COMP0009 Logic and Databases
Modal Logic

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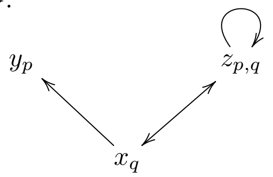
December 7, 2022

A modal propositional formula is either a proposition $p, q, r, \ldots$, a negated modal formula, a disjunction/conjunction/implication of two formulas, a box formula or a diamond formula. A Kripke frame $\mathcal{F} = (W, R)$ consists of a set W of worlds and a binary relation $R \subseteq W \times W$ of arrows. In other words, a Kripke frame is just a directed graph.

1. Consider the Kripke frame $\mathcal{F} = (\{x, y, z\}, \{(x, y), (x, z), (z, x), (z, z)\})$ and let v be the valuation $v(p) = \{y, z\},\ v(q) = \{x, z\}$.



   Are the following true?

   (a) $\mathcal{F}, v, x \models \Box q$

   (b) $\mathcal{F}, v, x \models (q \wedge (q \to \Box p))$

   (c) $\mathcal{F}, v, x \models \Diamond\Box\bot$, where $\bot$ is any unsatisfiable formula, e.g. $(p \wedge \neg p)$.

2. Write down a modal fromula that is true at a world w in a Kripke frame (regardless of the valuation) if and only if there are no outgoing edges from w (or no successors of w).

3. Let $\mathcal{F}$ be one of the three Kripke Frames: (i)$\mathcal{Q} = (\mathbb{Q}, <)$, (ii) $\mathcal{N} = (\mathbb{N}, <)$, (iii) $\mathcal{N}^- = (\mathbb{N}, >)$, where $\mathbb{N}$ is the set of natural numbers, $\mathbb{Q}$ is the set of rational numbers and $<$ denotes strict inequality (reversed for $>$). For each modal formula below, state whether the formula is valid in the frame $\mathcal{F}$.

   (a) $\Box p \to p$

   (b) $\Diamond(p \vee \neg p)$

   (c) $\Diamond\Diamond p \to \Diamond p$

   (d) $\Box\Box p \to \Box p$

   (e) $\Box\bot \vee \Diamond\Box\bot$.

4. (a) Suppose $\mathcal{F} \models \Diamond\top$ for some Kripke frame $\mathcal{F}$ where $\top$ is any valid formuula, e.g. $p \vee \neg p$. What does this tell you about $\mathcal{F}$? What if $\mathcal{F}$ is transitive and irreflexive?

   (b) Find a Kripke frame $\mathcal{F}$ such that $\mathcal{F} \models (\Diamond\Box p \to \Box\Diamond p)$. Find a Kripke frame $\mathcal{G}$ such that $\mathcal{G} \not\models (\Diamond\Box p \to \Box\Diamond p)$. Can you describe the frame property that this formula defines?