COMP0009, Compactness

November 15, 2023

Proofs are Finite

If

$$\Sigma \vdash \phi$$

then

$$\Sigma_0 \vdash \phi$$

for some finite subset Σ_0 of Σ.

Consistency and Inconcistency

Let Σ be a set of formulas. We say Σ is inconsistent if

$$\Sigma \vdash (p \wedge \neg p)$$

Otherwise (there is no such proof) we say Σ is consistent.

Compactness Theorem

Let Σ be a set of formulas.

Theorem

Σ has a model if and only if each finite subset of Σ has a model.

Proof of Compactness Theorem

If Σ has a model then trivially every finite subset has a model.
Now suppose Σ has no model. Then $\Sigma \models \bot$. By strong completeness

$$\Sigma \vdash \bot$$

Since proofs are finite, there is a finite subset $\Sigma_0 \subseteq \Sigma$ where $\Sigma_0 \vdash \bot$.
By soundness, $\Sigma_0 \models \bot$ so Σ_0 has no model. Hence, if every finite subset of Σ has a model then Σ has a model.

Problem

Let E be a binary relation denoting the edges of a graph, let $=$ denote equality. For $k \geq 0$ we say there is a path of length k from x to y if there is a sequence $(x_0, x_1, \ldots, x_{k-1})$, where $x = x_0,\ y = x_{k-1}$ and for all $i < k-1$ we have $(x_i, x_{i+1}) \in E$. Write down first order formulas $\phi_0(x, y),\ \phi_1(x, y),\ \phi_2(x, y), \ldots, \phi_k(x, y)$ with two free variables x, y, meaning

- There is a path of length 0 from node x to node y.
- There is a path of length 1 from node x to node y,
- There is a path of length 2 from x to y
- There is a path of length 3 from x to y
- There is a path of length k from x to y, where $k \geq 1$ is fixed.

Connected Graphs

A directed graph (G, E) consists of a set of nodes G and a binary relation $E \subseteq G \times G$. A directed graph $)G, E)$ is connected if for all $x, y \in G$ there is a path of length k from x to y where $k \geq 0$ is a finite integer.

First Order Logic Cannot Define Connectedness

Language

$$\begin{aligned} C &= \{c, d\} \\ F &= \emptyset \\ P &= \{=, E\} \text{ (both binary)} \end{aligned}$$

Suppose, for contradiction, that

$$G \models \Sigma \iff G \text{ is connected}$$

Let

$$\begin{aligned} \phi_0(x, y) &= (x = y) \\ \phi_{n+1}(x, y) &= \exists z(\phi_n(x, z) \wedge E(z, y)) \end{aligned}$$

"there is a path of length $n+1$ from x to y".
Consider

$$\Sigma \cup \{\neg\phi_1(c, d), \neg\phi_2(c, d), \ldots\}$$

Every finite subset has a model (what model?). By compactness, the whole set has a model, say G,

$$G \models \Sigma \cup \{\neg\phi_n(c, d) : n = 1, 2, 3, \ldots\}$$

G is therefore connected (since a model of Σ), but there is no path from c to d — a contradiction.

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By compactness, $\Sigma \cup F$ has a model — finite and infinite — contradiction.

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By compactness, $\Sigma \cup F$ has a model — finite and infinite — contradiction.
Therefore no such theory F exists.

Compactness theorem and non-standard analysis

Let

$$\Sigma = \{\text{all valid statements about } \mathbb{N}\}$$

in a language with constants $0, 1, 2, \ldots$ functions $+, \times$ and predicate $=$.
E.g. $2 + 2 = 4 \in \Sigma$.
Also $\forall x \forall y (x \times y = y \times x) \quad \in \Sigma$.
Let c be another constant symbol.
Every finite subset of

$$\Sigma^+ = \Sigma \cup \{c \neq 0, c \neq 1, c \neq 2, \ldots, c \neq n, \ldots\}$$

has a model (what model?).
Therefore Σ^+ has a model.

Non-standard real analysis

Let L be similar but with a constant for every real number. Let

$$\Sigma = \{\text{all valid statements about } \mathbb{R}\}$$

and

$$\Sigma^+ = \Sigma \cup \{\alpha > r : r \in \mathbb{R}\}$$

Every finite subset of Σ^+ has a model (just interpret α as a sufficiently big real number), therefore Σ^+ has a model M. Then $[\alpha]^M$ is an "infinitely big" real number and $[\frac{1}{\alpha}]^M$ is and "infinitesimally small" positive real number.

Can do calculus perfectly rigorously in this way. Can show that

$$\forall x((|x| < r) \rightarrow (x = St(x) + Inf(x)))$$

where r is a constant for any positive real, $St(x)$ is a "standard real" and $Inf(x)$ is an "infinitesimal real". Then let

$$f'(x) = St\left(\frac{f(x+\delta x) - f(x)}{\delta x}\right)$$

where x is any standard real and δx is any infinitesimal, provided this does not depend on the choice of δx.

Summary

- Σ is consistent iff Σ has a model (soundness and strong completeness)
- $\Sigma \vdash \phi \iff \exists \Sigma_0 \subseteq_f \Sigma,\ \Sigma_0 \vdash \phi$ (compactness)
- Σ has a model iff each finite subset of Σ has a model)
- No first order theory defines connected graphs
- $\mathbb{N}$ has a non-standard model.