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Question No. ....

Date .....

Exam No. ....

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## LOGICAL PARADOXES

What is a Paradox?

- The word 'paradox' has many uses in English language, but from logical point of view, a paradox results when one can give seemingly irrefutable reasons for a proposition  $P$  to be true, when it is clear to all that  $P$  is false (often for seemingly irrefutable reasons as well).

\* Read Zeno's Paradox of Achilles and the tortoise.

### TYPES OF PARADOXES

1. Syntactic paradoxes
2. Semantic paradoxes

#### SYNTACTIC PARADOXES

- In a first order predicate logic, there is no straightforward way to express the predication of properties to other properties. The usual way to express such predication is through the symbolism of the second-order (or higher) predicate logic.

But if we allow the predication of properties, (e.g., honest, comprehensible) to properties, then certain paradoxes called Syntactic (from syntax) paradoxes can be generated.

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Question No. .... Date .....

Exam No. ....

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If we can predicate properties to other properties, i.e.  
e.g., consider the sentence "Art is happy". This  
sentence asserts that some particular object or entity,  
Art has a certain property - namely, the property  
of being happy. ↓

If we let the capital letter " $H$ " denote the property  
of being happy, and the lower case letter " $a$ "  
name the individual, Art, we can symbolize this  
sentence as  $Ha$ . Similarly, the sentence "Betsy  
is happy" can be symbolized as  $Hb$ ; the sentence  
"Art is friendly" as  $Fa$ , and the sentence "Betsy  
is friendly" as  $Fb$ .

The sentences  $Fa$ ,  $Fb$ ,  $Ha$  and  $Hb$  are alike in  
that they have the same structure. In each  
of these sentences, a property is ascribed to some  
individual entity. This is one of the basic patterns  
of atomic sentences. ↓

However, it seems reasonable to suppose we can  
predicate properties of themselves. For example,  
it seems reasonable to suppose that the property  
of being comprehensible itself is comprehensible (in  
contrast to the property of being incomprehensible  
which itself is not incomprehensible), and reason-  
able to suppose that the property of being common  
(as opposed to rare) is itself common. But  
sometimes the predication of a property to itself  
yields trouble! The most famous example (of  
the trouble) is the so called impredicative paradox.

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Question No. .... Date .....

Exam No. ....

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Let us call any property that can be truly predicated of itself a predicable property, and any property that cannot be truly predicated of itself an impredicative property. Using this notation, we can say that the property of being common is a predicable property, because being common is a common property, and that the property of being rare is an impredicative property, because there are many kinds of rare things.

But what about the property of being impredicative? Can this property be truly predicated of itself? The unfortunate answer seems to be that if the property of being impredicative is predicated of itself, then it is not predicated of itself, and if it is not predicated of itself, then it is predicated of itself. Hence the paradox.

To make this clear, let us symbolize the property of being predicable as  $P$ , and the property of being impredicative as  $\bar{P}$ . Thus, to say that a given property  $F$  is  $P$  is to say that  $F P$ , and to say that a given property  $F$  is  $\bar{P}$  is to say that  $F \bar{P}$ .

To start with, either  $\bar{P}$  is ~~itself~~  $\bar{P}$  or else  $\bar{P}$  is  $P$ . Suppose  $\bar{P}$  is  $\bar{P}$ . If  $\bar{P}$  is  $\bar{P}$ , then  $\bar{P}$  is predicated of itself, and hence  $\bar{P}$  is  $P$ . So if  $\bar{P}$  is  $P$ , then  $\bar{P}$  is  $P$ . It follows that if  $\bar{P}$  is  $\bar{P}$ , then it is  $P$ , and if  $\bar{P}$  is  $P$ , then it is  $\bar{P}$ . Translating this back into plain English, what

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Question No. ....

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Exam No. ....

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We have shown is that if the property of being impredicative is impredicative, then it is predicated of itself, and hence it is predicable. And if the property of being impredicative is predicable, then it is not predicated of itself (impredicative would have to be impredicative to be predicated of itself), and hence is impredicative.

This contradictory result can be made more explicit by writing down the definition of  $\bar{P}$  and then constructing a simple argument as follows:

$$(1) \bar{P}F = df \sim FF$$

That is to say, that a property,  $F$ , is impredicative, is to say that it is not the case that  $F \in F$ . From which it follows that given any property  $F$ ,  $F \in \bar{P}$  if and only if it is not the case that  $F \in F$ . In other words,

$$(2) (F)(\bar{P}F \equiv \sim FF)$$

Hence, substituting  $\bar{P}$  for  $F$ , we get

$$(3) \bar{P}\bar{P} \equiv \sim \bar{P}\bar{P}$$

from which an explicit contradiction can be derived.

#### SOLUTIONS TO IMPREDICATIVE PARADOXES

Several solutions to paradoxes of this kind have been proposed. One of them is the theory of types (STT) and it is proposed by Bertrand Russell (see Principles of Mathematics).

According to this theory, all entities divide into a hierarchy of types, starting with individual entities, moving to properties of individual entities, then to properties of properties of individual entities and so on.

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Question No. .... Date .....

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for instance, Art is an individual entity; the property of being honest is a property of Art; Art may possess (hence honesty is a property of individuals); and the property of being rare is a property possessed by the property of being honest (hence rarity is a property of properties).

When entities are arranged in this way, the Simple Theory of Types (STT) requires that the type of a property be higher than any property of which it can be predicated. Example: If being old is predicated of Art, then it cannot be predicated either of itself or of any property.

It is customary to mark the distinction between properties and properties of properties by some notational device, such as the use of standard type to denote properties of individuals and bold face type to denote properties of properties of individuals. \* Using a notation of this kind, a sentence such as "Art is not old" will be symbolized as  $\sim Oa$ , and a sentence such as "Honesty is rare" will be symbolized as  $RH$ .

\* Notice that the sentence "Honesty is rare" is correctly symbolized as  $RH$ , and not as ~~RH~~.  $RH$ , for according to the Theory of Types, the property of being rare, which is predictable of properties, is of a type one level higher than properties that are predictable of individuals.

To summarize the Simple Theory of Types, requires first, that we arrange entities into a hierarchy of categories or types, starting with individuals, moving to properties of individuals, and then to properties of properties, properties of properties

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Question No. .... Date .....

Exam No. ....

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first, and so on; and second, that the type of a property be one type higher than any property or entity of which it can be predicated.

An obvious consequence of the Simple Theory of Types (STT) is that no property can be predicated of itself. And it is this consequence that solves the impredicative paradox, for if no property can be predicated of itself, then it becomes senseless to ask if the property of being impredicative is itself impredicative.

### OBJECTIONS TO THE STT

1. The Simple Theory of Types (STT) has been objected to as both ad hoc and counterintuitive.

Example: according to the STT, the raveness we can predicate of say, a postage stamp is different from the raveness we can predicate of the property of being honest. But it seems intuitively clear that it is the very same property of raveness that is predictable of postage stamps and of honesty.

2. Again, it has been argued that the very statement of the STT presupposes a violation of the theory itself. For instance, the STT presupposes that all individual properties of individual properties of properties, and so on have the property of being classifiable (i.e., have the property of belonging to exactly one category in the hierarchy of types). But the property of being type classifiable is not permitted by the STT. Hence, the theory presupposes what it will not permit.

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### SEMANTIC PARADOXES

- Semantic paradoxes are a collection of paradoxes involving the semantic notions of truth, predication and definability.

Although adoption of the Simple Theory of Types (STT) has its difficulties, it does solve syntactic paradoxes such as the impredicable paradox. But unfortunately it fails to solve the paradoxes usually referred to as semantic paradoxes.

The most famous semantic paradox is the so-called paradox of the Liar, which was first posed by the ancient Greeks. In fact, it was formulated by Eubilides as an objection to Aristotle's correspondence theory of truth. In its simplest form, the Liar paradox arises when we try to assess the truth of a sentence or proposition that asserts its own falsity. Example, it seems reasonable to suppose that every declarative sentence is either true or false, but consider this sentence below.



(1) Sentence (1) is false

Is Sentence (1) true or is it false? The unfortunate answer seems to be that if sentence (1) is true, then it is false, and if it is false, then it is true.



Take the first possibility - namely, that sentence (1) is true. If (1) is true, and (1) asserts that (1) is false, then it follows that (1) is false.



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Question No. .... Date .....

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Now suppose (1) is false. If (1) is false, and (1) asserts that (1) is false, then it follows that it is false that (1) is false, and therefore follows that (1) is true. So if (1) is false, then (1) is true. Either way, we have a contradiction, and hence a paradox.

An obvious thought is to solve the liar paradox by ruling out (as meaningless) any sentence that refers to itself. Note that the liar paradox is often considered erroneously as a paradox of self-reference. But unfortunately, the liar paradox can be generated with self-reference.

Example, consider the following two sentences:

(2) Sentence (3) is false

(3) Sentence (2) is true.

Sentence (2) refers to Sentence (3) and Sentence (3) refers to Sentence (2), but neither (2) nor (3) refers to itself. So both of these sentences satisfy the requirements that sentences should not be self-referential, and they seem to have the form required of legitimate declarative sentences.

But is sentence (2) true, or is it false? Again, the unfortunate answer seems to be that if it is true, then it is false, and if it is false, then it is true.

↓  
Take the first possibility - namely, that sentence (2) is true. If (2) is true, and

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Exam No. ....

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(2) asserts that (3) is false, it follows that (3) is false. But if (3) is false, and (3) asserts that (2) is true, it follows that it is false that (2) is true and hence (2) is false. So if (2) is true, then (2) is false.

Now suppose sentence (2) is false. If (2) is false, and (2) asserts that (3) is false, it follows that it is false that (3) is false, and hence that (3) is true. But if (3) is true, and (3) asserts that (2) is true, it follows that (2) is true. So if (2) is false, then (2) is true. Again, we have a contradiction, and hence again we have a paradox.

### SOLUTION TO SEMANTIC PARADOX

One way to solve the semantic paradoxes is to distinguish between levels of language—that is, between languages that are used to talk about non-linguistic things and those used to talk about other languages.

A language used to talk about some other language is considered to be on a higher level than the language talked about, so that sentences asserting the truth or falsity of a given sentence must be placed into a language at least one level higher than the given sentence. For instance, the sentence, "The sentence 'Art is tall' is true" must be placed into a language one level higher than the language in which the sentence

'Art is tall' occurs. Note that this solution

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Question No. .... Date .....

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was proposed by Russell in his 'Introduction to Ludwig Wittgenstein's Tractatus Logico Philosophicus'.

It is clear that the adoption of this solution solves that the War paradox. In the first place, all self-referential sentences, such as sentence (1) will be rejected as meaningless. And in the second place, at least one of the every pair of sentences such as sentences (2) and (3) will be rejected as meaningless. For instance, if (2) occurs in a given language, and (3) in a language one level higher, then (2) will be rejected as meaningless — whatever fate of (3), because no sentence can be permitted to assert the truth or falsity of a sentence in the same or higher-level language.