EE5605, Kernel Methods	CS21MDS14025
HW1 | 18th Nov	SAURADEEP DEBNATH

Q1: Show that the PRODUCT of kernels is a kernel.

SOLUTION:

Let $K(x, z) = k_1(x, z) * k_2(x, z)$

where k_1 and k_2 are valid kernels.

To show K is a valid kernel as well

As per Shawe-Taylor & ~~Chr~~ Cristianini, 2004, a valid & sufficient condition for a function $K(x, x')$ to be a valid kernel is that the Gram matrix K, whose elements are given by $K(x_n, x_m)$ should be Positive Semi Definite (PSD → for the sake of brevity, in this assignment) (& symmetric) for all possible choices of the set $\{x_n\}$

(i) SYMMETRIC: Since k_1 & k_2 are valid kernels, hence K_1 and K_2 corresponding Gram matrices are symmetrical.

$$\therefore k_1(x, z) = k_1(z, x) ; \; k_2(x, z) = k_2(z, x)$$

Let us denote the ith row & jth column element of a matrix by m_{ij}.

$\therefore K_{ij} = K1_{ij} * K2_{ij}$ (elementwise product)

(as per definition)

$= K1_{ji} * K_{2\,ji}$ ($\because$ K1, K2 symmetric)

$= K_{ji}$

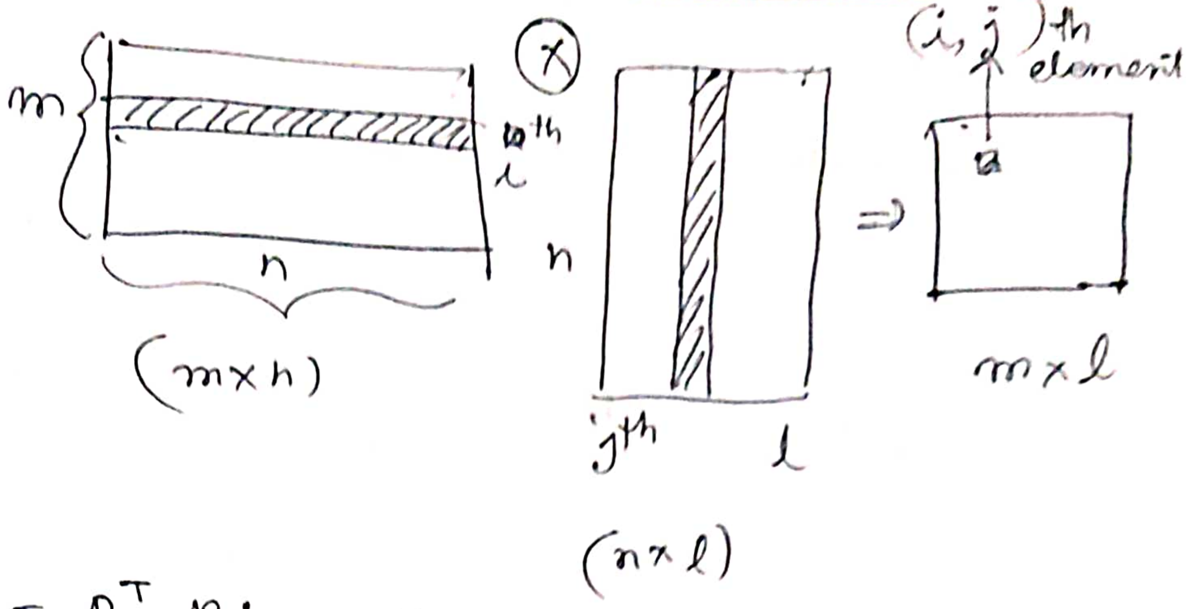
$\therefore$ K is symmetric. CONDITION 1 satisfied //

POSITIVE SEMI DEFINITE :

PART 1: We show that K1, K2 can be expressed as $\phi\phi^T$ form

PROOF: As per CHOLESKY DECOMPOSITION, A symmetric PSD matrix M can be factorized into a product $M = LL^T$ where L is a lower triangular matrix with positive diagonal elements. & L is unique.

Hence for K1, K2 there exist unique P, Q such that $K1 = PP^T$ & $K2 = QQ^T$



$$\therefore K1_{ij} = p_i^T p_j = \sum_k p_{ik} \, p_{kj}$$

(for ith row, jth column element of K1)

$$K2_{ij} = q_i^T q_j = \sum_l q_{il} \, q_{lj}$$

Now $u^T K u$ Needs to be ≥ 0 (FOR OUR PROOF)

$$u^T K u = \sum_i \sum_j u_i u_j K_{ij}$$

$$= \sum_i \sum_j u_i u_j K1_{ij} K2_{ij} \quad \text{(DEFINITON of K)}$$

$$= \sum_i \sum_j u_i u_j \left(\sum_k p_{ik} p_{kj}\right)\left(\sum_l q_{il} q_{lj}\right)$$

$$= \sum_k \sum_l \left(\sum_i u_i p_{ki} q_{il}\right)\left(\sum_j u_j p_{kj} q_{jl}\right)$$

(REARRANGING the terms)

Q2 TO SHOW:

$$k(x,y) \stackrel{def}{=} \exp(\langle x, y\rangle), \; x, y \in \mathbb{R}^d, \; d \geq 1$$

is a valid Kernel.

PROOF:

We have been given in the question →
provided a Taylor series $f(x) = \sum_{n=0}^{\infty} a_n x^n$, $|x| < \gamma$,
$x \in \mathbb{R}$ for $\gamma \in (0, \alpha]$ with $a_n \geq 0$ for all $n > 0$.

Then for $\mathcal{X}$ be a $\sqrt{\gamma}$-ball in $\mathbb{R}^d$,
$x, x' \in \mathbb{R}^d$ such that $\|x\| < \sqrt{\gamma}$.

$$K(x, x') = f(\langle x, x'\rangle)$$

$$= \sum_{n=0}^{\infty} a_n (\langle x, x'\rangle)^n \text{ is a Kernel.} \longrightarrow (1)$$

Now we know the Taylor series expansion
of e^x is:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots \text{ up to } \infty$$

$$= \sum_{n=0}^{\infty} \frac{x^n}{n!} \longrightarrow (2)$$

i.e. Comparing with eq^n (1) above

$$a_n = \frac{1}{n!}$$

Combining eq^n (1) & (2), we see :

$$K(x,y) = \sum_{n=0}^{\infty} \frac{\langle x, y\rangle^n}{n!}$$ is a valid Kernel.

$\therefore e(\langle x, y\rangle)$ is a Kernel $\boxed{\text{PROVED}}$

$\boxed{\text{ALTERNATIVELY}}$

eq(2) $\Rightarrow$ e^x is a sum of polynomials of various degrees from 0 to ∞

since both sums & products of valid Kernels are valid Kernel as well.

$\therefore$ from Taylor Series expansion of e^x

$$K(x,y) = \sum_{n=0}^{\infty} \frac{\langle x, y\rangle^n}{n!} = e(\langle x, y\rangle)$$

is a valid Kernel.

$\boxed{\text{PROVED}}$