GATE Assignment 4

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Download all python codes from

https://github.com/SavaranaDatta/EE3900/tree/main/GATE Assignment4/codes

and latex codes from

https://github.com/SavaranaDatta/EE3900/tree/main/GATE Assignment4/main.tex

1 Problem(GATE 2005(EC) 2.25)

A linear system is equivalently represented by two sets of state equations $\dot{X} = AX + BU$ and $\dot{W} = CW + DU$. The eigen values of the representations are also computed as $[\lambda]$ and $[\mu]$. Which of the following statements are true?

1) $[\lambda] = [\mu]$ and $X = W$
2) $[\lambda] = [\mu]$ and $X \neq W$
3) $[\lambda] \neq [\mu]$ and $X = W$
4) $[\lambda] \neq [\mu]$ and $X \neq W$

2 Solution

Definition 1 (State Space representation)**.** *It is a mathematical model of a physical system, as a set of input, output and state variables related by first order difference or differential equations. The most general state representation of a linear system with p inputs, q outputs, and n state variables can be written as*

$$\dot{\mathbf{X}} = \mathbf{AX} + \mathbf{BU} \tag{2.0.1}$$

$$\mathbf{Y} = \mathbf{CX} + \mathbf{DU} \tag{2.0.2}$$

where, $\mathbf{X} \in R^n$ *is the state vector,* $\mathbf{Y} \in R^q$ *is the output vector,* $\mathbf{U} \in R^p$ *is input vector,* $\mathbf{A} \in R^{n\times n}$ *is the state matrix,* $\mathbf{B} \in R^{n\times p}$ *is input matrix,* $\mathbf{C} \in R^{q\times n}$ *is output matrix,* $\mathbf{D} \in R^{q\times p}$ *is feedthrough matrix.*

Definition 2 (Eigen values of State Space representation)**.** *These are the solutions of the charecteristic equation*

$$\triangle(\lambda) = det(\lambda\mathbf{I} - \mathbf{A}) = 0 \tag{2.0.3}$$

where A is the state matrix.

Theorem 2.1. *Consider the n-dimensional continuous time linear system*

$$\dot{\mathbf{X}} = \mathbf{AX} + \mathbf{BU}, \mathbf{Y} = \mathbf{CX} + \mathbf{DU} \tag{2.0.4}$$

Let $\mathbf{T}$ *be an* $n \times n$ *real non-singular matrix and let* $\bar{\mathbf{X}} = \mathbf{TX}$*. Then the state equation*

$$\dot{\bar{\mathbf{X}}} = \bar{\mathbf{A}}\bar{\mathbf{X}} + \bar{\mathbf{B}}\mathbf{U}, \mathbf{Y} = \bar{\mathbf{C}}\bar{\mathbf{X}} + \bar{\mathbf{D}}\mathbf{U} \tag{2.0.5}$$

where $\bar{\mathbf{A}} = \mathbf{TAT}^{-1}, \bar{\mathbf{B}} = \mathbf{TB}, \bar{\mathbf{C}} = \mathbf{CT}^{-1}, \bar{\mathbf{D}} = \mathbf{D}$ *is said to be equivalent to* (2.0.4).

Proof. Given, $\dot{\mathbf{X}} = \mathbf{AX} + \mathbf{BU}$ and $\mathbf{Y} = \mathbf{CX} + \mathbf{DU}$, T is a non-singular matrix such that $\bar{\mathbf{X}} = \mathbf{TX}$. The same system can be defined using $\bar{\mathbf{X}}$ as the state,

$$\dot{\bar{\mathbf{X}}} = \mathbf{T}\dot{\mathbf{X}} = \mathbf{TAX} + \mathbf{TBU} \tag{2.0.6}$$

$$= \mathbf{TAT}^{-1}\bar{\mathbf{X}} + \mathbf{TBU} \tag{2.0.7}$$

$$\mathbf{Y} = \mathbf{CX} + \mathbf{DU} = \mathbf{CT}^{-1}\bar{\mathbf{X}} + \mathbf{DU} \tag{2.0.8}$$

□

Theorem 2.2. *Equivalent state space representations have same set of eigen values*

Proof. For the representation in (2.0.4), the eigen values $[\lambda]$ are such that

$$\mathbf{Ax} = \lambda\mathbf{x} \tag{2.0.9}$$

$$\Rightarrow (\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = 0 \tag{2.0.10}$$

$$\Rightarrow det(\mathbf{A} - \lambda\mathbf{I} = 0 \tag{2.0.11}$$

For the representation in (2.0.5), the eigen values $[\mu]$, are such that

$$\bar{\mathbf{A}}\mathbf{x} = \mu\mathbf{x} \tag{2.0.12}$$

$$\Rightarrow (\bar{\mathbf{A}} - \mu\mathbf{I})\mathbf{x} = 0 \tag{2.0.13}$$

$$\Rightarrow (\mathbf{TAT}^{-1} - \mu\mathbf{TT}^{-1})\mathbf{x} = 0 \tag{2.0.14}$$

$$\Rightarrow det(\mathbf{T}(\mathbf{A} - \mu\mathbf{I})\mathbf{T}^{-1}) = 0 \tag{2.0.15}$$

$$\Rightarrow det(\mathbf{A} - \mu\mathbf{I}) = 0 \tag{2.0.16}$$

Hence, equivalent state space representations have same set of eigen values. □

Here,

$$\dot{\mathbf{X}} = \mathbf{AX} + \mathbf{BU} \tag{2.0.17}$$

$$\dot{\mathbf{W}} = \mathbf{CW} + \mathbf{DU} \tag{2.0.18}$$

both the equations represent the same system. Hence, using 2.1 and 2.2, we can conclude that

$$[\lambda] = [\mu] \text{ and} \tag{2.0.19}$$

$$\mathbf{W} = \mathbf{TX} \tag{2.0.20}$$

where **T** can be any matrix(need not be an identity matrix). Hence, option 2 is the correct answer.