Challenging problem 19

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and latex-tikz code from

https://github.com/SavaranaDatta/AI1103/tree/main/Challenging_problem_19.tex

1 PROBLEM(CSIR UGC NET EXAM (Dec 2012) Q 51)

Suppose X_1, X_2, X_3, X_4 are i.i.d random variables taking values 1 and -1 with probability $1/2$ each. Then $E\left(X_1 + X_2 + X_3 + X_4\right)^4$ equals

1) 4
2) 76
3) 16
4) 12

2 SOLUTION

Definition 1. *The n^{th} moment of a random variable X is definied to be $E(X^n)$.*

Lemma 2.1. *The kth moment of* $\mathbf{X}$ *is the coefficient of $\frac{s^k}{k!}$ in the Taylor series expansion of $M_X(s)$.*

$$M_X(s) = E(e^{sX}) = \sum_{k=0}^{\infty} E(X^k)\frac{s^k}{k!}. \tag{2.0.1}$$

Let PMF of random variable $\boldsymbol{X}$ is given by

$$P_X(n) = \begin{cases} 0.5 & \text{n=1} \\ 0.5 & \text{n=-1} \end{cases} \tag{2.0.2}$$

$\boldsymbol{Z}$-transform of $\boldsymbol{X}$ is

$$P_X(z) = 0.5(z^{-1} + z) \tag{2.0.3}$$

Let Y be a random variable, defined as

$$Y = X_1 + X_2 + X_3 + X_4 \tag{2.0.4}$$

$\boldsymbol{Z}$-transform of Y is

$$\begin{aligned} P_Y(z) &= P_{X_1}(z)P_{X_2}(z)P_{X_3}(z)P_{X_4}(z) && (2.0.5) \\ &= (0.5)^4(z^{-1} + z)^4 && (2.0.6) \\ &= (0.5)^4\left(\sum_{k=0}^{4} {}^4\mathrm{C}_k z^{-2(k-2)}\right) && (2.0.7) \end{aligned}$$

$$P_Y(z) = \begin{cases} {}^4\mathrm{C}_0(0.5)^4 & \text{z=-4} \\ {}^4\mathrm{C}_1(0.5)^4 & \text{z=-2} \\ {}^4\mathrm{C}_2(0.5)^4 & \text{z=0} \\ {}^4\mathrm{C}_3(0.5)^4 & \text{z=2} \\ {}^4\mathrm{C}_4(0.5)^4 & \text{z=4} \end{cases} \tag{2.0.8}$$

The Moment Generating Function of $P_Y(z)$

$$\begin{aligned} M_Y(z) &= E[e^{zY}] && (2.0.9) \\ &= \sum_{r=0}^{4} {}^4\mathrm{C}_r(0.5)^4 e^{2(r-2)z} && (2.0.10) \end{aligned}$$

From the lemma 2.1 we can conclude that $E(Y^4)$= Coefficient of $\frac{z^4}{4!}$ in $M_Y(z)$.

$$\begin{aligned} &= {}^4\mathrm{C}_0(0.5)^4(-4)^4 + {}^4\mathrm{C}_1(0.5)^4(-2)^4 + \\ &\quad {}^4\mathrm{C}_3(0.5)^4(2)^4 + {}^4\mathrm{C}_4(0.5)^4(4)^4 && (2.0.11) \end{aligned}$$

$$\begin{aligned} E(Y^4) &= 16 + 4 + 4 + 16 && (2.0.12) \\ &= 40 && (2.0.13) \end{aligned}$$