COMP0009, First-order axiomatic proofs

November 2, 2023

Recap: Axiomatic Proofs for Propositional Logic

I. $p \to (q \to p)$

II. $(p \to (q \to r)) \to ((p \to q) \to (p \to r))$

III. $((\neg p \to \neg q) \to (q \to p))$

IV. $p \to \neg\neg p$ and $\neg\neg p \to p$

V. $(p \vee q) \leftrightarrow (\neg p \to q)$

VI. $(p \wedge q) \leftrightarrow \neg(p \to \neg q)$

Modus Ponens:

$$\frac{A, (A \to B)}{B}$$

Theorem Proving for Predicate Logic. Axiomatic Proofs

Take axioms schemas for propositional logic.

Quantifier Axioms:

VIII. $(\forall x \neg A \leftrightarrow \neg \exists x A)$

IX. $(\forall x A(x) \rightarrow A(t/x))$ if t is substitutable for x in A.

X. $(\forall x(A \rightarrow B) \rightarrow (\forall x A \rightarrow \forall x B))$.

Equality Axioms:

XI. $(x = x)$

XII. $(x = y) \rightarrow (y = x)$

XIII. $((x = y) \rightarrow (t(x) = t(y/x)))$

XIV. $((x = y) \rightarrow (A(x) \rightarrow A(y/x)))$ if y is substitutable for x in A.

An instance of any of the axioms above is obtained by replacing A, B, C etc. by arbitrary formulas.

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Q: Why don't we need $((x = y) \wedge (y = z)) \rightarrow (x = z)$?

Substitutability

IX. $(\forall x A(x) \to A(t/x))$ if t is substitutable for x in A.

t is substitutable for x in A if no variable in t becomes bound when replacing x in A by t.

The following is not an instance of Axiom **??**:

$$\forall x \exists y (x = y) \to \exists y (f(y) = y)$$

Here the variable y in $t := f(y)$ becomes bound.

Inference Rules

Modus Ponens

$$\frac{A, (A \to B)}{B}$$

Universal Generalisation

$$\frac{A(x)}{\forall x A(x)}$$

(N.B. $A(x) \to \forall x A(x)$ is not an axiom, as it is not valid. Universal generalisation says that if $A(x)$ is valid then $\forall x A(x)$ is also valid. This rule is sound.)

Proofs

A proof of ϕ is a finite sequence

$$\phi_0, \phi_1, \phi_2, \ldots, \phi_n = \phi$$

such that, for each $i \leq n$, either

- ϕ_i is an instance of one of the axioms or
- ϕ_i is obtained from ϕ_j (and maybe ϕ_k) where $j, k < i$, by an inference rule.

Write

$$\vdash \phi$$

in this case.

Proving from hypotheses

So far, this is all to do with validity over arbitrary models. If you want to find validities in a particular model, or a particularly type of model, then you can add hypotheses.
These hypotheses are formulas which are valid in the type of formula you want, and they define it.
E.g. Linearly Ordered Models
Hypotheses:

$$\forall x \forall y (x < y \lor y < x \lor x = y)$$
$$\forall x \neg (x < x)$$
$$\forall x \forall y \forall z ((x < y \land y < z) \to x < z)$$

Proofs with hypotheses

Let Γ be a set of hypotheses. Write

$$\Gamma \vdash \phi$$

if there is a sequence

$$\phi_0, \phi_1, \ldots, \phi_n = \phi$$

such that for each $i \leq n$ either

- ϕ_i is an axiom,
- ϕ_i is obtained from ϕ_j (ϕ_k) (some $j, k < i$) by an inference rule, or
- $\phi_i \in \Gamma$.

Example Proof using Hypotheses

Linear Order $\vdash \forall x \forall y \neg (x < y \wedge y < x)$

Proof

1. $\forall x \forall y \forall z ((x < y \wedge y < z) \rightarrow x < z)$ (Hypothesis)

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1. $\forall x \forall y \forall z ((x < y \land y < z) \to x < z)$ (Hypothesis)
2. $\forall x \forall y \forall z ((x < y \land y < z) \to (x < z)) \to$
   $\forall y \forall z ((x < y \land y < z) \to x < z)$ (Ax. **??**)

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5. $(x < y \land y < x) \to x < x)$ (Ax. **??**+Modus Ponens)
6. $((x < y \land y < x) \to x < x) \to (\neg(x < x) \to \neg(x < y \land y < x))$
   (instance of $(p \to q) \to (\neg q \to \neg p)$, provable in propositional logic)

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7. $\neg(x < x) \rightarrow \neg(x < y \wedge y < x)$ (Modus Ponens)

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8. $\forall x \neg (x < x)$ (Hypothesis)
9. $\neg (x < x)$ (Ax. **??**+Modus Ponens)
10. $\neg (x < y \wedge y < x)$ (Modus Ponens)

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10. $\neg(x < y \wedge y < x)$ (Modus Ponens)
11. $\forall x \forall y \neg(x < y \wedge y < x)$ ($2\times$ Universal Generalisation)