PAGE No.
DATE / /

Ordinary Differential Equation

O.D.E. : D.E. involving only one independent variable and one or more dependent variables and their derivative w.r.t. independent variable is called O.D.E.

P.D.E. : D.E. involving two or more independent variables and one or more dependent variables

Order of a D.E:
The order of highest derivative occuring in the D.E. is called its order.

Degree of a D.E. :
The degree or power or index of highest ordered derivative appearing in the D.E. (provided the D.E. should be free from radicals and rational terms, fractions) is known as degree of D.E.

eg: Determine order and degree of the D.E

① $\frac{dy}{dx} = 4x+y$ order = 1 degree = 1

② $\left(\frac{d^2y}{dx^2}\right)^3 + \left(\frac{dy}{dx}\right)^4 + y = 0$ order = 2 degree = 3

③ $\left[1+\frac{dy}{dx}\right]^{3/2} = \frac{d^3y}{dx^3}$ order = 3 degree = 2

④ $\left(\frac{dy}{dx}\right)^2 + \frac{k}{\frac{dy}{dx}} = 6$ order = 1 degree = 3

PAGE No.
DATE / /

Solution of a D.E. ⇒

Defn: Any relation betn dependent and independent variables which is free from derivatives and satisfies the given D.E. is called its solution.

① $\frac{dy}{dx} = k \Rightarrow y = kx$.

② $\frac{d^2y}{dx^2} + y = 0 \Rightarrow$ The relation $y = c_1 \cos x$ OR $y = c_2 \sin x$ be the sol

Also. $y = c_1 \cos x + c_2 \sin x$.

General solution: A relation betn dependent and independent variable and free from derivative, contains number of arbitary constant equal to the order of the D.E. which satisfies the D.E. is called the general solution.

Particular solution: A solution which is obtain from general soln by assigning particular values to the arbitary constants is called particular solution.

Formation of ordinary D.E. ⇒

An ordinary D.E. can be formed by eliminating number of arbitrary constants in its general solution.

eg: $y = c_1 x + c_2$, $\frac{dy}{dx} = c_1$, $\frac{d^2y}{dx^2} = 0$

PAGE No.
DATE / /

(2) Find the D.E. whose G.S. is $y = c_1x + c_2e^x$

$\Rightarrow \frac{dy}{dx} = c_1 + c_2e^x$

$\frac{d^2y}{dx^2} = c_2e^x$

$\frac{dy}{dx} = c_1 + \frac{d^2y}{dx^2} \Rightarrow c_1 = \frac{dy}{dx} - \frac{d^2y}{dx^2}$

$\therefore y = \left(\frac{dy}{dx} - \frac{d^2y}{dx^2}\right)x + \frac{d^2y}{dx^2}$.

$\therefore (x-1)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0$ is the required D.E.

(2) Find the D.E. of all circles touching Y-axis at the origin and centre on X-axis.

$(x-a)^2 + (y-0)^2 = a^2$

$x^2 + y^2 = 2ax$

$\cancel{2}x + \cancel{2}y\frac{dy}{dx} = \cancel{2}a$

$\therefore x^2 + y^2 = 2x\left[x + y\frac{dy}{dx}\right]$

(3) Find the D.E. of all circles of radius a

$\Rightarrow \left[1 + \left(\frac{dy}{dx}\right)^2\right]^3 = a^2\left(\frac{d^2y}{dx^2}\right)^2$

(1) Find the D.E. of which $Ax^2 + By^2 = 1$ is G.S

$\Rightarrow \cancel{2}xA + \cancel{2}Byy_1 = 0$

$\cancel{2}A + \cancel{2}B(y_1^2 + yy_2) = 0$

$$\begin{vmatrix} x^2 & y^2 & -1 \\ \cancel{2}x & yy_1 & 0 \\ \cancel{2} & y_1^2+yy_2 & 0 \end{vmatrix} = 0$$

$-1\left[x(y_1^2 + yy_2) - yy_1\right] = 0$

$xy_1^2 + xyy_2 - yy_1 = 0$.

PAGE No.
DATE / /

(5) $xy = ae^{x} + be^{-x} + x^2$

$y + xy_1 = ae^{x} - be^{-x} + 2x$

$y_1 + xy_2 + y_1 = ae^{x} + be^{-x} + 2$

$2y_1 + xy_2 = (xy - x^2) + 2$

(6) $y = ae^{-3t}\sin(2t+b)$

$y_1 = -3ae^{-3t}\sin(2t+b) + 2ae^{-3t}\cos(2t+b)$

$y_1 = -3y + 2e^{-3t}\cos(2t+b)$

$y_2 = -3y_1 - 6ae^{-3t}\cos(2t+b) - 4ae^{-3t}\sin(2t$

$y_2 = -3y_1 - 3(y_1 + 3y) - 4y$

$y_2 = -6y_1 - 9y - 4y$

$y_2 + 6y_1 + 13y = 0$

(7) $y = Ax + \frac{B}{x}$, $y_1 = A - \frac{B}{x^2}$

$y_2 = \frac{2B}{x^3}$ $A = y_1 + \frac{B}{x^2}$

$y = \left(y_1 + \frac{B}{x^2}\right)x + \frac{x^2 y_2}{2}$

$y = y_1 x + \frac{y_2 x}{2} \cdot x + \frac{x^2 y_2}{2}$

$y = y_1 x + \frac{y_2 x^2}{2} + \frac{x^2 y_2}{2} \Rightarrow$

$y = y_1 x + y_2 x^2$

(8) $y = ae^{4x} + be^{3x}$.

$\Rightarrow y_2 - 7y_1 + 12y = 0$

PAGE No.
DATE / /

Solution of ordinary D.E. of 1st order and 1st Degree:

A general form of 1st order & 1st degree O.D.E. is $Mdx + Ndy = 0$ where M & N are functions of x & y or constants.

Depending upon the nature of M & N we have different types of solution of 1st order & 1st degree O.D.E.

Types
1. Variable Separable form.
2. Reducible to V.S.F.
3. Homogeneous D.E.
4. Non — " —
5. Exact D.E.
6. Reducible to Exact D.E.
7. L.D.E. of 1st order, 1st degree.
8. Reducible to L.D.E.

1. Variable separable form:

$F_1(x)dx + F_2(y)dy = 0$ If the D.E. can be put in this form.

$F_1(x)g_1(y)\,dx + F_2(x)g_2(y)\,dy = 0$

1. Solve: $\frac{dy}{dx} + \frac{1+y^2}{1+x^2} = 0$

⇒ Given D.E. can be written as -

$\frac{dy}{1+y^2} + \frac{dx}{1+x^2} = 0 \rightarrow$ V.S.F.

$\tan^{-1}y + \tan^{-1}x = \tan^{-1}c$

$\tan^{-1}\left(\frac{x+y}{1-xy}\right) = \tan^{-1}c$

2. $dy/dx + x^2 = x^2e^{3y}$ $\quad \frac{dy}{dx} = x^2(e^{3y}-1)$

$\frac{dy}{e^{3y}-1} = x^2dx$

$+\frac{1}{3}\int \frac{3e^{-3y}}{1-e^{-3y}}\,dy = \frac{x^3}{3} + c_1$

$\log(1-e^{-3y}) = x^3 + c_1$

PAGE No.
DATE / /

③ $(e^y+1)\cos x\,dx + e^y \sin x\,dy = 0$

$$\frac{\cos x}{\sin x}dx + \frac{e^y}{e^y+1}dy = 0 \quad \text{v.s.f.}$$

$$\log \sin x + \log(e^y+1) = \log c$$

$$\sin x \cdot \log(e^y+1) = c$$

④ $\frac{dy}{dx} = \frac{1+y^2}{(1+x^2)xy}$

$$\frac{y\,dy}{1+y^2} = \frac{dx}{x(1+x^2)}$$

$$\frac{1}{2}\log(1+y^2) = 2\log x - \frac{1}{2}\log(1+x^2) + \log c$$

$$(1+y^2)(1+x^2) - x^2 = c$$

⑤ $\frac{dy}{dx} = \frac{x(2\log x+1)}{\sin y + y\cos y}$

$\Rightarrow (\sin y + y\cos y)\,dy = (2x\log x + x)\,dx$ → v.s.f.

by integrating, we get

$$\int \sin y\,dy + \int y\cos y\,dy = 2\int x\log x\,dx + \int x\,dx + c$$

$$-\cos y + y\sin y - \int \sin y\,dy = 2\int \log x \cdot \frac{x^2}{2} - \int \frac{1}{x}\cdot\frac{x^2}{2}dx + \frac{x^2}{2} + c$$

$$-\cos y + y\sin y + \cos y = 2\left[\log x \cdot \frac{x^2}{2} - \frac{x^2}{4}\right] + \frac{x^2}{2} + c$$

$$-\cancel{\cos y} + y\sin y + \cancel{\cos y} = x^2\log x - \frac{x^2}{2} + \frac{x^2}{2} + c$$

$$= x^2\log x + c$$

PAGE No.
DATE / /

Type 2) Differential Equations of Reducible to Variable Separable form by substitution.

1) The D.E. of the form $dy/dx = f(ax+by+c)$ can be reduced to v.s form by putting $ax+by+c = v$ then find dy/dx and separate the variables in x & v.

2) A D.E. is of the form $dy/dx = f(y/x)$ is reducible to v.s.f by using substitution $y/x = v$ and

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

① $(x+y)^2\left(x\frac{dy}{dx}+y\right) = xy\left(1+\frac{dy}{dx}\right)$

$\Rightarrow$ put $x+y = u$ & $xy = v$

$$1+\frac{dy}{dx} = \frac{du}{dx}, \quad x\frac{dy}{dx}+y = \frac{dv}{dx}$$

$$u^2\frac{dv}{dx} = v\left(\frac{du}{dx}\right)$$

$$u^2 dv = v\,du$$

$$\therefore \frac{dv}{v} = \frac{du}{u^2}$$ v.s.f by integrating

$$\int\frac{dv}{v} - \int\frac{du}{u^2} = c$$

$$\log v + \frac{1}{u} = c$$

$$\log xy + \frac{1}{x+y} = c$$

② $(y e^{xy} - \tan x)\,dx + (x e^{xy} - \sec y)\,dy = 0$

$\Rightarrow$ put $xy = u$

$$x\frac{dy}{dx} + y = \frac{du}{dx}$$

$$(y\,dx + x\,dy)\,e^{xy} = \tan x\,dx + \sec y\,dy$$

$$e^{xy}\,dxy = \tan x\,dx + \sec y\,dy \quad \text{vsf.}$$

by integrating

$$e^{xy} = \log \sec x + \log(\sec y + \tan y) + c$$

③ $x\,dx + y\,dy = y(x^2+y^2)\,dy$

~~$dxy = x\,dx + y\,d$~~

put $x^2 + y^2 = u$

$$2x\,dx + 2y\,dy = du$$

$$x\,dx + y\,dy = \frac{du}{2}$$

$$\frac{du}{2} = y u\,dy$$

$$\frac{du}{u} = 2y\,dy$$

$$\log u = y^2 + c$$

$$\log(x^2+y^2) = y^2 + c$$

PAGE No.
DATE / /

(4) $\frac{dy}{dx} = 1 - x\tan(x-y)$

$\Rightarrow$ $x-y = u$, $1 - dy/dx = \frac{du}{dx}$

$x\tan u = \frac{du}{dx}$

$x\,dx = \cot u\,du$

$\frac{x^2}{2} = \log\sin u + c$

$\frac{x^2}{2} = \log\sin(x-y) + c$

(5) $x^4\frac{dy}{dx} + x^3y - \sec(xy) = 0$

put $xy = u$

$y + x\frac{dy}{dx} = \frac{du}{dx}$

$\Rightarrow$ $x^3\frac{du}{dx} - \sec u = 0$

$x^3\,du/dx = \sec u$

$\frac{du}{\sec u} = \frac{dx}{x^3}$

$\cos u\,du = x^{-3}dx$

$\sin(xy) = \frac{-1}{2x^2} + c$

(6) $xy\log(x/y)dx + (y^2 - x^2\log(x/y))dy = 0$

$\Rightarrow$ $\frac{dx}{dy} = \frac{x^2\log(x/y) - y^2}{xy\log(x/y)}$

$x/y = u$, $x = uy$ $\quad dx/dy = u + y\frac{du}{dy}$

$u + y\frac{du}{dy} = \frac{u^2\log u - 1}{u\log u}$

PAGE No.
DATE / /

$$y\frac{du}{dy} = \frac{u\log u - 1}{\log u} - u$$

$$y\frac{du}{dy} = \frac{u\log u - 1 - u\log u}{\log u}$$

$$y\frac{du}{dy} = \frac{-1}{\log u}$$

$$\log u\, du = -\frac{dy}{y}$$

$$u\log u - \int \frac{1}{u} u\, du = -\log y + c$$

$$u\log u - u = -\log y + c$$

$$x/y \log x/y - x/y = \log y + c$$

(5) $xy\log\left(\frac{x}{y}\right)dx + \left(y^2 - x^2\log\frac{x}{y}\right)dy = 0$

$$\frac{dx}{dy} = \frac{x^2\log x/y - y^2}{xy\log x/y}$$

$$= \frac{x^2/y^2\log x/y - 1}{x/y\log x/y}$$

$x/y = u \qquad x = yu$

$$\frac{dx}{dy} = u + y\frac{du}{dy}$$

$$u + y\frac{du}{dy} = \frac{u^2\log u - 1}{u\log u}$$

$$y\frac{du}{dy} = \frac{u^2\log u - 1}{u\log u} - u$$

PAGE No.
DATE / /

$$y\frac{du}{dy} = \frac{u^2\log u - 1 - u^2\log u}{u\log u}$$

$$y\frac{du}{dy} = \frac{-1}{u\log u}$$

$$u\log u\, du = -\frac{dy}{y}$$

$$\frac{u^2}{2}\log u - \int \frac{1}{u}\cdot\frac{u^2}{2}\,du = -\log y + c$$

$$\frac{u^2}{2}\log u - \frac{1}{4}u^2 = -\log y + c$$

$$\frac{x^2}{2y^2}\log x/y - \frac{x^2}{4y^2} = -\log y + c$$

is the required D.E.

(6) $ye^{x/y}dx = (xe^{x/y} + y^2)\,dy$

$\Rightarrow$ $$\frac{dx}{dy} = \frac{xe^{x/y} + y^2}{ye^{x/y}}$$

$$\frac{dx}{dy} = \frac{x}{y} + ye^{-x/y}$$

$x/y = u \qquad x = yu, \quad \frac{dx}{dy} = u + y\frac{du}{dy}$

$$\cancel{u} + y\frac{du}{dy} = \cancel{u} + \cancel{y}e^{-u}$$

$$e^{u}du = dy$$

$$e^{x/y} = y + c$$

PAGE No.
DATE / /

Type 3) Solution of Homogeneous D.E.

D.E. of the form

$$\frac{dy}{dx} = \frac{a_1x+b_1y}{a_2x+b_2y}$$

put $y/x = u$ $\quad y = xu$

or $x/y = u$, $\quad x = yu$.

① $(y^4 - 2x^3y)\,dx + (x^4 - 2xy^3)\,dy = 0$

$$(y^4 - 2x^3y)\frac{dx}{dy} = 2xy^3 - x^4$$

$$\frac{dx}{dy} = \frac{2xy^3 - x^4}{y^4 - 2x^3y} = \frac{2\,y^3/x^3 - 1}{\frac{y^4}{x^4} - 2\,y/x}$$

$$\frac{dy}{dx} = \frac{y^4/x^4 - 2\,y/x}{2\,y^3/x^3 - 1}$$

$y/x = u$, $\quad y = xu$

$$\frac{du}{dx} = u + x\frac{du}{dx}$$

$$u + x\frac{du}{dx} = \frac{u^4 - 2u}{2u^3 - 1}$$

$$x\frac{du}{dx} = \frac{u^4 - 2u}{2u^3 - 1} - u = \frac{u^4 - 2u - 2u^4 + u}{2u^3 - 1}$$

$$x\frac{du}{dx} = \frac{u^4 + u}{1 - 2u^3}$$

$$\frac{1 - 2u^3}{u(1+u^3)}\,du = \frac{dx}{x}$$

PAGE No.
DATE / /

$$\int \left(\frac{1}{u} - \frac{3u^2}{1+u^3}\right) du = \frac{dx}{x}$$

$$\log u - \log(1+u^3) = \log x + \log c$$

$$\frac{y/x}{\left(1+ y^3/x^3\right)x} = c$$

$$\frac{y/x}{\frac{x^3+y^3}{x^2}} = c \Rightarrow xy = c(x^3+y^3)$$

(2) $x^3dx - y^3dy = 3xy(ydx - xdy)$
$= 3xy^2dx - 3x^2y\,dy$

$$(x^3 - 3xy^2)dx = +(y^3 - 3x^2y)dy.$$

$$\left(1 - 3\frac{y^2}{x^2}\right)dx = \left(y^3/x^3 - 3y/x\right)dy$$

$$\frac{dy}{dx} = \frac{1 - 3\,y^2/x^2}{y^3/x^3 - 3y/x}$$

$y/x = u \quad y = xu \quad dy/dx = u + x\,du/dx$

$$u + x\frac{du}{dx} = \frac{1-3u^2}{u^3-3u}$$

$$x\frac{du}{dx} = \frac{1-3u^2}{u^3-3u} - u = \frac{1-3u^2-u^4+3u^2}{u^3-3u}$$

$$\frac{u^3-3u}{1-u^4}du = \frac{dx}{x}$$

$$\frac{u^3}{1-u^4} - \frac{3u}{1-u^4}du = \log x + \log c$$

$$-\frac{1}{4}\log(1-u^4) - \int \frac{3u}{1-u^4}du = \log xc$$

$u^2 = t \quad 2u\,du = dt$

$$-\frac{1}{4}\log(1-u^4) - \frac{3}{2}\int \frac{dt}{1-t^2} = \log x$$

$$-\frac{1}{4}\log(1-u^4) - \frac{3}{2}\cdot\frac{1}{2}\log\left(\frac{1+t}{1-t}\right) = \log x + c$$

$$-\frac{1}{4}\log\left(1-\frac{y^4}{x^4}\right) + \frac{3}{4}\log\left(\frac{x+y}{x-y}\right) = \log x + c$$

$$\log\left(\frac{x^4-y^4}{x^4}\right) + 3\log\left(\frac{x+y}{x-y}\right) = \log\frac{1}{x^4} + c$$

$$\cancel{\log(x^4/y^4)}$$

$$\log(1-u^4) + \log\left(\frac{1+t}{1-t}\right)^3 + \log x^4 = c$$

$$\log(1-u^4) + \log\left(\frac{1+u^2}{1-u^2}\right)^3 + \log x^4 = c$$

$$\frac{(1-u^4)(1+u^2)^3 x^4}{(1-u^2)^3} = c$$

$$\frac{(1-u^2)(1+u^2)(1+u^2)^3 x^4}{(1-u^2)^3} = c$$

$$\frac{(1+u^2)^4}{(1-u^2)^2} x^4 = c$$

$$(x^2+y^2)^4 = c\,(x^2-y^2)^2$$

PAGE No.
DATE / /

Exact D.E. ⇒

A differential Equation of the form $M(x,y)\,dx + N(x,y)\,dy = 0$ is said to be exact D.E. if it satisfies necessary and sufficient condition i.e. $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Then its General solution is -

① $\int_{y=\text{constant}} M\,dx + \int [\text{Terms of } N \text{ free from } x]\,dy = c$

② $\int_{x=\text{constant}} N\,dy + \int [\text{Terms of } M \text{ free from } y]\,dx = c$

③ $\left(x\tan\frac{y}{x} - y\sec^2\frac{y}{x}\right)dx + x\sec^2\frac{y}{x}\,dy = 0$

$x\tan y/x - y\sec^2 y/x + x\sec^2 y/x \; dy/dx = 0$

$$\frac{dy}{dx} = \frac{y\sec^2 y/x - x\tan y/x}{x\sec^2 y/x}$$

$$\frac{dy}{dx} = y/x - \sin y/x \cos y/x \frac{\tan y}{\sec^2 u}.$$

$$\frac{dy}{dx} = \frac{y}{x} - \cancel{\tfrac{1}{2}\sin 2y/x}\,\frac{\tan y}{\sec^2 u}.$$

$y/x = u$, $y = xu$ $\quad dy/dx = u + x\,du/dx$

$$\cancel{u} + x\,du/dx = \cancel{u} - \tfrac{1}{2}\sin 2u$$

$$\frac{du}{\sin 2u} = -\tfrac{1}{2}\frac{dx}{x}.$$

$\frac{\sec^2 u}{\tan u}$ $\quad \cancel{u} + x\frac{du}{dx} = \cancel{u} - \frac{\tan u}{\sec^2 u}.$

$$\frac{\sec^2 u}{\tan u}\,du = -\frac{dx}{x}, \quad \log\tan u = -\log x + \log c$$

$x\tan y/x = c$

PAGE No.
DATE / /

(4) $(xy - x^2)\frac{dy}{dx} = y^2$

$\left(\frac{y}{x} - 1\right)\frac{dy}{dx} = y^2/x^2$

$\frac{dy}{dx} = \frac{y^2/x^2}{y/x - 1}$

put $y/x = u$ $\quad y = xu$

$\frac{dy}{dx} = u + x\frac{du}{dx}$

$u + x\frac{du}{dx} = \frac{u^2}{u-1}$

$x\frac{du}{dx} = \frac{u^2}{u-1} - u = \frac{\cancel{u^2} - \cancel{u^2} + u}{u-1}$

$x\frac{du}{dx} = \frac{u}{u-1}$

$\frac{u-1}{u}du = \frac{dx}{x} \rightarrow$ v.s.f.

by integrating

$\int\left(1 - \frac{1}{u}\right)du = \int\frac{dx}{x} + c$

$u - \log u = \log x + c$

$y/x - \log y/x \cdot x = c$

$y/x - \log y = c$

PAGE No
DATE / /

Type 4) Non-Homogeneous D.E. of first order in x and y.

Non-Homogeneous D.E. of first order in x and y can be written as

$$\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$$

case 1) If $\frac{a_1}{a_2} = \frac{b_1}{b_2}$ then in this case $a_2 = a_1k$ and $b_2 = b_1k$

$$\therefore \frac{dy}{dx} = \frac{(a_1x + b_1y) + c_1}{k(a_1x + b_1y) + c_2}$$

$\therefore$ $a_1x + b_1y$ is the repeated factor in this D.E.

$\therefore$ put $a_1x + b_1y = u$ and find $\frac{dy}{dx}$ value for replacement and V.S.F.

Case 2) $\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$

and $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$

then put $x = X + h$ and $y = Y + k$, where h, k are constants to be determined

$$\therefore \frac{dY}{dX} = \frac{a_1(X+h) + b_1(Y+k) + c_1}{a_2(X+h) + b_2(Y+k) + c_2}$$

$\therefore$ Now choose h & k s.t.

$a_1h + b_1k + c_1 = 0$ &

$a_2h + b_2k + c_2 = 0$

$\therefore \frac{dY}{dX} = \frac{a_1X + b_1Y}{a_2X + b_2Y}$ becomes Homogeneous D.E. in X and Y.

put $Y/X = u$.

PAGE No.
DATE / /

(1) $\frac{dy}{dx} = \frac{x-y+3}{2x-2y+5}$

$\therefore \frac{dy}{dx} = \frac{(x-y)+3}{2(x-y)+5}$

$\therefore$ put $x-y = u$

$1 - \frac{dy}{dx} = \frac{du}{dx}$

$\therefore \frac{dy}{dx} = 1 - \frac{du}{dx}$

$\therefore 1 - \frac{du}{dx} = \frac{u+3}{2u+5}$

$\therefore \frac{du}{dx} = 1 - \frac{u+3}{2u+5}$

$= \frac{2u+5-u-3}{2u+5}$

$\frac{du}{dx} = \frac{u+2}{2u+5}$

$\therefore \frac{2u+5}{u+2} du = dx$

$\left(2 + \frac{1}{u+2}\right) du = dx \rightarrow$ VSF

$\therefore$ by integrating we get

$\int \left(2 + \frac{1}{u+2}\right) du = \int dx + C$

$2u + \log(u+2) = x + C$

$2(x-y) + \log(x-y+2) = x + C$

$x - 2y + \log(x-y+2) = C$

PAGE No.
DATE / /

(2) $(3y+2x+4)\,dx-(4x+6y+5)\,dy=0$

$\rightarrow \dfrac{dy}{dx}=\dfrac{3y+2x+4}{6y+4x+5}$

$=\dfrac{(2x+3y)+4}{2(2x+3y)+5}$

put $2x+3y=u$

$2+3\dfrac{dy}{dx}=\dfrac{du}{dx}$

$\therefore \dfrac{dy}{dx}=\dfrac{1}{3}\left(\dfrac{du}{dx}-2\right)$

$\therefore \dfrac{1}{3}\left(\dfrac{du}{dx}-2\right)=\dfrac{u+4}{2u+5}$

$\dfrac{du}{dx}-2=\dfrac{3u+12}{2u+5}$

$\dfrac{du}{dx}=\dfrac{3u+12}{2u+5}+2$

$=\dfrac{3u+12+4u+10}{2u+5}$

$\dfrac{du}{dx}=\dfrac{7u+22}{2u+5}$

$\therefore \dfrac{2u+5}{7u+22}\,du=dx$

$\left(\dfrac{2}{7}-\dfrac{9/7}{7u+22}\right)du=dx \longrightarrow$ v.s.f.

by integrating we get

$\frac{2}{7}u-\frac{9}{7\cdot 7}\int\frac{7}{7u+22}\,du=x+c$

$\frac{2}{7}u-\frac{9}{49}\cdot\log(7u+22)=x+c$

$\frac{2}{7}(2x+3y)-\frac{9}{49}\log[14x+21y+22]=x+c$

PAGE No.
DATE / /

(3) $\frac{dy}{dx} = \frac{x+2y-3}{3x+6y-1}$

Ans:

$\frac{3(x+2y)}{5} + \frac{16}{25}\log[5(x+2y)-7] = x+c$

(4) $\frac{dy}{dx} = \frac{3x-4y-2}{6x-8y-5}$

Ans: $\log(6x-8y-7) = c-2x+4y$

(5) $\frac{dy}{dx} = \frac{x+y+1}{2x+2y+3}$

Ans: $6y-3x+\log(3x+3y+4) = c$

① $(3y-7x+7)\,dx+(7y-3x+3)\,dy=0$

$$\Rightarrow \frac{dy}{dx}=\frac{7x-3y-7}{7y-3x+3} \quad \text{①}$$

$$\frac{a_1}{a_2}=\frac{-7}{3}\neq\frac{b_1}{b_2}=\frac{-3}{7}$$

$\therefore$ put $x=X+h,\ y=Y+k,\ \frac{dy}{dx}=\frac{dY}{dX}$

$$\therefore \frac{dY}{dX}=\frac{7(X+h)-3(Y+k)-7}{7(Y+k)-3(X+h)+3} \quad \text{②}$$

$7h-3k-7=0 \Rightarrow 7h-3k=7$

$7k-3h+3=0 \Rightarrow -3h+7k=-3$

$\therefore h=1,\ k=0$

$\therefore$ eqn ② becomes

$$\frac{dY}{dX}=\frac{7X-3Y}{-3X+7Y} \quad \text{is a H.D.E.}$$

$$\therefore \frac{dY}{dX}=\frac{7-3Y/X}{-3+7Y/X}$$

put $Y/X=u \quad Y=Xu \quad \frac{dY}{dX}=u+X\frac{du}{dX}$

$$u+X\frac{du}{dX}=\frac{7-3u}{-3+7u}$$

$$X\frac{du}{dX}=\frac{7-3u}{-3+7u}-u=\frac{7-3u+3u-7u^2}{-3+7u}$$

$$\frac{-3+7u}{1-u^2}\,du=7\frac{dX}{X} \longrightarrow \text{v.s.f.}$$

Using partial fraction,

$$\frac{-3+7u}{1-u^2}=\frac{A}{1+u}+\frac{B}{1-u}$$

PAGE No.
DATE / /

$-3+7u = A(1-u) + B(1+u)$

put $u = -1$

$A = -5$ & put $u = 1$

$B = 2$

$\therefore \left(\frac{-5}{1+u} + \frac{2}{1-u}\right) du = 7\frac{dx}{x} \rightarrow$ vst

by integrating we get

$-5\log(1+u) - 2\log(1-u) = 7\log x + \log c$

$\log(1+u)^5(1-u)^2 x^7 = \log c$

$(1+y/x)^5(1-y/x)^2 x^7 = c$

$\frac{(x+y)^5}{\cancel{x^5}}\frac{(x-y)^2}{\cancel{x^2}}\cancel{x^7} = c$

$(x-h+y-k)^5(x-h-y+k)^2 = c$

$(x+y-1)^5(x-y-1)^2 = c$

② $(2x+y-3)dy = (x+2y-3)dx$

$\frac{dy}{dx} = \frac{x+2y-3}{2x+y-3}$ —— ①

$\frac{a_1}{a_2} = \frac{1}{2} \neq \frac{b_1}{b_2} = \frac{2}{1}$

$\therefore$ put $x = X+h$ $y = Y+k$ $\frac{dY}{dX} = \frac{dy}{dx}$

$\therefore \frac{dY}{dX} = \frac{X+h+2(Y+k)-3}{2(X+h)+(Y+k)-3}$ —— ②

PAGE No.
DATE / /

(27) $h + 2k = 3$

$2h + k = 3$

$h = 1$ & $k = 1$

$\therefore$ eqn (2) becomes $\frac{dy}{dx} = \frac{x+2y}{2x+y}$

$\frac{dy}{dx} = \frac{1+2y/x}{2+y/x}$

put $y/x = u$ $\quad y = xu \quad \frac{dy}{dx} = u + x\frac{du}{dx}$

$u + x\frac{du}{dx} = \frac{1+2u}{2+u}$

$x\frac{du}{dx} = \frac{1+2u}{2+u} - u = \frac{1+2u-2u-u^2}{2+u}$

$x\frac{du}{dx} = \frac{1-u^2}{2+u}$

$\frac{2+u}{1-u^2}du = \frac{dx}{x}$ → vst, using partial fraction

$\frac{2+u}{1-u^2} = \frac{A}{1-u} + \frac{B}{1+u}$

$2+u = A(1+u) + B(1-u)$

put $u = 1$ $\quad A = 3/2$

$u = -1$ $\quad B = 1/2$

$\therefore \left(\frac{3/2}{1-u} + \frac{1/2}{1+u}\right)du = \frac{dx}{x}$ by integration

$-3/2 \log(1-u) + 1/2 \log(1+u) = \log x + \log C$

$\log(1-u)^{-3}\cdot(1+u) = \log x^2 c$

$\frac{1+y/x}{(1-y/x)^3} = x^2 c \Rightarrow \frac{(x+y)/x}{\frac{(x-y)^3}{x^3}} = x^2 c$

$\frac{(x+y)x^2}{(x-y)^3} = x^2 c$

$\therefore (x+y) = c(x-y)^3$

$(x+y-2) = c(x-y)^3$

PAGE No.
DATE / /

(3) $\frac{dy}{dx} = \frac{y-x+1}{y+x-5}$

Ans: $\tan^{-1}\left(\frac{y-2}{x-3}\right) + \frac{1}{2}\log\left[(x-3)^2+(y-2)^2\right]$

(4) $\frac{dy}{dx} = \frac{x+2y-3}{2x+y-3}$

Ans: $(x-y)^3 = c(x+y-2)$

(5) $\frac{dy}{dx} = \frac{y+2}{x+y+1}$

Ans: $\log(y+2) - \frac{x-3}{y+2} = c$

(6) $(3x-2y+1)\,dx + (2x-3y+4)\,dy = 0$

Ans: $(x-y+1)(x+y-3)^5 = c$

(3) $\Rightarrow$ $\frac{a_1}{a_2} = -1 = \frac{b_1}{b_2} = -1$

$\therefore \frac{dY}{dX} = \frac{-(X+h)+(Y+k)+1}{(X+h)+(Y+k)-5}$

$\frac{dY}{dX} = \frac{-X+Y}{X+Y}$, $h=3, k=2$

$\frac{dY}{dX} = \frac{-1+Y/X}{1+Y/X}$

$Y = Xu$ $\quad \frac{dY}{dX} = u + X\frac{du}{dX}$

$u + X\frac{du}{dX} = \frac{-1+u}{1+u}$

$X\frac{du}{dX} = \frac{-1+u}{1+u} - u$

$X\frac{du}{dX} = \frac{-1+\cancel{u}-\cancel{u}-u^2}{1+u} = \frac{-1-u^2}{1+u}$

PAGE No.
DATE / /

$$\cancel{\frac{1+u}{\cdot}\,du = -2\frac{dx}{x}}$$

$$\cancel{u + \frac{u^2}{2} = -2\log x + c}$$

$$\cancel{\frac{y}{x} + \frac{y^2}{2x^2} = -2\log x + c}$$

$$\cancel{\frac{y-2}{x-3} + \frac{(y-2)^2}{2(x-3)^2} = -2\log(x-3)+c}$$

$$\frac{1+u}{1+u^2}\,du = -\frac{dx}{x} \longrightarrow \text{vst.}$$

$$\int\left(\frac{1}{1+u^2} + \frac{u}{1+u^2}\right)du = -\log x + \log c$$

$$\tan^{-1}u + \tfrac{1}{2}\log(1+u^2) = -\log xc$$

$$\tan^{-1}\frac{y}{x} + \frac{1}{2}\log\left[\frac{x^2+y^2}{x^2}\right] = -\log cx$$

$$\tan^{-1}\left[\frac{y-2}{x-3}\right] + \frac{1}{2}\log\left[\frac{(x-3)^2+(y-2)^2}{(x-3)^2}\right] = -\log(x-3)+c$$

$$\tan^{-1}\left[\frac{y-2}{x-3}\right] + \frac{1}{2}\log\left[(x-3)^2+(y-2)^2\right] - \log\cancel{(x-3)} = -\log\cancel{(x-3)} + c$$

PAGE No.
DATE / /

Exact Differential Equations:

1) $(y^2 e^{xy^2} + 4x^3)\,dx + (2xye^{xy^2} - 3y^2)\,dy = 0$

$\rightarrow$ let $M = y^2 e^{xy^2} + 4x^3$

$N = 2xye^{xy^2} - 3y^2$

$\frac{\partial M}{\partial y} = 2ye^{xy^2} + 2xy^3 e^{xy^2}$

$\frac{\partial N}{\partial x} = 2ye^{xy^2} + 2xy^3 e^{xy^2}$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow$ given D.E. is Exact

$\therefore$ using General solution formula -

$\int M\,dx + \int [\text{Terms of } N \text{ free from } x]\,dy$
$y = \text{constant}$

$\int (y^2 e^{xy^2} + 4x^3)\,dx - 3\int y^2\,dy = c$
$y = \text{const}$

$\frac{\cancel{y^2}\, e^{xy^2}}{\cancel{y^2}} + \frac{\cancel{4}x^4}{\cancel{4}} - \frac{\cancel{3}y^3}{\cancel{3}} = c$

$e^{xy^2} + x^4 - y^3 = c$

2) $(1 + e^{x/y})\,dx + e^{x/y}(1 - x/y)\,dy = 0$

let $M = 1 + e^{x/y}$, $N = e^{x/y}(1 - x/y)$

$\frac{\partial M}{\partial y} = \frac{-x}{y^2} e^{x/y} = \frac{-x}{y^2} e^{x/y}$

$\frac{\partial N}{\partial x} = \frac{1}{y} e^{x/y}(1 - x/y) + e^{x/y}\left(-\frac{1}{y}\right)$

$= \cancel{\frac{1}{y} e^{x/y}} - \frac{x}{y^2} e^{x/y} - \cancel{\frac{e^{x/y}}{y}}$

$= -\frac{x}{y^2} e^{x/y}$

PAGE No.
DATE / /

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow$ Given D.E. is Exact.

$\therefore$ by G.S. formula

$$\int_{y=\text{const}} M\,dx + \int (\text{Terms of } N \text{ free from } x)\,dy = c$$

$$\int_{y=\text{const}} (1+e^{x/y})\,dx + \int 0\,dy = c$$

$$x + ye^{x/y} = c$$

(3) $$\frac{dy}{dx} = -\left(\frac{4x^2y^2 + y\cos xy}{2x^4y + x\cos xy}\right)$$

Ans: $x^4y^2 + \sin xy = c$

(4) $$\frac{dy}{dx} = \frac{x+y-2}{y-x-4}$$ Ans: $x^2 + 2xy - y^2 - 4x + 8y = c$

(5) $$\left(\frac{1}{x^2} + \frac{3y^2}{x^4}\right)dx = \frac{2y}{x^3}\,dy$$, Ans: $\frac{1}{x} + \frac{y^2}{x^3} = c$

(6) $$\frac{dy}{dx} = \frac{4x-2y+1}{2x-6y+2}$$ Ans: $2xy + 2y - 3y^2 - x - 2x^2 = c$

PAGE No.
DATE / /

Type 6> D.E. reducible to Exact form

An I.F. (Integrating factor) is a multiplying factor by which the equation can be reduced to its Exact form

Rule 1> If the given D.E. is homogeneous and $xM + yN \neq 0$ then

$$I.F. = \frac{1}{Mx + Ny}$$

1> $(x^2y - 2xy^2)\,dx - (x^3 - 3x^2y)\,dy = 0$ — (1)

$\Rightarrow$ $\boxed{I.F. = 1/x^2y^2}$

Let $M = x^2y - 2xy^2$

$N = -(x^3 - 3x^2y)$

& given D.E. is Homogeneous

$$\therefore I.F. = \frac{1}{Mx + Ny}$$

$$= \frac{1}{x^3y - 2x^2y^2 - x^3y + 3x^2y^2}$$

$$I.F. = \frac{1}{x^2y^2}$$

$\therefore$ Multiplying the eqn (1) by $\frac{1}{x^2y^2}$ we get

$\therefore \left(\frac{1}{y} - \frac{2}{x}\right)dx - \left(\frac{x}{y^2} - 3/y\right)dy = 0$ is an exact D.E.

$\therefore$ by G.S. formula

$\int M\,dx + \int [\text{Terms of } N \text{ free from } x]\,dy = c$

$y = \text{const}$

$$\int \left(\frac{1}{y} - \frac{2}{x}\right) dx + \int 3/y \, dy = c$$

$y = \text{const.}$

$$\frac{x}{y} - 2\log x + 3\log y = c$$

$$x/y + \log y^3/x^2 = c$$

(2) $(x^3+y^3)\,dx - xy^2\,dy = 0$

Ans: $\log \frac{x}{y} - 1/3 \; y^3/x^3 = c$

(3) $x(x-y)\frac{dy}{dx} = y(x+y)$

Ans: $xy^2 = c$

Rule (2) If given D.E. is not homogeneous and can be written as

$$y\,f_1(x,y)\,dx + x\,f_2(x,y)\,dy = 0$$

then $I.F. = \dfrac{1}{mx - ny}$

eg (1) $y(1+xy)\,dx + x(1+xy+x^2y^2)\,dy = 0$

$\Rightarrow$ Given D.E. is not homogeneous & not exact i.e. $\partial m/\partial y \neq \partial N/\partial x$.

$$\therefore I.F. = \frac{1}{mx - ny}$$

$$= \frac{1}{xy + x^2y^2 - xy - x^2y^2 - x^3y^3}$$

$$= \frac{-1}{x^3y^3}$$

$\therefore$ Multiplying by $\frac{-1}{x^3y^3}$ to given eqn we get

$$\left(-\frac{1}{x^3y^2} - \frac{1}{x^2y}\right) dx + \left(\frac{-1}{x^2y^3} - \frac{1}{xy^2} - \frac{1}{y^2y}\right) = 0$$

is an exact D.E.

$\therefore$ by G.S. formula

$$\int_{y=\text{const}} M\,dx + \int [\text{Terms of } N \text{ free from } x]\,dy = C$$

$$\int_{y=\text{const}} \left(-\frac{1}{x^3y^2} - \frac{1}{x^2y}\right) dx - \int \frac{1}{y}\,dy = C$$

$$\frac{1}{2x^2y^2} + \frac{1}{xy} - \log y = C$$

(2) $y(xy + 2x^2y^2)\,dx + x(xy - x^2y^2)\,dy = 0$

$\rightarrow$ I.F. $= \frac{1}{3x^3y^3}$

G.S. $: -\frac{1}{3xy} + \frac{2}{3}\log x - \frac{1}{3}\log y = C$

(3) $y(xy - 3)\,dx + x(3xy - 3)\,dy = 0$

Ans: $\log(xy^3) + \frac{3}{xy} = C$

PAGE No.
DATE / /

Rule 3) Let $Mdx + Ndy = 0$ be the given D.E

If $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x)$ (fn of x only)

then $I.F. = e^{\int f(x)dx}$

① $(x^4 e^x - 2mxy^2)dx + 2mx^2y\,dy = 0$ — ①

Here

$\Rightarrow$ $M = x^4 e^x - 2mxy^2$, $N = 2mx^2y$

$\partial M/\partial y = -4mxy$ $\quad \partial N/\partial x = 4mxy$

$\partial M/\partial y \neq \partial N/\partial x \Rightarrow$ given D.E. is not exact

using Rules of finding I.F.

$\therefore \dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \dfrac{-4mxy - 4mxy}{2mx^2y}$

$= \dfrac{-8mxy}{2mx^2y} = \dfrac{-4}{x} = f(x)$

$I.F. = e^{\int f(x)dx}$

$= e^{-4\int 1/x\,dx} = e^{-4\log x} = e^{\log x^{-4}} = \dfrac{1}{x^4}$

$\therefore$ eqn ① becomes

$\left(e^x - 2m\,y^2/x^3\right)dx + 2m\,y/x^2\,dy = 0$

be a Exact D.E.

$\therefore \int_{y=\text{const}} M\,dx + \int$ (Terms of N free from x) $dy = c$

$\int_{y=\text{const}} \left(e^x - 2m\,y^2/x^3\right)dx + \int 0\,dy = c$

$e^x + \dfrac{\cancel{2}my^2}{\cancel{2}x^2} = c$

$e^x + my^2/x^2 = c$ is a required general solution

PAGE No.
DATE / /

2) $(xy^2 - e^{y/x^3})\,dx - x^2y\,dy = 0$

Ans: $I.F = 1/x^4$ $\quad \int e^{f(x)} f'(x) = e^{f}$

$$\left(-\frac{y^2}{2x^2}\right) + \left(\frac{e^{y/x^3}}{3}\right) = c$$

3) $(x^2+y^2+1)\,dx - 2xy\,dy = 0$

Ans: $x^2 - y^2 - 1 = cx$

4) $(y - 2x^3)\,dx - x(1-xy)\,dy = 0$

Ans. $y/x + x^2 - y^2/2 = c$

Rule 4) If $\dfrac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = g(y)$ then

$$I.F. = e^{\int g(y)\,dy}$$

① $\left(\frac{y}{x}\sec y - \tan y\right)dx + (\sec y \log x - x)\,dy = 0$ — (1)

$\Rightarrow$ Here $M = y/x \sec y - \tan y$, $N = \sec y \log x - x$

$\partial M/\partial y = 1/x \sec y + y/x \sec y \tan y - \sec^2 y$

$\partial N/\partial x = 1/x \sec y - 1$

$\partial M/\partial y \neq \partial N/\partial x$ $\therefore$ given D.E. is not exact

$\therefore$ by Rule ④

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{1/x \sec y - 1 - 1/x \sec y - y/x \sec y \tan y + \sec^2 y}{y/x \sec y - \tan y}$$

$$= \frac{-y/x \sec y \tan y + \tan^2 y}{y/x \sec y - \tan y}$$

$$= -\tan y \frac{(\cancel{y/x \sec y - \tan y})}{\cancel{y/x \sec y - \tan y}}$$

$$= -\tan y = g(y)$$

$$\therefore I.F. = e^{\int g(y)\,dy}$$

$$= e^{-\int \tan y\,dy} = e^{+\log \cos y} = \cancel{\cos y}$$

ans

$$= \cos y$$

$\therefore$ I.F. $= \cos y$.

eqn ① becomes.

$\therefore$

$$(y/x - \sin y)\,dx + (\log x - x \sec y)\,dy = 0$$

$$\int_{y=\text{const}} m\,dx + \int (\text{Terms of } N \text{ free from } x)\,dy = c$$

$$\int (y/x - \sin y)\,dx + \int 0\,dy = c$$

$y = \text{const}$.

$$y \log x - x \sin y = c.$$

① $(y^4 + 2y)\,dx + (xy^3 + 2y^4 - 4x)\,dy = 0$

Ans. $P.I. = 1/y^3$

$$(y + 2/p)\,x + y^2 = c.$$

② $\frac{dy}{dx}(x + 2y^3) = y + 2x^3y^2$

Ans $xy + x^4/2 - y^2 = c$

③ $y(x^2y + e^x)\,dx - e^x\,dy = 0$

Ans: $x^3/3 + e^x/y = c$

④ $y(2xy + e^x)\,dx = e^x\,dy$. Ans: $x^2 + e^x/y = c$

Rule 5) If the eqn $Mdx + Ndy = 0$ can be written as

$x^a y^b (my\,dx + nx\,dy) + x^{a_1} y^{b_1} (m_1 y\,dx + n_1 x\,dy) = 0$

then $I.F. = x^h y^k$.

① $(y^3 - 2x^2 y)\,dx + (2xy^2 - x^3)\,dy = 0$

Ans: $y^4 x^2 - y^2 x^4 = c$

$I.F. = x^h y^k = xy$

② $(x^2 y + y^4)\,dx + (2x^3 + 4xy^3)\,dy = 0$

Ans: $7x^{11/2} y^{11} + 11 x^{7/2} y^{14} = c$

$I.F. = x^{5/2} y^{10}$

③ $(2y + 6xy^2)\,dx + (3x + 8x^2 y)\,dy = 0$

$\Rightarrow h = 1, k = 2$

Ans: $x^2 y^3 + 2x^3 y^4 = c$.

Linear Differential Equations of first order and first degree ⇒

A differential equation is said to be linear if the degree of the differential equation is one.

General form:

i) $\frac{dy}{dx} + Py = Q$ linear in y

where P & Q are the functions of x or constants.

$I.F. = e^{\int P dx}$ and

general solution

$$y \times I.F. = \int Q \times I.F. dx + C$$

ii) $\frac{dx}{dy} + Px = Q$ linear in x

where P and Q are the functions of y or constants.

$I.F. = e^{\int P dy}$ and general solution

$$x \times I.F. = \int Q \times I.F. dy + C.$$

PAGE No.
DATE / /

① $\left(\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}\right)\frac{dx}{dy} = 1$

$\Rightarrow \frac{dy}{dx} = \frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}$

$\frac{dy}{dx} + \frac{y}{\sqrt{x}} = \frac{e^{-2\sqrt{x}}}{\sqrt{x}}$ is a L.D.E. in y

$\therefore$ I.F. $= e^{\int P\,dx}$

$P = \frac{1}{\sqrt{x}}$

$\therefore$ I.F. $= e^{\int x^{-1/2}dx} = e^{\frac{x^{-1/2+1}}{-1/2+1}}$

I.F. $= e^{2\sqrt{x}}$

$\therefore$ by using general solution formula

$y \times I.F. = \int Q \times I.F.\, dx + C$

$y\, e^{2\sqrt{x}} = \int \frac{e^{-2\sqrt{x}}}{\sqrt{x}} \cdot e^{2\sqrt{x}}\, dx + C$

$= \int x^{-1/2}\, dx + C$

$y\, e^{2\sqrt{x}} = \frac{x^{-1/2+1}}{-1/2+1} + C$

$y\, e^{2\sqrt{x}} = \frac{x^{1/2}}{1/2} + C$

$y\, e^{2\sqrt{x}} = 2\sqrt{x} + C$

is the required general solution

② $ye^y = (y^3 + 2xe^y)\, dy/dx$

$\Rightarrow \quad \dfrac{dx}{dy} = \dfrac{y^3 + 2xe^y}{ye^y} = y^2 e^{-y} + 2x/y$

$\therefore \quad \dfrac{dx}{dy} - 2\dfrac{x}{y} = y^2 e^{-y}$ is a L.D.E. in x

$\therefore \quad I.F. = e^{\int P\,dy}, \quad P = -2/y$

$= e^{-2\int 1/y\, dy} = e^{-2\log y} = e^{\log y^{-2}}$

$\therefore I.F. = e^{\log 1/y^2} = 1/y^2$

It's general soln is -

$x \times I.F. = \int Q \times I.F.\, dy + C$

$\dfrac{x}{y^2} = \int y^2 e^{-y} \cdot 1/y^2\, dy + C$

$= \dfrac{e^{-y}}{-1} + C$

$$\boxed{\frac{x}{y^2} + e^{-y} = C}$$

③ $\dfrac{dy}{dx} + y\cot x = \sin 2x$ is linear in y

$\therefore \quad I.F. = e^{\int P\,dx} = e^{\int \cot x\, dx} = e^{\log \sin x}$

$\therefore I.F. = \sin x$.

General soln is, $y \times IF = \int Q \times IF\, dx + C$

$y \sin x = \int \sin 2x \cdot \sin x\, dx + C$

$= 2\int \sin^2 x \cos x\, dx + C$

$= \int \sin^2 x\, (2\cos x)\, dx + C$

~~$y\sin x = 2\int(\sin^2 x \cos x)$~~

$$y \times \sin x = 2 \int \sin^2 x \cos x \, dx + c$$

put $\sin x = t$,

$\cos x \, dx = dt$

$$\therefore y \sin x = 2 \int t^2 dt + c$$

$$y \sin x = 2 \frac{t^3}{3} + c$$

$$\therefore \boxed{y \sin x = \frac{2}{3} \sin^3 x + c}$$

(4) $(x + 2y^3) \frac{dy}{dx} = y$

$$\Rightarrow \frac{dx}{dy} = \frac{x + 2y^3}{y}$$

$$\frac{dx}{dy} = \frac{x}{y} + 2y^2$$

$$\frac{dx}{dy} - \frac{x}{y} = 2y^2$$ in L.D.E. in x

$$\therefore I.F. = e^{\int P dy} = e^{-\int 1/y \, dy} = e^{-\log y} = e^{\log \frac{1}{y}}$$

$$I.F. = 1/y$$

General soln is

$$x \times I.F. = \int Q \times I.F. \, dy + c$$

$$\frac{x}{y} = \int 2y^2 \cdot 1/y \, dy + c$$

$$x/y = \cancel{2} y^2/\cancel{2} + c$$

$$\therefore \boxed{x/y = y^2 + c}$$

PAGE No.
DATE / /

(5) $(1+x^2)\frac{dy}{dx} + y = e^{\tan^{-1}x}$

Ans: $2y = e^{\tan^{-1}x} + c$ $\quad e^{\tan^{-1}x}$

(6) $x^2\frac{dy}{dx} = 3x^2 - 2xy + 1$

Ans: $x^2y = x^3 + x + c$

(7) $\frac{dy}{dx} = \frac{e^x - 3xy}{x^2}$, Ans: $x^3y = c + (x-1)e^x$

(8) $(2y + x^2)\,dx = x\,dy$, Ans: $y = x^2\log(cx)$

(9) $(1+x^2)\frac{dy}{dx} + xy = 1$

$\Rightarrow \frac{dy}{dx} + \frac{xy}{1+x^2} = \frac{1}{1+x^2}$ is L.D.E. in y

$I.F. = e^{\int P\,dx} = e^{\frac{1}{2}\int \frac{2x}{1+x^2}dx}$

$= e^{\frac{1}{2}\log(1+x^2)} = e^{\log(1+x^2)^{1/2}}$

$= (1+x^2)^{1/2}$

$\therefore$ G.S. is

$y \times I.F. = \int Q \times I.F\,dx + c$

$y\sqrt{1+x^2} = \int \frac{1}{1+x^2}\cdot\sqrt{1+x^2}\,dx + c$

$= \int \frac{1}{\sqrt{1+x^2}}\,dx + c$

$y\sqrt{1+x^2} = \log(x + \sqrt{1+x^2}) + c.$

PAGE No.
DATE / /

(10) $(e^{-y}\sec^2 y - x)\,dy = dx$

$\Rightarrow \dfrac{dx}{dy} + x = e^{-y}\sec^2 y$ is L.D.E. in x

$\therefore$ I.F. $= e^{\int P\,dy} = e^{\int dy} = e^{y}$

G.S. is

$x \times \text{I.F.} = \int Q \times \text{I.F.}\, dy + C$

$x e^{y} = \int e^{-y}\sec^2 y \cdot e^{y}\, dy + C$

$= \int \sec^2 y\, dy + C$

$\boxed{x e^{y} = \tan y + C}$

(8) Equation Reducible to Linear Form:

Bernoulli's Equation:

The D.E. of the form:

$\dfrac{dy}{dx} + Py = Q \cdot y^{n}$ — (1) where P & Q are funs of x or constants. is called Bernoulli's D.E. which is non Linear D.E.

To solve it, we divide (1) by y^{n}

$\therefore y^{-n}\dfrac{dy}{dx} + Py^{1-n} = Q$

put $y^{1-n} = u$, $(1-n)y^{-n}\dfrac{dy}{dx} = \dfrac{du}{dx}$

$\therefore y^{-n}\dfrac{dy}{dx} = \dfrac{1}{1-n}\dfrac{du}{dx}$

$\therefore \dfrac{1}{1-n}\dfrac{du}{dx} + Pu = Q$

$\therefore \dfrac{du}{dx} + P(1-n)u = Q(1-n)$

$\therefore \dfrac{du}{dx} + P_1 u = Q_1$

PAGE No.
DATE / /

$\therefore P_1 = P(1-n)$, $Q_1 = (1-n)Q$.

① $\frac{dy}{dx} - \frac{\tan y}{1+x} = (1+x)e^x \sec y$

$\Rightarrow$ Dividing $\sec y$ on both sides, we get

$\cos y \frac{dy}{dx} - \frac{\sin y}{1+x} = (1+x)e^x$

$\therefore \sin y = u$ $\quad \cos y \frac{dy}{dx} = \frac{du}{dx}$

$\therefore \frac{du}{dx} - \frac{u}{1+x} = (1+x)e^x$

$\therefore P = \frac{1}{1+x}$, $Q = (1+x)e^x$

$\therefore I.F. = e^{-\int P dx} = e^{-\int \frac{1}{1+x} dx} = e^{-\log(1+x)}$
$= (1+x)^{-1}$

$\therefore$ G.S. is, $\quad u \times I.F. = \int Q \times I.F. dx + c$

$(1+x)^{-1} \sin y = \int (1+x)e^x \cdot \frac{1}{1+x} dx + c$

$\frac{\sin y}{1+x} = e^x + c$

② $\frac{dy}{dx} - y\tan x = y^4 \sec x$

$\Rightarrow$ Dividing y^4 on both sides,

$y^{-4}\frac{dy}{dx} - y^{-3}\tan x = \sec x$

put $-y^{-3} = u$, $+3y^{-4}\frac{dy}{dx} = \frac{du}{dx}$

$\therefore \frac{du}{dx} + u\frac{\tan x}{3} = \frac{\sec x}{3}$

$P = e^{\int \frac{\tan x}{3} dx} = e^{\frac{1}{3}\log \sec x} = \sec^{\frac{3}{3}} x$

PAGE No.
DATE / /

G.S. is, $u \times IF = \int Q \times IF\, dx + c$

$$-y^{-3}\sec^3 x = \int \sec^4 x\, dx + c$$

$$= \int \sec^2 x \sec^2 x\, dx + c$$

$$= \int (1+\tan^2 x)\sec^2 x\, dx + c$$

$\therefore$ put $\tan x = t$

$\sec^2 x\, dx = dt$

$$= \int (1+t^2)\, dt + c$$

$$= t + t^3/3 + c$$

$$-y^{-3}\sec^3 x = \tan x + \frac{\tan^3 x}{3} + c.$$

③ $xy - dy/dx = y^3 e^{-x^2}$

$\Rightarrow$ $dy/dx - xy = -y^3 e^{-x^2}$

Dividing y^3 on both sides

$y^{-3}\, dy/dx - xy^{-2} = -e^{-x^2}$

put $y^{-2} = u$ $\quad -2y^{-3}\, dy/dx = du/dx$

$\therefore$ $+1/2\, du/dx + xu = +e^{-x^2}$

$du/dx + 2xu = 2e^{-x^2}$

$I.F. = e^{\int 2x\, dx} = e^{x^2}$

G.S. is, $u \times IF = \int Q \times IF\, dx + c$

PAGE No.
DATE / /

$$y^{-2} e^{x^2} = 2\int e^{-x^2} \cdot e^{x^2}\, dx + c$$

$$y^{-2} e^{x^2} = 2x + c$$

(4) $\frac{dy}{dx} + \frac{2y}{x} = y^2 x^2$

Ans: $1/x^2 y = -x + c$, I.F. $= 1/x^2$

(5) $e^y (1 + dy/dx) = e^x$

Ans: $e^y e^x = \frac{e^{2x}}{2} + c$

(6) $3y^2\, dy/dx + 2xy^3 = 4xe^{-x}$

Ans: $y^3 e^{x^2} = 2x^2 + c$

(7) $xy^2\, dy/dx - y^3 = x^2$

Ans: $y^3 = x^2(cx - x)$

PAGE No.
DATE / /

Orthogonal Trajectories:

Trajectory: A curve which cuts every member of a given family of curves according to some definite law, is called trajectory.

Orthogonal Trajectory: A curve which cuts every member of a given family of curves at right angles is called orthogonal trajectory.

Working Rule for finding Orthogonal Trajectories:

(1) For Cartesian curves: $f(x, y, c) = 0$ — (1)

Step 1) Differentiating eqn (1) w.r.t. x to eliminate c.

$$\therefore \quad \phi\left(x, y, \frac{dy}{dx}\right) = 0$$

Step 2) Replacing dy/dx by $-dx/dy$ &

Forming D.E. for family of orthogonal trajectories as -

$$\phi\left(x, y, -dx/dy\right) = 0 \quad \text{— (2)}$$

Solving D.E. (2), we get the family of orthogonal trajectories.

PAGE No.
DATE / /

(2) For polar curves $F(r, \theta, c) = 0$ — (1)

Step 1) Differentiating (1) w.r.t. θ and forming D.E. for the given family as,

$$\phi\left(r, \theta, \frac{dr}{d\theta}\right) = 0$$

Step 2) Replacing $\frac{dr}{d\theta}$ by $-r^2 \frac{d\theta}{dr}$ we get D.E. for the family of orthogonal trajectories as

$$\phi\left(r, \theta, -r^2 \frac{d\theta}{dr}\right) = 0 \quad \text{— (2)}$$

solving D.E. (2), we get the family of orthogonal trajectories.

(1) Find orthogonal trajectories of the family of parabolas $y = ax^2$ — (1)

→ Let $y = ax^2$

$$\frac{dy}{dx} = 2ax$$

$$\therefore \frac{dy}{dx} = \frac{2y}{x} \quad \text{using } a = y/x^2$$

is the D.E. for the family (1)

Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ we get

$$-dx/dy = 2y/x$$

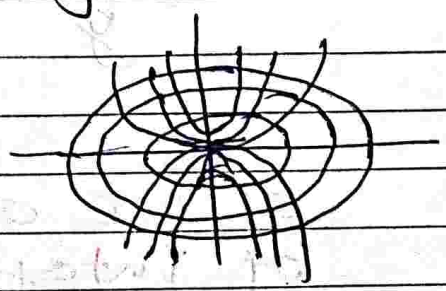
$\therefore$ $x\,dx + 2y\,dy = 0$ is the D.E. for the family of orthogonal trajectories.

$\therefore$ by integrating

$$\int x\,dx + 2\int y\,dy = c$$

$$x^2/2 + 2y^2/2 = c$$

$$x^2 + 2y^2 = c$$



$$\frac{x^2}{(\sqrt{2})^2} + y^2 = c \Rightarrow \frac{x^2}{(\sqrt{2c})^2} + \frac{y^2}{c^2} = 0$$

② Find the orthogonal trajectories of the family of

① $xy = c^2$ ——— (1)

$x\frac{dy}{dx} + y = 0$ is the D.E. of family curves (1)

Replacing dy/dx by $-dx/dy$

$\therefore -x\frac{dx}{dy} + y = 0$

$-x\,dx + y\,dy = 0$ be a D.E. of family of orthogonal trajectories

$\therefore$ by integrating we get

$-x^2/2 + y^2/2 = c$

$\therefore \boxed{x^2 - y^2 = C}$

③ $y^2 = 4ax$

$\rightarrow 2y\frac{dy}{dx} = 4a$

$2y\frac{dy}{dx} = \frac{y^2}{x}$ Replacing dy/dx by $-dx/dy$ we get

$\cancel{2}\cancel{y}\frac{dy}{\cancel{y}} = \frac{dx}{\cancel{x}}$

$-2\frac{dx}{dy} = \frac{y}{x}$

$+2x\,dx + y\,dy = 0$ is a D.E. of family of orthogonal trajectories

by integrating we get

$x^2 + y^2/2 = C$

$\boxed{2x^2 + y^2 = c}$

PAGE No.
DATE / /

① $r = a(1-\cos\theta)$ ② $r^2 = a^2\cos 2\theta$.
③ $r = a(1+\cos\theta)$ ④ $r^2 = a^2\sin 2\theta$.

⇒ ① let $r = a(1-\cos\theta)$

$\frac{dr}{d\theta} = a\sin\theta$ be a D.E. of family of given curves.

Replacing $dr/d\theta$ by $-r^2 d\theta/dr$

$\therefore -r^2 d\theta/dr = a\sin\theta$. $\because a = \frac{r}{1-\cos\theta}$.

$-r^2 d\theta/dr = \frac{r\sin\theta}{1-\cos\theta}$.

$-r\, d\theta/dr = \frac{\cancel{2}\sin\theta/2\cos\theta/2}{\cancel{2}\sin^2\theta/2}$

$-r\frac{d\theta}{dr} = \frac{\cos\theta/2}{\sin\theta/2}$

$-\frac{\sin\theta/2}{\cos\theta/2}d\theta = \frac{dr}{r}$ be a D.E. of family of orthogonal trajectories

$\therefore$ by integrating we get

$\log\cos\theta/2 = \log r + \log c$

$2\log\cos\theta/2 = \log r + \log c$

$\log\cos^2\theta/2 = \log rc$

$\frac{1+\cos\theta}{2} = rc$

$\therefore \boxed{r = c(1+\cos\theta)}$