Epistemic Logic

Knowledge and belief.
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E.g. $K_I(B_{\text{Pope}}(God))$ means 'I know that the Pope believes in God'.

Knowledge

True Belief.

Knowledge

Justified and True Belief.

Knowledge — modal view

Frame (W, R) where W is the set of all worlds that an agent believes are possible, and let $(w, w') \in R$ hold iff everything we know about w is also true of w' and vice versa.
Then R is an equivalence relation. The K_s modality is an S5 modality. Axioms:

- $K_s p \to p$
- $K_s p \to K_s K_s p$
- $p \to K_s(\neg K_s(\neg p))$.

Belief

Don't have $B_s p \rightarrow p$.

Belief

Don't have $B_s p \to p$.
Instead $\neg B_s(\bot)$.

Temporal Logic

Temporal formula::=

$$\mathbf{prop}|\neg\phi|(\phi \vee \phi')|\mathbf{F}\phi|\mathbf{P}\phi$$

$\mathbf{F}\phi$ means "ϕ will be true at some point in the future" and $\mathbf{P}\phi$ means "ϕ was true at some point in the past". Formally, in a Kripke frame $(\mathbf{W}, \mathbf{R})$ with valuation $\mathbf{v}$, world $\mathbf{x} \in \mathbf{W}$,

$$(\mathbf{W}, \mathbf{R}), \mathbf{v}, \mathbf{x} \models \mathbf{F}\phi \iff \exists \mathbf{y} \in \mathbf{W}((\mathbf{x}, \mathbf{y}) \in \mathbf{R} \wedge (\mathbf{W}, \mathbf{R}), \mathbf{v}, \mathbf{y} \models \phi)$$
$$(\mathbf{W}, \mathbf{R}), \mathbf{v}, \mathbf{x} \models \mathbf{P}\phi \iff \exists \mathbf{y} \in \mathbf{W}((\mathbf{y}, \mathbf{x}) \in \mathbf{R} \wedge (\mathbf{W}, \mathbf{R}), \mathbf{v}, \mathbf{y} \models \phi)$$

Other temporal connectives

Write $\mathbf{G}\phi$ as abbreviation for $\neg\mathbf{F}\neg\phi$ and write $\mathbf{H}\phi$ for $\neg\mathbf{P}\neg\phi$.
What do these formulas mean?

Basic temporal axioms (K_t)

- Axioms for propositional logic
- $G(p \to q) \to (Gp \to Gq)$
- $H(p \to q) \to (Hp \to Hq)$
- $p \to GPp$
- $p \to HFp$.

Linear Time

- K_t

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- $Gp \to GGp$ (transitivity)

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- $Gp \rightarrow GGp$ (transitivity)
- $(Fp \wedge Fq) \rightarrow (F(p \wedge Fq) \vee F(p \wedge q) \vee F(q \wedge Fp))$

Linear Time

- K_t
- $Gp \rightarrow GGp$ (transitivity)
- $(Fp \wedge Fq) \rightarrow (F(p \wedge Fq) \vee F(p \wedge q) \vee F(q \wedge Fp))$
- $(Pp \wedge Pq) \rightarrow (P(p \wedge Pq) \vee P(p \wedge q) \vee P(q \wedge Pp))$ (totality)

Multimodal Logic

$$\phi ::= prop|\neg\phi|(\phi_1 \vee \phi_2)|\Diamond_i\phi|\Box_i\phi \;\; (i = 0, 1, 2, \ldots, k-1)$$

Multimodal Kripke Frame

$$(W, R_0, R_1, \ldots, R_{k-1}) \;\; (R_i \subseteq W \times W)$$

Semantics $(W, R_0, \ldots, R_{k-1}), v, x \models \Diamond_i\phi \iff$

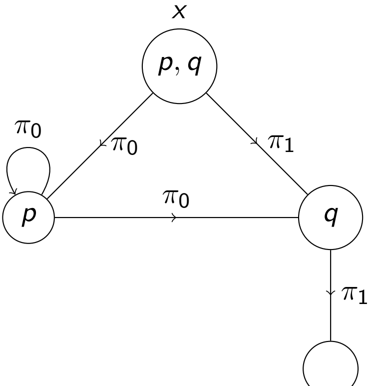
$$\exists y \in W((x, y) \in R_i \wedge (W, R_0, \ldots, R_{k-1}), v, y \models \phi)$$

Propositional Dynamic Logic (PDL)

- Propositions $P = \{p_0, p_1, p_2, \ldots\}$
- Atomic programs $\Pi = \{\pi_0, \pi_1, \ldots\}$
- Programs $\pi' ::= \pi_i | (\pi; \rho) | (\pi + \rho) | \pi^* | \phi?$ where π_i is an atomic program, π, ρ are programs, ϕ is a formula.
- Formulas $\phi ::= p | \neg\phi_1 | (\phi_1 \vee \phi_2) | \langle\pi\rangle\phi_1 | [\pi]\phi_1$ where ϕ_1, ϕ_2 are formulas and π is a program.

PDL Example

$$\mathcal{F}, v, x \models p \wedge q \wedge [\pi_0^*](p \vee (q \wedge \langle \pi_1 \rangle (\neg p \wedge \neg q)))$$



Programming constructs in PDL

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"While p do π" can be written as

$$(p?;\pi)^*;(\neg p)?$$

PDL Axioms and Rules

Axioms

- Propositional Axioms
- $[\pi](A \to B) \to ([\pi]A \to [\pi]B)$ (axiom K)
- $[\pi;\rho]A \leftrightarrow [\pi][\rho]A$
- $[\pi + \rho]A \leftrightarrow ([\pi]A \land [\rho]A)$
- $[\pi^*]A \leftrightarrow (A \land [\pi][\pi^*]A)$
- $[A?]B \leftrightarrow (A \to B)$

Rules

Modus Ponens From A and $A \to B$ deduce B

Necessitation From A deduce $[\pi]A$

Loop Invariance From $A \to [\pi]A$ deduce $A \to [\pi^*]A$.

Partial Program Correctness

Let A, B be formulas, let π be a program.
"Given precondition A, every terminating execution of π gives a state where B holds", can be written as the PDL formula

$$(A \to [\pi]B)$$