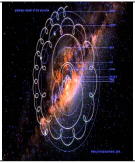
MODAL LOGIC

Some Theology

St. Anselm of Canterbury (1033–1109)

Ontological Proof of the Existence of God

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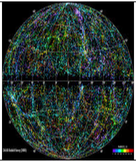

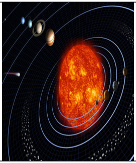

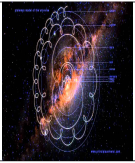

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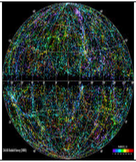		
		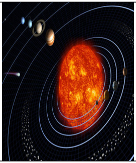
	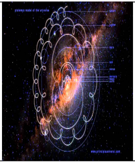	

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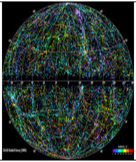
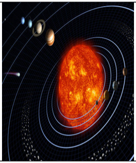
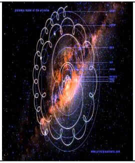
		
		
		

Ontological Proof of the Existence of God

Modal Operators: $\Diamond$ and $\Box$

$\Diamond p$ could mean

- "p is possible"
- "p might happen"
- "p will happen (at some time in the future)"
- "I think p could be true"

$\Box p = \neg\Diamond\neg p$. For each of the above, what does $\Box p$ mean?

Syntax

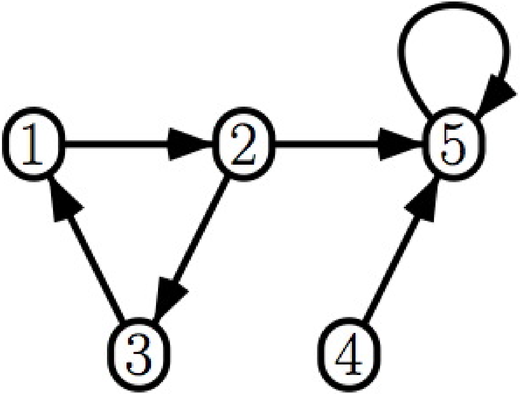
$$
\begin{aligned}
prop &::= p|q|r|\ldots \\
\phi &::= prop|\neg\phi|(\phi \wedge \phi)|(\phi \vee \phi)|(\phi \rightarrow \phi)|\Diamond\phi|\Box\phi
\end{aligned}
$$

e.g.

$$(\Diamond p \rightarrow \Box\Diamond p)$$

Frames

$\mathcal{F} = (W, R)$ where $R \subseteq W \times W$ (a directed graph).



Valuation:

$V : \mathit{prop} \to \wp(W)$ e.g.

$$
\begin{aligned}
V(p) &= \{1, 3, 5\} \\
V(q) &= \{1, 2\} \\
V(r) &= \emptyset
\end{aligned}
$$

Semantics

Model $\mathcal{M} = (W, R, V)$. Let $v \in W$.

$$
\begin{aligned}
\mathcal{M}, v \models p &\iff v \in V(p) \\
\mathcal{M}, v \models \neg\phi &\iff \mathcal{M}, v \not\models \phi \\
\mathcal{M}, v \models (\phi \wedge \theta) &\iff \mathcal{M}, v \models \phi \text{ and } \mathcal{M}, v, \models \theta \\
\mathcal{M}, v \models \Diamond\phi &\iff \text{there is } w \in W\ (v, w) \in R \text{ and } \mathcal{M}, w \models \phi \\
\mathcal{M}, v \models \Box\phi &\iff \text{for all } w \in W \text{ if } (v, w) \in R \text{ then } \mathcal{M}, w \models \phi
\end{aligned}
$$

Validity

- Valid in a model

$$(W, R, V) \models \phi \iff \text{for all } v \in W \ (W, R, V), v \models \phi$$

- Valid in a frame

$$(W, R) \models \phi \iff \text{for all valuations } V \ (W, R, V) \models \phi$$

- Valid over a class of frames

$$\mathcal{K} \models \phi \iff \text{for all frames } \mathcal{F} \in \mathcal{K} \ \mathcal{F} \models \phi$$

Valid Formulas

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- $\Box(p \to q) \to (\Box p \to \Box q)$

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- $\Box(p \wedge q) \to (\Box p \wedge \Box q)$
- $\Box(p \to q) \to (\Box p \to \Box q)$

But not

- $\Box(p \vee q) \to (\Box p \vee \Box q)$

Axioms and Rules for Modal Logic K

Rules:

- Modus Ponens ($\vdash A$ and $\vdash (A \to B)$ implies $\vdash B$).
-

Axioms

- Axioms for propositional logic
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- Necessitation ($\vdash A$ implies $\vdash \Box A$).

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Axioms and Rules for Modal Logic K

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- Modus Ponens ($\vdash A$ and $\vdash (A \rightarrow B)$ implies $\vdash B$).
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Axioms

- Axioms for propositional logic
- $\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$.

Defining Classes of Frames

ϕ defines $\mathcal{K}$ means "$\mathcal{F} \models \phi \iff \mathcal{F} \in \mathcal{K}$"

Property	First-order def	Modal Defining Formula
Reflexive	$\forall w Rww$	$\Box p \to p$
Transitive	$\forall u \forall v \forall w ((Ruv \wedge Rvw) \to Ruw)$	$\Box p \to \Box\Box p$
Symmetric	$\forall u \forall v (Ruv \to Rvu)$	$p \to \Box\Diamond p$
Dense	$\forall u \forall v (Ruv \to \exists w (Ruw \wedge Rwv))$	$\Box\Box p \to \Box p$

Equivalent Form

$$
\begin{aligned}
(\Box\Box p \to \Box p) &\equiv (\neg\Box p \to \neg\Box\Box p) \\
&\equiv (\Diamond q \to \Diamond\Diamond q)
\end{aligned}
$$

where $q = \neg p$.

Modal Logics

Class of frames	First-order def	Modal Logic
All frames	$w = w$	$K = \text{MP, Nec, Prop}, \Box(A \to B) \to (\Box A \to \Box B)$
Reflexive frames	$\forall w Rww$	$T = K + (\Box p \to p)$
Reflexive and Transitive	$\ldots \wedge \forall u \forall v \forall w((Ruv \wedge Rvw) \to Ruw)$	$S4 = T + (\Box p \to \Box\Box p)$
Equivalence Rel	$\ldots \wedge \forall u \forall v(Ruv \to Rvu)$	$S5 = S4 + (p \to \Box\Diamond p)$
$\vdots$		

Soundness and Completeness

Let $\mathcal{K}$ be one of those classes (all frames, reflexive, reflexive and transitive, reflexive symmetric transitive) let A be the conjunction of corresponding axioms, write $\vdash_A$ for 'provable using A' with Modus Ponens and Necessitation. For any frame $\mathcal{F} = (W, R)$

- $\mathcal{F} \models A \iff \mathcal{F} \in \mathcal{K}$,

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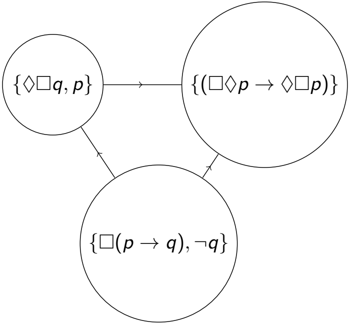
- $\mathcal{F} \models A \iff \mathcal{F} \in \mathcal{K}$, so A defines the class.
- $\mathcal{K} \models \phi \iff \vdash_A \phi$, so $\vdash_A$ is sound and complete for $\mathcal{K}$.

Excercise

Find a modal formula ϕ valid in exactly one of $(\mathbb{N}, <), (\mathbb{Q}, <)$.

Labelled Frames

$((W, E), \lambda)$ where $\lambda : W \to \wp(fmlas)$



Modal Tableau

To check satisfiability of ϕ, make a queue of labelled frames. Initially, one frame in queue, with one world $\{w\}$ and label $\lambda(w) = \{\phi\}$.

- While queue is not empty, dequeue $((W, E), \lambda)$.
- If there is $w \in W$ and $\{p, \neg p\} \subseteq \lambda(w)$ then don't enqueue.
- If there are only boxed formulas, negated diamond formulas, and literals in $range(\lambda)$ then halt and return YES, model found.
- Else, pick $w \in W$ and a formula $\theta \in \lambda(w)$, non-literal, not a box formula, not a negated diamond.
- If θ is an α-formula let $\lambda'(w) := (\lambda(w) \setminus \{\theta\}) \cup \{\alpha_1, \alpha_2\}$ (else $\lambda'(v) = \lambda(v),\ v \neq w$), enqueue $((W, E), \lambda')$.
- If θ is a β-formula let λ_1, λ_2 be same as λ except $\lambda_1(w) = (\lambda(w) \setminus \{\theta\}) \cup \{\beta_1\}$ and $\lambda_2(w) = (\lambda(w) \setminus \{\theta\}) \cup \{\beta_2\}$. Enqueue $((W, E), \lambda_1)$, enqueue $((W, E), \lambda_2)$.
- If $\theta = \Diamond A$ (or $\neg\Box A$) then let $W' = W \cup \{v\}$ (new), $E' = E \cup \{(w, v)\}$ and λ' be same as λ except $\lambda'(v) = \{A\} \cup \{B : \Box B \in \lambda(w)\} \cup \{\neg B : \neg\Diamond B \in \lambda(w)\}$ (respectively, $\lambda'(v) = \{\neg A\} \cup \{B : \Box B \in \lambda(w)\} \cup \{\neg B : \neg\Diamond B \in \lambda(w)\}$), and $\lambda'(w) = \lambda(w) \setminus \{\theta\}$. Enqueue $((W', E'), \lambda')$.
- If queue is empty return NO.

Tableau for reflexive frames

As before, but initially $E = \{(w, w)\}$ and whenever v is added to W also add (v, v) to E. Also, whenever $\Box A \in \lambda(w)$ also include $A \in \lambda(w)$, similarly if $\neg\Diamond A \in \lambda(w)$ also include $\neg A$ in $\lambda(v)$.

Tableau for symmetric frames

For diamond formulas in world w, when you add a new world v and an add an edge (w, v), also add an edge (v, w). Any box formulas (or negated diamonds) in v back propagate to w.

Tableau for transitive frames

When you have a diamond formula at w and add a new world v and add an edge (w, v), also add an edge (u, v) whenever $(u, w) \in E$. Include A in $\lambda(v)$ whenever $\Box A \in \lambda(u)$, include $\neg A$ in $\lambda(v)$ whenever $\neg\Diamond A \in \lambda(u)$, when $(u, w) \in E$.
NB. May not terminate over transitive frames.

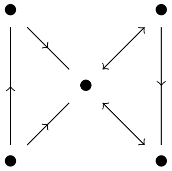
Frame and Model p-morphism

Let (W, R) and (W', R') be frames. A function $f : W \to W'$ is called a <u>frame p-morphism</u> if

- $(x, y) \in R$ implies $(f(x), f(y)) \in R'$ (a homomorphism), and
- if $(f(x), z) \in R'$ then there is $y \in W$ such that $z = f(y)$ and $(x, y) \in R$.

Let $v : prop \to \wp(W)$ and $v' : prop \to \wp(W')$ be valuations.
If f is a frame p-morphism from (W, R) to $(W', R")$ and $x \in v(p) \iff f(x) \in v'(p)$ then f is a <u>model p-morphism</u> of model (W, R, v) to (W', R', v').

Frame p-morphism



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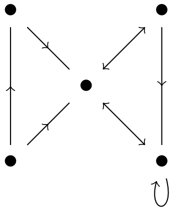

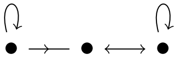

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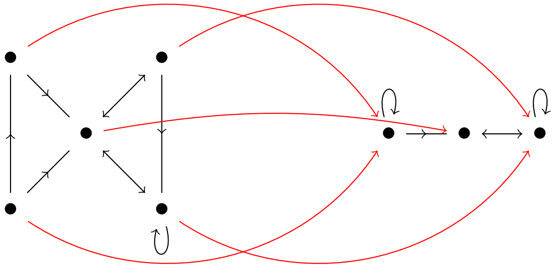

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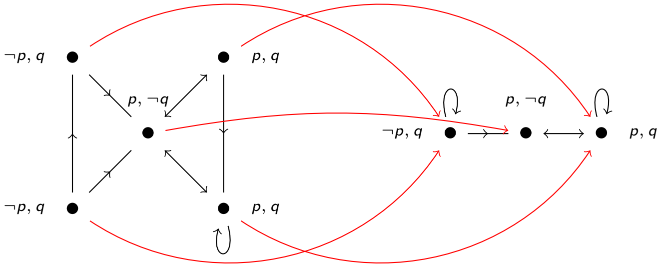
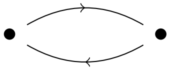


Frame p-morphism example

$\mathcal{F}_2 = (\{a, b\}, \{(a, b), (b, a)\})$



$\mathcal{F}_1 = (\{c\}, \{(c, c)\})$

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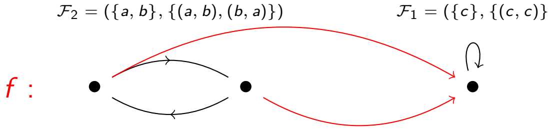

p-morphism theorem

If f is a model p-morphism from model (W, R, v) to model (W', R', v') and ϕ is any modal formula, for any $x \in W$ we have

$$W, R, v, x \models \phi \iff W', R', v', f(x) \models \phi$$

Proof. Base Case: $\phi = p$, by definition of p-morphism.

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Case $\phi = \Diamond A$.

$$\begin{aligned} W, R, v, w \models \Diamond A \iff & \exists u \in W,\ (w, u) \in R,\ W, R, v, u \models A \\ & \Rightarrow (f(w), f(u)) \in R' \wedge W', R', v', f(u) \models A \\ & \Rightarrow W', R', v', f(w) \models \Diamond A \\ & \Rightarrow \exists z \in W'((f(w), z) \in R' \wedge (W', R', v', z \models A)) \\ & \Rightarrow \exists x \in W(f(x) = z \wedge (w, x) \in R \wedge (W, R, v, x \models A)) \end{aligned}$$

Class of irreflexive frames is not modally definable

A frame (W, R) is irreflexive if for all $x \in W$ we have $(x, x) \notin R$. Suppose for contradiction that modal formula ϕ defines irreflexive frames, i.e. $(W, R) \models \phi \iff (W, R)$ is irreflexive.
Let $\mathcal{F}_2(\{a, b\}, \{(a, b), (b, a)\})$ be a two world irreflexive frame, let $\mathcal{F}_1 = (\{c\}, \{(c, c)\}$ be the one world reflexive frame, and let $f : \mathcal{F}_2 \to \mathcal{F}_1$ be p-morphism: $a, b \mapsto c$. $\mathcal{F}_1$ is not irreflexive, so $\mathcal{F}_1 \not\models \phi$. Hence there is a valuation v such that $\mathcal{F}_1, v, c \not\models \phi$. Define a valuation w over $\mathcal{F}_2$ by

$$w(p) = \begin{cases} \{a, b\} & \text{if } v(p) = \{c\} \\ \emptyset & \text{if } v(p) = \emptyset \end{cases}$$

Then f is a model p-morphism from $(\mathcal{F}_2, w)$ to the $(\mathcal{F}_1, v)$. Hence $\mathcal{F}_2, w, x \not\models \phi$ (all $x \in \{a, b\}$). So $\mathcal{F}_2 \not\models \phi$, but $\mathcal{F}_2$ is irreflexive, so this is a contradiction.