TRICKS IN DETERMINANT

① The determinant of the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 100 & 1 & 0 & 0 \\ 100 & 200 & 1 & 0 \\ 100 & 200 & 300 & 1 \end{bmatrix}$$ is

a) 100
b) 200
✓ c) 1
d) 300

Soln

Method 1: |Solving of 4x4 matrix|
Time consuming

Method 2: whenever there is a lower or upper triangular matrix or diagonal matrix. The determinant is the product of the diagonal elements.

∴ Here det = $1 \times 1 \times 1 \times 1 = 1$.

② Value of det of this matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -2 & 2 & 0 & 0 & 0 \\ 3 & 5 & 3 & 0 & 0 \\ 1 & 4 & 7 & 4 & 0 \\ -5 & -6 & 3 & 1 & 1 \end{bmatrix}$$

Ans = $1 \times 2 \times 3 \times 4 \times 1$

$= 24$ //

② For $A = \begin{bmatrix} 1 & \tan x \\ -\tan x & 1 \end{bmatrix}$ the determinant

of $A^T A^{-1}$ is (a) $\sec^2 x$ (b) $\cos 4x$ (c) 1 (d) 0

sol^n

Note A^T is reversing rows and columns

$A^{-1} = \frac{\text{adj } A}{|A|}$

Method 1 (time consuming)

$|A^T A^{-1}| = |A^T| \, |A^{-1}|$

Method 2

note, $|A^T| = |A|$

and $|A^{-1}| = \frac{1}{|A|}$

$\therefore |A^T A^{-1}| = |A| \cdot \frac{1}{|A|} = 1$ Ans

② The matrix $A = \begin{bmatrix} a & 0 & 3 & 7 \\ 2 & 5 & 1 & 3 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & b \end{bmatrix}$ has $\det(A) = 100$

and trace$(A) = 14$

The value of $|a-b|$ is ______ ?

To find the value of 2 unknowns a and b, we need 2 equations on a and b.

First, $\because$ $\det(A) = 100$

and I know the rule that

1) $A \to \det(A)$

* If rows are interchanged then

$$\det(A) = -|A|$$

so, $R_1 \leftrightarrow R_4$ we will get

$$A' = \begin{bmatrix} 0 & 0 & 0 & b \\ 2 & 5 & 1 & 3 \\ 0 & 0 & 2 & 4 \\ a & 0 & 3 & 7 \end{bmatrix} \quad \text{then } \underset{(A')}{\det} = -|A|$$

$$\therefore \quad -b \begin{vmatrix} 2 & 5 & 1 \\ 0 & 0 & 2 \\ a & 0 & 3 \end{vmatrix} = -|A|$$

$$= -100$$

$$\therefore \quad -b\{2(0) - 5(0 - 2a) + 1(0)\}$$

$$= -b\{0 + 10a + 0\} = -100$$

$$\cancel{-}10ab = \cancel{-}100$$

$$\therefore \boxed{ab = 10} \text{ — } (1)$$

Now given trace (A) = 14

$\therefore \quad a+5+2+b = 14$

$\therefore \quad \boxed{a+b=7} \quad \text{—— } ②$

$a = 7-b$

$\therefore \quad ab = 10$

$(7-b)b = 10$

$7b - b^2 = 10$

$b^2 - 7b + 10 = 0$

$\boxed{\cancel{b(b-7)}} \quad b^2 - (5+2)b + 10 = 0$

$b^2 - 5b - 2b + 10 = 0$

$b(b-5) - 2(b-5) = 0$

$b = 5, 2$

and $\quad a = 7-5 = 2$

or $\quad a = 7-2 = 5$

$\therefore \quad |a-b| = |5-2| = 3$

$\& \quad a-b = |2-5| = 3$ //

Either way $|a-b| = 3$

(Gate 2018: EE)

④ Let $A = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$ and $B = A^3 - A^2 - 4A + 5I$

where I is the 3×3 identity matrix.
The determinant of B is ______ (upto 1 decimal place).

soln

Use C-H (Cayley Hamilton theorem)
Given a matrix, get its characteristic eqn or C-E eqn

$|A - \lambda I| = 0$ (solve this to get char eqn)

Finally we can say that $\boxed{\lambda = A}$ by Cayley Hamilton theorem and arrive to your final solution.

Now given $A = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}_{3\times3}$

For a 3×3 matrix (standard result),

Char Eqn is : $\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0$

There are 3 unknowns λ, S_1 and S_2.

① $|A| = 1(-4) + 0 + 0 = -4$

② S_1 = Sum of diagonal elements

$= 1+2-2 = 1$

③ S_2 = Sum of minors of diagonal elements.

Here, since we have 3×3 matrix, there are 3 minor terms M_1, M_2 and M_3.

$$M_1 \Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix} \Rightarrow -4$$

$$M_2 \Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix} \Rightarrow -2$$

$$M_3 = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix} = 2$$

$\therefore \; S_2 = -4 - 2 + 2 = -4$

$\therefore$ Char Eqn is

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0$$

or $\boxed{\lambda^3 - \lambda^2 + (-4)\lambda - (-4) = 0}$

Now according to Cayley Hamilton theorem : $\boxed{\lambda = A}$

$\therefore \quad A^3 - A^2 - 4A + 4I = 0$

Now, given $B = A^3 - A^2 - 4A + 5I$

$\therefore \quad B = A^3 - A^2 - 4A + 4I + I$

$B = 0 + I$

$\therefore \quad |B| = |I|$ (Identity Matrix)

⑤ Rank of matrix $\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ (EC 2006)

(A) 0 (B) 1 (C) 2 (D) 3

soln Method 1 $\xrightarrow[R/C]{}$ Echelon form $\rightarrow$ # nonzero rows = Rank of Matrix.

(Time consuming)

Method 2 Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

then $\underset{3\times3}{|A|}$ or $\det(A) = 0$

$\therefore$ Row 1 and Row 3 are same

Now, $|A| = 0$

then Rank of matrix $\neq n$ or 3

If $|A| \neq 0$ then Rank $= 3$

But in our case $|A| = 0$

So, find the determinant of $|A|_{(n-1)(n-1)}$

which is

$$|A|_{2\times 2}$$

If $|A|_{(n-1)(n-1)} = 0$ Rank of matrix $\neq (n-1)$

$\neq 0$ Rank of matrix $= (n-1)$

Here, $|A|_{2\times 2} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -1-1$

$= -2 \neq 0$

$\therefore$ Rank of Matrix $= (n-1) = \underline{\underline{2}}$