Introduction to mathematical biology and application in R

Koissi Savi

December 13, 2020

zef
Center for
Development Research
University of Bonn

Ministerium für Innovation,
Wissenschaft und Forschung
des Landes Nordrhein-Westfalen

UNIVERSITÄT BONN

Outline

1. Motivation of the seminar-lecture

2. Dynamics of the population
   - Exponential growth
   - Logistic growth

History of mathematical biology

- **Leonardo Bonacci** [Fibonacci] created a series denoted **Fibonacci series** (each number is the sum of the two preceding ones) to explain the dynamics of rabbits around the 13th century

History of mathematical biology

- **Leonardo Bonacci** [Fibonacci] created a series denoted **Fibonacci series** (each number is the sum of the two preceding ones) to explain the dynamics of rabbits around the 13th century
- **Daniel Bernouilli** in the 18th century applied a system of two ordinary differential equations (ODEs) to describe the dynamics of smallpox on the human population and defined how vaccine could be used to prevent the spread of this disease.

History of mathematical biology

- **Leonardo Bonacci** [Fibonacci] created a series denoted **Fibonacci series** (each number is the sum of the two preceding ones) to explain the dynamics of rabbits around the 13th century
- **Daniel Bernouilli** in the 18th century applied a system of two ordinary differential equations (ODEs) to describe the dynamics of smallpox on the human population and defined how vaccine could be used to prevent the spread of this disease.
- **Thomas Malthus** in 1798 theorized on the **exponential growth** of the population.

History of mathematical biology

- **Leonardo Bonacci** [Fibonacci] created a series denoted **Fibonacci series** (each number is the sum of the two preceding ones) to explain the dynamics of rabbits around the 13th century
- **Daniel Bernouilli** in the 18th century applied a system of two ordinary differential equations (ODEs) to describe the dynamics of smallpox on the human population and defined how vaccine could be used to prevent the spread of this disease.
- **Thomas Malthus** in 1798 theorized on the **exponential growth** of the population.
- **Pierre François Verhulst** in 1836 formulated the **logistic growth** of the population

History of mathematical biology

- **Leonardo Bonacci** [Fibonacci] created a series denoted **Fibonacci series** (each number is the sum of the two preceding ones) to explain the dynamics of rabbits around the 13th century
- **Daniel Bernouilli** in the 18th century applied a system of two ordinary differential equations (ODEs) to describe the dynamics of smallpox on the human population and defined how vaccine could be used to prevent the spread of this disease.
- **Thomas Malthus** in 1798 theorized on the **exponential growth** of the population.
- **Pierre François Verhulst** in 1836 formulated the **logistic growth** of the population
- **Fritz Müller** in 1878 theorized on the mimicry of species that share a common predators. The theory was first described based species coloration eg: coral snakes vs scarlet kingsnake.

Motivation of the seminar-lecture

Some reasons motivate this seminar lecture,

- As a researcher (biologist/ ecologist) the mathematics helps to design your experiment or clinical trials so that your results will be meaningful (statistical significant)

Motivation of the seminar-lecture

Some reasons motivate this seminar lecture,

- As a researcher (biologist/ ecologist) the mathematics helps to design your experiment or clinical trials so that your results will be meaningful (statistical significant)
- Sometimes it is impossible or too difficult to answer a research question through traditional lab experiments, so biologists rely on math to develop models that represent the system they're studying. These approaches allow scientists to indicate the likelihood of certain outcomes as well as refine the research questions in an ethical manner.

Motivation of the seminar-lecture

Some reasons motivate this seminar lecture,

- As a researcher (biologist/ ecologist) the mathematics helps to design your experiment or clinical trials so that your results will be meaningful (statistical significant)
- Sometimes it is impossible or too difficult to answer a research question through traditional lab experiments, so biologists rely on math to develop models that represent the system they're studying. These approaches allow scientists to indicate the likelihood of certain outcomes as well as refine the research questions in an ethical manner.
- Mathematical modeling helps biologist to identify pattern in their studies.

Motivation of the seminar-lecture

Some reasons motivate this seminar lecture,

- As a researcher (biologist/ ecologist) the mathematics helps to design your experiment or clinical trials so that your results will be meaningful (statistical significant)
- Sometimes it is impossible or too difficult to answer a research question through traditional lab experiments, so biologists rely on math to develop models that represent the system they're studying. These approaches allow scientists to indicate the likelihood of certain outcomes as well as refine the research questions in an ethical manner.
- Mathematical modeling helps biologist to identify pattern in their studies.

Bull Math Biol (2015) 77:735–738
DOI 10.1007/s11538-015-0065-9

Society for Mathematical Biology

CrossMark

SPECIAL ISSUE ARTICLE

What Has Mathematics Done for Biology?

Michael C. Mackey · Philip K. Maini

Chap I: Dynamics of biological population

Exponential growth

Definition

This growth occurs when the instantaneous rate of change of a quantity is proportional to the quantity itself. It is also called **geometric growth** or **Malthusian growth model**.

Exponential growth

Definition

This growth occurs when the instantaneous rate of change of a quantity is proportional to the quantity itself. It is also called **geometric growth** or **Malthusian growth model**.

- This growth is observed for a colony of bacterial, of mitosis-borne cells.

Exponential growth

Definition

This growth occurs when the instantaneous rate of change of a quantity is proportional to the quantity itself. It is also called **geometric growth** or **Malthusian growth model**.

- This growth is observed for a colony of bacterial, of mitosis-borne cells.
- A virus (for example COVID-19, or smallpox) typically will spread exponentially at first, if no artificial immunization is available.

Exponential growth

Definition

This growth occurs when the instantaneous rate of change of a quantity is proportional to the quantity itself. It is also called **geometric growth** or **Malthusian growth model**.

- This growth is observed for a colony of bacterial, of mitosis-borne cells.
- A virus (for example COVID-19, or smallpox) typically will spread exponentially at first, if no artificial immunization is available.
- In finance the Pyramid schemes or Ponzi schemes also show this type of growth resulting in high profits for a few initial investors and losses among great numbers of investors.

Notation and computation

Let N be the number of individuals in a population at time t,
b the average per capita birth rate and
d be the average per capita death rate
After a short period dt, the fraction of change in the population N is given by:

$$\frac{dN}{dt} = bN - dN \tag{1}$$

Notation and computation

Let N be the number of individuals in a population at time t,
b the average per capita birth rate and
d be the average per capita death rate
After a short period dt, the fraction of change in the population N is given by:

$$\frac{dN}{dt} = bN - dN \tag{1}$$

$$(1) \leftrightarrow \frac{dN}{dt} = (b - d)N$$

Notation and computation

Let N be the number of individuals in a population at time t,
b the average per capita birth rate and
d be the average per capita death rate
After a short period dt, the fraction of change in the population N is given by:

$$\frac{dN}{dt} = bN - dN \tag{1}$$

$$(1) \leftrightarrow \frac{dN}{dt} = (b-d)N$$

$$\frac{dN}{N} = (b-d)dt$$

Notation and computation

Let N be the number of individuals in a population at time t,
b the average per capita birth rate and
d be the average per capita death rate
After a short period dt, the fraction of change in the population N is given by:

$$\frac{dN}{dt} = bN - dN \tag{1}$$

$$(1) \leftrightarrow \frac{dN}{dt} = (b-d)N$$

$$\frac{dN}{N} = (b-d)dt$$

$$\int_0^\infty \frac{dN}{N} = \int_0^\infty (b-d)dt$$

Notation and computation

Let N be the number of individuals in a population at time t,
b the average per capita birth rate and
d be the average per capita death rate
After a short period dt, the fraction of change in the population N is given by:

$$\frac{dN}{dt} = bN - dN \tag{1}$$

$$(1) \leftrightarrow \frac{dN}{dt} = (b-d)N$$

$$\frac{dN}{N} = (b-d)dt$$

$$\int_0^\infty \frac{dN}{N} = \int_0^\infty (b-d)dt$$

$$N = N_0 e^{(b-d)t} \tag{2}$$

Notation and computation

Let N be the number of individuals in a population at time t,
b the average per capita birth rate and
d be the average per capita death rate
After a short period dt, the fraction of change in the population N is given by:

$$\frac{dN}{dt} = bN - dN \tag{1}$$

$$(1) \leftrightarrow \frac{dN}{dt} = (b-d)N$$

$$\frac{dN}{N} = (b-d)dt$$

$$\int_0^{\infty} \frac{dN}{N} = \int_0^{\infty} (b-d)dt$$

$$N = N_0 e^{(b-d)t} \tag{2}$$

Assuming that $b - d = r$ with r representing the **per capita growth**;
$(2) \leftrightarrow N = N_0 e^{rt}$

Example of the dynamics of rabbit

Let's assume that we have a population of 1000 rabbit living in unlimited space and with unlimited resources. Besides, let's consider that the monthly growth rate of the rabbit is 10%. How will this population grow?

Months / time t	**Population** N_0	**Growth factor** r
0	1000	
1	1000	1.1
2	1000	1.1×1.1
3	1000	$(1.1)^3$
$\vdots$	$\vdots$	$\vdots$
t	1000	$(1.1)^t$

Thus, the population of rabbit is given by $N(t) = 1000 \times (1.1)^t$.
The population after 10 years (120 months) is given by
$N(120) = 1000 \times (1.1)^{120} == 93{,}000{,}000$ rabbits

Logistic growth

Definition

Contrary to the exponential growth that has a constant **per capita growth**, in **logistic growth** the per capita growth rate gets smaller as the population size approaches its maximum imposed by limited resource also called **carrying capacity** denoted K.

Logistic growth

Definition

Contrary to the exponential growth that has a constant **per capita growth**, in **logistic growth** the per capita growth rate gets smaller as the population size approaches its maximum imposed by limited resource also called **carrying capacity** denoted K.

- An exponential growth has a **J-shaped curve** while a logistic growth has an **s-shaped curve**.

Logistic growth

Definition

Contrary to the exponential growth that has a constant **per capita growth**, in **logistic growth** the per capita growth rate gets smaller as the population size approaches its maximum imposed by limited resource also called **carrying capacity** denoted K.

- An exponential growth has a **J-shaped curve** while a logistic growth has an **s-shaped curve**.

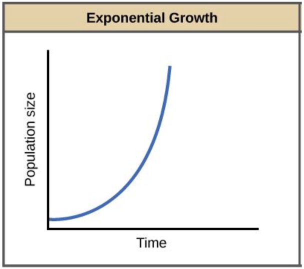

Image credit: "Environmental limits to population growth: Figure 1," by

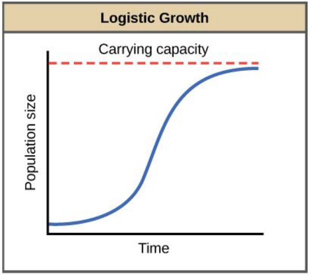

Image credit: "Environmental limits to population growth: Figure 1," by

Notation and computation

Let us modify the premises of the equation (1).

Notation and computation

Let us modify the premises of the equation (1).
Using an r (per capita growth rate) that depends on population size N and how close it is to carrying capacity K.

Notation and computation

Let us modify the premises of the equation (1).
Using an r (per capita growth rate) that depends on population size N and how close it is to carrying capacity K.
Assuming that the population has a base growth rate of r when it is very small, we can write the following equation:

Notation and computation

Let us modify the premises of the equation (1).
Using an r (per capita growth rate) that depends on population size N and how close it is to carrying capacity K.
Assuming that the population has a base growth rate of r when it is very small, we can write the following equation:

$$\frac{dN}{dt} = r\frac{(K-N)}{K}N \tag{3}$$

Notation and computation

Let us modify the premises of the equation (1).
Using an r (per capita growth rate) that depends on population size N and how close it is to carrying capacity K.
Assuming that the population has a base growth rate of r when it is very small, we can write the following equation:

$$\frac{dN}{dt} = r\frac{(K-N)}{K}N \tag{3}$$

$$\frac{dN}{dt} = rN(1-\frac{N}{K})$$

Notation and computation

Let us modify the premises of the equation (1).
Using an r (per capita growth rate) that depends on population size N and how close it is to carrying capacity K.
Assuming that the population has a base growth rate of r when it is very small, we can write the following equation:

$$\frac{dN}{dt} = r\frac{(K-N)}{K}N \tag{3}$$

$$\frac{dN}{dt} = rN(1-\frac{N}{K})$$

We can observe that this equation has two fixed points.

Notation and computation

Let us modify the premises of the equation (1).
Using an r (per capita growth rate) that depends on population size N and how close it is to carrying capacity K.
Assuming that the population has a base growth rate of r when it is very small, we can write the following equation:

$$\frac{dN}{dt} = r\frac{(K-N)}{K}N \tag{3}$$

$$\frac{dN}{dt} = rN(1-\frac{N}{K})$$

We can observe that this equation has two fixed points.

1. when $\lim_{N\to 0} \frac{dN}{dt} = rN ==>$ Exponential growth

Notation and computation

Let us modify the premises of the equation (1).
Using an r (per capita growth rate) that depends on population size N and how close it is to carrying capacity K.
Assuming that the population has a base growth rate of r when it is very small, we can write the following equation:

$$\frac{dN}{dt} = r\frac{(K-N)}{K}N \tag{3}$$

$$\frac{dN}{dt} = rN(1-\frac{N}{K})$$

We can observe that this equation has two fixed points.

1. when $\lim_{N\to 0}\frac{dN}{dt} = rN ==>$ Exponential growth
2. when $\lim_{N\to K}\frac{dN}{dt} = 0 ==>$ The slope of the curve is zero indicating an asymptote in K.

Notation and computation

Let us modify the premises of the equation (1).
Using an r (per capita growth rate) that depends on population size N and how close it is to carrying capacity K.
Assuming that the population has a base growth rate of r when it is very small, we can write the following equation:

$$\frac{dN}{dt} = r\frac{(K-N)}{K}N \tag{3}$$

$$\frac{dN}{dt} = rN(1-\frac{N}{K})$$

We can observe that this equation has two fixed points.

1. when $\lim_{N\to 0} \frac{dN}{dt} = rN$ ==> Exponential growth
2. when $\lim_{N\to K} \frac{dN}{dt} = 0$ ==> The slope of the curve is zero indicating an asymptote in K.

The general form of the solution to this ode is given by:

Notation and computation

Let us modify the premises of the equation (1).
Using an r (per capita growth rate) that depends on population size N and how close it is to carrying capacity K.
Assuming that the population has a base growth rate of r when it is very small, we can write the following equation:

$$\frac{dN}{dt} = r\frac{(K-N)}{K}N \tag{3}$$

$$\frac{dN}{dt} = rN(1-\frac{N}{K})$$

We can observe that this equation has two fixed points.

1. when $\lim_{N\to 0}\frac{dN}{dt} = rN$ ==> Exponential growth
2. when $\lim_{N\to K}\frac{dN}{dt} = 0$ ==> The slope of the curve is zero indicating an asymptote in K.

The general form of the solution to this ode is given by:

$$N = \frac{K}{1+be^{-rt}} \tag{4}$$

Notation and computation

Let us modify the premises of the equation (1).
Using an r (per capita growth rate) that depends on population size N and how close it is to carrying capacity K.
Assuming that the population has a base growth rate of r when it is very small, we can write the following equation:

$$\frac{dN}{dt} = r\frac{(K-N)}{K}N \tag{3}$$

$$\frac{dN}{dt} = rN(1-\frac{N}{K})$$

We can observe that this equation has two fixed points.

1. when $\lim_{N\to 0}\frac{dN}{dt} = rN ==>$ Exponential growth
2. when $\lim_{N\to K}\frac{dN}{dt} = 0 ==>$ The slope of the curve is zero indicating an asymptote in K.

The general form of the solution to this ode is given by:

$$N = \frac{K}{1+be^{-rt}} \tag{4}$$

With N_0 representing the initial population

Exercise

Demonstrate that the general form of a logistic growth model is given by the Eq (4)

$$\frac{dN}{dt} = r\frac{(K-N)}{K}N \rightarrow N = \frac{K}{1+be^{-rt}}$$

Solution

$$\frac{dN}{dt} = r\frac{(K-N)}{K}N \rightarrow N = \frac{K}{1+N_0e^{-rt}}$$

Solution

$$\frac{dN}{dt} = r\frac{(K-N)}{K}N \rightarrow N = \frac{K}{1+N_0e^{-rt}}$$

Let's now integrate both sides of above-mentioned equation

Solution

$$\frac{dN}{dt} = r\frac{(K-N)}{K}N \to N = \frac{K}{1+N_0e^{-rt}}$$

Let's now integrate both sides of above-mentioned equation

$$\int \frac{dN}{N(1-\frac{N}{K})} = \int r dt$$

Solution

$$\frac{dN}{dt} = r\frac{(K-N)}{K}N \to N = \frac{K}{1+N_0e^{-rt}}$$

Let's now integrate both sides of above-mentioned equation

$$\int \frac{dN}{N(1-\frac{N}{K})} = \int rdt$$

Let's apply the partial fraction decomposition property on the left hand side of the previous equation. We obtain

Solution

$$\frac{dN}{dt} = r\frac{(K-N)}{K}N \to N = \frac{K}{1+N_0e^{-rt}}$$

Let's now integrate both sides of above-mentioned equation

$$\int \frac{dN}{N(1-\frac{N}{K})} = \int rdt$$

Let's apply the partial fraction decomposition property on the left hand side of the previous equation. We obtain

$$\int \frac{1}{N} + \frac{1}{K-N}dN = \int rdt$$

Solution

$$\frac{dN}{dt} = r\frac{(K-N)}{K}N \to N = \frac{K}{1+N_0e^{-rt}}$$

Let's now integrate both sides of above-mentioned equation

$$\int \frac{dN}{N(1-\frac{N}{K})} = \int rdt$$

Let's apply the partial fraction decomposition property on the left hand side of the previous equation. We obtain

$$\int \frac{1}{N} + \frac{1}{K-N}dN = \int rdt$$

Solution

We can apply the additive property of integral and it is easy to see that:

Solution

We can apply the additive property of integral and it is easy to see that:

$$\int \frac{1}{N} + \frac{1}{K-N} dN = \int r dt \rightarrow ln(N) - ln(K-N) = rt + C$$

Solution

We can apply the additive property of integral and it is easy to see that:

$$\int \frac{1}{N} + \frac{1}{K-N} dN = \int r dt \to ln(N) - ln(K-N) = rt + C$$

With C be a constant.

Solution

We can apply the additive property of integral and it is easy to see that:

$$\int \frac{1}{N} + \frac{1}{K-N} dN = \int r dt \rightarrow ln(N) - ln(K-N) = rt + C$$

With C be a constant. Using the natural log property we have

Solution

We can apply the additive property of integral and it is easy to see that:

$$\int \frac{1}{N} + \frac{1}{K-N} dN = \int r dt \rightarrow ln(N) - ln(K-N) = rt + C$$

With C be a constant. Using the natural log property we have

$$ln(N) - ln(K-N) = rt + C \leftrightarrow ln(\frac{N}{K-N}) = rt + C$$

Solution

We can apply the additive property of integral and it is easy to see that:

$$\int \frac{1}{N} + \frac{1}{K-N} dN = \int rdt \to ln(N) - ln(K-N) = rt + C$$

With C be a constant. Using the natural log property we have

$$ln(N) - ln(K-N) = rt + C \leftrightarrow ln(\frac{N}{K-N}) = rt + C$$

We can then exponentiate the previous expression and as C is a constant let denoted the e^{-c} b. Thus, it is easy to see that

Solution

We can apply the additive property of integral and it is easy to see that:

$$\int \frac{1}{N} + \frac{1}{K-N} dN = \int r dt \to ln(N) - ln(K-N) = rt + C$$

With C be a constant. Using the natural log property we have

$$ln(N) - ln(K-N) = rt + C \leftrightarrow ln(\frac{N}{K-N}) = rt + C$$

We can then exponentiate the previous expression and as C is a constant let denoted the e^{-c} b. Thus, it is easy to see that

$$\frac{dN}{dt} = r\frac{(K-N)}{K}N \leftrightarrow N = \frac{K}{1+be^{-rt}}$$

Solution

We can apply the additive property of integral and it is easy to see that:

$$\int \frac{1}{N} + \frac{1}{K-N} dN = \int r dt \rightarrow ln(N) - ln(K-N) = rt + C$$

With C be a constant. Using the natural log property we have

$$ln(N) - ln(K-N) = rt + C \leftrightarrow ln(\frac{N}{K-N}) = rt + C$$

We can then exponentiate the previous expression and as C is a constant let denoted the e^{-c} b. Thus, it is easy to see that

$$\frac{dN}{dt} = r\frac{(K-N)}{K}N \leftrightarrow N = \frac{K}{1+be^{-rt}}$$

At $t = 0$ it is easy to see that $b = \frac{N_0}{L-N_0}$ with N_0 be the initial population. Thus,

$$N = \frac{N_0 K}{N_0 + (L-N_0)e^{-rt}}$$

Example of cows - Problem

100 cows were released on an island in 2005. By 2012, they were 324 cows. The island has a big pasture land that can carry a maximum of 5000 cows.

1. Assuming logistic growth, write the general equation that describes the population $N(t)$ at any time t?
2. How many dogs will be there by 2020?
3. In what year will the population reach 1000?

Example of cows - Solution

1- Equation describing the population

$N(t) = \frac{K}{1+be^{-rt}}$ In 2005 $t = 0 \rightarrow N(0) = 100$

In 2012 $t = 7 \rightarrow N(7) = 324$

The carrying capacity is $K = 5000$ $100 = \frac{5000}{1+be^{-r(0)}} \rightarrow b = 49$

$324 = \frac{5000}{1+49e^{-r(7)}} \rightarrow r = 0.1746$

$N(t) = \frac{5000}{1+49e^{-0.1746(t)}}$

2- Number of cows in 2020 (t=15)

1094 cows

3- Determination of t

$t = 14.35 \rightarrow 2019$

r and K-selected species

- Like the elephant, **K-selected** species have a logistic growth.
- They are defined by fewer offspring and they provide a long-term care after birth

r and K-selected species


- Like the elephant, **K-selected** species have a logistic growth.
- They are defined by fewer offspring and they provide a long-term care after birth

- **r-selected** species like jellyfish have an exponential growth
- They are defined by large number of offspring and short to no care after birth

r and K-selected species

- Like the elephant, **K-selected** species have a logistic growth.
- They are defined by fewer offspring and they provide a long-term care after birth

- **r-selected** species like jellyfish have an exponential growth
- They are defined by large number of offspring and short to no care after birth

- More debate in this category: The oak for instance has a large number of offspring but because a lot also survive they are defined as a K-selected species.

Lokta-Voltera model