COMP 580
Assignment 3

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Problem 1. Undirected chains: aperiodic non-bipartite

Assume the Markov chain satisfies $P(x, y) = P(y, x)$ so the underlying graph is undirected.

If the chain is aperiodic, the graph is non-bipartite

If the underlying graph were bipartite with partition $A \cup B$, then any walk must alternate between the two sides. Thus a return to any state requires an even number of steps. Hence

$$P^t(x, x) = 0 \quad \text{for all odd } t,$$

and the set of return times contains only even integers. Therefore

$$\text{period}(x) = \gcd\{t : P^t(x, x) > 0\} = 2.$$

This contradicts aperiodicity, so the graph must be non-bipartite.

If the graph is non-bipartite, the chain is aperiodic

A non-bipartite graph contains an odd cycle. Suppose the odd cycle has length k. This provides a return path of length k (odd). Using the same cycle twice gives a return path of length $2k$ (even). Since both odd and even return times exist,

$$\gcd(\text{return times}) = 1,$$

so the chain is aperiodic.

Problem 2. All States in an Irreducible Chain Have the Same Period

Let

$$d(x) = \gcd\{t : P^t(x, x) > 0\}.$$

Since the chain is irreducible, for any states x, y there exist integers m, n such that $P^m(x, y) > 0$ and $P^n(y, x) > 0$. Therefore,

$$P^{m+n}(x, x) > 0 \quad \Rightarrow \quad d(x) \mid (m + n),$$

and similarly,

$$P^{m+n}(y, y) > 0 \quad \Rightarrow \quad d(y) \mid (m + n).$$

Because this holds for every possible pair of paths and return times, $d(x)$ and $d(y)$ divide the same set of integers, thus $d(x) = d(y)$.

Hence, all states in an irreducible chain have the same period.

Problem 3. Aperiodic $\iff$ Existence of t with $P^t(x, y) > 0$ for All x, y

If the chain is aperiodic, such a t exists

A classical theorem for irreducible, aperiodic Markov chains states that there exists T such that for all $t \geq T$,

$$P^t(x, y) > 0 \quad \forall x, y.$$

Thus the desired t exists.

If such a t exists, the chain is aperiodic

If for some t,

$$P^t(x, y) > 0 \quad \forall x, y,$$

then in particular,

$$P^t(x, x) > 0.$$

Also $P^{t+1}(x, x) > 0$ since one may go $x \to y$ in t steps and then $y \to x$ in 1 step for some y. Hence both t and $t + 1$ are return times, and

$$\gcd(t, t + 1) = 1.$$

Thus the chain is aperiodic.

Problem 4. Irreducible Chain with a Self-Loop Is Aperiodic

If $P(x,x) > 0$ for some x, then 1 is a return time for x, so

$$\text{period}(x) = 1.$$

By irreducibility, all states share the same period (from Question 2). Therefore the chain is aperiodic.

Problem 5. The Matrix $P' = \alpha P + (1-\alpha)I$ is Irreducible, Aperiodic, and Shares the Same Stationary Distribution

Let $0 < \alpha < 1$ and P be irreducible.

Irreducibility of P'

Expanding $(P')^t$:

$$(P')^t = \sum_{k=0}^{t} \binom{t}{k} (1-\alpha)^{t-k} \alpha^k P^k.$$

Since P is irreducible, for any x, y there exists k with $P^k(x,y) > 0$. The term with that k contributes positively, so

$$(P')^t(x,y) > 0,$$

thus P' is irreducible.

Aperiodicity of P'

For any state x,

$$P'(x,x) = \alpha P(x,x) + (1-\alpha).$$

Even if $P(x,x) = 0$, the self-loop probability is at least $(1-\alpha) > 0$. Thus every state has a self-loop, and the period of every state is 1. Hence P' is aperiodic.

Stationary Distribution

Let π be stationary for P:

$$\pi P = \pi.$$

Then:

$$\pi P' = \pi(\alpha P + (1-\alpha)I) = \alpha \pi P + (1-\alpha)\pi = \alpha \pi + (1-\alpha)\pi = \pi.$$

Thus π is also stationary for P'.

Conclusion

The matrix $P' = \alpha P + (1 - \alpha) I$ defines an irreducible, aperiodic Markov chain that preserves the stationary distribution of the original chain.