- Sym break

- Topological order, topological phase, topological phase transition,
  $J_A - J_F$

- Alternating FM & AM Couplings

- Haldane spin chain

- determine phase boundary of the topological phase as a function of relative strength of the exchange coupling & the magnetic exchange anisotropy

- Jordan - Wigner transformation ✓
- non-local string operators

- mean-field approx
- coupled self-consistency eqⁿs

- simple ~~mode~~ model Hamiltonian ✓

- fermion creation & annihilation operators ✓

$\gamma \to 1$
- Strong anisotropy ⇒ Z-comp of spin frozen ⇒ fermions non-interacting
  have topological phase transition @ $J_A = J_F$

- weaker anisotropy ⇒ mean-field approx ⇒ numerical solution of coupled self-consistency equations

- $\gamma \to 0$

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1D spin chains :

⇒ Heisenberg chain (Bethe Ansatz)

⇒ XY chain (Jordan - Wigner transformation) [Pg. 82]

use periodic boundary conditions : Pg 28

Section 2.7 (An Intro to Quantum Spin systems)

~~end~~

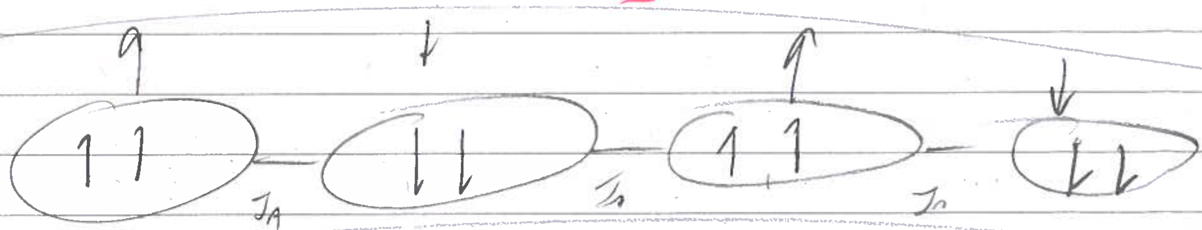
End of chain are joined

⇒ N^{th} site is a nearest - neighbour of the 1^{st}, as well as the $(N-1)^{th}$

taking $N \to \infty$

$$\mathcal{H} = J[\underline{S}_1 \cdot \underline{S}_2 + \underline{S}_2 \cdot \underline{S}_3 + \cdots + \underline{S}_N \cdot \underline{S}_1]$$

$$= J \sum_{i=1}^{N} S_i \cdot S_{i+1} \qquad i+N = i$$



A — B — C

A w/o B w/C

A + w/C

Pg 82. γ : anisotropy ⇒ Combination of Heisenberg & Ising

$= 0$ ⇒ XX model

$= 1$ ⇒ Ising case

correction factors for anisotropy ??

Hamiltonian ⇒

$$\mathcal{H} = J \sum_j [(1+\gamma) S_j^x S_{j+1}^x + (1-\gamma) S_j^y S_{j+1}^y]$$

Pg 82 ⇒ $\cancel{\mathcal{H} = J \sum \Delta S_j^z S_{j+1}^z + S}$

$$= J \sum_j [\underbrace{\gamma (S_j^x S_{j+1}^x - S_j^y S_{j+1}^y)}_{\Delta S^z S^z} + (S_j^x S_{j+1}^x + S_j^y S_{j+1}^y)]$$

nu: EVOLVE

Spin operator to Fermion Operator Pg 62

Pauli Spin Matricies

$S_i^x = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$S_i^y = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $\hbar = 1$

$S_i^z = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

raising operator

$a_i^\dagger = S_i^x + i S_i^y$

$a_i = S_i^x - i S_i^y$

For a single site i $+ \equiv \uparrow$
$- \equiv \downarrow$

$$\boxed{\begin{aligned} &\hat{S}^+ = \hat{S}^x + i\hat{S}^y \\ &\hat{S}^- = \hat{S}^x - i\hat{S}^y \\ &S^+|+\rangle = 0 \qquad S^-|-\rangle = 0 \\ &S^+|-\rangle = |+\rangle \qquad S^-|+\rangle = |-\rangle \end{aligned}}$$

] Page 60

$S_i^+|-\rangle = |+\rangle \qquad S_i^+|+\rangle = 0$

$S_i^-|-\rangle = 0 \qquad \cancel{S_i^-|+\rangle} = |{-}\rangle \quad S_i^-|+\rangle = |-\rangle$

$S_i^z|-\rangle = -\frac{1}{2}|-\rangle \qquad S_i^z|+\rangle = \frac{1}{2}|+\rangle$

$\Leftrightarrow S_i^- S_i^+ + S_i^+ S_i^- = 1$ $-\ (S_i^- S_i^+ + S_i^+ S_i^-)|+\rangle = |+\rangle$

$\Leftrightarrow S_i^{-2} = S_i^{+2} = 0$

Commutator relationship for a single sites 2, $i \neq j$

$$[S_i^-, S_j^+] \equiv S_i^- S_j^+ - S_j^+ S_i^- = 0$$

$$\begin{bmatrix} S_i^- S_j^+ + S_j^+ S_i^- = 2S_i^- S_j^+ \\ S_i^- S_j^- + S_j^- S_i^+ = 2S_i^- S_j^- \\ S_i^+ S_j^+ + S_j^+ S_i^+ = 2S_i^+ S_j^+ \end{bmatrix}$$

$\hookrightarrow$ ~~Fac~~

nu: EVOLVE

anticommutator for fermions
$\Updownarrow$ analogous
commutator for bose particles

Introducing Fermion operators: $\underline{c_i \;\&\; c_i^\dagger}$

- Fermions => anticommute

$$\{c_i, c_j^\dagger\} \equiv c_i c_j^\dagger + c_j^\dagger c_i = \delta_{ij} \begin{cases} i=j \Rightarrow 1 \\ i\neq j \Rightarrow 0 \end{cases}$$

$$\{c_i, c_j\} = 0$$

$$\boxed{\{c_i^\dagger, c_j^\dagger\} = 0} \Rightarrow \text{can be simultaneously measured}$$

For single site => can do $S_i^- = c_i \quad S_i^\dagger = c_i^\dagger$

BUT => For different sites

spin operator can be represented in terms of the fermions (c_i)

$$\Rightarrow \left[\begin{array}{ll} S_1^- = c_1 & S_1^\dagger = c_1^\dagger \\ S_2^- = [\exp(i\pi c_1^\dagger c_1)]c_2 & S_2^\dagger = c_2^\dagger[\exp(-i\pi c_1^\dagger c_1)] \\ S_i^- = Q_i c_i \quad i\geq 1 & S_i^\dagger = c_i^\dagger Q_i^\dagger \quad i\geq 1 \end{array}\right]$$

(6.13) Transformation to fermion operators

$$\boxed{Q_i = \exp\left[i\pi \sum_{l=1}^{i-1} c_l^\dagger c_l\right]}$$

=> Now, need to prove the spin operators from 6.13 commute when on different sites ($i\neq j$)

introduce new operators, ~~A_i and T_i~~ n_i, T_i, Q_i

Let $|+\rangle$ & $|-\rangle$ be the basis for the i^{th} state

Fermion operators acting on these basis functions

$$\boxed{\begin{array}{ll} c_i^\dagger|+\rangle = 0 & c_i^\dagger|-\rangle = |+\rangle \\ c_i|+\rangle = |-\rangle & c_i|-\rangle = 0 \end{array}}$$

Defining: n_i = number operator

$n_i \equiv c_i^+ c_i$ ↳ 'counts' the value of the z-component of th ang momenta relative to the state $|-\rangle$

$n_i^+ = n_i$

$n_i|+\rangle = c_i^+ c_i |+\rangle = c_i^+|-\rangle = |+\rangle$

$n_i|-\rangle = c_i^+ c_i|-\rangle = 0$

$n_i|-\rangle = 0|-\rangle$ spin dn

$n_i^+|+\rangle = 1|+\rangle$ spin up

defining $\boxed{T_i = e^{i\pi c_i^+ c_i} = e^{i\pi n_i}}$

$\Rightarrow T_i|+\rangle = e^{i\pi n_i}|+\rangle = e^{i\pi 1}|+\rangle = -|+\rangle$ $n_i = 1$ for spin up

$T_i|-\rangle = e^{i\pi n_i}|-\rangle = e^{i\pi(0)}|-\rangle = +|-\rangle$ $n_i = 0$ for spin down

$T_i^+ = e^{-i\pi n_i}$

$T_i^+|-\rangle = -|+\rangle$

$T_i|-\rangle = |-\rangle$

$\therefore$ effect of T_i^+ is the same as that of T_i on the two basis states

$\Rightarrow T_i^+ = T_i$

$\therefore T_i^2|+\rangle = |+\rangle$ & $T_i^2|-\rangle = |-\rangle$

↳ $T_i^2 = 1$

$n_i n_j = n_j n_i$

$\Rightarrow n_i n_j - n_j n_i = 0$ by commutator relationships

for ~~[illegible]~~ $= 5^+$

$\{c_i, c_j^+\}$

$$\boxed{\begin{aligned}[n_i, n_j] &= 0 \\ [T_i, T_j] &= 0\end{aligned}}$$

$$Q_i = \exp\left[i\lambda \sum_{l=1}^{i-1} n_l \right]$$ all n_l commute.

~~Pro provide~~
Provided that $[A,B]=0$
$e^{A+B} = e^A e^B$

$$Q_i = \prod_{l=1}^{i-1} e^{i\lambda n_l} = \cancel{Q} \prod_{l=1}^{i-1} T_l$$

$$Q_i^\dagger = \prod_{l=1}^{i-1} e^{-i\lambda n_l} = \prod_{l=1}^{i-1} T_l^\dagger = \prod_{l=1}^{i-1} T_l = Q_i$$

$$\boxed{Q_i^\dagger = Q_i}$$

$$[c_i, n_j] = c_i n_j - n_j n_i = c_i c_j^\dagger c_j - c_j^\dagger c_j c_i$$
$$= -c_j^\dagger c_i c_j + c_j^\dagger c_i c_j = 0$$

$[c_i, n_j] = 0$ $\qquad$ $\{c_i, T_i\} = 0$ $\qquad$ see 6.18 – 6.21

$[c_i, T_j] = 0$ $\qquad$ $[c_i, T_i] \neq 0$

$[c_i^\dagger, T_j] = 0$ $\qquad$ $[c_i, Q_i] = 0$ $\qquad$ Pg. 65

$[c_i^\dagger, Q_i] = 0$

$[S_i^-, S_j^+] = 0$

using 6.1

$$S_j^x S_{j+1}^x = \frac{1}{2}(S_j^+ + S_j^-)\frac{1}{2}(S_{j+1}^+ + S_{j+1}^-)$$
$$= \frac{1}{4}(S_j^+ S_{j+1}^+ + S_j^- S_{j+1}^-) + \frac{1}{4}(S_j^- S_{j+1}^+ + S_j^+ S_{j+1}^-)$$

??

$$S_j^y S_{j+1}^y = \frac{1}{2i}(S_j^+ - S_j^-)\frac{1}{2i}(S_{j+1}^+ - S_{j+1}^-)$$
$$= -\frac{1}{4}(S_j^+ S_{j+1}^+ + S_j^- S_{j+1}^-) + \frac{1}{4}(S_j^- S_{j+1}^+ + S_j^+ S_{j+1}^-)$$

Hamiltonian of the XY model in terms of the spin operators

6.1

$$\mathcal{H} = J \sum_{j=1}^{N} [(1+\gamma) S_j^x S_{j+1}^x + (1-\gamma) S_j^y S_{j+1}^y]$$

$$= \frac{J}{2} \sum_{j=1}^{N} [(S_j^- S_{j+1}^+ + S_j^+ S_{j+1}^-) + \gamma (S_j^+ S_{j+1}^+ + S_j^- S_{j+2}^-)]$$

$\gamma = 0 \Rightarrow$ XY-model

$\Downarrow$ in terms of ~~spin~~ fermion operators

$$S_j^- S_{j+1}^+ = Q_j c_j c_{j+1}^+ Q_{j+1}$$

$$= c_j T_j c_{j+1}^+ \quad \text{[using commutator relationships]}$$

$c_j T_j |+\rangle = -|-\rangle$; comp ; $c_j |+\rangle = |-\rangle$

$c_j T_j |-\rangle = 0$; ; $c_j |-\rangle = 0$

$$c_j T_j = -c_j$$

$$S_j^- S_{j+1}^+ = -c_j c_{j+1}^+ = c_{j+1}^+ c_j$$

similarly

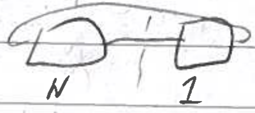
$$S_j^+ S_{j+1}^- = c_j^+ c_{j+1}$$

$$S_j^+ S_{j+1}^+ = c_j^+ c_{j+1}^+$$

$$S_j^- S_{j+1}^- = c_{j+1} c_j$$

For $j = N$, at the end of the series, for periodic boundary conditions:

$$S_N^+ S_1^- = Q_N c_N^+ c_1 \neq c_N^+ c_1 = Q_N c_N^+ c_1 \neq c_N^+ c_1$$



Subbing to the Hamiltonian

$$\mathcal{H} = \frac{J}{2} \sum_{j=1}^{N} [(c_{j+1}^+ c_j + c_j^+ c_{j+1}) + \gamma (c_j^+ c_{j+1}^+ + c_{j+1} c_j)]$$

$$- \frac{J}{2} [(c_1^+ c_N + c_N^+ c_1) + \gamma (c_N^+ c_1^+ + c_1 c_N)]$$

$$+ \frac{J}{2} Q_N [(c_N c_1^+ + c_N^+ c_1) + \gamma (c_N^+ c_1^+ + c_N c_1)]$$

can be neglected in the limit that $N \to \infty$

C-cyclic problem

in Ann Phy 16, 407 1961

it is in matrix form

In textbook An intro to Quantum Spin Systems

§4.12 - spins can be arranged in a definite order,

- only neighboring spins interact
- z-components of spins cannot be entered.

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$\cos \quad \frac{e^{ik}+e^{-ik}}{2} = \cos(k)$

Eq 6.27:

$$\mathcal{H} = \frac{1}{2}\sum_{j=1}^{N}\left[(c_{j+1}^{\dagger}c_j + c_j^{\dagger}c_{j+1}) + \gamma(c_j^{\dagger}c_{j+1}^{\dagger} + c_{j+1}c_j)\right]$$

- ~~fuck~~ quadratic Hamiltonian involving only fermion operators

Diagonalise to make use of the translational invariance by introducing discrete Fourier transformed operators d_k & $d_k^{\dagger}$ → diagonalize the Hamiltonian — find Eigenvalue

$$d_k = \frac{1}{\sqrt{N}}\sum_{j=1}^{N} e^{-ikj}c_j$$

$k \in [-\pi, \pi]$

~~let~~

$k = \lambda \frac{2\pi}{N}$

$$d_k^{\dagger} = \frac{1}{\sqrt{N}}\sum_{j=1}^{N} e^{ikj}c_j^{\dagger}$$

$\lambda = (-\frac{N}{2}+1), \dots (\frac{N}{2})$

$d_{k_1}d_{k_2} = -d_{k_2}d_{k_1}$

$d_{k_1}d_{k_2} + d_{k_2}d_{k_1} = 0$

↓ reverse transform

$\{d_{k_1}, d_{k_2}\} = \{d_{k_1}^{\dagger}, d_{k_2}^{\dagger}\} = 0$

↓

$$c_j = \frac{1}{\sqrt{N}}\sum_k e^{ikj}d_k$$

using the properties of c_j & $c_j^{\dagger}$ → $\therefore \Rightarrow \{d_{k_1}, d_{k_2}^{\dagger}\} = \delta_{k_1 k_2}$

$$c_j^{\dagger} = \frac{1}{\sqrt{N}}\sum_k e^{-ikj}d_k^{\dagger}$$

d's are fermions

~~using~~ using $\sum_j e^{i(k_1-k_2)j} = N\delta_{k_1 k_2}$

[orthogonality relationships]

↓ rewriting each term of the Hamiltonian

$$\sum_j c_{j+1}^{\dagger}c_j = \sum_j \frac{1}{N}\sum_{k_1}\sum_{k_2} e^{-ik_1(j+1)}e^{ik_2 j}d_{k_1}^{\dagger}d_{k_2}$$

$$= \frac{1}{N}\sum_{k_1}\sum_{k_2} e^{-ik_1} N\delta_{k_1 k_2} d_{k_1}^{\dagger}d_{k_2}$$

↓ picking $k_1 = k_2 = k$

$$\Rightarrow \sum_k e^{-ik}d_k^{\dagger}d_k$$

Similarly: $\sum_j c_j^{\dagger}c_{j+1} = \sum_k e^{ik}d_k^{\dagger}d_k \Rightarrow \sum_j c_{j+1}c_j^{\dagger} = \sum_k e^{-ik}d_k^{\dagger}d_k$

$$\sum_j c_j^{\dagger}c_{j+1}^{\dagger} = \sum_k e^{ik}d_k^{\dagger}d_{-k}^{\dagger}$$

$$\sum_j c_{j+1}c_j = \sum_k e^{ik}d_k d_{-k}$$

lin comb of fermion operators

↳ Bogoliubov transformation

⇓

quasi-particle operators

[in momentum space] ??

$$\longrightarrow \mathcal{H} = J\sum_k \left[\cos k\,(d_k^\dagger d_k) + \gamma \frac{e^{ik}}{2}\,\overbrace{(d_k^\dagger d_{-k}^\dagger + d_k d_{-k})}\right]$$

- no coupling between states w/ diff $|k|$
- still need to diagonalize the coupled k & $-k$ terms

instead of summing $k \in [-\pi, \pi]$

⇓ combine k & $-k$

↳ summing $k \in [0, \pi]$

$$\mathcal{H} = J\sum_{k=0}^{\pi}\Big[\cos(k)(d_k^\dagger d_k + d_{-k}^\dagger d_{-k}) + \gamma\Big[\frac{e^{ik}}{2}(d_k^\dagger d_{-k}^\dagger + d_k d_{-k}) + \frac{e^{-ik}}{2}(d_{-k}^\dagger d_k^\dagger + d_{-k} d_k)\Big]\Big]$$

using anticommutation relationships

$$\mathcal{H} = J\sum_{k=0}^{\pi}\left[\cos k\,(d_k^\dagger d_k + d_{-k}^\dagger d_{-k}) + \gamma\left[i\sin k\,(d_k^\dagger d_{-k}^\dagger + d_k d_{-k})\right]\right]$$

⇓ diagonalize the Hamiltonian

- only d_k and d_{-k}, $d_k^\dagger$ and $d_{-k}^\dagger$

↳ we need to look for two diff lin-comb η_k & η_{-k}

?? also fermion operators

$$\eta_k = A_k d_k + B_k d_{-k}^\dagger$$
$$\eta_{-k} = C_k d_{-k} + D_k d_k^\dagger$$

w/ ~~commutation~~ anticommutation relationships

$$\{\eta_k, \eta_k\} = \{\eta_k^\dagger, \eta_k^\dagger\} = 0$$
$$\{\eta_{-k}, \eta_{-k}\} = \{\eta_{-k}^\dagger, \eta_{-k}^\dagger\} = 0$$
$$\{\eta_k, \eta_{-k}^\dagger\} = \{\eta_{-k}, \eta_k^\dagger\} = 0$$
$$\{\eta_k, \eta_k^\dagger\} = \{\eta_{-k}, \eta_{-k}^\dagger\} = 1$$
$$\{\eta_k, \eta_{-k}\} = \{\eta_k^\dagger, \eta_{-k}^\dagger\} = 0$$

$$|A_k|^2 + |B_k|^2 = 1$$
$$|C_k|^2 + |D_k|^2 = 1$$

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$$\mathcal{H} = J\sum_{k>0}^{k} [\cos k\,(d_k^\dagger d_k + d_{-k}^\dagger d_{-k}) + \gamma\,[i \sin k\,(d_k^\dagger d_{-k}^\dagger + d_k d_{-k})]]$$

tgt w/ lin comb of η_k, η_{-k} & d_k, d_{-k}

has the form below:

$$\mathcal{H} = \sum_{k>0}^{\pi} [\Lambda_{1k}\, \eta_k^\dagger \eta_k + \Lambda_{2k}\, \eta_{-k}^\dagger \eta_{-k} + \underbrace{X_k}_{\text{constant}}]$$

7/6/2021

CMMP intern
notes

- Classical $\Longrightarrow$ quantum

$$\underline{S} = \begin{pmatrix} S_x \\ S_y \\ S_z \end{pmatrix} \Longrightarrow \underline{\hat{S}} = \begin{pmatrix} \hat{S}_x \\ \hat{S}_y \\ \hat{S}_z \end{pmatrix}$$

- acting on Hilbert space

$|i\rangle$ (ket)
$\langle j|$ (bra)

- dot product
$\langle j | i \rangle \in \mathbb{C}$

- $\langle i | = (|i\rangle)^\dagger$ trans pose + complex conjugate

- Matrix element : $A_{ij} = \langle j | \hat{A} | i \rangle \in \mathbb{C}$

- Commutator Relationships $\Rightarrow$ $[\hat{A}, \hat{B}] = 0$ (commute)

$$[\hat{A}, \hat{B}] = -[\hat{B}, \hat{A}]$$
$$[\hat{A}, \hat{A}] = 0$$
$$[\hat{A}, \hat{B}+\hat{C}] = [\hat{A}, \hat{B}] + [\hat{A}, \hat{C}] \quad \text{(linearity)}$$
$$[\hat{A}, \hat{B}\hat{C}] = [\hat{A}, \hat{B}]\hat{C} + \hat{B}[\hat{A}, \hat{C}]$$

- spin commutator relations (cyclic permutations)

$$[\hat{S}_x, \hat{S}_y] = i\hbar \hat{S}_z$$
$$[\hat{S}_z, \hat{S}_x] = i\hbar \hat{S}_y$$
$$[\hat{S}_y, \hat{S}_z] = i\hbar \hat{S}_x$$

$$[\hat{S}_\alpha, \hat{S}_\beta] = i\hbar\, \underbrace{\epsilon_{\alpha\beta\gamma} \hat{S}_\gamma}$$

↳ Einstein sum coeff

↳ $\epsilon_{\alpha\beta\gamma}$ $\Rightarrow$ non-zero only if no two indicies are the same

$$[\hat{S}_z, \hat{S}^2] = 0$$

e.g. $\epsilon_{xxy} = 0$

$\epsilon_{xyz} = \epsilon_{zxy} = \epsilon_{yzx} = 1$

$\epsilon_{yxz} = \epsilon_{zyx} = \epsilon_{xzy} = -1$

$$\hat{S}^2 |S, m\rangle = \hbar^2 S(S+1) |S, m\rangle$$
$$\hat{S}_z |S, m\rangle = \hbar m |S, m\rangle$$

Focusing on $s = \frac{1}{2}$

$|s, m\rangle$

$:= |\frac{1}{2}, \frac{1}{2}\rangle = |\uparrow\rangle$

$|\frac{1}{2}, -\frac{1}{2}\rangle = |\downarrow\rangle$

(Hilbert space spanned by two states [both : 2 dim space])

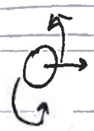
Pauli spin matricies :

what is the significance of $\hat{S}_x$ & $\hat{S}_y$ components?

$\hat{S}_x |\uparrow\rangle = \frac{\hbar}{2} |\downarrow\rangle$

$\hat{S}_x |\downarrow\rangle = \frac{\hbar}{2} |\uparrow\rangle$

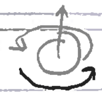
$$\hat{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$



$\hat{S}_y |\uparrow\rangle = -i\frac{\hbar}{2} |\downarrow\rangle$

$\hat{S}_y |\downarrow\rangle = i\frac{\hbar}{2} |\uparrow\rangle$

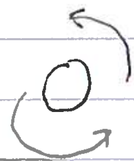
$$\hat{S}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$



$\hat{S}_z |\uparrow\rangle = \frac{\hbar}{2} |\uparrow\rangle$

$\hat{S}_z |\downarrow\rangle = -\frac{\hbar}{2} |\downarrow\rangle$

$$\hat{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$



defining : what does this mean ??

$\hat{S}_+ := \hat{S}_x + i\hat{S}_y$ $\qquad S_+ = \hbar\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$\hat{S}_- := \hat{S}_x - i\hat{S}_y$ $\qquad S_- = \hbar\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

$\hat{S}_+ |\downarrow\rangle = \hbar |\uparrow\rangle$

$\hat{S}_- |\downarrow\rangle = 0$

$\hat{S}_+ |\uparrow\rangle = \hat{S}_x |\uparrow\rangle + i\hat{S}_y |\uparrow\rangle = \frac{\hbar}{2}|\downarrow\rangle + i(-i\frac{\hbar}{2}|\downarrow\rangle) = \hbar|\downarrow\rangle$

$\hat{S}_- |\uparrow\rangle = \hat{S}_x |\downarrow\rangle - i\hat{S}_y |\downarrow\rangle = 0$

Two-site problem:

$\underset{1}{\uparrow} \quad \underset{2}{\downarrow} \Rightarrow$ 2 arrangements

classically

AFM: ↑↓↑↓↑↓↑↓

FM: ↑↑↑↑↑↑↑

for two-interacting $S = \frac{1}{2}$ spins

$$\hat{H} = J \, \hat{\underline{S}}_1 \cdot \hat{\underline{S}}_2 \qquad J > 0 : \text{AFM coupling}$$

Hilbert space is spanned by:

$$|\uparrow,\uparrow\rangle, \quad |\uparrow,\downarrow\rangle, \quad |\downarrow,\uparrow\rangle, \quad |\downarrow,\downarrow\rangle$$
$$|1\rangle, \quad |2\rangle, \quad |3\rangle, \quad |4\rangle$$

4D Hilbert space

Hilbert space grows exponentially w/ # of lattice sites, N
↳ 2^N for spin $\frac{1}{2}$

$$\hat{H} = J\left(\frac{1}{2}\hat{S}_1^+\hat{S}_2^- + \frac{1}{2}\hat{S}_1^-\hat{S}_2^+ + \hat{S}_1^z\hat{S}_2^z\right)$$

$$\hat{H}|1\rangle = \hat{H}|\uparrow,\uparrow\rangle = \frac{\hbar^2}{2} J |\uparrow,\uparrow\rangle$$

$$\hat{H}|4\rangle = \hat{H}|\downarrow,\downarrow\rangle = \frac{\hbar^2}{4} J |\downarrow,\downarrow\rangle$$

$$\hat{H}|2\rangle = \hat{H}|\uparrow,\downarrow\rangle = \frac{\hbar^2}{2} J |\downarrow,\uparrow\rangle - \frac{\hbar^2}{4} J |\uparrow,\downarrow\rangle$$

$$\hat{H}|3\rangle = \hat{H}|\downarrow,\uparrow\rangle = \frac{\hbar^2}{2} J |\uparrow,\downarrow\rangle - \frac{\hbar^2}{4} J |\downarrow,\uparrow\rangle$$

$$\underline{\underline{H}} = \frac{\hbar^2}{4} J \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

haven't used for a while, took some time to familiarize

Diagonalizing : $A = \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix}$

$$\det \begin{pmatrix} -1-\lambda & 2 \\ 2 & -1-\lambda \end{pmatrix} = 0$$

$$(1+\lambda)^2 - 4 = 0$$

$$(1+\lambda)^2 = 4$$

$$1+\lambda = \pm 2$$

$$\lambda = \pm 2 - 1$$

$$\lambda_1 = 1 \qquad D = \begin{pmatrix} 1 & 0 \\ 0 & -3 \end{pmatrix}$$

$$\lambda_2 = -3$$

$AK = \lambda K$

$\lambda_1 = 1$

$$\begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\left. \begin{array}{l} -a + 2b = a \\ 2a - b = b \end{array} \right] \quad a = b = 1 \qquad v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$L = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$L^{-1} = -\frac{1}{2} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$$

$\lambda_2 = -3$

$$\left. \begin{array}{l} -a + 2b = -3a \\ 2a - b = -3b \end{array} \right] \quad \begin{array}{l} 2b = -2a \\ 2a = -2b \\ a = 1 \\ b = -1 \end{array}$$

$$v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

LDL^{-1}

$$= \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -3 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \\ & \end{pmatrix}$$

$$AL = \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} -1+2 & -1-2 \\ 2-1 & 2+1 \end{pmatrix} = \begin{pmatrix} 1 & -3 \\ 1 & 3 \end{pmatrix}$$

$$LD = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -3 \end{pmatrix} = \begin{pmatrix} 1 & -3 \\ 1 & 3 \end{pmatrix}$$

$$AL = LD \Rightarrow A = LDL^{-1}$$

Resulting Hamiltonian for the two site, AFM coupling problem:

$$\underline{H} = \frac{\hbar^2}{4} J \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -3 & \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

shouldn't ~~|2>~~

$\underline{H}|2\rangle = \underline{H}|3\rangle$???

$E_1\ \hat{H}|1\rangle = \frac{\hbar^2}{4} J |\uparrow,\uparrow\rangle$

$E_4\ \hat{H}|4\rangle = \frac{\hbar^2}{4} J |\downarrow,\downarrow\rangle$

$E_2 : \hat{H}|2\rangle = \frac{\hbar^2}{4} J |\uparrow,\downarrow\rangle$

$E_3\ \hat{H}|3\rangle = -\frac{3\hbar^2}{4} J |\downarrow,\uparrow\rangle$

$\cancel{\frac{\hbar^2}{4}} J \overset{|2\rangle}{|\uparrow,\downarrow\rangle} = 2\cancel{\frac{\hbar^2}{4}} J \overset{|3\rangle}{|\downarrow,\uparrow\rangle} - \cancel{\frac{\hbar^2}{4}} J \overset{|2\rangle}{|\uparrow,\downarrow\rangle}$

$|2\rangle = 2|3\rangle - |2\rangle$

$2|2\rangle = 2|3\rangle$

$|2\rangle = |3\rangle$ => so need normalization?

\+

i think not, coz assuming eigenbasis all normalized.

or

$-\frac{3}{4}|3\rangle = \frac{1}{2}|2\rangle - \frac{1}{4}|3\rangle$

$-2|3\rangle = 2|2\rangle$

$|2\rangle = -|3\rangle$

$\underline{|2\rangle = \pm|3\rangle}$

PhD

ground state of some non-integrable model

— (↑↓) — (↑↓) — (↑↓)

need field theory

eigenvectors:

$$\psi_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \psi_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\psi_2 : \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} \quad \psi_3 = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$$

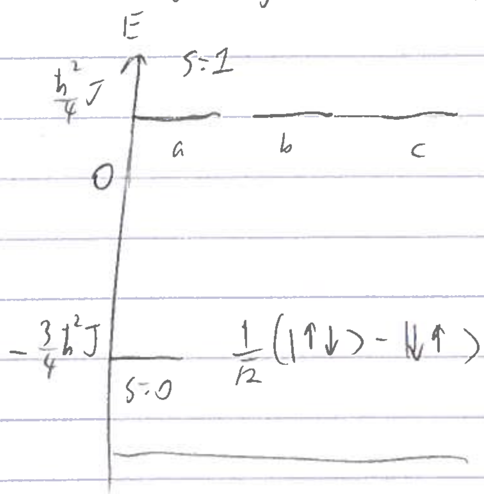
$\hookrightarrow \frac{1}{\sqrt{2}} |\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle$ $\qquad \hookrightarrow \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)$

1. QM oscillator $a^\dagger a$

2. 2nd Quantisation

Zacheri, Kancha?

Resulting diagonalized Hamiltonian given as:



$|S, m_s\rangle$

a: $|\uparrow\uparrow\rangle$, $|1,1\rangle$

b: $\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$, $|1,0\rangle$

c: $|\downarrow\downarrow\rangle$, $|1,-1\rangle$

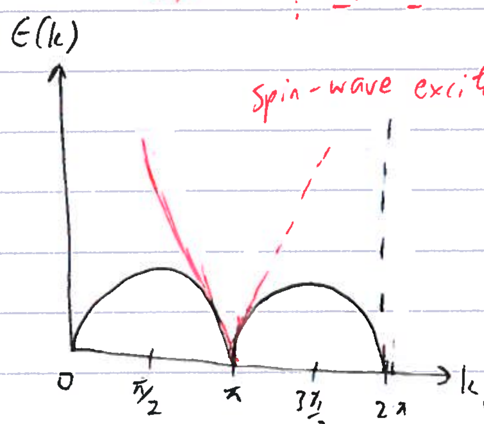
$\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$, $|0,0\rangle$

Heisenberg Antiferromagnetic Chain

AFM : $J>0$

$i \quad i+1 \quad i+2 \quad i+3$

$\hat{\mathcal{H}} = J \sum \hat{S}_i \cdot \hat{S}_{i+1}$



↘ AFM ordering vector

$e^{iqr} = e^{iqn} = e^{i\pi n} = (-1)^n$

↳ Néel order (AFM order)

$\uparrow\downarrow \uparrow\downarrow \uparrow\downarrow \uparrow\downarrow$

=>

Haldane Conjecture:

$S = \frac{1}{2}, \frac{3}{2}, \frac{5}{2} \dots \Rightarrow$ excitation is gapless

$S = 1, 2, 3, \dots \Rightarrow$ excitation spectrum is gapped

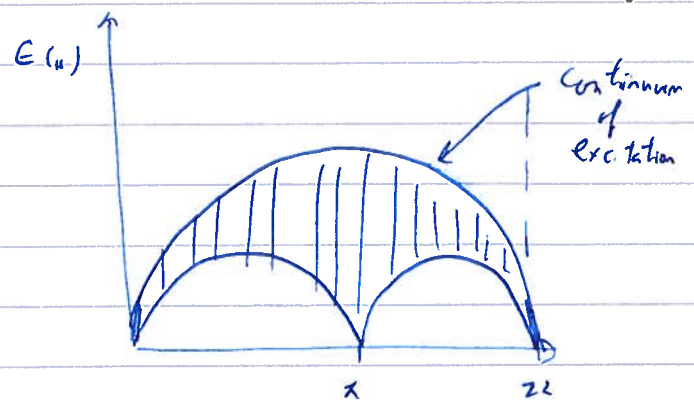
* Spin $\frac{1}{2}$ is special => no sharp spin-wave excitations

there is an exact spectrum

↳ Bethe Ansatz (see other sheets)

* Spin wave excitations

↳ Quasiparticles (magnons)



spin-wave excitations break up into pair of "spinons" (domain walls)

↑ ↓ ↑ ↓ ↑ ↓ ↑

⇓ => S=1 excitation, $\Delta E = 4J$

↑ ↓ ↑ ↑ ↑ ↓ ↑

⇓

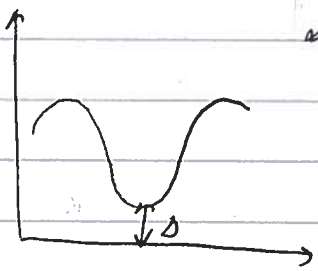
↑ ↓ ↓ ↑ ↓ ↓ ↑

$S = \frac{1}{2}$ $S = \frac{1}{2}$

domain walls are like fractionalised quasiparticles that can propagate freely (in a continuum.)

For the S=1 case:

determined expts numerically & other methods



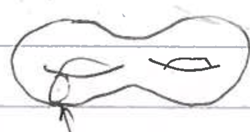
Δ: excitation => caused by topological order
↳ topological gap.

Topological indicies: (see other sheets)

Torus g=1

rubber ring : w=0

pretzel g=2



w=1

winding number (w)
genus of a manifold (g)

topological quantum state are potentially useful for quantum computing
↳ ∵ topological protection
↳ small, smooth perturbations cannot change the topology

Generalisation of the Haldane model :

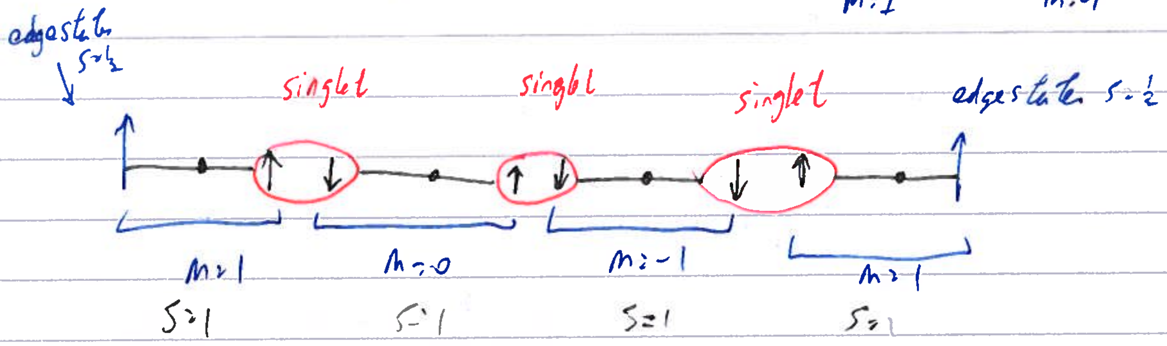
$$\hat{\mathcal{H}} = J \sum_i \underline{\hat{S}}_i \cdot \underline{\hat{S}}_{i+1} + \beta J \underbrace{\sum_i (\underline{\hat{S}}_i \cdot \underline{\hat{S}}_{i+1})^2}$$

↳ biquadratic coupling

$\beta = 0$: Haldane model is non-integrable

$\beta = \frac{1}{3}$: AKLT model is integrable, w/ exact known ground state.

now we can write $S=1$ in terms of triplet state of two spin $\frac{1}{2}$: $\uparrow\uparrow$ ($S=\frac{1}{2}$, $S=\frac{1}{2}$) $m=1$; $\downarrow\downarrow$ $m=-1$; $\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$ $m=0$



possible states (as a rule of thumb) + − 0 0 + − 0 + −

+ 0 0 − 0 0 + − + − 0 0 0 +

↳ Néel state + − + −
w/ any number of "0" insertions ⇒ hidden order

↳ can be linked to a string order parameter (see other notes)

↳ (product of spin operators)
non-local operator
↳ infinite product

↳ can be used to map out the spins

⊛ ground state is non-magnetic (product of singlets)

⊛ magnetic excitations are gapped.

EXTRA!

The Harmonic Oscillator => Annihilation & Creation Operators

Modern QM by Prof. J.J. Sakurai

Pg 89.

$$\mathcal{H} = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2} \qquad \omega = \sqrt{\frac{k}{\mu}}$$

Def: two non-Hermitian operators: x & p are operators

$$a = \sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{ip}{m\omega}\right) \quad ; \quad a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(x - \frac{ip}{m\omega}\right)$$

(annihilation operator) (creation operator)

$$[a, a^\dagger] = aa^\dagger - a^\dagger a = \frac{m\omega}{2\hbar}\left[\left(x + \frac{ip}{m\omega}\right)\left(x - \frac{ip}{m\omega}\right) - \left(x - \frac{ip}{m\omega}\right)\left(x + \frac{ip}{m\omega}\right)\right]$$

$$= \frac{m\omega}{2\hbar}\left[\left(\cancel{x^2} - \frac{xip}{m\omega} + \frac{ipx}{m\omega}\right) - \left(\cancel{x^2} + \frac{xip}{m\omega} - \frac{ipx}{m\omega}\right)\right]$$

$$= \frac{\cancel{m\omega}}{2\hbar}\left[-xip + ipx - xip + ipx\right]$$

$$= \frac{i}{\hbar}\left[px - xp\right] = \frac{i}{\hbar}[p, x] = -\frac{i(i\hbar)}{\hbar} = 1$$

$$\boxed{[a, a^\dagger] = 1}$$

Number operator $N = a^\dagger a$

$$a^\dagger a = \frac{m\omega}{2\hbar}\left[\left(x^2 + \frac{p^2}{m^2\omega^2}\right) + \frac{xip}{m\omega} - \frac{ipx}{m\omega}\right]$$

$$= \frac{m\omega^2 x^2}{2\hbar\omega} + \frac{p^2}{2\hbar m\omega} + \frac{i}{2\hbar}(xp - px)$$

$$= \frac{1}{\hbar\omega}\underbrace{\left(\frac{m\omega^2 x^2}{2} + \frac{p^2}{2m}\right)}_{H} + \frac{i}{2\hbar}\underbrace{[x, p]}_{i\hbar}$$

$$N = a^\dagger a = \frac{H}{\hbar\omega} - \frac{1}{2} \Rightarrow H = \hbar\omega\left(N + \frac{1}{2}\right)$$

Some commutative relationship aside:

$$[N, a] = [a^\dagger a, a] = a^\dagger\underbrace{[a, a]}_{0} + \underbrace{[a^\dagger, a]}_{-1}a$$

$$[N, a] = -a$$

$$[N, a^\dagger] = a^\dagger$$

for some eigenstate $|n\rangle$ w/ eigenvalue n

$$N|n\rangle = n|n\rangle$$

$$H|n\rangle = (n+\tfrac{1}{2})\hbar\omega|n\rangle$$

☆ $$Na^\dagger|n\rangle = (\underbrace{[N,a^\dagger]}_{a^\dagger} + a^\dagger N)|n\rangle$$

$$= a^\dagger(1+N)|n\rangle = (n+1)a^\dagger|n\rangle$$

$\Rightarrow$ $a^\dagger|n\rangle$ is an eigenket of N w/ eigenvalue ↑↑ by 1 [creation]

∴ ↑↑ one unit of $\hbar\omega$

☆ $$Na|n\rangle = ([N,a]+aN)|n\rangle = (n-1)a|n\rangle$$ $\Rightarrow$ $a|n\rangle$ is an eigenket of N w/ eigenvalue ↓↓ by 1

∴ ↓↓ one unit of $\hbar\omega$ [annihilation]

↳ ~~implies $a|n\rangle$ & $|n-1\rangle$~~ are the same, up to a multiplicative constant, c.

$$a|n\rangle = c|n-1\rangle$$

for both $|n\rangle$ & $|n-1\rangle$ to be normalized.

$$\langle n|a^\dagger a|n\rangle = |c|^2$$

$$n = |c|^2 \Rightarrow$$

for $c>0$ and $c \in \mathbb{R}$ (by convention)

$$c = \sqrt{n}$$

$$\boxed{\begin{aligned} a|n\rangle &= \sqrt{n}\,|n-1\rangle \\ a^\dagger|n\rangle &= \sqrt{n+1}\,|n+1\rangle \end{aligned}}$$

in the condition that

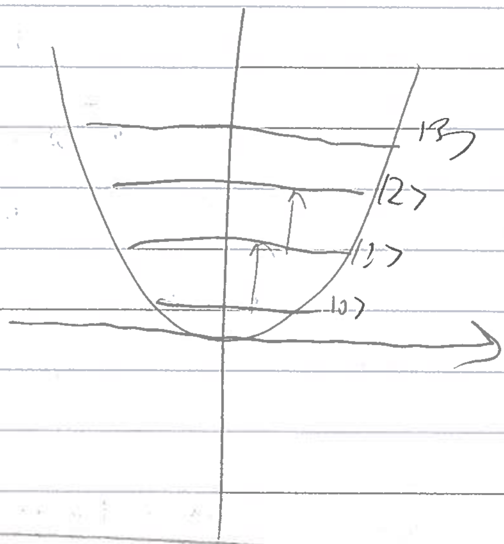
☆ $$n = \langle n|N|n\rangle = (\langle n|a^\dagger).(a|n\rangle) \geq 0$$

EXTRA:

for $n=0$

$E_0 = \frac{\hbar\omega}{2}$ $\Rightarrow$ $H|n\rangle = E_0|n\rangle = E|0\rangle$

applying the creation operator to $|0\rangle$



$|1\rangle = a^\dagger|0\rangle$

$|2\rangle = \left(\frac{a^\dagger}{\sqrt{2}}\right)|1\rangle = \left(\frac{(a^\dagger)^2}{\sqrt{2}}\right)|0\rangle$

$\vdots$

$|n\rangle = \left[\frac{(a^\dagger)^n}{\sqrt{n!}}\right]|0\rangle$

2nd Quantisation (by Frank's notes + book: Condensed Matter Field Theory; A Altland & B. Simons, 2010)

- for many body systems
- for interpreting creation of quanta of energy by ladder operators ($a^\dagger$, a)

* indistinguishability of fermion & bosons:

under particle exchange w.r.t. position x_1 & x_2 (λ_1, λ_2)

$|\lambda_1, \lambda_2\rangle(x_1, x_2) = |\lambda_1, \lambda_2\rangle_{F(B)} \equiv \frac{1}{\sqrt{2}}\left(\langle x_1|\lambda_1\rangle\langle x_2|\lambda_2\rangle + \zeta\langle x_1|\lambda_2\rangle\langle x_2|\lambda_1\rangle\right)$

$= \frac{1}{\sqrt{2}}\left(|\lambda_1\rangle \otimes |\lambda_2\rangle + \zeta|\lambda_2\rangle \otimes |\lambda_1\rangle\right)$

$\zeta = +1$ for bosons

$\zeta = -1$ for fermions

$\otimes$: antisymmetrised / symmetrised product

* Permutation operator

$\hat{P}_{ij}\,\psi(x_1, \dots x_i \dots x_j \dots x_n) = \zeta\psi(x_1, \dots, x_i, \dots, x_j \dots x_n)$

$\hat{P}_{ij}\,\hat{P}_{ji} = 1$

Basis state for many particle state:

$$\Psi(x_1, ..., x_N) = N \sum_P (\pm)^P \underbrace{\psi_{\alpha_1}(x_1) \cdots \psi_{\alpha_N}(x_N)}_{\text{Orthonormal basis states of a single particle}}$$

order of permutation

$N = \frac{1}{\sqrt{N!}}$ for normalization

- Occupational No.

generally: $|n_1, n_2 \ldots\rangle$

Fermions: $0, 1$

Bosons: $0, 1, 2, \ldots, \infty$

- Fock space: $\mathcal{F}$ => set of states w/ All possible combinations of occupation numbers
Hilbert direct sum of tensor product of copies of a single-particle Hilbert space H

$$\mathcal{F} \equiv \bigoplus_{N=0}^{\infty} \mathcal{F}^N$$

direct sum

- some important commutation (anticommutation) relationships for fermions (bosons)

from: $\hat{c}_i^{\dagger} \hat{c}_j^{\dagger} |0\rangle = \pm c_j^{\dagger} c_i^{\dagger} |0\rangle$

$[c_i, c_j^{\dagger}] = \delta_{ij}$ $\quad \{c_i, c_j^{\dagger}\} = \delta_{ij}$

$[c_i, c_j] = 0$ $\quad \{c_i, c_j\} = 0$

$[c_i^{\dagger}, c_j^{\dagger}] = 0$ $\quad \{c_i^{\dagger}, c_j^{\dagger}\} = 0$

- number operator: $n_\eta = c_\eta^{\dagger} c_\eta$

Signature of topology order:

finite system: surface states which are topologically protected

=> for the Haldane chain: unpaired $s=\frac{1}{2}$ at the ends of the chain (edge states)

↳ topological edge states are protected by the gap, Δ, of the bulk

✱ topological order cannot be destroyed by perturbation that do not close the gap

@ topological phase ~~the~~ transition: $\Delta = 0$

Important: going from $\beta = \frac{1}{3}$ $(\Delta > 0)$ to $\beta = 0$, (Haldane model)

Δ remains finite all the way

=> Haldane model is topological

✱ In the project:

spin-$\frac{1}{2}$ model: using Jordan-Wigner Transformation to map onto a model of spinless ~~f~~ fermions

$D=1$

- contains a string operator

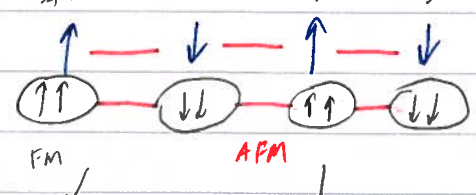
[related to string operator which measures topological order]

- for the AKLT ($\beta = \frac{1}{3}$, $S=1$) model, ground state is written in $S=\frac{1}{2}$ degrees of freedom

↳ why not start w/ $S=\frac{1}{2}$ model?

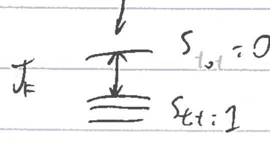
=> cook up a $S=\frac{1}{2}$ model, that in a certain regime, behaves as the Haldane model ($S=1$)

For $J_F \gg J_A$: (behaves like a spin-1 object)



Haldane spin chain physics

✱ topological?



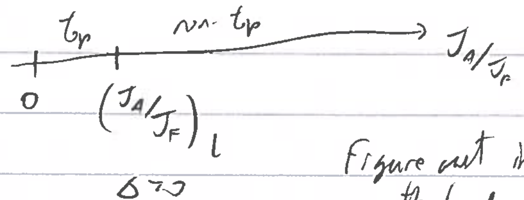


Figure out in which regime the topological order breaks

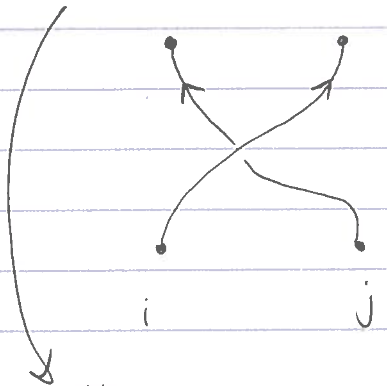
First quantised: $\Rightarrow$ S.E.

$\Rightarrow$ fixed particle no.

$\rightarrow$ N-particle wavefunctions that are eigenstates w/ energies

$$\epsilon_0 \leq \epsilon_1 \leq \epsilon_2 \leq \epsilon_3 \ \dots \ \epsilon_N$$

$$|\psi(\dots, r_i, \dots r_j)|^2 = |\psi(\dots, r_j, \dots r_i)|^2$$



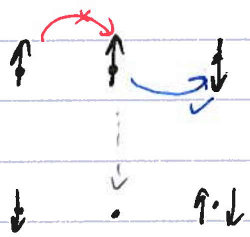
$$\psi(\dots, r_i \dots r_j \dots) = \underset{\text{fermions}}{\overset{\text{bosons}}{\pm}} \psi(\dots, r_j \dots, r_i \dots)$$

Pauli principle:

$$\psi(\dots, r_i, \dots r_i, \dots) = 0$$

* No two fermions can occupy the same quantum state (as from chem)

For fermions on a lattice $(\underset{\text{position}}{r}, \underset{\text{spin}}{\sigma} = \uparrow, \downarrow)$



$$\psi(r + L\hat{e}_i) = \psi(r)$$

$V = L^3$ w/ periodic boundary cond^{ns}

$$e^{ikr} e^{iLk_i} = e^{ikr}$$

$$e^{iLk_i} = 1$$

$$k_i = \frac{2\pi}{L} n \qquad n \in \mathbb{Z}$$

in momentum space: $(\underline{k}, \sigma = \uparrow, \downarrow)$

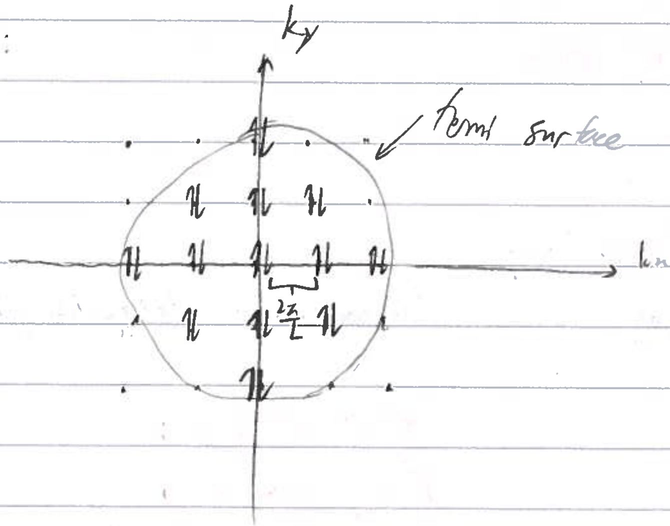
$$-\frac{\hbar^2}{2m} \nabla^2 \psi(r) = \epsilon \psi(r)$$

$$\psi(r) \propto e^{i\underline{k}\cdot\underline{r}} \quad \text{(plane wave state)}$$

$$\epsilon = \frac{\hbar^2 k^2}{2m}$$

4

$d=2$:



$L \to \infty$

density fixed

$\therefore$ Pauli principle $\Rightarrow$ ground state of N fermions is $\therefore$ obtained by filling momentum states of $\uparrow\uparrow$ energies (filling the fermi sea)

As $L \to \infty$, density is fixed

density is given by:

$V = L^3$ $\qquad \frac{2\pi}{L} = k$

$$\rho = \frac{N}{V} = \frac{N_\uparrow + N_\downarrow}{V} = \frac{2}{V}\sum_{\substack{k \\ |k|\le k_F}} = \frac{2}{(2\pi)^3}\sum_{\substack{k \\ |k|\le k_F}} \Delta k^3$$

$k^3 = \left(\frac{2\pi}{L}\right)^3$

$L^3 = \frac{2\pi}{k^3}$

$$\xrightarrow[L\to\infty]{} \frac{2}{(2\pi)^3}\int_{|k|\le k_F} d^3k = \frac{2}{(2\pi)^3}\,\frac{4}{3}\pi k_F^3 = \frac{1}{3\pi^2}k_F^3$$

spherical corr

$k_F = (3\pi^2 \rho)^{\frac{1}{3}}$

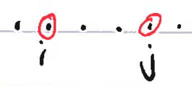
$$\epsilon_F = \frac{\hbar^2 k^2}{2m} = (3\pi^2)^{\frac{2}{3}}\frac{\hbar^2}{2m}\rho^{\frac{2}{3}}$$

2nd Quantisation $\qquad$ wavefunctions $\longrightarrow$ operators

$\therefore$ Hilbert space for all possible particle no.

$\Rightarrow$ harmonic oscillator (0-dim) $\qquad \hat{H}|n\rangle = \hbar\omega(n+\frac{1}{2})|n\rangle$

$|0\rangle, |1\rangle, |2\rangle, \ldots,$ $\qquad = \hbar\omega(b^\dagger b + \frac{1}{2})|n\rangle \qquad n = b^\dagger b$



for the vacuum state:

$|0\rangle = |0\rangle_1 \otimes |0\rangle_2 \otimes |0\rangle_3 \ldots$

(Fock space ??)

$[b, b^\dagger] = 1 \qquad \Leftrightarrow \qquad bb^\dagger - b^\dagger b = 1$

$[b, b] = 0, [b^\dagger, b^\dagger] = 0$

$$b_j^\dagger b_i^\dagger |0\rangle = \overset{\text{bosons}}{\underset{\text{fermions}}{\pm}} b_i^\dagger b_j^\dagger |0\rangle$$

$\Downarrow$ i & j exchange

$$b_i^\dagger b_j^\dagger \cancel{|0\rangle} = \pm b_j^\dagger b_i^\dagger \cancel{|0\rangle}$$ translating to

Commutator relationships for **bosons** $\longleftrightarrow$ Anticommutator relationship for **fermions**

$$[b_i^\dagger, b_j^\dagger] = 0 \qquad \{c_i^\dagger, c_j^\dagger\} = 0$$

for all i, j

$$[b_i, b_j] = 0 \qquad \{c_i, c_j\} = 0$$

$$[b_i, b_j^\dagger] = \delta_{ij} \qquad \{c_i, c_j^\dagger\} = \delta_{ij}$$

eg. $c_i^\dagger c_i^\dagger |0\rangle = -c_i^\dagger c_i^\dagger |0\rangle = 0$

(Pauli)

Hamiltonian in 2nd Quantisation

(d=1, fermions) $\sigma = \uparrow, \downarrow$; $r = i$

$i \in \mathbb{Z}$ (lattice site)

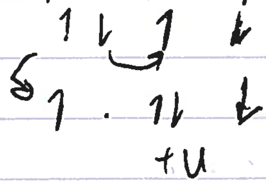
$$\hat{H} = \pm \sum_\sigma \sum_i \left(\underbrace{c_{i+1,\sigma}^\dagger c_{i\sigma}}_{} + \underbrace{c_{i\sigma}^\dagger c_{i+1,\sigma}}_{+h.c.} \right)$$

hermitian conjugate

interactions :

$$\hat{H}_{int} = U \sum_i \hat{n}_{i\uparrow} \hat{n}_{i\downarrow} = U \sum_i c_{i\uparrow}^\dagger c_{i\uparrow} c_{i\downarrow}^\dagger c_{i\downarrow} \Rightarrow$$

Hubbard Repulsion



potential :

$$\hat{H} = \sum_{i\sigma} V_i \hat{n}_{i\sigma}$$

$$= \sum_{i\sigma} V_i c_{i\sigma}^\dagger c_{i\sigma}$$

- Bases of transformation:

  高 常 x ∂_p

  position & momentum:

Jordan-Wigner Transformation: (fermionize)

↳ for pairs of spins of diff sites.

* spin operators @ diff site commute
* fermion operators @ diff sites anticommute

* main idea: writing spin operators in terms of number, creation & annihilation operators

$$\hat{S}^+_m = \hat{c}^+_m \prod_{l<m} (1-2\hat{n}_l) \quad ; \quad \hat{S}^+_m = \hat{c}^+_m e^{i\pi \sum_{l<m} \hat{n}_l} \quad , \quad f^+$$

$$\hat{S}^-_m = \prod_{l<m} (1-2\hat{n}_l)\hat{c}_m \quad ; \quad \hat{S}^-_m = e^{-i\pi \sum_{l<m} \hat{n}_l} \hat{c}_m \quad , \quad f$$

$$\hat{S}^z_m = \hat{n}_m - \frac{1}{2} = c^+_m c_m - \frac{1}{2} = f^+ f - \frac{1}{2}$$

$$\hat{c}^+_m (1-2\hat{n}_m) = -(1-2\hat{n}_m)\hat{c}^+_m$$

$$\hat{c}_m (1-2\hat{n}_m) = -(1-2\hat{n}_m)\hat{c}_m$$

2nd Quantisation ⇒ the Quantum Antiferromagnet (P. 79 Condensed Matter Field Theory)

$$\hat{H} = J \sum_{\langle mn \rangle} \hat{S}_m \cdot \hat{S}_n$$

$J > 0$

$$* \quad [\hat{S}^+, \hat{S}^-] = \hat{S}^+\hat{S}^- - \hat{S}^-\hat{S}^+ = \hat{c}^+_m \prod_{l<m}(1-2\hat{n}_l) \prod_{l<m}(1-2\hat{n}_l)\hat{c}_m - \prod_{l<m}(1-2\hat{n}_l)\hat{c}_m \hat{c}^+_m \prod_{l<m}(1-2\hat{n}_l) \Rightarrow$$

→

$$[\hat S_m^+, \hat S_m^-] = \overbrace{c_m^\dagger \prod_{l<m}(1-2\hat n_l)}^{f^\dagger}\overbrace{\prod_{l<m}(1-2\hat n_l)\hat c_m}^{f} - \prod_{l<m}(1-2\hat n_l)\hat c_m c_m^\dagger \prod_{l<m}(1-2\hat n_l)$$

$$= \left[- c_m^\dagger \prod_{l<m}(1-2\hat n_l)\, \hat c_m \prod_{l<m}(1-2\hat n_l)\right] - \left[- \hat c_m \prod_{l<m}(1-2\hat n_l)\, \hat c_m^\dagger \prod_{l<m}(1-2\hat n_l)\right]$$

$= f^\dagger f - f f^\dagger$ $\quad = \cancel{2f^\dagger f}$ $\quad f^\dagger f - 1 + f^\dagger f = 2f^\dagger f = 2\hat n = 1 = 2\hat S^z$

$\{f_i, f_i^\dagger\} = \delta_{ii} = 1$

$\Rightarrow f_i f_i^\dagger - f^\dagger f = 1$

$\cancel{-f^\dagger f = 1} \to f f^\dagger$

$f_i f^\dagger = 1 + f^\dagger f$

$[\hat S_m^+, \hat S_m^-] = 2\hat S^z$

(b) $\hat S_m^+ \hat S_{m+1}^- = f_m^\dagger f_{m+1}$

$\hookrightarrow$ $\hat S_m^+ \hat S_{m+1}^- = \cancel{f_m^\dagger e^{i\pi \sum_{j<m} \hat n_m} e^{-i\pi\sum_{j<m+1}\hat n_{m+1}} f_{m+1}}$ $\quad = 1$ (sum till m)

~~Aware of the different~~ indicies

$$= f_m^\dagger e^{i\pi \sum_{j<m} \hat n_l} e^{-i\pi \sum_{k<m+1} \hat n_2} f_{m+1} = f_m^\dagger e^{i\pi\sum_{j<m}\hat n_j} e^{-i\pi\sum_{l<m}\hat n_l} e^{-i\pi \hat n_m} f_{m+1}$$

$[\hat n_j, \hat n_l] = 0$

only $e^{-i\pi \hat n_m}$ remains

$$= f_m^\dagger e^{-i\pi \hat n_m} f_{m+1} = f_m^\dagger f_{m+1}$$

$\hat n_m \in \mathbb{Z}$

$e^{-i\pi \hat n_m}$ $\quad 0$

$= \underbrace{\cos(\pi \hat n_m)}_{=1,-1} - i\sin(\pi\hat n_m)$

$\hat n_m = \{0, 1\}$

c)

Fermionic hopping & mapping

$\vec{S} = \begin{pmatrix} S_x \\ S_y \\ S_z \end{pmatrix}$

Assuming the following form

$$\hat{H} = J \sum_{\langle m \rangle} \hat{S}_m \cdot \hat{S}_{m+1} + \beta J \sum_{m} (\hat{S}_m \cdot \hat{S}_{m+1})^2$$

[biquadratic coupling using 1st order perturbation]

$$= \sum_m J[\hat{S}^x_m \hat{S}^x_{m+1} + \hat{S}^y_m \hat{S}^y_{m+1} + \hat{S}^z_m \hat{S}^z_{m+1}]$$
$$+ \beta J [\hat{S}^x_m \hat{S}^x_{m+1} + \hat{S}^y_m \hat{S}^y_{m+1} + \hat{S}^z_m$$

$S^+_m = \hat{S}^x + i\hat{S}^y$

$S^-_m = \hat{S}^x - i\hat{S}^y$

$$\hat{H} = \sum_n \left[J_z \hat{S}^z_n \hat{S}^z_{n+1} + \frac{J_\perp}{2} (\hat{S}^+_n \hat{S}^-_{n+1} + \hat{S}^-_n \hat{S}^+_{n+1}) \right]$$

$\hat{S}^+\hat{S}^- - \hat{S}^-\hat{S}^+ = 2\hat{S}^z$

using:

$\Rightarrow$ $[\hat{S}^+, \hat{S}^-] = 2\hat{S}^z$

$\hat{S}^+ \hat{S}^-_{m+1} = f^+_m f_{m+1}$

& Jordan-Wigner transformation

$\{f_i, f^\dagger_{i+1}\} = 0$

$f_i f^\dagger_{i+1} = -f^\dagger_i f_{i+1}$

$$\hat{H} = \sum_n \left[\left[J_z \left(f^\dagger_n f_{n+1} - \frac{1}{2}\right)\left(f^\dagger_n f_{n+1} - \frac{1}{2}\right)\right] + \left[\frac{J_\perp}{2} \left(f^\dagger_n f_{n+1} + \underbrace{f_n f^\dagger_{n+1}}_{h.c.} \right) \right] \right]$$

$$\hat{H} = \sum_n J_z \left(f^\dagger_n f_{n+1} f^\dagger_n f_{n+1} - f^\dagger_n f_{n+1} + \frac{1}{4} \right)$$
$$+ \frac{J_\perp}{2} (f^\dagger_n f_{n+1} + \overset{f_n f^\dagger_{n+1}}{h.c.})$$

$\Downarrow$ But how to map?

for XY-model, non-interacting tight-binding

n $\quad$ $n+1$

$$\hat{H} = \sum_n \frac{J_\perp}{2} (f^\dagger_n f_{n+1} + f_n f^\dagger_{n+1})$$

$\hat{H} =$ Diagonalising using ~~discrete Fourier Transform~~.

finding eigen~~vects~~ value in fourier k space

$$\hat{H} = \frac{J}{2} \sum_n (f^\dagger_n f_{n+1} + f_n f^\dagger_{n+1})$$

~~$\hat{H} = \frac{J}{2}\sum_n (f^\dagger_n f^\dagger_{n+1}) \begin{pmatrix} 0 & \epsilon(k) \\ \epsilon(k) & 0 \end{pmatrix} \begin{pmatrix} f_n \\ f_{n+1} \end{pmatrix}$~~ ??

$$d_k = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} e^{-ik_n} f_n \quad \Rightarrow \quad f_n = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} e^{ik_n} d_k$$

$$d_k^\dagger = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} e^{-ik_n} f^\dagger_n \quad \text{reverse transform} \quad f^\dagger_n = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} e^{-ik_n} d^\dagger_n$$

$$\sum_j e^{i(k_1-k_2)j} = N\delta_{k_1,k_2}$$

~~$\hat{H} = \frac{J}{2}\sum_n ($~~

rewriting each term of the Hamiltonian

$$\sum_n f^\dagger_n f_{n+1} = \sum_k e^{ik} d_k^\dagger d_k$$

$$\sum_n f_n f^\dagger_{n+1} = \sum_k e^{-ik} d_k^\dagger d_k$$

$$\epsilon(k) = J\cos(k \underbrace{d_k^\dagger d_k}_{a}) = J\cos(ka)$$

OK

$$\hat{H} = \sum_{ij} H_{ij} \hat{a}^\dagger_i \hat{a}_j = \sum_{ij} H_{ij} a^\dagger_i a_j$$

Hermitian matrix ⇒ is real & can be diagonalized by a unitary transformation to a new basis

⇒

$$\hat{\alpha}^\dagger_i = U_{ij} \hat{a}^\dagger_j \Rightarrow \text{creation operator on a basis: } \hat{a}^\dagger_j = U^\dagger_{ji} \alpha^\dagger_i$$

(23)

$$\hat{H} = \sum_{ij} H_{ij} a^\dagger_i a_j$$

$$= \sum_{ijkl} H_{ij} U_{ik} \alpha^\dagger_k \alpha_l U^\dagger_{lj} = \sum_{kl} \underbrace{\sum_{ij} H_{ij} U_{ik} U^\dagger_{lj}}_{\Rightarrow (U^\dagger H U)_{kl}} \alpha^\dagger_k \alpha_l \overset{k=l}{=} \sum_l \epsilon_l \hat{\alpha}^\dagger_l \hat{\alpha}_l$$

& finding a Unitary rotation that ~~diago~~ is diagonal diagonalises $\hat{H}$

Diagonalise it:

$$\hat{\mathcal{H}} = \frac{J}{2} \sum_n \left(f_n^\dagger f_{n+1} + f_n f_{n+1}^\dagger \right)$$

$$\Rightarrow \hat{\mathcal{H}} = \frac{J}{2} \sum_n \left(f_n^\dagger \; f_{n+1}^\dagger \right) \begin{pmatrix} 0 & \epsilon(k) \\ \epsilon(k) & 0 \end{pmatrix} \begin{pmatrix} f_n \\ f_{n+1} \end{pmatrix}$$??

side proof

$$\cancel{\left(f_n^\dagger f_{n+1}^\dagger\right)\begin{pmatrix} \epsilon_k f_{n+1} \\ \epsilon_k f_n \end{pmatrix}}$$

$$\cancel{\Rightarrow f_n^\dagger f_{n+1} + f_{n+1}^\dagger f_n}$$

$$\left(f_{n+1}^\dagger \epsilon(k) \quad f_n^\dagger \epsilon(k) \right) \begin{pmatrix} f_n \\ f_{n+1} \end{pmatrix}$$

$$= \left(f_{n+1}^\dagger f_n + f_n^\dagger f_{n+1} \right) \epsilon(k)$$

kind of like a dot product

try: $\det \begin{vmatrix} 0 & \epsilon(k) \\ \epsilon(k) & 0 \end{vmatrix} = 0$

$$-\epsilon^2(k) = 0$$

$$\epsilon^2(k) = 0$$

Trivially $\epsilon(k) = 0$

non-trivial solution:

$$\epsilon(k) = \left(e^{ika}\right) = 0$$

$$= \left(\cos(ka) + i\sin(\tfrac{i}{2}k)\right)$$

$k = \frac{\pi n}{2}$ $\quad n = 1, 2, 3$

$\epsilon(k) = \cos(ka)$

$$ka = \frac{\pi}{2} n$$

$$k = \frac{\pi n}{2a}$$

13

Jordan - Wigner Transformation

1D, spin $\frac{1}{2}$

Taking the toy Hamiltonian

$$\hat{\mathcal{H}} = J \sum_i \left(\hat{\underline{S}}_i \cdot \hat{\underline{S}}_{i+1} + \alpha \hat{\underline{S}}_i^z \hat{\underline{S}}_{i+1}^z \right)$$

$\alpha \Rightarrow$ anisotropy term

$\alpha = 0$ isotropic case

$\Downarrow$

$J > 0$ AFM

XXZ chain

ground state: Néel AFM ↑↓↑↓

$\alpha > 0$: spins along z-axis ↑↓↑↓

(easy-axis anisotropy)

$J_x = J_y = J \quad J_z = (1+\alpha) J$

$\alpha < 0$: spins in the xy plane →←→←

(eg. x-axis)

(easy-plane anisotropy)

→ ← → ← → ←

unit cell contains two spins

we want to "flip"/"rotate" every other 2nd spin

↓

→ → → → → → (1 site per unit cell)

Using transformations of operators, which also satisfies with the commutator relationships

$$T_i^x = \underbrace{(-1)^i}_{e^{i\pi n} \; n \in \mathbb{Z}} \hat{S}_i^x$$

$$T_i^y = \hat{S}_y^y$$

$$T_i^z = (-1)^i \hat{S}_z^z$$

$\Rightarrow T^\alpha$ are proper spin $\frac{1}{2}$ operators

$*[\hat{S}_i^x, \hat{S}_i^y] = i\hbar \hat{S}_i^z$

$$\Rightarrow [T_i^x, T_i^y] = (-1)^i [\hat{S}_i^x, \hat{S}_i^y] = (-1)^i i\hbar \hat{S}_i^z = i\hbar \hat{T}_i^z$$

$$\left.\begin{aligned} &\text{I,} \quad \hat{T}^+ = \hat{T}^x + i \hat{T}^y \\ &\quad\;\; \hat{T}^- = \hat{T}^x - i \hat{T}^y \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} \hat{T}^x &= \tfrac{1}{2}(\hat{T}^+ + \hat{T}^-) \\ \hat{T}^y &= \tfrac{1}{2i}(\hat{T}^+ - \hat{T}^-) \end{aligned} \right] (***)$$

Spin $\frac{1}{2}$ operators $\Longleftrightarrow$ Spinless fermions

$\hat{T}^z |\frac{1}{2}, m\rangle = m |\frac{1}{2}, m\rangle$

$m = \pm\frac{1}{2}$ $(\hbar = 1)$

$\hat{T}^z |\Uparrow\rangle = \frac{1}{2} |\Uparrow\rangle$
$\hat{T}^z |\Downarrow\rangle = -\frac{1}{2} |\Downarrow\rangle$

state of spinless fermions

$|0\rangle$: no fermion
$|1\rangle$: one fermion

nothing else $\rightarrow$ Pauli principle

$\hat{T}^+ |\Uparrow\rangle = 0$
$\hat{T}^+ |\Downarrow\rangle = |\Uparrow\rangle$

$\hat{T}^- |\Downarrow\rangle = 0$
$\hat{T}^- |\Uparrow\rangle = |\Downarrow\rangle$

$\hat{c}^+ |1\rangle = 0$
$\hat{c}^+ |0\rangle = |1\rangle$

$\hat{c}^- |0\rangle = 0$
$\hat{c}^- |1\rangle = |0\rangle$

$n = \hat{c}^+\hat{c}$ (occupation number operator)

identifying: $|\Uparrow\rangle \hat{=} |0\rangle$
$|\Downarrow\rangle \hat{=} |1\rangle$

$\hat{T}^z = \frac{1}{2} - \hat{c}^+\hat{c}$
$\hat{T}^+ = \hat{c}$
$\hat{T}^- = \hat{c}^+$

$\Rightarrow$ works on single site: & assuming c, c^+ satisfy:

$\{c, c\} = \{c^+, c^+\} = 0$, $\{c_i, c_j^+\} = \delta_{ij}$

$\{c, c^+\} = 1$

single site @ the moment

$[\hat{T}^x, \hat{T}^y] = \frac{1}{4i}[\hat{T}^+ + \hat{T}^-, \hat{T}^+ - \hat{T}^-] \underset{\substack{\text{operators}\\\text{commute}\\\text{with itself}}}{\overset{\text{linearity}}{=}} \frac{1}{4i}[-[\hat{T}^+, \hat{T}^-] + [\hat{T}^-, \hat{T}^+]]$

$= \frac{1}{2i}[\hat{T}^-, \hat{T}^+]$

$= \frac{1}{2i}[\hat{T}^-\hat{T}^+ - \hat{T}^+\hat{T}^-] = \frac{1}{2i}[\hat{c}^+\hat{c} - \underbrace{\hat{c}\hat{c}^+}_{1-\hat{c}^+\hat{c}}] = \frac{1}{2i}(2\hat{c}^+c - 1) \times \frac{i}{i}$

$= i(\frac{1}{2} - \hat{c}^+\hat{c}) = i\hat{T}^z$ ✓

15

* spin operators on different site commute
  fermion operators anti commute.

$$(**)\left[\begin{array}{l}[\hat{T}_i^\alpha, \hat{T}_j^\beta] = i\delta_{ij}\epsilon_{\alpha\beta\gamma}\hat{T}_i^\gamma \\ [\hat{T}_i^+, \hat{T}_j^-] = [\hat{T}_i^x + i\hat{T}_i^y, \hat{T}_j^x - i\hat{T}_j^y] = -i[\hat{T}_i^x, \hat{T}_j^y] + i[\hat{T}_i^y, \hat{T}_j^x] \\ \quad = -i[\hat{T}_i^x, \hat{T}_j^y] - i[\hat{T}_i^x, \hat{T}_j^y] = 2\delta_{ij}\hat{T}_i^z\end{array}\right.$$

$$(*)\left[\begin{array}{l}\{c_i, c_j\} = \{c_i^+, c_j^+\} = 0 \\ \{c_i, c_j^+\} = \delta_{ij}\end{array}\right.$$

for different sites:

$$[\hat{T}_i^\alpha, \hat{T}_j^\beta] = 0 \qquad [\hat{T}_i^+, \hat{T}_j^-] = 0$$

$$\{c_i, c_j\} = \{c_i^+, c_j^+\} = \{c_i, c_j^+\} = 0$$

repairing w/ string operators: (JW transformation)

$$\hat{T}_i^z = \frac{1}{2} - \hat{c}_i^+\hat{c}_i$$
$$\hat{T}_i^+ = \prod_{l<i}(1 - 2\hat{c}_l^+\hat{c}_l)\hat{c}_i$$
$$\hat{T}_i^- = \hat{c}_i^+\prod_{l<i}(1 - 2\hat{c}_l^+\hat{c}_l)$$

string operator

$$\hat{D}_i = \prod_{l<i}(1 - 2\hat{c}_l^+\hat{c}_l)$$

$\hat{D}_i$

same site:

$$[\hat{c}_i, \hat{D}_i] = \hat{c}_i\hat{D}_i - \hat{D}_i\hat{c}_i = \hat{c}_i\prod_{l<i}(1 - 2c_l^+c_l) - \hat{D}_i\hat{c}_i$$

$$\underset{(*)}{=} \underbrace{\prod_{l<i}(1 - 2\hat{c}_l^+\hat{c}_l)}_{D_i}c_i - \hat{D}_i\hat{c}_i = 0$$

$$[\hat{c}_i, \hat{D}_i] = 0 \; ; \; [c_i^+, \hat{D}_i] = 0 \; ; \; \hat{D}_i^+ = \hat{D}_i$$

$$\hat{c}_i\prod_{l<i}(1 - 2\hat{c}_l^+\hat{c}_l) = \prod_{l<i}(c_i - 2\hat{c}_i\underbrace{\hat{c}_l^+}_{=\delta_{il} - \hat{c}_l^+\hat{c}_i}\hat{c}_l)$$

$$= \prod_{l<i}\hat{c}_i + 2\hat{c}_l^+\hat{c}_i\underbrace{\hat{c}_l}_{-\hat{c}_l\hat{c}_i} = \prod_{l<i}\hat{c}_i - 2\hat{c}_l^+\hat{c}_l\hat{c}_i$$

$$= \prod_{l<i}(1 - 2\hat{c}_l^+\hat{c}_l)c_i$$

$$[\hat{D}_i]^2 = \prod_{l<i}(1 - 2\hat{c}_l^+\hat{c}_l)\prod_{l<i}(1 - 2\hat{c}_l^+\hat{c}_l)$$

$$= \prod_{l<i}(1 - 4c_l^+c_l + 4\hat{c}_l^+\underbrace{\hat{c}_l c_l^+}_{1 - \hat{c}_l^+\hat{c}_l}\hat{c}_l)$$

$$= \prod_{l<i}(1 - 4\cancel{c_l^+c_l} + 4\cancel{\hat{c}_l^+\hat{c}_l} - 4\underbrace{\hat{c}_l^+\hat{c}_l^+\hat{c}_l\hat{c}_l}_{=0})$$

$$= 1 \qquad \boxed{\hat{D}_i^2 = 1}$$

$$\hat{D}_i\hat{D}_j = \prod_{l=i}^{j}(1 - 2\hat{c}_l^+\hat{c}_l)$$

i j

$\hat{D}_i$

$\hat{D}_j$

For neighboring lattice sites:

$$\hat{D}_i\hat{D}_{i+1} = 1 - 2\hat{c}_i^+c_i$$

* if $\hat{c}_i, c_i^+$ satisfy fermion anti-commutator relations (*), then the operators $\hat{T}_i^\alpha$ defined by JW => satisfies the spin commutator relationships (**)

7/7/2021 : Things to do:

① For $i<j$, check that $[\hat{T}_i^+, \hat{T}_j^-] = 0$

② Rewrite Hamiltonian in terms of c_i^+, c_i, & simplify as much as possible.

Rewriting Hamiltonian :

$$\hat{\mathcal{H}} = J\sum_i \left(\hat{\underline{S}}_i \cdot \hat{\underline{S}}_{i+1} + \alpha \hat{S}_i^z \cdot \hat{S}_{i+1}^z \right)$$

$$= J\sum_i \left((-1)^i \hat{T}_i^x (-1)^{i+1} \hat{T}_i^x + \hat{T}_i^y \hat{T}_{i+1}^y - (1+\alpha) \hat{T}_i^z \hat{T}_{i+1}^z \right)$$

$$= -J\sum_i \left(\hat{T}_i^x \hat{T}_{i+1}^x - T_i^y T_{i+1}^y + (1+\alpha) T_i^z T_{i+1}^z \right)$$

$$(***) \Longrightarrow -J\sum_i \left\{ \tfrac{1}{2}\big(\hat{T}_i^+ \hat{T}_{i+1}^- + \underbrace{\hat{T}_i^- \hat{T}_{i+1}^+}_{h.c.?} \big) + (1+\alpha) \hat{T}_i^z \hat{T}_{i+1}^z \right\}$$

$$= -J\sum_i \Big\{ \tfrac{1}{2} \big(\underbrace{\hat{D}_i \hat{c}_i \hat{c}_{i+1}^\dagger \hat{D}_{i+1}}_{A} + \underbrace{\hat{c}_{i+1}^\dagger \hat{D}_{i+1} \hat{D}_i \hat{c}_i}_{B} \big) + (1+\alpha) \underbrace{\underbrace{\big(\tfrac{1}{2} - \hat{c}_i^\dagger \hat{c}_i\big)}_{\hat{T}_i^z} \underbrace{\big(\tfrac{1}{2} - \hat{c}_{i+1}^\dagger \hat{c}_{i+1}}_{\hat{T}_{i+1}^z}}_{C}$$

$$[\hat{T}_i^+, \hat{T}_{i+1}^-] = 0$$

$$\hat{T}_i^+ \hat{T}_{i+1}^- = \hat{T}_{i+1}^- \hat{T}_i^+ \quad \Big\| \quad \hat{T}_i^- \hat{T}_{i+1}^+ = \hat{T}_{i+1}^+ \hat{T}_i^-$$

$$C = \frac{1}{4} - \frac{1}{2}\left(\hat{c}_i^\dagger \hat{c}_i + \hat{c}_{i+1}^\dagger \hat{c}_{i+1} \right) + c_i^\dagger \underbrace{c_i c_{i+1}^\dagger}_{-c_{i+1}^\dagger c_i} c_{i+1}$$

$$= \frac{1}{4} - \frac{1}{2}\left(c_i^\dagger c_i + c_{i+1}^\dagger c_{i+1} \right) - c_i^\dagger c_{i+1}^\dagger c_i c_{i+1}$$

$$A = \hat{D}_i \hat{c}_i \hat{c}_{i+1}^\dagger \hat{D}_{i+1}$$

$\cancel{c_i, c_i^\dagger}$

$$c_i c_i^\dagger + c_i^\dagger c_i = \cancel{0}\,1$$

$$= c_i \underbrace{D_i D_{i+1}}_{1-2c_i^\dagger c_i} c_{i+1}^\dagger$$

$$= \cancel{c_i(1-2c_i^\dagger c_i) c_{i+1}^\dagger}$$

$$= \cancel{(c_i - 2c_i c_i^\dagger c_i) c_{i+1}^\dagger}$$

$$= c_i c_{i+1}^\dagger - 2 \underbrace{c_i c_i^\dagger}_{1-c_i^\dagger c_i} c_i c_{i+1}^\dagger = c_i c_{i+1}^\dagger - \cancel{2(1-c_i^\dagger c_i)(c_i c_{i+1}^\dagger)}$$

$$\cancel{= c_i c_{i+1}^\dagger - 2 + 2 c_i^\dagger \underbrace{c_i c_i c_{i+1}^\dagger}_{=0}}$$

$$c_i c_i^\dagger + c_i^\dagger c_i = 1$$

$$A = \cancel{c_i c_{i+1}^\dagger} - 2$$

$$B = c_i^\dagger \underbrace{D_i D_{i+1}}_{(1-2c_i^\dagger c_i)} c_{i+1}$$

$$= \left(c_i^\dagger - 2 c_i^\dagger c_i^\dagger c_i \right) c_{i+1}$$

$$= c_i^\dagger c_{i+1} - 2 \underbrace{c_i^\dagger c_i^\dagger}_{=0} c_i c_{i+1}$$

$$B = c_i^\dagger c_{i+1}$$

$$\cancel{C = \frac{1}{4} - \frac{1}{2} c_i^\dagger c_i - \frac{1}{2} c_{i+1}^\dagger c_{i+1}} - c_i^\dagger c_{i+1}^\dagger c_i c_{i+1}$$

$$\cancel{= \frac{1}{4} - \frac{1}{2}(c_i c_i^\dagger - 1) - \frac{1}{2}(c_{i+1} c_{i+1}^\dagger - 1)} - c_i^\dagger c_{i+1}^\dagger c_i c_{i+1}$$

$$\cancel{= \frac{1}{4} - \frac{1}{2} c_i c_i^\dagger + \frac{1}{2} - \frac{1}{2} c_{i+1} c_{i+1}^\dagger + \frac{1}{2}} - c_i^\dagger \underbrace{c_{i+1}^\dagger c_i}_{1} c_{i+1}$$

$$= \frac{1}{4} + \cancel{1} - \frac{1}{2}(c_i c_i^\dagger + c_{i+1} c_{i+1}^\dagger) -$$

Checking $[T_i^+, T_j^-] = 0 \quad i<j$

$$[D_i c_i, D_j c_j^+] = \underbrace{D_i c_i D_j}c_j^+ - \underbrace{D_j c_j^+} D_i c_i$$

$$\to (1-2c_i^+c_i)c_j^+ = (c_j^+ - 2c_i^+ c_i c_j^+)$$

$$c_i(\overset{\neq}{1 \cdot 2c_i^+ c_i}) = c_i - 2c_i + \underbrace{2c_i^+ c_i c_i}_{=0} = -c_i$$

if $i \neq j \Rightarrow c_i(1-2c_j^+c_j) = (1-2c_j^+c_j)c_i$

$\cancel{\{c_i, c_j\}} = c_i$

$$= -D_i c_i c_j^+ - (c_j^+ - 2c_i^+ c_i c_j^+) D_i c_i$$

$\{c_i, c_j^+\} = c_i c_j^+ + c$

$$= -D_i c_i c_j^+ - (c_j^+ D_i c_i - 2c_i^+ c_i c_j^+ D_i c_i)$$

$$= -D_i c_i c_j^+ - (+D_i \underbrace{c_j^+ c_i} + 2c_i^+ \underbrace{c_i c_j^+} c_i D_i)$$

$\neq$

$$= \underbrace{-D_i c_i c_j^+ - (+D_i c_i c_j^+}_{=0} \underbrace{-2c_i^+ c_j^+ c_i c_i D_i}_{=0})$$

$$\frac{1}{4} + \underbrace{\sum_i (\hat{n}_i \hat{n}_{i+1}}_{\text{replace}} - \frac{1}{2}(\hat{n}_{i+1} + \hat{n}_i))$$

should get degen bands for new Hamiltonian

$$\sum_i \hat{n}_i \hat{n}_{i+1} = n_1 n_2 + n_2 n_3 \cdots n_i n_{i+1}$$

$$\sum_i^N \hat{n}_{i+1} = n_2 + n_3 + n_4 + \cdots n_{N+1} = n_1 + n_2 + n_3 + n_4 + n_5 \cdots n_{N+1} - n_1$$

$$\sum_i$$

$$A = \sum_i n$$

~~$i+1 \to k$~~

$k = i+1$

$i = k-1$

$$\sum_i^N \hat{n}_i + \hat{n}_{N+1}$$

$$\frac{N}{4} + \sum_i \hat{n}_i(n_i + n_{N+1}) - \frac{1}{2}\hat{n}_i - \frac{1}{2}\hat{n}_{N+1} - \frac{\hat{n}_1}{2}$$

$$A = c_i c_{i+1}^{\dagger} - 2 c_i \underbrace{c_i^{\dagger} c_i}\, c_{i+1}^{\dagger} = c_i c_{i+1}^{\dagger} - 2 c_i (1 - c_i c_i^{\dagger}) c_{i+1}^{\dagger} = c_i c_{i+1}^{\dagger} - (2 c_i + 2 c_i c_i^{\dagger}) c_{i+1}^{\dagger}$$

$$= c_i c_{i+1}^{\dagger} - 2 c_i c_{i+1}^{\dagger} - 2 c_i c_i c_i^{\dagger} c_{i+1}^{\dagger}$$

$$= - c_i c_{i+1}^{\dagger} - 2 c_i c_i^{\dagger} c_{i+1}^{\dagger}$$

$$c_i c_i^{\dagger} + c_i^{\dagger} c_i = 1$$

$$= 1 - c_i c_i^{\dagger}$$

Final Hamiltonian:

$$\hat{\mathcal{H}} = -\frac{J}{2} \sum_i \overbrace{\left(c_{i+1}^{\dagger} c_i + c_i^{\dagger} c_{i+1} \right)}^{\text{fermionic hopping}}$$

$$+ (1+\alpha) J \sum_i \underbrace{\left(\tfrac{1}{2} - \hat{n}_i\right)\left(\tfrac{1}{2} - \hat{n}_{i+1}\right)}_{\text{contains } \hat{n}_i \hat{n}_{i+1}}$$

non-interacting terms:

- hopping terms: $\hat{c}_i^{\dagger} \hat{c}_{i+1} + h.c.$
- density terms: $\hat{c}_i^{\dagger} \hat{c}_i = \hat{n}_i$
- anomalous terms: $c_i^{\dagger} c_{i+1}^{\dagger} + c_{i+1} c_i$ (don't conserve particle number)

interactions:

$\hat{n}_i \hat{n}_{i+1}$: cannot solve the problem analytically:

→ variational method

→ perturbation theory

→ mean-field theory

$\alpha = -1$: non-interacting system only x & y component of spin : XY spins

$$\Rightarrow \hat{\mathcal{H}} = -\frac{J}{2} \sum_i \left(c_{i+1}^{\dagger} c_i + c_i^{\dagger} c_{i+1} \right) \quad [\text{tight binding Hamiltonian}]$$

$$\hat{H} = -\frac{J}{2}\sum_i (c_{i+1}^\dagger c_i + c_i^\dagger c_{i+1})$$

in this case $a=1$ for simplicity

F.T $c_x^\dagger$ & c_x

to find energy eigenval. by diagonalisation

changing from position space to momentum space, k & k'

$$\hat{c}_x = \underbrace{\int_{-\pi/a}^{\pi/a} \frac{dk}{2\pi}}_{\int_k} \hat{c}_k^\dagger e^{ikx}$$

$$c_x = \int_{k'} \hat{c}_{k'} e^{-ik'x}$$

$$\hat{H} = -\frac{J}{2}\sum_x (c_{x+a}^\dagger c_x + c_x^\dagger c_{x+a}) = -\frac{J}{2}\sum_x \int_k\int_{k'} \{\hat{c}_k^{*} e^{ik(x+a)} \hat{c}_{k'} e^{-ik'x} + h.c.\}$$

Combining exp

$$= -\frac{J}{2}\sum_x \int_k\int_{k'} \left[\hat{c}_k^\dagger \hat{c}_{k'} \underbrace{e^{i(k-k')x}}_{\substack{=1 \text{ if } k=k' \\ =0 \text{ if } k\neq k'}} e^{ika} + h.c.\right]$$

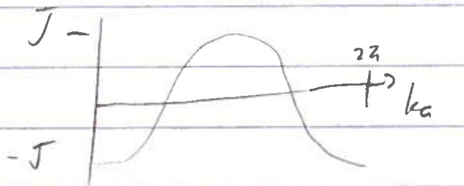
$$\sum_x e^{i(k-k')x} = \int_{-\infty}^{\infty} e^{i(k-k')x} dx = \delta_{kk'}$$

$$= -\frac{J}{2}\int_k\int_{k'} \delta_{kk'} \left[\hat{c}_k^\dagger \hat{c}_{k'} e^{ika} + h.c.\right]$$

$$\overset{k=k'}{=} -\frac{J}{2}\int_k (e^{ika} + e^{-ika})(\hat{c}_k^\dagger c_k)$$

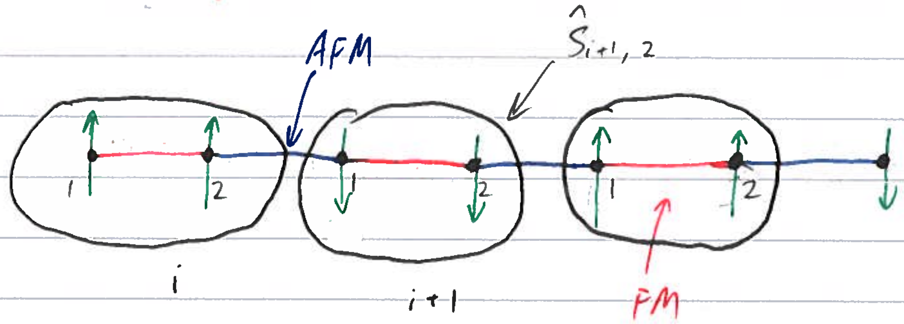
$$= -J\int_k \cos(ka)\, c_k^\dagger c_k \overset{\text{finite chain}}{=} -J\sum_k \cos(ka)\, c_k^\dagger c_k$$

Energy dispersion: $-J\cos(ka)$



Main

The Model:



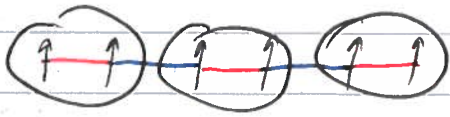
$$\hat{\mathcal{H}} = -J_F \sum_i \left(\hat{\underline{S}}_{i,1}\cdot\hat{\underline{S}}_{i,2} + \alpha \hat{S}^z_{i,1}\hat{S}^z_{i,2}\right)$$
$$+ J_A \sum_i \left(\hat{\underline{S}}_{i,2}\cdot\hat{\underline{S}}_{i+1,1} + \alpha \hat{S}^z_{i,2}\hat{S}^z_{i+1,1}\right)$$

$\alpha = 0$ Isotropic model (Heisenberg model)

$\alpha = -1$ xy model $\therefore S^z$ does not enter

$$= -J_F \sum_i \left[S^x_{i,1} S^x_{i,2} + S^y_{i,1} S^y_{i,2} + (1+\alpha)(S^z_{i,1} S^z_{i,2}) \right]$$
$$+ J_A \sum_i \left[S^x_{i,2} S^x_{i+1,1} + S^y_{i,2} S^y_{i+1,1} + (1+\alpha)(S^z_{i,2} S^z_{i+1,1}) \right]$$

spin flip



$$\left.\begin{aligned} \hat{T}^x_{i,m} &= (-1)^i \hat{S}^x_{i,m} \\ \hat{T}^y_{i,m} &= \hat{S}^y_{i,m} \\ \hat{T}^z_{i,m} &= (-1)^i \hat{S}^z_{i,m} \end{aligned}\right] \Rightarrow$$

$$H = -J_F \sum_i \Big[\overbrace{T^x_{i,1} T^x_{i,2}}^{A} + \overbrace{T^y_{i,1} T^y_{i,2}}^{B} + (1+\alpha) \overbrace{T^z_{i,1} T^z_{i,2}}^{C} \Big]$$
$$- J_A \sum_i \Big[\overbrace{T^x_{i,2} T^x_{i+1,1}}^{D} - \overbrace{T^y_{i,2} T^y_{i+1,1}}^{E} + (1+\alpha) \overbrace{T^z_{i,2} T^z_{i+1,1}}^{F} \Big]$$

$\Downarrow$ 1) $T^x, T^y \longrightarrow T^+, T^-$

2) JW ~~tranf~~ trans

$i \to i,m$

JW:

$$\hat{T}^z_{i,m} = \frac{1}{2} - c^\dagger_{i,m} c_{i,m}$$

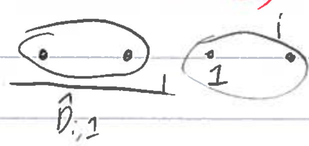
$$\hat{T}^+_{i,m} = \hat{c}_{i,m} \hat{D}_{i,m}$$

$$\hat{T}^-_{i,m} = \hat{c}^\dagger_{i,m} \hat{D}_{i,m}$$

$$\hat{D}^2_{i,m} = 1$$

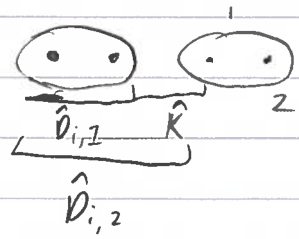
$$T^x = \frac{1}{2}(T^+ + T^-)$$
$$T^y = \frac{1}{2i}(T^+ - T^-)$$

$$\hat{D}_{i,1} = \prod_{l<i} (1 - 2c^\dagger_{l,1} c_{l,1})(1 - 2c^\dagger_{l,2} c_{l,2})$$



$$\hat{D}_{i,2} = \prod_{l<i} (1 - 2c^\dagger_{l,1} c_{l,1})(1 - 2c^\dagger_{l,2} c_{l,2}) \overbrace{(1 - 2c^\dagger_{i,1} c_{i,1})}^{K}$$

$$= \hat{D}_{i,1} (1 - 2c^\dagger_{i,1} c_{i,1})$$



Double checking the calculation

?

A+B: $T_{i,1}^x T_{i,2}^x + T_{i,1}^y T_{i,2}^y = \frac{1}{2}(T_{i,1}^+ T_{i,2}^- + T_{i,1}^- T_{i,2}^+)$

$= \frac{1}{2}(c_{i,1} D_{i,1} D_{i,2} c_{i,2}^\dagger + D_{i,1} c_{i,1}^\dagger D_{i,2} c_{i,2})$

$= \frac{1}{2}(c_{i,1} \underbrace{D_{i,1} D_{i,1}}_{1} (1-2c_{i,1}^\dagger c_{i,1}) c_{i,2}^\dagger + c_{i,1}^\dagger \underbrace{D_{i,1} D_{i,1}}_{1} (1-2c_{i,1}^\dagger c_{i,1}) c_{i,2}$

$= \frac{1}{2}(c_{i,1} c_{i,2}^\dagger - 2 \underbrace{c_{i,1} c_{i,1}^\dagger}_{1-c_{i,1}^\dagger c_{i,1}} c_{i,1} c_{i,2}^\dagger + c_{i,1}^\dagger c_{i,2} - \underbrace{2 c_{i,1}^\dagger c_{i,1}^\dagger c_{i,1} c_{i,2}})$

$= \frac{1}{2}(c_{i,1} c_{i,2}^\dagger - 2 c_{i,1} c_{i,2}^\dagger + 2 c_{i,1}^\dagger c_{i,1} c_{i,1} c_{i,2}^\dagger + c_{i,1}^\dagger c_{i,2})$

$= \frac{1}{2}(c_{i,1}^\dagger c_{i2} + c_{i2}^\dagger c_{i1})$

$H_F = -\frac{J_F}{2} \sum_i [c_{i,1}^\dagger c_{i2} + c_{i2}^\dagger c_{i1} + 2(1+\alpha)(\frac{1}{2} - c_{i1}^\dagger c_{i1})(\frac{1}{2} - c_{i2}^\dagger c_{i2})]$

D−E:

$T_{i2}^x T_{i+1,1}^x - T_{i,2}^y T_{i+1,1}^y = \frac{1}{4}(T_{i,2}^+ + T_{i,2}^-)(T_{i+1,1}^+ + T_{i+1,1}^-)$
$+ \frac{1}{4}(T_{i,2}^+ - T_{i,2}^-)(T_{i+1,1}^+ - T_{i+1,1}^-)$

$\Rightarrow \frac{1}{4}(T_{i,2}^+ T_{i+1,1}^+ + \cancel{T_{i,2}^+ T_{i+1,1}^-} + \cancel{T_{i,2}^- T_{i+1,1}^+} + T_{i,2}^- T_{i+1,1}^-)$

$+ \frac{1}{4}(T_{i,2}^+ T_{i+1,1}^+ - \cancel{T_{i,2}^+ T_{i+1,1}^-} - \cancel{T_{i,2}^- T_{i+1,1}^+} + T_{i,2}^- T_{i+1,1}^-$

$= \frac{1}{2}(T_{i,2}^+ T_{i+1,1}^+ + T_{i,2}^- T_{i+1,1}^-) = \frac{1}{2}(c_{i2} D_{i2} D_{i+1,1} c_{i+1,1} + c_{i2}^\dagger D_{i2} D_{i+1,1} c_{i+1,1}^\dagger)$

$$D-E = \frac{1}{2}\left(c_{i,2} D_{i,2} D_{i+1,1} c_{i+1,1} + c_{i,2}^\dagger D_{i,2} D_{i+1,1} c_{i+1,1}^\dagger\right)$$

(*) $$= \frac{1}{2}\Big(c_{i,2} \overbrace{D_{i,2} D_{i,2}}^{1} (1-2c_{i,2}^\dagger c_{i,2})\, c_{i+1,1}$$

$$+ c_{i,2}^\dagger \overbrace{D_{i,2} D_{i,2}}^{1} (1-2c_{i,2}^\dagger c_{i,2})\, c_{i+1,1}^\dagger\Big)$$

$$= \frac{1}{2}\Big(c_{i,2} c_{i+1,1} - 2 \overbrace{c_{i,2} c_{i,2}^\dagger}^{1-c_{i,2}^\dagger c_{i,2}} c_{i,2} c_{i+1,1}$$

$$+ c_{i,2}^\dagger c_{i+1,1}^\dagger - 2 \overbrace{c_{i,2}^\dagger c_{i,2}^\dagger}^{=0} c_{i,2} c_{i+1,1}^\dagger\Big)$$

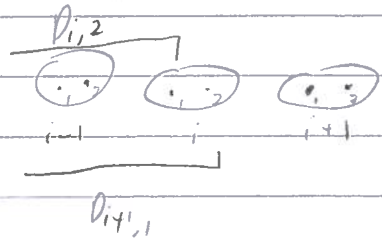
$$= \frac{1}{2}\Big(c_{i,2} c_{i+1,1} - 2(1-c_{i,2}^\dagger c_{i,2})\, c_{i,2} c_{i+1,1}$$

$$+ c_{i,2}^\dagger c_{i+1,1}^\dagger\Big)$$

$$= \frac{1}{2}\left(c_{i,2} c_{i+1,1} - 2 c_{i,2} c_{i+1,1} + 2 c_{i,2}^\dagger \cancel{c_{i,2}} c_{i,2} c_{i+1,1} + c_{i,2}^\dagger c_{i+1,1}^\dagger\right)$$

$$= \frac{1}{2}\left(-\overbrace{c_{i,2} c_{i+1,1}}^{\{c_i, c_j\}=0} + c_{i,2}^\dagger c_{i+1,1}^\dagger\right) = \frac{1}{2}\left(c_{i+1,1} c_{i,2} + c_{i,2}^\dagger c_{i+1,1}^\dagger\right)$$

(*) $D_{i+1,1} = D_{i,2}(1-2c_{i,2}^\dagger c_{i,2})$



(*) $D_{i+1,1} = D_{i,2}(1-2c_{i,2}^\dagger c_{i,2})$

$J_F \beta / 2$ || using $\beta := \frac{J_A}{J_F}$

$$\hat{H}_A = -\frac{J_A}{2}\sum_i \left[c_{i+1,1} c_{i,2} + c_{i,2}^\dagger c_{i+1,1}^\dagger + 2(1+\alpha)\left(\tfrac{1}{2} - c_{i,2}^\dagger c_{i,2}\right)\left(\tfrac{1}{2} - c_{i+1,1}^\dagger c_{i+1,1}\right)\right)$$

$$\hat{H} = \hat{H}_F + \hat{H}_A$$

$$\hat{H} = -\frac{J_F}{2}\sum_i \left[c_{i,1}^\dagger c_{i,2} + c_{i,2}^\dagger c_{i,1} + \beta\left(c_{i+1,1} c_{i,2} + c_{i,2}^\dagger c_{i+1,1}^\dagger\right)\right]$$

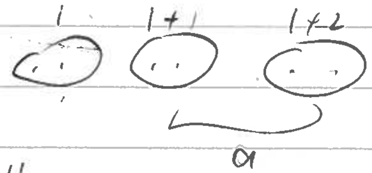
$$- J_F(1+\alpha)\sum_i \left(\tfrac{1}{2} - c_{i,1}^\dagger c_{i,1}\right)\left(\tfrac{1}{2} - c_{i,2}^\dagger c_{i,2}\right)$$

$$- J_F \beta (1+\alpha) \sum_i \left(\tfrac{1}{2} - c_{i,2}^\dagger c_{i,2}\right)\left(\tfrac{1}{2} - c_{i+1,1}^\dagger c_{i+1,1}\right)$$

Easy case: XY case $\Rightarrow \alpha = -1$

$$\hat{H}_{XY} = -\frac{J_F}{2}\sum_i \left(c_{i,1}^\dagger c_{i,2} + c_{i,2}^\dagger c_{i,1} + \beta\left(c_{i+1,1} c_{i,2} + c_{i,2}^\dagger c_{i+1,1}^\dagger\right)\right)$$

FT to change to momentum space
& diagonalize



Hopping terms:

$$\sum_i (c_{i,1}^\dagger c_{i,2} + h.c.) \Leftrightarrow \sum_x (c_1^\dagger(x) c_2(x) + h.c.)$$

$$c_1^\dagger(x) = \int_k c_1^\dagger(k) e^{ikx} \qquad c_1(x) = \int_k c_1(k) e^{-ikx}$$

$$= \sum_x \iint_{kk'} c_1^\dagger(k) e^{ikx} c_2(k') e^{-ik'x} + c_2^\dagger(k) e^{-ikx} c_1(k') e^{ikx}$$

$$= \sum_x \iint_{kk'} c_1^\dagger(k) c_2(k') e^{i(k-k')x} + c_2^\dagger(k) c_1(k') e^{-i(k-k')x} \qquad \sum_x e^{i(k-k')x} = \delta_{kk'} \quad (**)$$

$$\overset{(**)(k=k')}{=} \int_k c_1^\dagger(k) c_2(k) + c_2^\dagger(k) c_1(k)$$

$$\overset{(***)}{=} \int_0^{\pi} \frac{dk}{2\pi} \left[c_1^\dagger(k) c_2(k) + c_2^\dagger(k) c_1(k) - c_2(-k) c_1^\dagger(-k) - c_1(-k) c_2^\dagger(-k) \right]$$

$$k \in [-\pi, \pi) \quad \int_{-\pi}^{\pi} \frac{dk}{2\pi} f(k) \underset{\substack{\tilde{k}=-k \\ d\tilde{k}=-dk}}{\Rightarrow} = -\int_{\pi}^{-\pi} \frac{d\tilde{k}}{2\pi} f(-\tilde{k}) = \int_{-\pi}^{\pi} \frac{d\tilde{k}}{2\pi} f(-\tilde{k})$$

$$= \frac{1}{2} \int_{-\pi}^{\pi} \frac{dk}{2\pi} (f(k) + f(-k))$$

$$\overset{\text{even fn}}{=} \int_0^{\pi} \frac{dk}{2\pi} (f(k) + f(-k))$$

or odd $= 0$

$$\therefore \int_{-\pi}^{\pi} \frac{dk}{2\pi} f(k) = \int_0^{\pi} \frac{dk}{2\pi} (f(k) + f(-k)) \quad (***)$$

Anomalous terms: $\sum_i c_{i,2}^\dagger c_{i+1,1}^\dagger + c_{i+1,1} c_{i,2}$

$$= \sum_x c_2^\dagger(x) c_1^\dagger(x+a) + c_1(x+a) c_2(x)$$

$$\overset{FT}{=} \sum_x \iint_{kk'} c_2^\dagger(k) e^{ikx} e^{+ik'(x+a)} c_1^\dagger(k') + e^{-ik(x+a)} c_1(k) e^{-ik'x} c_2(k')$$

$$= \sum_x \iint_{kk'} c_2^\dagger(k) c_1^\dagger(k') e^{i(k+k')x} e^{ik'a} + c_1(k) c_2(k') e^{-i(k+k')x} e^{-ika}$$

$$= \sum_x \iint_{kk'} \delta_{k,k'} \left[e^{-ik'a} c_2^\dagger(k) c_1^\dagger(k') + e^{-ika} c_1(k) c_2(k') \right]$$

$$\overset{-k'=k}{=} \int_k e^{-ika} c_2^\dagger(k) c_1^\dagger(-k) + e^{-ika} c_1(k) c_2(-k)$$

$$= \int_{-\pi}^{\pi} \frac{dk}{2\pi}\left[e^{-ika} c_2^\dagger(k) c_1^\dagger(-k) + e^{-ika} c_1(k) c_2(-k)\right]$$

$$\overset{(※※※)}{=} \int_{0}^{\pi} \frac{dk}{2\pi}\Big[e^{-ika} c_2^\dagger(k) c_1^\dagger(-k) + e^{-ika} \overbrace{c_1(k) c_2(-k)}^{-c_2(-k)c_1(k)} + e^{ika} \underbrace{c_2^\dagger(-k) c_1^\dagger(k)}_{-c_1^\dagger(k) c_2^\dagger(-k)} + e^{ika} c_1(-k) c_2(k)\Big]$$

$$= \int_{0}^{\pi} \frac{dk}{2\pi}\Big[e^{-ika} c_2^\dagger(k) c_1^\dagger(-k) - e^{-ika} c_2(-k) c_1(k) - e^{ika} c_1^\dagger(k) c_2^\dagger(-k) + e^{ika} c_1(-k) c_2(k)\Big] //$$

$$\hat{H}_{xy} = -\frac{J_F}{2} \int_k \left(c_1^\dagger(k), c_2^\dagger(k), c_1(-k), c_2(-k)\right) \underline{\underline{M}} \begin{pmatrix} c_1(k) \\ c_2(k) \\ c_1^\dagger(-k) \\ c_2^\dagger(-k) \end{pmatrix} \Rightarrow$$ to incorporate anomalous & hopping terms

$$\underline{\underline{M}} = \left(\begin{array}{cc|cc} & \square & & \times \\ \square & & \times & \\ \hline & \times & & \square \\ \times & & \square & \end{array}\right)$$

$$\hat{H}_{xy} = -\frac{J_F}{2} \int_k c_1^\dagger(k) c_2(k) + c_2^\dagger(k) c_1(k) + e^{-ika} \beta \left(c_2^\dagger(k) c_1^\dagger(-k) + c_1(k) c_2(-k)\right)$$

The Hamiltonian:

$$\hat{H} = -\frac{J_F}{2}\int_0^{\pi}\frac{dk}{2\pi}\Big[c_1^+(k)c_2(k) + c_2^+(k)c_1(k) - c_2(-k)c_1^+(-k) - c_1(-k)c_2^+(-k)$$

$$+ e^{-ika}\beta\, c_2^+(k)c_1^+(-k) - e^{-ika}\beta\, c_2(-k)c_1(k)$$

$$- e^{ika}\beta\, c_1^+(k)c_2^+(-k) + \beta e^{ika} c_1(-k)c_2(k)\Big]$$

By doubling dimensions:

$$\hat{H}_{xy} = -\frac{J_F}{2}\int_k \left(c_1^+(k), c_2^+(k), c_1(-k), c_2(-k)\right)\underline{\underline{M}}\begin{pmatrix} c_1(k) \\ c_2(k) \\ c_1^+(-k) \\ c_2^+(-k)\end{pmatrix}$$

$$\underline{\underline{M}} = \left(\begin{array}{cc|cc} 0 & 1 & 0 & -\beta e^{ika} \\ 1 & 0 & \beta e^{-ika} & 0 \\ \hline 0 & \beta e^{ika} & 0 & -1 \\ -\beta e^{ika} & 0 & -1 & 0 \end{array}\right) \checkmark$$

diagonalizing using Mathematica

$$\epsilon_1(k) = \pm\frac{J_F}{2}\sqrt{\beta e^{-ika} + (\beta^2+1)\beta e^{ika}} = \pm\frac{J_F}{2}\sqrt{\beta^2+1+2\beta\cos(ka)}$$

$$\epsilon_2(k) = \pm\frac{J_F}{2}\sqrt{(\beta^2+1)e^{2ika}e^{-2ika} - \beta e^{ika-2ika} + \beta e^{ika}}$$

$k \in [0,\pi]$

$\beta = J_A/J_F$

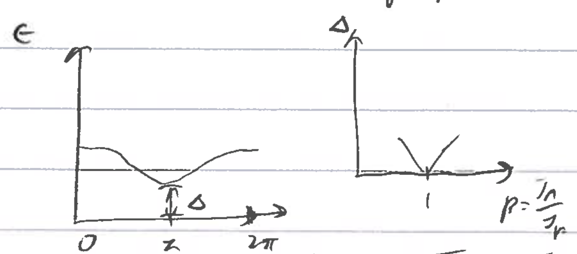
$$= \pm\frac{J_F}{2}\sqrt{\beta^2+1-2\beta\cos(ka)}$$

For $a=1$

$$\epsilon(k) = \frac{J_F}{2}\sqrt{\beta^2+1+2\beta\cos(k)}$$

Plot[$\epsilon(k)$, {k, 0, 2π}] 2π periodic

See Mathematica for plot



$\Delta = 0$ for $\beta = 1 \iff J_A = J_F$

$\Rightarrow$ topological phase transition

$$\Delta = \epsilon(\pi) = \frac{J_F}{2}\sqrt{\beta^2+1-2\beta} = \frac{J_F}{2}\sqrt{(\beta-1)^2}$$

$$= \frac{J_F}{2}|\beta - 1|$$

28

To do: do a easier 2×2 matrix

introducing new fermion operators: then do the Bogoliubov transformation

1. Check for:

$$d_1(k) = \frac{1}{\sqrt{2}}\left(c_1(k) + c_2(k)\right)$$

$$d_2(k) = \frac{1}{\sqrt{2}}\left(c_1(k) - c_2(k)\right) \qquad k \in [-\pi, \pi]$$

$$\hat{H} = \int_k \left(d_1^\dagger(k)\, d_1(-k)\right)\begin{pmatrix} 2\times 2 \end{pmatrix}\begin{pmatrix} d_1(k) \\ d_1^\dagger(-k)\end{pmatrix} + \int_k \left(d_2^\dagger(k)\, d_2(-k)\right)\begin{pmatrix} 2\times 2 \end{pmatrix}\begin{pmatrix} d_2(k) \\ d_2^\dagger(-k)\end{pmatrix}$$

$$\hat{H} = \int_k \left(d_1^\dagger(k)\, d_1(-k)\, d_2^\dagger(k)\, d_2(-k)\right) \underset{=}{\tilde{\Omega}} \begin{pmatrix} d_1(k) \\ d_1^\dagger(-k) \\ d_2(k) \\ d_2^\dagger(-k)\end{pmatrix}$$

$$\underset{=}{\tilde{\Omega}} = \left(\begin{array}{cc|cc} \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & 0 & 0 \\ \hline 0 & 0 & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot \end{array}\right)$$

$\to \pm\epsilon_1$, $\to \pm\epsilon_2$

2. new "arrangement"

$$\hat{H} = \int_k \left(c_1^\dagger(k), c_1(-k), c_2^\dagger(k), c_2(-k)\right) \underset{=}{\Omega} \begin{pmatrix} c_1(k) \\ c_1^\dagger(-k) \\ c_2(k) \\ c_2^\dagger(-k)\end{pmatrix}$$

$k \in [0, \pi]$

3. See Bogoliubov transformation

Mean field approx. MFA.

Let $\hat{A}, \hat{B}$ be operators

We have $\hat{A}\cdot\hat{B}$ in the Hamiltonian

for eg

$$-\cancel{\tfrac{J_z}{2}} - J_z\beta(1+\alpha)\sum_i \underbrace{\left(\tfrac{1}{2} - c_{i2}^{\dagger} c_{i2}\right)}_{\hat{A}}\underbrace{\left(\tfrac{1}{2} - c_{i+1,1}^{\dagger} c_{i+1,1}\right)}_{\hat{B}}$$

for

We write $\hat{A}\,\hat{B}$ as:

$$\hat{A} = \langle \hat{A} \rangle + \delta\hat{A} \qquad \langle \hat{O} \rangle \text{ are just numbers}$$
$$\hat{B} = \langle \hat{B} \rangle + \delta\hat{B}$$

$$\hat{A}\cdot\hat{B} = \langle\hat{A}\rangle\langle\hat{B}\rangle + \langle\hat{A}\rangle\delta\hat{B} + \langle\hat{B}\rangle\delta\hat{A} + \cancel{\delta\hat{A}\,\delta\hat{B}}$$ ← neglect for mean-field approx

$$= \langle\hat{A}\rangle\langle\hat{B}\rangle + \langle\hat{A}\rangle(\hat{B} - \langle\hat{B}\rangle) + \langle\hat{B}\rangle(\hat{A} - \langle\hat{A}\rangle)$$

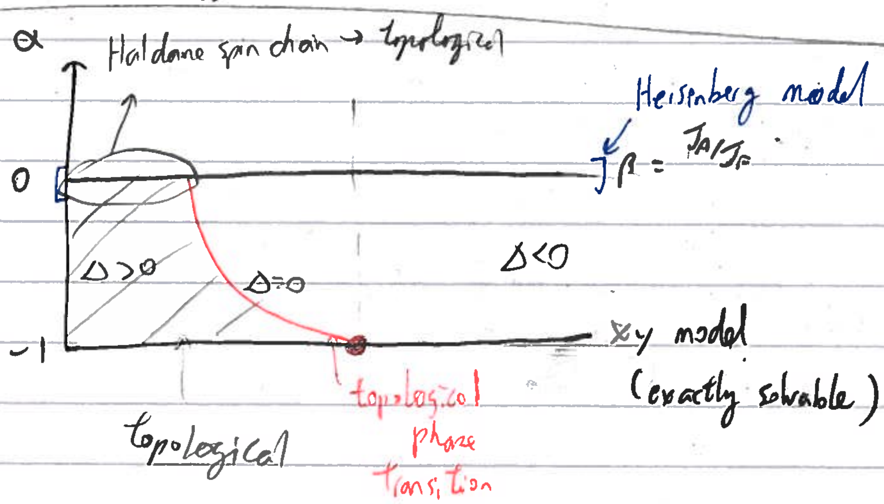
$$= \cancel{\langle\hat{A}\rangle\langle\hat{B}\rangle} + \langle\hat{A}\rangle\hat{B} - \cancel{\langle\hat{A}\rangle\langle\hat{B}\rangle} + \hat{A}\langle\hat{B}\rangle - \langle\hat{A}\rangle\langle\hat{B}\rangle$$

$$\hat{A}\cdot\hat{B} \approx \langle\hat{A}\rangle\hat{B} + \hat{A}\langle\hat{B}\rangle - \langle\hat{A}\rangle\langle\hat{B}\rangle$$

For the Hamiltonian:

do the same

Aim:



28

$$\hat{H} = -\frac{J_F}{2}\sum_i\Big[\overbrace{c_{i1}^\dagger c_{i2}}^{\hat{E}} + \overbrace{c_{i2}^\dagger c_{i1}}^{\hat{F}} + \beta(\overbrace{c_{i+1,1} c_{i2}}^{\hat{G}} + \overbrace{c_{i2}^\dagger c_{i+1,1}^\dagger}^{\hat{H}})\Big] \quad [\mathrm{I}]$$

$$- J_F(1+\alpha)\sum_i\left(\tfrac{1}{2} - c_{i1}^\dagger c_{i1}\right)\left(\tfrac{1}{2} - c_{i2}^\dagger c_{i2}\right) \quad \mathrm{II}$$

$$- J_F\,\beta(1+\alpha)\sum_i\left(\tfrac{1}{2} - c_{i2}^\dagger c_{i2}\right)\left(\tfrac{1}{2} - c_{i+1,1}^\dagger c_{i+1,1}\right)\Big] \quad \mathrm{III}$$

Not quite right

Rewriting for mean-field approx to make interacting terms vanish for to make it exactly solvable & make it into non-interacting

$$\overbrace{c_{i1}^\dagger c_{i1}}^{\hat{D}}\,\overbrace{c_{i2}^\dagger c_{i2}}^{\hat{A}} \qquad \hat{E},\ \hat{F} \qquad c_{i1}^\dagger c_{i2}$$

$$\overbrace{c_{i2}^\dagger c_{i2}}^{\hat{A}}\,\overbrace{c_{i+1,1}^\dagger c_{i+1,1}}^{\hat{B}} \qquad \hat{H},\ \hat{G}$$

$\hat{A} = c_{i2}^\dagger c_{i2}$;	$\hat{D} = c_{i1}^\dagger c_{i1}$
$\hat{B} = c_{i+1,1}^\dagger c_{i+1,1}$;	$\hat{E} = c_{i1}^\dagger c_{i2}$
$\hat{F} = c_{i2}^\dagger c_{i1}$;	$\hat{G} = c_{i+1,1}\, c_{i2}$
$H = c_{i2}^\dagger c_{i+1,1}^\dagger$	

$$\mathrm{III}\ \tfrac{1}{N}\sum_i\ \frac{1}{4} - \frac{1}{2}c_{i+1,1}^\dagger c_{i+1,1} - \frac{1}{2}c_{i2}^\dagger c_{i2} + c_{i2}^\dagger c_{i2} c_{i+1,1}^\dagger c_{i+1,1}$$

$$\overset{\text{MFA}}{\approx}\ \frac{1}{4} - \frac{1}{2}\hat{B} + \frac{1}{2}\hat{A} + \left(\hat{A}\langle\hat{B}\rangle + \langle\hat{A}\rangle\hat{B} - \langle\hat{A}\rangle\langle\hat{B}\rangle\right)$$

similarly

$$\mathrm{II}:\quad \frac{1}{4} - \frac{1}{2}c_{i2}^\dagger c_{i2} - \frac{1}{2}c_{i1}^\dagger c_{i1} + c_{i1}^\dagger c_{i1} c_{i2}^\dagger c_{i2}$$

$$\overset{\text{MFA}}{\approx}\ \frac{1}{4} - \frac{1}{2}\hat{D} - \frac{1}{2}\hat{A} + \left(\hat{D}\langle\hat{A}\rangle + \hat{A}\langle\hat{D}\rangle - \langle\hat{D}\rangle\langle\hat{A}\rangle\right)$$

$$\hat{H} = -\frac{J_F}{2}\sum_i\Big[\hat{E} + \hat{F} + \beta\hat{G} + \beta\hat{H} + 2(1+\alpha)\left(\frac{1}{4} - \frac{1}{2}(\hat{D}+\hat{A}) + \hat{D}\langle\hat{A}\rangle + \hat{A}\langle\hat{D}\rangle - \langle\hat{D}\rangle\langle\hat{A}\rangle\right)$$
$$+ 2\beta(1+\alpha)\left(\frac{1}{4} - \frac{1}{2}(\hat{B}+\hat{A}) + \hat{A}\langle\hat{B}\rangle + \langle\hat{A}\rangle\hat{B} - \langle\hat{A}\rangle\langle\hat{B}\rangle\right)\Big]$$

$$= -\frac{J_F}{2}\sum_i\Big[\hat{E} + \hat{F} + \beta\hat{G} + \beta\hat{H} + (2+2\alpha)\Big(\frac{1}{4} - \frac{1}{2}(\hat{A}+\hat{D}) + \hat{D}\langle\hat{A}\rangle + \hat{A}\langle\hat{D}\rangle - \langle\hat{D}\rangle\langle\hat{A}\rangle$$
$$+ \frac{\beta}{4} - \frac{\beta}{2}(\hat{B}+\hat{A}) + \beta\hat{A}\langle\hat{B}\rangle + \beta\langle\hat{A}\rangle\hat{B} - \beta\langle\hat{A}\rangle\langle\hat{B}\rangle\Big)$$

MFA continued

~~A~~:

$$\hat{H} = -\frac{J_F}{2}\sum_i \Big[\hat{E} + \hat{F} + \beta\hat{G} + \beta\hat{H} + \frac{1}{2} - \hat{A} - \hat{D} + 2\hat{D}\langle\hat{A}\rangle + 2\hat{A}\langle\hat{D}\rangle - 2\langle\hat{D}\rangle\langle\hat{A}\rangle$$

$$+ \frac{\beta}{2} - \beta(\hat{B}+\hat{A}) + 2\beta\hat{A}\langle\hat{B}\rangle + 2\beta\langle\hat{A}\rangle\hat{B} - 2\beta\langle\hat{A}\rangle\langle\hat{B}\rangle$$

$$+ \frac{\alpha}{2} - \alpha\hat{A} - \alpha\hat{D} + 2\alpha\hat{D}\langle\hat{A}\rangle + 2\alpha\hat{A}\langle\hat{D}\rangle - 2\alpha\langle\hat{D}\rangle\langle\hat{A}\rangle$$

$$+ \frac{\alpha\beta}{2} - \alpha\beta\hat{B} - \alpha\beta\hat{A} + 2\alpha\beta\hat{A}\langle\hat{B}\rangle + 2\alpha\beta\langle\hat{A}\rangle\hat{B} - 2\alpha\beta\langle\hat{A}\rangle\langle\hat{B}\rangle$$

$$= -\frac{J_F}{2}\sum_i \Big[\hat{E} + \hat{F} + \frac{1}{2} - \hat{A} - \hat{D} + 2\hat{D}\langle\hat{A}\rangle + 2\hat{A}\langle\hat{D}\rangle - 2\langle\hat{D}\rangle\langle\hat{A}\rangle$$

$$+\beta(\hat{G}+\hat{H}) + \beta\Big(\frac{1}{2} - \hat{B} - \hat{A} + 2\hat{A}\langle\hat{B}\rangle + 2\langle\hat{A}\rangle\hat{B} - 2\langle\hat{A}\rangle\langle\hat{B}\rangle\Big)$$

$$+\alpha\Big(\frac{1}{2} - \hat{A} - \hat{D} + 2\hat{D}\langle\hat{A}\rangle + 2\langle\hat{A}\rangle\langle\hat{D}\rangle - 2\langle\hat{D}\rangle\langle\hat{A}\rangle$$

$$+\alpha\beta\Big(\frac{1}{2} - \hat{B} - \hat{A} + 2\hat{A}\langle\hat{B}\rangle + 2\langle\hat{A}\rangle\hat{B} - 2\langle\hat{A}\rangle\langle\hat{B}\rangle\Big]$$

$$= -\frac{J_F}{2}\sum_i \Big[\hat{E} + \hat{F} + (1+\alpha)\Big(\frac{1}{2} - \hat{A} - \hat{D} + 2\hat{D}\langle\hat{A}\rangle + 2\hat{A}\langle\hat{D}\rangle - 2\langle\hat{D}\rangle\langle\hat{A}\rangle\Big)$$

$$\beta(\hat{G}+\hat{H}) + \beta\Big(\frac{1}{2} - \hat{B} - \hat{A} + 2\hat{A}\langle\hat{B}\rangle + 2\langle\hat{A}\rangle\hat{B} - 2\langle\hat{A}\rangle\langle\hat{B}\rangle\Big)(1+\alpha)\Big]$$

$$\hat{H} = -\frac{J_F}{2}\sum_i \Big[\hat{E} + \hat{F} + \beta(\hat{G}+\hat{H}) + (1+\alpha)\Big(\frac{1}{2} - \hat{A} - \hat{D} + 2\hat{D}\langle\hat{A}\rangle + 2\hat{A}\langle\hat{D}\rangle - 2\langle\hat{D}\rangle\langle\hat{A}\rangle$$

$$+\beta\Big(\frac{1}{2} - \hat{B} - \hat{A} + 2\hat{A}\langle\hat{B}\rangle + 2\langle\hat{A}\rangle\hat{B} - 2\langle\hat{A}\rangle\langle\hat{B}\rangle\Big)\Big]$$

OR

$\cancel{\hat{H} = -\frac{J_F}{2}\sum_i [\hat{E} + \hat{F}}$

$$\frac{\partial H}{\partial \alpha} = 0 \quad ??$$

for minimizing ?

$$\frac{\partial H}{\partial \beta} = 0 \quad ??$$

50

$\cancel{\hat H = -\frac{J}{2}\sum_i}$

$$\frac{\partial \hat H}{\partial \alpha} = -\frac{J}{2}\sum_i \Big(\frac12 - \hat A - \hat D + 2\hat D\langle \hat A\rangle + 2\hat A\langle \hat D\rangle - 2\langle \hat D\rangle\langle \hat A\rangle$$
$$+\beta\big(\frac12 - \hat B - \hat A + 2\hat A\langle\hat B\rangle + 2\langle \hat A\rangle \hat B - 2\langle\hat A\rangle\langle\hat B\rangle\big)\Big)$$

$$\frac{\partial \hat H}{\partial\beta} = -\frac{J}{2}\sum_i \hat G + \hat H + (1+\alpha)\big(\frac12 - \hat B - \hat A + 2\hat A\langle\hat B\rangle + 2\langle\hat A\rangle\hat B - 2\langle\hat A\rangle\langle\hat B\rangle\big)$$

$$\frac{\partial \hat H}{\partial\alpha} = \frac{\partial \hat H}{\partial\beta} = 0$$

$$\Rightarrow \frac12 - \hat A - \hat D + 2\hat D\langle\hat A\rangle + 2\hat A\langle\hat D\rangle - 2\langle\hat D\rangle\langle\hat A\rangle$$
$$+\beta\big(\frac12 - \hat B - \hat A - 2\hat A\langle\hat B\rangle + 2\langle\hat A\rangle\hat B - 2\langle\hat A\rangle\langle\hat B\rangle\big)$$

$$\cancel{\hat G + \hat H} + (1+\alpha)\big(\frac12 - \hat B - \hat A + 2\hat A\langle\hat B\rangle + 2\langle\hat A\rangle\hat B - 2\langle\hat A\rangle\langle\hat B\rangle\big)$$

$$= \hat G + \hat H$$

$\in A$

$$\Rightarrow (\beta - 1 - \alpha)\big(\frac12 - \hat B - \hat A + 2\hat A\langle\hat B\rangle + 2\langle\hat A\rangle\hat B - 2\langle\hat A\rangle\langle\hat B\rangle\big)$$
$$= \hat G + \hat H - \frac12 + \hat A + \hat D - 2\hat D\langle\hat A\rangle + 2\hat A\langle\hat D\rangle - 2\langle\hat D\rangle\langle\hat A\rangle$$

$\langle\hat A\rangle := a \in \mathbb{R}$
$\langle\hat D\rangle := d \in \mathbb{R}$
$\langle\hat B\rangle := b \in \mathbb{R}$ $\Rightarrow$

$$(\beta-1-\alpha)\big(\frac12 - \hat B - \hat A + 2b\hat A + 2a\hat B - 2ab\big) = \hat G + \hat H - \frac12 + \hat A + \hat D - 2a\hat D + 2d\hat A - 2ad$$

$$\Leftrightarrow (\beta-\alpha)\big(\frac12 - \hat B - \hat A + 2b\hat A + 2a\hat B - 2ab\big) - \cancel{\frac12} + \hat B + \cancel{\hat A} - 2b\hat A - 2a\hat B + 2ab$$

$$= \hat G + \hat H - \cancel{\frac12} + \cancel{\hat A} + \hat D - 2a\hat D + 2d\hat A - 2ad$$

~~29~~ 30

New arrangements

Introducing new fermion operators:

$$d_1(k) = \frac{1}{\sqrt{2}}\left(c_1(k) + c_2(k)\right) \qquad d_1^+(k) = \frac{1}{\sqrt{2}}\left(c_1^+(k) + c_2^+(k)\right)$$

$$d_2(k) = \frac{1}{\sqrt{2}}\left(c_1(k) - c_2(k)\right) \qquad d_2^+(k) = \frac{1}{\sqrt{2}}\left(c_1^+(k) - c_2^+(k)\right)$$

Checking the anti-commutator relationships

✓ $$\{d_1(k), d_1^+ k\} = d_1(k) d_1^+(k) + d_1^+(k) d_1(k)$$

$$= \frac{1}{2}\left[\left(c_1(k) + c_2(k)\right)\left(c_1^+(k) + c_2^+(k)\right) + \left(c_1^+(k) + c_2^+(k)\right)\left(c_1(k) + c_2(k)\right)\right]$$

$$= \frac{1}{2}\Big[\overbrace{\{c_1(k), c_1^+(k)\}}^{1} + \overbrace{\{c_1(k), c_2^+(k)\}}^{0} + \underbrace{\{c_2(k), c_1^+(k)\}}_{0} + \overbrace{\{c_2(k), c_2^+(k)\}}^{1} \Big] = 1$$

✓ $$\{d_2(k), d_2^+(k)\} = \frac{1}{2}\left[\left(c_1(k) - c_2(k)\right)\left(c_1^+(k) - c_2^+(k)\right) + \left(c_1^+(k) - c_2^+(k)\right)\left(c_1(k) - c_2(k)\right)\right]$$

$$= \frac{1}{2}\Big[\{c_1(k), c_1^+(k)\} - \{c_1(k), c_2^+(k)\} - \{c_2(k), c_1^+(k)\} + \{c_2(k), c_2^+(k)\} \Big] = 1$$

✓ $$\{d_1(k), d_2^+(k)\} = \frac{1}{2}\left[\left(c_1(k) + c_2(k)\right)\left(c_1^+(k) - c_2^+(k)\right) + \left(c_1^+(k) - c_2^+(k)\right)\left(c_1(k) + c_2(k)\right)\right]$$

$$= \frac{1}{2}\Big[\overbrace{\{c_1(k), c_1^+(k)\}}^{1} + \overbrace{\{c_2(k), c_1^+(k)\}}^{0} - \underbrace{\left(c_1(k) c_2^+(k) + c_2^+(k) c_1(k)\right)}_{0} - \underbrace{\{c_2(k), c_2^+(k)\}}_{1} \Big] = 0$$

✓ $$\{d_1^+(k), d_2(k)\} = \frac{1}{2}\left[\left(c_1^+(k) + c_2^+(k)\right)\left(c_1(k) - c_2(k)\right) + \left(c_1(k) - c_2(k)\right)\left(c_1^+(k) + c_2^+(k)\right)\right]$$

$$= \frac{1}{2}\Big[\{c_1^+(k), c_1(k)\} - \{c_1^+(k), c_2(k)\} + \{c_2^+(k), c_1(k)\} - \{c_2(k), c_2^+(k)\} \Big]$$

$$= 0$$

~~therefore d_1~~ d_1, d_1^+, d_2, d_2^+ all satisfy the anticommutator relationship

Checking :

$$\hat{H} = \int_k \overbrace{\left(d_1^+(k)\ d_1(-k) \right) \underline{\underline{M}}_1 \begin{pmatrix} d_1(k) \\ d_1^+(-k) \end{pmatrix}}^{H_1}$$

$$+ \int_k \underbrace{\left(d_2^+(k)\ d_2(-k) \right) \underline{\underline{M}}_2 \begin{pmatrix} d_2(k) \\ d_2^+(-k) \end{pmatrix}}_{H_2}$$

1.

$$H_1 = \left(d_1^+(k)\ d_1(-k) \right) \underline{\underline{M}}_1 \begin{pmatrix} d_1(k) \\ d_1^+(-k) \end{pmatrix}$$

$$\left(d_1^+(k)\ d_1(-k) \right) \underline{\underline{M}}_1 = \frac{1}{\sqrt{2}} \left(\left(c_1^+(k) + c_2^+(k) \right), \left(c_1(-k) + c_2(-k) \right) \right) \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} m_1 (c_1^+(k) + c_2^+(k)) + m_3 (c_1(-k) + c_2(-k)) \\ m_2 (c_1^+(k) + c_2^+(k)) + m_? (c_1(-k) + c_2(-k)) \end{bmatrix}^T$$

OR

$$H_1 = \left(m_1 d_1^+(k) + m_3 d_1(-k),\ m_2 d_1^+(k) + m_4 d_1(-k) \right) \begin{pmatrix} d_1(k) \\ d_1^+(-k) \end{pmatrix}$$

$$= m_1 d_1^+(k) d_1(k) + m_3 d_1(-k) d_1(k) + m_2 d_1^+(k) d_1^+(-k) + m_4 d_1(-k) d_1^+(-k)$$

similarly

$$H_2 = p_1 d_2^+(k) d_2(k) + p_3 d_2(-k) d_2(k) + p_2 d_2^+(k) d_2^+(-k) + p_4 d_2(-k) d_2^+(-k)$$

"doesn't really work"

$$\hat{H} = \int_k \left(d_1^\dagger(k), d_1(-k), d_2^\dagger(k), d_2(-k) \right) \tilde{\Omega} \begin{pmatrix} d_1(k) \\ d_1^\dagger(-k) \\ d_2(k) \\ d_1^\dagger(-k) \end{pmatrix}$$

$$d_1^\dagger(k) d_1(k) = \frac{1}{2}\left[\overset{(0)}{c_1^\dagger(k) c_1(k)} + \overset{(1)}{\underline{c_2^\dagger(k) c_1(k)}} + \overset{(1)}{\underline{c_1^\dagger(k) c_2(k)}} + \overset{(2)}{c_2^\dagger(k) c_2(k)} \right]$$

$$d_2^\dagger(k) d_2(k) = \frac{1}{2}\left[c_1^\dagger(k) c_1(k) - \underline{c_2^\dagger(k) c_1(k)} - \underline{c_1^\dagger(k) c_2(k)} + c_2^\dagger(k) c_2(k) \right)$$

$$d_1(-k) d_1^\dagger(-k) = \frac{1}{2}\left[c_1(-k) c_1^\dagger(-k) + \underline{c_1(-k) c_2^\dagger(-k)} + \underline{c_2(-k) c_1^\dagger(-k)} + c_2(-k) c_2^\dagger(-k) \right]$$

$$d_2(-k) d_2^\dagger(-k) = \frac{1}{2}\left[c_1(-k) c_1^\dagger(-k) - \underline{c_1(-k) c_2^\dagger(-k)} - \underline{c_2(-k) c_1^\dagger(-k)} + c_2(-k) c_2^\dagger(-k) \right]$$

$$\left(d_1^\dagger(k), d_1(-k), d_2^\dagger(k), d_2(-k) \right) \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ m_1 & m_2 & m_3 & m_4 \\ n_1 & n_2 & n_3 & n_4 \end{pmatrix} \begin{pmatrix} d_1(k) \\ d_1^\dagger(-k) \\ d_2(k) \\ d_2^\dagger(-k) \end{pmatrix}$$

$$\begin{pmatrix} d_1^\dagger(k) a_1 + d_1(-k) b_1 + d_2^\dagger(k) m_1 + d_2(-k) n_1 \\ d_1^\dagger(k) a_2 + d_1(-k) b_2 + d_2^\dagger(k) m_2 + d_2(-k) n_2 \\ d_1^\dagger(k) a_3 + d_1(-k) b_3 + d_2^\dagger(k) m_3 + d_2(-k) n_3 \\ d_1^\dagger(k) a_4 + d_1(-k) b_4 + d_2^\dagger(k) m_4 + d_2(-k) n_4 \end{pmatrix}^T \begin{pmatrix} d_1(k) \\ d_1^\dagger(-k) \\ d_2(k) \\ d_1^\dagger(-k) \end{pmatrix}$$

$$\begin{aligned} = & \ a_1 d_1^\dagger(k) d_1(k) + b_1 d_1(-k) d_1(k) + m_1 d_2^\dagger(k) d_1(k) + n_1 d_2(-k) d_1(k) && \text{JA} \\ & + a_2 d_1^\dagger(k) d_1^\dagger(-k) + b_2 d_1(-k) d_1^\dagger(-k) + m_2 d_2^\dagger(k) d_1^\dagger(-k) + n_2 d_2(-k) d_1^\dagger(-k) && \text{JB} \\ & + a_3 d_1^\dagger(k) d_2(k) + b_3 d_1(-k) d_2(k) + m_3 d_2^\dagger(k) d_2(k) + m_4 d_2(-k) d_2(k) && \text{JC} \\ & + a_4 d_1^\dagger(k) d_2^\dagger(-k) + b_4 d_1(-k) d_2^\dagger(-k) + m_4 d_2^\dagger(k) d_2^\dagger(-k) + m_4 d_2(-k) d_2^\dagger(-k) && \text{JD} \end{aligned}$$

$$\left(c_1^\dagger(k)\; c_1(-k)\; c_2^\dagger(k)\; c_2(-k)\right)^{T} \underline{\Omega} \begin{pmatrix} c_1(k) \\ c_1^\dagger(-k) \\ c_2(-k) \\ c_2^\dagger(-k) \end{pmatrix}$$

$$\downarrow$$

$$\begin{pmatrix} 0 & 0 & \gamma & -\mu \\ 0 & 0 & \mu & -\gamma \\ \gamma & \mu^* & 0 & 0 \\ -\mu^* & -\gamma & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \gamma c_2^\dagger(k) - \mu^* c_2(-k) \\ \mu^* c_2^\dagger(k) - \gamma c_2(-k) \\ \gamma c_1^\dagger(k) + \mu c_1(-k) \\ -\mu c_1^\dagger(k) - \gamma c_1(-k) \end{pmatrix}^{T} \begin{pmatrix} c_1(k) \\ c_1^\dagger(-k) \\ c_2(k) \\ c_2^\dagger(-k) \end{pmatrix}$$

$$\mu = \tilde{\mu} e^{ik a}$$

$$= \gamma c_2^\dagger(k) c_1(k) - \mu^* c_2(-k) c_1(k)$$

$$+ \mu^* c_2^\dagger(k) c_1^\dagger(-k) - \gamma c_2(-k) c_1^\dagger(-k)$$

$$+ \gamma c_1^\dagger(k) c_2(k) + \mu c_1(-k) c_2(k)$$

$$- \mu c_1^\dagger(k) c_2^\dagger(-k) - \cancel{\gamma c_1(-k)}\, \gamma c_1(-k)\, c_2^\dagger(-k)$$

$$= \gamma\left(c_2^\dagger(k) c_1(k) - c_2(-k) c_1^\dagger(-k) + c_1^\dagger(k) c_2(k) - c_1(-k) c_2^\dagger(-k)\right)$$

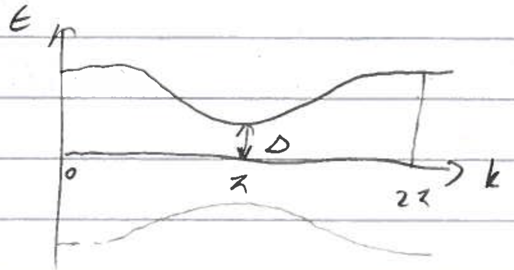
$$+ \mu^*\left(c_2^\dagger(k) c_1^\dagger(-k) - c_2(-k) c_1(k)\right)$$

$$+ \mu\left(c_1(-k) c_2(k) - c_1^\dagger(k) c_2^\dagger(-k)\right)$$

Can use the previous matrix, but switch the ones to γ, $\beta \to \tilde{\beta}$

$$\underline{\underline{M}}(k) = \left(\begin{array}{cc|cc} 0 & 0 & \gamma & -\mu \\ 0 & 0 & \mu & -\gamma \\ \hline \gamma & \mu^* & 0 & 0 \\ -\mu^* & -\gamma & 0 & 0 \end{array}\right) \qquad \mu = \tilde{\beta} e^{ika}$$

$$\epsilon(k) = \frac{J_F}{2}\sqrt{\tilde{\beta}^2 + \gamma^2 + 2\tilde{\beta}\gamma\cos(k)} \qquad k \in [0, 2\pi]$$



The energy density (energy per unit volume per Ferromagnetic coupling) [dimensionless]

$$\epsilon := \frac{E}{N J_F} = -\frac{1}{2}\int_0^{2\pi}\frac{dk}{2\pi}\sqrt{\tilde{\beta}^2 + \gamma^2 + 2\tilde{\beta}\gamma\cos(k)}$$

$$+ (1+\alpha)\,(\beta\lambda^2 - t^2)$$

$$\boxed{\begin{aligned}\epsilon = -\frac{1}{4\pi}\int_0^{2\pi} dk \Big\{ & \beta^2[1+2\lambda(1+\alpha)]^2 + [1-2t(1+\alpha)]^2 \\ & + 2\beta[1+2\lambda(1+\alpha)][1-2t(1+\alpha)]\cos(k)\Big\}^{\frac{1}{2}} \\ & + (1+\alpha)(\beta\lambda^2 - t^2)\end{aligned}}$$

* α & β are given parameters

$\left.\begin{array}{l}\alpha \in [-1, 0] \\ \beta \in [0, 1]\end{array}\right\}$ need to be fixed first, then vary

* λ, t need to be determined self-consistently:

$\epsilon = \epsilon(\lambda, t)$

I. $\frac{\partial \epsilon}{\partial \lambda} = 0$

II. $\frac{\partial \epsilon}{\partial t} = 0$

$\Rightarrow \underset{=0}{\frac{\partial \epsilon}{\partial \lambda}} - \underset{=0}{\frac{\partial \epsilon}{\partial t}} = 0$

* take random starting value (λ_0, t_0)

$(\lambda_1, t_1) = (\lambda_0, t_0) - \delta\, \nabla\epsilon|_{\lambda_0, t_0}$ ($\delta = 10^{-3}$)

$\epsilon_1 < \epsilon_0$

$\Rightarrow (\lambda_{n+1}, t_{n+1}) = (\lambda_n, t_n) - \delta\, \nabla\epsilon|_{\lambda_n, t_n}$

$\epsilon_{n+1} < \epsilon_n$

where $\nabla\epsilon|_{\lambda_0, t_0} = \begin{pmatrix} \frac{\partial \epsilon}{\partial \lambda}\big|_{\lambda_0, t_0} \\ \frac{\partial \epsilon}{\partial t}\big|_{\lambda_0, t_0}\end{pmatrix}$

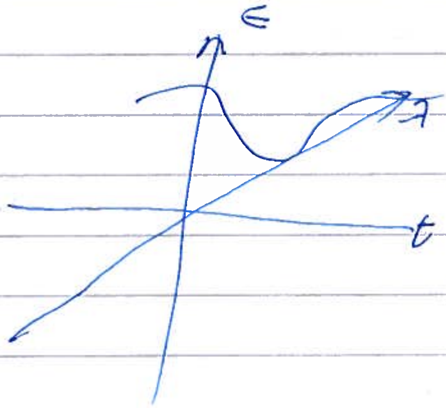
$\hookrightarrow \frac{\partial \epsilon}{\partial \lambda}\Big|_{\lambda_0, t_0} = \lim_{a \to 0}\frac{\epsilon(\lambda_0 + a, t_0) - \epsilon(\lambda_0, t_0)}{a} \approx \frac{\epsilon(\lambda_0 + a, t_0) - \epsilon(\lambda_0, t_0)}{a}$

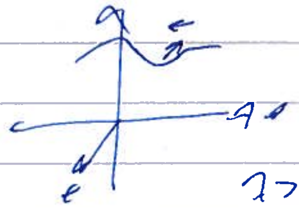
for a sufficiently small a

$\hookrightarrow$ need numerical integration as well. (elliptical integral)

~~when the int~~

Wig's theorem





$\lambda > 0$

shift left

else shift right

$\frac{\sqrt{3}}{2}\sqrt{\tilde{\beta}^2+\gamma^2+2\gamma\tilde{\beta}\cos(k)}$

$= \frac{\sqrt{3}}{2}\sqrt{\tilde{\beta}^2+\gamma(1+2\tilde{\beta}\cos(k))}$

$\cancel{\cos^2(k)=1}$

$\cancel{\cos(2k)=1 \Rightarrow 2\cos^2(k)-1}$

non-convergence => stepsize

saddle point

$\frac{\partial E}{\partial k}$

calculate the gap

same line where the gap is zero

use new E which has an elliptic function

$a=-0.1$; $b=0.9$

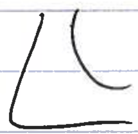
$E(a,b)$

phase boundary by bisection

$t_c = -0.13394$ //

$\lambda_c = 0.466761$ //

density 1/6



work out where the slope is

bisection

to 2 m.d.p.

Notes for meeting

$$\hat{H}_{mf} = -\frac{J_P}{2}\sum_i \{ \overbrace{(1-2t(1+\alpha))}^{\gamma}(c_{i2}^\dagger c_{i1} + c_{i1}^\dagger c_{i2}) + \overbrace{\beta(1+2\lambda(1+\alpha))}(c_{i+1,1}^\dagger c_{i2} + c_{i2}^\dagger c_{i+1,1}) \} + N J_P (1+\alpha)(\beta\lambda^2 - t^2)$$

$\gamma := (1-2t(1+\alpha))$

$\tilde{\beta} := \beta(1+2\lambda(1+\alpha))$

hard dono stay order parameter

$$\hat{H}_{mf} = -\frac{J_P}{2}\sum_i \gamma(c_{i2}^\dagger c_{i1} + c_{i1}^\dagger c_{i2}) + \tilde{\beta}\, c_{i1,+}$$

$\Delta = \Delta(\lambda, t)$
check gap.

determine λ & t range myself
λ & $t < 1$ maybe

integral numerically

try different $\lambda_0 t_0$
to see if converge to local / global minimum

steepest gradient descent

elliptic integral

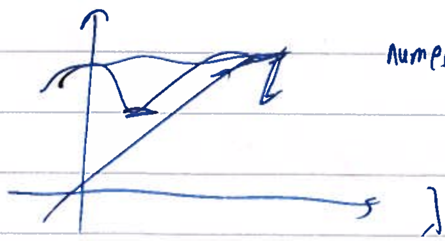
see optimized algorithms for numeric integration

I. fix $\alpha, \beta \overset{fix}{\Rightarrow}$ then loop
$\alpha \in [-1, 0]$
$\beta \in [0, 1)$

OR $\dfrac{\epsilon(\lambda_0 + a, t_0) - \epsilon(\lambda_0, t_0)}{a}$ for sufficiently small a.

II, correct λ & t
$\Rightarrow \frac{\partial \epsilon}{\partial \lambda} = \frac{\partial \epsilon}{\partial t} = 0$ $\quad \approx 10^{-2} / 10^{-3}$ $\quad \epsilon$

III take some starting value (λ_0, t_0)



numerical 2 comp vector

IV numerically integrate the integral of each

converge to local / global minimum
hopefully

there will be recursion

32

$$\hat{H} = -\frac{J_F}{2}\sum_i \left[c_{i1}^\dagger c_{i2} + c_{i2}^\dagger c_{i1} + \beta\left(c_{i+1,1}^\dagger c_{i2} + c_{i2}^\dagger c_{i+1,1}\right)\right]$$
$$- J_F (1+\alpha) \sum_i \left(\tfrac{1}{2} - c_{i1}^\dagger c_{i1}\right)\left(\tfrac{1}{2} - c_{i2}^\dagger c_{i2}\right) \quad \mathrm{I}$$
$$- J_F\, \beta (1+\alpha) \sum_i \left(\tfrac{1}{2} - c_{i2}^\dagger c_{i2}\right)\left(\tfrac{1}{2} - c_{i+1,1}^\dagger c_{i+1,1}\right) \quad \mathrm{II}$$

using all possible decoupling

$$c_{i1}^\dagger c_{i1} c_{i2}^\dagger c_{i2} = - c_{i1}^\dagger c_{i2} c_{i2}^\dagger c_{i1} \overset{MFA}{\approx} -\langle c_{i1}^\dagger c_{i2}\rangle c_{i2}^\dagger c_{i1} - c_{i1}^\dagger c_{i2}\langle c_{i2}^\dagger c_{i1}\rangle + \langle c_{i1}^\dagger c_{i2}\rangle\langle c_{i2}^\dagger c_{i1}\rangle$$
$$= -t\left(c_{i2}^\dagger c_{i1} + c_{i1}^\dagger c_{i2}\right) + t^2$$

$t = \langle c_{i1}^\dagger c_{i2}\rangle = \langle c_{i2}^\dagger c_{i1}\rangle \in \mathbb{R}$

constraint of ϱ:

$$c_{i1}^\dagger c_{i1} c_{i2}^\dagger c_{i2} \simeq \varrho\left(c_{i1}^\dagger c_{i1} + c_{i2}^\dagger c_{i2}\right) - \varrho^2$$
$$- t\left(c_{i2}^\dagger c_{i1} + c_{i1}^\dagger c_{i2}\right) + t^2$$

$\Rightarrow \varrho = \langle c_{i1}^\dagger c_{i1}\rangle = \langle c_{i2}^\dagger c_{i2}\rangle = \frac{1}{2} \in \mathbb{R}$

$\hat{T}_{im}^z = \frac{1}{2} - c_{im}^\dagger c_{im} \quad m = 1,2$

$\alpha < 0$: XY anisotropy $\Rightarrow$ moments develop on XY plane

$\langle T_{in}^z\rangle = 0 \Leftrightarrow \langle c_{im}^\dagger c_{im}\rangle = \frac{1}{2} = \varrho$

$$c_{i2}^\dagger c_{i2} c_{i+1,1}^\dagger c_{i+1,1} \simeq \varrho\left(c_{i2}^\dagger c_{i2} + c_{i+1,1}^\dagger c_{i+1,1}\right) - \varrho^2$$
$$+ \lambda\left(c_{i2}^\dagger c_{i+1,1}^\dagger + c_{i+1,1} c_{i2}\right) - \lambda^2$$

$\Rightarrow$ ~~$\lambda = \langle c_{i+1,1} c_{i2}\rangle = \langle c_{i2}^\dagger c_{i+1,1}^\dagger\rangle \in \mathbb{R}$~~

$$\mathrm{I.}\ \sum_i \left(\tfrac{1}{2} - c_{i1}^\dagger c_{i1}\right)\left(\tfrac{1}{2} - c_{i2}^\dagger c_{i2}\right) = \cancel{\tfrac{N}{4}} - \cancel{\tfrac{1}{2}\sum_i\left(c_{i1}^\dagger c_{i1} + c_{i2}^\dagger c_{i2}\right)} - t\left(c_{i2}^\dagger c_{i1} + c_{i1}^\dagger c_{i2}\right) + Nt^2$$
$$+ \cancel{\tfrac{1}{2}\sum_i\left(c_{i1}^\dagger c_{i1} + c_{i2}^\dagger c_{i2}\right)} - \cancel{\tfrac{N}{4}}$$
$$= Nt^2 - t\sum_i\left(c_{i2}^\dagger c_{i1} + c_{i1}^\dagger c_{i2}\right)$$

$$\mathrm{II.}\ \sum_i \left(\tfrac{1}{2} - c_{i2}^\dagger c_{i2}\right)\left(\tfrac{1}{2} - c_{i+1,1}^\dagger c_{i+1,1}\right) = \cancel{\tfrac{N}{4}} - \cancel{\tfrac{1}{2}\sum_i\left(c_{i2}^\dagger c_{i2} + c_{i+1,1}^\dagger c_{i+1,1}\right)} + \cancel{\tfrac{1}{2}\sum_i\left(c_{i2}^\dagger c_{i2} + c_{i+1,1}^\dagger c_{i+1,1}\right)} - \cancel{\tfrac{N}{4}}$$
$$+\lambda \sum_i\left(c_{i2}^\dagger c_{i+1,1}^\dagger + c_{i+1,1} c_{i2}\right) - N\lambda$$

$\lambda = \langle c_{i+1,1} c_{i2}\rangle = \langle c_{i2}^\dagger c_{i+1,1}^\dagger\rangle \in \mathbb{R}$

$$= \lambda \sum_i\left(c_{i2}^\dagger c_{i+1}^\dagger + c_{i+1,1} c_{i2}\right) - N\lambda^2$$

$$\hat{H}_{MF} = -\frac{J_F}{2}\sum_i\left\{\left[1 - 2t(1+\alpha)\right]\left(c_{i1}^\dagger c_{i2} + c_{i2}^\dagger c_{i1}\right)\right.$$
$$\left. + \beta\left[1 + 2\lambda(1+\alpha)\right]\left[c_{i+1,1} c_{i2} + c_{i2}^\dagger c_{i+1,1}^\dagger\right]\right\}$$
$$+ N J_F (1+\alpha)\left(\beta\lambda^2 - t^2\right)$$

Setting $\gamma := 1 - 2t(1+\alpha)$

$\tilde{\beta} := \beta\left[1 + 2\lambda(1+\alpha)\right]$

$$\hat{H}_{MF} = -\frac{J_F}{2}\int_k \left(c_1^\dagger(k)\ c_1(-k)\ c_2^\dagger(k)\ c_2(-k)\right) \underline{\underline{M}}(k) \begin{pmatrix} c_1(k) \\ c_2^\dagger(-k) \\ c_2(k) \\ c_2^\dagger(-k) \end{pmatrix}$$
$$+ \frac{J_F}{4}(1+\alpha)\left(\beta\lambda^2 - t^2\right)$$

35

$$\int_0^{2\pi} dk \sqrt{A^2 + B^2 + 2AB\cos(k)}$$

mathematica

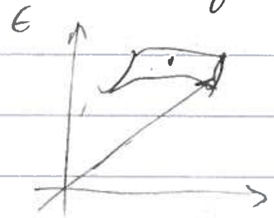
$$= 4\,|A+B|\ \text{EllipticE}\left(\frac{4AB}{(A+B)^2}\right)$$

$A = \beta[1 + 2\lambda(1+\alpha)]$ $A = A(\beta, \lambda, \alpha)$

$B = 1 - 2t(1+\alpha)$ $B = B(t, \alpha)$

$$\rightarrow\ \epsilon = -\frac{1}{\pi}|A+B|\ \text{EllipticE}\left(\frac{4AB}{(A+B)^2}\right) + (1+\alpha)(\beta\lambda^2 - t^2)$$

\# actually finding a saddle point



steepest grad descent for λ (−=)

steepest grad ascent for t (+=)

problems with the first code:

- energy's wrong for some reason
- need use a new ϵ which has an elliptic function
- if non-convergence over => use smaller steps

model answer for comparison:

for $a = -0.1$; $b: 0.8$

$t = 0.153394$

$\lambda: 0.464764$

Next step:

work out phase boundaries by bisection

Report ~~[illegible]~~ draft //

Method.

2nd Quantised form

Derivation:

Need a Picture

Start w/ Hamiltonian //

In 2nd Quantised form

w/ spin operators

↓

defining new operators

↓

JW transform

to map spinless fermions

↓ for XY case: $\alpha = -1$

Fourier Transform to

change to momentum space

↓ Double dimension → (4×4) matrix

~~Boto~~ Bogoliubov transformation

to incorporate anomalous term

↓

obtaining matrix (4×4)

↳ diagonalise to get eigenvalue

and double checked by MATLAB (??)

& Mathematica

↓

Mean field approx to make

Interaction terms vanish,

by using all possible decouplings,

and introducing ~~[illegible]~~ 3 more

expectation values, with constraints

↳ similar to XY case, with

some extra substitutions

diagonalisation is exactly the

same as the $\alpha = -1$ case

↳ obtaining energy density

equations

Computation:

$\alpha \in [0,1]$; $\beta \in [-1,0]$

~~[illegible]~~ $\lambda_0 = \mathrm{rand}(0,1)$

$t_0 = \mathrm{random}(0,1)$

2 determine λ & t self-consistently

w/ $\frac{\partial \epsilon}{\partial \lambda} = 0$; $\frac{\partial \epsilon}{\partial t} \to 0$

↳ description of code

Results:

1. energy density plot

2. phase transition diagram

Appendix: GitHub link

$$\hat{\mathcal{H}} = -J\sum_i \frac{1}{2}\left((T_i^+ T_{i+1}^- + \hat{T}_i^- T_{i+1}) + (1+\alpha)\underbrace{T_i^z T_{i+1}^z}\right)$$

$A = T_i^+ T_{i+1}^-$

$= D_i c_i c_{i+1}^\dagger D_{i+1}$

$= c_i \underbrace{D_i D_{i+1}}_{1-2c_i^\dagger c_i} c_{i+1}^\dagger$

$= c_i (1 - 2c_i^\dagger c_i) c_{i+1}^\dagger$

$= (c_i - 2c_i c_i^\dagger c_i) c_{i+1}^\dagger$

$= c_i c_{i+1}^\dagger - 2 c_i \underbrace{c_i^\dagger c_i} c_{i+1}^\dagger$

$\{c_i, c_i^\dagger\} = 1$

$c_i c_i^\dagger + c_i^\dagger c_i = 1$

$c_i^\dagger c_i = 1 - c_i c_i^\dagger$

$= c_i c_{i+1}^\dagger - 2 c_i (1 - c_i c_i^\dagger) c_{i+1}^\dagger$

$= c_i c_{i+1}^\dagger - (2c_i - 2\underbrace{c_i c_i}_{=0} c_i^{+1}) c_{i+1}^\dagger$

$-\theta$

$\{c_i, c_{i+1}^\dagger\} = 0$

$c_i c_{i+1}^\dagger = -c_{i+1}$

$A = -c_i c_{i+1}^\dagger$

$B = c_i^\dagger c_{i+1}$

$C = T_i^z T_{i+1}^z$

$= (\frac{1}{2} - c_i^\dagger c_i)(\frac{1}{2} - c_{i+1}^\dagger c_{i+1})$

$= \frac{1}{4} - \frac{1}{2} c_{i+1}^\dagger c_{i+1} - \frac{1}{2} c_i^\dagger c_i + c_i^\dagger c_i c_{i+1}^\dagger c_{i+1}$

$= \frac{1}{4} - \frac{1}{2}(c_{i+1}^\dagger c_{i+1} + c_i^\dagger c_i)$

$+ c_i^\dagger \underbrace{c_i c_{i+1}^\dagger}_{-c_{i+1}^\dagger c_i} c_{i+1}$

$= \frac{1}{4} - \frac{1}{2}(c_{i+1}^\dagger c_{i+1} + c_i^\dagger c_i) - c_i^\dagger c_{i+1}^\dagger c_i c_{i+1}$

$$\mathcal{H} = -\frac{J}{2}\sum_i \left(A + B + 2(1+\alpha) C \right)$$

$$= -\frac{J}{2}\sum_i \left(-c_i c_{i+1}^\dagger + c_i^\dagger c_{i+1} + 2(1+\alpha)\left(\frac{1}{2} - c_i^\dagger c_i\right)\left(\frac{1}{2} - c_{i+1}^\dagger c_{i+1}\right) \right)$$

$$= -\frac{J}{2}\sum_i \left((c_{i+1}^\dagger c_i + c_i^\dagger c_{i+1}) + 2(1+\alpha)\left(\frac{1}{2} - c_i^\dagger c_i\right)\left(\frac{1}{2} - c_{i+1}^\dagger c_{i+1}\right) \right)$$