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OPTGAME 1.0: A NUMERICAL ALGORITHM TO DETERMINE SOLUTIONS FOR TWO-PERSON DIFFERENCE GAMES [1]

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Abstract: We present an algorithm to analytically solve two-person affine-quadratic difference games, and to numerically calculate nonlinear quadratic difference games using an appropriate local linearization procedure for a time-structure relevant for economic modeling. The numerical approximation and the derivation of different solutions of the underlying affine-quadratic dynamic games are explained; in particular, we determine open-loop and feedback Nash and Stackelberg equilibrium solutions and Pareto-optimal solutions. *Copyright © 2001 IFAC*

Keywords: Dynamic game theory, numerical algorithm, Nash equilibrium, Stackelberg equilibrium, Pareto-optimal solutions

1. INTRODUCTION

The importance of the application of intertemporal optimization techniques with more than one decision-maker in economics has grown during the last few years. Several studies have used dynamic game theory as a mathematical instrument to illustrate questions of strategic interdependencies between decision- and policy-makers, in particular in the analysis of oligopolistic markets and of international policy coordination (see, for example, de Zeeuw and van der Ploeg, 1991; Engwerda *et al.*, 1999; Fershtman and Kamien, 1987; Petit, 1990).

Different solution concepts are distinguished in the literature on dynamic games according to the information patterns and the strategy spaces of the players, which correspond to the degree of commitment assumed for the players (see, for example, Başar and Olsder, 1999; Leitmann, 1974; Mehlmann, 1991). In this paper we consider both non-coopera–tive and cooperative solution concepts for dynamic games. In the former case, we exclude binding agreements between the players, while these agreements are assumed to be established and to hold in the latter case. Intermediate possibilities and the question as to how cooperation can evolve in a non-cooperative context are ignored. Among non-cooperative solutions, we distinguish between Nash equilibrium solutions, where no player can improve his performance by one-sided de-

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viations from the equilibrium strategy, and Stackelberg equilibrium solutions, where the players have asymmetric roles: in the case of a two-person game, one player is the leader and has the ability or power to declare his strategy first and to impose it on the other player, the follower, who reacts rationally to the leader's strategy.

For both the Nash and the Stackelberg solution of a dynamic game, additional assumptions concerning the information pattern of the players can be made. Here, we consider open-loop information patterns, where the players' strategies depend only on the initial state of the dynamic system, and feedback information patterns, where the strategies depend on the current state of the system (but not on the initial conditions). In accordance with the interpretation of the distinction between open-loop and feedback Nash equilibrium solutions suggested by Reinganum and Stokey (1985), with the open-loop information pattern, each player can be imagined as committing himself at the start of the game a priori to all future actions he will take ("path strategies"). Under the feedback information pattern, players can be imagined as observing the current values of the state variable and reacting to them by choosing their actions according to a "decision rule", i.e. a strategy specified in the initial period which makes the values of the control variables dependent on the current values of the state variables at each (future) point in time.[2]

Here, we introduce and describe the OPTGAME 1.0 algorithm[3] for solving two-person discrete-time affine-quadratic games and for approximating the solutions of two-person nonlinear quadratic difference games, i.e. dynamic games where the objective function is assumed to be a polynomial of degree two in states and control variables and is to be optimized for a pre-specified period of time subject to a nonlinear autonomous discrete-time system. The core part of the algorithm is the "combination" of a local linearization procedure along a reference path (see Chow, 1975; Matulka and Neck, 1992) and elements taken from the theory of affine-quadratic games (see, for example, Başar and Olsder, 1999) – modified to meet the requirements of complex economic problems (involving both difference equations and identities).

The paper is structured as follows: Section 2 is devoted to the formulation of the (non-)linear-quadratic difference game to be considered, while Section 3 includes a detailed description of some elements of the OPTGAME 1.0 algorithm. These include lists of in- and output of the algorithm, illustration of the local linearization procedure of the nonlinear constraints along the reference path, and a brief description of the entire OPTGAME 1.0 algorithm by means of flow charts. Section 4 lists the algorithm's formal details with respect to the numerical approximation of the open-loop and feedback Nash and Stackelberg equilibrium solutions and the Pareto solutions. We conclude the paper in Section 5 with a short summary of the technical results, insights generated during the process of determining the equilibrium solutions, and some suggestions for further research.

2. THE DYNAMIC GAME PROBLEM

OPTGAME 1.0 approximates solutions for game-theoretic problems with a quadratic objective function and a nonlinear dynamic model in discrete time.[4] Thus, we consider the following intertemporal quadratic loss functions of both players ($i = 1,2$),

$$J_i = \sum_{t=1}^{T} L_{it} = \sum_{t=1}^{T} \tfrac{1}{2}\left(\mathbf{X}_t - \tilde{\mathbf{X}}_{it}\right)'\mathbf{Q}_{it}\left(\mathbf{X}_t - \tilde{\mathbf{X}}_{it}\right), \qquad (1)$$

which can be expanded to yield a polynomial of degree 2 in $\mathbf{X}_t$, i.e. for $i = 1,2$

$$L_{it} = \tfrac{1}{2}\mathbf{X}_t'\mathbf{Q}_{it}\mathbf{X}_t + \mathbf{X}_t'\mathbf{\Omega}_{it} + \omega_{it},$$

where

$$\mathbf{\Omega}_{it} := -\mathbf{Q}_{it}\tilde{\mathbf{X}}_{it} \quad \text{and} \quad \omega_{it} := \tfrac{1}{2}\tilde{\mathbf{X}}_{it}'\mathbf{Q}_{it}\tilde{\mathbf{X}}_{it}.$$

T denotes the terminal period of the finite planning horizon. $\mathbf{X}_t$ denotes an r-dimensional stacked state vector consisting of n_s state variables (summarizing the information available about the dynamical system), n_1 control variables determined by player 1, and n_2 control variables determined by player 2,

$$\mathbf{X}_t = \begin{pmatrix} \mathbf{x}_t \\ \mathbf{u}_{1t} \\ \mathbf{u}_{2t} \end{pmatrix}_{r\times 1}, \quad r = n_s + n_1 + n_2. \qquad (2)$$

$\tilde{\mathbf{X}}_{it}$ ($i = 1,2$) denotes the r-dimensional stacked "desired targets" vector which contains the "ideal levels" of the state and control variables for player i ($i = 1,2$),

$$\tilde{\mathbf{X}}_{it} = \begin{pmatrix} \tilde{\mathbf{x}}_{it} \\ \tilde{\mathbf{u}}_{i1t} \\ \tilde{\mathbf{u}}_{i2t} \end{pmatrix}_{r\times 1}, \quad r = n_s + n_1 + n_2. \qquad (3)$$

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[2] The importance of feedback Nash and feedback Stackelberg equilibrium solutions derives from the fact that both have the desirable property of subgame perfection (Markov perfection) or strong time consistency (see, for example, Dockner and Neck, 1988; Başar, 1989). Note that the open-loop Nash equilibrium is weakly time consistent but not subgame perfect, while the open-loop Stackelberg equilibrium solution is not even weakly time consistent and, hence, not subgame perfect either.

[3] The OPTGAME 1.0 algorithm allows the numerical calculation of the feedback and the open-loop Nash and Stackelberg equilibrium solutions and the cooperative Pareto-optimal solutions for two players.

[4] Note that from now on all equations, definitions, and abbreviations in this paper are valid for $t = 1,\ldots,T$.

Furthermore, the *symmetric* $r \times r$–dimensional penalty matrices $\mathbf{Q}_{it}$ ($i = 1,2$) are defined as

$$\mathbf{Q}_{it} = \begin{pmatrix} \mathbf{Q}_{it}^{\mathbf{x}} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{Q}_{it}^{\mathbf{u}_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{Q}_{it}^{\mathbf{u}_2} \end{pmatrix}_{r \times r}, \quad r = n_s + n_1 + n_2 . \quad (4)$$

(These matrices might, for example, be defined according to the rule $\mathbf{Q}_{it} = \vartheta^{t-1}\mathbf{Q}_{i0}$, $0 < \vartheta \leq 1$ (i=1,2), where $\mathbf{Q}_{i0}$ (i=1,2) is predetermined and the scalar ϑ in equation (1) is interpreted as a discount factor.) The positive semi-definite matrices $\mathbf{Q}_{it}^{\mathbf{x}}$, $\mathbf{Q}_{2t}^{\mathbf{u}_1}$, and $\mathbf{Q}_{1t}^{\mathbf{u}_2}$ and the positive definite matrices $\mathbf{Q}_{1t}^{\mathbf{u}_1}$ and $\mathbf{Q}_{2t}^{\mathbf{u}_2}$ describe the penalties for deviations of the state and control variables in time period t from their desired target values.[5]

Additionally, the matrices containing the penalties for deviations of the control variables from their desired values are assumed to have full rank. Furthermore, we have to guarantee the existence of the following set of inverted matrices for i,j=1,2, $i \neq j$, where $\mathbf{H}_{it}$ (i=1,2) and $\mathbf{h}_{it}$ (i=1,2) and $\mathbf{H}_t$ and $\mathbf{h}_t$ denote the respective Riccati matrices for the different solutions of the affine-quadratic game, $\mathbf{B}_{it}$ (i=1,2) are given by (17) below, and $\mathbf{E}_{2t}$ is defined below by equation (30) for i=2.

$$\mathbf{N}_{it} := \left(\mathbf{Q}_{it}^{\mathbf{u}_i} + \mathbf{B}_{it}{}'\mathbf{H}_{it}\mathbf{B}_{it}\right)^{-1} \quad (5)$$

$$\overline{\mathbf{N}}_{it} := \left(\mu\mathbf{Q}_{1t}^{\mathbf{u}_i} + (1-\mu)\mathbf{Q}_{2t}^{\mathbf{u}_i} + \mathbf{B}_{it}{}'\mathbf{H}_t\mathbf{B}_{it}\right)^{-1} \quad (6)$$

$$\mathbf{M}_{it} := \left(\mathbf{I} - \mathbf{N}_{it}\mathbf{B}_{it}{}'\mathbf{H}_{it}\mathbf{B}_{jt}\mathbf{N}_{jt}\mathbf{B}_{jt}{}'\mathbf{H}_{jt}\mathbf{B}_{it}\right)^{-1} \quad (7)$$

$$\overline{\mathbf{M}}_{it} := \left(\mathbf{I} - \overline{\mathbf{N}}_{it}\mathbf{B}_{it}{}'\mathbf{H}_t\mathbf{B}_{jt}\overline{\mathbf{N}}_{jt}\mathbf{B}_{jt}{}'\mathbf{H}_t\mathbf{B}_{it}\right)^{-1} \quad (8)$$

$$\overline{\overline{\mathbf{M}}}_t := \left(\mathbf{I} + \sum_{j=1}^{2}\mathbf{B}_{jt}\left(\mathbf{Q}_{jt}^{\mathbf{u}_j}\right)^{-1}\mathbf{B}_{jt}{}'\mathbf{H}_{jt}\right)^{-1} \quad (9)$$

$$\underline{\mathbf{M}}_{1t} := \left(\mathbf{Q}_{1t}^{\mathbf{u}_1} + \mathbf{B}_{1t}{}'\left(\mathbf{E}_{2t}{}'\mathbf{H}_{1t}\mathbf{E}_{2t} + \mathbf{H}_{2t}\mathbf{B}_{2t}\mathbf{N}_{2t}\mathbf{Q}_{1t}^{\mathbf{u}_2}\mathbf{N}_{2t}\mathbf{B}_{2t}{}'\mathbf{H}_{2t}\right)\mathbf{B}_{1t}\right)^{-1} \quad (10)$$

$$\underline{\underline{\mathbf{M}}}_t := \left(\mathbf{I} + \mathbf{H}_t\begin{pmatrix} \mathbf{B}_{1t}\left(\mathbf{Q}_{1t}^{\mathbf{u}_1}\right)^{-1}\mathbf{B}_{1t}{}' & \mathbf{B}_{2t}\left(\mathbf{Q}_{2t}^{\mathbf{u}_2}\right)^{-1}\mathbf{B}_{2t}{}' \\ \mathbf{B}_{2t}\left(\mathbf{Q}_{2t}^{\mathbf{u}_2}\right)^{-1}\mathbf{B}_{2t}{}' & \mathbf{0} \end{pmatrix}\right)^{-1} . \quad (11)$$

To calculate the cooperative (Pareto-optimal) solution, we define a joint objective functional by

$$J = \mu J_1 + (1-\mu)J_2 = \sum_{t=1}^{T}\{\mu L_{1t} + (1-\mu)L_{2t}\} . \quad (12)$$

[5] The existence of unique solutions of the Riccati matrices for all periods $t \in \{1,...,T\}$ and all solution concepts of affine quadratic games discussed in Section 4 can be readily verified according to Başar and Olsder (1999) by assuming that the penalty matrices for the states are nonnegative definite and that the penalty matrices for the controls are positive definite.

The parameters $\mu \in [0,1]$ and $(1-\mu) \in [0,1]$ reflect the associated player's "importance" or influence on the joint objective.

The dynamic system, which will usually be an economic model for the applications we have in mind, is assumed to be given by a system of nonlinear difference equations,

$$\mathbf{x}_t = \mathbf{f}\left(\mathbf{x}_{t-1}, \mathbf{x}_t, \mathbf{u}_{1t}, \mathbf{u}_{2t}, \mathbf{z}_t\right), \quad (13)$$

with the initial condition $\mathbf{x}_0$ given. The n_z–dimensional vector $\mathbf{z}_t$ contains exogenous variables not subject to control. $\mathbf{f}$ is a vector-valued function, where $\mathbf{f}^{\alpha}$ ($\alpha = 1,...,n_s$) denotes the αth component of $\mathbf{f}$.

The dynamic representation described by equation (13) indicates that at the beginning of time period t, the state vector $\mathbf{x}_t$ is still unknown, while $\mathbf{x}_{t-1}$ is known to the decision-makers when specifying the optimal control vectors $\mathbf{u}_{1t}$ and $\mathbf{u}_{2t}$. These control variables and the lagged state vector allow calculation of the state vector at the end of period t. Note that the assumption of a first-order system of difference equations in (13) is not really restrictive, as higher-order difference equations can be reduced to systems of first-order difference equations by suitably redefining variables as new state variables and augmenting the state vector. Also the assumption of a quadratic objective function, although of a special form, can be interpreted as a second-order Taylor-series approximation to a more general objective function. Thus, the class of problems to be treated by the OPTGAME 1.0 algorithm is rather broad.

3. ELEMENTS OF THE OPTGAME 1.0 ALGORITHM

3.1 Overview

As depicted in Figure 1, the OPTGAME 1.0 algorithm starts with two initialization modules and proceeds by computing a tentative state path (see module 3) from the nonlinear system equations with given tentative paths for the control variables and the lagged multivariable state vector (see module 2) using the Gauss-Seidel algorithm. Then the OPTGAME algorithm starts the nonlinearity loop, including (*A*) linearization of the system equations at the reference values, replacing the nonlinear autonomous system with a linear non-autonomous one (see module 4a and Section 3.2), (*B*) calculation of the Nash, Stackelberg, and Pareto solutions of the locally linearized deterministic game (with a finite planning horizon) under open-loop and feedback information patterns and calculation of the corresponding "optimal" state paths according to the nonlinear system equations (13) (see module 4b and Section 4), and (*C*) adaptation of the tentative state and control paths for the next iteration (see module 4c). Computation of the welfare losses determined by equations (1) and (12), evaluated along the five types of "optimal"

control and state paths derived from the iterative process associated with the nonlinearity loop (see module 5) and the output of the associated results in tables and in graphical form (see module 6) concludes the OPTGAME 1.0 algorithm.

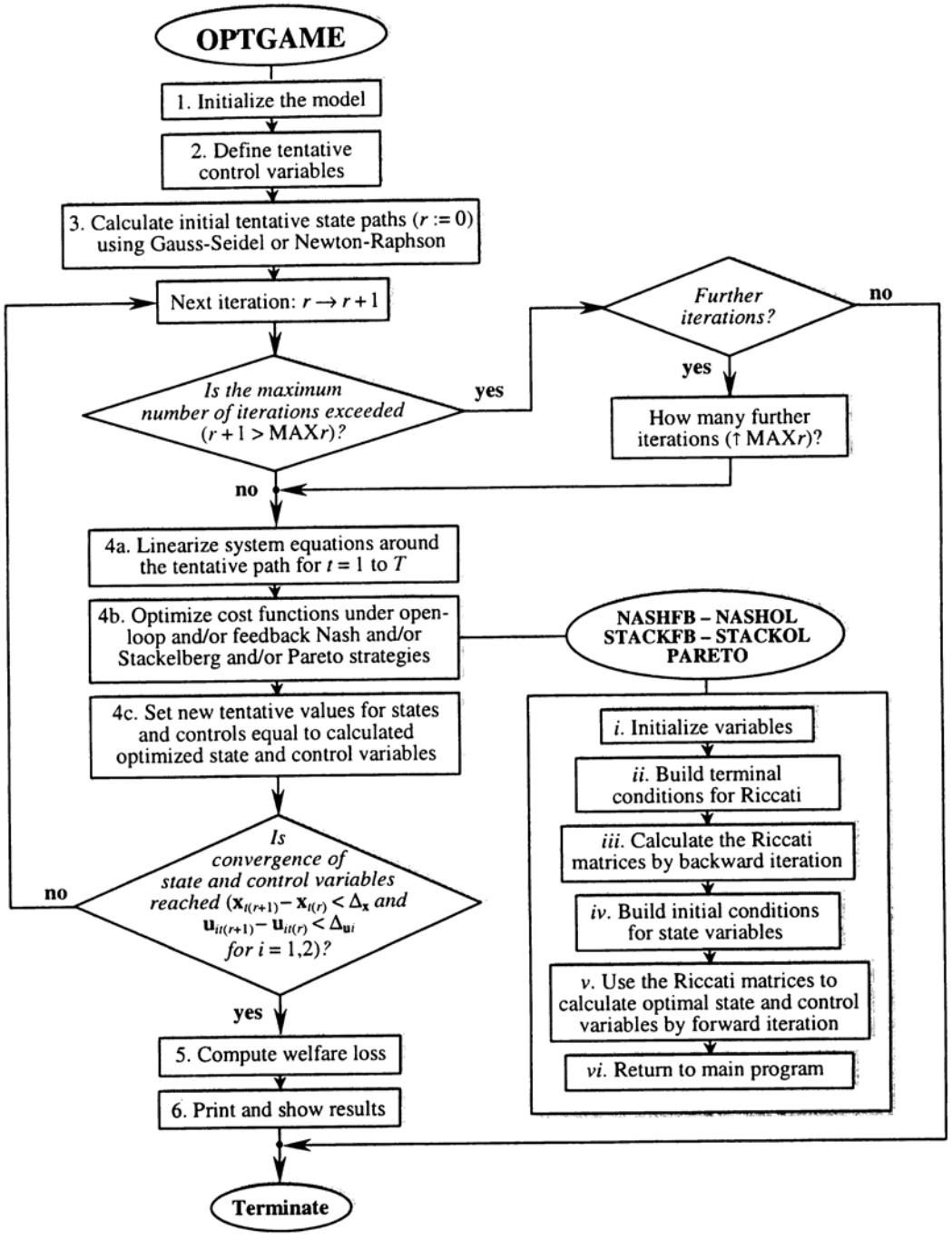

Fig. 1. Flow chart of OPTGAME 1.0, where r represents the number of iterations

The algorithm contains the following elements to be provided by the user and results delivered by the program:

<u>*Input of the algorithm*</u> for $t=1,...,T$ and $i,j=1,2$ $(i \neq j)$:

T	length of planning horizon
$\mathbf{f}$	system function
$\mathbf{x}_0$	initial value of the state vector
$\mathbf{z}_t$	path of exogenous variables
$\mathbf{Q}_{it}$	weighting matrices
$\tilde{\mathbf{x}}_{it}$	desired path for state variables
$\tilde{\mathbf{u}}_{ijt}$	desired path for control variables
($\bar{\mathbf{x}}_{t-1}$	historical values for state variables)
($\bar{\mathbf{u}}_{it}$	initial tentative control paths)
μ	weight for Pareto solution

<u>*Output of the algorithm*</u> for $t = 1,...,T$ and $i,j = 1,2$ (for each of the five solution concepts considered in Section 4):

$\mathbf{u}_{it}^*$	optimal paths of the control variables
$\mathbf{x}_t^*$	optimal paths of the state variables
J_i^* , J_i	quadratic loss functions

The term "OPTGAME" denotes both the numerical algorithm presented here and the corresponding computer program (available from the authors on request), where the software consists of a set of procedures implemented in the GAUSS[6] programming language and in C++. Among the benefits of the more recent C++ version of OPTGAME 1.0 are the increased run-time speed, extended capability for handling large data sets and matrices of variable sizes, and the support of Microsoft Windows and UNIX operating systems.

Although the above lists of inputs and outputs give a vague impression of the OPTGAME algorithm, they do not explain how the outputs are generated and how the underlying numerical algorithms are derived. All these items will be dealt with in the following sections.

3.2 Linearization of the System Equation

In most game-theoretical models, the system dynamics of the form (13) are a convenient starting point. For the calculation of the Nash, Stackelberg, and Pareto-optimal equilibrium strategies, however, another state-space representation which does not include $\mathbf{x}_t$ at the right-hand side of the implicit function (13) is more adequate. Following Chow (1975) and Matulka and Neck (1992), it is easy to eliminate $\mathbf{x}_t$ in the course of linearizing system (13).

- With the given exogenous non-controlled variables, $\mathbf{z}_t$, assumed tentative control paths, $\bar{\mathbf{u}}_{it}$ (i=1,2; which are determined either by historical values or by previous optimization runs), and the lagged tentative state variables, $\bar{\mathbf{x}}_{t-1}$, starting with $\mathbf{x}_0$, the autonomous nonlinear system (13) can be solved for all t using the numerical Gauss-Seidel approximation. Thus, a reference path is derived for the state variables. That is, for known values of $\bar{\mathbf{x}}_{t-1}$, $\bar{\mathbf{u}}_{1t}$, $\bar{\mathbf{u}}_{2t}$ and $\mathbf{z}_t$, we can compute a value $\bar{\mathbf{x}}_t$ such that

$$\bar{\mathbf{x}}_t - \mathbf{f}(\bar{\mathbf{x}}_{t-1}, \bar{\mathbf{x}}_t, \bar{\mathbf{u}}_{1t}, \bar{\mathbf{u}}_{2t}, \mathbf{z}_t) = \mathbf{0} \tag{14}$$

  by straightforward application of the Gauss-Seidel technique – a well-known nonlinear equation-solving method.

  In the nonlinearity loop (see Figure 1) the tentative paths are varied by replacing the old reference paths by the "optimized" (=equilibrium) paths of the various optimization runs until convergence is reached, i.e. until the state and the control variables do not differ by more than a small amount between two successive linearizations.

- Then, we numerically linearize the vector-valued system function, $\mathbf{f}$, around the reference paths $\bar{\mathbf{x}}_{t-1}$, $\bar{\mathbf{x}}_t$, $\bar{\mathbf{u}}_{1t}$, $\bar{\mathbf{u}}_{2t}$, and the given reference path $\mathbf{z}_t$ according to a first-order Taylor series approximation and obtain the following approximate vector-valued non-autonomous linear system equation:

$$\mathbf{x}_t = \mathbf{A}_t \mathbf{x}_{t-1} + \mathbf{B}_{1t}\mathbf{u}_{1t} + \mathbf{B}_{2t}\mathbf{u}_{2t} + \mathbf{s}_t , \tag{15}$$

with

$$\mathbf{A}_t := (\mathbf{I} - \mathbf{F}_{\mathbf{x}_t})^{-1}\mathbf{F}_{\mathbf{x}_{t-1}} , \tag{16}$$

$$\mathbf{B}_{it} := (\mathbf{I} - \mathbf{F}_{\mathbf{x}_t})^{-1}\mathbf{F}_{\mathbf{u}_{it}} , \tag{17}$$

$$\mathbf{s}_t := \bar{\mathbf{x}}_t - \mathbf{A}_t\bar{\mathbf{x}}_{t-1} - \mathbf{B}_{1t}\bar{\mathbf{u}}_{1t} - \mathbf{B}_{2t}\bar{\mathbf{u}}_{2t} , \tag{18}$$

  for i=1,2, where $\mathbf{I}$ denotes the $n_s \times n_s$ identity matrix. The $n_s \times n_s$–dimensional matrix $\mathbf{A}_t$, the $n_s \times n_1$–dimensional matrix $\mathbf{B}_{1t}$, the $n_s \times n_2$–dimensional matrix $\mathbf{B}_{2t}$, and the n_s–dimensional vector $\mathbf{s}_t$ are time-dependent functions of the reference paths along which they are evaluated. Here and in the following section we require that the first and second derivatives of the system function $\mathbf{f}$ with respect to $\bar{\mathbf{x}}_{t-1}$, $\bar{\mathbf{x}}_t$, $\bar{\mathbf{u}}_{1t}$, $\bar{\mathbf{u}}_{2t}$ exist and that they are continuous using the following abbreviations: $\mathbf{F}_{\mathbf{x}_{t-1}}$ and $\mathbf{F}_{\mathbf{x}_t}$ denote $n_s \times n_s$–matrices whose elements are defined by

$$(\mathbf{F}_{\mathbf{x}_{t-1}})_{ij} := \frac{\partial \mathbf{f}^i}{\partial \mathbf{x}_{t-1,j}}, \quad i,j = 1,\ldots,n_s , \tag{19}$$

$$(\mathbf{F}_{\mathbf{x}_t})_{ij} := \frac{\partial \mathbf{f}^i}{\partial \mathbf{x}_{t,j}}, \quad i,j = 1,\ldots,n_s , \tag{20}$$

  and $\mathbf{F}_{\mathbf{u}_{kt}}$ denotes an $n_s \times n_k$ – matrix for $k = 1,2$ whose elements are defined by

$$(\mathbf{F}_{\mathbf{u}_{kt}})_{ij} := \frac{\partial \mathbf{f}^i}{\partial \mathbf{u}_{kt,j}}, \quad \begin{matrix} i = 1,\ldots,n_s, \\ j = 1,\ldots,n_k. \end{matrix} \tag{21}$$

  Here $\mathbf{x}_{t-1,j}$ denotes the j-th element of the lagged state vector $\mathbf{x}_{t-1}$, etc. It has to be assumed that the matrix $\mathbf{I} - \mathbf{F}_{\mathbf{x}_t}$ is non-singular.

4. COMPUTATION OF THE GAME SOLUTIONS

As is known from Başar and Olsder (1999), for example, using the appropriate optimization technique corresponding to the assumed information structure of the game yields different Riccati equations which can be solved by backward integration, where the terminal conditions for the Riccati matrices, $\mathbf{H}_{1T}$, $\mathbf{H}_{2T}$, $\mathbf{h}_{1T}$, $\mathbf{h}_{2T}$, and $\mathbf{H}_T$ and $\mathbf{h}_T$, are determined in a way

[6] GAUSS is a high-level matrix programming language specializing in commands, functions, and procedures for data analysis and statistical applications which is of special interest for the application of OPTGAME 1.0 in various fields of economics, e.g. macroeconomic policy problems.

similar to the one used, for example, by Kendrick (1981) in control theory.

For the feedback Nash and Stackelberg equilibrium solutions and the Pareto-optimal solutions, we need to define the so-called feedback matrices, $\mathbf{G}_{it}$ $(i = 1,2)$ and $\mathbf{g}_{it}$ $(i = 1,2)$. By substituting these matrices into linear functions of the lagged state vector $\mathbf{x}_{t-1}$, it is possible to determine the values of the optimal control variables for each player in feedback form, $\mathbf{u}_{it}^*$ $(i = 1,2)$,

$$\mathbf{u}_{it}^* = \mathbf{G}_{it}\mathbf{x}_{t-1}^* + \mathbf{g}_{it}\,, \tag{22}$$

for $\mathbf{x}_0^* = \mathbf{x}_0$. The controls are differently derived in the open-loop cases as stated below. For all but the open-loop Stackelberg equilibrium solution, we can calculate the optimal state values, $\mathbf{x}_t^*$, by forward iteration, i.e.

$$\mathbf{x}_t^* = \mathbf{K}_t\,\mathbf{x}_{t-1}^* + \mathbf{k}_t\,, \tag{23}$$

using the initial values of the states, $\mathbf{x}_0^* = \mathbf{x}_0$. The matrices $\mathbf{K}_t$ and $\mathbf{k}_t$ are given separately for each solution concept below (with the exception of the open-loop Stackelberg equilibrium solution).

After having derived (and stored) the optimal control values for the feedback Nash and Stackelberg and the Pareto-optimal games as determined by equation (22), and for the open-loop Nash and Stackelberg game as reported below, we calculate the "optimal" state vectors according to the nonlinear system equations (13). (Note that these "optimal" state and control path will serve as tentative paths for the next iteration, (hopefully) converging towards the solution as returned by the OPTGAME 1.0 program.)

4.1 The Feedback Nash Equilibrium Solution

The solution of the feedback Nash game with two players is generated by applying the dynamic programming technique following Kydland (1975), de Zeeuw and van der Ploeg (1991), Hatheway (1992), Hager (1994), and Başar and Olsder (1999): *Starting with the terminal period $t = T$, stepping recursively back in time towards the initial period, i.e. for $t = T,\ldots,2$, the Riccati equations*

$$\mathbf{H}_{i(t-1)} = \mathbf{K}_t'\mathbf{H}_{it}\mathbf{K}_t + \mathbf{Q}_{i(t-1)}^{\mathbf{x}} + \sum_{j=1}^{2}\mathbf{G}_{jt}'\mathbf{Q}_{it}^{\mathbf{u}_j}\mathbf{G}_{jt}\,, \tag{24}$$

$$\mathbf{H}_{iT} = \mathbf{Q}_{iT}^{\mathbf{x}}\,,$$

$$\mathbf{h}_{i(t-1)} = \mathbf{K}_t'(\mathbf{h}_{it} - \mathbf{H}_{it}\mathbf{k}_t) + \mathbf{Q}_{i(t-1)}^{\mathbf{x}}\tilde{\mathbf{x}}_{i(t-1)} + \\ + \sum_{j=1}^{N}\mathbf{G}_{jt}\mathbf{Q}_{it}^{\mathbf{u}_j}(\tilde{\mathbf{u}}_{ijt} - \mathbf{g}_{jt})\,, \tag{25}$$

$$\mathbf{h}_{iT} = \mathbf{Q}_{iT}^{\mathbf{x}}\tilde{\mathbf{x}}_{iT}$$

can be solved, where

$$\mathbf{K}_t := \mathbf{A}_t + \mathbf{B}_{1t}\mathbf{G}_{1t} + \mathbf{B}_{2t}\mathbf{G}_{2t}\,, \tag{26}$$

$$\mathbf{k}_t := \mathbf{s}_t + \mathbf{B}_{1t}\mathbf{g}_{1t} + \mathbf{B}_{2t}\mathbf{g}_{2t}\,, \tag{27}$$

yielding the Riccati matrices, $\mathbf{H}_{it}$ $(i = 1,2)$ and $\mathbf{h}_{it}$ $(i = 1,2)$, for either of the players. The feedback matrices, $\mathbf{G}_{it}$ and $\mathbf{g}_{it}$ for $i,j = 1,2$ and $i \neq j$, are determined by

$$\mathbf{G}_{it} := -\mathbf{M}_{it}\mathbf{N}_{it}\mathbf{B}_{it}'\mathbf{H}_{it}\mathbf{E}_{jt}\mathbf{A}_t\,, \tag{28}$$

$$\mathbf{g}_{it} := -\mathbf{M}_{it}\mathbf{N}_{it}\left(\mathbf{v}_{it} - \mathbf{B}_{it}'\mathbf{H}_{it}\mathbf{B}_{jt}\mathbf{N}_{jt}\mathbf{v}_{jt}\right), \tag{29}$$

where the matrices $\mathbf{N}_{it}$ $(i = 1,2)$ and $\mathbf{M}_{it}$ $(i = 1,2)$ are defined by equations (5) *and* (7)*, respectively, and where*

$$\mathbf{E}_{it} := \mathbf{I} - \mathbf{B}_{it}\mathbf{N}_{it}\mathbf{B}_{it}'\mathbf{H}_{it}\,, \tag{30}$$

$$\mathbf{v}_{it} := \mathbf{B}_{it}'(\mathbf{H}_{it}\mathbf{s}_t - \mathbf{h}_{it}) - \mathbf{Q}_{it}^{\mathbf{u}_i}\tilde{\mathbf{u}}_{iit}\,. \tag{31}$$

For a proof, see Appendix 1.

4.2 The Open-Loop Nash Equilibrium Solution

Conceptually, the open-loop Nash solution can be viewed as follows. Each of the policy-makers chooses optimal values for his controls for the entire duration of the planning period under the assumption that the other sovereign policy authority is acting in a similar fashion, where the decisions are performed simultaneously by all players. Thus, at the beginning of the game each of the players makes a binding commitment to stick to a chosen policy. As long as these commitments hold, the solution is optimal in the sense that none of the players can improve his welfare by one-sided deviations from the equilibrium path.

Under the assumption of a linear relation between state and costate variables, the open-loop Nash equilibrium solution is generated using the discrete-time minimum principle as illustrated in de Zeeuw and van der Ploeg (1991), Hatheway (1992), Hager (1994) or Başar and Olsder (1999): *The Riccati matrices for both players $i = 1,2$ and for all time periods $t \in \{1,\ldots,T\}$, $\mathbf{H}_{1T}$, $\mathbf{H}_{2T}$, $\mathbf{h}_{1T}$, and $\mathbf{h}_{2T}$, are derived by backward iteration starting with their terminal conditions, $\mathbf{H}_{iT}$ $(i = 1,2)$ and $\mathbf{h}_{iT}$ $(i = 1,2)$, according to the following system of recursive matrix equations,*

$$\mathbf{H}_{i(t-1)} = \mathbf{A}_t'\mathbf{H}_{it}\mathbf{K}_t + \mathbf{Q}_{i(t-1)}^{\mathbf{x}}, \tag{32}$$

$$\mathbf{H}_{iT} = \mathbf{Q}_{iT}^{\mathbf{x}}\,,$$

$$\mathbf{h}_{i(t-1)} = \mathbf{A}_t'(\mathbf{H}_{it}\mathbf{k}_t + \mathbf{h}_{it}) - \mathbf{Q}_{i(t-1)}^{\mathbf{x}}\tilde{\mathbf{x}}_{i(t-1)}\,, \tag{33}$$

$$\mathbf{h}_{iT} = -\mathbf{Q}_{iT}^{\mathbf{x}}\tilde{\mathbf{x}}_{iT},$$

where $\overline{\overline{\mathbf{M}}}_t$ *is defined by equation* (9), *and where*

$$\mathbf{K}_t := \overline{\overline{\mathbf{M}}}_t\mathbf{A}_t, \tag{34}$$

$$\mathbf{k}_t := \overline{\overline{\mathbf{M}}}_t\left(\mathbf{s}_t + \sum_{j=1}^{2}\mathbf{B}_{jt}\left(\tilde{\mathbf{u}}_{jjt} - \left(\mathbf{Q}_{jt}^{\mathbf{u}_j}\right)^{-1}\mathbf{B}_{jt}'\mathbf{h}_{jt}\right)\right). \tag{35}$$

Both players' open-loop Nash equilibrium controls can be determined by $\mathbf{u}_{it}^* = \mathbf{\Gamma}_{it}\mathbf{x}_t^* + \gamma_{it}$, *where for* $i = 1,2$

$$\mathbf{\Gamma}_{it} := -\left(\mathbf{Q}_{it}^{\mathbf{u}_i}\right)^{-1}\mathbf{B}_{it}'\mathbf{H}_{it}, \tag{36}$$

$$\gamma_{it} := -\left(\mathbf{Q}_{it}^{\mathbf{u}_i}\right)^{-1}\mathbf{B}_{it}'\mathbf{h}_{it} + \tilde{\mathbf{u}}_{iit}. \tag{37}$$

For a proof, see Appendix 2.

4.3 The Pareto-Optimal Solutions

We derive the Pareto solution in a similar fashion to the feedback Nash equilibrium solution in Section 4.1 using the dynamic programming technique, the only difference being that a convex combination of the players' objective functions (equation (12) with the quadratic tracking form of the objective function (1)) is to be optimized jointly by both players. (The appropriate application of the minimum principle, however, would also yield the optimal cooperative solution, as this corresponds to a classical optimal control problem.)

The solution of the cooperative Pareto-optimal game with two players is given by the following procedure: *Starting with the specifications for their terminal conditions,* $\mathbf{H}_T$ *and* $\mathbf{h}_T$, *the Riccati equations*

$$\mathbf{H}_{t-1} = \mathbf{K}_t'\mathbf{H}_t\mathbf{K}_t + \mathbf{P}_{t-1}^{\mathbf{x}} + \sum_{i=1}^{2}\mathbf{G}_{it}'\mathbf{P}_t^{\mathbf{u}_i}\mathbf{G}_{it}, \tag{38}$$

$$\mathbf{H}_T = \mathbf{P}_T^{\mathbf{x}},$$

$$\mathbf{h}_{t-1} = \mathbf{K}_t'(\mathbf{h}_t - \mathbf{H}_t\mathbf{k}_t) + \tilde{\mathbf{P}}_{t-1}^{\mathbf{x}} + \sum_{i=1}^{2}\mathbf{G}_{it}\left(\tilde{\mathbf{P}}_t^{\mathbf{u}_i} - \mathbf{P}_t^{\mathbf{u}_i}\mathbf{g}_{it}\right), \tag{39}$$

$$\mathbf{h}_T = \tilde{\mathbf{P}}_T^{\mathbf{x}}$$

can be solved recursively backwards in time, i.e. for $t = T,\ldots,2$, *where* $\mathbf{K}_t$ *is defined by equation* (26), $\mathbf{k}_t$ *by equation* (27), *and where for* $i = 1,2$

$$\mathbf{P}_t^{\mathbf{x}} := \mu\mathbf{Q}_{1t}^{\mathbf{x}} + (1-\mu)\mathbf{Q}_{2t}^{\mathbf{x}}, \tag{40}$$

$$\tilde{\mathbf{P}}_t^{\mathbf{x}} := \mu\mathbf{Q}_{1t}^{\mathbf{x}}\tilde{\mathbf{x}}_{1t} + (1-\mu)\mathbf{Q}_{2t}^{\mathbf{x}}\tilde{\mathbf{x}}_{2t}, \tag{41}$$

$$\mathbf{P}_t^{\mathbf{u}_i} := \mu\mathbf{Q}_{1t}^{\mathbf{u}_i} + (1-\mu)\mathbf{Q}_{2t}^{\mathbf{u}_i}, \tag{42}$$

$$\tilde{\mathbf{P}}_t^{\mathbf{u}_i} := \mu\mathbf{Q}_{1t}^{\mathbf{u}_i}\tilde{\mathbf{u}}_{1it} + (1-\mu)\mathbf{Q}_{2t}^{\mathbf{u}_i}\tilde{\mathbf{u}}_{2it}. \tag{43}$$

Player i's feedback matrices, $\mathbf{G}_{it}$ $(i = 1,2)$ *and* $\mathbf{g}_{it}$ $(i = 1,2)$, *are determined by the equations*

$$\mathbf{G}_{it} := -\overline{\mathbf{M}}_{it}\overline{\mathbf{N}}_{it}\mathbf{B}_{it}'\mathbf{H}_t\overline{\mathbf{E}}_{jt}\mathbf{A}_t, \tag{44}$$

$$\mathbf{g}_{it} := -\overline{\mathbf{M}}_{it}\overline{\mathbf{N}}_{it}\left(\overline{\mathbf{v}}_{it} - \mathbf{B}_{it}'\mathbf{H}_t\mathbf{B}_{jt}\overline{\mathbf{N}}_{jt}\overline{\mathbf{v}}_{jt}\right), \tag{45}$$

where the matrices $\overline{\mathbf{N}}_{it}$ $(i = 1,2)$ and $\overline{\mathbf{M}}_{it}$ $(i = 1,2)$ *are defined by equations* (6) *and* (8), *respectively, and where*

$$\overline{\mathbf{E}}_{it} := \mathbf{I} - \mathbf{B}_{it}\overline{\mathbf{N}}_{it}\mathbf{B}_{it}'\mathbf{H}_t, \tag{46}$$

$$\overline{\mathbf{v}}_{it} := \mathbf{B}_{it}'\mathbf{H}_t\mathbf{s}_t - \mathbf{B}_{it}'\mathbf{h}_t - \mu\mathbf{Q}_{1t}^{\mathbf{u}_i}\tilde{\mathbf{u}}_{1it} - (1-\mu)\mathbf{Q}_{2t}^{\mathbf{u}_i}\tilde{\mathbf{u}}_{2it}. \tag{47}$$

For a proof, see Appendix 3.

4.4 The Feedback Stackelberg Equilibrium Solution

The non-cooperative feedback Stackelberg equilibrium solution is derived in a similar way to the feedback Nash equilibrium solution with one important difference: While the actions of the players are performed symmetrically in a Nash game, the players act hierarchically in a Stackelberg game. This leads to the additional consideration of the *follower's* (player 2) reaction to the announced strategy of the *leader* (player 1). This kind of hierarchical game may be relevant if one of the players has a dominant position, i.e. the feedback Stackelberg equilibrium solution may be imagined to be valid if the leader announces his decision rule, $\mathbf{u}_{1t} = \phi_1(\mathbf{x}_{t-1})$, whereas the follower bases his actions on the current state and on the decision of the leader according to the reaction function $\mathbf{u}_{2t} = \phi_2(\mathbf{x}_{t-1}, \mathbf{u}_{1t})$. The leader in turn considers the reaction coefficient, $\partial\mathbf{u}_{2t}/\partial\mathbf{u}_{1t}$, as a rational reaction of the follower in the optimization process (see, for example, de Zeeuw, 1984; de Zeeuw and van der Ploeg, 1991; Başar and Olsder, 1999).

The solution of the feedback Stackelberg game with player 1 as leader and player 2 as follower is given by the following procedure: *Using the principle of dynamic programming allows us to derive the Riccati matrices,* $\mathbf{H}_{1T}$, $\mathbf{H}_{2T}$, $\mathbf{h}_{1T}$, *and* $\mathbf{h}_{2T}$, *which are given as the solutions of the system of Riccati equations,* (24) *and* (25), *where the feedback matrices,* $\mathbf{G}_{it}$ $(i = 1,2)$ *and* $\mathbf{g}_{it}$ $(i = 1,2)$, *are determined by*

$$\begin{aligned}\mathbf{G}_{1t} := &-\underline{\mathbf{M}}_{1t}\mathbf{B}_{1t}'\mathbf{E}_{2t}'\mathbf{H}_{1t}\mathbf{E}_{2t}\mathbf{A}_t - \\ &-\underline{\mathbf{M}}_{1t}\mathbf{B}_{1t}'\mathbf{H}_{2t}\mathbf{B}_{2t}\mathbf{N}_{2t}\mathbf{Q}_{1t}^{\mathbf{u}_2}\mathbf{N}_{2t}\mathbf{B}_{2t}'\mathbf{H}_{2t}\mathbf{A}_t,\end{aligned} \tag{48}$$

$$\begin{aligned}\mathbf{g}_{1t} := &-\underline{\mathbf{M}}_{1t}\mathbf{B}_{1t}'\mathbf{E}_{2t}(\mathbf{H}_{1t}(\mathbf{s}_t - \mathbf{B}_{2t}\mathbf{N}_{2t}\mathbf{v}_{2t}) - \mathbf{h}_{1t}) - \\ &-\underline{\mathbf{M}}_{1t}\mathbf{B}_{1t}'\mathbf{H}_{2t}\mathbf{B}_{2t}\mathbf{N}_{2t}\mathbf{Q}_{1t}^{\mathbf{u}_2}(\mathbf{N}_{2t}\mathbf{v}_{2t} + \tilde{\mathbf{u}}_{12t}) + \\ &+\underline{\mathbf{M}}_{1t}\mathbf{Q}_{1t}^{\mathbf{u}_1}\tilde{\mathbf{u}}_{11t},\end{aligned} \tag{49}$$

and

$$\mathbf{G}_{2t} := -\mathbf{N}_{2t}\mathbf{B}_{2t}'\mathbf{H}_{2t}(\mathbf{A}_t + \mathbf{B}_{1t}\mathbf{G}_{1t}), \tag{50}$$

$$\mathbf{g}_{2t} := -\mathbf{N}_{2t}\left(\mathbf{v}_{2t} + \mathbf{B}_{2t}'\mathbf{H}_{2t}\mathbf{B}_{1t}\mathbf{g}_{1t}\right), \tag{51}$$

where the vector $\mathbf{v}_{it}$ $(i = 1,2)$ *is defined by equation* (31), *the matrices* $\mathbf{N}_{it}$ $(i = 1,2)$ *and* $\mathbf{E}_{it}$ $(i = 1,2)$ *are determined by equations* (5) *and* (30), *respectively, and* $\underline{\mathbf{M}}_{1t}$ *by equation* (10).

For a proof, see Appendix 4.

4.5 The Open-Loop Stackelberg Equilibrium Solution

In contrast to the feedback Stackelberg game, the open-loop Stackelberg game assumes that the leader (player 1) announces his strategy for the entire planning horizon. Hence, the open-loop Stackelberg equilibrium solution may be derived in analogy to the open-loop Nash equilibrium solution, but under the additional assumption of an asymmetric information structure of the game as sketched by Başar and Older (1999). The leader makes binding commitments about future policy actions taking the follower's rational reaction function into account. Note that the leader is always better off (and the follower worse off) than in the corresponding open-loop Nash equilibrium solution, because the leader forces his strategy upon the follower.

The solution of the open-loop Stackelberg game with player 1 as leader and player 2 as follower is given by the following procedure: *Starting with their terminal conditions for* $\mathbf{H}_T$ *and* $\mathbf{h}_T$, *the Riccati equations*

$$\mathbf{H}_{t-1} = \underline{\mathbf{A}}_t'\underline{\underline{\mathbf{M}}}_t\mathbf{H}_t\underline{\mathbf{A}}_t + \underline{\mathbf{Q}}_{t-1}^{\mathbf{x}}, \tag{52}$$

$$\mathbf{H}_T = \underline{\mathbf{Q}}_T^{\mathbf{x}}, \tag{53}$$

$$\mathbf{h}_{t-1} = \underline{\mathbf{A}}_t'\underline{\underline{\mathbf{M}}}_t(\mathbf{H}_t\mathbf{d}_t + \mathbf{h}_t) - \mathbf{q}_{t-1}, \tag{54}$$

$$\mathbf{h}_T = -\mathbf{q}_T, \tag{55}$$

can be solved by backward integration yielding the Riccati matrices, $\mathbf{H}_t$ *and* $\mathbf{h}_t$. *The generalized state vector, which consists of the "original" state vector and the costate vector of player* 1 *corresponding to the costate equation of player* 2, *is given by*

$$\xi_t^* = \underline{\underline{\mathbf{M}}}_t(\underline{\mathbf{A}}_t\xi_{t-1}^* - \underline{\underline{\mathbf{E}}}_t\mathbf{h}_t + \mathbf{d}_t), \tag{56}$$

where $\underline{\underline{\mathbf{M}}}_t$ *is defined by equation* (11), *and where*

$$\underline{\mathbf{Q}}_t^{\mathbf{x}} := \begin{pmatrix} \mathbf{Q}_{1t}^x & \mathbf{Q}_{2t}^x \\ \mathbf{Q}_{2t}^x & \mathbf{0} \end{pmatrix}, \tag{57}$$

$$\mathbf{q}_t := \begin{pmatrix} \mathbf{Q}_{1t}^{\mathbf{x}}\tilde{\mathbf{x}}_{1t} \\ \mathbf{Q}_{2t}^{\mathbf{x}}\tilde{\mathbf{x}}_{2t} \end{pmatrix}, \tag{58}$$

$$\mathbf{d}_t := \begin{pmatrix} \mathbf{s}_t + \mathbf{B}_{1t}\tilde{\mathbf{u}}_{11t} + \mathbf{B}_{2t}\tilde{\mathbf{u}}_{22t} \\ \mathbf{0} \end{pmatrix}, \tag{59}$$

$$\underline{\mathbf{A}}_t := \begin{pmatrix} \mathbf{A}_t & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_t \end{pmatrix}, \tag{60}$$

$$\underline{\underline{\mathbf{E}}}_t := \begin{pmatrix} \mathbf{B}_{1t}\left(\mathbf{Q}_{1t}^{\mathbf{u}_1}\right)^{-1}\mathbf{B}_{1t}' & \mathbf{B}_{2t}\left(\mathbf{Q}_{2t}^{\mathbf{u}_2}\right)^{-1}\mathbf{B}_{2t}' \\ \mathbf{B}_{2t}\left(\mathbf{Q}_{2t}^{\mathbf{u}_2}\right)^{-1}\mathbf{B}_{2t}' & \mathbf{0} \end{pmatrix}. \tag{61}$$

The control vector is determined by

$$\begin{pmatrix} \mathbf{u}_{1t}^* \\ \mathbf{u}_{2t}^* \end{pmatrix} = \begin{pmatrix} \tilde{\mathbf{u}}_{11t} \\ \tilde{\mathbf{u}}_{22t} \end{pmatrix} - \underline{\mathbf{Q}}_t^{\mathbf{u}}\left(\mathbf{H}_t\xi_t^* + \mathbf{h}_t\right), \tag{62}$$

where

$$\underline{\mathbf{Q}}_t^{\mathbf{u}} := \begin{pmatrix} \left(\mathbf{Q}_{1t}^{\mathbf{u}_1}\right)^{-1}\mathbf{B}_{1t}' & \mathbf{0} \\ \mathbf{0} & \left(\mathbf{Q}_{2t}^{\mathbf{u}_2}\right)^{-1}\mathbf{B}_{2t}' \end{pmatrix}. \tag{63}$$

For a proof, see Appendix 5.

5. CONCLUDING REMARKS

OPTGAME 1.0, an algorithm to solve two-person affine-quadratic difference games exactly, and to approximate the solutions of discrete-time nonlinear quadratic games by means of an appropriate linearization procedure, is presented in this paper. Applying this technique of approximating intertemporal interrelated optimization problems with two decision-makers yields a real improvement with respect to decision-making support in the field of economics. Applications can easily be implemented to dynamic oligopoly and other forms of imperfect competition in microeconomics and to national or international policy coordination problems in macroeconomics. For an example with two institutions of fiscal and monetary policies, see, for example, Neck (1999) and Hager *et al.* (2001). Applications from other fields of research include international technology transfer pollution control carried out by Kaitala *et al.* (1991), Long (1992), Dockner and Hager (1993), for example, but also international drug control (Behrens *et al.*, 2002), to mention only two of them. Extensions of the algorithm are topics of further research. In particular, we will extend OPTGAME 1.0 to an algorithm which allows us to investigate game-theoretic problems with an arbitrary number of decision-makers.

REFERENCES

Başar, T. (1989). Time consistency and robustness of equilibria in non-cooperative dynamic games. In: *Dynamic Policy Games in Economics* (F. van der Ploeg and A. de Zeeuw, Eds.), pp. 9–54. North-Holland, Amsterdam.

Başar, T. and G.J. Olsder (1999). *Dynamic Noncoop-*

erative Game Theory. 2nd edition. SIAM, Philadelphia, Pennsylvania.

Behrens, D.A., J.P. Caulkins and G. Tragler (2002). *A Dynamic Game of Two Regions with Interacting Drug Epidemics*. Working Paper, Vienna University of Technology, Vienna.

Chow, G.C. (1975). *Analysis and Control of Dynamic Economic Systems*. John Wiley & Sons, New York, NY.

de Zeeuw, A.J. (1984). *Difference Games and Linked Econometric Policy Models*. PhD Thesis, Tilburg University, Netherlands.

de Zeeuw, A.J. and F. van der Ploeg (1991). Difference games and policy evaluation: a conceptual framework. *Oxford Economic Papers*, **43**, 612–636.

Dockner, E.J. and M. Hager (1993). *International Coordination of Pollution Control*. Working Paper, Vienna University of Economics and Business Administration, Vienna.

Dockner, E.J. and R. Neck (1988). *Time-Consistency, Subgame-Perfectness, Solution Concepts and Information Patterns in Dynamic Models of Stabilization Policies*. Discussion Paper, University of Saskatchewan, Saskatoon.

Engwerda, J.C., B. van Aarle and J.E.J. Plasmans (1999). The (in)finite horizon open-loop Nash LQ game: An application to EMU. *Annals of Operations Research*, **88**, 251–273.

Fershtman, C. and I. Kamien (1987). Dynamic duopolistic competition with sticky prices. *Econometrica*, **55**, 1151–1164.

Hager, M. (1994). *Nonlinear Dynamic Games: Theory, Numerical Solutions and Applications in Economics*. PhD Thesis, Vienna Univ. of Technology, Vienna.

Hager, M., R. Neck and D.A. Behrens (2001): Solving dynamic macroeconomic policy games using the algorithm OPTGAME 1.0. *Optimal Control Applications and Methods*, **22**, 301–332.

Hatheway, L.H. (1992). *Modeling International Economic Interdependence: An Application of Feedback Nash Dynamic Games*. PhD Thesis, University of Texas at Austin, Austin, Texas.

Kaitala, V., M. Pohjola and O. Tahvonen (1991): Transboundary air pollution between Finland and the USSR – a dynamic acid rain game. In: *Dynamic Games in Economic Analysis* (R.P. Hämäläinen *et al.*, Eds.), pp. 183–191. *Lecture Notes in Control and Information Sciences* **157**. Springer, Heidelberg.

Kendrick, D. (1981). *Stochastic Control for Economic Models*. McGraw-Hill, New York, NY.

Kydland, F. (1975). Noncooperative and dominant player solutions in discrete dynamic games. *International Economic Review*, **16**, 321–335.

Leitmann, G. (1974). *Cooperative and Non-Cooperative Many Players Differential Games*. Springer, Vienna.

Léonard, D. and N.V. Long (1992). *Optimal Control Theory and Static Optimization in Economics*. CUP, Cambridge.

Long, N.V. (1992). Pollution control: a differential game approach. *Annals of Operations Research*, **37**, 283–296.

Matulka, J. and R. Neck (1992). OPTCON: an algorithm for the optimal control of nonlinear stochastic models. *Annals of Operations Research*, **37**, 375–401.

Mehlmann, A. (1991). *Applied Differential Games*. Plenum Press, New York, NY.

Neck, R. (1999). Dynamic Games of Fiscal and Monetary Policies for Austria. *Annals of Operations Research*, **88**, 233–249.

Petit, M.L. (1990). *Control Theory and Dynamic Games in Economic Policy Analysis*. Cambridge University Press, Cambridge.

Reinganum, J.F. and N.S. Stokey (1985). Oligopoly Extraction of a Common Property Natural Resource: The Importance of the Period of Commitment in Dynamic Games. *International Economic Review*, **26**, 161–173.

APPENDIX

We omit the asterisks throughout the entire appendix to simplify notation.

1. Derivation of the Feedback Nash Equilibrium Solution

We derive the feedback Nash equilibrium solution according to the procedure described by Başar and Olsder (1999, pp. 279–281): period-by-period, recursively backwards in time, either of the players minimizes his cost function, given the information about the state and the other player's feedback rule for this period. We start with the quadratic tracking form of the objective function (1) in period $t = T$ for $i = 1,2$ and proceed by determining the terminal conditions for the Riccati matrices,

$$\left.\begin{aligned} \mathbf{H}_{iT} &= \mathbf{Q}_{iT}^{\mathbf{x}} \\ \mathbf{h}_{iT} &= \mathbf{Q}_{iT}^{\mathbf{x}} \tilde{\mathbf{x}}_{iT} \end{aligned}\right\} \quad i = 1,2 . \tag{A1}$$

Then, the cost-to-go function of player $i = 1,2$ for the terminal period may be written in the following form

$$\begin{aligned} J_{iT} = L_{iT} = \tfrac{1}{2}\mathbf{x}_T' \mathbf{H}_{iT}\mathbf{x}_T - \mathbf{x}_T' \mathbf{h}_{iT} + \mathbf{c}_{iT} + \\ + \sum_{j=1}^{2} \mathbf{u}_{jT}' \mathbf{Q}_{iT}^{\mathbf{u}_j} \left(\tfrac{1}{2}\mathbf{u}_{jT} - \tilde{\mathbf{u}}_{ijT}\right), \end{aligned} \tag{A2}$$

where the constant terms, $\mathbf{c}_{iT}$ $(i = 1,2)$, are without any relevance for further calculations. According to the system equations (15) we can derive the state vector, $\mathbf{x}_T$, using the optimized lagged state vector and the control variables, $\mathbf{u}_{iT}$ $(i = 1,2)$ which are also (linear) functions of the optimized lagged state vector. Hence, the optimal cost-to-go function is given by the minimization of J_{iT} $(i = 1,2)$ with respect to $\mathbf{u}_{iT}$ $(i = 1,2)$ using Bellman's principle of optimality. (Note that J_{iT}

($i = 1,2$) is a strictly convex function of $\mathbf{u}_{iT}$ ($i = 1,2$) since $\mathbf{Q}_{iT}$ ($i = 1,2$) is positively semi-definite and the matrices containing the penalties for deviations of the control variables from their desired values are positive definite.)

Due to the symmetry of the penalty matrices $\mathbf{Q}_{iT}$ (i=1,2), minimizing equation (A2) (where the state vector $\mathbf{x}_T$ is replaced by the right hand side of (15)) with respect to $\mathbf{u}_{iT}$ (i=1,2) yields the following (necessary *and* sufficient) first-order conditions for i=1,2,

$$\begin{aligned}\mathbf{B}_{iT}'\mathbf{H}_{iT}\left(\mathbf{A}_T\mathbf{x}_{T-1}+\mathbf{B}_{1T}\mathbf{u}_{1T}+\mathbf{B}_{2T}\mathbf{u}_{2T}+\mathbf{s}_T\right)-\\ -\mathbf{B}_{iT}'\mathbf{h}_{iT}+\mathbf{Q}_{iT}^{\mathbf{u}_i}\left(\mathbf{u}_{iT}-\tilde{\mathbf{u}}_{iiT}\right)=\mathbf{0}.\end{aligned} \tag{A3}$$

Solving equation (A3) for the control vector $\mathbf{u}_{iT}$ ($i = 1,2$) yields equation (22) where the feedback matrices, $\mathbf{G}_{iT}$ ($i = 1,2$) and $\mathbf{g}_{iT}$ ($i = 1,2$), are determined by equations (28) and (29), respectively. With equation (22) and the feedback matrices we can determine the optimal state vector, $\mathbf{x}_T$, as a linear function of $\mathbf{x}_{T-1}$ as given by (23) using $\mathbf{K}_t$ and $\mathbf{k}_t$ as defined by (26) and (27) for $t = T$, respectively.

[*] Then we step back in time, performing the optimization for period t=T–1. Therefore, we have to substitute the optimized state vector, $\mathbf{x}_T$=$\mathbf{K}_T\mathbf{x}_{T-1}$+$\mathbf{k}_T$, and the optimized control vectors, $\mathbf{u}_{iT}$ =$\mathbf{G}_{iT}\mathbf{x}_{T-1}$+$\mathbf{g}_{iT}$ (i=1,2), into the cost-to-go functions $J_{i(T-1)}$ =L_{iT}+$L_{i(T-1)}$ (i=1,2). By collecting all terms containing $\mathbf{x}_{T-1}$ we can identify the Riccati matrices for period T–1, $\mathbf{H}_{i(T-1)}$ (i=1,2) and $\mathbf{h}_{i(T-1)}$ (i=1,2), as defined by equations (24) and (25), respectively. Then, we minimize the objective functions $J_{i(T-1)}$ =L_{iT}+$L_{i(T-1)}$ (i=1,2) analogously to the procedure described above: Substitution of the system dynamics (15) for period T–1 into $J_{i(T-1)}$ and taking the first derivatives with respect to either of the controls, $\mathbf{u}_{i(T-1)}$ (i=1,2), yields condition (A3) for T→T–1. Then we determine optimal controls and states as done for period t=T. [**].

Then, the entire procedure, [*]–[**], can be extended straightforwardly to any other period $t = \tau$ (T–$2 > \tau > 1$) by induction, giving the feedback Nash equilibrium solution. The existence and uniqueness of the solutions of the Riccati equations for all periods $t \in \{1,...,T\}$ of the affine-quadratic game described in Section 2 and, consequently, the existence of the unique feedback Nash equilibrium solution can be verified due to the strict convexity of the objective function (1) with respect to the controls. ■

2. *Derivation of the Open-Loop Nash Equilibrium Solution*

The solution of the open-loop Nash game with two players is given by applying the discrete-time minimum principle (see, for example, Léonard and Long, 1992, pp. 130–131) to perform the minimization of each player's loss function subject to the system equations, (15), the initial conditions, and the assumption that the other player acts identically. We define the (strictly convex) Hamiltonian functions, $\mathcal{H}_{it}$ ($i = 1,2$) for any time period $t \in \{1,...,T\}$ as

$$\begin{aligned}\mathcal{H}_{it} = {} & \mathbf{x}_t'\mathbf{Q}_{it}^{\mathbf{x}}\left(\tfrac{1}{2}\mathbf{x}_t-\tilde{\mathbf{x}}_{it}\right)+\mathbf{c}_{it}+\\ & +\sum_{j=1}^{2}\mathbf{u}_{jt}'\mathbf{Q}_{it}^{\mathbf{u}_j}\left(\tfrac{1}{2}\mathbf{u}_{jt}-\tilde{\mathbf{u}}_{ijt}\right)+\\ & +\boldsymbol{\lambda}_{it}'\left(\mathbf{A}_t\mathbf{x}_{t-1}-\mathbf{x}_t+\mathbf{s}_t+\mathbf{B}_{1t}\mathbf{u}_{1t}+\mathbf{B}_{2t}\mathbf{u}_{2t}\right),\end{aligned} \tag{A4}$$

where $\boldsymbol{\lambda}_{it}$ ($i = 1,2$) is the n_s–dimensional costate vector for player i, and $\mathbf{c}_{it}$ ($i = 1,2$) denotes constant terms. Appropriate differentiation yields the adjoint equations for $i = 1,2$

$$\boldsymbol{\lambda}_{i(t-1)} = \mathbf{Q}_{i(t-1)}^{\mathbf{x}}\left(\mathbf{x}_{t-1}-\tilde{\mathbf{x}}_{i(t-1)}\right)+\mathbf{A}_t'\boldsymbol{\lambda}_{it}, \tag{A5}$$

and necessary (and sufficient) conditions for the control variables. Since we know the terminal condition for the vector of costate variables, we can express the first-order condition of the optimization problem as a hypothetical reaction function for both players. Since we, furthermore, assume a linear relationship between the adjoint and the optimal state variables, $\boldsymbol{\lambda}_{it} = \mathbf{H}_{it}\mathbf{x}_t + \mathbf{h}_{it}$ ($i = 1,2$) where $\mathbf{H}_{it}$ and $\mathbf{h}_{it}$ denote the Riccati matrices, these costate variables can be substituted into the hypothetical reaction functions yielding for $i = 1,2$

$$\mathbf{u}_{it} = \tilde{\mathbf{u}}_{iit}-\left(\mathbf{Q}_{it}^{\mathbf{u}_i}\right)^{-1}\mathbf{B}_{it}'\left(\mathbf{H}_{it}\mathbf{x}_t+\mathbf{h}_{it}\right). \tag{A6}$$

Then the optimal state equation follows from the substitution of (A6) into (15) and is determined by

$$\mathbf{x}_t = \overline{\overline{\mathbf{M}}}_t\left(\mathbf{A}_t\mathbf{x}_{t-1}+\overline{\overline{\mathbf{v}}}_t\right), \tag{A7}$$

where the $\overline{\overline{\mathbf{M}}}_t$ is defined by equation (9), and

$$\overline{\overline{\mathbf{v}}}_t := \mathbf{s}_t+\sum_{j=1}^{2}\mathbf{B}_{jt}\left(\tilde{\mathbf{u}}_{jjt}-\left(\mathbf{Q}_{jt}^{\mathbf{u}_j}\right)^{-1}\mathbf{B}_{jt}'\mathbf{h}_{jt}\right). \tag{A8}$$

Substituting $\boldsymbol{\lambda}_{it} = \mathbf{H}_{it}\mathbf{x}_t + \mathbf{h}_{it}$ ($i = 1,2$) together with (A15) into the equations for the costates and rearranging terms with respect to $\mathbf{x}_{t+1}$ yields

$$\boldsymbol{\lambda}_{i(t-1)} = \mathbf{H}_{i(t-1)}\mathbf{x}_{t-1}+\mathbf{h}_{i(t-1)}, \tag{A9}$$

where the Riccati matrices, $\mathbf{H}_{i(t-1)}$ ($i = 1,2$) and $\mathbf{h}_{i(t-1)}$ ($i = 1,2$), are determined by equations (32) and (33), respectively. Using $\boldsymbol{\lambda}_{i(T+1)} = \mathbf{0}$ the equations for the terminal conditions of the Riccati matrices can be derived:

$$\left.\begin{aligned}\mathbf{H}_{iT} &= \mathbf{Q}_{iT}^{\mathbf{x}}\\ \mathbf{h}_{iT} &= -\mathbf{Q}_{iT}^{\mathbf{x}}\tilde{\mathbf{x}}_{iT}\end{aligned}\right\} \quad i = 1,2. \tag{A10}$$

Iterating backwards in time yields the Riccati matrices for all time periods $t \in \{1,...,T\}$. Then the optimal

control vectors can be determined for both players:

$$\mathbf{u}_{it} = -\left(\mathbf{Q}_{it}^{\mathbf{u}_i}\right)^{-1}\mathbf{B}_{it}'\left(\mathbf{H}_{it}\mathbf{x}_t + \mathbf{h}_{it}\right) + \tilde{\mathbf{u}}_{iit}, \qquad \text{(A11)}$$

alternating with the derivation of the optimal state variables according to relationship (23) with $\mathbf{K}_t$ and $\mathbf{k}_t$ as defined by (34) and (35), respectively. ■

3. Derivation of the Pareto-Optimal Solution

The dynamic programming method requires a recursive period-by-period minimization of the joint objective function (12) subject to (15), starting with the terminal period and the determination of the terminal conditions for the Riccati matrices,

$$\mathbf{H}_T = \mu\mathbf{Q}_{1T}^{\mathbf{x}} + (1-\mu)\mathbf{Q}_{2T}^{\mathbf{x}} =: \mathbf{P}_T^{\mathbf{x}}, \qquad \text{(A12)}$$

$$\mathbf{h}_T = \mu\mathbf{Q}_{1T}^{\mathbf{x}}\tilde{\mathbf{x}}_{1T} + (1-\mu)\mathbf{Q}_{2T}^{\mathbf{x}}\tilde{\mathbf{x}}_{2T} =: \tilde{\mathbf{P}}_T^{\mathbf{x}}. \qquad \text{(A13)}$$

In a similar fashion to the feedback Nash game, the optimization starts with minimizing the strictly convex cost-to-go function for the terminal period,

$$J_T = \mathbf{x}_T'\left(\tfrac{1}{2}\mathbf{H}_T\mathbf{x}_T - \mathbf{h}_T\right) + \mathbf{c}_T + \\ + \sum_{j=1}^{N}\mathbf{u}_{jT}'\left(\tfrac{1}{2}\mathbf{P}_T^{\mathbf{u}_j}\mathbf{u}_{jT} - \tilde{\mathbf{P}}_T^{\mathbf{u}_j}\right), \qquad \text{(A14)}$$

(where $\mathbf{P}_T^{\mathbf{u}_i}$ $(i = 1,2)$ and $\tilde{\mathbf{P}}_T^{\mathbf{u}_i}$ $(i = 1,2)$ are defined by (42) and (43) for $t = T$, respectively, and $\mathbf{c}_T$ denotes constant terms) with respect to $\mathbf{u}_{iT}$ $(i = 1,2)$ which delivers the first-order conditions for both players' control variables. We derive these control variables as linear functions of the lagged state vector as determined by equation (22), where the feedback matrices, $\mathbf{G}_{iT}$ $(i = 1,2)$ and $\mathbf{g}_{iT}$ $(i = 1,2)$, are determined by equations (44) and (45), respectively. With $\mathbf{u}_{iT} = \mathbf{G}_{iT}\mathbf{x}_{T-1} + \mathbf{g}_{iT}$ $(i = 1,2)$ we can determine the optimal state vector $\mathbf{x}_T$ as a linear function of $\mathbf{x}_{T-1}$, as expressed by equation (23) using definitions (26) and (27). As in the feedback Nash game, we have to substitute the optimized state and control vectors into the appropriate cost-to-go function for determining the Riccati matrices of time period $t = T-1$ as given by (38) and (39), respectively. These equations can be solved recursively in time, obtaining the Riccati matrices for period $T-1$ as a function of the Riccati matrices of the terminal period. This procedure can be extended in a straightforward way to period $t = T-2$ and to any other period t by induction – as already performed in the feedback Nash case (including existence and uniqueness). ■

4. Derivation of the Feedback Stackelberg Equilibrium Solution

As outlined by Başar and Olsder (1999, pp. 373–375), two different optimization problems occur for the derivation of the Stackelberg game's solution. The first one consists in determining the rational reaction of the *follower* (player 2), $\mathbf{u}_{2t} = \phi_2(\mathbf{x}_{t-1}, \mathbf{u}_{1t})$, to the announced decision rule of the *leader* (player 1), $\mathbf{u}_{1t} = \phi_1(\mathbf{x}_{t-1})$. Following the calculation of the feedback Nash equilibrium strategies according to the dynamic programming principle yields the first-order condition for the follower with the terminal conditions for the Riccati matrices, as determined by (A1) for $i = 2$. The uniquely defined control variable, $\mathbf{u}_{2T}$, expressed as a function of the leader's control vector, $\mathbf{u}_{1T}$, is determined by

$$\mathbf{u}_{2T}(\mathbf{u}_{1T}) = \mathbf{R}_T\mathbf{u}_{1T} - \\ - \mathbf{N}_{2T}\mathbf{B}_{2T}'\mathbf{H}_{2T}\mathbf{A}_T\mathbf{x}_{T-1} - \mathbf{N}_{2T}\mathbf{v}_{2T}, \qquad \text{(A15)}$$

where the matrix $\mathbf{N}_{2T}$ is defined by equation (5), the vector $\mathbf{v}_{2T}$ by equation (31), and where the so-called reaction coefficient of the follower for the terminal period is determined by the derivative of $\mathbf{u}_{2T}$ with respect to $\mathbf{u}_{1T}$, i.e.

$$\mathbf{R}_T := \frac{\partial\mathbf{u}_{2T}}{\partial\mathbf{u}_{1T}} = -\mathbf{N}_{2T}\mathbf{B}_{2T}'\mathbf{H}_{2T}\mathbf{B}_{1T}. \qquad \text{(A16)}$$

The second optimization problem determines the optimal action of the leader, given the rational reaction of the follower as expressed by the reaction coefficient $\mathbf{R}_T$ defined by equation (A16), and we start to solve the leader's optimization problem by determining the terminal conditions for the Riccati matrices as given by (A1) for $i = 1$. (In complete analogy to the feedback Nash equilibrium solution, minimization starts with the terminal period using Bellman's principle of optimality.) The crucial difference between the first-order condition in the Stackelberg game and the corresponding equation for the feedback Nash game is the occurrence of terms containing the reaction coefficient, $\mathbf{R}_T$, in (A17):

$$\left(\mathbf{B}_{1T} + \mathbf{B}_{2T}\mathbf{R}_T\right)'\left(\mathbf{H}_{1T}\left(\mathbf{A}_T\mathbf{x}_{T-1} + \mathbf{s}_T\right) - \mathbf{h}_{1T}\right) + \\ + \left(\mathbf{B}_{1T} + \mathbf{B}_{2T}\mathbf{R}_T\right)'\mathbf{H}_{1T}\left(\mathbf{B}_{1T}\mathbf{u}_{1T} + \mathbf{B}_{2T}\mathbf{u}_{2T}\right) + \\ + \mathbf{Q}_{1T}^{\mathbf{u}_1}\left(\mathbf{u}_{1T} - \tilde{\mathbf{u}}_{11T}\right) + \mathbf{R}_T'\mathbf{Q}_{1T}^{\mathbf{u}_2}\left(\mathbf{u}_{2T} - \tilde{\mathbf{u}}_{12T}\right) = \mathbf{0}. \qquad \text{(A17)}$$

We solve equation (A17) with respect to $\mathbf{u}_{1T}$ by using the definition of the reaction function of the follower, (A15). This leads to equation (22) for $i = 1$ where the feedback matrices, $\mathbf{G}_{1T}$ and $\mathbf{g}_{1T}$, are determined by equations (48) and (49), respectively. Immediately from (A15) it follows that $\mathbf{u}_{2T} = \mathbf{G}_{2T}\mathbf{x}_{T-1} + \mathbf{g}_{2T}$, where the feedback matrices, $\mathbf{G}_{2T}$ and $\mathbf{g}_{2T}$, are determined by equations (50) and (51), respectively. Structurally, all the rest happens in total analogy to the feedback Nash game (including existence and uniqueness of the solutions, identification of the Riccati matrices, and calculation of the equilibrium path of the state variables). ■

5. Derivation of the Open-Loop Stackelberg Equilibrium Solution

The open-loop Stackelberg strategy may be obtained

analogously to the open-loop Nash case but augmented with an asymmetric information structure, requiring the follower (player 2) to minimize his loss function according to Pontryagin's minimum principle by solving a classical optimal control problem with the given strategy of the leader. With the strictly convex Hamiltonian function, (A4) for $i=2$, we can derive the follower's canonical equation as defined by (A5) for $i=2$ and the follower's first-order condition as hypothetical reaction function as expressed by equation (A6) for $i=2$.

The leader (player 1) in turn faces a non-classical optimal control problem because he has to take into account the dynamics of the follower's optimal decision (expressed in terms of the costate variable and determined by equation (A5) for $i=2$). Therefore, the leader has to deal with a system of state equations which includes the adjoint system of the follower. Hence, optimization for the leader is performed by minimizing

$$\begin{aligned}\mathcal{H}_{it} &= \tfrac{1}{2}\left(\mathbf{X}_t - \tilde{\mathbf{X}}_{1t}\right)'\mathbf{Q}_{1t}\left(\mathbf{X}_t - \tilde{\mathbf{X}}_{1t}\right)+\\ &+\boldsymbol{\lambda}_{1t}'\left(\mathbf{A}_t\mathbf{x}_{t-1} - \mathbf{x}_t + \mathbf{s}_t + \mathbf{B}_{1t}\mathbf{u}_{1t}\right)+\\ &+\boldsymbol{\lambda}_{1t}'\left(\mathbf{B}_{2t}\tilde{\mathbf{u}}_{22t} - \mathbf{B}_{2t}\left(\mathbf{Q}_{2t}^{\mathbf{u}_2}\right)^{-1}\mathbf{B}_{2t}'\boldsymbol{\lambda}_{2t}\right)+\\ &+\boldsymbol{\lambda}_{12t}'\left(\mathbf{A}_{t+1}'\boldsymbol{\lambda}_{2(t+1)} - \boldsymbol{\lambda}_{2t} + \mathbf{Q}_{2t}^{\mathbf{x}}\left(\mathbf{x}_2 - \tilde{\mathbf{x}}_{2t}\right)\right),\end{aligned} \tag{A18}$$

for all time periods $t \in \{1,\ldots,T\}$, where $\boldsymbol{\lambda}_{1t}$ is the leader's costate variable with respect to the state vector $\mathbf{x}_t$, and $\boldsymbol{\lambda}_{12t}$ represents the leader's costate variables with respect to the "additional state variable" $\boldsymbol{\lambda}_{2t}$. Appropriate differentiation yields the rule for the leader's control variable,

$$\mathbf{u}_{1t} = \tilde{\mathbf{u}}_{11t} - \left(\mathbf{Q}_{1t}^{\mathbf{u}_1}\right)^{-1}\mathbf{B}_{1t}'\boldsymbol{\lambda}_{1t}, \tag{A19}$$

and the adjoint equations (with $\boldsymbol{\lambda}_{1(T+1)} = \mathbf{0}$),

$$\begin{aligned}\boldsymbol{\lambda}_{1(t-1)} &= \mathbf{A}_t'\boldsymbol{\lambda}_{1t} +\\ &+\mathbf{Q}_{1(t-1)}^{\mathbf{x}}\left(\mathbf{x}_{t-1} - \tilde{\mathbf{x}}_{1(t-1)}\right) + \mathbf{Q}_{2(t-1)}^{\mathbf{x}}\boldsymbol{\lambda}_{12(t-1)},\end{aligned} \tag{A20}$$

$$\boldsymbol{\lambda}_{12t} = \mathbf{A}_t\boldsymbol{\lambda}_{12(t-1)} - \mathbf{B}_{2t}\left(\mathbf{Q}_{2t}^{\mathbf{u}_2}\right)^{-1}\mathbf{B}_{2t}'\boldsymbol{\lambda}_{1t}. \tag{A21}$$

Using equation (A19) and (A6) for $i=2$ we can rewrite the state equations, (15). Then, we obtain a system of four coupled difference equations. Hence, we have to decouple these equations, specify the initial and terminal conditions, and solve two of these equations by forward iteration and the remaining ones by backward iteration. Note that

$$\xi_t := \begin{pmatrix}\mathbf{x}_t\\ \boldsymbol{\lambda}_{12t}\end{pmatrix} \quad \text{and} \quad \boldsymbol{\lambda}_t := \begin{pmatrix}\boldsymbol{\lambda}_{1t}\\ \boldsymbol{\lambda}_{2t}\end{pmatrix}, \tag{A22}$$

are defined as supplementary state and costate vectors of the leader. The initial condition for one of the components of the vector ξ_t is already provided by $\mathbf{x}_0$. The initial condition for $\boldsymbol{\lambda}_{12t}$ is $\boldsymbol{\lambda}_{120} = \mathbf{0}$; cf. for example, Dockner and Neck (1988) or de Zeeuw and van der Ploeg (1991). The terminal condition for the supplementary costate can be obtained from the terminal conditions of the leader's first costate and the follower's costate. This procedure yields the following system

- to be solved by forward iteration:

$$\xi_t = \underline{\mathbf{A}}_t\xi_{t-1} - \underline{\underline{\mathbf{E}}}_t\boldsymbol{\lambda}_t + \mathbf{d}_t, \tag{A23}$$

$$\xi_0 = \begin{pmatrix}\mathbf{x}_0\\ \mathbf{0}\end{pmatrix}, \tag{A24}$$

- to be solved by backward iteration:

$$\boldsymbol{\lambda}_{t-1} = \underline{\mathbf{A}}_t'\boldsymbol{\lambda}_t + \underline{\mathbf{Q}}_{t-1}^{\mathbf{x}}\xi_{t-1} - \mathbf{q}_{t-1}, \tag{A25}$$

$$\boldsymbol{\lambda}_T = \underline{\mathbf{Q}}_T^{\mathbf{x}}\xi_T - \mathbf{q}_T. \tag{A26}$$

where $\mathbf{q}_t$ is defined by (58), $\mathbf{d}_t$ is given by (59), $\underline{\mathbf{A}}_t$ by (60), $\underline{\underline{\mathbf{E}}}_t$ by (61), and $\underline{\mathbf{Q}}_t^{\mathbf{x}}$ by (57).

Now, we proceed in analogy to the open-loop Nash game. We assume that the supplementary state and costate vectors are related linearly, i.e. $\boldsymbol{\lambda}_t = \mathbf{H}_t\xi_t + \mathbf{h}_t$. This allows us to derive Riccati equations to yield Riccati matrices, $\mathbf{H}_t$ and $\mathbf{h}_t$, for all periods by recursively iterating backwards in time. By substituting (A23) into $\boldsymbol{\lambda}_t = \mathbf{H}_t\xi_t + \mathbf{h}_t$, rearranging terms and solving the resulting equation with respect to $\boldsymbol{\lambda}_t$ we get

$$\boldsymbol{\lambda}_t = \underline{\underline{\mathbf{M}}}_t\left(\mathbf{H}_t\underline{\mathbf{A}}_t\xi_{t-1} + \mathbf{H}_t\mathbf{d}_t + \mathbf{h}_t\right), \tag{A27}$$

where $\mathbf{d}_t$ is given by (59), $\underline{\mathbf{A}}_t$ by (60), and $\underline{\underline{\mathbf{M}}}_t$ by (11). Substituting (A27) into the costate equation (A25) allows identification of the lagged Riccati matrices, $\mathbf{H}_{t-1}$ and $\mathbf{h}_{t-1}$, as defined by equations (52) and (54), respectively. The terminal conditions for the Riccati equations are determined by equations (53) and (55), respectively. Substituting the linear relation between states and costates, $\boldsymbol{\lambda}_t = \mathbf{H}_t\xi_t + \mathbf{h}_t$, into the supplementary state equation, (A23), allows us to write ξ_t as a function of ξ_{t-1}. As stated by equation (56) the supplementary state vector can, thus, be developed as a function of ξ_0 (given by (A24)). Finally, both players' control vectors $\mathbf{u}_{it}$ $(i=1,2)$ can be calculated as determined by equation (62). ■