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Stability Analysis of Marginally Outermost Trapped Surfaces Using Spectral Methods

by

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Abstract

Black holes are among the most enigmatic phenomena in the universe, with many questions surrounding their stability and behavior. This thesis investigates the stability of Marginally Outer Trapped Surfaces (MOTS), critical structures associated with black holes, using spectral methods to analyze the stability operator. The principal eigenvalue of this operator serves as a key indicator of MOTS stability, with positive values implying stability and negative values suggesting instability.

The study is divided into two phases. First, a symbolic framework is developed to solve simpler differential equations, validating the approach by comparing calculated eigenvalues with exact solutions. The results show excellent accuracy, with relative errors mostly of magnitude 10^{-3}, while some of the very first eigenvalues achieve an even higher precision of magnitude 10^{-5}. This demonstrates the framework's reliability. Second, the analysis is extended to numerical eigenvalue approximation, using cosine basis functions to discretize the stability operator. By varying a parameter (α) in the operator, the eigenvalue spectrum is analyzed, revealing exponential growth for higher eigenvalues and critical stability trends in the principal eigenvalue.

The findings highlight the sensitivity of MOTS stability to spacetime conditions and offer insights into black hole behavior during dynamic events such as mergers. Limitations of the study include a focus on lower-order eigenvalues, computational constraints, and simplified geometric assumptions. Future research could explore higher-order modes, dynamic space-times, and extend computational capabilities using high-performance languages like Mojo for more accurate and efficient analyses.

This thesis establishes a robust framework for studying MOTS stability, contributing to our understanding of black hole dynamics and their role in the universe. By combining mathematical theory with computational tools, it provides a foundation for future investigations into one of nature's most compelling mysteries.

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Lastly, I want to thank my family and friends for their constant encouragement and understanding during this journey. Your support means the world to me.

Statement of contribution

This thesis represents the culmination of my independent research on the stability of Marginally Outer Trapped Surfaces (MOTS) using spectral methods. The analysis, methodology, and computational framework presented here were developed under the supervision of Professor Ivan Booth. His expert guidance, insightful feedback, and constructive discussions were integral to the success of this work. While the primary research and implementation were my responsibility, Professor Booth's mentorship provided the necessary foundation and direction to navigate the complexities of this project.

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Chapter 1

Introduction

Black holes are among the most fascinating and mysterious objects in the universe. Despite years of research, many fundamental questions about their nature remain unanswered. One of the most important questions is about their stability: are black holes stable structures, or can they change over time, potentially evolving or even disappearing? Understanding this aspect is key to unraveling their role in the universe. This thesis focuses on exploring the stability of black holes using mathematical and computational tools.

To study this, we use a mathematical technique called spectral analysis. This method is similar to breaking down a piece of music into its individual notes to understand its structure. In our case, the "notes" represent specific properties of black holes, encoded in their mathematical description. By analyzing the equations that describe black holes, we aim to extract important information, particularly the principal eigenvalue. This number is critical because it can provide insights into whether a black hole is stable or prone to change under certain conditions.

The computational part of this research involves using Python along with advanced mathematical libraries to calculate these eigenvalues. By varying parameters in the equations, we aim to identify patterns and establish connections between the principal eigenvalue and the stability of black holes. Such predictions could enhance our understanding of how black holes form, evolve, and interact with the universe around them.

This thesis is structured to provide a detailed exploration of these ideas. We begin by introducing the mathematical concepts behind spectral analysis and explain how it applies to black holes. Next, we describe the computational framework developed for this study, including the methods used to test and validate the approach. Finally, we present the results of our analysis, discuss their implications, and highlight their potential impact on future research in black hole physics.

Chapter 2

Background

In this chapter, we provide a comprehensive foundation for understanding the stability of Marginally Outer Trapped Surfaces (MOTS) and their significance in black hole physics. We begin by tracing the evolution of black hole theory, from its theoretical roots to its observational breakthroughs. Key concepts such as trapped surfaces, the stability operator, and spectral analysis are discussed in detail. These topics collectively set the stage for the methodological framework and analysis presented in subsequent chapters, bridging the gap between mathematical theory and physical phenomena.

2.1 Black Holes: A Journey from Theory to Reality

Black holes, once relegated to the realm of theoretical speculation, have emerged as central players in our understanding of the cosmos. Their roots can be traced back to the early 20th century, following the publication of Albert Einstein's groundbreaking theory of general relativity in 1915 [6]. This revolutionary theory proposed that gravity is not a force acting at a distance, but rather the curvature of spacetime itself, caused by the presence of mass and energy.

Mere months after Einstein's publication, Karl Schwarzschild, a brilliant German physicist stationed on the front lines of World War I, took on the challenge of solving Einstein's complex field equations. In a remarkable feat, he derived an exact solution that described the spacetime curvature around a spherically symmetric, non-rotating, and uncharged mass [19]. This solution, now known as the Schwarzschild metric, is represented mathematically as:

$$ds^2 = -\left(1 - \frac{2GM}{rc^2}\right)dt^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{2.1}$$

where ds is the spacetime interval, G is the gravitational constant, M is the mass of the black hole, c is the speed of light, and t, r, θ, and ϕ are the time, radial, polar, and azimuthal coordinates, respectively. When Karl Schwarzschild first derived this solution in 1916, the true physical significance of the radius $r = 2M$ was not fully understood. At that time, many believed that $r = 2M$ represented a genuine physical singularity, rather than a coordinate artifact or a boundary for a black hole. The modern interpretation—that a sufficiently compact mass distribution would create a region of spacetime from which no light can escape—only emerged more clearly much later, particularly in the mid-20th century, as theoretical and observational advances improved our grasp of gravitational collapse and horizon formation [7].

Skepticism and debate persisted for decades following Schwarzschild's work. The notion of a singularity, implying infinite density at the center of a massive object, challenged physical intuition. It was not until astrophysical research matured, coupled with observational hints and theoretical breakthroughs, that the idea of black holes became more widely accepted. Important contributions came from scientists like Subrahmanyan Chandrasekhar and Robert Oppenheimer, who established the conditions under which stellar remnants inevitably collapse. Chandrasekhar's calculations showed that there is a maximum mass limit beyond which a white dwarf star cannot prevent collapse [4]. Oppenheimer and his student Hartland Snyder later demonstrated that a sufficiently massive star undergoing continual gravitational contraction would form an object so dense that not even light could escape, thus solidifying the black hole concept in mainstream astrophysics.

2.2 Beyond Schwarzschild: Kerr and Other Black Holes

While the Schwarzschild metric provides a valuable description of non-rotating black holes, most astrophysical black holes are expected to possess significant angular momentum. In 1963, Roy Kerr discovered an exact solution to Einstein's field equations that describes a rotating black hole, now known as the Kerr metric [14]. The Kerr metric is considerably more complex than the Schwarzschild metric, but it captures the essential features of rotating black holes, including the presence of an ergosphere, a region outside the event horizon where objects are dragged along by the black hole's rotation. The Kerr metric can be expressed as:

$$\begin{aligned} ds^2 = -\left(1 - \frac{2GMr}{\rho^2 c^2}\right) dt^2 - \frac{4GMar\sin^2\theta}{\rho^2 c} dt d\phi + \frac{\rho^2}{\Delta} dr^2 \\ + \rho^2 d\theta^2 + \left(\frac{(r^2 + a^2)^2 - a^2 \Delta \sin^2\theta}{\rho^2}\right) \sin^2\theta d\phi^2 \end{aligned} \tag{2.2}$$

where $\Delta = r^2 - 2GMr/c^2 + a^2$, $\rho^2 = r^2 + a^2 \cos^2\theta$ and $a = J/Mc$.

In these expressions, ds is the spacetime interval, G is the gravitational constant, M is the mass of the black hole, c is the speed of light, J is the angular momentum of the black hole, and t, r, θ, and ϕ are the time, radial, polar, and azimuthal coordinates, respectively.

In addition to Schwarzschild and Kerr black holes, there are two other theoretical types: Reissner-Nordström black holes, which possess electric charge but no angular momentum, and Kerr-Newman black holes, which possess both electric charge and angular momentum. However, it is believed that astrophysical black holes are unlikely to possess significant electric charge, as any charge would quickly be neutralized by surrounding matter.

2.3 Early Evidence and the Event Horizon

The first observational hints of black holes emerged from the study of X-ray binaries, systems in which a compact object, either a neutron star or a black hole, accretes matter from a companion star. As matter spirals towards the compact object, it heats up and emits X-rays, providing a telltale sign of its presence. By carefully measuring the orbital parameters of these systems, astronomers could infer the mass of the compact object. The discovery of compact objects with masses exceeding the theoretical maximum for neutron stars provided strong evidence for the existence of black holes [16].

Further evidence came from the observation of active galactic nuclei (AGNs), incredibly luminous regions at the centers of galaxies believed to be powered by supermassive black holes. These black holes, millions or even billions of times more massive than the Sun, accrete vast amounts of matter, releasing tremendous amounts of energy in the process [15]. Observations of the motions of stars and gas near the centers of galaxies revealed the presence of compact, massive objects, further bolstering the case for black holes [10].

The event horizon, a fundamental feature of black holes, is a boundary in spacetime beyond which nothing, not even light, can escape. Mathematically, the radius of the event horizon for a non-rotating black hole is given by the Schwarzschild radius:

$$R_s = \frac{2GM}{c^2} \tag{2.3}$$

where G is the gravitational constant, M is the mass of the black hole, and c is the speed of light. Although observational evidence for black holes had been accumulating for decades, direct confirmation of event horizons remained elusive. The Event Horizon Telescope (EHT), a global array of millimeter-wave radio telescopes, achieved a resolution on the order of the black hole's event horizon scale, providing the first direct image of the shadow cast by the supermassive black hole at the center of M87 [8]. By resolving structures at roughly the size of the event horizon itself, the EHT observations offer compelling, nearly direct evidence that these extremely compact

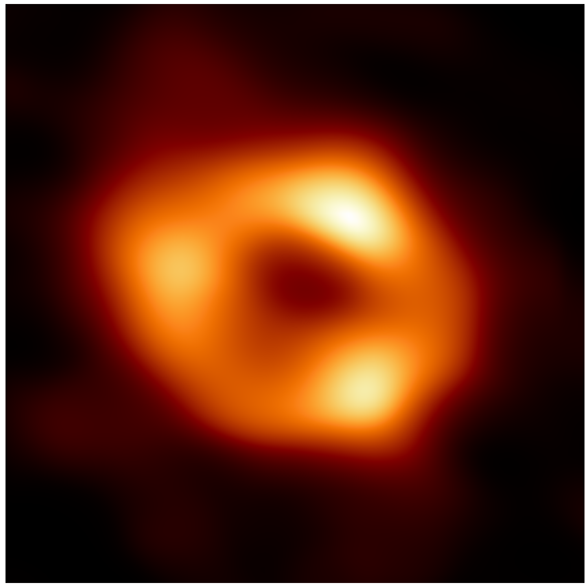

Figure 2.1: First ever direct image of black hole's shadow [5]

regions are indeed as small and dense as predicted by general relativity. This landmark achievement not only affirms the existence of event horizons but also paves the way for probing the predictions of general relativity in the strong-field regime with unprecedented detail.

2.4 Trapped Surfaces in General Relativity

2.4.1 Introduction to Trapped Surfaces

Trapped surfaces are critical structures in general relativity, introduced by Roger Penrose in his pioneering work on gravitational collapse.[18] A **trapped surface** is a two-dimensional spacelike surface embedded in a spacetime, with the unique property that light rays normal to it converge irrespective of their direction. These surfaces indicate regions where gravity is so strong that no signal can escape, playing a vital role in characterizing black holes.

Mathematically, a trapped surface S is a compact, smooth hypersurface in a spacetime (M, g) where the expansions θ_+ (outgoing null rays) and θ_- (ingoing null rays) satisfy:

$$\theta_+ \leq 0, \quad \theta_- \leq 0.$$

In simpler terms, both sets of light rays are converging. This behavior is fundamentally linked to the formation of singularities and black hole horizons.

2.4.2 Null Expansions and Geometric Description

The null expansions $\theta_\pm$ describe how the cross-sectional area of a congruence of null geodesics evolves as they move away from the surface. They are given by:

$$\theta_\pm = h^{ab} \nabla_a \ell_b^\pm,$$

where h^{ab} is the induced metric on the surface, ∇_a is the covariant derivative in the spacetime (M, g), $\ell_b^\pm$ are the null vectors orthogonal to the surface S. For trapped surfaces, the condition $\theta_+ \leq 0$ implies that even outward-pointing null rays are converging, which occurs when the surface lies within a region of strong gravitational influence.

2.5 Marginally Outer Trapped Surfaces (MOTS)

Marginally Outer Trapped Surfaces (MOTS) are critical structures in the study of black holes and spacetime geometry. While trapped surfaces generally indicate regions where light rays are forced to converge due to intense gravitational fields, MOTS represent a special, finely balanced category. They form a crucial link between the interior dynamics of a black hole and its boundary, providing insights into the evolving shape and nature of horizons.[18]

2.5.1 Definition and Properties

To understand a MOTS, it can be helpful to think of a precise balance point. Imagine placing a small marble on a perfectly symmetrical hilltop: if placed exactly at the summit, the marble does not roll away in any direction. Similarly, consider outward-moving light rays on a closed, two-dimensional surface in spacetime. In most cases, these light rays either move outward freely or are drawn inward by strong gravity. A Marginally Outer Trapped Surface is like that hilltop point where the outward-directed light rays do not expand outwards nor contract inwards. In mathematical terms, a MOTS is a two-dimensional spacelike surface S on which the outgoing null expansion θ_+ vanishes:

$$\theta_+ = 0.$$

This condition means the surface sits delicately between being fully trapped and being untrapped. It marks a geometric threshold, indicating that the surface is at the brink of enclosing a region so gravitationally intense that even outward-moving light cannot escape.[13]

When such a surface is found, it often corresponds to what can be understood as a locally defined black hole boundary. Unlike the event horizon, which depends on the entire future evolution of spacetime to define its boundary, a MOTS gives a more immediate, quasi-local handle on where the black hole's influence becomes so dominant that it traps light rays. This makes MOTSs invaluable in numerical and theoretical studies where one wants a manageable, local indicator of horizon formation and behavior.

2.5.2 Physical Significance

MOTS are strongly tied to apparent horizons. An apparent horizon, often the outermost MOTS in a given spatial slice, serves as a practical and instantaneous marker of a black hole boundary within that slice. While the event horizon is a global concept, the presence of a MOTS provides a snapshot of horizon geometry at a given moment, helping researchers track how horizons emerge, grow, merge, and settle. In dynamic scenarios such as binary black hole merger or matter accretion, the position and shape of MOTSs can change rapidly, offering insights into how the black hole responds to external perturbations and how the gravitational field adjusts.[3]

2.5.3 Stability and Spectral Analysis

The stability of a MOTS hinges on how it responds to small deformations. If we slightly change the shape of the MOTS, do we find that it returns to its original equilibrium form, or does it deform further, indicating an unstable configuration? To address this question, one introduces a linearized stability operator $\mathcal{L}$. Acting on small perturbations ϕ of the surface, $\mathcal{L}$ encodes how these perturbations evolve. For a non-rotating MOTS, $\mathcal{L}$ often takes a form like:

$$\mathcal{L}\phi = -\Delta\phi + 2K\phi + \nabla_a(\alpha\nabla^a\phi),$$

where Δ is the Laplace-Beltrami operator on the surface S, K is its Gaussian curvature, and α is a function encoding geometric information from the ambient spacetime.[11]

The behavior of the perturbations is governed by the eigenvalues of $\mathcal{L}$. Consider the principal eigenvalue, which is the lowest (or most dominant) eigenvalue of this operator. If this principal eigenvalue is positive, then small disturbances decay, and the MOTS is stable. This is analogous to a ball resting at the bottom of a shallow bowl: if tapped, it returns to equilibrium. On the other hand, if the principal eigenvalue is negative, the smallest perturbations can grow, much like a ball perched on a peak that rolls away from its original point. In this scenario, the MOTS is unstable, signaling that the horizon configuration may shift into a new geometric state. Tracking how these eigenvalues vary as parameters of the black hole solution change (such as mass distribution or spin proxies) is a key aspect of spectral analysis. By studying the eigenvalue spectrum, one gains quantitative insight into the underlying physics that dictates whether a MOTS remains stable or undergoes transitions as the spacetime evolves.

2.5.4 Applications in Black Hole Physics

MOTS have proven indispensable in numerical relativity, where one must often identify black hole boundaries in complicated and evolving spacetimes without relying on information about the entire future. By locating MOTSs, researchers can approximate apparent horizons, track how horizons move and morph during binary black hole mergers, and connect the geometric features of horizons to observable phenomena like gravitational waves.[3] As computational techniques improve and we probe more extreme regimes—rotating black holes, matter-rich environments, or exotic field configurations—the role of MOTS as practical and insightful tools only increases.

2.6 Stability Operator

The stability operator is central to understanding how Marginally Outer Trapped Surfaces (MOTS) respond to small disturbances. It provides a mathematical framework for identifying whether a given MOTS configuration will remain steady or transition into a new state when subjected to perturbations. By examining its eigenvalues, one can determine if the surface is dynamically stable or if it tends to deform, indicating instability.[1]

2.6.1 Mathematical Definition and Stability Parameter

Consider a Marginally Outer Trapped Surface (MOTS) embedded in a spacetime (M, g), and let ϕ represent a scalar perturbation of this surface. The stability operator $\mathcal{L}$ is a second-order elliptic differential operator defined on the surface. It governs how the MOTS reacts to perturbations and, in its general form, is given by:

$$\mathcal{L}\phi = -\Delta\phi + Q\phi,$$

where Δ is the Laplacian intrinsic to the MOTS and Q is a potential term that encodes geometric contributions from the Gaussian curvature and extrinsic curvature of the MOTS. Specifically, $Q = 2K + \|\mathcal{X}\|^2 + \mathrm{div}(\mathcal{X})$, where K is the Gaussian curvature, and $\mathcal{X}$ is a vector field related to the extrinsic curvature of the MOTS. This operator encapsulates the influence of both the intrinsic and extrinsic geometry of the MOTS.[17]

The principal eigenvalue λ_0 of $\mathcal{L}$ serves as a key measure of how the null expansion θ_+ of a MOTS responds to small perturbations. The null expansion θ_+ characterizes how bundles of outgoing light rays behave near the surface: if θ_+ is zero, the surface is marginally trapped, balancing on the brink between trapped and untrapped configurations. A positive principal eigenvalue $\lambda_0 > 0$ indicates that any attempt to deform the MOTS—thereby altering the null expansion—will be counteracted, causing small perturbations to subside and return the surface to its marginally trapped state. In other words, the MOTS resists changes in its shape and remains stable.

On the other hand, if $\lambda_0 \leq 0$, perturbations to the surface geometry can push the null expansion into regions where it no longer remains zero. In this situation, the MOTS is susceptible to evolving into a truly trapped surface with negative null expansion, or moving away from being marginally trapped, indicating instability in its geometric configuration. Physically, this can be thought of as a delicate balance—if the MOTS is like a ball resting on a hilltop, even a small shift in its null expansion properties will send it rolling off into a new, potentially more trapped or more untrapped configuration. In this way, λ_0 provides direct insight into how robustly the MOTS maintains the delicate balance implied by $\theta_+ = 0$ against small deformations.[2, 1, 9]

2.6.2 Physical Significance of the Stability Operator

The stability operator emerges when considering how the null expansions θ_+ and related geometric quantities vary under small changes to the MOTS. Physically, it serves as a diagnostic tool that links the intrinsic geometry of the surface to its dynamic behavior in the spacetime. The eigenfunctions associated with the stability operator's eigenvalues describe different deformation patterns of the MOTS, with low-order eigenfunctions representing large-scale changes and higher-order ones capturing more subtle variations.[3]

In practical terms, analyzing the stability operator helps distinguish robust black hole boundaries from those that are transient or evolving. A MOTS with a positive principal eigenvalue is more likely to correspond to a physically meaningful and persistent boundary, often associated with an apparent horizon. As conditions in the spacetime change—such as during black hole mergers or accretion events—the stability operator's spectrum can shift, revealing transitions from stable to unstable MOTS configurations and influencing the identification and interpretation of horizons in dynamic scenarios.[1, 12]

2.7 Spectral Analysis for Eigenvalue Approximation: Application to MOTS Stability

Spectral analysis is a vital technique for studying the stability of MOTS. By finding the eigenvalues and eigenfunctions of $\mathcal{L}$, it becomes possible to assess stability in both simple and complex geometric settings. In theoretical analyses, one might consider highly symmetric cases where the operator simplifies and exact solutions can be derived. Such solutions offer benchmarks against which more sophisticated numerical methods can be tested.[13]

When dealing with realistic scenarios—curved geometries, intricate boundary conditions, and varying spacetime parameters—symbolic solutions often become intractable. Here, numerical approaches come into play. Methods like finite-difference approximations or spectral expansions in terms of orthogonal basis functions transform the continuous operator into a discrete matrix problem. The resulting eigenvalue approximations, although numerical, provide crucial insights. They allow researchers to determine whether a MOTS remains stable under certain conditions or transitions as the spacetime evolves.[12]

In black hole physics, applying spectral analysis to the stability operator clarifies how MOTS respond to changes in mass, spin, or external perturbations. By carefully examining the principal eigenvalue and its variations, it is possible to predict when horizons form, move, merge, or disappear. This understanding not only contributes to theoretical knowledge but also guides the interpretation of numerical simulations and observational data, linking the mathematical structure of the stability operator to the tangible astrophysical processes shaping black hole horizons.[3]

2.8 The Ongoing Quest

The study of black holes continues to be a vibrant and exciting field of research. Many questions remain, including the formation and growth of supermassive black holes in the early universe, the precise nature of the singularity at the heart of a black hole, and the interplay between black holes and their host galaxies. The Event Horizon Telescope, along with other cutting-edge observational and theoretical tools, is poised to unlock further secrets of these enigmatic objects, deepening our understanding of the universe and its fundamental laws.

Chapter 3

Methods

This chapter details the framework developed for analyzing the stability of Marginally Outer Trapped Surfaces (MOTS) through spectral methods. The methodology is divided into two primary components: symbolic integration for analytical verification and numerical approximation for practical application. This section focuses on the symbolic approach, which establishes a proof of concept and lays the foundation for complex numerical integrations.

3.1 Symbolic Integration as a Proof of Concept

Symbolic integration serves as the first step in the methodology, leveraging analytical solutions to validate the computational framework. This approach uses symbolic computation tools to solve simple differential equations with known solutions and compares them to analytical results.

3.1.1 Goals and Rationale

The primary objectives of this phase are:

1. To verify the accuracy of the framework by solving equations symbolically.

2. To demonstrate that the framework can handle operators and boundary conditions relevant to MOTS stability.

3. To provide a benchmark for testing the accuracy of numerical methods in subsequent stages.

By focusing on simpler cases, this phase ensures the robustness of the computational setup and its compatibility with spectral methods.

3.2 Differential Equations and Basis Functions

The analysis focuses on differential equations that incorporate standard second-order linear operators commonly encountered in spectral problems. These operators appear in a wide range of physical and mathematical applications, and solving their associated eigenvalue problems provides valuable insight into the properties of the systems they describe. In general, the eigenvalue problem can be expressed as:

$$\mathcal{L}\,\phi(x) = \lambda\phi(x),$$

where $\mathcal{L}$ is a differential operator, λ is an eigenvalue, and $\phi(x)$ is the corresponding eigenfunction. To verify the robustness of the framework, various basis functions are employed. When the problem involves periodic domains or conditions, sine and cosine functions often prove convenient due to their natural orthogonality and compatibility with the boundary conditions. Polynomial bases can be introduced for problems defined on bounded intervals, while Chebyshev polynomials are frequently chosen in spectral methods for their favorable numerical stability and rapid convergence. By exploring different bases, one can evaluate how effectively the computational scheme adapts to diverse scenarios.

3.3 Implementation

The implementation relies on symbolic integration tools available in Python, particularly through the `SymPy` library. Symbolic approaches provide exact expressions for solutions and enable direct comparison with known analytical results. This comparison establishes a baseline of accuracy and guides the refinement of numerical methods for more intricate problems. Initially, simpler boundary value problems with known solutions serve as benchmarks. Once the symbolic results are validated against these exact solutions, the framework can be confidently extended to more complex and less tractable scenarios.

3.3.1 Example: Solving a Stability Operator Eigenvalue Problem

Consider the eigenvalue problem:

$$L(y) = \lambda y,$$

with

$$L = -\frac{d^2}{dx^2} - 3\frac{d}{dx} - 2,$$

on the interval $[0, 1]$ subject to Dirichlet boundary conditions $y(0) = y(1) = 0$.

Beginning with the differential equation:

$$-y''(x) - 3y'(x) - (2 + \lambda)y(x) = 0,$$

we seek solutions of the form $y(x) = e^{ikx}$. Substituting this trial solution yields a characteristic equation for k:

$$k^2 - 3ik - (2 + \lambda) = 0.$$

Solving for k gives:

$$k = \frac{3i}{2} \pm \frac{\sqrt{-1 + 4\lambda}}{2}.$$

To ensure the solution respects the boundary conditions, we rewrite the general solution in terms of real functions. One convenient representation is:

$$y(x) = e^{-\frac{3}{2}x}\left(A\cos\left(\tfrac{\sqrt{4\lambda-1}}{2}x\right) + B\sin\left(\tfrac{\sqrt{4\lambda-1}}{2}x\right)\right).$$

Applying $y(0) = 0$ forces $A = 0$. Applying $y(1) = 0$ requires:

$$\frac{\sqrt{4\lambda - 1}}{2} = n\pi, \quad n \in \mathbb{Z}.$$

This yields:

$$4\lambda - 1 = 4\pi^2 n^2 \implies \lambda = \frac{1}{4} + \pi^2 n^2.$$

Thus, the exact eigenvalues are:

$$\lambda_n = \pi^2 n^2 + \frac{1}{4}.$$

This exact solution provides a critical benchmark. In practice, instead of solving the differential equation directly, one may construct a matrix eigenvalue problem using a chosen basis. For example, selecting basis functions $\sin(m\pi x)$ and employing the trial function:

$$yX(x) = e^{-\frac{3x}{2}} \sin(n\pi x),$$

one projects the operator L onto the subspace spanned by these sine functions. This leads to integrals of the form:

$$\int_0^1 L(\sin(n\pi x)) \sin(m\pi x)\, dx,$$

which produce a finite-dimensional matrix. The eigenvalues of this matrix approximate those of the continuous operator.

After constructing the operator L in the chosen sine basis, each matrix element is computed via:

$$L_{mn} = \int_0^1 L(\sin(n\pi x)) \sin(m\pi x)\, dx.$$

For a small subset of modes ($m, n = 1, \dots, 5$), we first write down the integral expressions. For instance, the top-left 2×2 portion of the matrix would be:

$$\begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} = \begin{bmatrix} \int_0^1 L(\sin(\pi x)) \sin(\pi x)\, dx & \int_0^1 L(\sin(\pi x)) \sin(2\pi x)\, dx \\ \int_0^1 L(\sin(2\pi x)) \sin(\pi x)\, dx & \int_0^1 L(\sin(2\pi x)) \sin(2\pi x)\, dx \end{bmatrix}.$$

Evaluating these integrals for the first five modes yields the following 5×5 matrix:

$$\begin{bmatrix} 7.8696044 & -8. & 0. & -3.2 & 0. \\ 8. & 37.4784176 & -14.4 & 0. & -5.71428571 \\ 0. & 14.4 & 86.82643961 & -20.57142857 & 0. \\ 3.2 & 0. & 20.57142857 & 155.91367042 & -26.66666667 \\ 0. & 5.71428571 & 0. & 26.66666667 & 244.74011003 \end{bmatrix}.$$

In the actual computation, the code constructs a much larger 50×50 matrix, using the same procedure. By comparing the eigenvalues obtained from this larger matrix to the known exact eigenvalues, we compute relative errors. These relative errors directly assess how accurately the chosen basis size, integration techniques, and computational approach reproduce the exact solutions. This ensures confidence in both the symbolic framework and subsequent numerical implementations, paving the way for more complex and parameter-dependent operators.

3.3.2 Relative Error Calculation

After obtaining the computed eigenvalues, it is essential to quantify their accuracy. The relative error between the calculated $\lambda_{\text{calculated}}$ and exact λ_{exact} eigenvalues is defined as:

$$\text{Relative Error} = \frac{|\lambda_{\text{calculated}} - \lambda_{\text{exact}}|}{\lambda_{\text{exact}}}.$$

Evaluating this measure for a range of modes and comparing the results under different basis choices or operator complexities reveals how closely the methodology approximates the true solutions. As a result, the relative error analysis provides both a diagnostic tool for troubleshooting and a metric for improvement.

3.4 Numerical Approximation of Eigenvalues

In this phase, the methodology moves from the symbolic foundations established earlier to a more complex, parameter-dependent operator. While the first phase demonstrated accuracy and flexibility using symbolic integration and known solutions as benchmarks, this second phase focuses on a differential operator $\mathcal{L}$ that depends explicitly on a parameter α. By investigating how the eigenvalue spectrum varies as α changes, we gain insights into the operator's stability properties under more challenging conditions. The connection to the first phase is clear: having validated the approach using simpler operators and readily verifiable solutions, we now build on that confidence to tackle a more intricate scenario that cannot be as easily solved by pure symbolic methods.

3.4.1 The Differential Operator $\mathcal{L}$

The operator $\mathcal{L}\psi$ is expressed as:

$$\mathcal{L}\psi = \left[\left(\frac{1}{4} - 2\sin^2(\theta)\cos^2(\theta)\alpha^2 + \left(-\frac{5}{2}\cos^2(\theta) + \frac{1}{2}\right)\alpha\right)\psi + \left(-\sin(\theta)\cos(\theta)\alpha - \frac{\cos(\theta)}{4\sin(\theta)}\right)\psi' - \frac{\psi''}{4}\right]\exp\left(-2\alpha(\cos^2(\theta) - 2)\right), \tag{3.1}$$

Here, $\psi(\theta)$ is the eigenfunction, ψ' and ψ'' are the first and second derivatives of ψ with respect to θ, α is a parameter that influences the behavior of the operator.

The operator $\mathcal{L}$ acts on a function $\psi(\theta)$ and is defined in the code as `Lpsi`. It is parameterized by α, which allows for a family of operators to be considered. This general form can represent various types of problems, such as Sturm–Liouville systems, quantum mechanical operators, or stability operators arising in differential equations.

The essential point is that $\mathcal{L}$ is linear and acts on $\psi(\theta)$ in a way that, when posed as an eigenvalue problem $\mathcal{L}(\psi) = \lambda\psi$, the eigenvalues λ and eigenfunctions $\psi(\theta)$ characterize the operator's behavior.

3.4.2 Basis Functions and Projection

Rather than attempting to solve $\mathcal{L}(\psi) = \lambda\psi$ directly in its continuous form, the operator is projected onto a set of chosen basis functions. By doing so, the continuous eigenvalue problem is converted into a discrete matrix eigenvalue problem. In this implementation, cosine basis functions $\psi_n(\theta) = \cos(n\theta)$ are employed. These functions are well-suited for the domain $\theta \in [0, \pi]$ and integrate cleanly into the construction of the matrix representation of $\mathcal{L}$. While the initial phase considered sine functions for simpler problems, the switch to cosine functions here is motivated by numerical considerations that make the computations more stable and efficient for the given operator.

3.4.3 Matrix Representation

Once the operator $\mathcal{L}$ is projected onto the cosine basis, each matrix element L_{mn} is computed as an inner product of the form:

$$L_{mn} = \langle \cos(m\theta), \mathcal{L}(\cos(n\theta)) \rangle,$$

with the inner product defined by

$$\langle f, g \rangle = \int_0^\pi f(\theta) g(\theta) \, d\theta.$$

Evaluating these integrals yields a matrix that approximates the action of $\mathcal{L}$ within the chosen subspace spanned by the cosine functions. By carefully computing these integrals and assembling the resulting matrix, the eigenvalue problem reduces to a finite-dimensional linear algebra problem, enabling the use of standard numerical methods to find λ.

3.4.4 Parameter α and Spectral Analysis

A central aspect of this phase is the parameter α, which influences the form of the operator $\mathcal{L}$. By varying α, we effectively study a continuum of operators and thus observe how the eigenvalue spectrum changes in response. For each fixed α, the code constructs the corresponding matrix representation of $\mathcal{L}$ and computes its eigenvalues. By examining these eigenvalues across different values of α, we can detect trends, transitions, or critical points that signal modifications in the underlying stability properties of the system under investigation.

This approach, built upon the validated procedures of the first phase, now extends to a richer setting. The specification of the operator, the use of cosine basis functions, the domain $\theta \in [0, \pi]$, and the integral-based construction of matrix elements all align with the previously established framework. The new complexity introduced by α and by a more intricate operator $\mathcal{L}$ demonstrates the adaptability of the initial symbolic methods to more general and challenging numerical scenarios.

Chapter 4

Results

4.1 Symbolic Validation and Relative Errors

The first phase of the methodology focused on verifying the accuracy of the symbolic framework against known exact solutions. By using Equation (3.1) as a reference equation and computing both analytical and symbolic eigenvalues for well-understood boundary value problems, the approach set out to establish a baseline of reliability and precision. This step aligned with the goals outlined in the methodology section, where achieving low relative errors and ensuring a stable foundation for subsequent numerical endeavors were primary objectives.

As shown in Figure 4.1, the relative errors remain below 10^{-3} for the majority of the eigenvalues and approach 10^{-5} for the principal eigenvalue, indicating a high degree of accuracy. The slight increase in error observed for higher eigenvalue indices can be attributed to numerical precision limits rather than conceptual shortcomings. These results confirm that the symbolic computations not only meet the initial criteria set forth in the methodology but also provide a reliable platform from which more

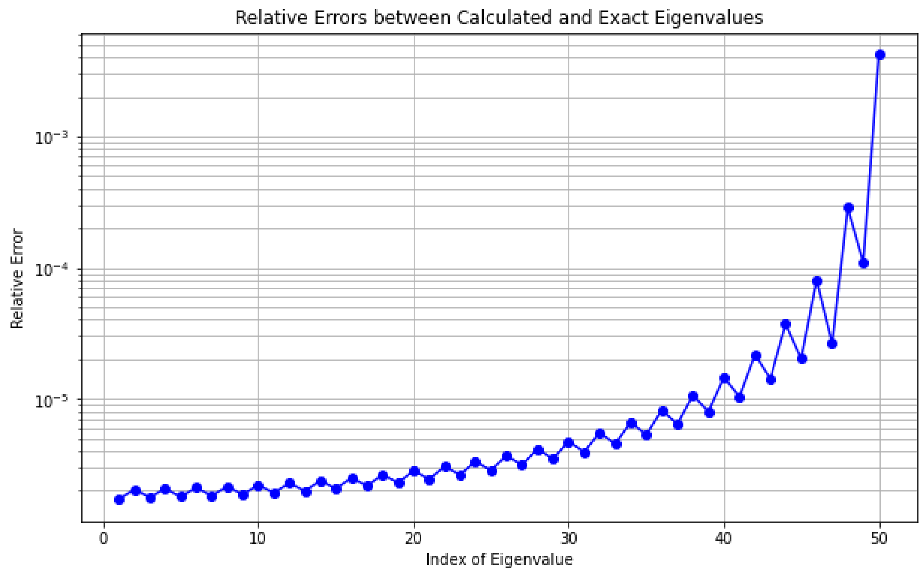

Figure 4.1: Relative errors between the computed and exact eigenvalues for the benchmark cases of Phase 1, obtained using 50 basis functions. These results correspond to the test problem defined by Equation (3.1).

complex numerical analyses can be confidently pursued.

4.2 Numerical Eigenvalue Approximation

This second phase of the study shifts from the symbolic validations established earlier to the numerical approximation of eigenvalues for a parameter-dependent operator. Having confirmed the reliability and precision of the methodology in Phase 1—where reference solutions and Equation (3.1) served as benchmarks—this phase explores Equation (3.2) and how the eigenvalue spectrum evolves as the parameter α changes. In doing so, it addresses the goals set forth in the methodology, demonstrating not only the ability to solve more complex problems numerically, but also how the previously validated symbolic foundation supports the interpretation of increasingly intricate results.

4.2.1 Eigenvalue Trends Across Parameters

Figure 4.2 illustrates the behavior of twenty eigenvalues as a function of α. Initially, for small values of α, the eigenvalues remain relatively close to their baseline values, reflecting conditions similar to those analyzed in Phase 1. As α increases, however, the eigenvalues deviate significantly from their initial states. Many show substantial growth, with some modes increasing rapidly, indicating heightened sensitivity of the operator to parameter variations. This sensitivity aligns with the understanding gained from the simpler problems studied previously: when parameters in the operator are altered, the spectrum responds in ways that reveal underlying stability or instability.

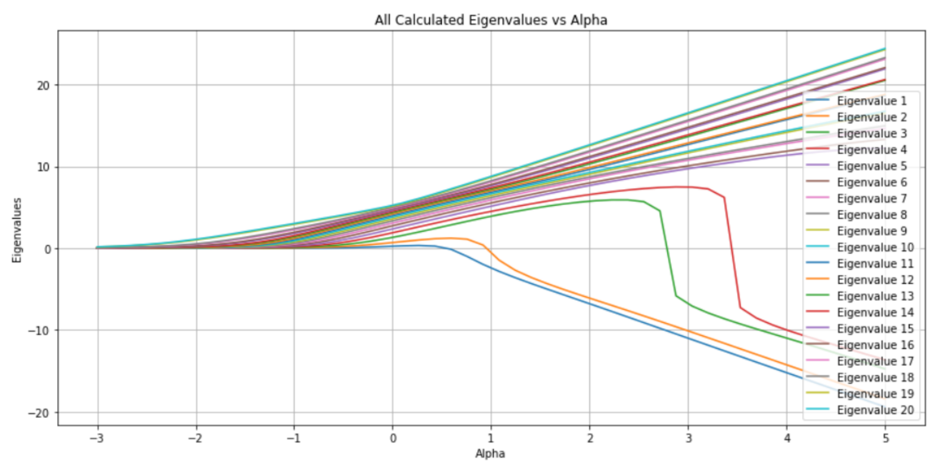

Figure 4.2: All calculated eigenvalues as a function of α. The range and complexity of the eigenvalue responses highlight how the operator's stability characteristics evolve with changing parameters.

To examine the finer details, Figure 4.3 presents the first five eigenvalues across the same range of α. These lower-order modes are often the most physically significant, since they tend to dominate the system's response. At small α, they remain relatively stable, mirroring the conditions tested using Equation (3.1) and thus reaffirming the framework's initial successes. As α becomes larger, distinct divergences appear: some eigenvalues rise steadily, others dip below zero, and all depart from their stable baselines. Such divergent behavior suggests that even small changes in the operator's parameters can push the system from a stable configuration into a regime where certain modes become destabilized. The impact on these first few eigenvalues underscores the operator's parameter sensitivity and how the complexity introduced by α influences both large- and small-scale dynamics.

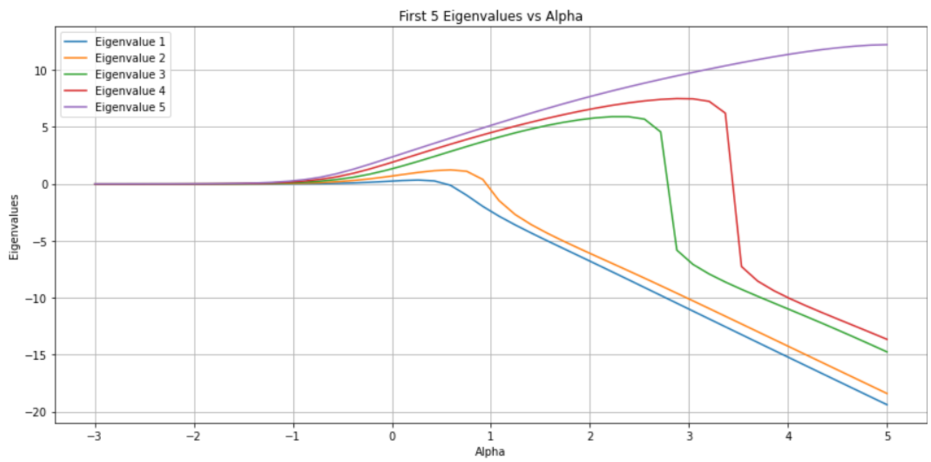

Figure 4.3: First five eigenvalues as a function of α. These modes, stable and well-understood at low α, display significant shifts as α grows, indicating transitions away from the stable conditions established in Phase 1.

4.2.2 Principal Eigenvalue Analysis

Figure 4.4 provides a closer look at the principal eigenvalue, the lowest mode which often serves as a key indicator of overall stability. Initially near zero and relatively unaffected by small parameter changes, this eigenvalue plummets into negative territory as α increases. A negative principal eigenvalue implies that the operator's corresponding mode is no longer stable; in physical terms, even slight perturbations could now be amplified. This transition is precisely the kind of behavior the methodology aimed to capture when extending from the baseline tests to a parameter-dependent operator.

The initial symbolic phase ensured that calculations were accurate for well-defined, simpler scenarios, thus allowing these more complex, parameter-driven results to be confidently interpreted. The observed shift of the principal eigenvalue into negative values stands as compelling evidence that changes in α can create conditions conducive to instability, aligning with the broader objective of understanding how parameters govern the stability landscape.

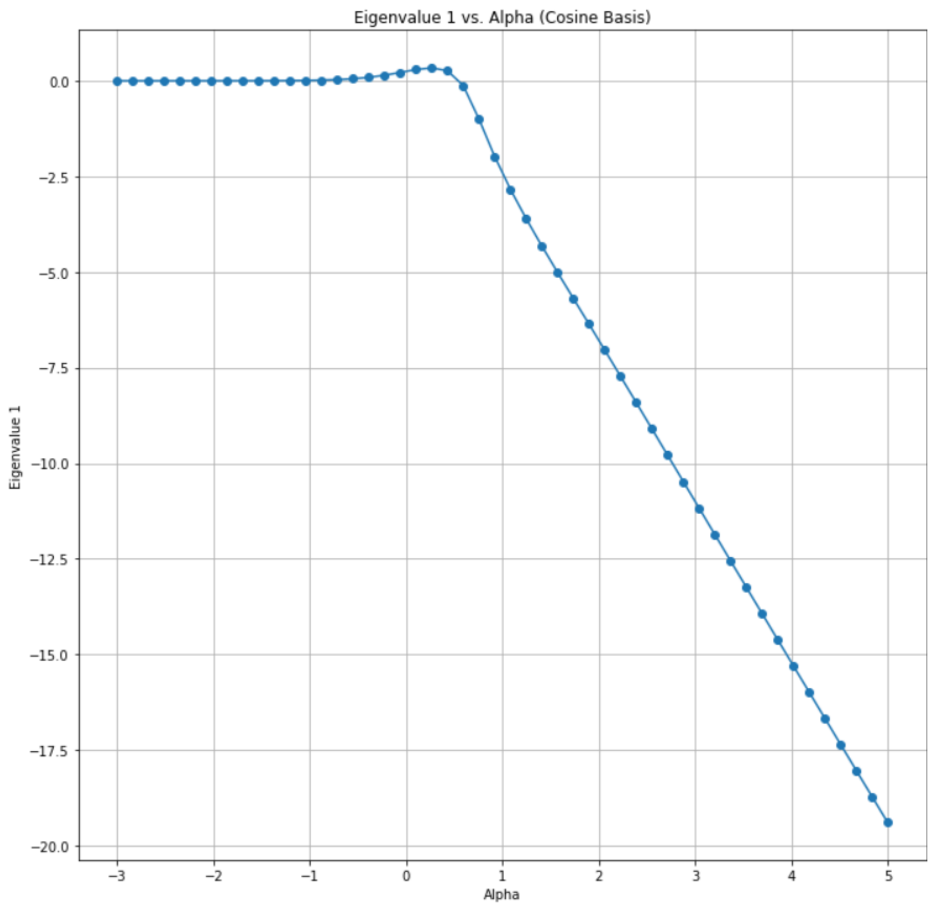

Figure 4.4: Principal eigenvalue as a function of α. Starting with a gradual increase then a decisive drop below zero indicates a transition from stable configurations into regimes of instability, reflecting the operator's sensitivity to parameter changes.

Chapter 5

Conclusion

In this thesis, we explored the stability of Marginally Outer Trapped Surfaces (MOTS) in black holes using spectral methods for analysis. By investigating the eigenvalue spectrum of the stability operator, we provided a framework to assess whether a black hole or its trapped surface is stable under perturbations. The results of this work offer both mathematical rigor and physical insights into the behavior of black holes and their boundaries.

5.1 Limitations

This work relies heavily on trigonometric basis functions, such as sines and cosines, which presume certain symmetries and can handle only relatively simple geometric configurations. Realistic black hole merger scenarios often break these simplifying assumptions, introducing asymmetries and complex field distributions that cannot be easily captured by such a limited set of basis functions. Consequently, the accuracy and fidelity of the current approach may diminish when confronted with highly non-symmetric or dynamically evolving spacetimes.

Furthermore, the finite-dimensional truncation of the basis introduces another layer of limitation. By including only a finite number of modes, subtle instabilities and fine-grained geometric details may remain unresolved. The computational cost of higher-order mode calculations can grow rapidly, restricting how many modes can be retained without compromising efficiency. In addition, parameter studies—such as scanning through a range of α—are computationally expensive, and high-precision integrals or large matrix eigenproblems may push the limits of available computational resources.

A final caveat arises from the simplified physics underlying the model scenarios. Many analyses exclude matter fields, rotation, charge, and other physically relevant factors. Although this simplification lays a strong foundation, it may not faithfully represent the complexity of actual astrophysical environments. Real black holes, especially those that form or evolve dynamically, may exhibit behaviors poorly approximated by these idealized conditions.

5.2 Future Scope

Future research can adopt more sophisticated basis sets better aligned with the actual geometry and complexities of MOTSs. Shifting from simple trigonometric functions to spherical harmonics, spin-weighted harmonics, or other specialized bases promises to handle non-symmetric horizons and finer stability features more accurately. Increasing both the number of eigenvalues examined and the size of the basis expansion can yield a richer spectral picture, capturing secondary instabilities, bifurcations, and delicate geometric phenomena.

Expanding beyond static or simplified scenarios to fully time-dependent and three-dimensional simulations represents another natural step. Dynamical spacetimes, such as those involving merging black holes, accretion flows, or gravitational wave sources, would bridge the gap between theory and numerical relativity, enabling the direct comparison of stability predictions with observed phenomena.

Incorporating additional physical ingredients—such as rotation, charge, scalar fields, or magnetic fields—will illuminate how stability is influenced by extreme astrophysical conditions. By exploring these more intricate parameter spaces, one can chart out how black holes transition through various stable and unstable phases.

Finally, algorithmic and computational improvements, including the adoption of high-performance programming languages and the utilization of GPU acceleration or parallel computing architectures, will enhance the efficiency and scalability of these calculations. Such advancements will enable larger basis expansions, finer parameter resolutions, and more complex scenarios, ultimately deepening our understanding of black hole stability in realistic astrophysical environments.

5.3 Final Remarks

This research provides a stepping stone for future studies on MOTS stability and its implications for black hole physics. By combining mathematical theory, computational tools, and numerical analysis, we have developed a framework that is both rigorous and flexible. As computational tools evolve and new programming paradigms like Mojo emerge, the methods explored in this thesis can be refined and extended to tackle even more challenging problems in general relativity.

Black holes remain one of the most mysterious phenomena in the universe, and this research contributes to unraveling their secrets. By understanding their stability, we not only gain insight into their behavior but also advance our understanding of the fundamental laws that govern our universe. The journey of discovery continues, with each step bringing us closer to unveiling the true nature of these enigmatic objects.

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