

Landmark-based Registration



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Medical Image Analysis

Koen Van Leemput

Fall 2023

Image registration

Combine information contained in different scans

Images need to be spatially aligned!

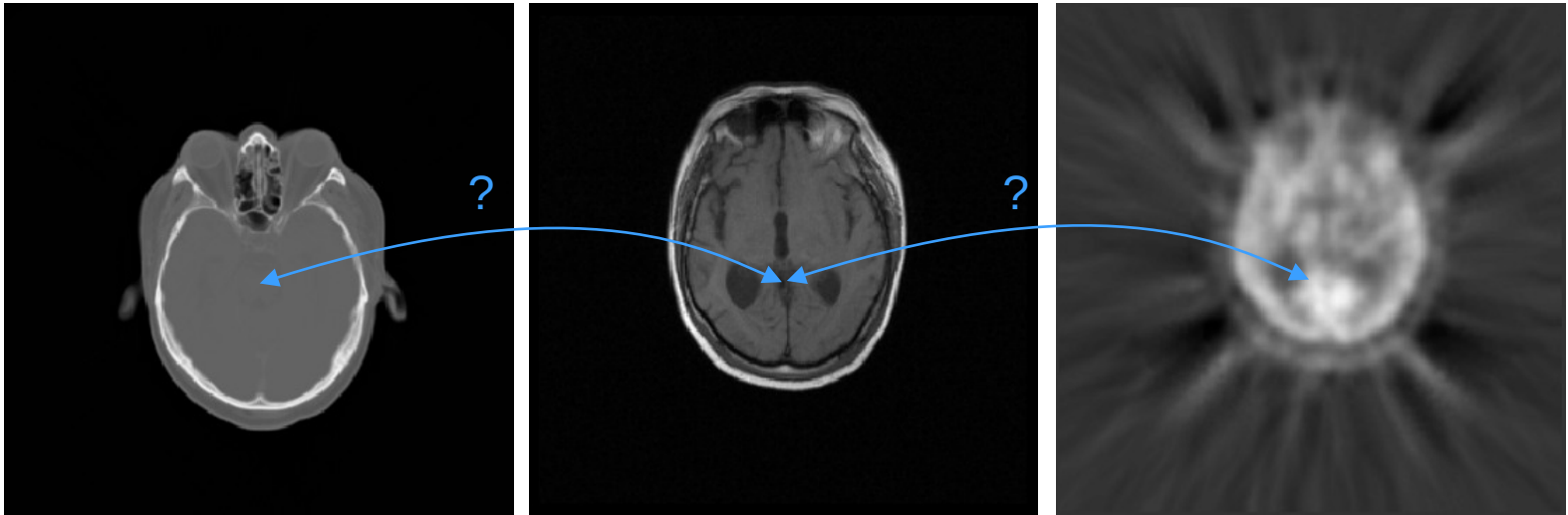
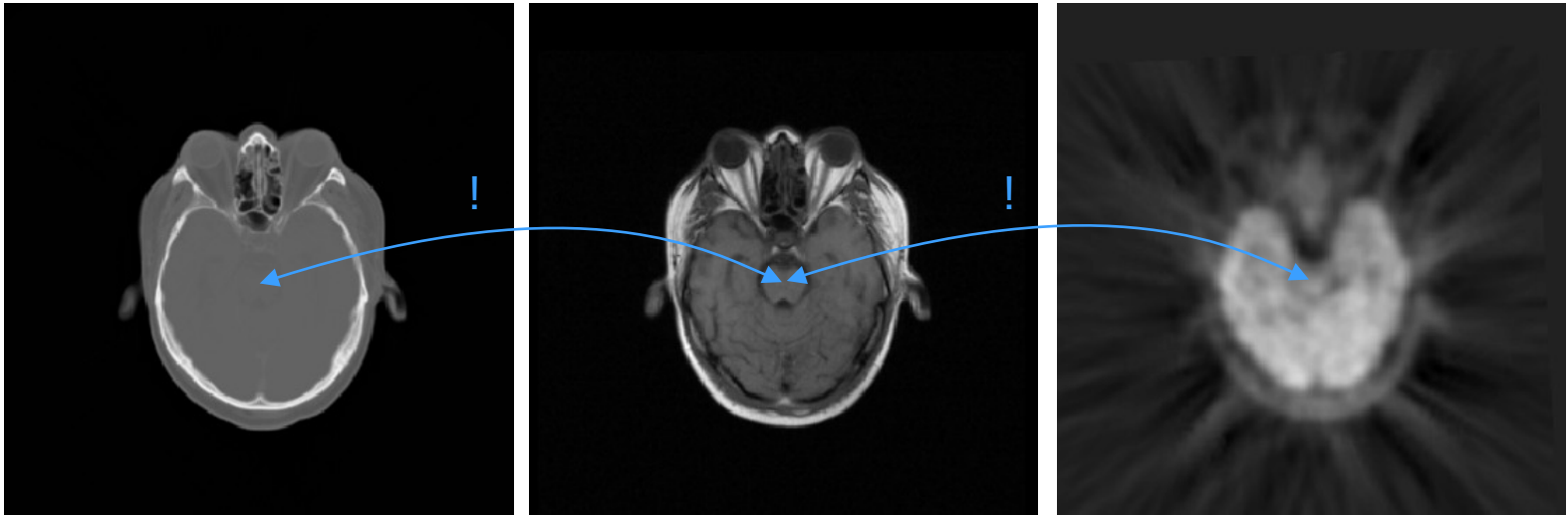


Image registration

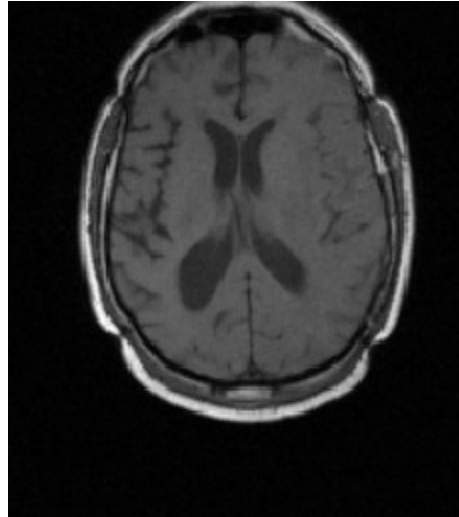
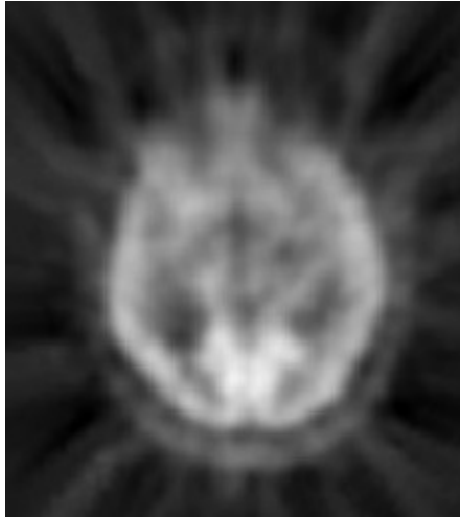
Combine information contained in different scans

Images need to be spatially aligned!



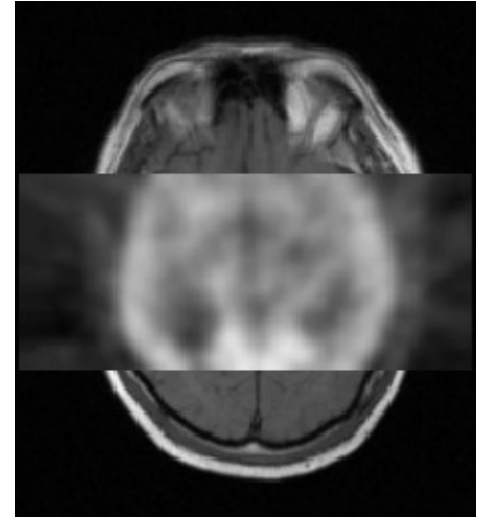
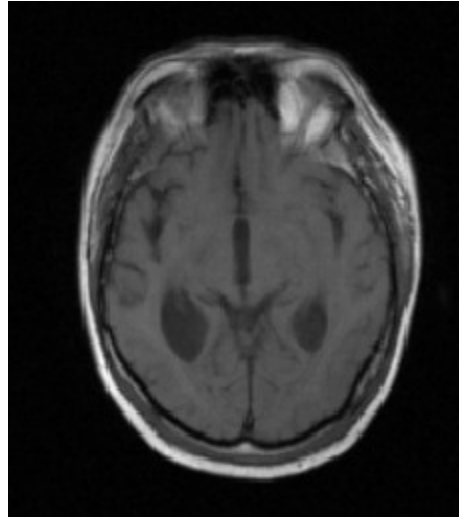
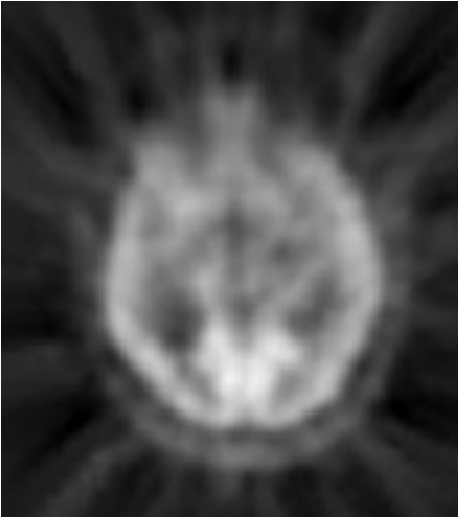
Example: PET/MR

Before registration...



Example: PET/MR

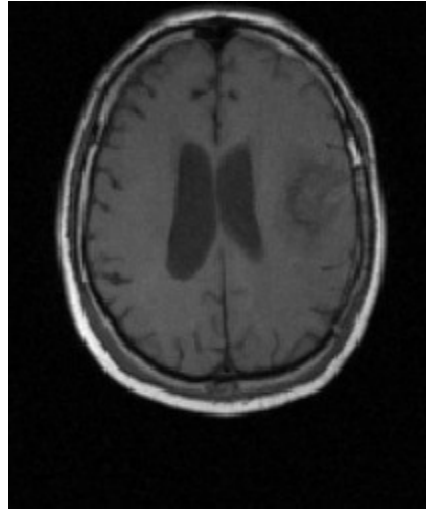
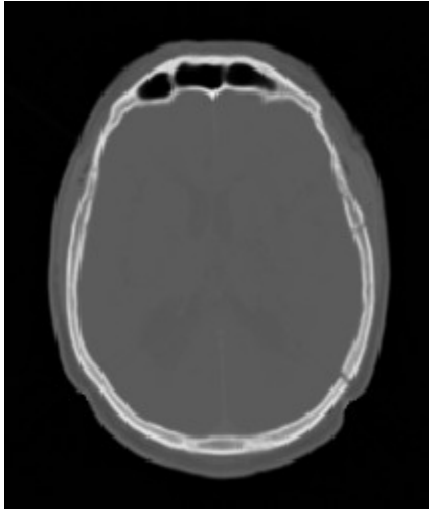
... after registration



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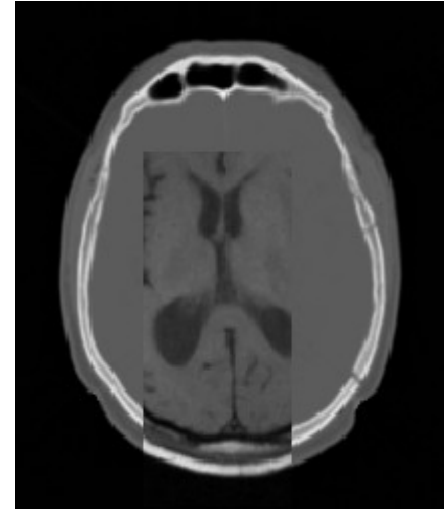
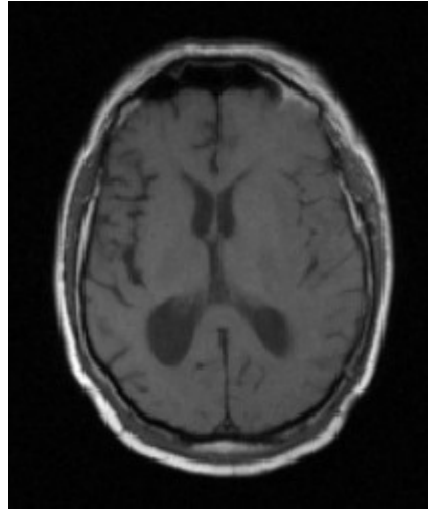
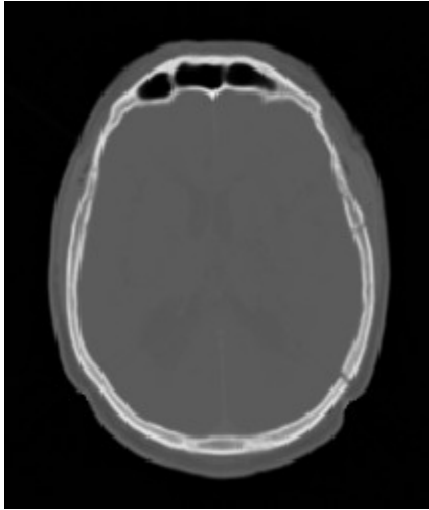
Example: CT/MR

Before registration...



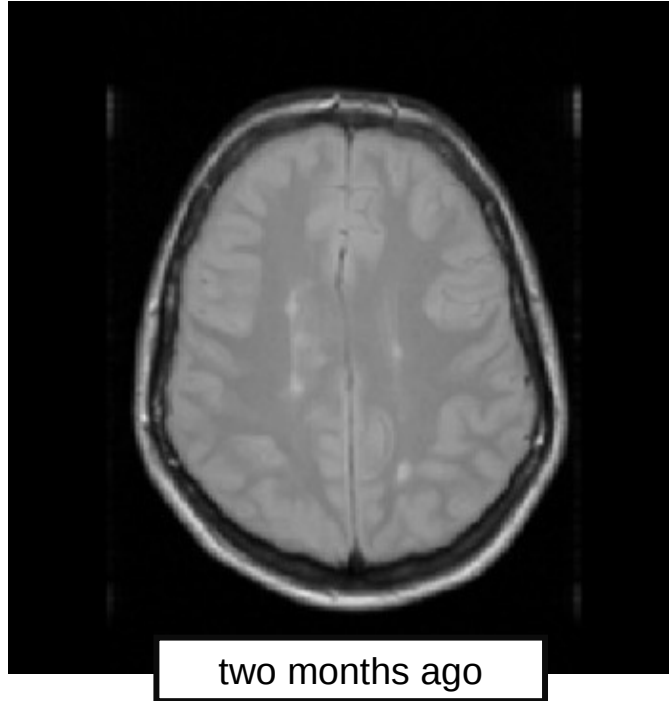
Example: CT/MR

... after registration



Example: longitudinal scans

Patient with multiple sclerosis (MS)

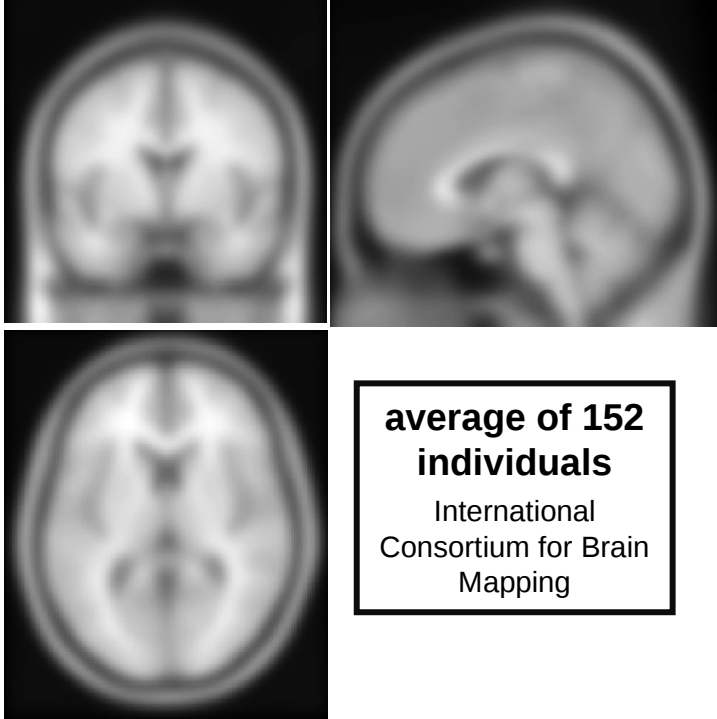


Example: longitudinal scans

Patient with multiple sclerosis (MS)

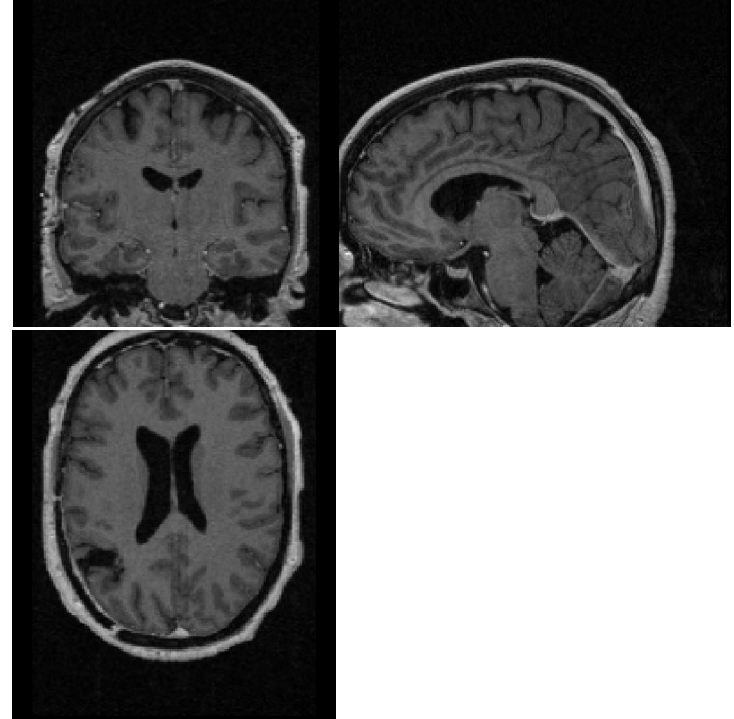


Example: population studies

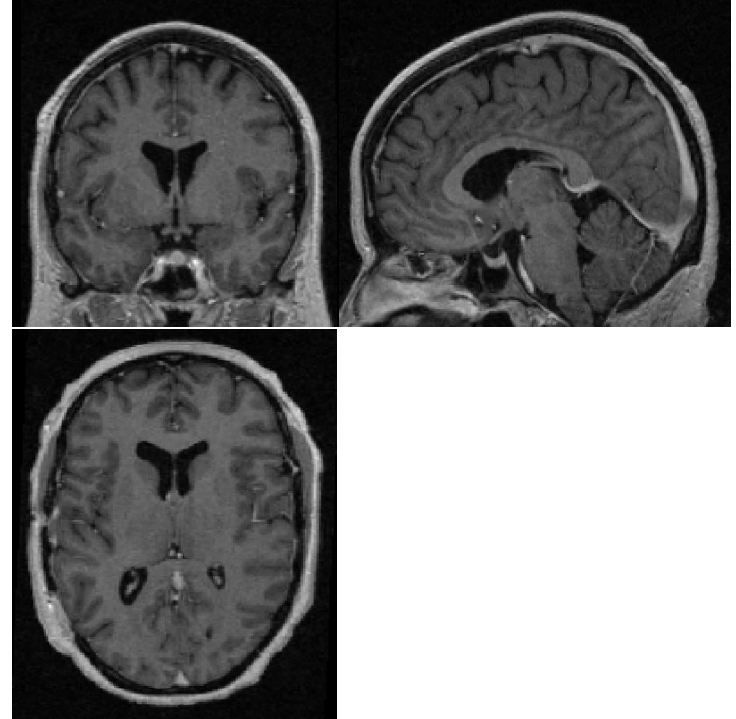
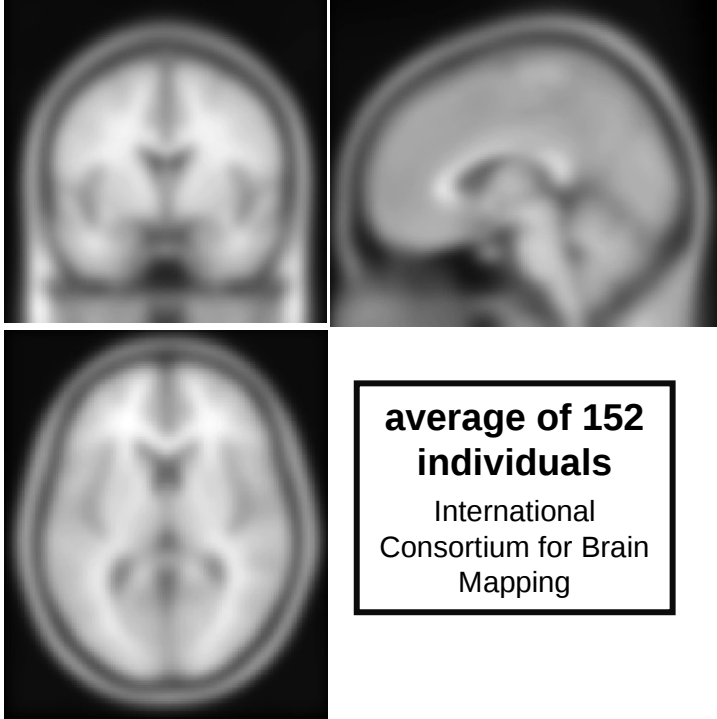


**average of 152
individuals**

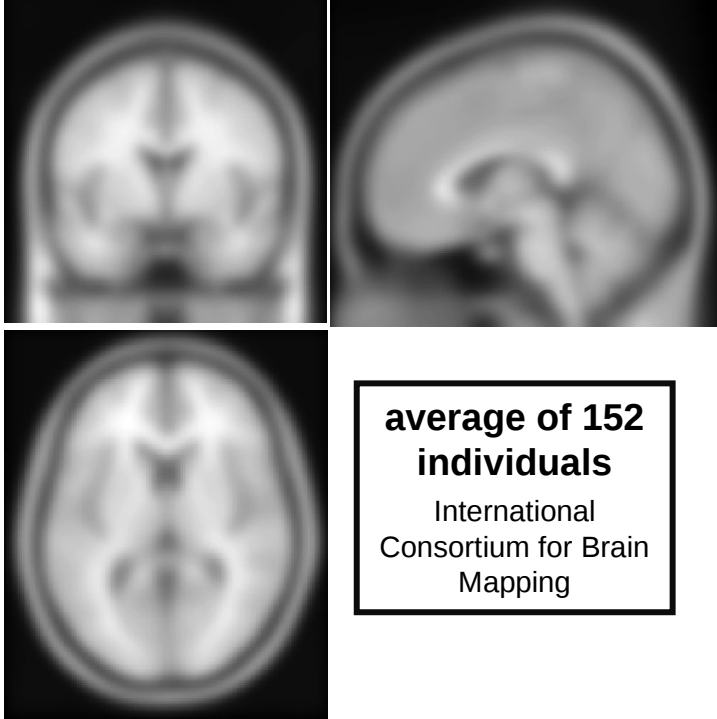
International
Consortium for Brain
Mapping



Example: population studies

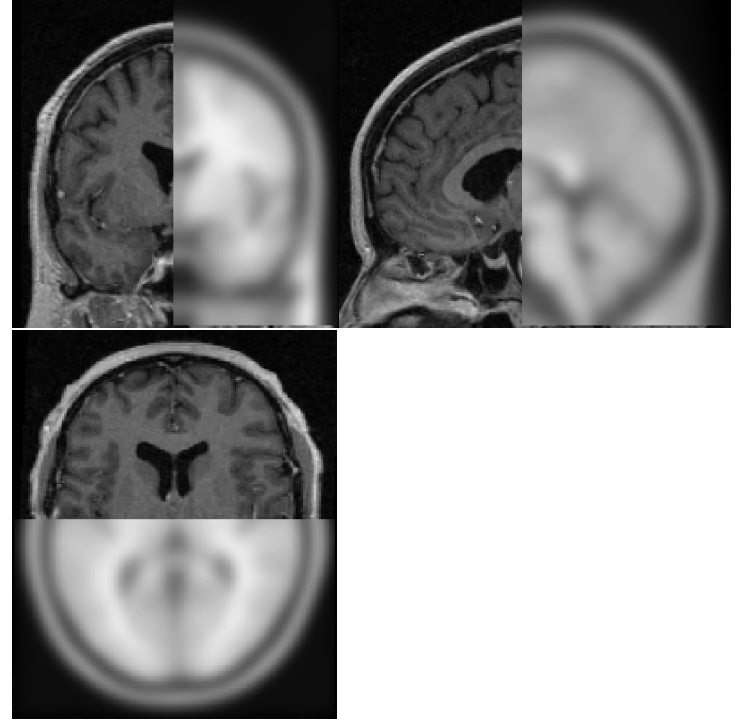


Example: population studies



**average of 152
individuals**

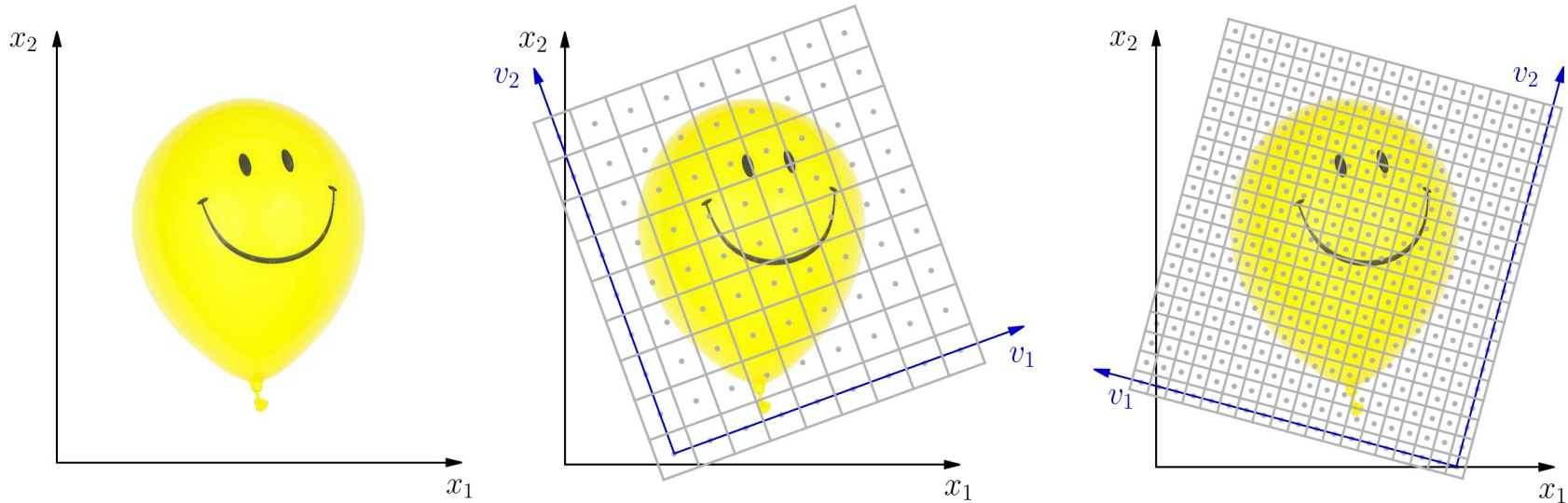
International
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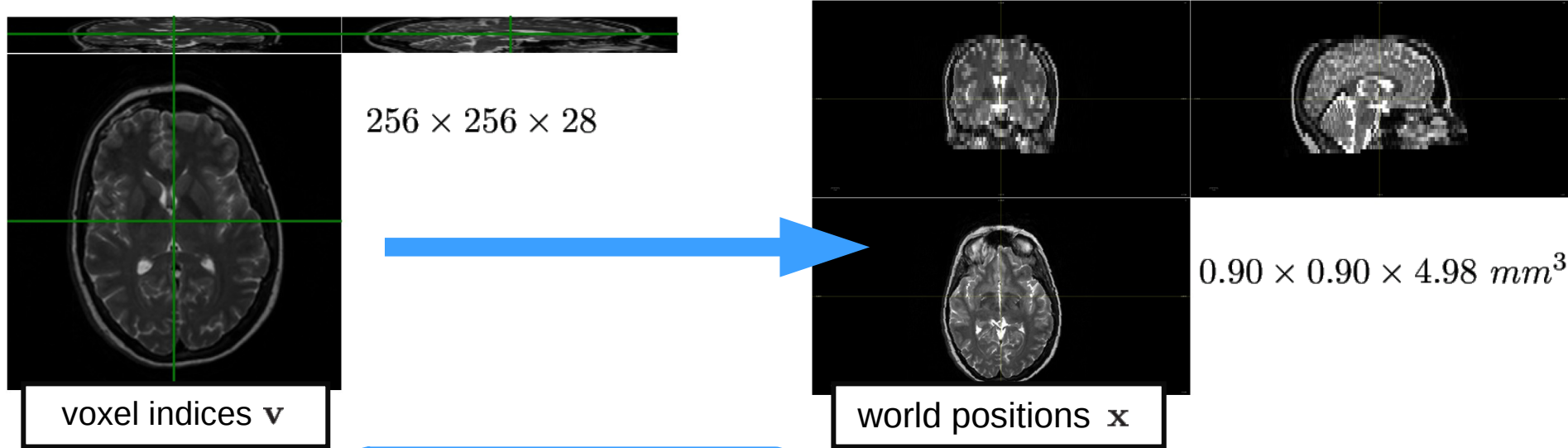
Coordinate systems

For each image, there are two coordinate systems:

- ✓ Voxel coordinates $\mathbf{v} = (v_1, v_2, v_3)^T$ (integer indices)
- ✓ World coordinates $\mathbf{x} = (x_1, x_2, x_3)^T$ (in mm)



Coordinate systems



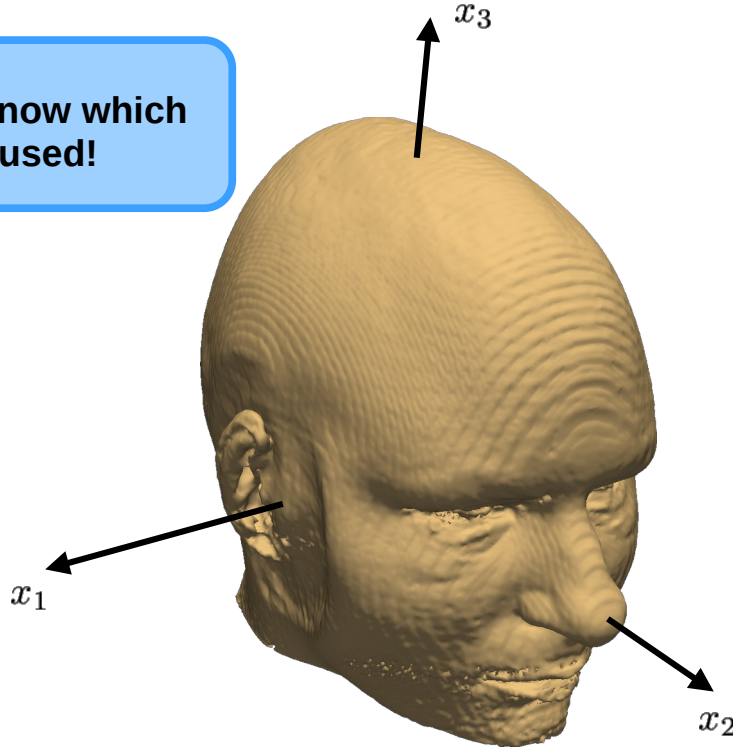
Conversion: $\mathbf{x} = \mathbf{A}\mathbf{v} + \mathbf{t}$

$$\mathbf{A} = \begin{pmatrix} -0.8923 & -0.0802 & -0.3732 \\ -0.0850 & 0.8921 & 0.3528 \\ -0.0612 & -0.0696 & 4.9512 \end{pmatrix}$$

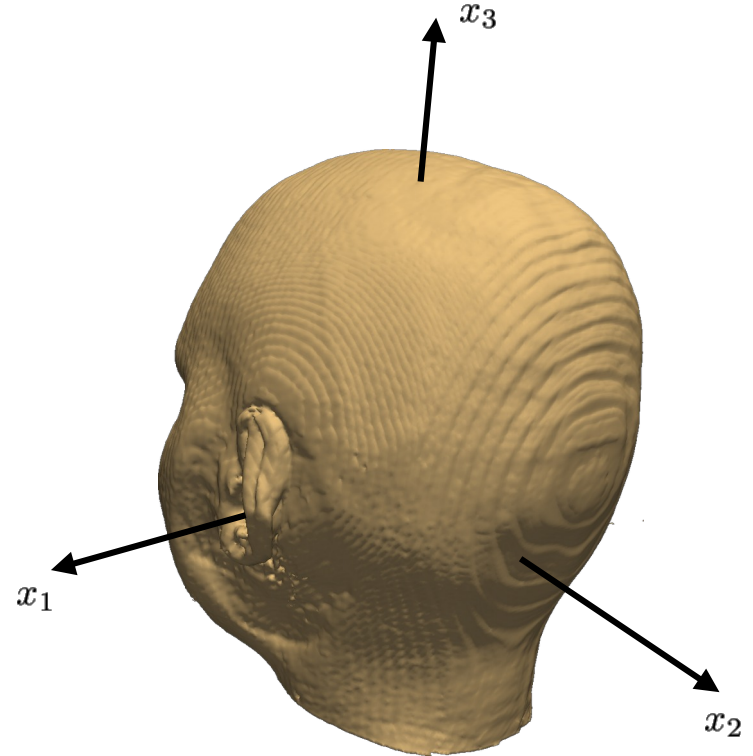
$$\mathbf{t} = \begin{pmatrix} 129.2834 \\ -98.7363 \\ -27.6911 \end{pmatrix}$$

World coordinates = convention

Important to know which convention is used!



Right – Anterior – Superior
(RAS)



Left – Posterior – Superior
(LPS)



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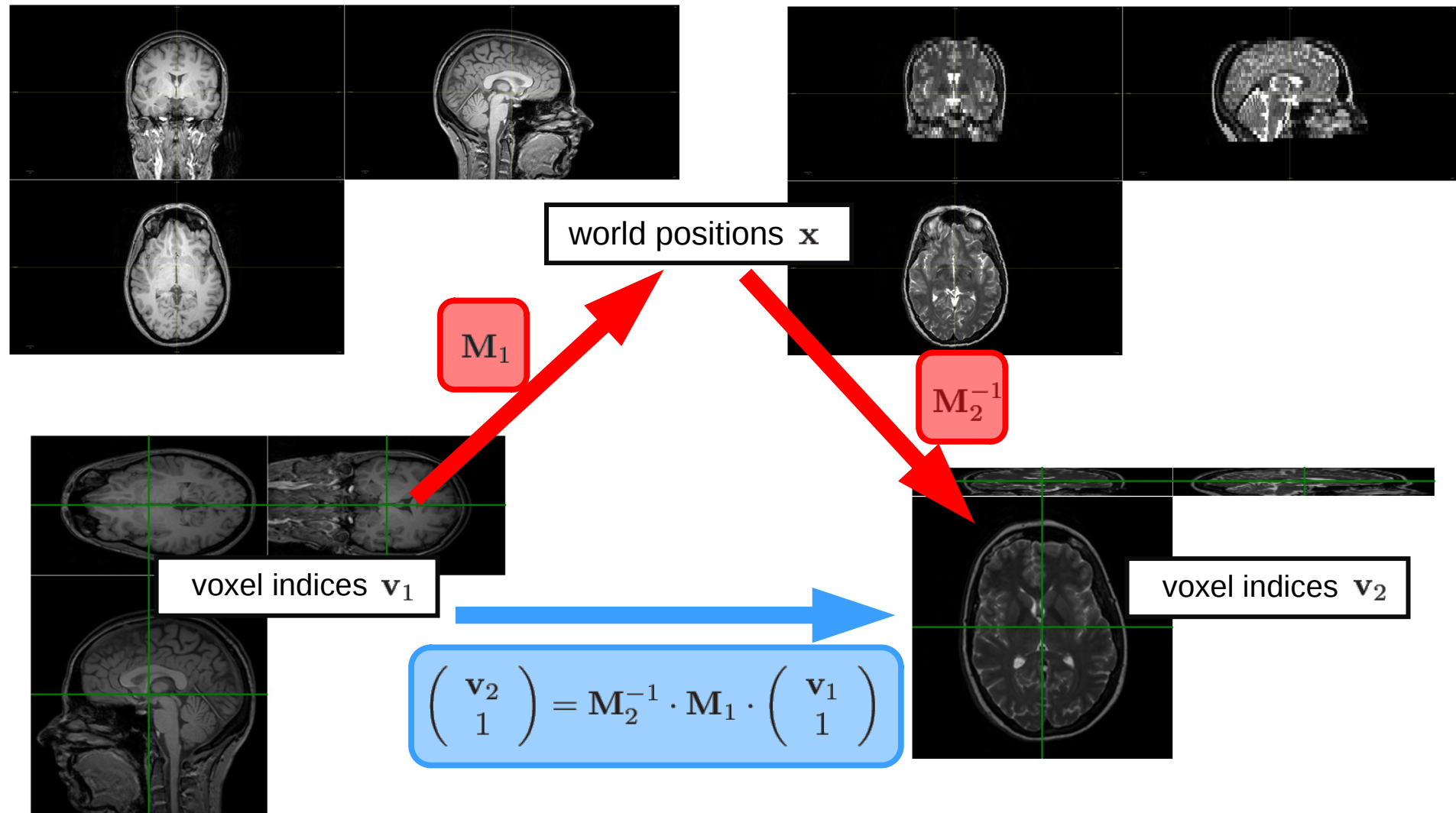
Homogeneous coordinates

Vectors are augmented with a 1 at the end

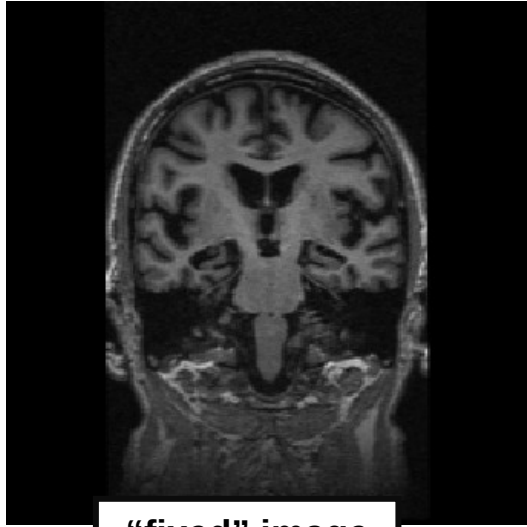
✓ **Idea:** Rewrite $\mathbf{x} = \mathbf{A}\mathbf{v} + \mathbf{t}$, i.e.,
$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} + \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix}$$

as:
$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ 1 \end{pmatrix} = \underbrace{\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & t_1 \\ a_{2,1} & a_{2,2} & a_{2,3} & t_2 \\ a_{3,1} & a_{3,2} & a_{3,3} & t_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{\mathbf{M}} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ 1 \end{pmatrix}$$

✓ **Benefit:** map voxel indices using only matrix multiplications

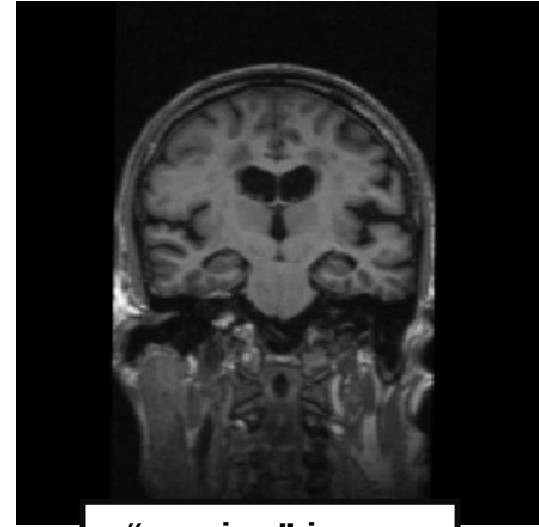


Spatial transformations



“fixed” image

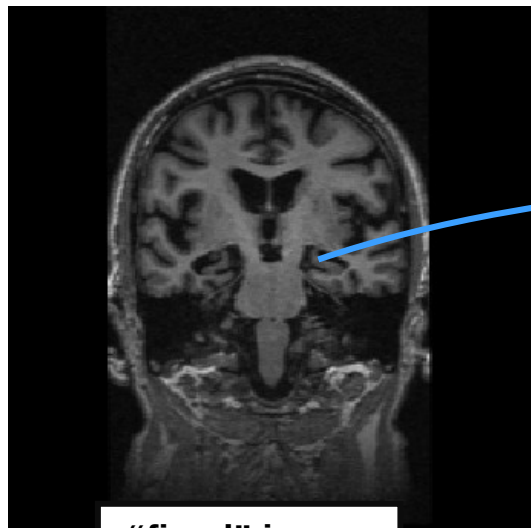
$$\mathbf{x} = (x_1, \dots, x_D)^T$$



“moving” image

$$\mathbf{y} = (y_1, \dots, y_D)^T$$

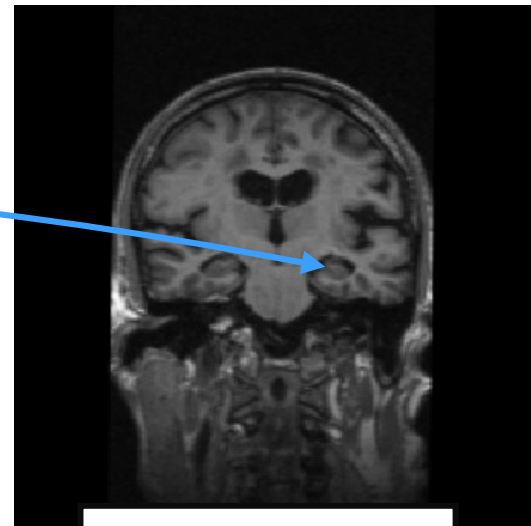
Spatial transformations



"fixed" image

$$\mathbf{x} = (x_1, \dots, x_D)^T$$

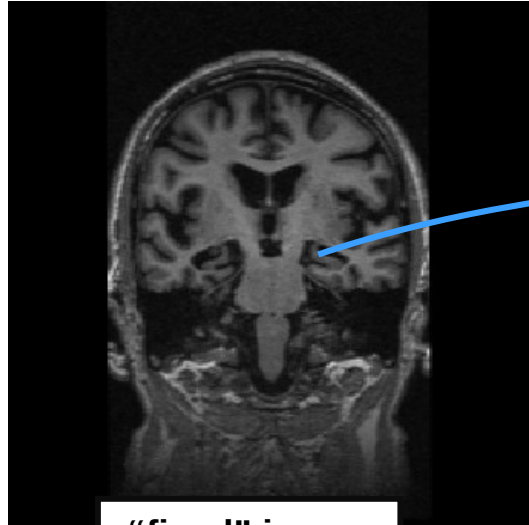
$$\mathbf{y}(\mathbf{x}, \mathbf{w}) = \begin{pmatrix} y_1(\mathbf{x}, \mathbf{w}) \\ \vdots \\ y_D(\mathbf{x}, \mathbf{w}) \end{pmatrix}$$



"moving" image

$$\mathbf{y} = (y_1, \dots, y_D)^T$$

Spatial transformations



"fixed" image

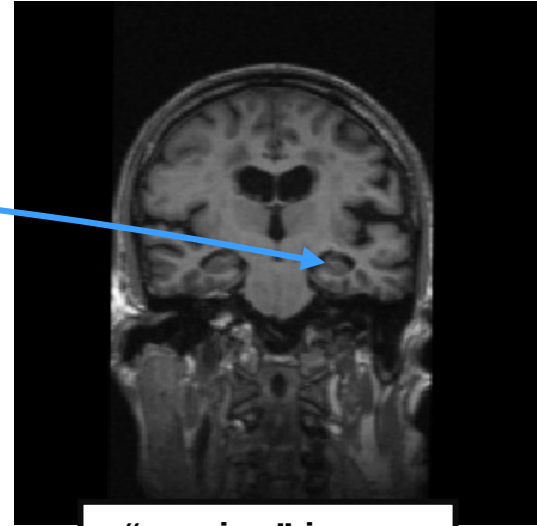
$$\mathbf{x} = (x_1, \dots, x_D)^T$$

$$\mathbf{y}(\mathbf{x}, \mathbf{w}) = \begin{pmatrix} y_1(\mathbf{x}, \mathbf{w}) \\ \vdots \\ y_D(\mathbf{x}, \mathbf{w}) \end{pmatrix}$$



$$y_d(\mathbf{x}, \mathbf{w})$$

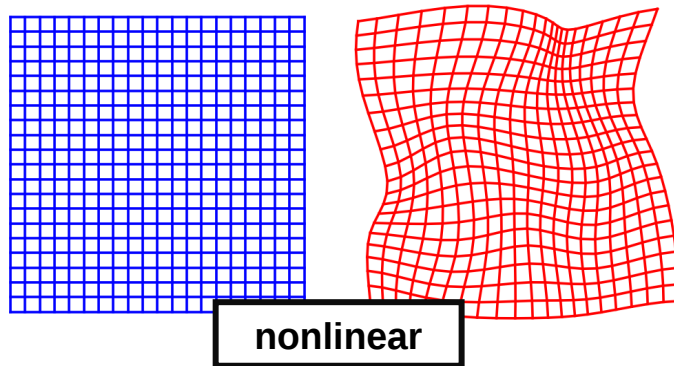
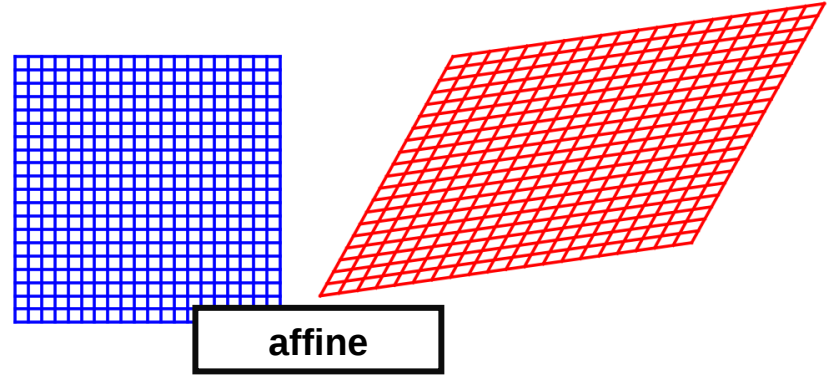
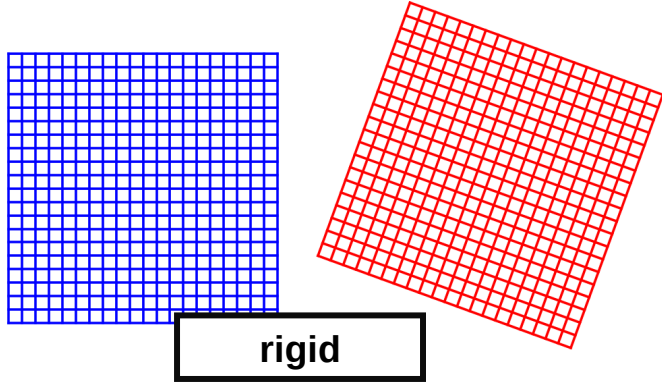
controls how points \mathbf{x} in the fixed image
move along the d -th direction in the moving image
as the parameters \mathbf{w} are varied



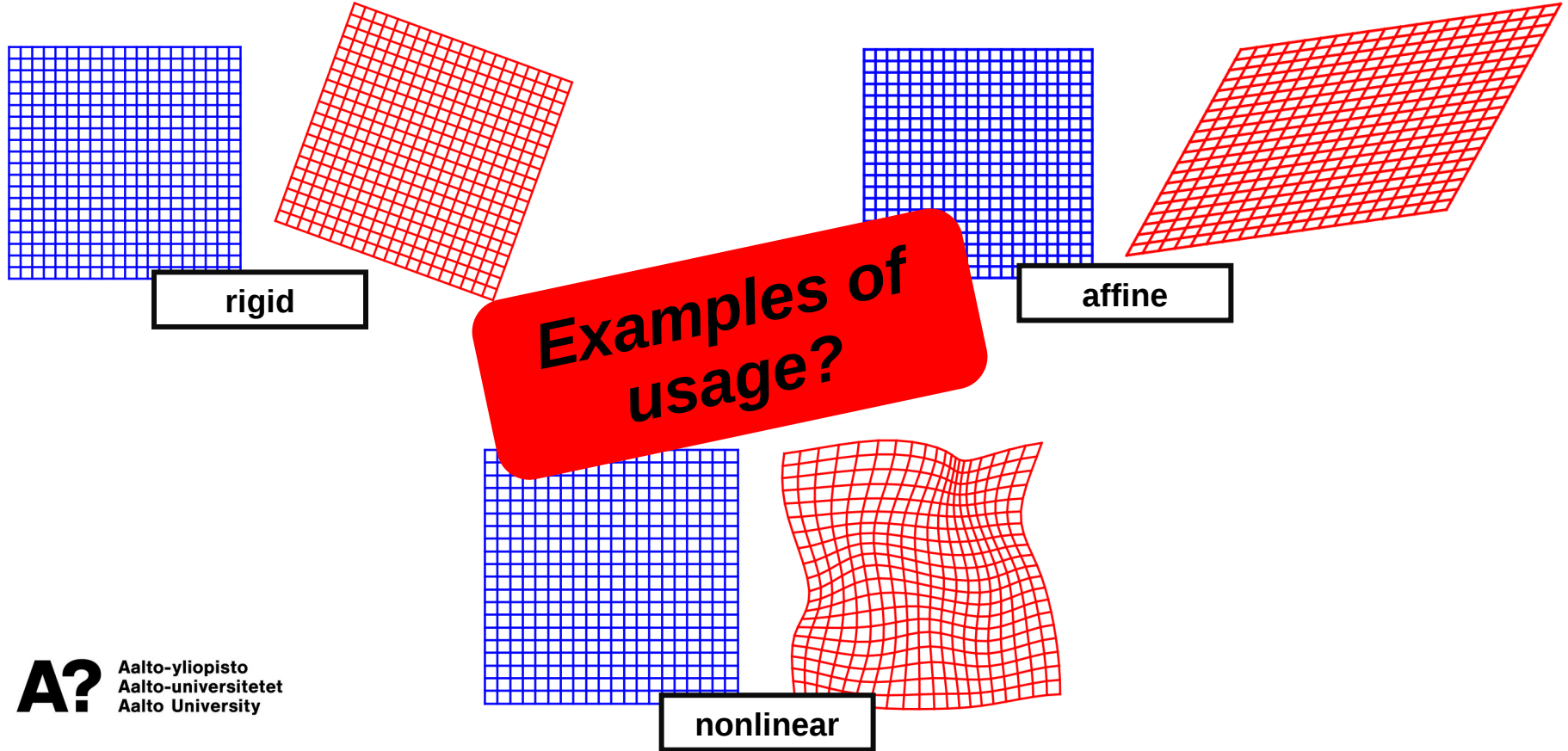
"moving" image

$$\mathbf{y} = (y_1, \dots, y_D)^T$$

Spatial transformations



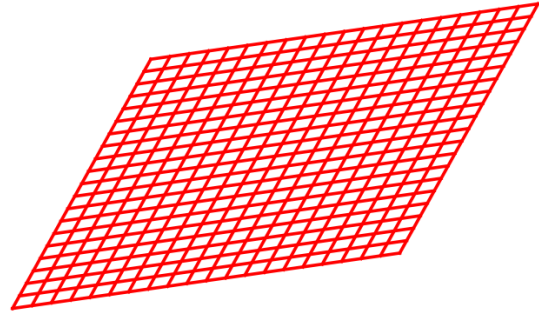
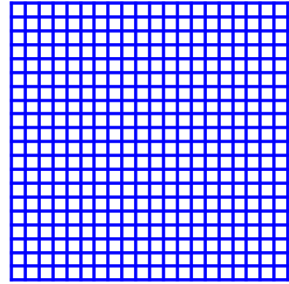
Spatial transformations



Affine transformation

$$\mathbf{y}(\mathbf{x}, \mathbf{w}) = \mathbf{A}\mathbf{x} + \mathbf{t}$$

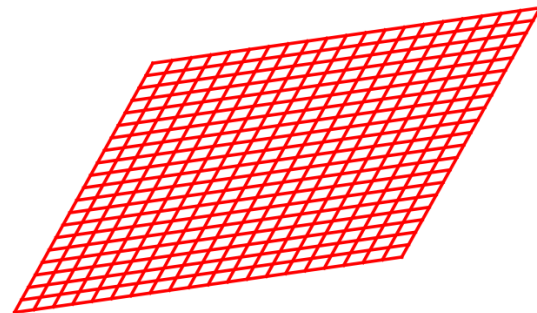
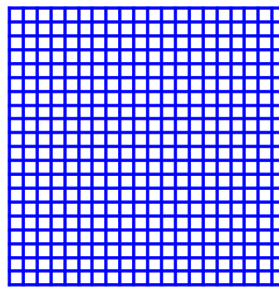
$$\mathbf{A} = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \quad \text{and} \quad \mathbf{t} = \begin{pmatrix} t_1 \\ t_2 \end{pmatrix}$$



Affine transformation

$$\mathbf{y}(\mathbf{x}, \mathbf{w}) = \mathbf{A}\mathbf{x} + \mathbf{t}$$

$$\mathbf{A} = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \quad \text{and} \quad \mathbf{t} = \begin{pmatrix} t_1 \\ t_2 \end{pmatrix}$$



$$y_d(\mathbf{x}, \mathbf{w})$$

controls how points \mathbf{x} in the fixed image move along the d -th direction in the moving image as the parameters \mathbf{w} are varied

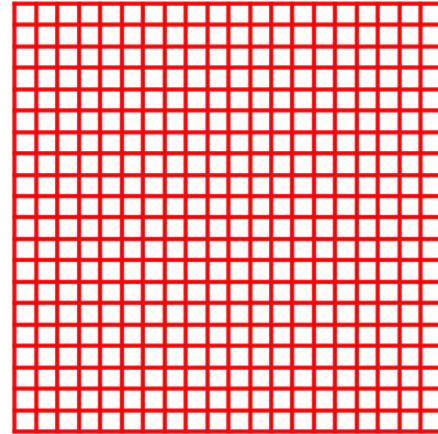
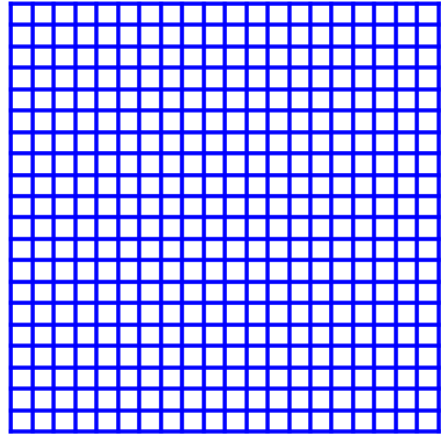
$$y_d(\mathbf{x}, \mathbf{w}_d) = t_d + a_{d,1}x_1 + \dots + a_{d,D}x_D$$

$$\mathbf{w}_d = (t_d, a_{d,1}, \dots, a_{d,D})^T$$

$$\mathbf{w} = (\mathbf{w}_1^T, \dots, \mathbf{w}_D^T)^T$$

Affine transformation

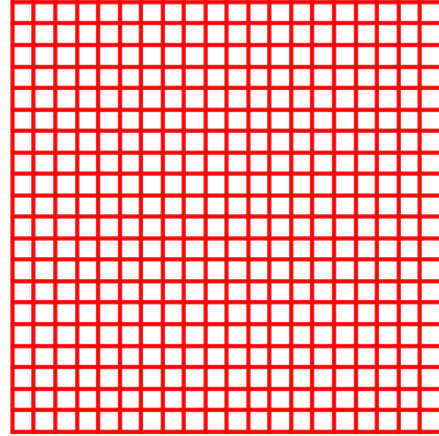
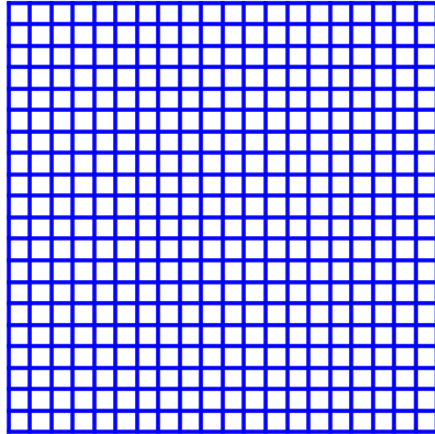
$$\mathbf{y}(\mathbf{x}, \mathbf{w}) = \mathbf{A}\mathbf{x} + \mathbf{t}$$



$$\mathbf{A} = \begin{pmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \end{pmatrix}, \quad \mathbf{t} = \begin{pmatrix} 23 \\ 0 \end{pmatrix}$$

Affine transformation

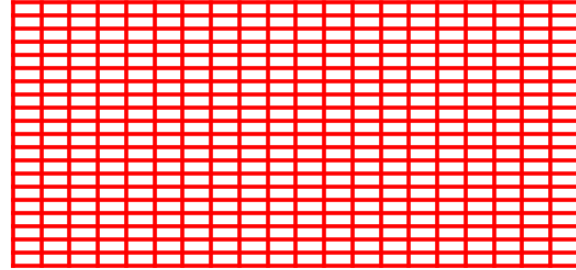
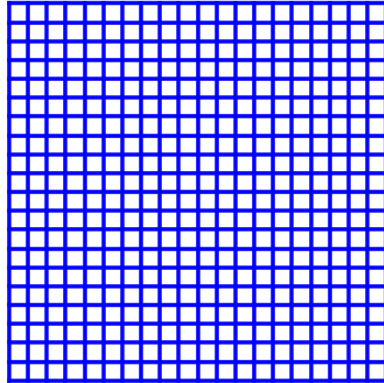
$$\mathbf{y}(\mathbf{x}, \mathbf{w}) = \mathbf{A}\mathbf{x} + \mathbf{t}$$



$$\mathbf{A} = \begin{pmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \end{pmatrix}, \quad \mathbf{t} = \begin{pmatrix} 23 \\ 16 \end{pmatrix}$$

Affine transformation

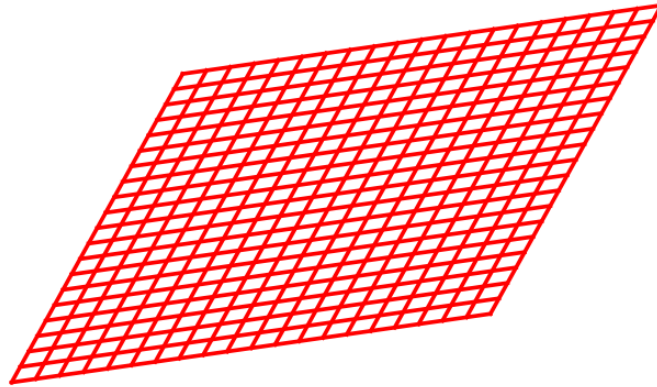
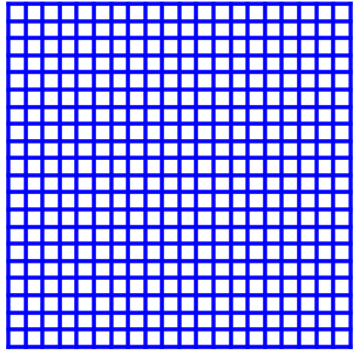
$$\mathbf{y}(\mathbf{x}, \mathbf{w}) = \mathbf{A}\mathbf{x} + \mathbf{t}$$



$$\mathbf{A} = \begin{pmatrix} 1.5 & 0.0 \\ 0.0 & 0.7 \end{pmatrix}, \quad \mathbf{t} = \begin{pmatrix} 23 \\ 16 \end{pmatrix}$$

Affine transformation

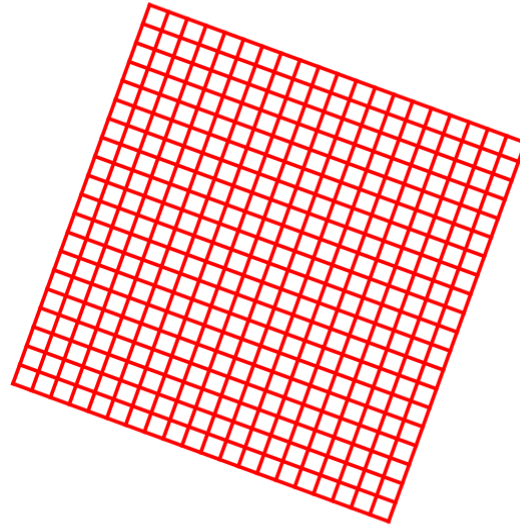
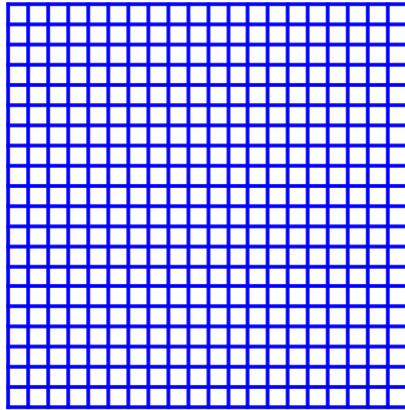
$$\mathbf{y}(\mathbf{x}, \mathbf{w}) = \mathbf{A}\mathbf{x} + \mathbf{t}$$



$$\mathbf{A} = \begin{pmatrix} 1.4 & 0.5 \\ 0.2 & 0.9 \end{pmatrix}, \quad \mathbf{t} = \begin{pmatrix} 23 \\ 2 \end{pmatrix}$$

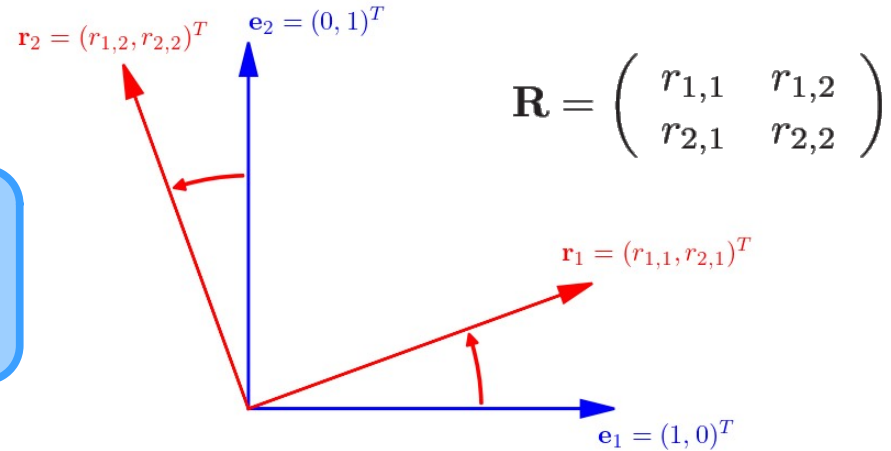
Rigid transformation

$$\mathbf{y}(\mathbf{x}, \mathbf{w}) = \mathbf{R}\mathbf{x} + \mathbf{t}, \quad \mathbf{R}^T \mathbf{R} = \mathbf{I} \text{ and } \det(\mathbf{R}) = 1$$



Rigid transformation

Task: why do $\mathbf{R}^T \mathbf{R} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
and $\det(\mathbf{R}) = 1$ impose a rotation?



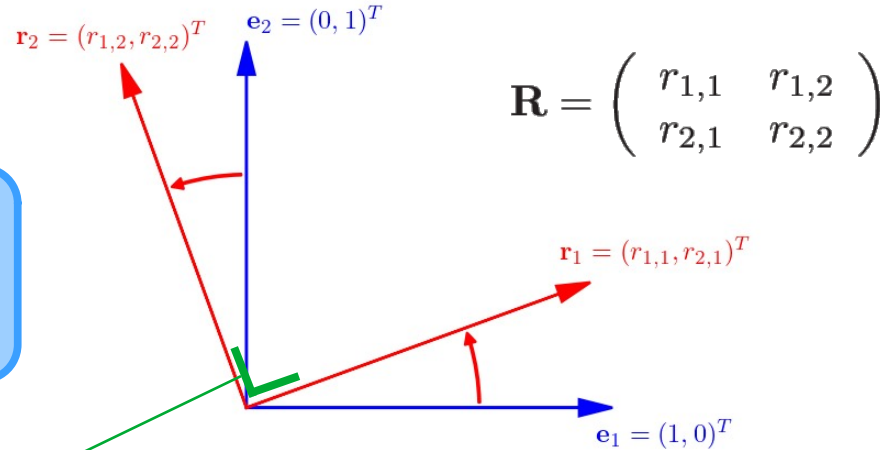
Rigid transformation

$$\|\mathbf{r}_1\| = \sqrt{\mathbf{r}_1^T \mathbf{r}_1} = 1$$

Task: why do $\mathbf{R}^T \mathbf{R} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
and $\det(\mathbf{R}) = 1$ impose a rotation?

$$\|\mathbf{r}_2\| = \sqrt{\mathbf{r}_2^T \mathbf{r}_2} = 1$$

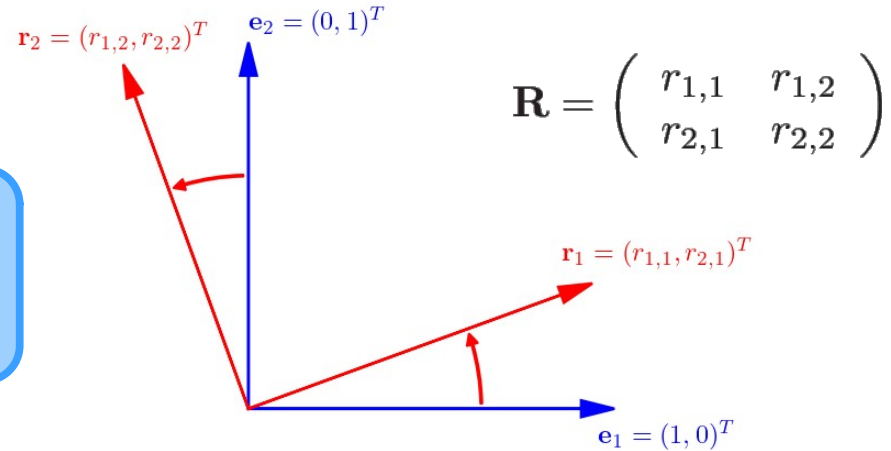
$$\mathbf{r}_1^T \mathbf{r}_2 = 0$$



Rigid transformation

Task: why do $\mathbf{R}^T \mathbf{R} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
and $\det(\mathbf{R}) = 1$ impose a rotation?

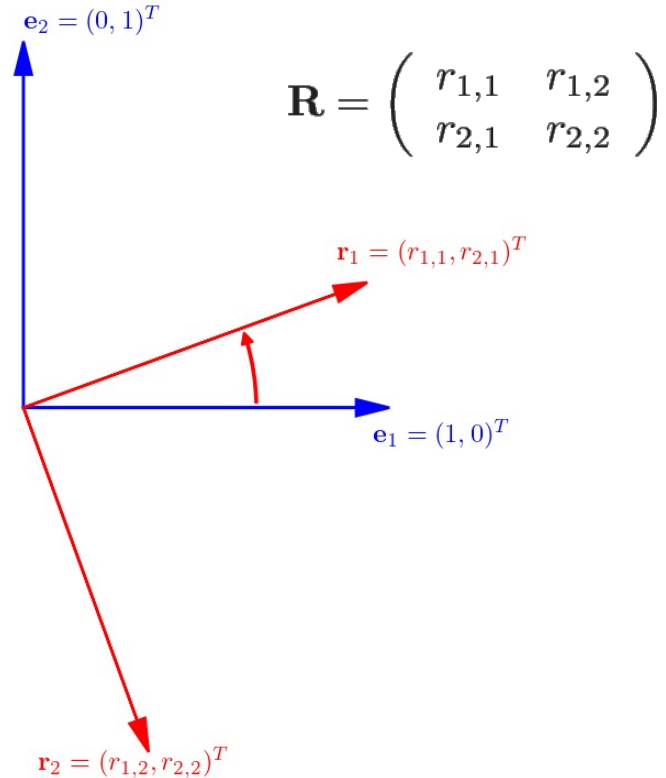
?



Rigid transformation

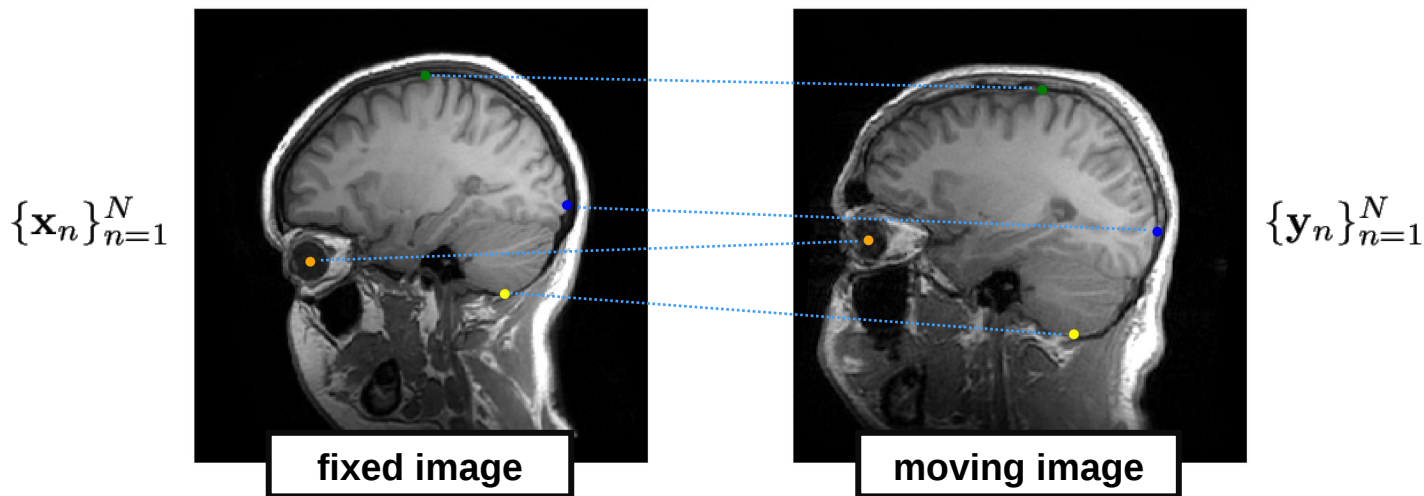
Task: why do $\mathbf{R}^T \mathbf{R} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
and $\det(\mathbf{R}) = 1$ impose a rotation?

?



Landmark-based registration

- ✓ Manually annotate N corresponding points in two images:



- ✓ Register the images by minimizing the distance between matching point pairs:

$$E(\mathbf{w}) = \sum_{n=1}^N \|\mathbf{y}_n - \mathbf{y}(\mathbf{x}_n, \mathbf{w})\|^2$$

Landmark-based registration

Applied to affine registration: $E(\mathbf{w}) = \sum_{n=1}^N \|\mathbf{y}_n - \mathbf{A}\mathbf{x}_n - \mathbf{t}\|^2$

Task 1: if $\mathbf{A} = \mathbf{I}$, what is \mathbf{t} ?

Hint: remember that $\|\mathbf{a} - \mathbf{b}\|^2 = \sum_{d=1}^D (a_d - b_d)^2$

Task 2: in the general case, what are \mathbf{A} and \mathbf{t} ?

Landmark-based registration

Applied to affine registration: $E(\mathbf{w}) = \sum_{n=1}^N \|\mathbf{y}_n - \mathbf{A}\mathbf{x}_n - \mathbf{t}\|^2 = \sum_{n=1}^N \sum_d^D (y_{n,d} - t_n - \sum_{d'=1}^D x_{n,d'} a_{d,d'})^2$


Task 1: if $\mathbf{A} = \mathbf{I}$, what is \mathbf{t} ?

Hint: remember that $\|\mathbf{a} - \mathbf{b}\|^2 = \sum_{d=1}^D (a_d - b_d)^2$

Task 2: in the general case, what are \mathbf{A} and \mathbf{t} ?

Landmark-based registration

Applied to affine registration: $E(\mathbf{w}) = \sum_{n=1}^N \|\mathbf{y}_n - \mathbf{A}\mathbf{x}_n - \mathbf{t}\|^2 = \sum_{n=1}^N \sum_d^D (y_{n,d} - t_n - \sum_{d'=1}^D x_{n,d'} a_{d,d'})^2$



Task 1: if $\mathbf{A} = \mathbf{I}$, what is \mathbf{t} ?

Hint: remember that $\|\mathbf{a} - \mathbf{b}\|^2 = \sum_{d=1}^D (a_d - b_d)^2$

Task 2: in the general case, what are \mathbf{A} and \mathbf{t} ?

Landmark-based registration

Applied to affine registration: $E(\mathbf{w}) = \sum_{n=1}^N \|\mathbf{y}_n - \mathbf{A}\mathbf{x}_n - \mathbf{t}\|^2$

Task 1: if $\mathbf{A} = \mathbf{I}$, what is \mathbf{t} ?

Hint: remember that $\|\mathbf{a} - \mathbf{b}\|^2 = \sum_{d=1}^D (a_d - b_d)^2$

Task 2: in the general case, what are \mathbf{A} and \mathbf{t} ?

$$\begin{aligned} &= \sum_{n=1}^N \sum_d^D (y_{n,d} - t_n - \sum_{d'=1}^D x_{n,d'} a_{d,d'})^2 \\ &= \sum_d^D \sum_{n=1}^N (y_{n,d} - t_n - \sum_{d'=1}^D x_{n,d'} a_{d,d'})^2 \end{aligned}$$

Landmark-based registration

Applied to affine registration: $E(\mathbf{w}) = \sum_{n=1}^N \|\mathbf{y}_n - \mathbf{A}\mathbf{x}_n - \mathbf{t}\|^2$

Task 1: if $\mathbf{A} = \mathbf{I}$, what is \mathbf{t} ?

Hint: remember that $\|\mathbf{a} - \mathbf{b}\|^2 = \sum_{d=1}^D (a_d - b_d)^2$

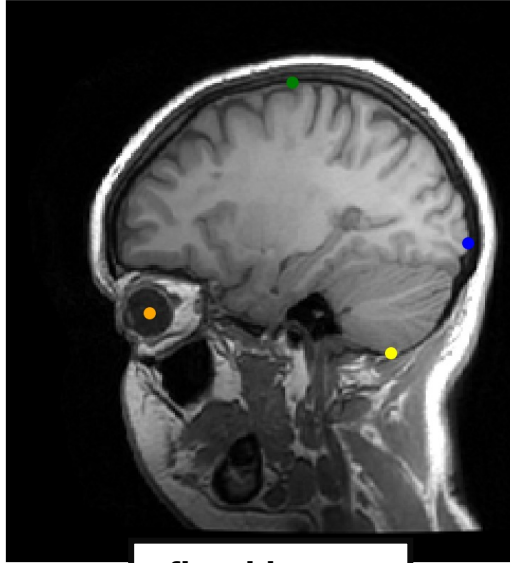
Task 2: in the general case, what are \mathbf{A} and \mathbf{t} ?

$$\begin{aligned} E(\mathbf{w}) &= \sum_{n=1}^N \sum_d^D (y_{n,d} - t_n - \sum_{d'=1}^D x_{n,d'} a_{d,d'})^2 \\ &= \sum_d^D \sum_{n=1}^N (y_{n,d} - t_n - \sum_{d'=1}^D x_{n,d'} a_{d,d'})^2 \end{aligned}$$

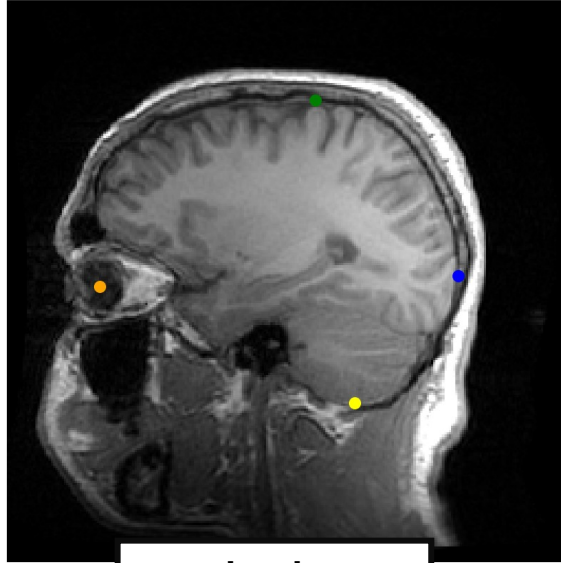
$$\begin{pmatrix} t_d \\ a_{d,1} \\ \vdots \\ a_{d,D} \end{pmatrix} = (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \begin{pmatrix} y_{1,d} \\ \vdots \\ y_{N,d} \end{pmatrix}$$

$$\text{where } \mathbf{X} = \begin{pmatrix} 1 & x_{1,1} & \cdots & x_{1,D} \\ 1 & x_{2,1} & \cdots & x_{2,D} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_{N,1} & \cdots & x_{N,D} \end{pmatrix}$$

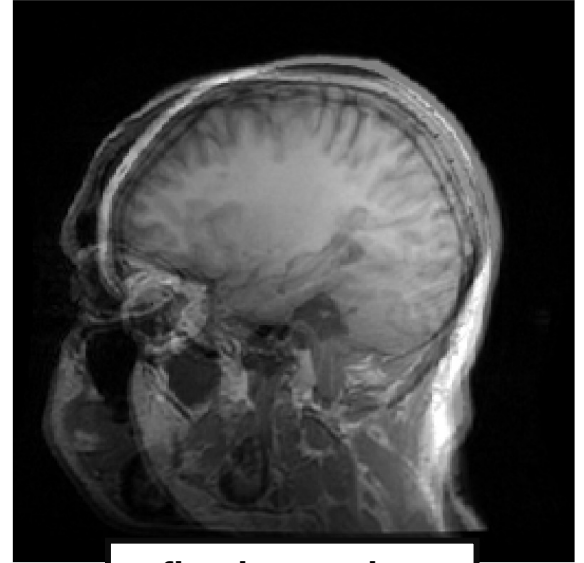
Landmark-based registration



fixed image



moving image



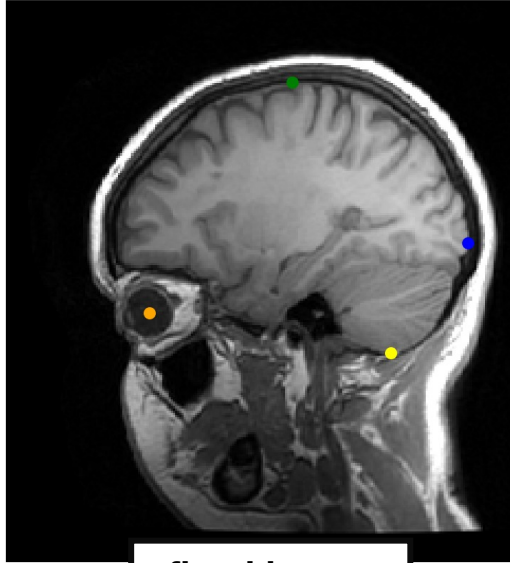
fixed + moving

Before registration

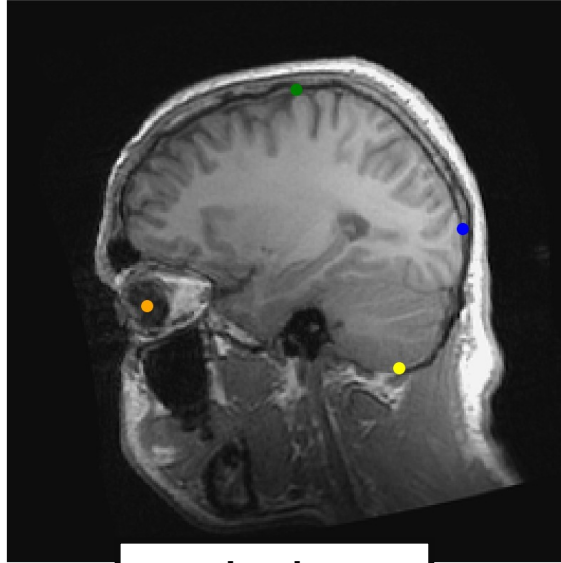


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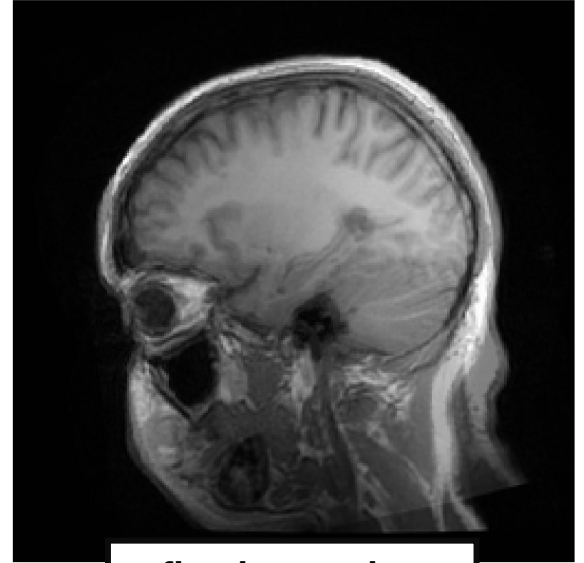
Landmark-based registration



fixed image



moving image

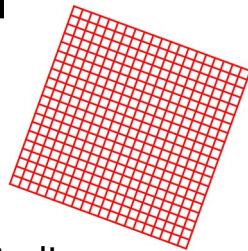
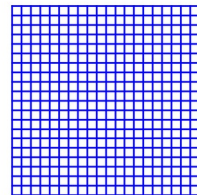


fixed + moving

After registration

Landmark-based registration

Applied to rigid registration: $E(\mathbf{w}) = \sum_{n=1}^N \|\mathbf{y}_n - \mathbf{R}\mathbf{x}_n - \mathbf{t}\|^2$



- ✓ Constraints $\mathbf{R}^T \mathbf{R} = \mathbf{I}$ and $\det(\mathbf{R}) = 1$ make the math much more complicated!
- ✓ Solution:

$$\mathbf{R} = \mathbf{V}\mathbf{U}^T, \quad \sum_{n=1}^N \tilde{\mathbf{x}}_n \tilde{\mathbf{y}}_n^T = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T, \quad \mathbf{U}^T \mathbf{U} = \mathbf{I}, \quad \mathbf{V}^T \mathbf{V} = \mathbf{I}$$

$$\mathbf{t} = \bar{\mathbf{y}} - \mathbf{R}\bar{\mathbf{x}},$$

$$\text{where } \tilde{\mathbf{x}}_n = \mathbf{x}_n - \bar{\mathbf{x}} \quad \text{and} \quad \tilde{\mathbf{y}}_n = \mathbf{y}_n - \bar{\mathbf{y}}$$

$$\text{with } \bar{\mathbf{x}} = \frac{1}{N} \sum_{n=1}^N \mathbf{x}_n \quad \text{and} \quad \bar{\mathbf{y}} = \frac{1}{N} \sum_{n=1}^N \mathbf{y}_n$$

(“flip” a column of \mathbf{R} if $\det(\mathbf{R}) = -1$)