COMP0009 Logic and Databases
Exercises 6: First order compactness.

Robin Hirsch

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Let $\vdash$ be a proof system for first order logic (denoting either proof by an axiom system, by tableau or by some other method, sound and complete for first order validities.).

1. Let Γ be a set of first order sentences and let ϕ be a single sentence. Explain what $\Gamma \vdash \phi$ means.
2. Explain the notation $\Gamma \models \phi$.
3. Explain what it means when we say that a set Σ of L-formulas is *inconsistent*.
4. What does it mean when we say $\vdash$ is sound? What does it mean when we say $\vdash$ is strongly complete?
5. State the compactness theorem for first order logic.

Let $L = L(C, F, P)$ be a signature with constants $C = \{0, 1\}$, functions $F = \{+, \times\}$ and predicates $\{<, =\}$. Let $N = (\mathbb{N}, I)$ be the L-structure whose domain is the set of natural numbers, where $I(0) = 0,\ I(1) = 1$, where $I(+), I(\times)$ denote ordinary addition and multiplication of natural numbers (respectively) and where $I(<), I(=)$ denote ordinary less than or equal (respectively) on natural numbers.

6. Write down a closed term t in this language that denotes 7.
7. Which elements of the domain $\mathbb{N}$ are named by closed terms in N?

Let Σ be the set of all L-sentences true in N.

8. Write down an L-sentence in Σ. Write down an L sentence *not* in Σ.

Now let L^+ be the same signature as L, but also including one new constant symbol ω. Consider the infinite theory

$$\Sigma^+ = \Sigma \cup \{t < \omega : t \text{ is a closed } L\text{-term}\}$$

9. Let F be a finite subset of Σ^+. Prove that F is consistent. [Hint: find a *model* of F based on N, but with a suitable interpretation of ω.]
10. Use the compactness theorem to prove that Σ^+ has a model $N^+ = (\mathbb{N}^+, I^+)$.
11. If $m \in \mathbb{N}^+$ is named by a closed L-term t, then we say that m is a *standard* number, other elements of $\mathbb{N}^+$ are *non-standard*. Write down several closed L^+-terms denoting distinct, non-standard numbers.

12. Is $\forall x \forall y(x \times y = y \times x)$ true in N^+?

13. Is there a number $m \in \mathbb{N}^+$ such that $\mathbb{N}^+ \models m+1 = \omega$? If so, is m interpretted as a standard number?

14. We say that m is a predecessor of n if $m+1 = n$. Which elements of $\mathbb{N}^+$ have predecessors?

15. The principal of induction can be written as the second order formula $\forall P((P(0) \wedge \forall x(P(x) \rightarrow P(x+1))) \rightarrow \forall x P(x))$, where P is a unary predicate. Does the principal of induction hold in N^+?