COMP0009 Predicate Tableaus

October 17, 2023

First Order Tableaus

- A **literal** is an atom or its negation (i.e. $r^n(t_1, \ldots, t_n)$ or $\neg r^n(t_1, \ldots, t_n)$ where r is a predicate and t_i is a term)
- A **closed term** is a term that contains no variables (e.g. constants, functions over constants, etc.)
- We use the same kind of tableau construction that we used for propositional logic...

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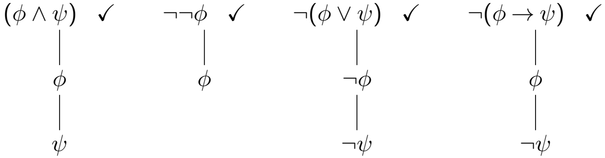
- A **closed term** is a term that contains no variables (e.g. constants, functions over constants, etc.)
- We use the same kind of tableau construction that we used for propositional logic...
- ...but we need new expansion rules to deal with the quantifiers!

Expansion Rules

α formulas Add both formulas in one branch at each leaf below current node. Tick current node.



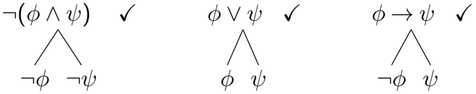
$(\phi \wedge \psi)$ ✓ — ϕ — ψ

$\neg\neg\phi$ ✓ — ϕ

$\neg(\phi \vee \psi)$ ✓ — $\neg\phi$ — $\neg\psi$

$\neg(\phi \rightarrow \psi)$ ✓ — ϕ — $\neg\psi$

β formulas Make two separate branches (one with each formula) at each leaf below current node. Tick current node.



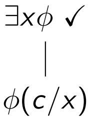
$\neg(\phi \wedge \psi)$ ✓ — $\neg\phi$ | $\neg\psi$

$\phi \vee \psi$ ✓ — ϕ | ψ

$\phi \rightarrow \psi$ ✓ — $\neg\phi$ | ψ

δ formulas

Choose new constant c (not included in tableau so far). Add formula at each leaf below current node. Tick current node.



$$\exists x \phi \ \checkmark$$

$$\phi(c/x)$$

δ formulas

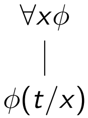
Choose new constant c (not included in tableau so far). Add formula at each leaf below current node. Tick current node.

$$\begin{array}{c} \exists x \phi \checkmark \\ | \\ \phi(c/x) \end{array} \qquad \begin{array}{c} \neg \forall x \phi \checkmark \\ | \\ \neg \phi(c/x) \end{array}$$

Observe: $\neg \forall x \phi \equiv \exists x \neg \phi$

γ formulas

Pick any closed term t. Add formula at each leaf below current node. Do not tick the node.



$\forall x \phi$

$\phi(t/x)$

γ formulas

Pick any closed term t. Add formula at each leaf below current node. Do not tick the node.

$$\begin{array}{c} \forall x \phi \\ | \\ \phi(t/x) \end{array} \qquad \begin{array}{c} \neg \exists x \phi \\ | \\ \neg \phi(t/x) \end{array}$$

Observe: $\neg \exists x \phi \equiv \forall x \neg \phi$

Which expansion rule should be applied to the first-order formulae below?

- $\forall x \, \neg p(x)$
- $H(a) \to F(a)$
- $\neg\neg\exists y \; p(y)$
- $\neg(\forall x \, \neg p(x) \to \neg\exists y \; p(y))$
- $\neg(\forall x \neg q(x) \lor \exists x \forall y \neg(x < y))$
- $G(a) \to H(a)$
- $\neg\neg(\forall x \, (G(x) \to H(x)) \land \forall x \, (H(x) \to F(x)) \land G(a) \land \neg\exists x \, (G(x) \land F(x)))$
- $\neg\exists x \, (G(x) \land F(x))$
- $\neg\forall y \neg(c < y)$

Which expansion rule should be applied to the first-order formulae below?

- $\forall x \, \neg p(x)$ **(γ rule)**
- $H(a) \rightarrow F(a)$
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- $\neg\exists x \, (G(x) \wedge F(x))$ **(γ rule)**
- $\neg\forall y \neg(c < y)$ **(δ rule)**

Example 3: Is formula $(\forall x \neg p(x) \rightarrow \neg \exists y\ p(y))$ valid?

$$(1)\ \neg(\forall x\ \neg p(x) \rightarrow \neg \exists y\ p(y))$$

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(1) $\neg(\forall x\ \neg p(x) \rightarrow \neg \exists y\ p(y))$ ✓

$\alpha(1)$

(2) $\forall x\ \neg p(x)$

(3) $\neg\neg\exists y\ p(y)$

Example 3: Is formula $(\forall x \neg p(x) \to \neg\exists y\ p(y))$ valid?

$$(1)\ \neg(\forall x\ \neg p(x) \to \neg\exists y\ p(y)) \quad \checkmark$$

$\alpha(1)$

$$(2)\ \forall x\ \neg p(x)$$

$$(3)\ \neg\neg\exists y\ p(y) \quad \checkmark$$

$\alpha(3)$

$$(4)\ \exists y\ p(y)$$

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$(1)\ \neg(\forall x\ \neg p(x) \to \neg\exists y\ p(y))$ ✓

$\alpha(1)$

$(2)\ \forall x\ \neg p(x)$

$(3)\ \neg\neg\exists y\ p(y)$ ✓

$\alpha(3)$

$(4)\ \exists y\ p(y)$ ✓

$\delta(4, c)$

$(5)\ p(c)$

Example 3: Is formula $(\forall x \neg p(x) \rightarrow \neg \exists y\ p(y))$ valid?

$(1)\ \neg(\forall x\ \neg p(x) \rightarrow \neg \exists y\ p(y))\quad \checkmark$

$\alpha(1)$

$(2)\ \forall x\ \neg p(x)\quad \checkmark$

$(3)\ \neg\neg \exists y\ p(y)\quad \checkmark$

$\alpha(3)$

$(4)\ \exists y\ p(y)\quad \checkmark$

$\delta(4, c)$

$(5)\ p(c)$

$\gamma(2, c)$

$(6)\ \neg p(c)$

$\oplus$

Closed tableau — **root formula is unsatisfiable, original formula is valid.**

Example 4: Is formula $\neg(\forall x\,(Gx \to Hx) \land \forall x\,(Hx \to Fx) \land Ga \land \neg\exists x\,(Gx \land Fx))$ valid?

$$(1)\ \neg\neg(\forall x\,(Gx \to Hx) \land \forall x\,(Hx \to Fx) \land Ga \land \neg\exists x\,(Gx \land Fx))$$

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$$(1)\, \neg\neg(\forall x\,(Gx \to Hx) \land \forall x\,(Hx \to Fx) \land Ga \land \neg\exists x\,(Gx \land Fx)) \quad \checkmark$$

$$\scriptstyle \alpha(1) \displaystyle \,\big|$$

$$(2)\, \forall x\,(Gx \to Hx) \land \forall x\,(Hx \to Fx) \land Ga \land \neg\exists x\,(Gx \land Fx)$$

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(1) $\neg\neg(\forall x\,(Gx \to Hx) \land \forall x\,(Hx \to Fx) \land Ga \land \neg\exists x\,(Gx \land Fx))$ ✓

$\alpha(1)$

(2) $\forall x\,(Gx \to Hx) \land \forall x\,(Hx \to Fx) \land Ga \land \neg\exists x\,(Gx \land Fx)$ ✓

$\alpha(2)$

(3) $\forall x\,(Gx \to Hx)$

(4) $\forall x\,(Hx \to Fx)$

(5) Ga

(6) $\neg\exists x\,(Gx \land Fx)$

Example 4: Is formula $\neg(\forall x\,(Gx \to Hx) \land \forall x\,(Hx \to Fx) \land Ga \land \neg\exists x\,(Gx \land Fx))$ valid?

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$\alpha(1)$

$$(2)\ \forall x\,(Gx \to Hx) \land \forall x\,(Hx \to Fx) \land Ga \land \neg\exists x\,(Gx \land Fx) \quad \checkmark$$

$\alpha(2)$

$$(3)\ \forall x\,(Gx \to Hx)$$

$$(4)\ \forall x\,(Hx \to Fx)$$

$$(5)\ Ga$$

$$(6)\ \neg\exists x\,(Gx \land Fx)$$

$\gamma(6, a)$

$$(7)\ \neg(Ga \land Fa)$$

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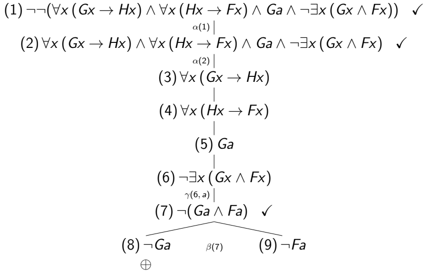


(1) $\neg\neg(\forall x\,(Gx \rightarrow Hx) \land \forall x\,(Hx \rightarrow Fx) \land Ga \land \neg\exists x\,(Gx \land Fx))$ ✓

$\alpha(1)$

(2) $\forall x\,(Gx \rightarrow Hx) \land \forall x\,(Hx \rightarrow Fx) \land Ga \land \neg\exists x\,(Gx \land Fx)$ ✓

$\alpha(2)$

(3) $\forall x\,(Gx \rightarrow Hx)$

(4) $\forall x\,(Hx \rightarrow Fx)$

(5) Ga

(6) $\neg\exists x\,(Gx \land Fx)$

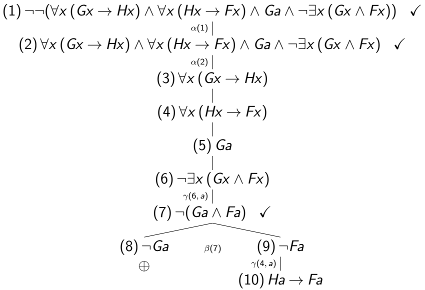
$\gamma(6, a)$

(7) $\neg(Ga \land Fa)$ ✓

(8) $\neg Ga$ $\beta(7)$ (9) $\neg Fa$

$\oplus$

Example 4: Is formula $\neg(\forall x\,(Gx \rightarrow Hx) \wedge \forall x\,(Hx \rightarrow Fx) \wedge Ga \wedge \neg\exists x\,(Gx \wedge Fx))$ valid?



(1) $\neg\neg(\forall x\,(Gx \rightarrow Hx) \wedge \forall x\,(Hx \rightarrow Fx) \wedge Ga \wedge \neg\exists x\,(Gx \wedge Fx))$ ✓

$\alpha(1)$

(2) $\forall x\,(Gx \rightarrow Hx) \wedge \forall x\,(Hx \rightarrow Fx) \wedge Ga \wedge \neg\exists x\,(Gx \wedge Fx)$ ✓

$\alpha(2)$

(3) $\forall x\,(Gx \rightarrow Hx)$

(4) $\forall x\,(Hx \rightarrow Fx)$

(5) Ga

(6) $\neg\exists x\,(Gx \wedge Fx)$

$\gamma(6, a)$

(7) $\neg(Ga \wedge Fa)$ ✓

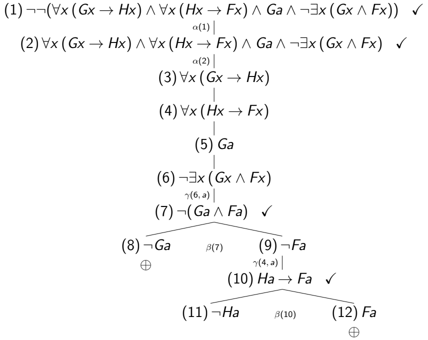
(8) $\neg Ga$ $\beta(7)$ (9) $\neg Fa$

$\oplus$

$\gamma(4, a)$

(10) $Ha \rightarrow Fa$

Example 4: Is formula $\neg(\forall x\,(Gx \rightarrow Hx) \wedge \forall x\,(Hx \rightarrow Fx) \wedge Ga \wedge \neg\exists x\,(Gx \wedge Fx))$ valid?



(1) $\neg\neg(\forall x\,(Gx \rightarrow Hx) \wedge \forall x\,(Hx \rightarrow Fx) \wedge Ga \wedge \neg\exists x\,(Gx \wedge Fx))$ ✓

$\alpha(1)$

(2) $\forall x\,(Gx \rightarrow Hx) \wedge \forall x\,(Hx \rightarrow Fx) \wedge Ga \wedge \neg\exists x\,(Gx \wedge Fx)$ ✓

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(3) $\forall x\,(Gx \rightarrow Hx)$

(4) $\forall x\,(Hx \rightarrow Fx)$

(5) Ga

(6) $\neg\exists x\,(Gx \wedge Fx)$

$\gamma(6, a)$

(7) $\neg(Ga \wedge Fa)$ ✓

(8) $\neg Ga$ $\quad\beta(7)\quad$ (9) $\neg Fa$

$\oplus$

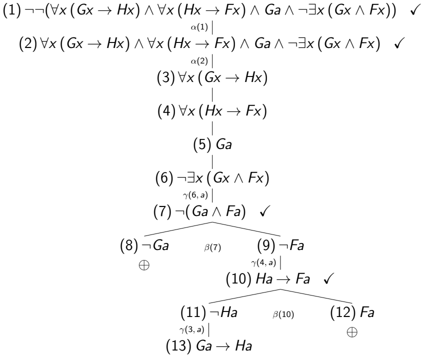
$\gamma(4, a)$

(10) $Ha \rightarrow Fa$ ✓

(11) $\neg Ha$ $\quad\beta(10)\quad$ (12) Fa

$\oplus$

Example 4: Is formula $\neg(\forall x\,(Gx \to Hx) \land \forall x\,(Hx \to Fx) \land Ga \land \neg\exists x\,(Gx \land Fx))$ valid?



(1) $\neg\neg(\forall x\,(Gx \to Hx) \land \forall x\,(Hx \to Fx) \land Ga \land \neg\exists x\,(Gx \land Fx))$ ✓

$\alpha(1)$

(2) $\forall x\,(Gx \to Hx) \land \forall x\,(Hx \to Fx) \land Ga \land \neg\exists x\,(Gx \land Fx)$ ✓

$\alpha(2)$

(3) $\forall x\,(Gx \to Hx)$

(4) $\forall x\,(Hx \to Fx)$

(5) Ga

(6) $\neg\exists x\,(Gx \land Fx)$

$\gamma(6, a)$

(7) $\neg(Ga \land Fa)$ ✓

(8) $\neg Ga$

$\oplus$

$\beta(7)$

(9) $\neg Fa$

$\gamma(4, a)$

(10) $Ha \to Fa$ ✓

(11) $\neg Ha$

$\gamma(3, a)$

(13) $Ga \to Ha$

$\beta(10)$

(12) Fa

$\oplus$

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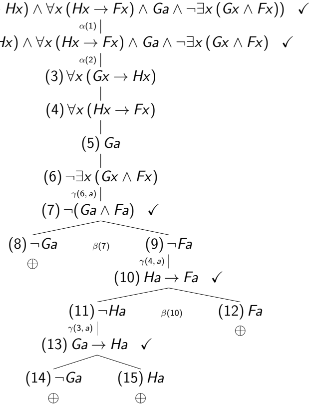

(1) $\neg\neg(\forall x\,(Gx \to Hx) \land \forall x\,(Hx \to Fx) \land Ga \land \neg\exists x\,(Gx \land Fx))$ ✓

$\alpha(1)$

(2) $\forall x\,(Gx \to Hx) \land \forall x\,(Hx \to Fx) \land Ga \land \neg\exists x\,(Gx \land Fx)$ ✓

$\alpha(2)$

(3) $\forall x\,(Gx \to Hx)$

(4) $\forall x\,(Hx \to Fx)$

(5) Ga

(6) $\neg\exists x\,(Gx \land Fx)$

$\gamma(6, a)$

(7) $\neg(Ga \land Fa)$ ✓

(8) $\neg Ga$ $\beta(7)$ (9) $\neg Fa$

$\oplus$

$\gamma(4, a)$

(10) $Ha \to Fa$ ✓

(11) $\neg Ha$ $\beta(10)$ (12) Fa

$\gamma(3, a)$ $\oplus$

(13) $Ga \to Ha$ ✓

(14) $\neg Ga$ (15) Ha

$\oplus$ $\oplus$

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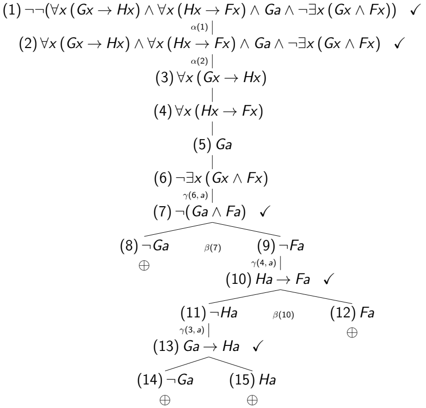

(1) $\neg\neg(\forall x\,(Gx \to Hx) \wedge \forall x\,(Hx \to Fx) \wedge Ga \wedge \neg\exists x\,(Gx \wedge Fx))$ ✓

$\alpha(1)$

(2) $\forall x\,(Gx \to Hx) \wedge \forall x\,(Hx \to Fx) \wedge Ga \wedge \neg\exists x\,(Gx \wedge Fx)$ ✓

$\alpha(2)$

(3) $\forall x\,(Gx \to Hx)$

(4) $\forall x\,(Hx \to Fx)$

(5) Ga

(6) $\neg\exists x\,(Gx \wedge Fx)$

$\gamma(6, a)$

(7) $\neg(Ga \wedge Fa)$ ✓

(8) $\neg Ga$ $\beta(7)$ (9) $\neg Fa$

$\oplus$

$\gamma(4, a)$

(10) $Ha \to Fa$ ✓

(11) $\neg Ha$ $\beta(10)$ (12) Fa

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(13) $Ga \to Ha$ ✓

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$\oplus$ $\oplus$

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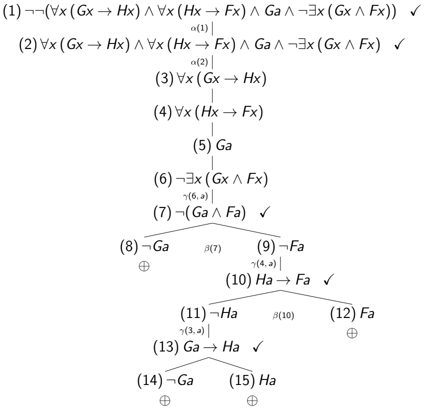

(1) $\neg\neg(\forall x\,(Gx \to Hx) \land \forall x\,(Hx \to Fx) \land Ga \land \neg\exists x\,(Gx \land Fx))$ ✓

$\alpha(1)$

(2) $\forall x\,(Gx \to Hx) \land \forall x\,(Hx \to Fx) \land Ga \land \neg\exists x\,(Gx \land Fx)$ ✓

$\alpha(2)$

(3) $\forall x\,(Gx \to Hx)$

(4) $\forall x\,(Hx \to Fx)$

(5) Ga

(6) $\neg\exists x\,(Gx \land Fx)$

$\gamma(6, a)$

(7) $\neg(Ga \land Fa)$ ✓

(8) $\neg Ga$ $\oplus$ — $\beta(7)$ — (9) $\neg Fa$

$\gamma(4, a)$

(10) $Ha \to Fa$ ✓

(11) $\neg Ha$ — $\beta(10)$ — (12) Fa $\oplus$

$\gamma(3, a)$

(13) $Ga \to Ha$ ✓

(14) $\neg Ga$ $\oplus$ — (15) Ha $\oplus$

Closed tableau — **root formula is not satisfiable and the original formula is valid.**

Example 5: Is formula $\forall x \neg q(x) \vee \exists x \forall y \neg(x < y)$ valid?

$(1)\ \neg(\forall x \neg q(x) \vee \exists x \forall y \neg(x < y)) \quad \checkmark$

$\alpha(1)$

$(2)\ \neg \forall x \neg q(x) \quad \checkmark$

$(3)\ \neg \exists x \forall y \neg(x < y)$

$\delta(2, c)$

$(4)\ \neg\neg q(c) \quad \checkmark$

$\alpha(4)$

$(5)\ q(c)$

$\gamma(3, c)$

$(6)\ \neg \forall y \neg(c < y) \quad \checkmark$

$\delta(6, d)$

$(7)\ \neg\neg(c < d) \quad \checkmark$

$\alpha(7)$

$(8)\ (c < d)$

$\gamma(3, d)$

$(9)\ \neg \forall y \neg(d < y) \quad \checkmark$

$\delta(9, e)$

$(10)\ \neg\neg(d < e) \quad \checkmark$

$\alpha(10)$

$(11)\ (d < e)$

...

Open tableau — the tableau will never close, hence the **root formula is satisfiable and the original formula is not valid.**

Alternative: tableaux as lists

Recall

literal	$p, \neg p$
α	$(\phi_1 \wedge \phi_2), \neg(\phi_1 \vee \phi_2), \neg(\phi_1 \to \phi_2), \neg\neg\phi$
β	$(\phi_1 \vee \phi_2), (\phi_1 \to \phi_2), \neg(\phi_1 \wedge \phi_2)$

Formula expansions

α	α_1	α_2
$(A \wedge B)$	A	B
$\neg(A \vee B)$	$\neg A$	$\neg B$
$\neg(A \to B)$	A	$\neg B$
$\neg\neg A$	A	—

β	β_1	β_2
$(A \vee B)$	A	B
$(A \to B)$	$\neg A$	B
$\neg(A \wedge B)$	$\neg A$	$\neg B$

Propositional Tableaux

- A **theory** Σ is a set of propositional formulas.
- If $p, \neg p \in \Sigma$ theory is **contradictory**, write $C(\Sigma)$
- If each formula in Σ is a literal, theory is <u>fully expanded</u>, write $\mathrm{Exp}(\Sigma)$
- A **tableau** is a list of theories. Think of these as alternative theories.

Tableau(ϕ)

```
Initialise Tab $= [\{\phi\}]$
while Not empty Tab do
  $\Sigma =$ Dequeue(Tab)
  if Exp($\Sigma$) and NOT C($\Sigma$) then
    Output SATISFIABLE
  else
    Pick non-literal $\psi \in \Sigma$
    switch ($\psi$)
    case $\alpha$:
      $\Sigma = \Sigma[\alpha/\{\alpha_1, \alpha_2\}]$, if NOT C($\Sigma$) and $\Sigma \notin$ Tab then enqueue $\Sigma$
    case $\beta$:
      $\Sigma_1 = \Sigma[\beta/\beta_1]$, if $\Sigma_1 \notin$ Tab and NOT C($\Sigma_1$) then enqueue $\Sigma_1$
      $\Sigma_2 = \Sigma[\beta/\beta_2]$, if $\Sigma_2 \notin$ Tab and NOT C($\Sigma_2$) then enqueue $\Sigma_2$
    end switch
  end if
end while
(Empty Tab) Output UNSATISFIABLE
```

Predicate Tableaux

Under switch statement add cases δ and γ.

```
**switch** $(\psi)$
**case** $\delta = \exists x\theta(x)$:
    $\Sigma := \Sigma[\exists x\theta(x)/\theta(c)]$ (new const $c$),
**case** $\delta = \neg\forall x\theta(x)$:
    $\Sigma := \Sigma[\neg\forall x\theta(x)/\neg\theta(c)]$ (new const $c$),
**case** $\gamma = \forall x\theta(x)$:
    pick closed term $t$ occurring in $\Sigma$,  $\Sigma = \Sigma \cup \{\theta(t)\}$.
**case** $\gamma = \neg\exists x\theta(x)$:
    fill in
**end switch**
If $\Sigma \not\in Tab$ and NOT $C(\Sigma)$ enqueue $\Sigma$.
```

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