - F_{\mathcal{E}}$ where $F_{\mathcal{E}}$ is the fidelity) on the y-axis and $\langle \hat{n} \rangle$ on the x-axis. Each code has two associated data points: code error without any recovery, and code error with optimal recovery. The no recovery points all overlap.

work in this thesis would be to try and implement some sort of genetic algorithm to select and "breed" winning random codes in an effort to generate better codes. Such a process may be instructive in determining if there are other structures that generate good encoding schemes. It would additionally be fruitful to see how experimentally viable and producible specific random codes are in real physical systems are too.

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Appendices

A Implementation Details

This project was done in Python 3.10, mainly using the Qutip library. The specific dependencies and their versions are available at `https://github.com/SaurabhTotey/Random-Quantum-Codes/blob/main/requirements.txt`. A significant portion of the time spent on this thesis was spent on migrating some old code from using an SDP solver written in Matlab to using CVXPy, which is a Python library. However, the SDP solvers start diverging on their results for numbers on the orders of magnitude of around 10^{-6} or 10^{-7}. A last implementation detail is that a lot of time was spent on making the code able to cache results for non-random parts of the code. This saved much time when running trials.

B Full Code

The full repository for this work is available at `https://github.com/SaurabhTotey/Random-Quantum-Codes`. The specific experiment results are available at `https://github.com/SaurabhTotey/Random-Quantum-Codes/blob/main/DemoHaarRandomCodes.ipynb`. Project dependencies and their versions are available at `https://github.com/SaurabhTotey/Random-Quantum-Codes/blob/main/requirements.txt`.