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1. If $f:\mathbb{R}\to\mathbb{R}$, let $g:\mathbb{R}\to\mathbb{R}$ and $h:\mathbb{R}\to\mathbb{R}$ where $g(x)\leq f(x)\leq h(x)$.

$$f(x)=\begin{cases} x \text{ if } x\in\mathbb{Q} \\ -x \text{ if } x\notin\mathbb{Q}\end{cases}$$

We define $g(x)=-|x|$ and $h(x)=|x|$ so that $g(x)\leq f(x)\leq h(x)$.

If $g(x)\leq f(x)\leq h(x)$ for all $\mathbb{R}\to\mathbb{R}$, and $\lim_{x\to 0} g(x)=\lim_{x\to 0} h(x)=L$, then $\lim_{x\to 0} f(x)=L$

$$\lim_{x\to 0} g(x)=\lim_{x\to 0} -|x| = \left|\lim_{x\to 0} -x\right| = |-0| = 0$$

$$\lim_{x\to 0} h(x)=\lim_{x\to 0} |x| = \left|\lim_{x\to 0} x\right| = |0| = 0$$

Therefore as $\lim_{x\to 0} g(x) = \lim_{x\to 0} h(x)=0$, then $\lim_{x\to 0} f(x)=0$

2.a) ~~For $x\neq 8$,~~

$$x-8=(\sqrt[3]{x})^3-(2^3)=(\sqrt[3]{x}-2)(\sqrt[3]{x^2}+2\sqrt[3]{x}+4)$$

For $x\neq 8$,

$$\lim_{x\to 8}\frac{x-8}{\sqrt[3]{x}-2}=\lim_{x\to 8}\frac{(\sqrt[3]{x}-2)(\sqrt[3]{x^2}+2\sqrt[3]{x}+4)}{\sqrt[3]{x}-2}$$

$$=\lim_{x\to 8}(\sqrt[3]{x^2}+2\sqrt[3]{x}+4)=4+4+4=12.$$

520002888

2. b) $\dfrac{x^3-4x}{|x^3-2x^2-x+2|} = \dfrac{x(x+2)(x-2)}{|(x-2)(x+1)(x-1)|} = \dfrac{x(x+2)(x-2)}{|x-2||x+1||x-1|}$

Therefore $\displaystyle\lim_{x\to 2} \frac{x^3-4x}{|x^3-2x^2-x+2|} = \lim_{x\to 2} \frac{x(x+2)(x-2)}{|x-2||x-1||x+1|}$

Consider the cases when $x>2$ and when $x<2$, where $x\neq 2$ as $f(2)$ is undefined.

Case 1: $x>2 \Leftrightarrow x-2>0 \Rightarrow |x-2| = (x-2)$
Case 2: $x<2 \Leftrightarrow x-2<0 \Rightarrow |x-2| = -(x-2)$

Therefore for Case 1:

$$\lim_{x\to 2^+} \frac{x(x+2)(x-2)}{|x-2||x-1||x+1|} = \lim_{x\to 2^+} \frac{x(x+2)(x-2)}{(x-2)|x-1||x+1|}$$
$$= \lim_{x\to 2^+} \frac{x(x+2)}{|x-1||x+1|} = \frac{2\cdot 4}{|1|\cdot|3|} = \frac{8}{3}$$

And for Case 2:

$$\lim_{x\to 2^-} \frac{x(x+2)(x-2)}{|x-2||x+1||x-1|} = \lim_{x\to 2^-} \frac{x(x+2)(x-2)}{-(x-2)|x-1||x+1|}$$
$$= \lim_{x\to 2^-} \frac{x(x+2)}{-|x-1||x+1|} = \frac{2\cdot 4}{-|1|\cdot|3|} = \frac{-8}{3}$$

Hence $\displaystyle\lim_{x\to 2^+} \frac{x^3-4x}{|x^3-2x^2-x+2|} \neq \lim_{x\to 2^-} \frac{x^3-4x}{|x^3-2x^2-x+2|}$

As the limit of f as x approaches 2 from the left is not equal to the limit of f as x approaches 2 from the right, the two sided limit $\displaystyle\lim_{x\to 2} \frac{x^3-4x}{|x^3-2x^2-x+2|}$ does not exist.

520002888

3a) $f(x) = \frac{x^2+2x-3}{x^2-x-6} = \frac{(x+3)(x-1)}{(x+2)(x-3)}$

We know that $\lim_{x\to0^+} \frac{1}{x} = \infty$ and $\lim_{x\to0^-} \frac{1}{x} = -\infty$, then:

$$\lim_{x\to-2^+} \frac{(x+3)(x-1)}{(x+2)(x-3)} = \frac{\lim_{x\to-2^+}(x+3)(x-1)}{\lim_{x\to-2^+}(x+2)(x-3)} = \infty$$

$$\lim_{x\to-2^-} \frac{(x+3)(x-1)}{(x+2)(x-3)} = \frac{\lim_{x\to-2^-}(x+3)(x-1)}{\lim_{x\to-2^-}(x+2)(x-3)} = -\infty$$

Therefore $f(x)$ has a vertical assymptote at $x=-2$.

$$\lim_{x\to3^+} \frac{(x+3)(x-1)}{(x+2)(x-3)} = \frac{\lim_{x\to3^+}(x+3)(x-1)}{\lim_{x\to3^+}(x+2)(x-3)} = \infty$$

$$\lim_{x\to3^-} \frac{(x+3)(x-1)}{(x+2)(x-3)} = \frac{\lim_{x\to3^-}(x+3)(x-1)}{\lim_{x\to3^-}(x+2)(x-3)} = -\infty$$

Therefore $f(x)$ has a vertical assymptote at $x=3$

A function has a horizontal assymptote ~~when~~ at $f(x)=L$ when $\lim_{x\to\infty} f(x) = L$ where $L \in \mathbb{R}$, then:

$$\lim_{x\to\infty} \frac{x^2+2x-3}{x^2-x-6} = \lim_{x\to\infty} \frac{x^2/x^2 + 2x/x^2 - 3/x^2}{x^2/x^2 - x/x^2 - 6/x^2}$$

$$= \lim_{x\to\infty} \frac{1 + 2/x - 3/x^2}{1 - 1/x - 6/x^2} = \frac{1+0-0}{1-0-0} = \frac{1}{1} = 1$$

Therefore ~~the~~ $f(x)$ has a horizontal assymptote at $f(x)=1$

520002888

3b.) $f(x) = \dfrac{x^3-4x}{|x^3-2x^2-x+2|} = \dfrac{x(x+2)(x-2)}{|(x-2)(x+1)(x-1)|}$

Consider $\lim_{x\to 1} \dfrac{x(x+2)(x-2)}{|(x-2)(x+1)(x-1)|}$

~~For $x \to 1$, $(x-2)$~~

For $1<x<2$, $(x-2)(x-1)(x+1)<0$, therefore: ~~and we are looking at $x \to 1$~~

$$\lim_{x\to 1^+} \frac{x(x+2)(x-2)}{|(x-2)(x+1)(x-1)|} = \lim_{x\to 1^+} \frac{-x(x+2)(x-2)}{(x-2)(x+1)(x-1)} = -\infty$$

For $x<-1$, $(x-2)(x-1)(x+1)<0$, therefore:

$$\lim_{x\to -1^-} \frac{x(x+2)(x-2)}{|(x-2)(x+1)(x-1)|} = \cancel{x(x+2)} \lim_{x\to -1^-} \frac{-x(x+2)(x-2)}{(x-2)(x+1)(x-1)} = \infty$$

~~As~~ ~~$\lim_{x\to 1^+}$~~ As $\lim_{x\to 1^+} f(x) = -\infty$ and $\lim_{x\to -1^-} f(x) = \infty$, we know that $f(x)$ has vertical assymptotes at $x=1$ and $x=-1$.

To find the horizontal assymptote(s) of $f(x)$, we consider ~~2 cases: when $x>2$ and when $x<-1$.~~ $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$.

~~Case 1 when $x>2$, $(x-2)$, $(x-1)$, $(x+1)$~~

- $\lim_{x\to\infty} \dfrac{x^3-4x}{|x^3-2x^2-x+2|} = \lim_{x\to\infty} \dfrac{x^3-4x}{x^3-2x^2-x+2}$ As when $x>2$, $(x-2)(x+1)(x-1)>0$

$$\lim_{x\to\infty} \frac{x^3-4x}{x^3-2x^2-x+2} = \lim_{x\to\infty} \frac{x^3/x^3 - 4x/x^3}{x^3/x^3 - 2x^2/x^3 - x/x^3 + 2/x^3}$$

$$= \lim_{x\to\infty} \frac{1-4/x^2}{1-2/x-1/x^2+2/x^3} = \frac{1-0}{1-0-0+0} = 1.$$

- $\lim_{x\to-\infty} \dfrac{x^3-4x}{|x^3-2x^2-x+2|} = \lim_{x\to-\infty} \dfrac{x^3-4x}{-x^3+2x^2+x-2}$ As when $x<-1$, $(x-2)(x+1)(x-1)<0$.

$$\lim_{x\to-\infty} \frac{x^3-4x}{-x^3+2x^2+x-2} = \lim_{x\to-\infty} \frac{x^3/x^3 - 4x/x^3}{-x^3/x^3 + 2x^2/x^3 + x/x^3 - 2/x^3}$$

$$= \lim_{x\to-\infty} \frac{1-4/x^2}{-1+2/x+1/x^2-2/x^3} = \frac{1-0}{-1+0+0-0} = -1$$

Therefore, $f(x)$ has a horizontal assymptote at:

$f(x)=1$ so that $f(x)\to 1$ as $x\to\infty$

$f(x)=-1$ so that $f(x)\to -1$ as $x\to-\infty$

520002888

3 c) $f(x) = \sqrt{x^2+2x+2} = \sqrt{(x+1)^2+1}$

As x becomes large, $\sqrt{(x+1)^2+1}$ looks like $\sqrt{(x+1)^2} = |x+1|$, therefore we select $|x+1|$ as our candidate for an inclined assymptote

Now, $\sqrt{x^2+2x+2} - |x+1| = \left(\sqrt{x^2+2x+2} - |x+1|\right)\left(\dfrac{\sqrt{x^2+2x+2} + |x+1|}{\sqrt{x^2+2x+2} + |x+1|}\right)$

$$= \frac{x^2+2x+2-x^2-2x-1}{\sqrt{x^2+2x+2} + |x+1|} = \frac{1}{\sqrt{x^2+2x+2} + |x+1|}$$

~~We know that $0 \le \sqrt{x^2+2x+2} - |x+1| = \frac{1}{\sqrt{x^2+2x+2}+|x+1|}$~~

We know that $\sqrt{x^2+2x+2} \ge |x+1|$

We also know that $0 \le \sqrt{x^2+2x+2} - |x+1| = \dfrac{1}{\sqrt{x^2+2x+2}+|x+1|}$

Hence $\dfrac{1}{|x+1|} \ge \dfrac{1}{\sqrt{x^2+2x+2}} \ge \dfrac{1}{\sqrt{x^2+2x+2}+|x+1|}$

Therefore

$$0 \le \frac{1}{\sqrt{x^2+2x+2}+|x+1|} \le \frac{1}{|x+1|}$$

So $\lim\limits_{x\to\pm\infty} 0 \le \lim\limits_{x\to\pm\infty} \sqrt{x^2+2x+2} - |x+1| \le \left|\lim\limits_{x\to\pm\infty} \frac{1}{x+1}\right|$

Therefore $0 \le \lim\limits_{x\to\pm\infty} \sqrt{x^2+2x+2} - |x+1| \le 0$

By the squeeze theorem: $\lim\limits_{x\to\pm\infty} \sqrt{x^2+2x+2} - |x+1| = 0$

~~Therefore~~ For $x > 0$, $|x+1| = x+1$.

Therefore $y = x+1$ is an inclined assymptote for f (as $x \to \infty$)

For $x < 0$, $|x+1| = -x-1$.

Therefore $y = -x-1$ is an inclined assymptote for f (as $x \to -\infty$)

5200002888

4. $$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \lim_{h\to 0} \frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h}$$ where $g(x) \neq 0$

$$= \lim_{h\to 0} \frac{1}{h} \frac{f(x+h)g(x) - f(x)g(x+h)}{g(x+h)g(x)}$$

$$= \lim_{h\to 0} \frac{1}{h} \frac{f(x+h)g(x) - f(x)g(x) + f(x)g(x) - f(x)g(x+h)}{g(x+h)g(x)}$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{f(x+h)g(x) - f(x)g(x)}{g(x+h)g(x)} - \frac{f(x)g(x+h) - f(x)g(x)}{g(x+h)g(x)}\right]$$

$$= \lim_{h\to 0}\left[\frac{1}{g(x+h)}\right]\lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}\right] - \lim_{h\to 0}\left[\frac{f(x)}{g(x+h)g(x)}\right]\lim_{h\to 0}\left[\frac{g(x+h)-g(x)}{h}\right]$$ assuming $f'(x)$ and g' exist.

$$= \frac{1}{g(x)} \cdot f'(x) - \frac{f(x)}{[g(x)]^2} \cdot g'(x)$$

$$= \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

5. $$\frac{d(f(x))}{dx} = (\sec(x)\tan(x)+1)e^{\sec(x)+x}$$

$$= \left(\frac{\sin(x)}{\cos^2(x)} + 1\right)e^{\sec(x)+x}$$

$$= \left(\frac{\sin(x)}{\cos^2(x)} + 1\right)e^{\frac{1}{\cos(x)}+x}$$

$\frac{d(f(x))}{dx}$ is undefined when $\cos(x)=0$

$\cos^{-1}(0) = \frac{\pi}{2}$

~~Hence $x \neq \frac{\pi}{2} \pm 2n\pi$ where $n \in \mathbb{Z}$~~

~~Therefore $df: \{x \in \mathbb{R}: x \neq \frac{\pi}{2} \pm 2n\pi$ where $n \in \mathbb{Z}\} \to \mathbb{R}$~~

Hence $x \neq 2n\pi \pm \frac{\pi}{2}$ where $n \in \mathbb{Z}$

Therefore $Df: \{x \in \mathbb{R}: x \neq 2n\pi \pm \frac{\pi}{2}$ where $n \in \mathbb{Z}\} \to \mathbb{R}$.

520002888

6. For $t=0$, we cannot use the central difference scheme to find the velocity as we do not have a data point for $t<0$. Therefore we will find the velocity using ~~the~~ forward difference

For $t=0$,

$$v(0) \approx \frac{f(x+h)-f(x)}{h}$$

let x=displacement.

$$v(0) \approx \frac{x(0.1)-x(0)}{0.1}$$

$$v(0) \approx \frac{0.50-0}{0.1} = 5$$

$$v(0) \approx 5ms^{-1}$$

For $t=0.2$ (using the central difference scheme),

$$v(0.2) \approx \frac{f(x+h)-f(x-h)}{2h}$$

$$v(0.2) \approx \frac{x(0.3)-x(0.1)}{2(0.1)}$$

$$v(0.2) \approx \frac{1.00-0.50}{0.2} = \frac{0.50}{0.2} = 2.5$$

$$v(0.2) \approx 2.5ms^{-1}$$

For $t=0.3$ we cannot use the central difference scheme to find velocity as we do not have a data point for $t>0.3$, so we can use backward difference.

For $t=0.3$,

$$v(0.3) \approx \frac{f(x)-f(x-h)}{h}$$

$$v(0.3) \approx \frac{x(0.3)-x(0.2)}{0.1}$$

$$v(0.3) \approx \frac{1.00-0.80}{0.1} = \frac{0.20}{0.1} = 2$$

$$v(0.3) \approx 2ms^{-1}$$

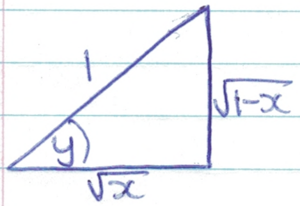
520002888

7. $\arccos(\sqrt{x}) = \cos^{-1}(\sqrt{x})$

$$y = \arccos(\sqrt{x}) \Leftrightarrow \cos(y) = \sqrt{x}$$

$$\frac{d}{dx}(\cos(y)) = \frac{d}{dx}(\sqrt{x})$$

$$-\sin(y)\frac{dy}{dx} = \frac{1}{2\sqrt{x}} \qquad \frac{dy}{dx} = \frac{1}{-2\sqrt{x}\sin(y)}$$



$$\cos(y) = \frac{\sqrt{x}}{1} = \sqrt{x}$$

$$\sin(y) = \frac{\sqrt{1-x}}{1} = \sqrt{1-x}$$

Therefore $\frac{dy}{dx} = \frac{1}{-2\sqrt{x}\sqrt{1-x}} = \frac{-1}{2\sqrt{x-x^2}}$ provided that $2\sqrt{x-x^2} \neq 0$.

~~y = arccos~~ $\frac{d}{dx}(\arccos(\sqrt{x}))$ is defined for $x \in (0,1)$.

or $\{x \in \mathbb{R} : 0 < x < 1\}$.