

Landmark-based Registration



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Medical Image Analysis

Koen Van Leemput

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Image registration

Combine information contained in different scans

Images need to be spatially aligned!

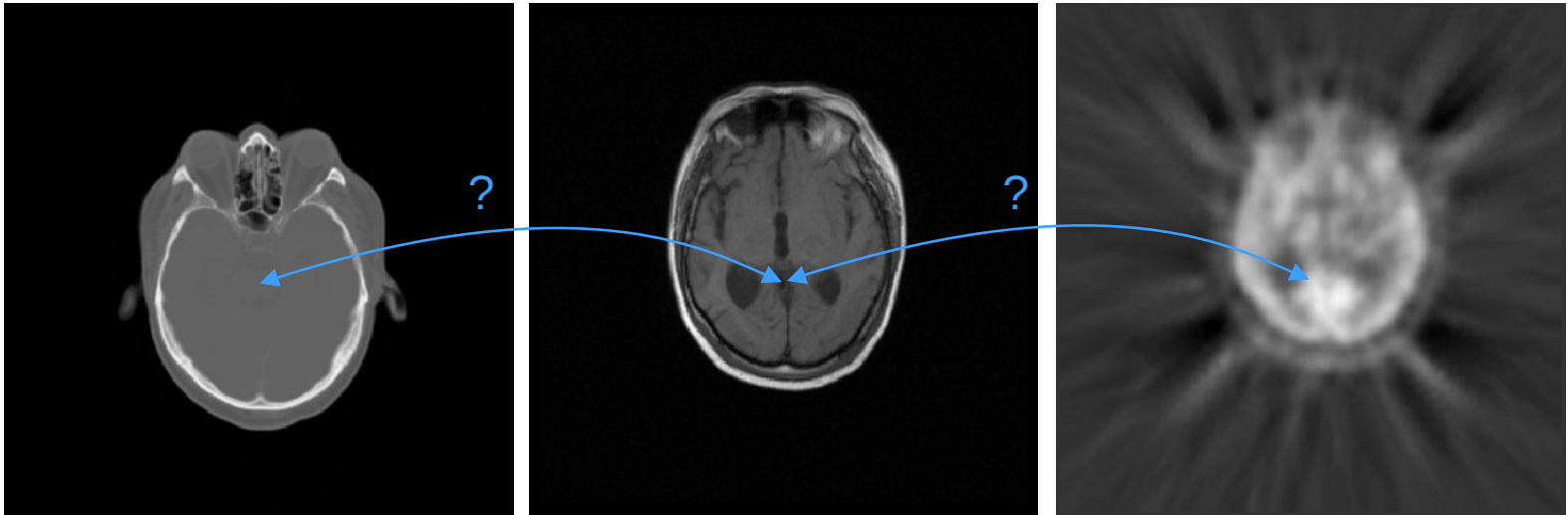
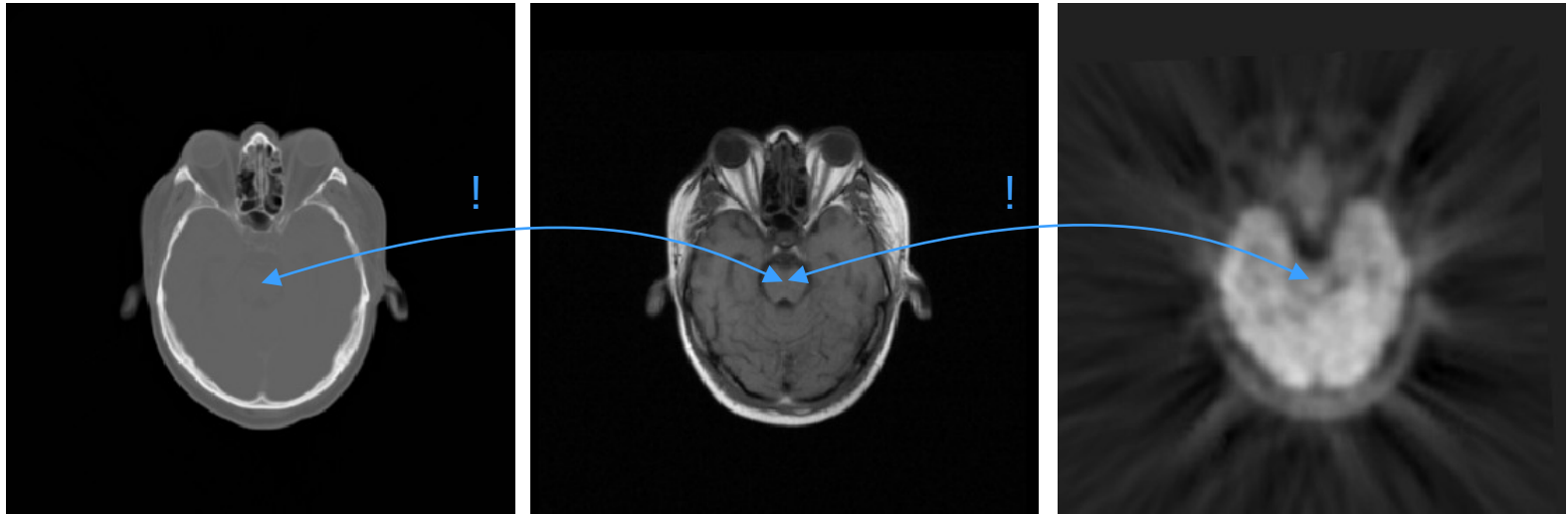


Image registration

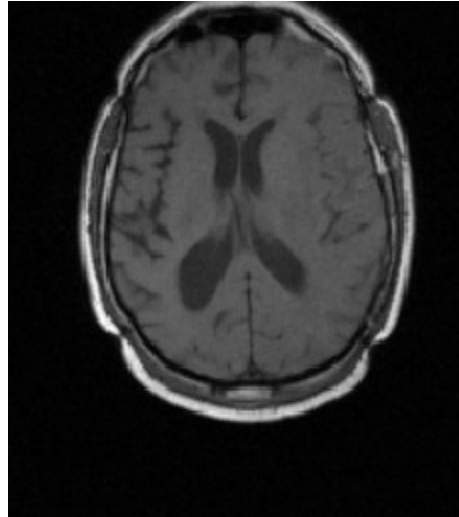
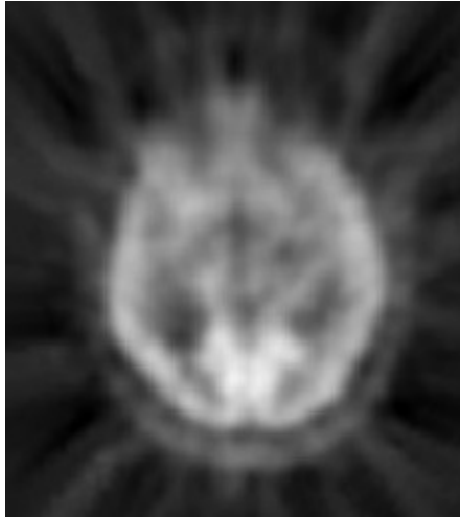
Combine information contained in different scans

Images need to be spatially aligned!



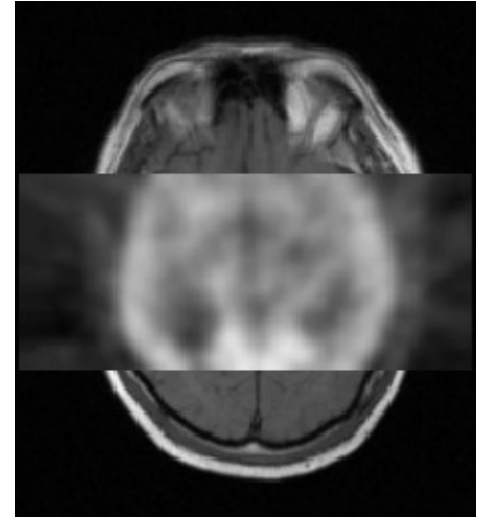
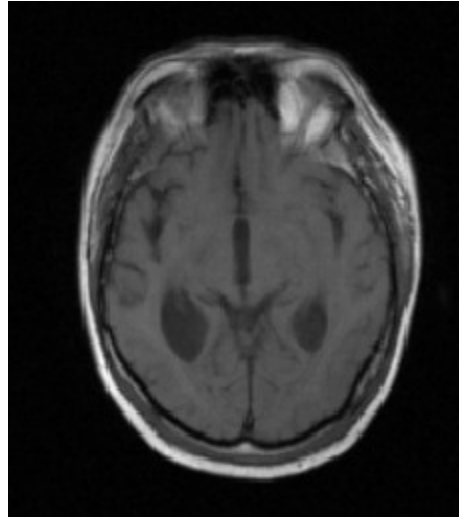
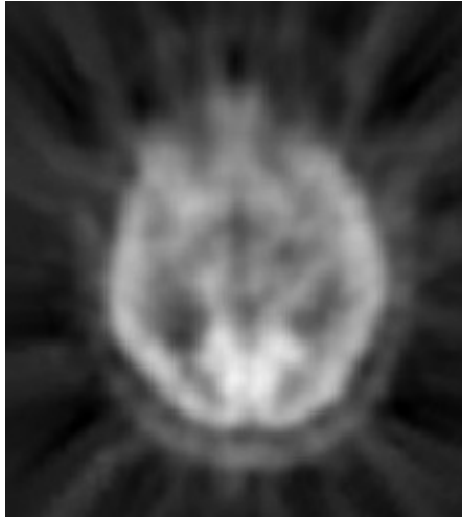
Example: PET/MR

Before registration...



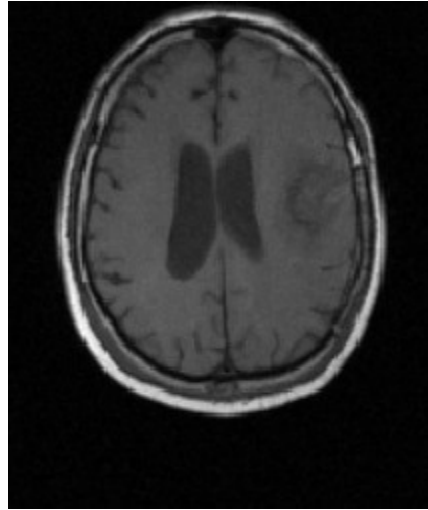
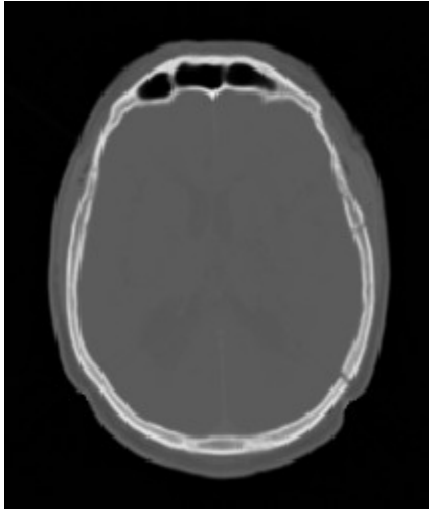
Example: PET/MR

... after registration



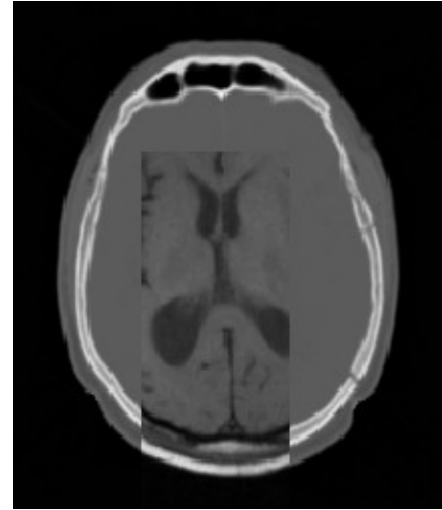
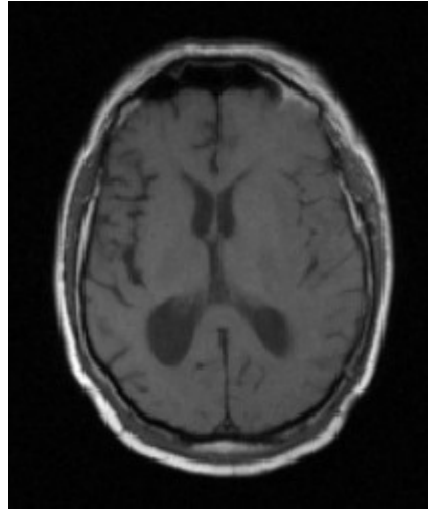
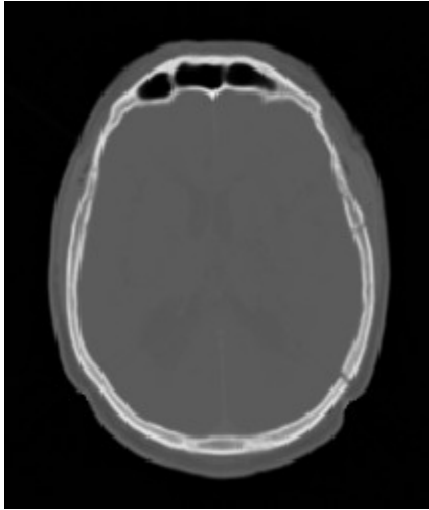
Example: CT/MR

Before registration...



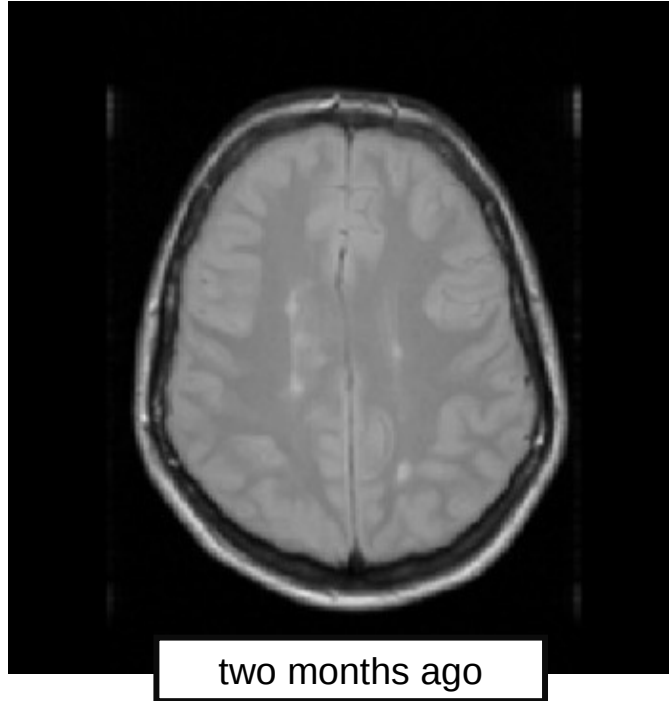
Example: CT/MR

... after registration



Example: longitudinal scans

Patient with multiple sclerosis (MS)

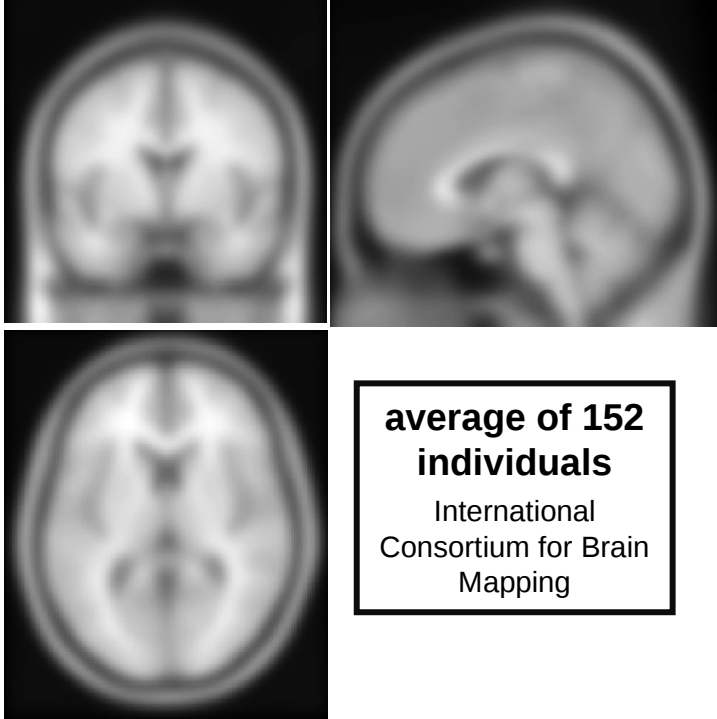


Example: longitudinal scans

Patient with multiple sclerosis (MS)

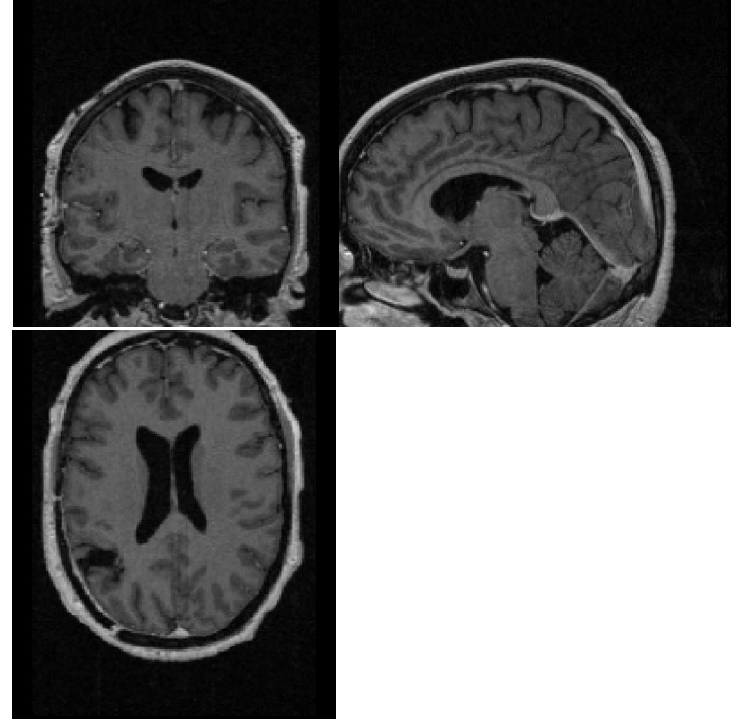


Example: population studies

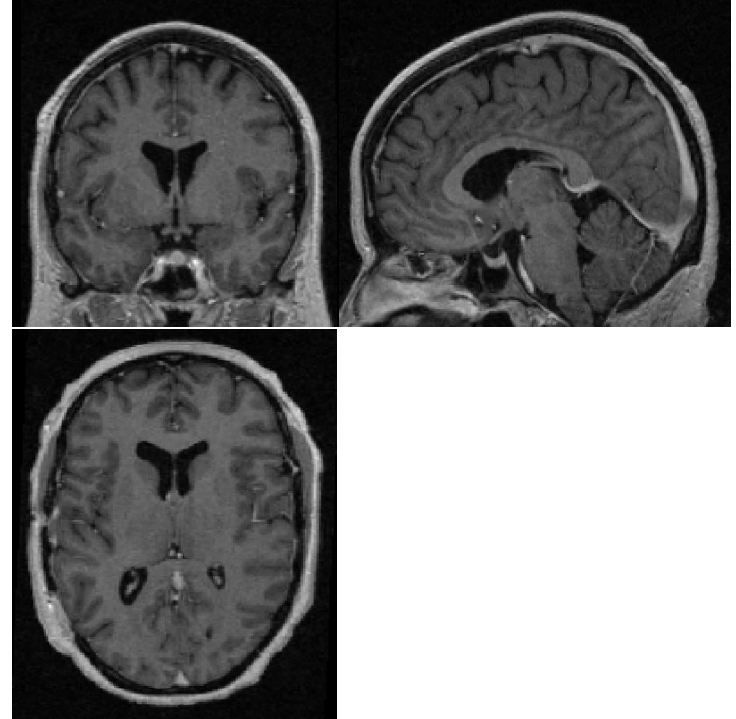
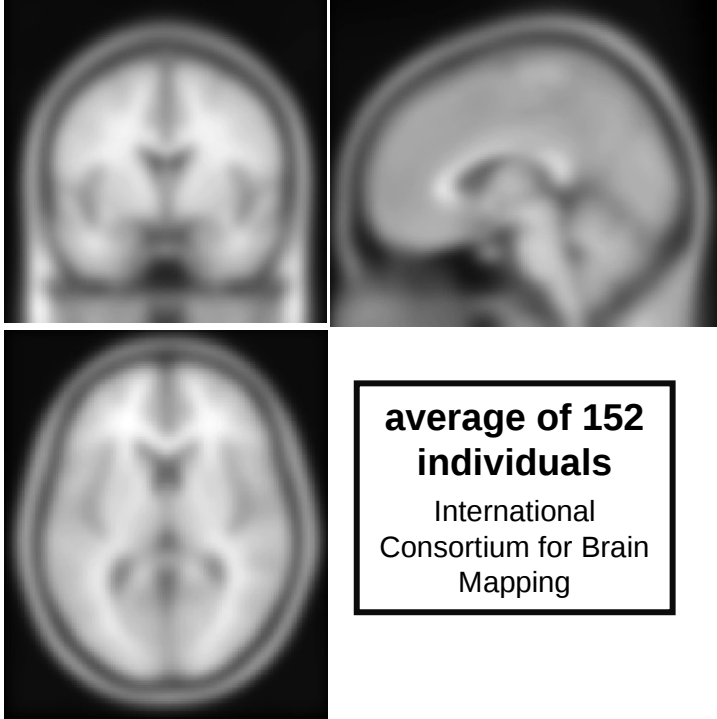


**average of 152
individuals**

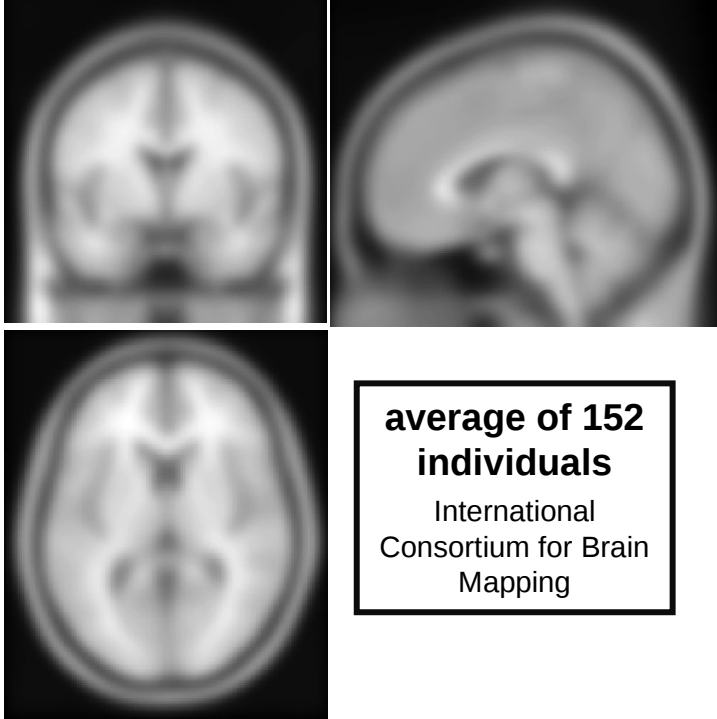
International
Consortium for Brain
Mapping



Example: population studies

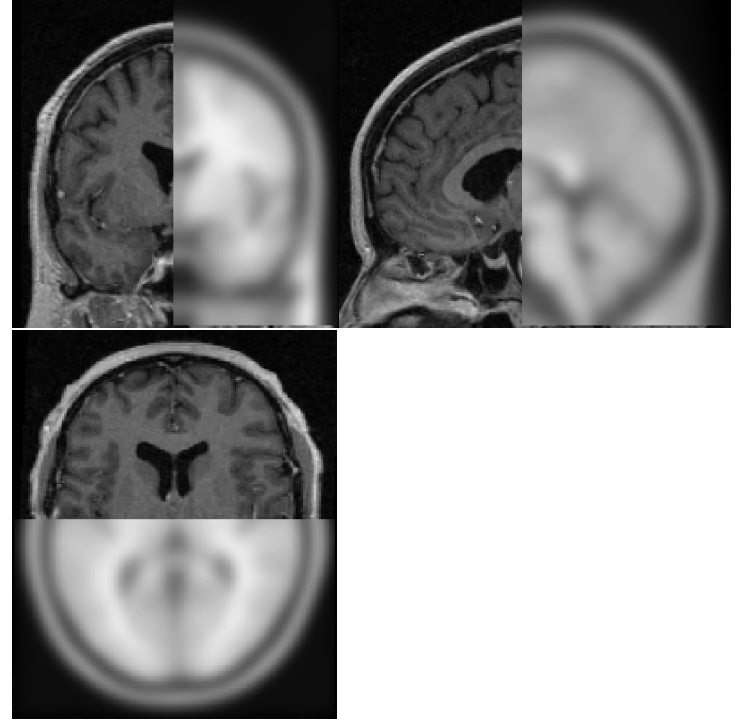


Example: population studies



**average of 152
individuals**

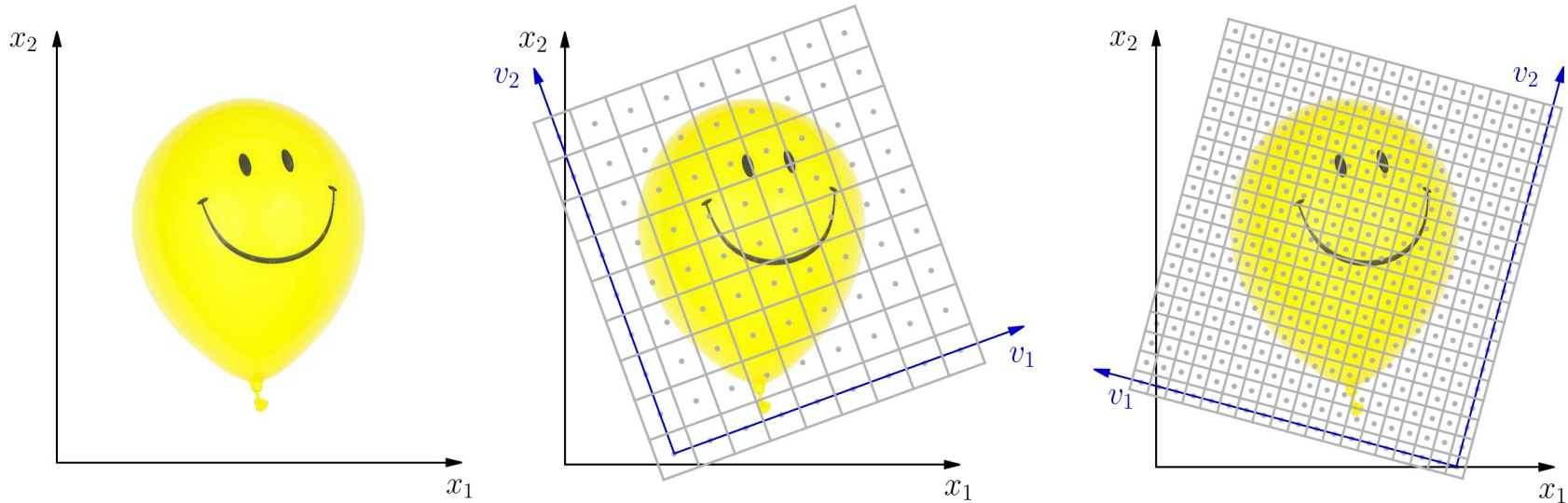
International
Consortium for Brain
Mapping



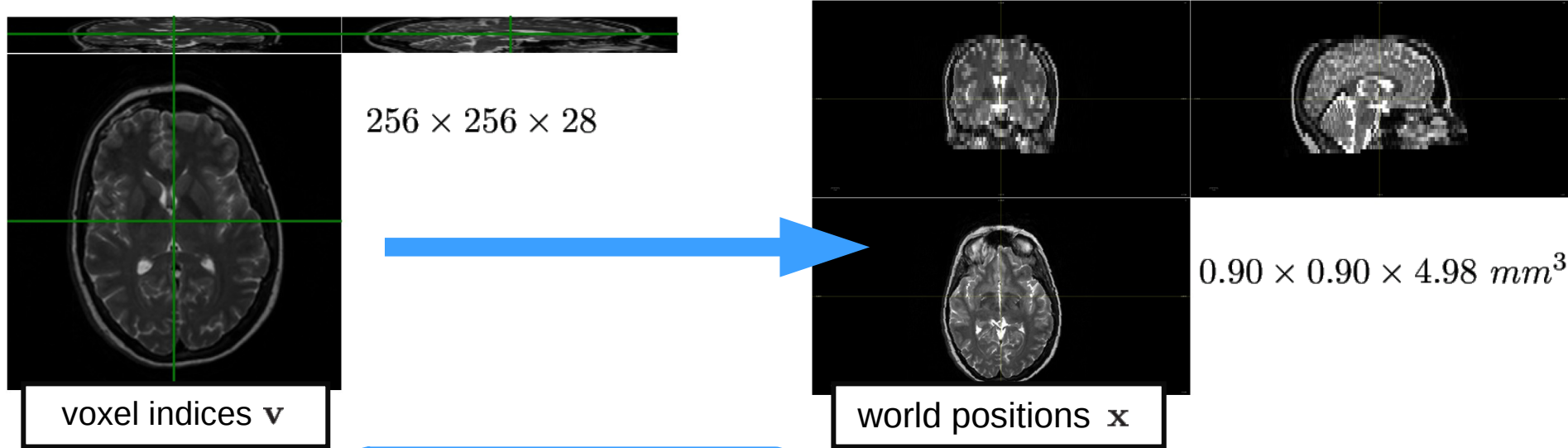
Coordinate systems

For each image, there are two coordinate systems:

- ✓ Voxel coordinates $\mathbf{v} = (v_1, v_2, v_3)^T$ (integer indices)
- ✓ World coordinates $\mathbf{x} = (x_1, x_2, x_3)^T$ (in mm)



Coordinate systems



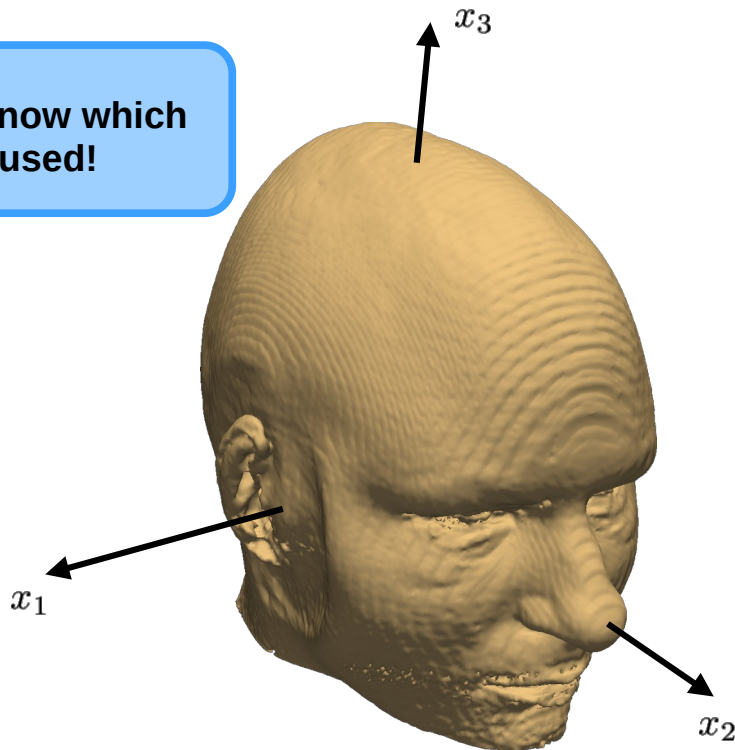
Conversion: $\mathbf{x} = \mathbf{A}\mathbf{v} + \mathbf{t}$

$$\mathbf{A} = \begin{pmatrix} -0.8923 & -0.0802 & -0.3732 \\ -0.0850 & 0.8921 & 0.3528 \\ -0.0612 & -0.0696 & 4.9512 \end{pmatrix}$$

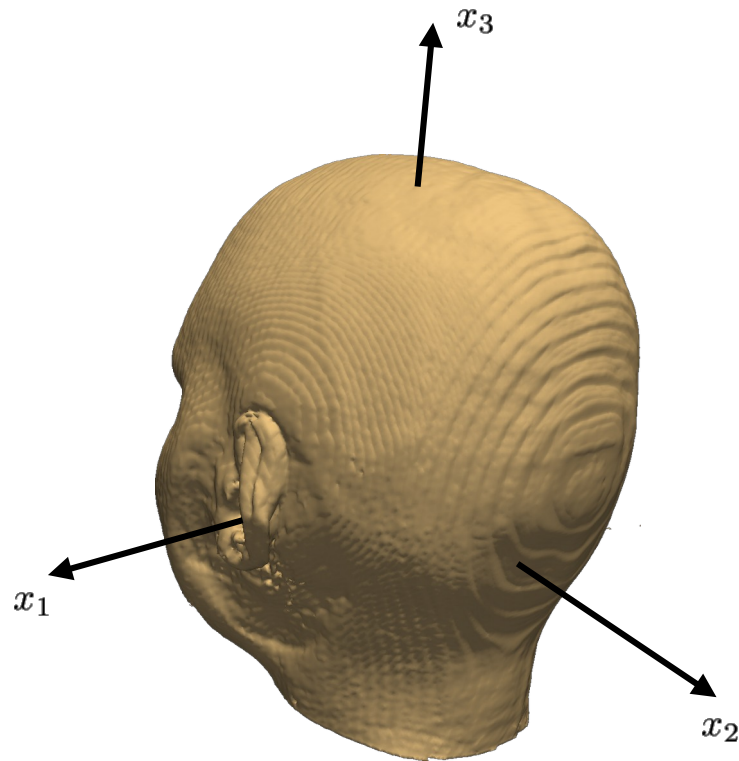
$$\mathbf{t} = \begin{pmatrix} 129.2834 \\ -98.7363 \\ -27.6911 \end{pmatrix}$$

World coordinates = convention

Important to know which convention is used!



Right – Anterior – Superior
(RAS)



Left – Posterior – Superior
(LPS)

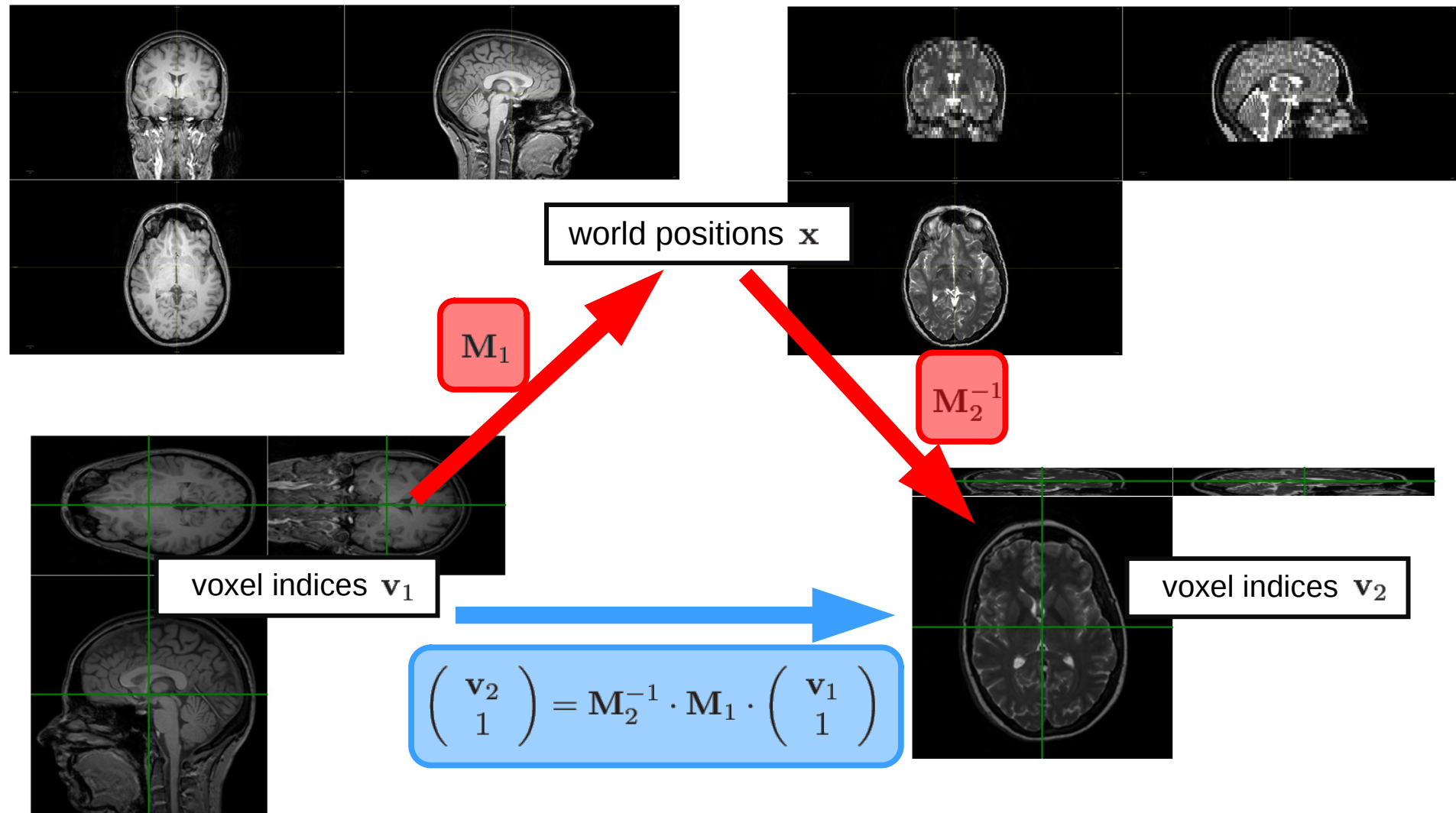
Homogeneous coordinates

Vectors are augmented with a 1 at the end

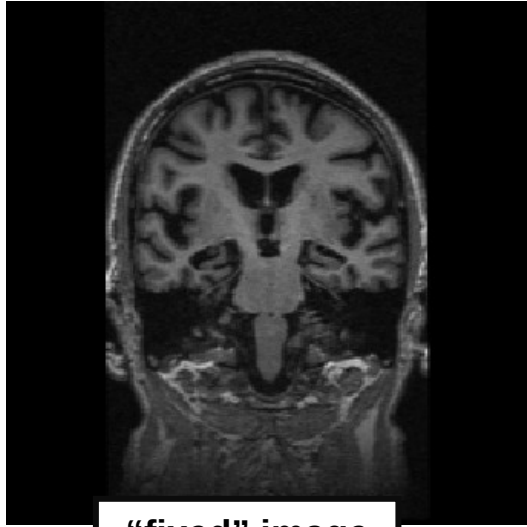
✓ **Idea:** Rewrite $\mathbf{x} = \mathbf{A}\mathbf{v} + \mathbf{t}$, i.e.,
$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} + \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix}$$

as:
$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ 1 \end{pmatrix} = \underbrace{\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & t_1 \\ a_{2,1} & a_{2,2} & a_{2,3} & t_2 \\ a_{3,1} & a_{3,2} & a_{3,3} & t_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{\mathbf{M}} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ 1 \end{pmatrix}$$

✓ **Benefit:** map voxel indices using only matrix multiplications

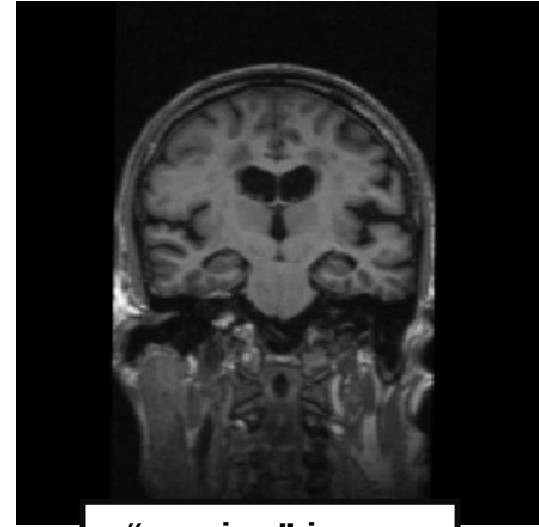


Spatial transformations



“fixed” image

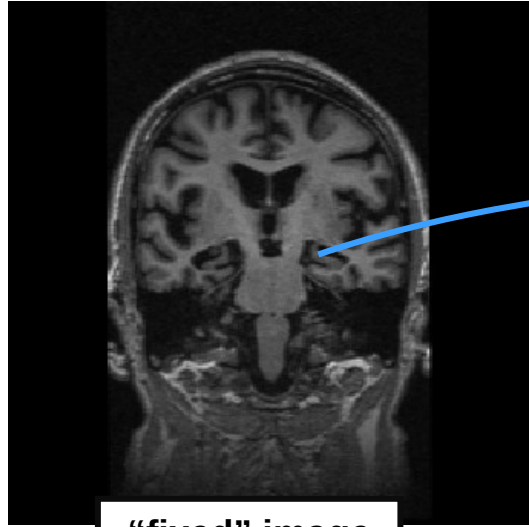
$$\mathbf{x} = (x_1, \dots, x_D)^T$$



“moving” image

$$\mathbf{y} = (y_1, \dots, y_D)^T$$

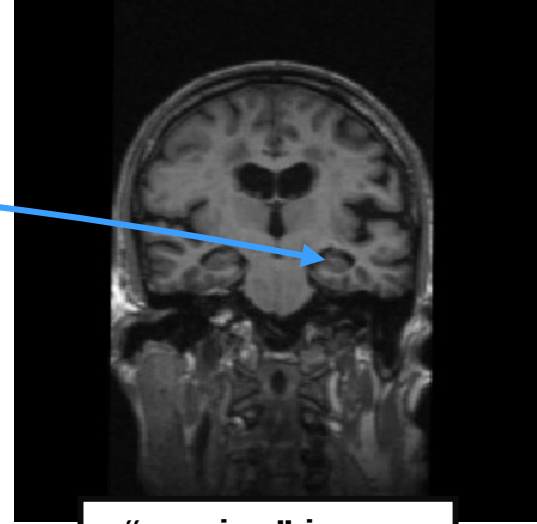
Spatial transformations



"fixed" image

$$\mathbf{x} = (x_1, \dots, x_D)^T$$

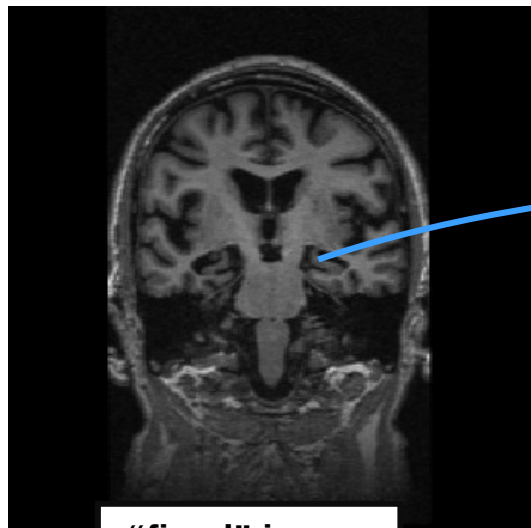
$$\mathbf{y}(\mathbf{x}, \mathbf{w}) = \begin{pmatrix} y_1(\mathbf{x}, \mathbf{w}) \\ \vdots \\ y_D(\mathbf{x}, \mathbf{w}) \end{pmatrix}$$



"moving" image

$$\mathbf{y} = (y_1, \dots, y_D)^T$$

Spatial transformations



"fixed" image

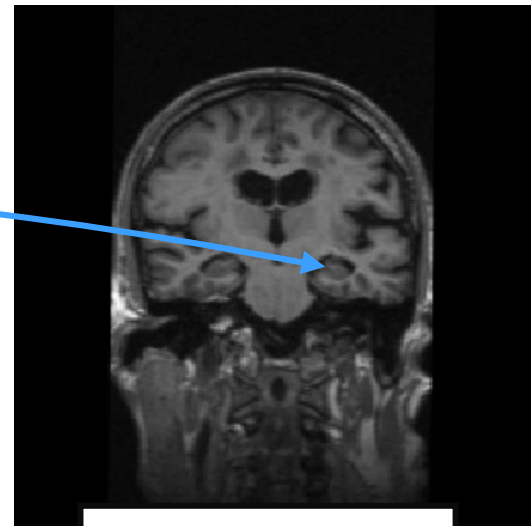
$$\mathbf{x} = (x_1, \dots, x_D)^T$$

$$\mathbf{y}(\mathbf{x}, \mathbf{w}) = \begin{pmatrix} y_1(\mathbf{x}, \mathbf{w}) \\ \vdots \\ y_D(\mathbf{x}, \mathbf{w}) \end{pmatrix}$$



$$y_d(\mathbf{x}, \mathbf{w})$$

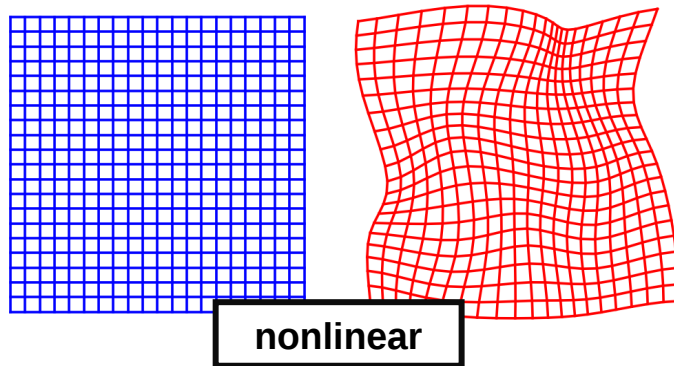
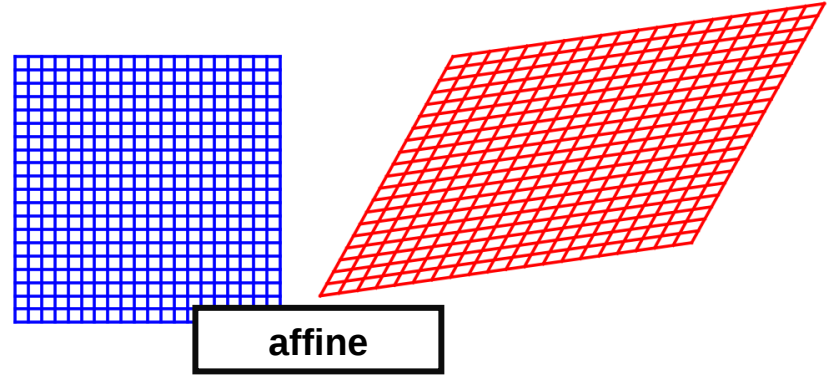
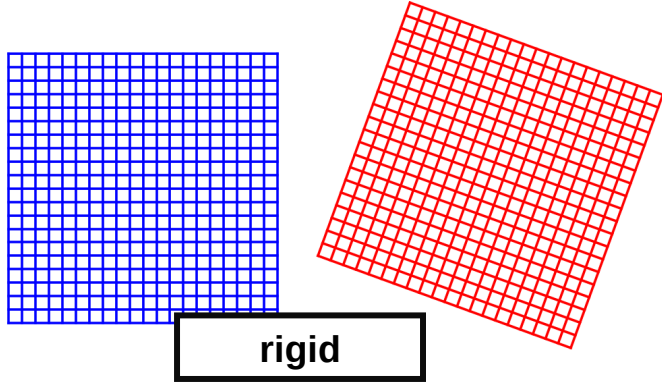
controls how points \mathbf{x} in the fixed image
move along the d -th direction in the moving image
as the parameters \mathbf{w} are varied



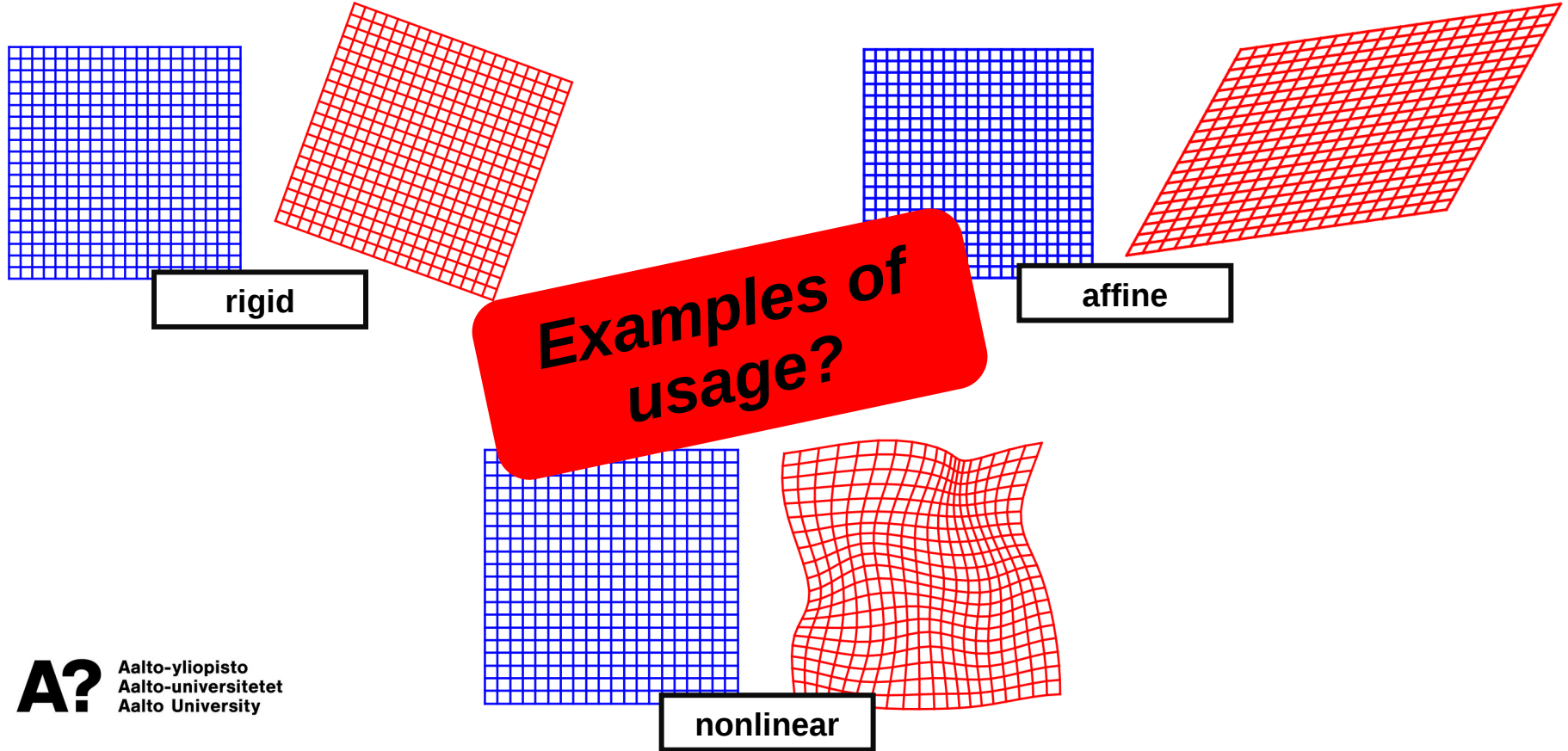
"moving" image

$$\mathbf{y} = (y_1, \dots, y_D)^T$$

Spatial transformations



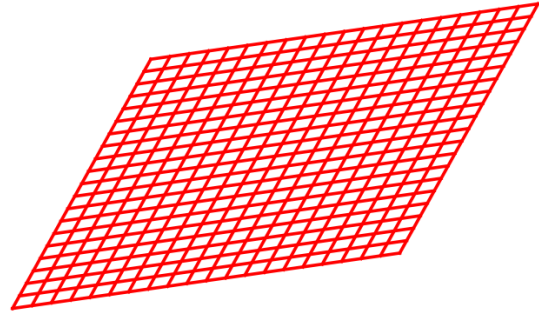
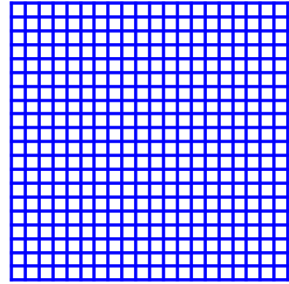
Spatial transformations



Affine transformation

$$\mathbf{y}(\mathbf{x}, \mathbf{w}) = \mathbf{A}\mathbf{x} + \mathbf{t}$$

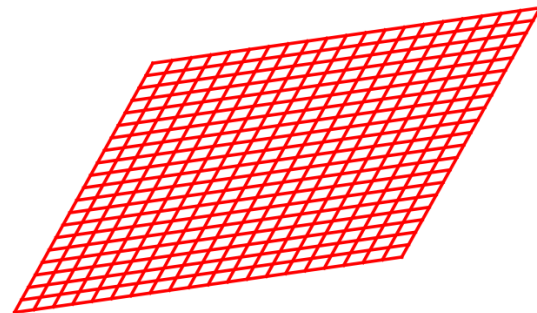
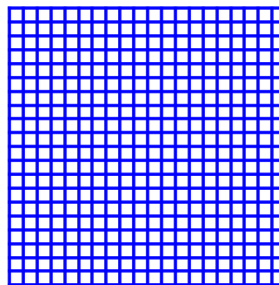
$$\mathbf{A} = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \quad \text{and} \quad \mathbf{t} = \begin{pmatrix} t_1 \\ t_2 \end{pmatrix}$$



Affine transformation

$$\mathbf{y}(\mathbf{x}, \mathbf{w}) = \mathbf{A}\mathbf{x} + \mathbf{t}$$

$$\mathbf{A} = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \quad \text{and} \quad \mathbf{t} = \begin{pmatrix} t_1 \\ t_2 \end{pmatrix}$$



$$y_d(\mathbf{x}, \mathbf{w})$$

controls how points \mathbf{x} in the fixed image move along the d -th direction in the moving image as the parameters \mathbf{w} are varied

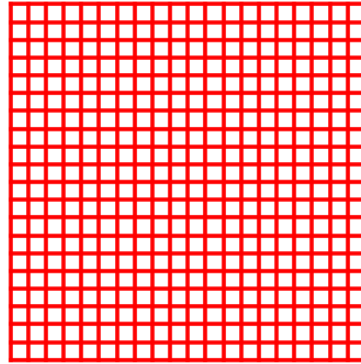
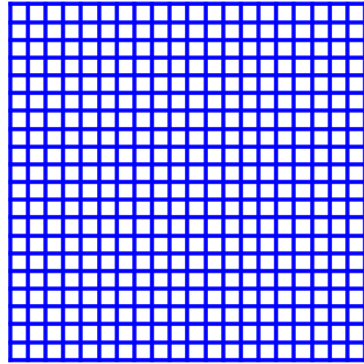
$$y_d(\mathbf{x}, \mathbf{w}_d) = t_d + a_{d,1}x_1 + \dots + a_{d,D}x_D$$

$$\mathbf{w}_d = (t_d, a_{d,1}, \dots, a_{d,D})^T$$

$$\mathbf{w} = (\mathbf{w}_1^T, \dots, \mathbf{w}_D^T)^T$$

Affine transformation

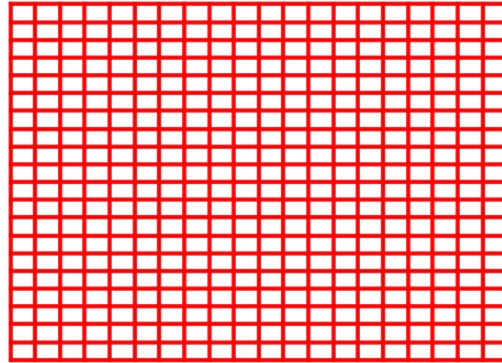
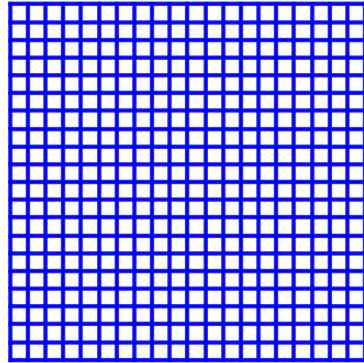
$$\mathbf{y}(\mathbf{x}, \mathbf{w}) = \mathbf{A}\mathbf{x} + \mathbf{t}$$



$$\mathbf{A} = \begin{pmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \end{pmatrix}, \mathbf{t} = \begin{pmatrix} 23 \\ 0 \end{pmatrix}$$

Affine transformation

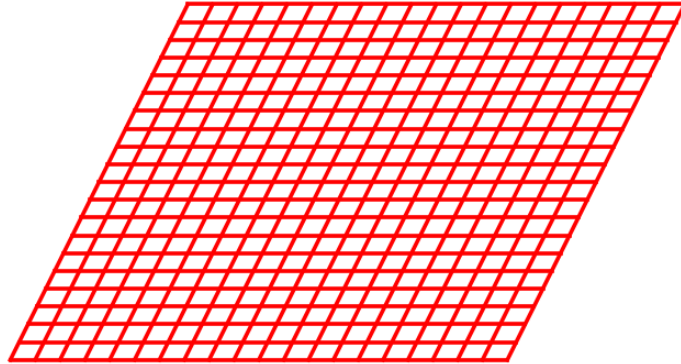
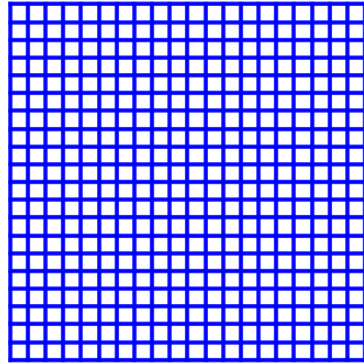
$$\mathbf{y}(\mathbf{x}, \mathbf{w}) = \mathbf{A}\mathbf{x} + \mathbf{t}$$



$$\mathbf{A} = \begin{pmatrix} 1.4 & 0.0 \\ 0.0 & 1.0 \end{pmatrix}, \mathbf{t} = \begin{pmatrix} 23 \\ 0 \end{pmatrix}$$

Affine transformation

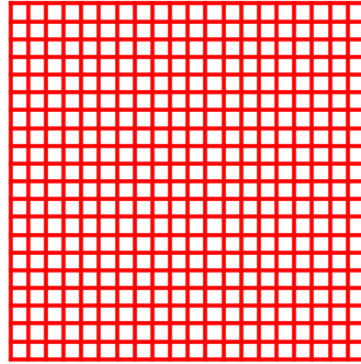
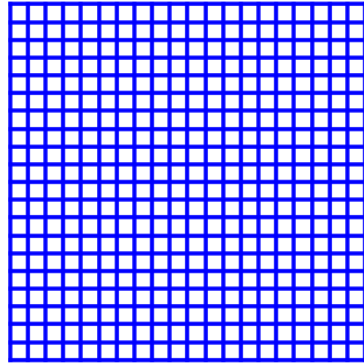
$$\mathbf{y}(\mathbf{x}, \mathbf{w}) = \mathbf{A}\mathbf{x} + \mathbf{t}$$



$$\mathbf{A} = \begin{pmatrix} 1.4 & 0.5 \\ 0.0 & 1.0 \end{pmatrix}, \mathbf{t} = \begin{pmatrix} 23 \\ 0 \end{pmatrix}$$

Affine transformation

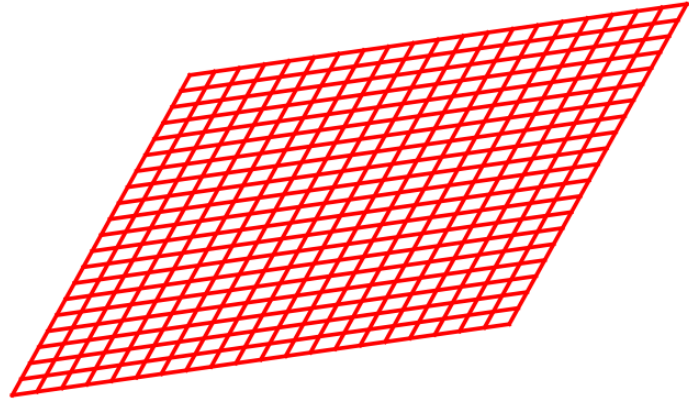
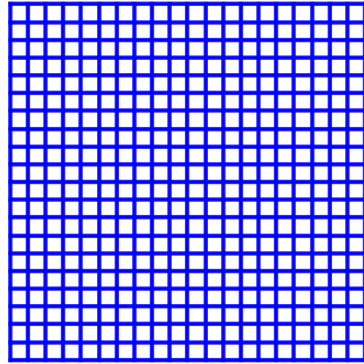
$$\mathbf{y}(\mathbf{x}, \mathbf{w}) = \mathbf{A}\mathbf{x} + \mathbf{t}$$



$$\mathbf{A} = \begin{pmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \end{pmatrix}, \quad \mathbf{t} = \begin{pmatrix} 23 \\ 6 \end{pmatrix}$$

Affine transformation

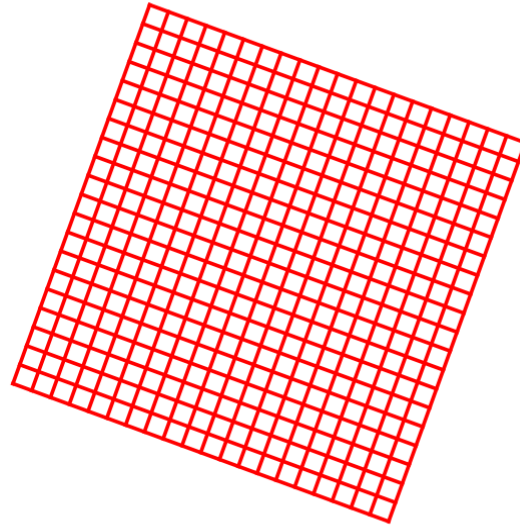
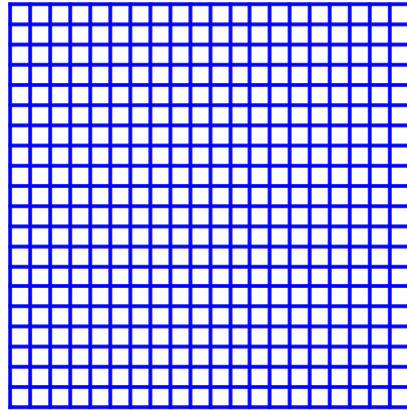
$$\mathbf{y}(\mathbf{x}, \mathbf{w}) = \mathbf{A}\mathbf{x} + \mathbf{t}$$



$$\mathbf{A} = \begin{pmatrix} 1.4 & 0.5 \\ 0.2 & 0.9 \end{pmatrix}, \mathbf{t} = \begin{pmatrix} 23 \\ 6 \end{pmatrix}$$

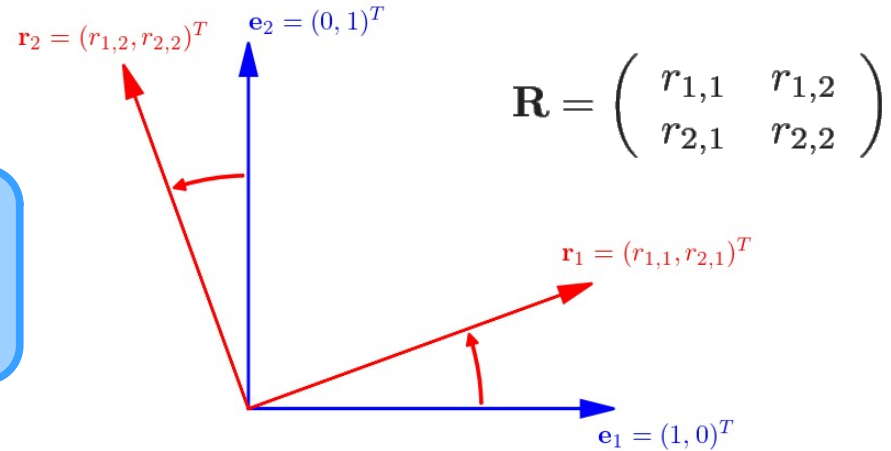
Rigid transformation

$$\mathbf{y}(\mathbf{x}, \mathbf{w}) = \mathbf{R}\mathbf{x} + \mathbf{t}, \quad \mathbf{R}^T\mathbf{R} = \mathbf{I} \text{ and } \det(\mathbf{R}) = 1$$



Rigid transformation

Task: why do $\mathbf{R}^T \mathbf{R} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
and $\det(\mathbf{R}) = 1$ impose a rotation?



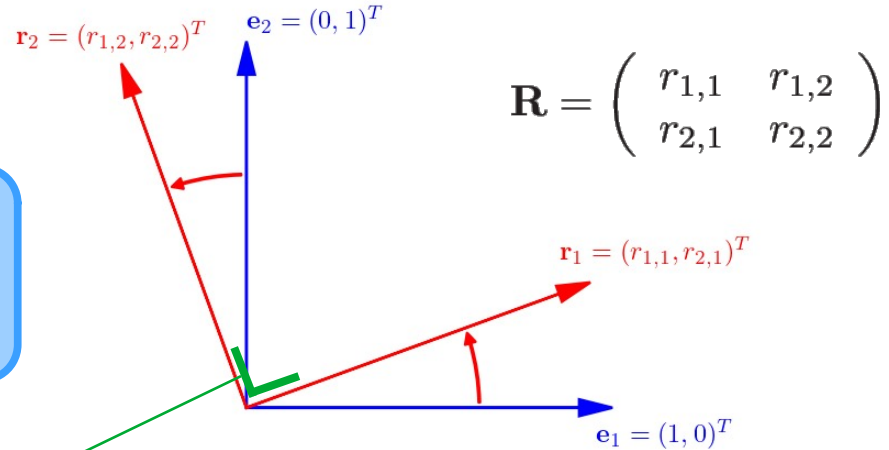
Rigid transformation

$$\|\mathbf{r}_1\| = \sqrt{\mathbf{r}_1^T \mathbf{r}_1} = 1$$

Task: why do $\mathbf{R}^T \mathbf{R} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
and $\det(\mathbf{R}) = 1$ impose a rotation?

$$\|\mathbf{r}_2\| = \sqrt{\mathbf{r}_2^T \mathbf{r}_2} = 1$$

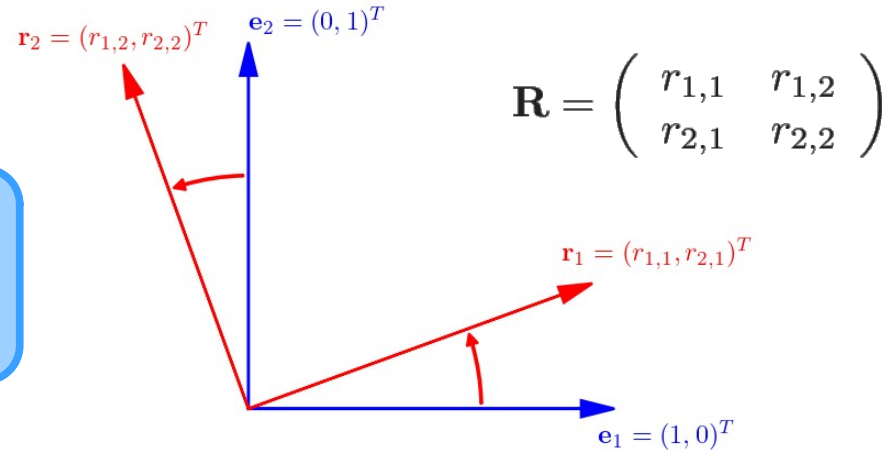
$$\mathbf{r}_1^T \mathbf{r}_2 = 0$$



Rigid transformation

Task: why do $\mathbf{R}^T \mathbf{R} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
and $\det(\mathbf{R}) = 1$ impose a rotation?

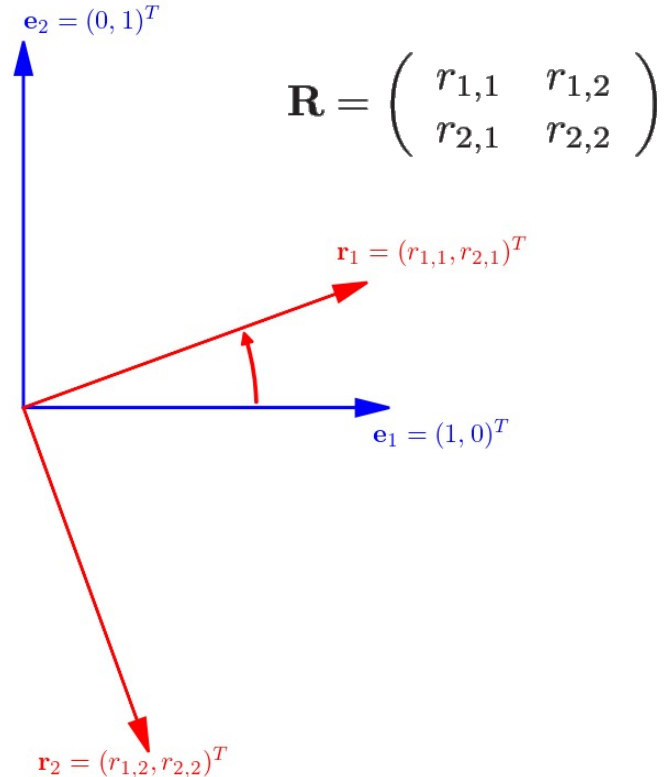
?



Rigid transformation

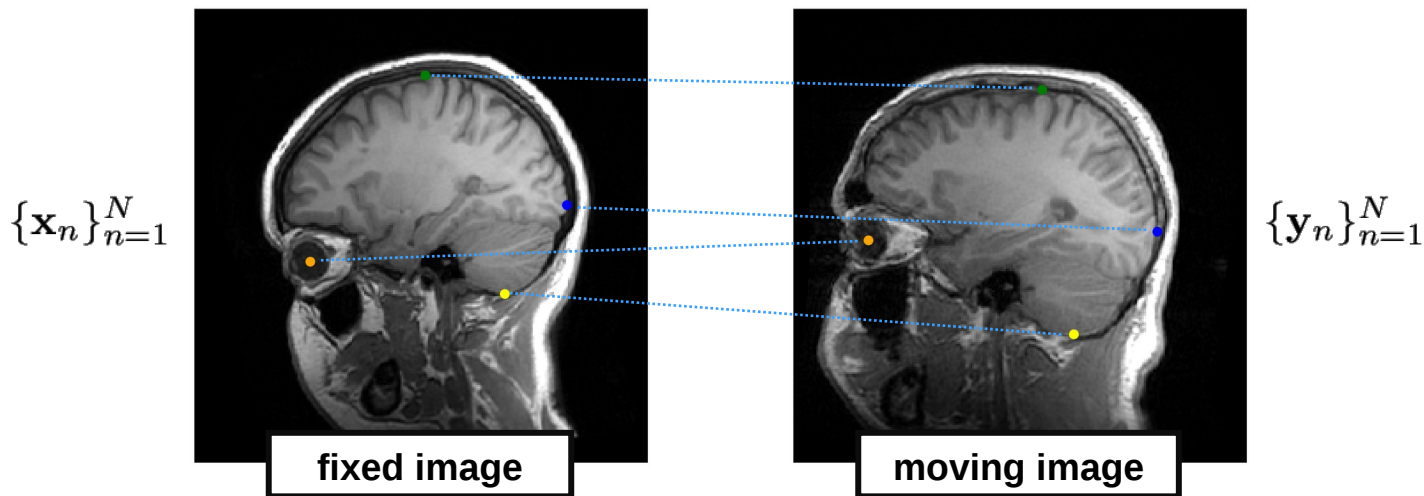
Task: why do $\mathbf{R}^T \mathbf{R} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
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?



Landmark-based registration

- ✓ Manually annotate N corresponding points in two images:



- ✓ Register the images by minimizing the distance between matching point pairs:

$$E(\mathbf{w}) = \sum_{n=1}^N \|\mathbf{y}_n - \mathbf{y}(\mathbf{x}_n, \mathbf{w})\|^2$$

Landmark-based registration

Applied to affine registration: $E(\mathbf{w}) = \sum_{n=1}^N \|\mathbf{y}_n - \mathbf{A}\mathbf{x}_n - \mathbf{t}\|^2$

Task 1: if $\mathbf{A} = \mathbf{I}$, what is \mathbf{t} ?

Hint: remember that $\|\mathbf{a} - \mathbf{b}\|^2 = \sum_{d=1}^D (a_d - b_d)^2$

Task 2: in the general case, what are \mathbf{A} and \mathbf{t} ?

Landmark-based registration

Applied to affine registration: $E(\mathbf{w}) = \sum_{n=1}^N \|\mathbf{y}_n - \mathbf{A}\mathbf{x}_n - \mathbf{t}\|^2 = \sum_{n=1}^N \sum_d^D (y_{n,d} - t_n - \sum_{d'=1}^D x_{n,d'} a_{d,d'})^2$


Task 1: if $\mathbf{A} = \mathbf{I}$, what is \mathbf{t} ?

Hint: remember that $\|\mathbf{a} - \mathbf{b}\|^2 = \sum_{d=1}^D (a_d - b_d)^2$

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Landmark-based registration

Applied to affine registration: $E(\mathbf{w}) = \sum_{n=1}^N \|\mathbf{y}_n - \mathbf{A}\mathbf{x}_n - \mathbf{t}\|^2 = \sum_{n=1}^N \sum_d^D (y_{n,d} - t_n - \sum_{d'=1}^D x_{n,d'} a_{d,d'})^2$



Task 1: if $\mathbf{A} = \mathbf{I}$, what is \mathbf{t} ?

Hint: remember that $\|\mathbf{a} - \mathbf{b}\|^2 = \sum_{d=1}^D (a_d - b_d)^2$

Task 2: in the general case, what are \mathbf{A} and \mathbf{t} ?

Landmark-based registration

Applied to affine registration: $E(\mathbf{w}) = \sum_{n=1}^N \|\mathbf{y}_n - \mathbf{A}\mathbf{x}_n - \mathbf{t}\|^2$

Task 1: if $\mathbf{A} = \mathbf{I}$, what is \mathbf{t} ?

Hint: remember that $\|\mathbf{a} - \mathbf{b}\|^2 = \sum_{d=1}^D (a_d - b_d)^2$

Task 2: in the general case, what are \mathbf{A} and \mathbf{t} ?

$$\begin{aligned} &= \sum_{n=1}^N \sum_d^D (y_{n,d} - t_n - \sum_{d'=1}^D x_{n,d'} a_{d,d'})^2 \\ &= \sum_d^D \sum_{n=1}^N (y_{n,d} - t_n - \sum_{d'=1}^D x_{n,d'} a_{d,d'})^2 \end{aligned}$$

Landmark-based registration

Applied to affine registration: $E(\mathbf{w}) = \sum_{n=1}^N \|\mathbf{y}_n - \mathbf{A}\mathbf{x}_n - \mathbf{t}\|^2$

Task 1: if $\mathbf{A} = \mathbf{I}$, what is \mathbf{t} ?

Hint: remember that $\|\mathbf{a} - \mathbf{b}\|^2 = \sum_{d=1}^D (a_d - b_d)^2$

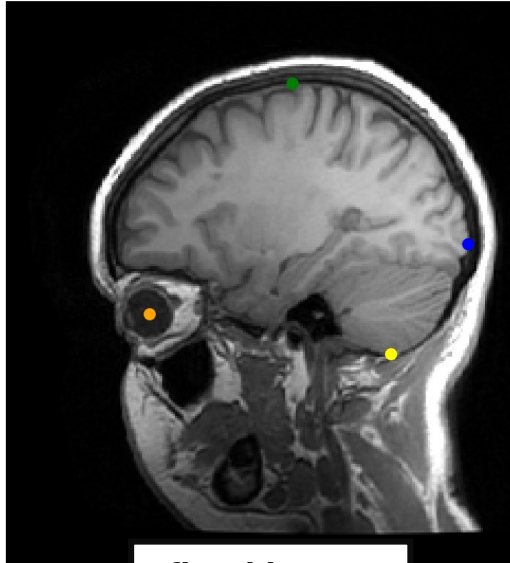
Task 2: in the general case, what are \mathbf{A} and \mathbf{t} ?

$$\begin{aligned} E(\mathbf{w}) &= \sum_{n=1}^N \sum_d^D (y_{n,d} - t_n - \sum_{d'=1}^D x_{n,d'} a_{d,d'})^2 \\ &= \sum_d^D \sum_{n=1}^N (y_{n,d} - t_n - \sum_{d'=1}^D x_{n,d'} a_{d,d'})^2 \end{aligned}$$

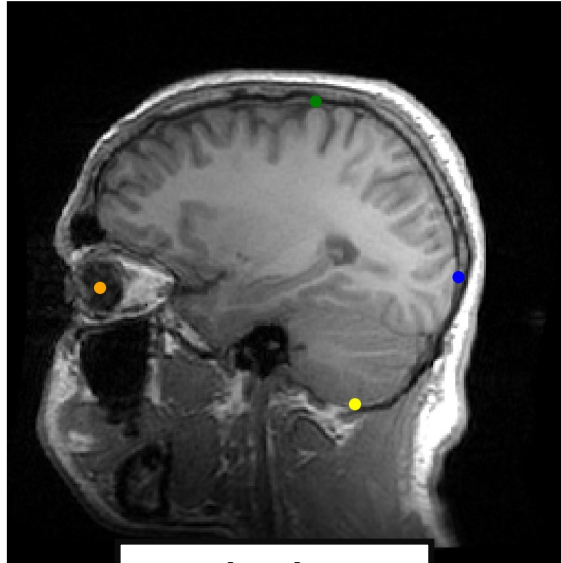
$$\begin{pmatrix} t_d \\ a_{d,1} \\ \vdots \\ a_{d,D} \end{pmatrix} = (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \begin{pmatrix} y_{1,d} \\ \vdots \\ y_{N,d} \end{pmatrix}$$

$$\text{where } \mathbf{X} = \begin{pmatrix} 1 & x_{1,1} & \cdots & x_{1,D} \\ 1 & x_{2,1} & \cdots & x_{2,D} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_{N,1} & \cdots & x_{N,D} \end{pmatrix}$$

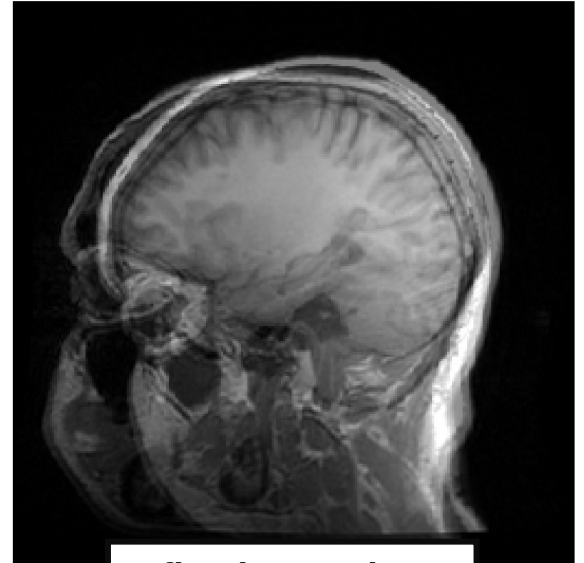
Landmark-based registration



fixed image



moving image



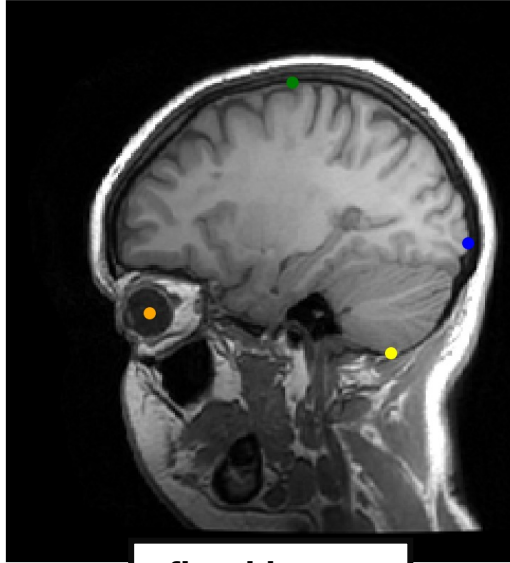
fixed + moving

Before registration

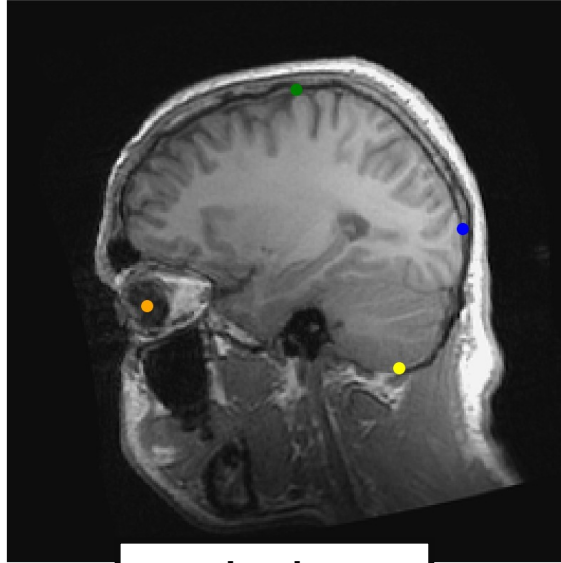


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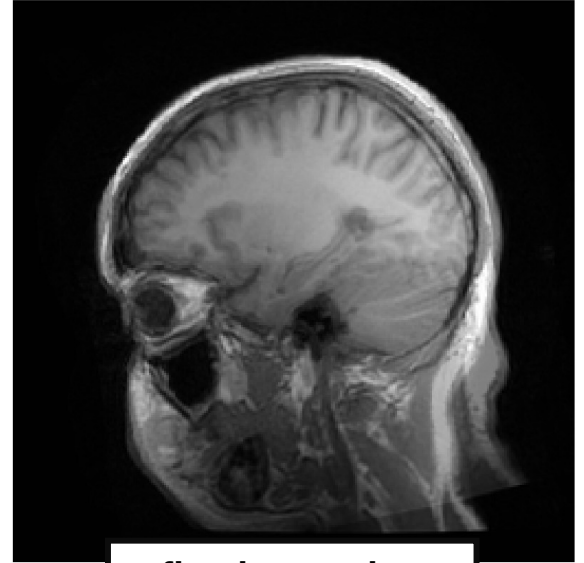
Landmark-based registration



fixed image



moving image



fixed + moving

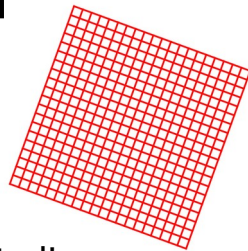
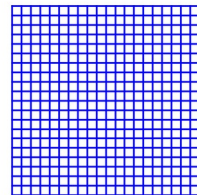
After registration



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Landmark-based registration

Applied to rigid registration: $E(\mathbf{w}) = \sum_{n=1}^N \|\mathbf{y}_n - \mathbf{R}\mathbf{x}_n - \mathbf{t}\|^2$



- ✓ Constraints $\mathbf{R}^T \mathbf{R} = \mathbf{I}$ and $\det(\mathbf{R}) = 1$ make the math much more complicated!
- ✓ Solution:

$$\mathbf{R} = \mathbf{V}\mathbf{U}^T, \quad \sum_{n=1}^N \tilde{\mathbf{x}}_n \tilde{\mathbf{y}}_n^T = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T, \quad \mathbf{U}^T \mathbf{U} = \mathbf{I}, \quad \mathbf{V}^T \mathbf{V} = \mathbf{I}$$

$$\mathbf{t} = \bar{\mathbf{y}} - \mathbf{R}\bar{\mathbf{x}},$$

$$\text{where } \tilde{\mathbf{x}}_n = \mathbf{x}_n - \bar{\mathbf{x}} \quad \text{and} \quad \tilde{\mathbf{y}}_n = \mathbf{y}_n - \bar{\mathbf{y}}$$

$$\text{with } \bar{\mathbf{x}} = \frac{1}{N} \sum_{n=1}^N \mathbf{x}_n \quad \text{and} \quad \bar{\mathbf{y}} = \frac{1}{N} \sum_{n=1}^N \mathbf{y}_n$$

(“flip” a column of \mathbf{R} if $\det(\mathbf{R}) = -1$)