Algebraic Logic

- Logic: strings, syntax/semantics, axioms, rules.
- Algebra: set, functions, equations.

Group $\mathcal{G} = (G, e, {}^{-1}, \circ)$

G is a set, $e \in G$, $\quad {}^{-1} : G \to G$, $\quad \circ : (G \times G) \to G$.
$\mathcal{G} = (G, e, {}^{-1}, \circ)$ is a <u>group</u> if for all $a, b, c \in G$

Associative	$(a \circ b) \circ c = a \circ (b \circ c)$
Identity	$e \circ a = a \circ e = a$
Inverse	$a \circ a^{-1} = a^{-1} \circ a = e$

Groups, examples

- $(\mathbb{Z}, 0, -, +)$
- $(\mathbb{Q} \setminus \{0\}, 1, (q \mapsto \frac{1}{q}), \times)$
- $(GL(n, \mathbb{R}), \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}, ^{-1}, \circ)$
- $(\{0, 1, \dots, n-1\}, 0, -_n, +_n)$ (integers, modulo n)
- $\mathcal{P}_n = (Perms_n, Id_n, ^{-1}, \circ)$, permutations of n elements.

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Proof. Let $\mathcal{G} = (G, e, ^{-1}, \circ)$ be any group. Define $\theta : G \to \mathcal{P}_G$ by

$$\theta(g) = \{(h, h \circ g) : h \in G\}$$

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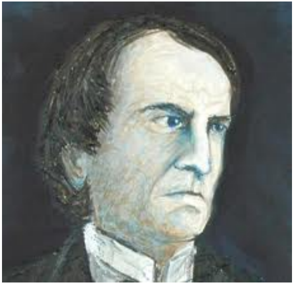
- $(\theta(g) = \theta(h)) \to (g = h)$ (injective)
- $\theta(e) = Id_G,\ \ \theta(g^{-1}) = (\theta(g))^{-1}$ and $\theta(g \circ h) = \theta(g) \circ \theta(h)$

Boolean Algebra



Boolean Algebra Axioms for $\mathcal{B} = (B, 0, 1, +, \cdot, -)$

Axioms for $+$.

$$\begin{array}{rcll} a + (b + c) & = & (a + b) + c & \text{associativity} \\ a + b & = & b + a & \text{commutativity} \\ a + a & = & a & \text{idempotency} \\ 0 + a & = & a & \text{zero law} \end{array}$$

Axioms for $-$.

$$\begin{array}{cc} - - a = a & a + \bar{a} = 1 \\ -1 = 0 & -(a + b) = \bar{a} \cdot \bar{b} \ \text{(de Morgan)} \end{array}$$

Distribution law, and Absorption law

$$a \cdot (b + c) = a \cdot b + a \cdot c, \qquad a + (a \cdot b) = a$$

Any $(B, 0, 1, +, \cdot, -)$ obeying these equations is a boolean algebra.

Example of Boolean Algebra

$$(\{0,1\}, 0, 1, +, \cdot, -)$$

$+$	0	1
0	0	1
1	1	1

$\cdot$	0	1
0	0	0
1	0	1

$$- : \begin{array}{ccc} 0 & \mapsto & 1 \\ 1 & \mapsto & 0 \end{array}$$

Boolean Algebra, Example 2

Take a non-empty set X.

$$\wp(X) = \{\text{all subsets of } X\}$$

So

$$S \in \wp(X) \iff S \subseteq X$$

Then

$$\mathcal{B}(X) = (\wp(X), \emptyset, X, \cup, \cap, \setminus)$$

is a boolean algebra.
Abstract $0, 1, +, \cdot$ are replaced by concrete $\emptyset, X, \cup, \cap$ and $-a$ is replaced by $X \setminus a$.
A <u>boolean set algebra</u> is any subalgebra of $\mathcal{B}(X)$.

Representation

Let $\mathcal{B} = (B, 0, 1, +, \cdot, -)$ be a boolean algebra. An injection

$$\theta : B \to \wp(X)$$

for some base set X is called a representation if

- $\theta(0) = \emptyset$
- $\theta(1) = X$
- $\theta(a + b) = \theta(a) \cup \theta(b)$
- $\theta(a \cdot b) = \theta(a) \cap \theta(b)$
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So θ is an isomorphism from $\mathcal{B}$ into some set algebra $(\wp(X), \emptyset, X, \cup, \cap, \setminus)$.

Order $\leq$

Write $a \leq b \iff a + b = b$.
Questions:

1. in a BA, is $\leq$ (a) reflexive? (b) symmetric? (c) transitive?

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Write $a < b \iff (a \leq b \wedge b \not\leq a)$.

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- In the boolean algebra freely generated by $\{a, b, c\}$ there are eight atoms $a \cdot b \cdot c$, $a \cdot b \cdot \bar{c}$, $\ldots$, $\bar{a} \cdot \bar{b} \cdot \bar{c}$ (here $\bar{a}$ denotes $(-a)$)
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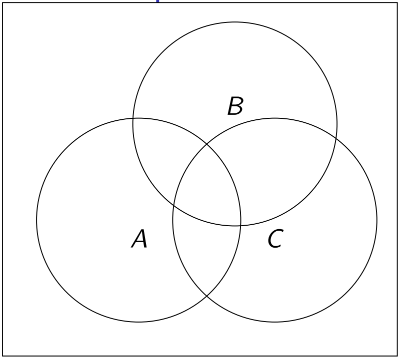
- Write $At(\mathcal{B})$ for the set of all atoms of $\mathcal{B}$.
- A boolean algebra $\mathcal{B}$ is <u>atomic</u> if every non-zero element of $\mathcal{B}$ is above an atom.
- Every finite boolean algebra is atomic.

Atoms, example



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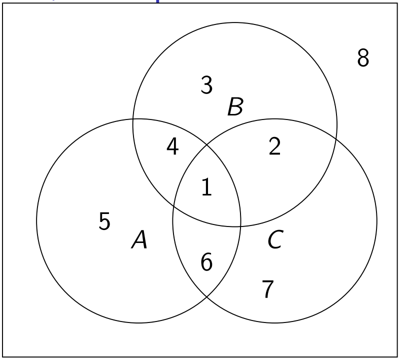


8 Atoms

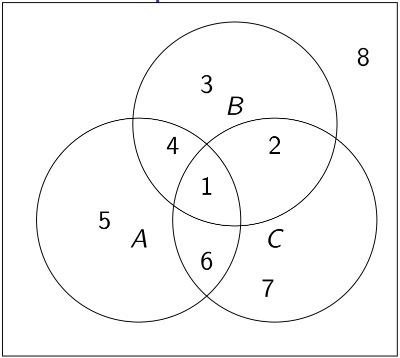
$1 = \quad A \cdot B \cdot C$

$2 = \quad \bar{A} \cdot B \cdot C$

$\vdots$

$8 = \quad \bar{A} \cdot \bar{B} \cdot \bar{C}$

Atoms, example



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$$\vdots$$
$$8 = \quad \bar{A} \cdot \bar{B} \cdot \bar{C}$$

$$A = \quad 1 + 4 + 5 + 6$$
$$B = \quad 1 + 2 + 3 + 4$$
$$C = \quad 1 + 2 + 6 + 7$$

BA has 2^8 elements

All finite boolean algebras are atomic

But there are infinite boolean algebras with no atoms at all.

Suprema, infima

- Let $S \subseteq \mathcal{B},\ b \in \mathcal{B}$.
- b is an upper bound of S if for all $s \in S$ we have $s \leq b$. If so, write $S \leq b$.
- b is least upper bound of S if $S \leq b$ and for all $u \in \mathcal{B}$ if $S \leq u$ then $b \leq u$. If so, b is unique and we write $b = \sum S$.
- If S has no least upper bound $\sum S$ is undefined,
- Similarly $\prod S$ is greatest lower bound if it exists, undefined otherwise.

An atomless boolean algebra

Consider $\wp(\mathbb{N})$. For $S, T \subseteq \mathbb{N}$, let $S \nabla T = (S \setminus T) \cup (T \setminus S)$, the symmetric difference. Define an equivalence relation $\sim$ by

$$S \sim T \iff (S \nabla T) \text{ is finite}$$

Write $[S] = \{T \subseteq \mathbb{N} : T \sim S\}$.
Q: which sets are in $[\emptyset]$? Which are in $[\mathbb{N}]$?
Write $\mathbb{N}/\sim$ for the set of all equivalence classes. Consider

$$\mathcal{B} = (\wp(\mathbb{N})/\sim, [\emptyset], [\mathbb{N}], +, \cdot, -)$$

where $[S] + [T] = [S \cup T]$, $-[S] = [\mathbb{N} \setminus S]$.
$\mathcal{B}$ is an atomless BA.

Atomic Boolean Algebra

If $\mathcal{B}$ is an atomic boolean algebra then for any $b \in \mathcal{B}$ we have

$$b = \sum \{\alpha \in At(\mathcal{B}) : \alpha \leq b\}.$$

Every atomic boolean algebra is representable

Let $\mathcal{B}$ be a boolean algebra, atomic.
The map $\theta : \mathcal{B} \to \wp(At(\mathcal{B}))$ defined by

$$\theta(b) = \{\alpha \in At(\mathcal{B}) : \alpha \leq b\}$$

is a representation of $\mathcal{B}$ over $At(\mathcal{B})$.

Proof that θ is injective

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Claim: $b \cdot \bar{c} \neq 0$. For contradiction, suppose $b \cdot \bar{c} = 0$.

$$b \cdot \bar{c} = 0 \Rightarrow \qquad \begin{aligned} b + c &= (b \cdot 1) + c \\ &= ((b \cdot c) + (b \cdot \bar{c})) + c \\ &= (b \cdot c + 0) + c = (b \cdot c) + c = c \end{aligned}$$

by BA axioms (exercise: which axioms?), contrary to assumption.

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So $b \cdot \bar{c} \neq 0$ hence there is an atom $\alpha \leq b \cdot \bar{c}$, so $\alpha \leq b$, $\alpha \not\leq c$.
Hence $\alpha \in \theta(b) \setminus \theta(c)$, so $\theta(b) \neq \theta(c)$.

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  $\theta(a + b) = \theta(a) \cup \theta(b),\ \theta(a \cdot b) = \theta(a) \cap \theta(b)$
- Preserves negation: $\theta(-a) = At(\mathcal{B}) \setminus \theta(a)$.

Stone's Theorem (1936)

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To make the argument work, (even when the boolean algebra is not atomic) Stone used <u>ultrafilters</u> instead of atoms.

Sums of Products

Boolean Term::= $Var|0|1|(s+t)|(s\cdot t)|-s$. E.g. $-((x+y)\cdot\bar{z})$.
Every boolean term is equal to a sum of products. Let t be a boolean term. Three alternative methods of finding equivalent sum of products.

1. Treat t as if it were a propositional formula (replacing $+,\cdot,-$ by $\vee,\wedge,\neg$). Use tableau to find a DNF equivalent. This is a sum of products.
2. Make a truth table. Then t equals the sum of the products of variables/negated variables in rows that evaluate to 1.
3. Drive down negations. Replace

$$\begin{aligned} -(a+b) &\quad \text{by} \quad \bar{a}\cdot\bar{b} \\ -(a\cdot b) &\quad \text{by} \quad \bar{a}+\bar{b} \\ --a &\quad \text{by } a \end{aligned}$$

   If $\cdot$ is above $+$ then use distribution law. Replace

$$(a+b)\cdot(c+d) \text{ by } a\cdot c+a\cdot d+b\cdot c+b\cdot d$$

Boolean Algebra, Summary

Boolean Algebra is defined by a few axioms.
Every Boolean Algebra is representable as a concrete subalgebra of a boolean set algebra $(\wp(X), \emptyset, X, \cup, \cap, \setminus)$.

Binary Relations

$$a \subseteq X \times X$$

a is a binary relation over base X.

$$\mathsf{Rel}(X) = \{a : a \subseteq X \times X\} = \wp(X \times X)$$

Note

$$(\mathsf{Rel}(X), \emptyset, X \times X, \cup, \cap, (X \times X) \setminus \_)$$

is a boolean set algebra.
Identity relation $\mathsf{Id}_X = \{(x, x) : x \in X\}$.
Converse operator $a^\smile = \{(x, y) : (y, x) \in a\}$.
Composition operator denoted ;,

$$a; b = \{(x, y) : \exists z((x, z) \in a \wedge (z, y) \in b)\}$$

Proper Relation Algebra

A set of relations A $\subseteq$ Rel(X) is called a <u>proper relation algebra</u> if

- A contains a biggest relation $U \in Rel(X)$ and a smallest relation $\emptyset$, and $(A, \emptyset, U, \cup, \cap, U \setminus \_)$ is a boolean set algebra, and
- A includes Id_X, and if $a, b \in A$ then $a^{\smile} \in A$ and $a; b \in A$.

Abstract Relation Algebra

A relation algebra $\mathcal{A}$ has the form

$$\mathcal{A} = (A, 0, 1, +, \cdot, -, 1', {}^\smile, ;)$$

(constants $0, 1, 1'$ unary operations $-, {}^\smile$, binary operations $+, \cdot, ;$) satisfying these axioms

1. $(A, 0, 1, +, \cdot, -)$ is a boolean algebra
2. $(A, 1', ;)$ is a monoid ($;$ is associative and $1'; a = a; 1' = a$)
3. Additivity: $(a+b);(c+d) = a;c + a;d + b;c + b;d$, $(a+b)^\smile = a^\smile + b^\smile$
4. Zero: $0^\smile = 0; a = a; 0 = 0$ (all $a \in \mathcal{A}$)
5. Convolution: $(a^\smile)^\smile = a$, $(a;b)^\smile = b^\smile; a^\smile$
6. Triangle law:

$$(a;b) \cdot c^\smile = 0 \iff (b;c) \cdot a^\smile = 0$$

Representation

An isomorphism θ from abstract $\mathcal{A} = (A, 0, 1, +, \cdot, -, 1', \smile, ;)$ into a proper relation algebra $Rel(X)$ (some non-empty set X) is called a <u>representation</u> of $\mathcal{A}$.
RRA denotes the class of all representable relation algebras. I.e. the closure under isomorphism of the class of proper relation algebras.

RA example $\mathcal{P}$

Three atoms $1', <, >$ (so 8 elements)
Identity $1'$
Converses $<^{\smile} = >, \quad >^{\smile} = <, \; (1')^{\smile} = 1'$,
and composition defined for atoms by

;	$1'$	$<$	$>$
$1'$	$1'$	$<$	$>$
$<$	$<$	$<$	$(1' + < + >)$
$>$	$>$	$(1' + < + >)$	$>$

Converse and composition defined on sums of atoms by additivity.

Representation of $\mathcal{P}$

Base $\mathbb{Q}$, the rational numbers.
Representation θ

$$\theta(1') = Id_{\mathbb{Q}} = \{(q, q) : q \in \mathbb{Q}\}$$
$$\theta(<) = \{(q, r) : q < r,\ q, r \in \mathbb{Q}\}$$
$$\theta(>) = \{(q, r) : q > r,\ q, r \in \mathbb{Q}\}$$

$\mathcal{P}$ has no representation over a finite base

Let $\theta : \mathcal{P} \to Rel(X)$ be a representation, we have to prove that X is infinite.

- There are $x, y \in X$ where $(x, y) \in <^{\theta}$ (since θ is injective)
- $(x, y) \in <^{\theta}$ implies $(x, y) \notin (1')^{\theta}$ hence $x \neq y$ (since $< \cdot 1' = 0$)
- There is z such that $(x, z) \in <^{\theta}$ and $(z, y) \in <^{\theta}$ (since $<=<;<$)
- For any $n \in \mathbb{N}$ there are $z_0, z_1, \ldots, z_{n-1}$ where $(x, z_0), (z_i, z_{i+1}), (z_{n-1}, y) \in <^{\theta}$ (for $i < n-1$), all distinct
- So X is infinite.

Monk Algebra *Mon*

Atoms: $1'$, r, b, self-converse.
Composition:

$;$	$1'$	r	b
$1'$	$1'$	r	b
r	r	$(1' + b)$	$(r + b)$
b	b	$(r + b)$	$(1' + r)$

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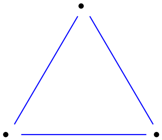
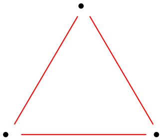
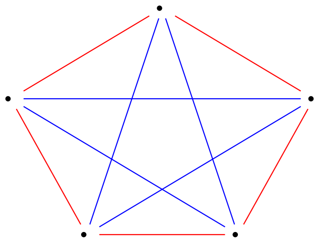
Forbidden:





Representation of *Mon*



Mon has no representation with six or more points

Reason:

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- Between six points there are 15 edges.

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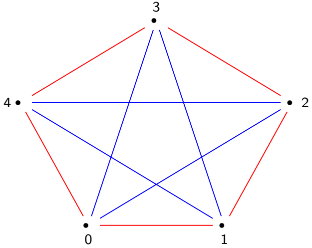
Reason:

- Between six points there are 15 edges.
- If all edges are coloured red or blue then there is a monochrome triangle (forbidden).

Representations of *Mon* are base isomorphic to each other



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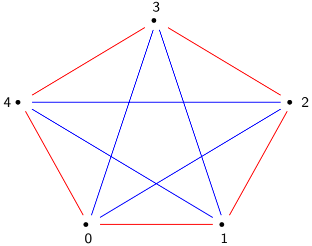

$$\{(0,0),(1,2),(2,4),(3,1),(4,3)\}$$

Monk Algebra *Mon'*

Atoms: $1'$, r_1, r_2, b, all self-converse.

Composition:

;	$1'$	r_1	r_2	b
$1'$	$1'$	r_1	r_2	b
r_1	r_1	$(1' + b)$	b	$r_1 + r_2 + b$
r_2	r_2	b	$1' + b$	$r_1 + r_2 + b$
b	b	$(r_1 + r_2 + b)$	$(r_1 + r_2 + b)$	$(1' + r_1 + r_2)$

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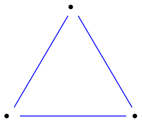
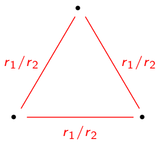
$1'$	$1'$	r_1	r_2	b
r_1	r_1	$(1' + b)$	b	$r_1 + r_2 + b$
r_2	r_2	b	$1' + b$	$r_1 + r_2 + b$
b	b	$(r_1 + r_2 + b)$	$(r_1 + r_2 + b)$	$(1' + r_1 + r_2)$

Forbidden:





Mon' has no representation

Hence $Mon' \in RA \setminus RRA$, and so $RRA \subsetneq RA$.

Problem: is a finite RA $\mathcal{A}$ representable?

- This problem is undecidable.

Problem: is a finite RA $\mathcal{A}$ representable?

- This problem is undecidable.
- We devise a two-player game, to test representability of atomic RAs.

Characterising representability by games

Two players Abelarde and Héloïse written $\forall, \exists$. Fix some finite RA $\mathcal{A}$. $\exists$ claims that $\mathcal{A}$ is representable, $\forall$ tries to show that it is not.

$$\exists \text{ has a winning strategy in } G(\mathcal{A}) \iff \mathcal{A} \text{ is representable}$$

Networks

Let At be the set of atoms of an atomic relation algebra $\mathcal{A}$.
Let $G = (X, X \times X)$ be the complete directed graph with nodes X.
A map $N : (X \times X) \to At$ is called a <u>network</u> if

$$\begin{aligned} N(x,y) \leq 1' &\iff (x = y) \\ N(y,x) &= (N(x,y))^{\smile} \\ N(x,y); N(y,z) &\geq N(x,z) \end{aligned}$$

A play of the game $G_n(\mathcal{A})$

There are n rounds. If n is finite, then $\forall$ and $\exists$ play a sequence of networks

$$N_0 \subseteq N_1 \subseteq N_n$$

A play of $G_\omega(\mathcal{A})$ is

$$N_0 \subseteq N_1 \subseteq \cdots$$

Initial round of $G_n(\mathcal{A})$

$\forall$ picks atom a_0

$$\bullet \xrightarrow{a_0} \bullet$$

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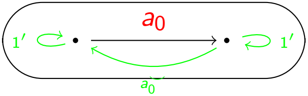
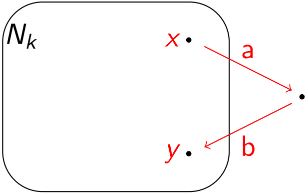
$\forall$ picks atom a_0



$\exists$ completes labelling (forced)

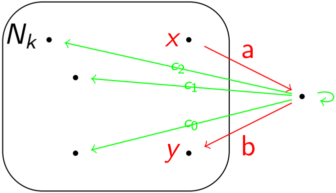
Later round of $G_n(\mathcal{A})$, current network N_k

$\forall$ picks $x, y \in N$ and atoms a, b such that $a; b \geq N_k(x, y$



Later round of $G_n(\mathcal{A})$, current network N_k

$\exists$ chooses atoms $c_0, c_1, \ldots$ to complete the labelling of N_{k+1}



Who wins?

In any round, if $\exists$ cannot play, or if she plays a labelled graph that fails to be an atomic network, then $\forall$ wins.
If $\exists$ plays a legitimate atomic network in each round then she wins.

Characterising representability for finite RAs, by games

Theorem

Let $\mathcal{A}$ be a finite relation algebra.

1. $\mathcal{A} \in RRA$ *iff* $\exists$ *has a winning strategy in* $G_\omega(\mathcal{A})$.
2. $\exists$ *has a winning strategy in* $G_\omega(\mathcal{A})$ *iff she has one in* $G_n(\mathcal{A})$ *for all finite* n.
3. *One can construct first-order sentences* σ_n *for* $n < \omega$ *such that* $\mathcal{A} \models \sigma_n$ *iff* $\exists$ *has a winning strategy in* $G_n(\mathcal{A})$.

Conclude that for a finite relation algebra $\mathcal{A}$,

$$\mathcal{A} \in RRA \iff \mathcal{A} \models \{\sigma_n : n \in \mathbb{N}\}.$$

For infinite atomic relation algebras $\mathcal{A}$

$$\mathcal{A} \models \{\sigma_n : n \in \mathbb{N}\} \Rightarrow (\mathcal{A} \in RRA)$$

Ramsay Numbers

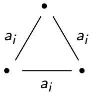
Let K_n be the complete irreflexive undirected graph on n nodes $nodes = \{0, 1, \ldots, n-1\}$.
$E(K_n) = \{\{x, y\} : x \neq y \in nodes\}$.
Given n colours $Col_k = \{C_0, C_1, \ldots, C_{k-1}\}$, a k colour edge colouring of K_n is a map $f : E(K_n) \to Col_k$ such that for all $x, y, z \in nodes$, we have $|\{f\{x, y\}, f\{x, z\}, f\{y, z\}\}| > 1$.

n	$M(n)$
0	2
1	3
2	6
3	17
$\vdots$	
n	$2 + n \cdot (M(n-1) - 1)$

K_{M_n} has no n colour edge colouring.

$Mon_n : n \geq 0$

- $n+1$ atoms $At = \{1', a_i : i < n\}$ all self-converse.
- $1'; x = x; 1' = x$ (all $x \in At$)
- $$a_i ; a_j = \begin{cases} -a_i & = 1' + \sum_{j \neq i < n} a_j \quad \text{if } i = j \\ -1' & = \sum_{k<n} a_k \quad \text{if } i \neq j < n \end{cases}$$
- Forbid

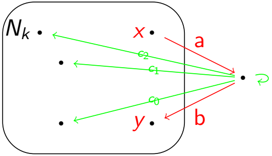
  $: i < n$
- No representation can have $M(n)$ points or more.

Mon'_n is not representable

Split atom a_0 into M_n parts, so atoms of Mon'_n are $\{a_0^i : i < M_n\} \cup \{1', a_i : 1 \leq i < n\}$.
The maximum possible size of a representation of Mon'_n is at most $M(n) - 1$, not big enough to witness all the atoms.
Hence Mon'_n has no representation at all.

$\exists$ has a w.s. in $G_n(Mon'_n)$

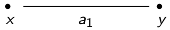
$\forall$ picks $x, y \in N$ and atoms a, b such that $a; b \geq N_k(x, y)$



Note: $|N_k| \leq n$

$\exists$ chooses atoms $c_0, c_1, \ldots$ with distinct colours (distinct from colours of a, b too)

$\forall$ has a w.s. in $G_{M(n)}(Mon'_n)$



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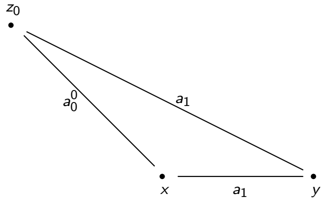

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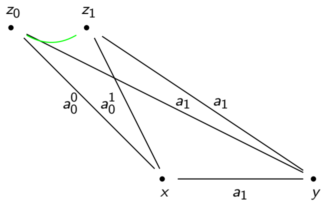

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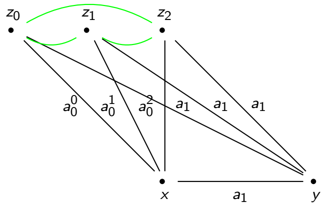

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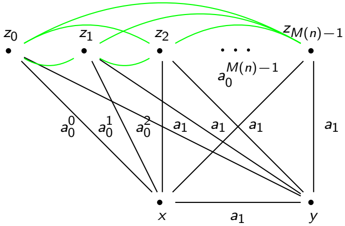

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- Hence RRA cannot be defined by finitely many axioms.