Landmark-based Registration

Aalto-yliopisto
Aalto-universitetet
Aalto University

Medical Image Analysis

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Fall 2023

Examples of registration

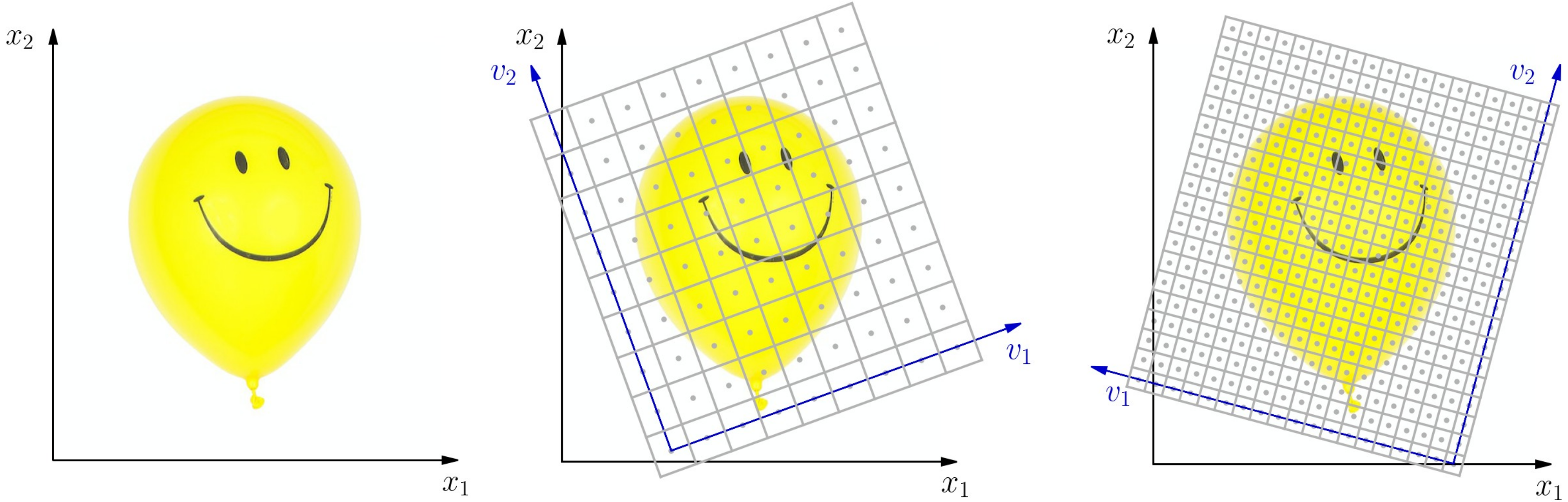
xxx

- ✔ xxx

Coordinate systems

For each image, there are *two* coordinate systems:

- ✔ Voxel coordinates $\mathbf{v} = (v_1, v_2, v_3)^{\mathrm{T}}$ *(integer indices)*
- ✔ World coordinates $\mathbf{x} = (x_1, x_2, x_3)^{\mathrm{T}}$ *(in mm)*



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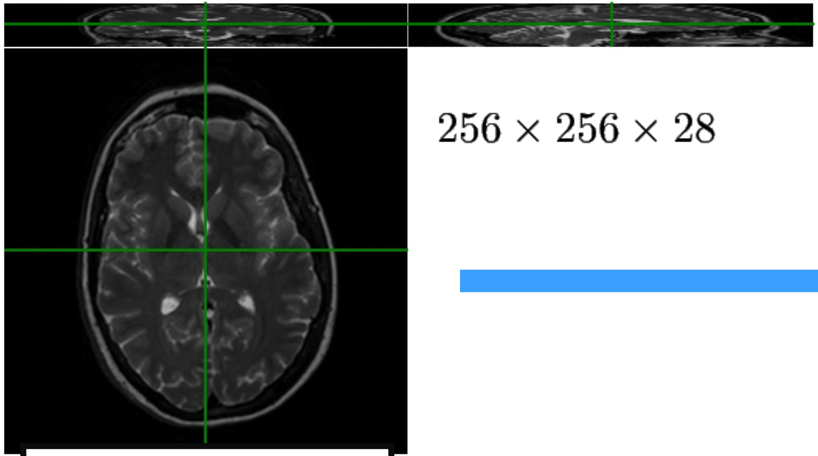
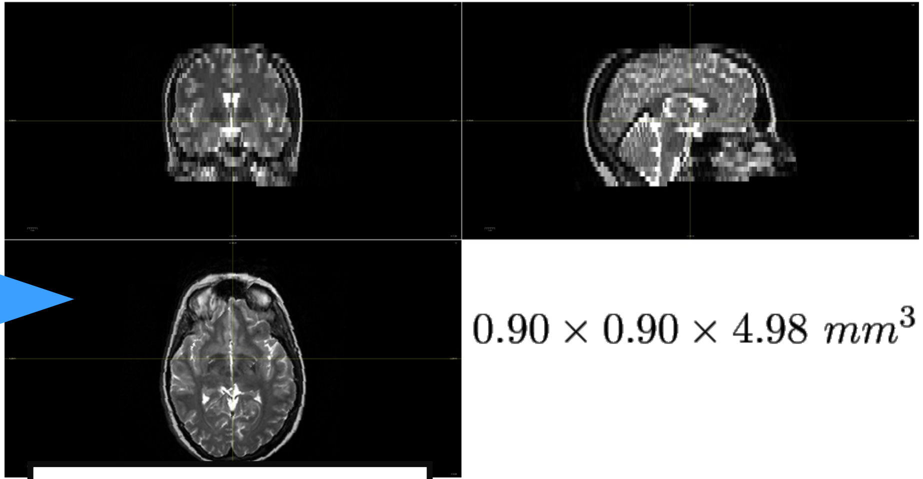
Conversion: $\mathbf{x} = \mathbf{A}\mathbf{v} + \mathbf{t}$

Coordinate systems



$256 \times 256 \times 28$

voxel indices $\mathbf{v}$



$0.90 \times 0.90 \times 4.98\ mm^3$

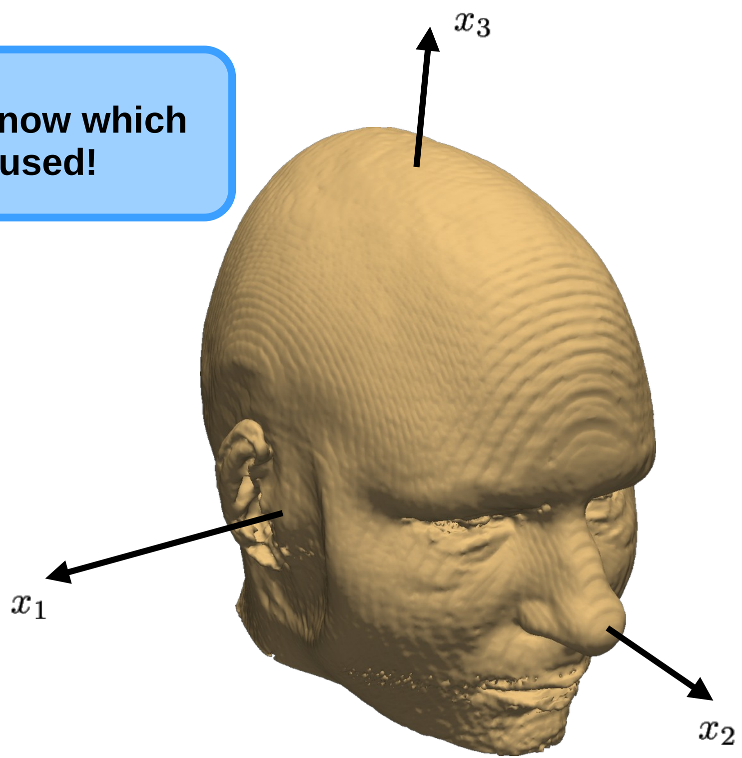
world positions $\mathbf{x}$

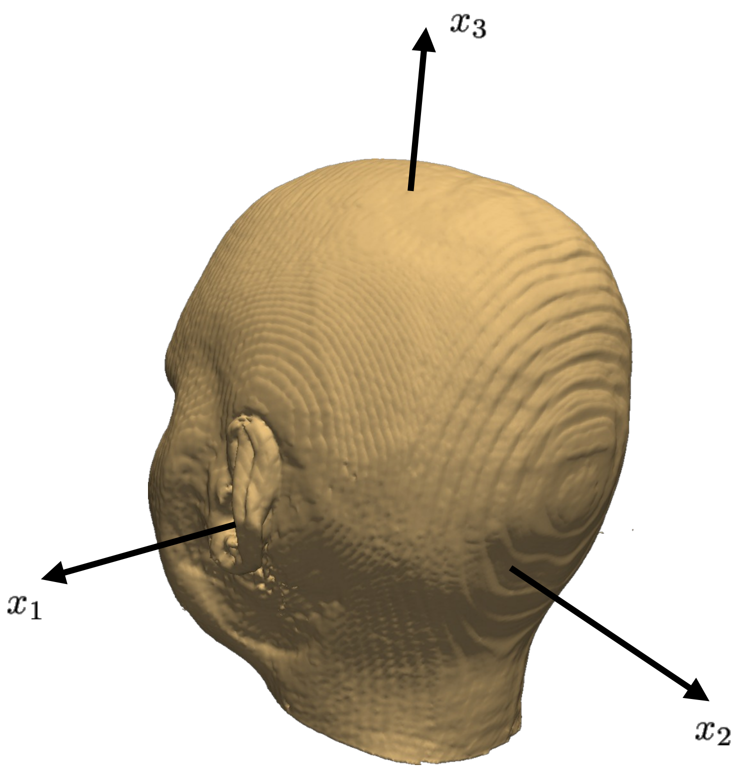
Conversion: $\mathbf{x} = \mathbf{A}\mathbf{v} + \mathbf{t}$

$$\mathbf{A} = \begin{pmatrix} -0.8923 & -0.0802 & -0.3732 \\ -0.0850 & 0.8921 & 0.3528 \\ -0.0612 & -0.0696 & 4.9512 \end{pmatrix} \qquad \mathbf{t} = \begin{pmatrix} 129.2834 \\ -98.7363 \\ -27.6911 \end{pmatrix}$$

World coordinates = convention

Important to know which convention is used!





<u>R</u>ight – <u>A</u>nterior – <u>S</u>uperior
(RAS)

<u>L</u>eft – <u>P</u>osterior – <u>S</u>uperior
(LPS)

Homogeneous coordinates

Vectors are augmented with a 1 at the end

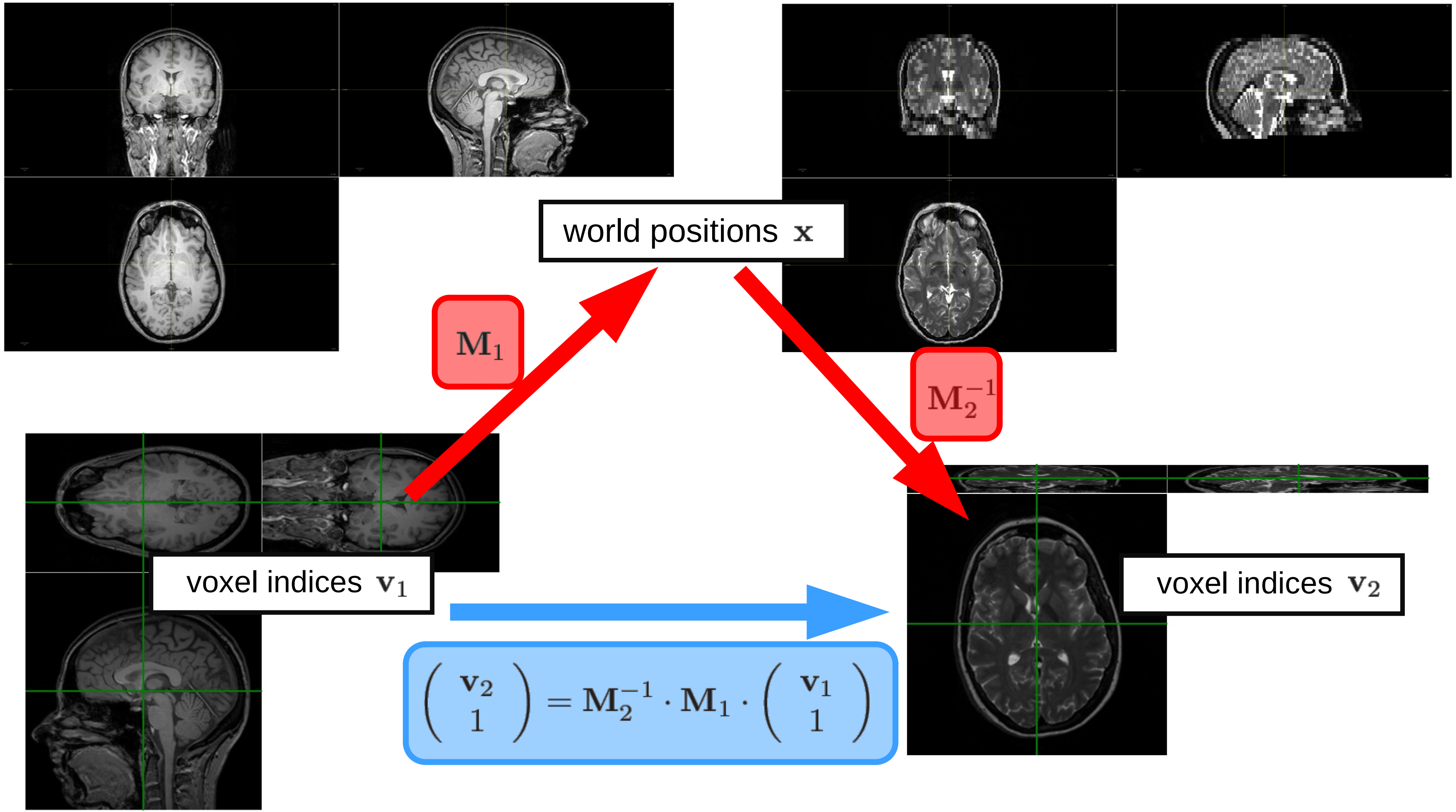
- ✔ **Idea:** Rewrite $\mathbf{x} = \mathbf{A}\mathbf{v} + \mathbf{t}$, i.e,

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} + \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix}$$

as:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ 1 \end{pmatrix} = \underbrace{\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & t_1 \\ a_{2,1} & a_{2,2} & a_{2,3} & t_2 \\ a_{3,1} & a_{3,2} & a_{3,3} & t_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{\mathbf{M}} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ 1 \end{pmatrix}$$

- ✔ **Benefit:** map voxel indices using only matrix multiplications

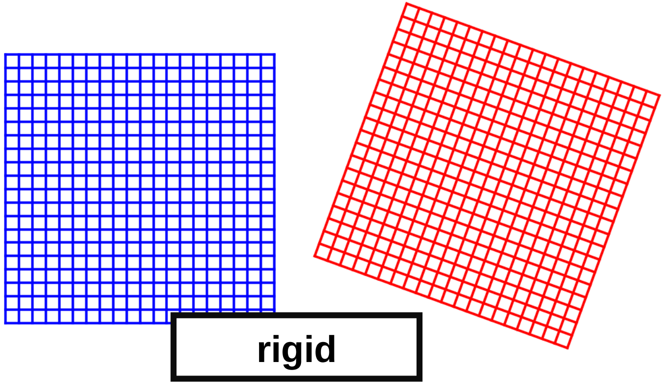

world positions $\mathbf{x}$
$\mathbf{M}_1$
$\mathbf{M}_2^{-1}$
voxel indices $\mathbf{v}_1$
voxel indices $\mathbf{v}_2$
$\begin{pmatrix} \mathbf{v}_2 \\ 1 \end{pmatrix} = \mathbf{M}_2^{-1} \cdot \mathbf{M}_1 \cdot \begin{pmatrix} \mathbf{v}_1 \\ 1 \end{pmatrix}$

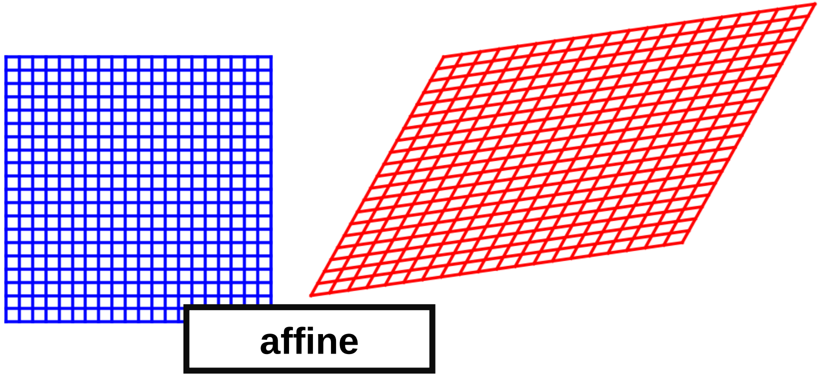
Spatial transformations

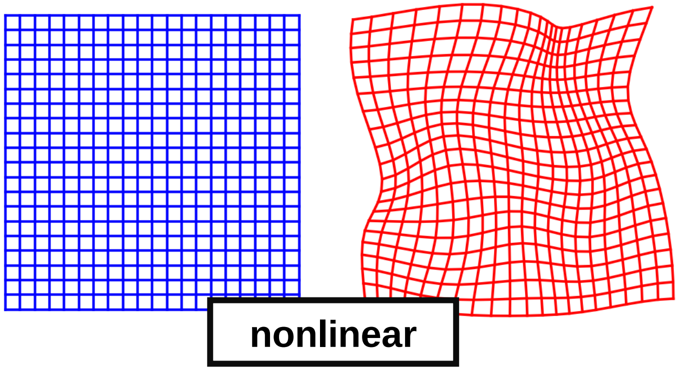
TODO: here stuff about x, y, and y(x, w)

- ✔ XXX

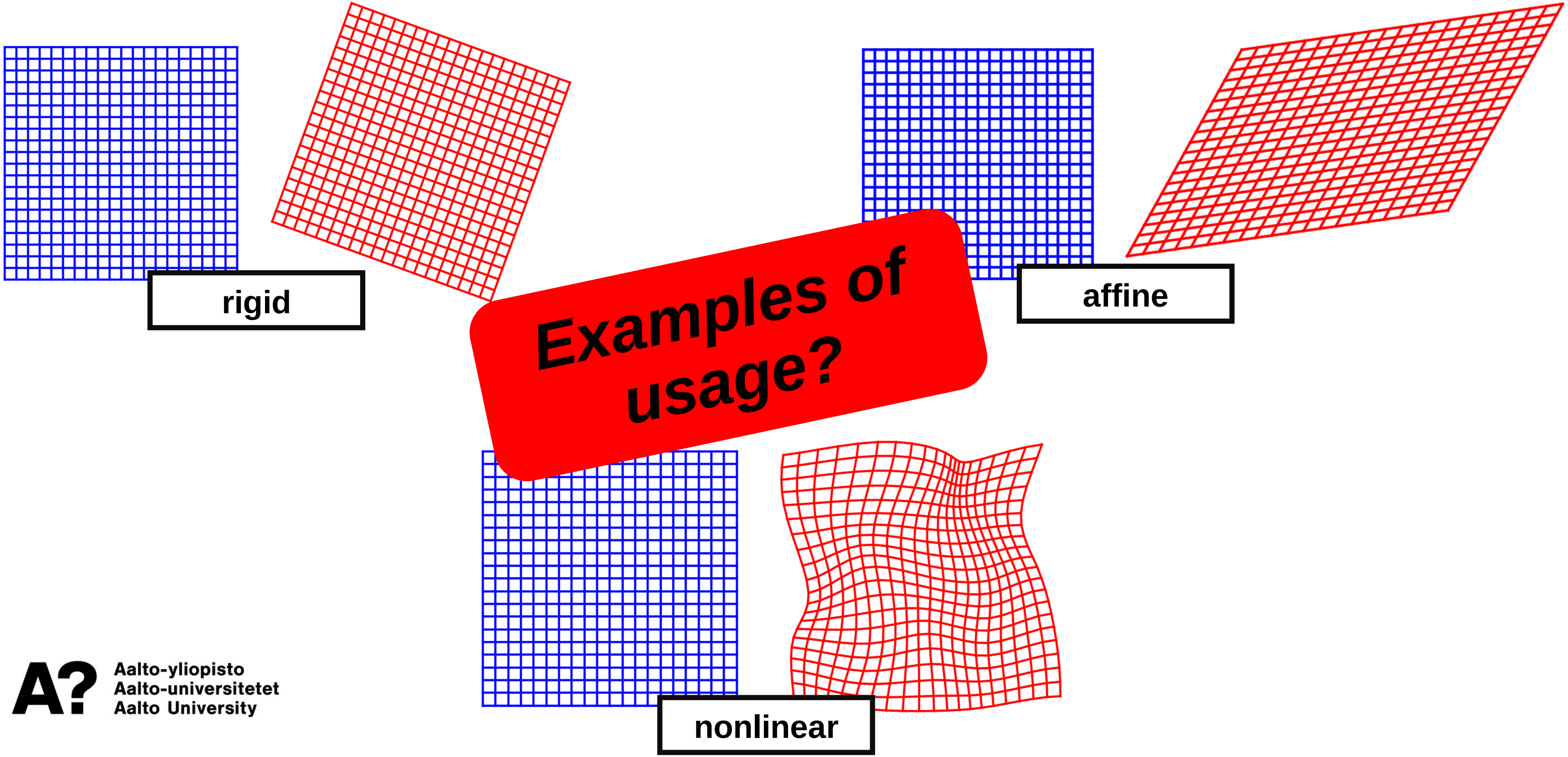
Spatial transformations





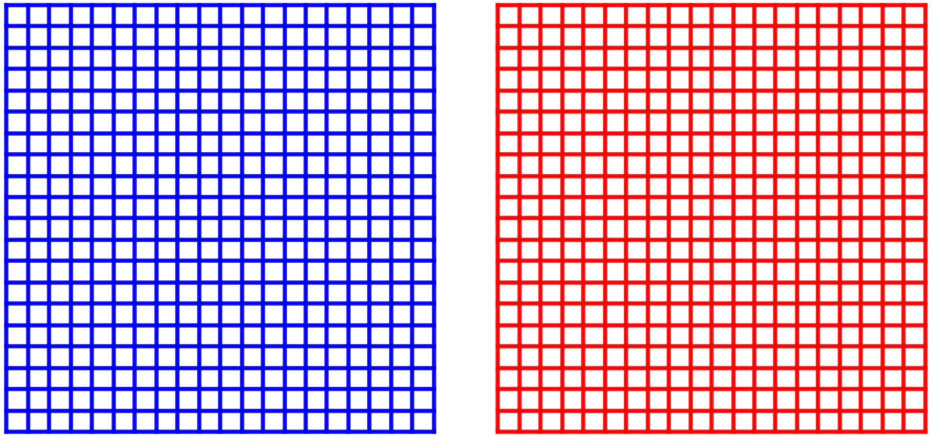


Spatial transformations



Affine transformation

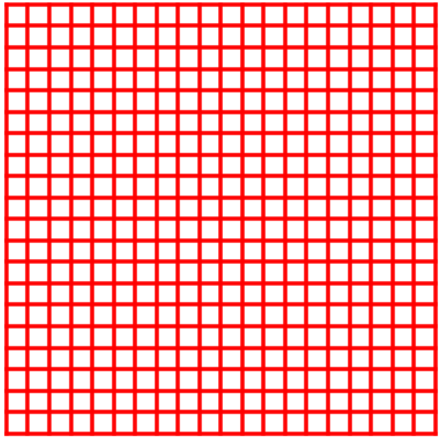
$$\mathbf{y}(\mathbf{x}, \mathbf{w}) = \mathbf{A}\mathbf{x} + \mathbf{t}$$

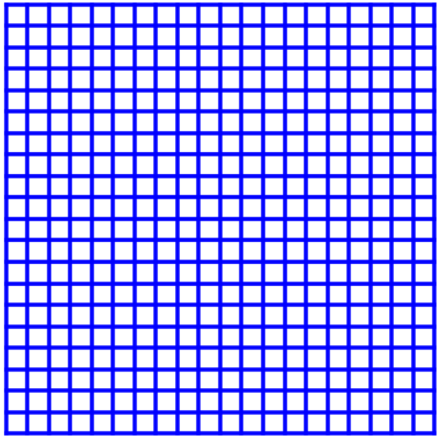


$$\mathbf{A} = \begin{pmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \end{pmatrix}, \quad \mathbf{t} = \begin{pmatrix} 23 \\ 0 \end{pmatrix}$$

Affine transformation

$$\mathbf{y}(\mathbf{x},\mathbf{w}) = \mathbf{A}\mathbf{x} + \mathbf{t}$$

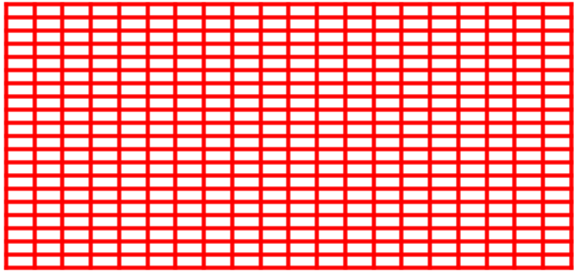


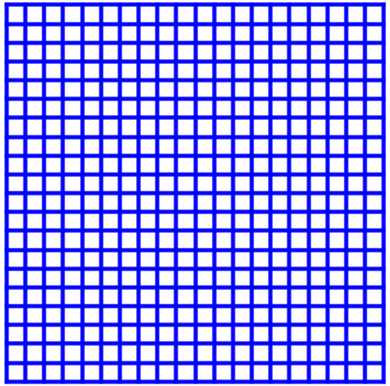


$$\mathbf{A} = \begin{pmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \end{pmatrix}, \quad \mathbf{t} = \begin{pmatrix} 23 \\ 16 \end{pmatrix}$$

Affine transformation

$$\mathbf{y}(\mathbf{x}, \mathbf{w}) = \mathbf{A}\mathbf{x} + \mathbf{t}$$

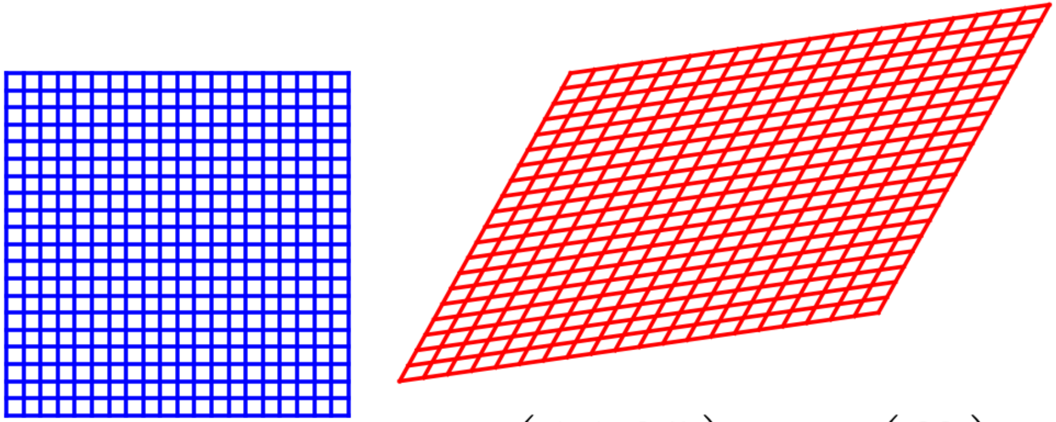




$$\mathbf{A} = \begin{pmatrix} 1.5 & 0.0 \\ 0.0 & 0.7 \end{pmatrix}, \quad \mathbf{t} = \begin{pmatrix} 23 \\ 16 \end{pmatrix}$$

Affine transformation

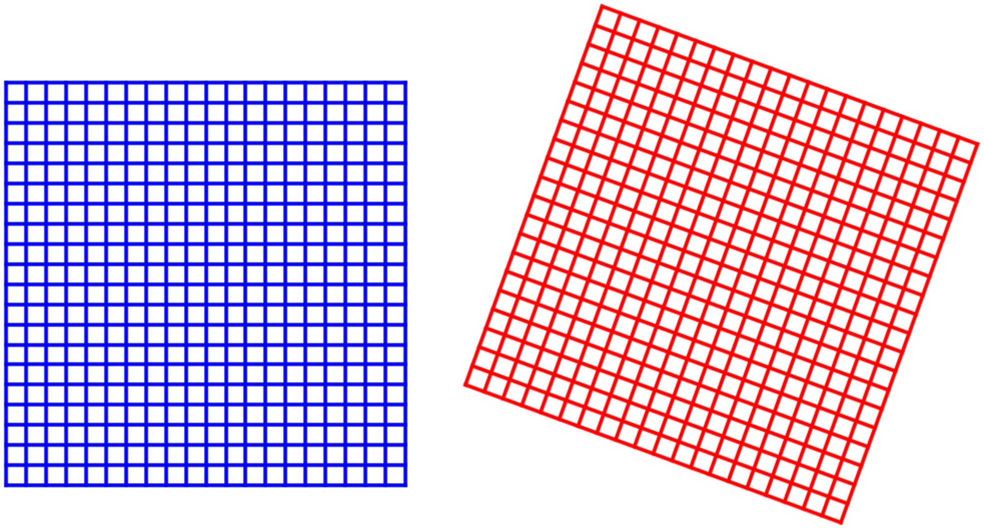
$$\mathbf{y}(\mathbf{x}, \mathbf{w}) = \mathbf{A}\mathbf{x} + \mathbf{t}$$



$$\mathbf{A} = \begin{pmatrix} 1.4 & 0.5 \\ 0.2 & 0.9 \end{pmatrix}, \quad \mathbf{t} = \begin{pmatrix} 23 \\ 2 \end{pmatrix}$$

Rigid transformation

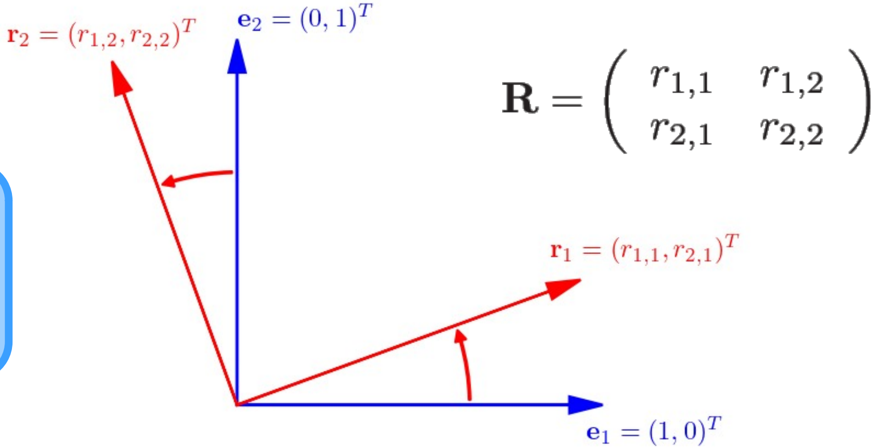
$$\mathbf{y}(\mathbf{x}, \mathbf{w}) = \mathbf{R}\mathbf{x} + \mathbf{t}, \qquad \mathbf{R}^{\mathrm{T}}\mathbf{R} = \mathbf{I} \text{ and } \det(\mathbf{R}) = 1$$



Rigid transformation

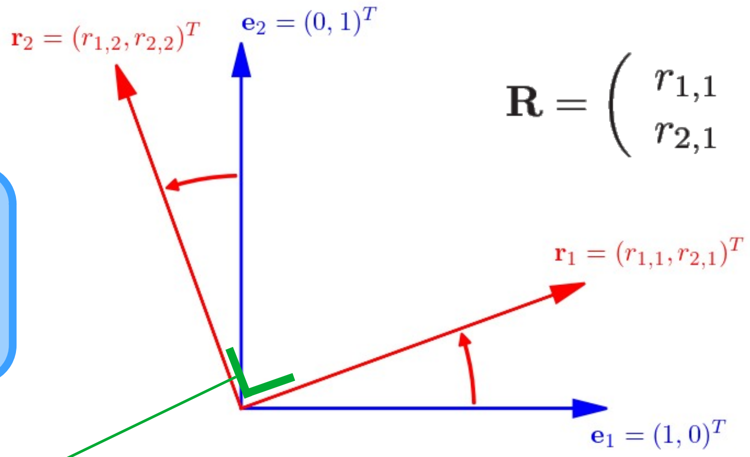
Task: why do $\mathbf{R}^T\mathbf{R} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

and $\det(\mathbf{R}) = 1$ impose a rotation?



Rigid transformation

$$\|\mathbf{r}_1\| = \sqrt{\mathbf{r}_1^T\mathbf{r}_1} = 1$$



$$\mathbf{R} = \begin{pmatrix} r_{1,1} & r_{1,2} \\ r_{2,1} & r_{2,2} \end{pmatrix}$$

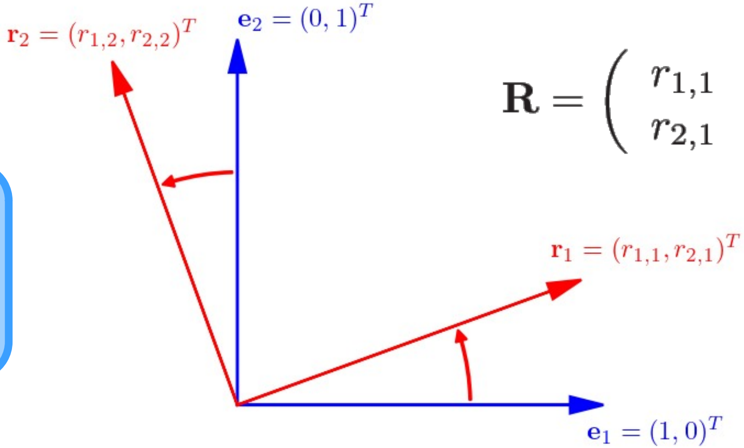
Task: why do $\mathbf{R}^T\mathbf{R} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

and $\det(\mathbf{R}) = 1$ impose a rotation?

$$\mathbf{r}_1^T\mathbf{r}_2 = 0$$

$$\|\mathbf{r}_2\| = \sqrt{\mathbf{r}_2^T\mathbf{r}_2} = 1$$

Rigid transformation



$$\mathbf{R} = \begin{pmatrix} r_{1,1} & r_{1,2} \\ r_{2,1} & r_{2,2} \end{pmatrix}$$

Task: why do $\mathbf{R}^T\mathbf{R} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\det(\mathbf{R}) = 1$ impose a rotation?

?

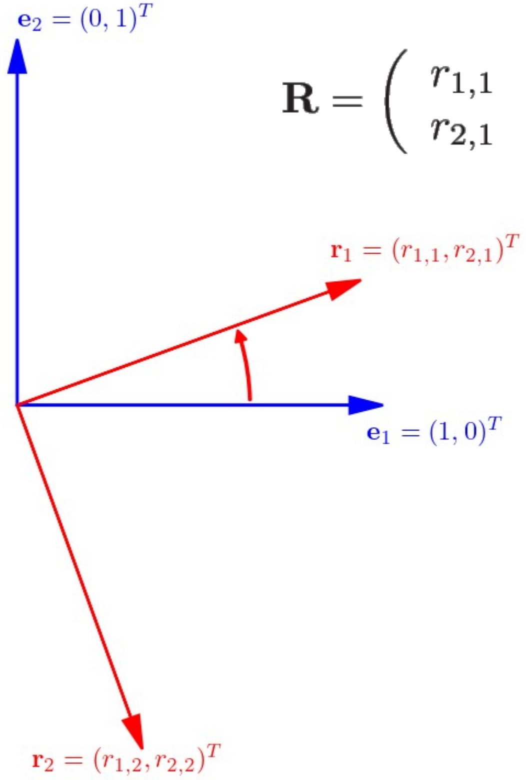
Rigid transformation

Task: why do $\mathbf{R}^T\mathbf{R} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

and $\det(\mathbf{R}) = 1$ impose a rotation?

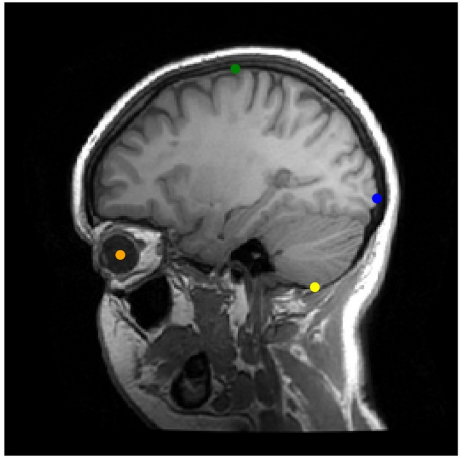
?

$$\mathbf{R} = \begin{pmatrix} r_{1,1} & r_{1,2} \\ r_{2,1} & r_{2,2} \end{pmatrix}$$

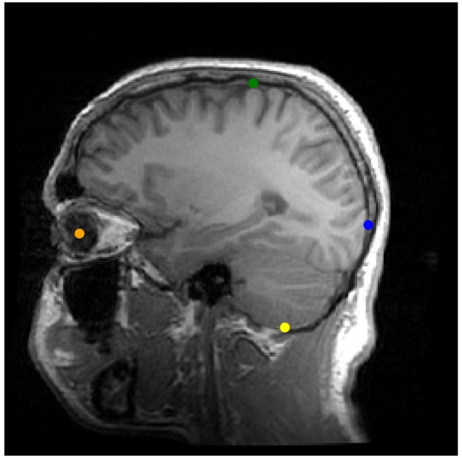


Landmark-based registration

- ✔ Manually annotate N corresponding points in two images:



$\{\mathbf{x}_n\}_{n=1}^{N}$



$\{\mathbf{y}_n\}_{n=1}^{N}$

- ✔ Register the images by minimizing the distance between matching point pairs:

$$E(\mathbf{w}) = \sum_{n=1}^{N} \|\mathbf{y}_n - \mathbf{y}(\mathbf{x}_n, \mathbf{w})\|^2$$

Landmark-based registration

Applied to affine registration: $E(\mathbf{w}) = \sum_{n=1}^{N} \|\mathbf{y}_n - \mathbf{A}\mathbf{x}_n - \mathbf{t}\|^2$

Task 1: if $\mathbf{A} = \mathbf{t}$, what is $\mathbf{t}$?

<u>Hint:</u> remember that $\|\mathbf{a} - \mathbf{b}\|^2 = \sum_{d=1}^{D} (a_d - b_d)^2$

Task 2: in the general case, what are $\mathbf{A}$ and $\mathbf{t}$?

Landmark-based registration

Applied to affine registration: $E(\mathbf{w}) = \sum_{n=1}^{N} \|\mathbf{y}_n - \mathbf{A}\mathbf{x}_n - \mathbf{t}\|^2$ $= \sum_{n=1}^{N} \sum_{d}^{D} (y_{n,d} - t_n - \sum_{d'=1}^{D} x_{n,d'} a_{d,d'})^2$

Task 1: if $\mathbf{A} = \mathbf{t}$, what is $\mathbf{t}$?

Hint: remember that $\|\mathbf{a} - \mathbf{b}\|^2 = \sum_{d=1}^{D} (a_d - b_d)^2$

Task 2: in the general case, what are $\mathbf{A}$ and $\mathbf{t}$?

Landmark-based registration

Applied to affine registration: $E(\mathbf{w}) = \sum_{n=1}^{N} \|\mathbf{y}_n - \mathbf{A}\mathbf{x}_n - \mathbf{t}\|^2 = \sum_{n=1}^{N}\sum_{d}^{D}(y_{n,d} - t_n - \sum_{d'=1}^{D} x_{n,d'}a_{d,d'})^2$

Task 1: if $\mathbf{A} = \mathbf{t}$, what is $\mathbf{t}$?

Hint: remember that $\|\mathbf{a} - \mathbf{b}\|^2 = \sum_{d=1}^{D}(a_d - b_d)^2$

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Landmark-based registration

Applied to affine registration: $E(\mathbf{w}) = \sum_{n=1}^{N} \|\mathbf{y}_n - \mathbf{A}\mathbf{x}_n - \mathbf{t}\|^2$

$$= \sum_{n=1}^{N} \sum_{d}^{D} (y_{n,d} - t_n - \sum_{d'=1}^{D} x_{n,d'} a_{d,d'})^2$$

$$= \sum_{d}^{D} \sum_{n=1}^{N} (y_{n,d} - t_n - \sum_{d'=1}^{D} x_{n,d'} a_{d,d'})^2$$

Task 1: if $\mathbf{A} = \mathbf{t}$, what is $\mathbf{t}$?

Hint: remember that $\|\mathbf{a} - \mathbf{b}\|^2 = \sum_{d=1}^{D} (a_d - b_d)^2$

Task 2: in the general case, what are $\mathbf{A}$ and $\mathbf{t}$?

Landmark-based registration

Applied to affine registration: $E(\mathbf{w}) = \sum_{n=1}^{N} \|\mathbf{y}_n - \mathbf{A}\mathbf{x}_n - \mathbf{t}\|^2$

$$= \sum_{n=1}^{N} \sum_{d}^{D} \left(y_{n,d} - t_n - \sum_{d'=1}^{D} x_{n,d'} a_{d,d'}\right)^2$$

$$= \sum_{d}^{D} \sum_{n=1}^{N} \left(y_{n,d} - t_n - \sum_{d'=1}^{D} x_{n,d'} a_{d,d'}\right)^2$$

$$\begin{pmatrix} t_d \\ a_{d,1} \\ \vdots \\ a_{d,D} \end{pmatrix} = \left(\mathbf{X}^{\mathrm{T}}\mathbf{X}\right)^{-1} \mathbf{X}^{\mathrm{T}} \begin{pmatrix} y_{1,d} \\ \vdots \\ y_{N,d} \end{pmatrix}$$

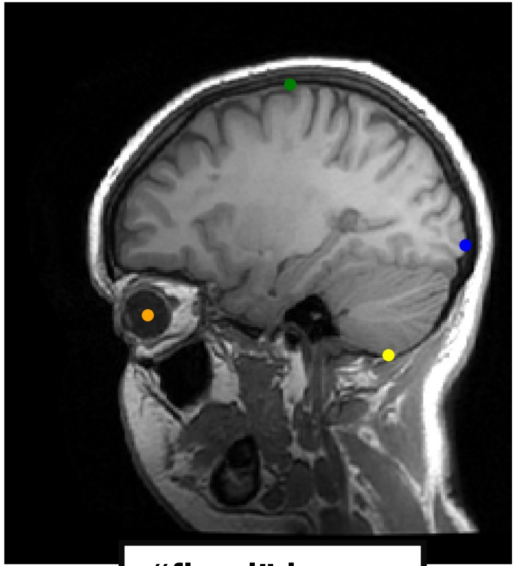
where $\mathbf{X} = \begin{pmatrix} 1 & x_{1,1} & \cdots & x_{1,D} \\ 1 & x_{2,1} & \cdots & x_{2,D} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_{N,1} & \cdots & x_{N,D} \end{pmatrix}$

Task 1: if $\mathbf{A} = \mathbf{t}$, what is $\mathbf{t}$?

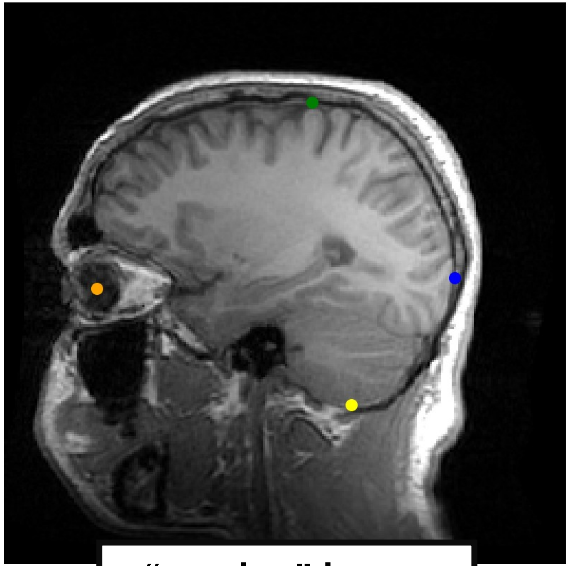
Hint: remember that $\|\mathbf{a} - \mathbf{b}\|^2 = \sum_{d=1}^{D} (a_d - b_d)^2$

Task 2: in the general case, what are $\mathbf{A}$ and $\mathbf{t}$?

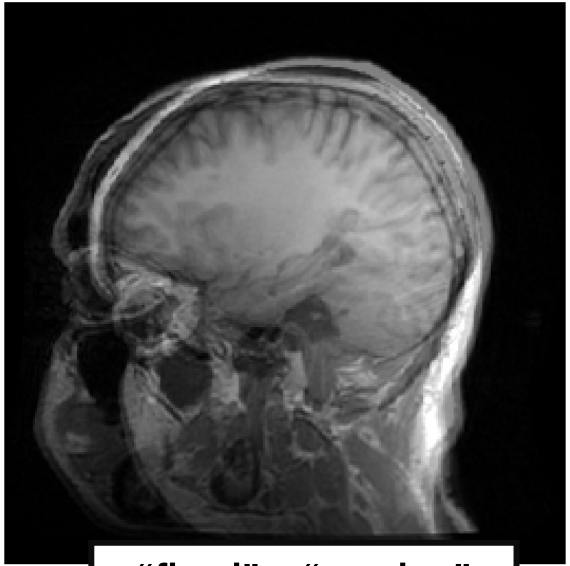
Landmark-based registration



“fixed” image



“moving” image



“fixed” + “moving”

Before registration

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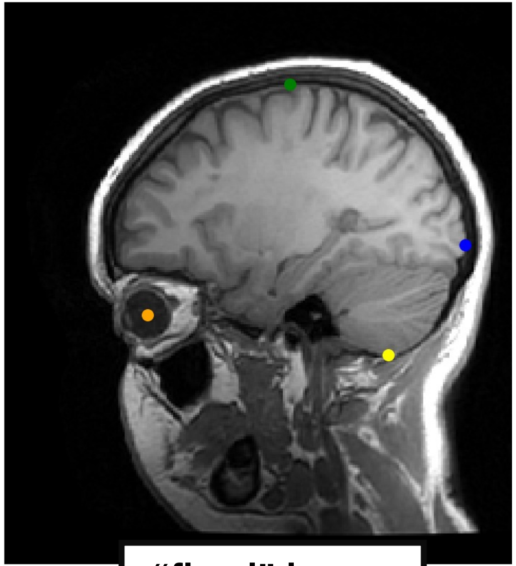
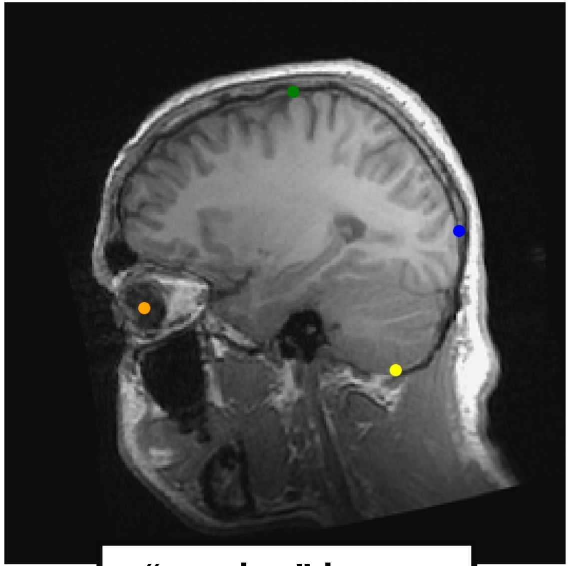
Aalto-universitetet
Aalto University

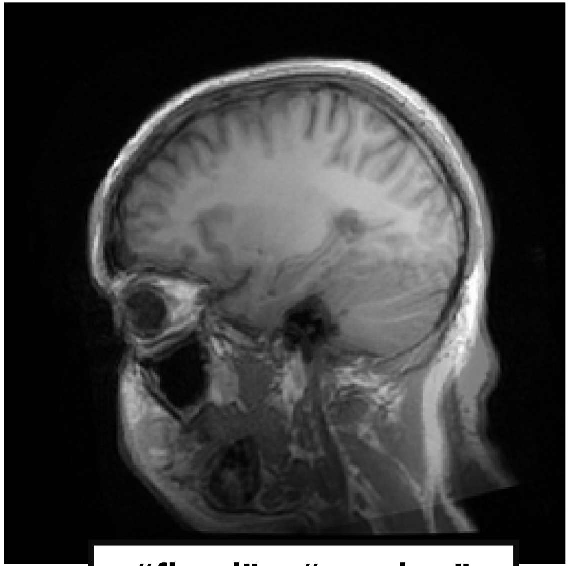
Landmark-based registration



“fixed” image



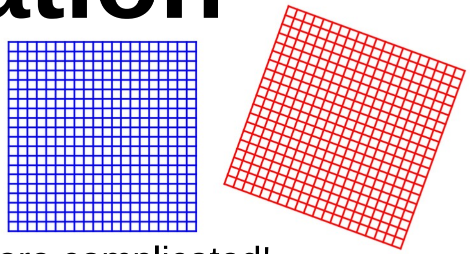
“moving” image



“fixed” + “moving”

After registration

Landmark-based registration



Applied to *rigid* registration: $E(\mathbf{w}) = \sum_{n=1}^{N} \|\mathbf{y}_n - \mathbf{R}\mathbf{x}_n - \mathbf{t}\|^2$

- ✔ Constraints $\mathbf{R}^{\mathrm{T}}\mathbf{R} = \mathbf{I}$ and $\det(\mathbf{R}) = 1$ make the math much more complicated!
- ✔ Solution:

$$\mathbf{R} = \mathbf{V}\mathbf{U}^{\mathrm{T}}, \quad \sum_{n=1}^{N} \tilde{\mathbf{x}}_n \tilde{\mathbf{y}}_n^{\mathrm{T}} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^{\mathrm{T}}, \quad \mathbf{U}^{\mathrm{T}}\mathbf{U} = \mathbf{I}, \quad \mathbf{V}^{\mathrm{T}}\mathbf{V} = \mathbf{I}$$

$$\mathbf{t} = \bar{\mathbf{y}} - \mathbf{R}\bar{\mathbf{x}},$$

where $\tilde{\mathbf{x}}_n = \mathbf{x}_n - \bar{\mathbf{x}}$ and $\tilde{\mathbf{y}}_n = \mathbf{y}_n - \bar{\mathbf{y}}$

with $\bar{\mathbf{x}} = \frac{1}{N}\sum_{n=1}^{N} \mathbf{x}_n$ and $\bar{\mathbf{y}} = \frac{1}{N}\sum_{n=1}^{N} \mathbf{y}_n$

(“flip” a column of $\mathbf{R}$ if $\det(\mathbf{R}) = -1$)