From Dr. Kruger notes

Lagrangian & Hamiltonian

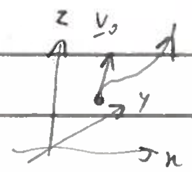
Completed in Munich

Eq^n of motion:

Newtonian Mechanics: Equations that determine the properties of a system in the next instant given their values at the preceding instant (causality)
— based on differential eq^n & initial conditions

- ez stuff: $\underline{r} = (x, y, z) \Rightarrow$ position — Newton's 2nd Law
  $\underline{v} = \dot{\underline{r}} = \frac{dr}{dt} \Rightarrow$ velocity $\qquad m\ddot{\underline{r}} = \underline{F}$
  $a = \ddot{r} = \frac{d^2 r}{dt^2} \Rightarrow$ acceleration

- 2nd ~~order~~ order d.f ODE requires 2 initial conditions:
  eg. & $\underline{r}(t_0) = \underline{r}_0$ — ①
  $\underline{v}(t_0) = \underline{v}_0$ — ②

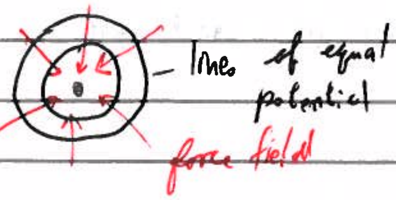


↳ trajectory of the particle is completely unique
↳ ~~described by~~ determined by diff eq^n (Newton's 2nd Law)
& initial conditions.

- Conservative forces: $\underline{F} = -\underline{\nabla} U$
  $U = U(\underline{r}, t) \Rightarrow$ potential energy

  eg. Coulomb potential
  ↳ $U(r) = -\frac{\alpha}{r}$ $\left(r = |\underline{r}| = \sqrt{\sum x_i^2} \right)$
  $\left(\underline{e}_r = \underline{r}/|\underline{r}| \right)$

$$F = -\underline{\nabla} U = \underline{\nabla} \frac{\alpha}{r} = -\frac{\alpha}{r^2} \underline{e}_r$$



Conservation of energy:

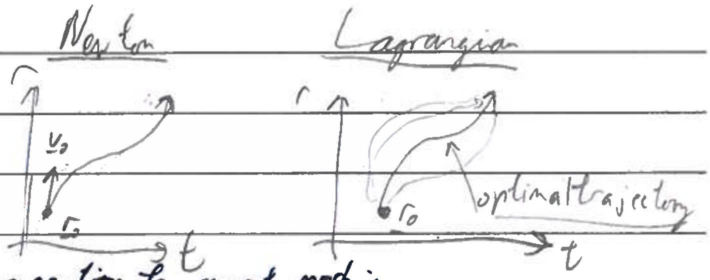
$$\frac{dE}{dt} = \frac{d}{dt}(T + U)$$
$$= \frac{d}{dt}\left(\frac{m}{2}\dot{\underline{r}}^2 + U(\underline{r}, t) \right)$$
$$= m\dot{\underline{r}} \cdot \ddot{\underline{r}} + \underbrace{\frac{\partial U}{\partial \underline{r}}}_{\nabla U} \cdot \dot{\underline{r}} + \frac{\partial U}{\partial t}$$
$$= \dot{\underline{r}} \underbrace{(m\ddot{\underline{r}} + \nabla U)}_{= 0 \text{ (Newton's 2nd Law)}} + \frac{\partial U}{\partial t}$$

$\frac{\partial U}{\partial t} = 0 \Leftrightarrow$ energy conserved

$\Rightarrow$ Symmetries $\longleftrightarrow$ Conservation laws
(invariance under time translation)

Lagrangian mechanics:

- math equivalent
- based on destiny (philosophy)
- looks @ global trajectories & finds the optimal one.



→ connection to quant mech:
[summation over all possible trajectories w/ certain weights;
classical trajectory has max weight]

- The Principle of Least Action [ONLY works for conservative systems)
  - where all forces can be gotten from a potential function

– Trajectory lies in the space of generalised coordinate

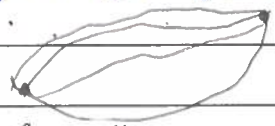
$$\cancel{q} = q = (q_1, q_2, \cdots, q_N)$$

$q = (x, y, z) \Rightarrow$ cartesian coordinates in $d=3$, one particle

$q = (\underbrace{x_1^{(1)}, \cdots, x_d^{(1)}}, \underbrace{x_1^{(2)} \cdots x_d^{(2)}}, \cdots, \underbrace{x_1^{(n)}, \cdots, x_d^{(n)}})$ (cartesian coordinates of n particles in d dimensions

$\cancel{q}\ q = (r, \theta, z) \Rightarrow$ cylindrical
$q = (r, \theta, \phi) \Rightarrow$ spherical] $n=1$, $d=3$

– Suppose particle is destined to travel from $q_0 = q(t_0)$ to $q_1 = q(t_1)$
What is the optimal trajectory?



Trajectory that is optimal is the one that minimizes the functional

– teleological principle (τέλος)(end, purpose) rather than

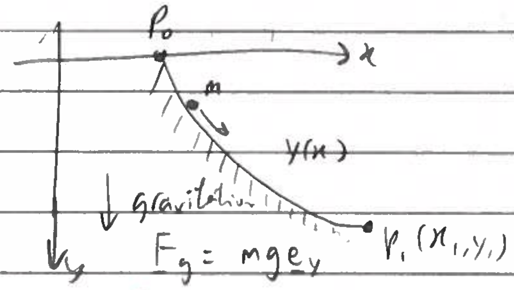
Best way to move in space for infinitesimal units of time

$$\underbrace{S}_{\text{action}} = \int_{t_0}^{t_1} dt\ \underbrace{L}_{\text{Lagrangian}}(q(t), \dot{q}(t), t)$$

[magic function with $q(t)$, $\dot{q}(t)$, t as the input

An example of variational problem:

Brachistochrone problem (1696 - Bernoulli)



- slide connecting P_0 & P_1
- no friction

Question: for which slide $(y(x))$ is the sliding time T, minimal?

- Solution is the function $y(x)$
- determine the functional $T(y, y')$ & minimize it to find the optimal slide

$$T = \int_{P_0 \to P_1} dt = \int \frac{ds}{v} = \int_{x_0}^{x_1} \frac{\sqrt{1+(y')^2}}{v}\, dx$$

$$ds = \sqrt{(dx)^2 + (dy)^2} = dx\sqrt{1+(y')^2}$$

By energy conservation

$$mgy = \frac{1}{2}mv^2$$

$$v = \sqrt{2gy}$$

Functional $T(y, y')$ takes different functions for diff functions $(y(x))$

$$T(y, y') = \int_{x_0}^{x_1} \frac{\sqrt{1+(y')^2}}{\sqrt{2gy}}\, dx = \frac{1}{\sqrt{2g}} \int_{x_0}^{x_1} \sqrt{\frac{1+(y')^2}{y}}\, dx$$

using $S(q, \dot{q}) = \int_{t_0}^{t_1} dt\, L(q, \dot{q}, t)$ $\Rightarrow$ $T(y, y') = \int_{x_0}^{x_1} dx \underbrace{\sqrt{\frac{1+(y')^2}{y}}}\left(\frac{1}{\sqrt{2g}}\right)$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q}$$

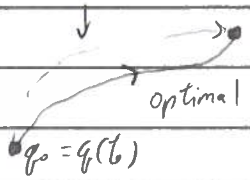
$$\boxed{\frac{d}{dt}\frac{\partial f}{\partial y'} = \frac{\partial f}{\partial y}} *$$

solve the 2nd order diff eq^n in the case the Lagrangian L does not depend on t, we get one integration for free

L derivatives

Euler - Lagrange Equations

- Considering infinitesimal variations of trajectories



think it as $L \cdot L' L$

variation, change in action:

change in function

$$\delta S = \int_{t_0}^{t_1} dt \left[L(q+\delta q, \dot q + \delta \dot q, t) - L(q, \dot q, t) \right]$$

1. $q(t) \to q(t) + \underbrace{\delta q(t)}_{\text{this is small.}}$

2. $\delta q(t_0) = \delta q(t_1) = 0$

Implicit: All functions have continuous 2^{nd} derivative

$$= \int_{t_0}^{t_1} dt \left[\frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot q} \delta \dot q + \underbrace{O(\delta q^2, \delta \dot q^2)}_{\text{Taylor expand up to degree of 2}} \right]$$

$$= \int_{t_0}^{t_1} dt \left[\frac{\partial L}{\partial q} \delta q + \frac{d}{dt}\left[\frac{\partial L}{\partial \dot q} \delta q \right] - \left(\frac{d}{dt} \frac{\partial L}{\partial \dot q} \right) \delta q \right]$$

$q(t) = q(t_0) + \dot q(t_0)(t-t_0) + \frac{\ddot q(t_0)}{2}(t-t_0)^2$

$\dot q(t) = \dot q(t_0) + \ddot q(t_0)(t-t_0)$

$$\delta S = \int_{t_0}^{t_1} dt\, \delta q \left[\frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot q} \right] + \underbrace{\left[\frac{\partial L}{\partial \dot q} \delta q \right]_{t_0}^{t_1}}_{=0}$$

$\because \delta q(t_0) = \delta q(t_1) = 0$

For optimal trajectory

$\delta S = 0$ for all infinitesimal variations δq

$L(q, \dot q, t) = L(q_0, \dot q, t) + L'(q_0, \dot q_0, t_0)(q - q_0)$

$= L + \frac{\partial L}{\partial q_0}$ ... $\dot q - \dot q_0$ ... $t - t_0$

$a = q_0$; $b = \dot q_0$; $c = t$

$$\boxed{\frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot q}} \Rightarrow \text{Euler-Lagrange equations}$$

For $q = (q_1, \ldots, q_N)$, we have a set of N differential equations, 2nd order, coupled

$\forall \alpha = 1, \ldots, N : \frac{d}{dt} \frac{\partial L}{\partial \dot q_\alpha} = \frac{\partial L}{\partial q_\alpha}$

~~t~~ time var

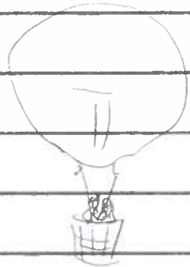
~~t~~ varying velocity

$\delta \dot q(t)$

Same as taking the time derivative of varied path $q(t)$

$\Rightarrow \delta \dot q(t) = \frac{d}{dt} q(t)$

Me when things don't let me draw



- Determine trajectory $q(t)$ in configuration space for given $(q(t_0), q$ ... causality restored ??

7^{th} Oct - 9^{th} Nov - 23^{rd} Dec

Euler - Lagrange Eqn (Benny ver)

~~Ac~~

Action:

$$S[x,\dot{x}] = \int dt\, L(x,\dot{x})$$

magic function

→ Best way to move in space for infinitesimal unit of time

Eqn of motion takes place when the action is minimized : $\delta S = 0$

$$\delta S = \delta \int dt\, L(x,\dot{x}) = 0$$

$$= \int dt \quad \delta L(x,\dot{x})$$

$$= \int dt \left(\frac{\delta L}{\delta x}\delta x + \frac{\delta L}{\delta \dot{x}}\delta \dot{x} \right) \quad \left[\text{only take first order} \right]$$

$$= \int dt \left(\frac{\delta L}{\delta x}\delta x + \frac{\delta L}{\delta \dot{x}}\delta\left(\frac{dx}{dt}\right)\right)$$

$$= \int dt \left(\frac{\delta L}{\delta x}\delta x + \underbrace{\frac{\delta L}{\delta \dot{x}}\left(\frac{d}{dt}\delta x\right)}_{\text{the variation } \delta}\right)$$

$$= \underbrace{\int dt \frac{\delta L}{\delta x}\delta x}_{①} + \underbrace{\int dt \frac{\delta L}{\delta \dot{x}}\left(\frac{d}{dt}\delta x\right)}_{②}$$ time derivative can swap places (small trick accepted for meantime)

Part ②: use integration by parts

$$u = \frac{\delta L}{\delta \dot{x}} \quad v' = \frac{d}{dt}\delta x$$

$$u' = \frac{d}{dt}\frac{\delta L}{\delta \dot{x}} \quad v = \delta x$$

$$\underbrace{\frac{\delta L}{\delta \dot{x}}\delta x \Big|_{t_0}^{t_1}}_{=0} - \int \frac{d}{dt}\frac{\delta L}{\delta \dot{x}}\delta x\, dt$$

$\because \delta x(t_1) = \delta x(t_0) = 0$
Boundary conditions set

$$\int dt \frac{\delta L}{\delta x}\delta x - \int \frac{d}{dt}\frac{\delta L}{\delta \dot{x}}\delta x\, dt$$

$$\frac{d}{dt}\frac{\delta L}{\delta \dot{x}} = \frac{\delta L}{\delta x}$$

Computationally

$$\Rightarrow \underbrace{\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{\delta L}{\partial x}}$$

$$\delta S = \int \delta x \underbrace{\left(\frac{\delta L}{\delta x} - \frac{d}{dt}\frac{\delta L}{\delta \dot{x}}\right)}_{0} dt$$

$\delta S = 0 \rightarrow$ implies ~~for~~ $\left(\text{or } \frac{\delta S}{\delta x} = 0\right)$

$O(\delta_q{}^2, \delta_{\dot q}{}^2)$:

$\left(\frac{d}{dt}\frac{\partial L}{\partial \dot q}\right)\delta q$

~~$f(x,y,z)$~~ $\qquad l = L' - L$

$$L(q,\dot q,t) = L(q_0,\dot q_0,t_0) + \frac{\partial L}{\partial q}(q-q_0) + \frac{\partial L}{\partial \dot q}(\dot q-\dot q_0) + \frac{\partial L}{\partial t}(t-t_0)$$

$$+ \frac{1}{2}\Big[\frac{\partial^2 L}{\partial q^2}(q-q_0)^2 + \cancel{\frac{\partial^2 L}{\partial q\partial \dot q}(q-q_0)(\dot q-\dot q_0)} + \cancel{\frac{\partial^2 L}{\partial q\partial t}(q-q_0)(t-t_0)}$$

$$+ \cancel{\frac{\partial^2 L}{\partial \dot q\partial q}(q-q_0)(\dot q-\dot q_0)} + \frac{\partial^2 L}{\partial \dot q^2}(\dot q-\dot q_0)^2 + \cancel{\frac{\partial^2 L}{\partial \dot q\partial t}(\dot q-\dot q_0)(t-t_0)}$$

$$+ \cancel{\frac{\partial^2 L}{\partial t\partial q}(t-t_0)(q-q_0)} + \cancel{\frac{\partial^2 L}{\partial t\partial \dot q}(t-t_0)(\dot q-\dot q_0)}$$

$$+ \frac{\partial^2 L}{\partial t^2}(t-t_0)^2$$

?? ~~try~~

$\Rightarrow \boxed{q-q_0 \approx 0 \quad t-t_0\approx 0 \quad \dot q-\dot q_0 \approx 0}$

try:

~~$q-q_0 =$~~

drop $\partial_q\partial_{\dot q}$

$$L(q,\dot q,t) = L(q_0,\dot q_0,t_0) + \frac{\partial L}{\partial q}(q-q_0) + \frac{\partial L}{\partial \dot q}(\dot q-\dot q_0) + \frac{\partial L}{\partial t}(t-t_0)$$
$$+\frac{1}{2}\left[\frac{\partial^2 L}{\partial q^2}(q-q_0)^2 + \frac{\partial^2 L}{\partial \dot q^2}(\dot q-\dot q_0)^2 + \frac{\partial^2 L}{\partial t^2}(t-t_0)\right]$$

$$= L(q_0,\dot q_0,t_0) + (q-q_0)\left[\frac{\partial L}{\partial q} + \frac{1}{2}\frac{\partial^2 L}{\partial q^2}(q-q_0)\right]$$
$$+ (\dot q-\dot q_0)\left[\frac{\partial L}{\partial \dot q} + \frac{1}{2}\frac{\partial^2 L}{\partial \dot q^2}(\dot q-\dot q_0)\right]$$
$$+ (t-t_0)\left[\frac{\partial L}{\partial t} + \frac{1}{2}\frac{\partial^2 L}{\partial t^2}(t-t_0)\right]$$

Solving Brachistochrone problem:

Solving 2nd Order Diff eq^n

$$\frac{d}{dx}\frac{\partial f}{\partial y'} = \frac{\partial f}{\partial y}$$

for $\frac{df}{dx} = \frac{d}{dx} f(y, y') = \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' = \left(\frac{d}{dx}\frac{\partial f}{\partial y'}\right) y' + \frac{\partial f}{\partial y'} y'' = \frac{d}{dx}\left(\frac{\partial f}{\partial y'} y''\right)$

$\Rightarrow \frac{d}{dx}\left(f - \frac{\partial f}{\partial y'} y'\right) = 0$

int 2边 一次 d.f.f eq^n 1st order

$f - \frac{\partial f}{\partial y'} y' = C_1$

$f = \sqrt{\frac{1+(y')^2}{y}}$ | using $f - \frac{\partial f}{\partial y'} y' = C_1$

multivariate chain rule →

$\frac{df}{dx} = \frac{d}{dx} f(y, y') = \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''$

$= \left(\frac{d}{dx}\frac{\partial f}{\partial y'}\right) y' + \frac{\partial f}{\partial y'} y''$ | sub $\frac{d}{dx}\frac{\partial f}{\partial y'} = \frac{\partial f}{\partial y}$

on reverse product

$= \frac{d}{dx}\left(\frac{\partial f}{\partial y'} y'\right) = \frac{df}{dx}$

$\Rightarrow \int \frac{d}{dx}\left(f - \frac{\partial f}{\partial y'} y'\right) = \int 0$

$f - \frac{\partial f}{\partial y'} y' = C_1$

$\frac{\partial}{\partial y'}\left(\sqrt{\frac{1+(y')^2}{y}}\right)$

$= \frac{\partial}{\partial y'}\left(\frac{1+(y')^2}{y}\right)^{\frac{1}{2}}$

$= \frac{\partial}{\partial y'}\left(\frac{1}{y} + \frac{(y')^2}{y}\right)^{\frac{1}{2}}$

$= \frac{2y'}{y}\frac{1}{2}\left(\sqrt{\frac{1+(y')^2}{y}}\right)^{-\frac{1}{2}}$

$f = \sqrt{\frac{1+(y')^2}{y}}$

$\sqrt{\frac{1+(y')^2}{y}} - \frac{\partial f}{\partial y'} y' = C_1$

$\sqrt{\frac{1+(y')^2}{y}} - \sqrt{\frac{1+(y')^2}{y}}^{\,-1} \frac{(y')^2}{y} = C_1$

$\frac{1+(y')^2}{y} - \frac{(y')^2}{y} = C_1\sqrt{\frac{1+(y')^2}{y}}$

$\frac{1}{y} = C_1\sqrt{\frac{1+(y')^2}{y}}$

$\frac{\cancel{y}}{y^2 C_1^2} = \frac{1+(y')^2}{\cancel{y}}$

$y' = \sqrt{\frac{1}{yC_1^2} - 1} \Rightarrow y' = \sqrt{\frac{2r_0 - y}{y}}$ $\quad r_0 = \frac{1}{2C_1^2}$

Solving $y' = \sqrt{\frac{2r_0 - y}{y}}$

$$\frac{dy}{dx} = \sqrt{\frac{2r_0 - y}{y}}$$

$$\int dy \sqrt{\frac{y}{2r_0 - y}} = \int dx = x - c_2$$

$\sin 2\theta = 2\sin\theta\cos\theta$

$\cos 2\theta = 1 - 2\sin^2\theta$

$2\sin^2\theta = 1 - \cos 2\theta$

$\sin^2\theta = \frac{1-\cos 2\theta}{2}$

by substitution:

$$y = 2r_0 \sin^2\frac{\phi}{2} = r_0(1 - \cos\phi)$$

$$\frac{dy}{d\phi} = \frac{d}{d\phi}(r_0 - r_0\cos\phi)$$

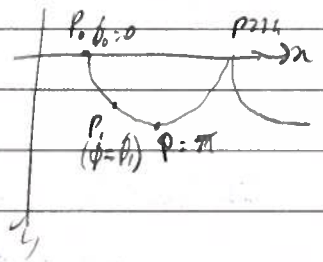
$$dy = r_0 \sin\phi \, d\phi = 2r_0 \sin\frac{\phi}{2}\cos\frac{\phi}{2}\, d\phi$$

$$\int \sqrt{\frac{2r_0 \sin^2\frac{\phi}{2}}{2r_0 - 2r_0\sin^2\frac{\phi}{2}}}\, 2r_0 \sin\frac{\phi}{2}\cos\frac{\phi}{2}\, d\phi$$

$$= 2r_0 \int \sqrt{\frac{\sin^2\frac{\phi}{2}}{1-\sin^2\frac{\phi}{2}}}\, \sin\frac{\phi}{2}\cos\frac{\phi}{2}\, d\phi = 2r_0 \int \sqrt{\frac{\sin^2\frac{\phi}{2}}{\cos^2\frac{\phi}{2}}}\, \sin\frac{\phi}{2}\cos\frac{\phi}{2}\, d\phi$$

$$= 2r_0 \int \sin^2\frac{\phi}{2}\, d\phi = r_0 \int d\phi\,(1-\cos\phi) = r_0(\phi - \sin\phi)$$

$\therefore$ solution $r(\phi) = \begin{pmatrix} x(\phi) \\ y(\phi) \end{pmatrix} = \underbrace{\begin{pmatrix} c_2 \\ 0 \end{pmatrix} + r_0 \begin{pmatrix} \phi - \sin\phi \\ 1 - \cos\phi \end{pmatrix}}_{\text{parametric of a cycloid}}$



initial conditions:

$$r(0) = \begin{pmatrix} c_2 \\ 0 \end{pmatrix} + r_0 \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} c_2 \\ 0 \end{pmatrix} \Rightarrow x_0 = c_2$$

$$r(\phi_1) = \begin{pmatrix} x(\phi_1) \\ y(\phi_1) \end{pmatrix} = \begin{pmatrix} x_0 \\ 0 \end{pmatrix} + r_0 \begin{pmatrix} \phi_1 - \sin\phi_1 \\ 1 - \cos\phi_1 \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

$$\frac{x_1 - x_0}{y_1} = \frac{\phi_1 - \sin\phi_1}{1 - \cos\phi_1} \Rightarrow \text{determines } \phi_1$$

$$r_0 = \frac{y_1}{1 - \cos\phi_1} \quad \text{determines } r_0$$

E.L.

Connection of Euler Lagrang eq^n to Newtonian mechanics.

- Engineering Lagrange (L) such that E.L. eq^n is identical to Newton's 2nd Law

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q} \overset{?}{\Longleftrightarrow} m\ddot{\underline{r}} = -\underline{\nabla} U = \underline{F}$$

$$L = \frac{m}{2}\dot{\underline{r}}^2 - U(\underline{r})$$

each coordinate eg. x: ($\dot{\underline{r}}^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2$)

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x}$$

$$m\ddot{x} = -\frac{\partial U}{\partial x} = F_x$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\underline{r}}} = \frac{d}{dt} m\dot{\underline{r}} = m\ddot{\underline{r}}$$

$$\frac{\partial L}{\partial r} = -\frac{\partial U}{\partial r} = -\underline{\nabla} U = \underline{F}$$

=> expressing Newton's 2nd Law w/ principle of least action.

Minimizing:

$$S = \int_{t_0}^{t_1} dt \left[\frac{m\dot{\underline{r}}^2(t)}{2} - U(\underline{r}(t)) \right]$$

=> gives newton's second Law of motion.

General propertie of the Action.

Say we have 2 independent systems : (A & B)

system A: $q_A = (q_{A,1}, \cdots, q_{A,N})$

system B: $q_B = (q_{B,1}, \cdots, q_{B,N})$

Addition of Lagrangians:

$$L(q_A, \dot{q}_A, q_B, \dot{q}_B, t) = L_A(q_A, \dot{q}_A, t) + L_B(q_B, \dot{q}_B, t)$$

↳ leads to 2 independent equations of motion

$$\frac{d}{dt}\frac{\partial L_A}{\partial \dot{q}_A} = \frac{\partial L_A}{\partial q_A} \quad // \quad \frac{d}{dt}\frac{\partial L_B}{\partial \dot{q}_B} = \frac{\partial L_B}{\partial q_B}$$

1. ↳ interacting subsystems:

$$L = L_A(q_A, \dot{q}_A, t) + L_B(q_B, \dot{q}_B, t) + L_{AB}(q_A, q_B, \dot{q}_A, \dot{q}_B, t)$$

↳ differential equations for A & B are no longer decoupled due to L_{AB}

2.2. Invariance under the multiplication w/ constants (λ)

$L' = \lambda L$, λ = const => same eq^n of motion

$$\frac{d}{dt}\frac{\partial L'}{\partial \dot{q}} = \lambda\frac{\partial L}{\partial \dot{q}}, \quad \frac{\partial L'}{\partial q} = \lambda\frac{\partial L}{\partial q}$$

$$\frac{d}{dt}\lambda\frac{\partial L}{\partial \dot{q}} = \lambda\frac{\partial L}{\partial q} \Longleftrightarrow \frac{d}{dt}\frac{\partial L'}{\partial \dot{q}} = \frac{\partial L'}{\partial q}$$

3. Invariance under adding total derivative

$$L' = L(q,\dot q,t) + \frac{d}{dt} f(q,t)$$

why?

$$S' = \int_{t_0}^{t_1} dt\, L'(q,\dot q,t) = S + \underbrace{f(q(t_1),t_1) - f(q(t_0),t_0)}_{\text{unchanged by a variation with } \delta q(t_0) = \delta q(t_1) = 0}$$

$\Rightarrow$ identical eq^{ns} of motion

invariance can also ~~be~~ be seen by explicit derivation of eq^{ns} of motion:

$$L' = L + \frac{d}{dt} f(q,t) = L + \frac{\partial f}{\partial q}\dot q + \frac{\partial f}{\partial t} \qquad \frac{d}{dt}\frac{\partial f}{\partial \dot q} = \frac{\partial f}{\partial q}$$

Euler Lagrange eq^{n}

using

$$\frac{d}{dt}\frac{\partial L}{\partial \dot q} = \frac{\partial L}{\partial q}$$

$$\Rightarrow \quad 0 = \frac{d}{dt}\frac{\partial L'}{\partial \dot q} - \frac{\partial L'}{\partial q} = \frac{d}{dt}\frac{\partial}{\partial \dot q}\left(L + \frac{\partial f}{\partial q}\dot q + \frac{\partial f}{\partial t}\right) - \frac{\partial}{\partial q}\left(L + \frac{\partial f}{\partial q}\dot q + \frac{\partial f}{\partial t}\right)$$

$$= \frac{d}{dt}\frac{\partial L}{\partial \dot q} + \frac{d}{dt}\frac{\partial f}{\partial q} + \frac{d}{dt}\frac{\partial}{\partial \dot q}\frac{\partial f}{\partial t} - \frac{\partial L}{\partial q} - \frac{\partial^2 f}{\partial q^2}\dot q - \frac{\partial}{\partial q}\frac{\partial f}{\partial t}$$

$$= \underbrace{\frac{d}{dt}\frac{\partial L}{\partial \dot q} - \frac{\partial L}{\partial q}}_{\checkmark} + \underbrace{\frac{d}{dt}\frac{\partial f}{\partial q} - \left(\frac{\partial^2 f}{\partial q^2}\dot q + \frac{\partial^2 f}{\partial q \partial t}\right)} + \left(\frac{d}{dt}\frac{\partial}{\partial \dot q}\frac{\partial f}{\partial t}\right)$$

$\Rightarrow$ same eq^{ns} of motion $= 0$

Galilean invariance

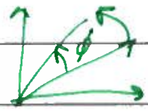
- deriving Lagrangian L from first principles

- free particle in homogeneous + isotropic space (+ time homogeneous)

- invariance $\Rightarrow$ equation of motion remains unchanged
- homogeneous $\Rightarrow$ invariance wrt spatial shift

$$\underline{r} \longrightarrow \underline{r}'$$

- isotropic $\Rightarrow$ invariance wrt rotations



$$\Rightarrow L = L(v^2) = \alpha_0 + \frac{\alpha_2}{2} v^2 + \frac{\alpha_4}{4!} v^4 + \ldots$$

only possibility!

We can set $\alpha_0 = 0$ w/o loss of generality
as α_0 drops out in the Euler-Lagrange eq^n

E.L eq^n : $\frac{d}{dt}\frac{\partial L}{\partial \underline{r}} = 0$

$$\Rightarrow \forall_{i=1,\ldots,d} : \frac{\partial L}{\partial v_i} = \frac{\partial L}{\partial v^2} \underbrace{\frac{\partial v^2}{\partial v_i}}_{2v_i} = \text{const}$$

$\Rightarrow \underline{v} = \text{const} \rightarrow$ "inertial frame of reference"

- Galilean principle of relativity:

$\underline{r} \mapsto \underline{r}' = \underline{r} + \underline{u}t \ , \ t' = t$ [Galilean transform]

should leave the equations of motion invariant (valid for $v \ll c$)

$\underline{v} = \frac{d\underline{r}'}{dt'} = \frac{d\underline{r}'}{dt} = \frac{d\underline{r}}{dt} + \underline{u} = \underline{v} + \underline{u} \Rightarrow \underline{v}' = \underline{v} + \underline{u}$, consider infinitesimally small $\underline{u}$

$$L' = L(v'^2) = L((v+u)^2) = L(v^2 + 2uv + u^2)$$
$$= L(v^2) + \frac{\partial L}{\partial v^2} 2\underline{u}v + O(\underline{u}^2)$$

From invariance of eq^n of motion we require that : $L' = L + \frac{df(\underline{r},t)}{dt}$

$$\Rightarrow \quad \frac{df(\underline{r},t)}{dt} = 2\,\underline{u}\cdot\underline{v}\,\frac{\partial L}{\partial v^2}$$

$$\parallel$$

$$\frac{\partial f}{\partial \underline{r}}\underline{v} + \frac{\partial f}{\partial t}$$

$$\Rightarrow \quad \frac{\partial f}{\partial t} = 0 \quad \& \quad \frac{\partial f}{\partial \underline{r}} = 2\frac{\partial L}{\partial v^2}\underline{u}$$

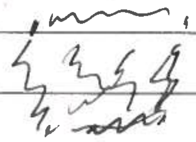
$\Rightarrow$ $\frac{\partial f}{\partial \underline{r}}$ independent of $\underline{v}$

$$\Rightarrow \quad \boxed{L = \frac{\alpha}{2}v^2 = \frac{m}{2}v^2}, \quad f = m\underline{u}\cdot\underline{r}$$

m: mass

– Interaction:

Add $L_I = -U(\underline{r}_1,\dots,\underline{r}_N)$



$$L = \sum_i L_i(\underline{r}_i, \dot{\underline{r}}_i, t) + L_I$$

require:

- instantaneous
- only position dependent L

$$\rightarrow \boxed{L = \sum_i \frac{m_i}{2}\dot{\underline{r}}_i^2 - U(\underline{r}_1,\dots,\underline{r}_N)}$$

$$L = T - U \quad (KE - PE)$$

We derived Newton's 2nd Law:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\underline{r}}_i} = \frac{\partial L}{\partial \underline{r}_i} \iff m_i\ddot{\underline{r}}_i = -\nabla_i U(\underline{r}_1\cdots\underline{r}_N)$$

$$(i = 1,\dots,N)$$

$\Rightarrow$ complete set of 2nd order Diff eq^n

In general: potential consists of external potential & interaction potential:

$$U(r_1\cdots r_N) = \sum_i U_{ext}(r_i) + \frac{1}{2}\underbrace{\sum_{\substack{(i\neq j)\\ ij}} U_{int}(r_i - r_j)}_{\text{pair interactions}}$$

2. Conservation Laws: Symmetries $\Leftrightarrow$ conservation laws (Noether's theorem)

— Energy:

$\ast$ Time invariance : $L = L(q, \dot{q})$ [no explicit dependence on time]

$\ast$ ~~cmb Lagrange~~ E.L.:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\underline{r}}} = \frac{\partial L}{\partial \underline{r}} \quad / \quad \underline{\underline{\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q}}}$$

$$\frac{dL}{dt} = \frac{\partial L}{\partial q}\frac{dq}{dt} + \frac{\partial L}{\partial \dot{q}}\frac{d\dot{q}}{dt}$$

$$= \frac{\partial L}{\partial q}\dot{q} + \frac{\partial L}{\partial \dot{q}}\ddot{q} \underset{\text{E.L.}}{=} \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}}\right)\dot{q} + \frac{\partial L}{\partial \dot{q}}\ddot{q} = \frac{d}{dt}\left(\dot{q}\frac{\partial L}{\partial \dot{q}}\right)$$

inverse chain rule

2/19/2020

[redacted]

Microsoft office

Macbook

on fire

$$\frac{dL}{dt} = \frac{d}{dt}\left(\dot{q}\frac{\partial L}{\partial \dot{q}}\right)$$

$$\frac{d}{dt}\left(\dot{q}\frac{\partial L}{\partial \dot{q}} - L\right) = 0$$

$$\int \frac{d}{dt}\left(\dot{q}\frac{\partial L}{\partial \dot{q}} - L\right) = \int 0$$

(used when solving the Brachistochrone problem)

[redacted]

$$\dot{q}\frac{\partial L}{\partial \dot{q}} - L = \text{const} \Rightarrow \text{constant of motion}$$

↳ what is the meaning

Evaluate for interacting mass points:

$$L = \sum_i \frac{m_i}{2}\dot{\underline{r}}_i^2 - U(\underline{r}_1, \ldots \underline{r}_N)$$

$$\dot{r}\frac{\partial L}{\partial \dot{r}} - L = \dot{r}\frac{\partial}{\partial \dot{r}}\left(\sum \frac{m_i}{2}\dot{\underline{r}}_i^2\right) - \left(\sum_i \frac{m_i}{2}\dot{\underline{r}}_i^2 - U(\underline{r}_1, \ldots, \underline{r}_N)\right)$$

$$= \sum_i m_i \dot{\underline{r}}_i^2 - \sum_i \frac{m_i}{2}\dot{\underline{r}}_i^2 + U(\underline{r}_1, \ldots, \underline{r}_N)$$

$$= \underbrace{\sum_i \frac{m_i}{2}\dot{\underline{r}}_i^2}_{\text{KE}} + \underbrace{U(\underline{r}_1, \ldots \underline{r}_N)}_{\text{PE}} = \text{Total energy}$$

Time invariance $\longleftrightarrow$ energy conservation

$$\boxed{E = \underline{\dot{q}}\frac{\partial L}{\partial \underline{\dot{q}}} - L = \sum_\alpha \dot{q}_\alpha \frac{\partial L}{\partial \dot{q}_\alpha} - L, \quad \frac{dE}{dt} = 0}$$

- Conservation of Momentum:

- Homogeneous space: => doesn't matter where you put a system of interacting particles

→ invariance under spatial conditions

↳ equations of motion unchanged

$$\forall i = 1, \dots, n : \underline{r}_i' = \underline{r}_i + \underbrace{\delta \underline{r}}_{\uparrow \text{ small \& constant}}$$

$\delta \underline{r}$

$$L' = L(\underline{r}_i + \delta\underline{r}, \dot{\underline{r}}_i, t)$$

$$= L + \sum_i \frac{\partial L}{\partial \underline{r}_i}\delta\underline{r} + \mathcal{O}(\delta \underline{r}^2)$$

$$\Rightarrow \delta L = \delta\underline{r} \cdot \sum_i \frac{\partial L}{\partial \underline{r}_i} \overset{!}{=} 0$$

denoting a useful, important, but unexpected equality

we want to show this equality is true

- principle of least action?

This should be equal to zero for all δr! => why?

$$-\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = \frac{\partial L}{\partial r}$$

$$\Rightarrow \sum_i \frac{\partial L}{\partial \underline{r}_i} \underset{E.L.}{=} \underbrace{\frac{d}{dt}\sum_i \frac{\partial L}{\partial \dot{\underline{r}}_i}}_{\text{constant of motion}} = 0$$

$$\underline{P} = \sum_i \frac{\partial L}{\partial \dot{\underline{r}}_i} = \sum_i \underline{p}_i, \ \frac{d\underline{P}}{dt} = 0$$

$$\frac{d}{dt}\sum_i \frac{\partial L}{\partial \dot{\underline{r}}_i} = 0$$

- interacting mass points: $L = \sum_i \frac{m_i}{2}\dot{\underline{r}}_i^2 - U(\underline{r}_1 \dots \underline{r}_n)$

$$\underline{P} = \sum_i \underline{p}_i, \quad \underline{p}_i = \frac{\partial L}{\partial \dot{\underline{r}}_i} = m_i \dot{\underline{r}}_i$$

- also deriving Newton's 3rd Law (action = reaction):

rmb: $F = m\ddot{r} = -\nabla U$

$$0 \overset{!}{=} U(\underline{r}_1 + \delta\underline{r}, \dots \underline{r}_n + \delta\underline{r}) - U(\underline{r}_1, \dots, \underline{r}_n)$$

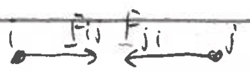
$$= \sum_i \frac{\partial U}{\partial \underline{r}_i} \cdot \delta \underline{r}$$

$\Rightarrow \delta\underline{r}$ ~~arbitrarita~~ $\delta\underline{r}$ arbitrary $\Rightarrow$

$$0 = \sum_i \frac{\partial U}{\partial \underline{r}_i} = 0 \Rightarrow \sum_i \underline{F}_i = 0$$

$$\underline{F}_i = -\nabla_i U$$

no net force! (only internal forces)

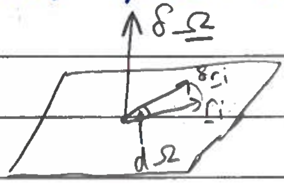


$$\underline{F}_i = \sum_{j \neq i} \underline{F}_{ij}$$

$$\sum_i \underline{F}_i = 0 \Rightarrow \underline{F}_{ij} = -\underline{F}_{ji}$$

- Conservation of Angular Momentum:

Isotropic space: Invariance wrt arbitrarily infinitesimal rotations



$d\Omega = |d\underline{\Omega}| \Rightarrow$ rotation angle

$\delta\Omega$ orthogonal to rotation plane

$$\delta\underline{r}_i = \delta\underline{\Omega} \times \underline{r}_i$$

$L' = L + \delta L$

$= L(\underline{r}_i + \delta\underline{r}_i, \dot{\underline{r}}_i + \delta\dot{\underline{r}}_i, t) \approx L + \sum_i \left(\frac{\partial L}{\partial \underline{r}_i} \delta\underline{r}_i + \frac{\partial L}{\partial \dot{\underline{r}}_i} \delta\dot{\underline{r}}_i \right)$

$\Rightarrow \delta L = \sum_i \left(\frac{\partial L}{\partial \underline{r}_i} \delta\underline{r}_i + \frac{\partial L}{\partial \dot{\underline{r}}_i} \delta\dot{\underline{r}}_i \right) = \sum_i \left(\frac{\partial L}{\partial \underline{r}_i} (\delta\underline{\Omega} \times \underline{r}_i) + \frac{\partial L}{\partial \dot{\underline{r}}_i} (\delta\underline{\Omega} \times \dot{\underline{r}}_i) \right)$

$\begin{pmatrix} \underline{a}(\underline{b}\times\underline{c}) \\ = \underline{c}(\underline{a}\times\underline{b}) \\ = \underline{b}(\underline{c}\times\underline{a}) \end{pmatrix}$ $= \delta\Omega \sum_i \left(\underline{r}_i \times \frac{\partial L}{\partial \underline{r}_i} + \dot{\underline{r}}_i \times \frac{\partial L}{\partial \dot{\underline{r}}_i} \right) \overset{!}{=} 0$

E.L.

$\Rightarrow = \delta\Omega \sum_i \left(\frac{d}{dt}\underline{r}_i \times \frac{\partial L}{\partial \dot{\underline{r}}_i} + \underline{r}_i \times \frac{d}{dt} \frac{\partial L}{\partial \dot{\underline{r}}_i} \right)$

$= \delta\Omega \sum_i \frac{d}{dt} \left(\underline{r}_i \times \underbrace{\frac{\partial L}{\partial \dot{\underline{r}}_i}}_{p_i} \right) \overset{!}{=} 0$

$\Rightarrow$ conservation of angular momentum

$\Leftarrow$ $\delta\Omega$ arbitrarily

angular momentum $\Rightarrow \underline{L} = \sum_i \underline{L}_i = \sum_i \underline{r}_i \times \underline{p}_i$, $\frac{d\underline{L}}{dt} = 0$

- Centre of mass

centre of mass: $\underline{R} = \frac{1}{M} \sum_i m_i \underline{r}_i$

↳ total mass: $M = \sum_i m_i$

total momentum: $\underline{P} = \sum_i m_i \dot{\underline{r}}_i$

Galilean invariance:

$\underline{R}_0 = \underline{R}(t) - \dot{\underline{r}} t$

$\Rightarrow \Rightarrow \underline{R}_0 = \underline{R}(t) - \frac{\underline{P}}{M} t$

$\frac{d\underline{R}_0}{dt} = \frac{1}{M} \left(\sum_i m_i \dot{\underline{r}}_i - \underline{P} \right) = 0$

$\Rightarrow$ free motion of centre of mass

conservation of initial centre of mass (relating to Galilean invariance)

Symmetries & conservation laws

Symmetry	Conservation Laws	Components (dimensions ??)
time	Energy $E = \sum_i \dot{\underline{r}}_i \frac{\partial L}{\partial \dot{\underline{r}}_i} - L$	1
homogeneous space [spatial translations]	momentum $\underline{P} = \sum_i \underline{p}_i = \sum_i \frac{\partial L}{\partial \dot{\underline{r}}_i}$	3
isotropic space (rotations)	angular momentum $\underline{L} = \sum_i \underline{L}_i = \sum_i \underline{r}_i \times \underline{p}_i = \sum_i \underline{r}_i \times \frac{\partial L}{\partial \dot{\underline{r}}_i}$	3
Galilean invariance	centre of mass $\underline{R}_0 = \underline{R} - \frac{\underline{P}}{M} t$, $\underline{R} = \frac{1}{M} \sum_i m_i \underline{r}_i$	3

10 fundamental mechanical quantities

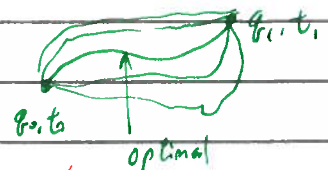
=> 10 fundamental spacetime symmetries

Canonical Formalism

Hamilton-Jacobi Theory

- Consider action S as a function of $q(q_1, \dots, q_n)$ & time t of the end point

Principle of Least Action



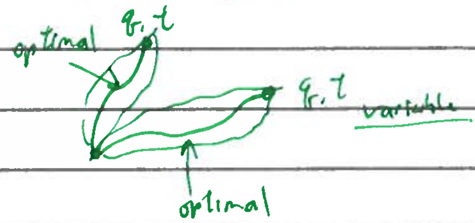
$$S = \int_{t_0}^{t_1} L(q(t), \dot{q}(t), t)\, dt$$

Boundary conditions

$q(t_0) = q_0$, $q(t_1) = q_1$

$\delta q(t_0) = \delta q(t_1) = 0$

Action $S(q, t)$



each trajectory w/ ~~minimum~~

w/ optimal S

$$S(q, t) = \int_{t_0}^{t} dt'\, L(q(t'), \dot{q}(t'), t')$$

Boundary conditions

~~$q(t_0) = q_0$~~

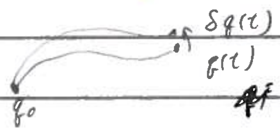
$q(t_0) = q_0$ $\quad q(t) = q$

~~$q(t_1) = q_1 +$~~ $\quad q - q_0 = \delta q$

$q = \cancel{\delta q +} q_0 + \delta q$

- consider infinitesimal of the end q:

$$\delta S = \frac{\partial S}{\partial q} \delta q(t)$$



We can also work with:

$$\delta S = \int_{t_0}^{t_1} dt' \left(\frac{\partial L}{\partial q} \delta q(t') + \frac{\partial L}{\partial \dot{q}} \delta \dot{q}(t') \right) = \int_{t_0}^{t} dt' \frac{\partial L}{\partial q} \delta q(t') + \int_{t_0}^{t} \frac{\partial L}{\partial \dot{q}} \frac{d}{dt} \delta q(t')\, dt'$$

$\Rightarrow$ integrate by parts.

$u = \frac{\partial L}{\partial \dot{q}}$ $\quad v' = \frac{d}{dt} \delta q(t')$

$u' = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}$ $\quad v = \delta q(t')$

$$= \int_{t_0}^{t} dt' \frac{\partial L}{\partial q} \delta q(t') + \left(\frac{\partial L}{\partial \dot{q}} \delta q(t') \Big|_{t_0}^{t} \right) - \int_{t_0}^{t} \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} \delta q(t')\, dt'$$

$$= \int_{t_0}^{t} dt' \underbrace{\left(\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} \right)}_{=0 \ \because \text{ only choose optimal trajectories}} \delta q(t') + \left[\frac{\partial L}{\partial \dot{q}} \delta q(t') \Big|_{t_0}^{t} \right]$$

different boundary conditions

$$\delta S = \frac{\partial L}{\partial \dot{q}} \delta q(t)$$

Examples for canonical momentum:

From BY

$$\delta S = \frac{\partial L}{\partial \dot{q}} \delta q(t)$$

$$\frac{\delta S}{\delta q(t)} = \underbrace{\frac{\partial L}{\partial \dot{q}} = p}_{\text{canonical momentum}}$$

a)
$$L = \frac{m}{2}\dot{\underline{r}}^2 - V(\underline{r}) \qquad \left(L = \frac{m}{2}\sum \dot{x}_i^2 - V(\underline{r}) \quad p_i = \frac{\partial L}{\partial \dot{x}_i} = m\dot{x}_i \right)$$

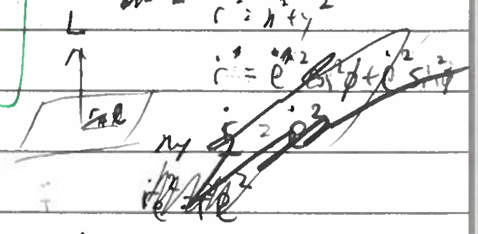
$$\underline{p} = \frac{\partial L}{\partial \dot{\underline{r}}} = m\dot{\underline{r}}$$

b) Central force problems in polar coordinates:

$$L = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - U(r)$$

$$x' = e'\cos\phi \qquad r = r(x, y)$$
$$y' = e'\sin\phi \qquad r' = r'(x', y')$$
$$r^2 = x^2 + y^2$$

$$\boxed{p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r} \qquad p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} = L_z}$$



$$\dot{x}_i = r\omega = r\dot{\theta}$$

∅ from $S(q,t) = \int_{t_0}^{t} dt' L(q(t'), \dot{q}(t'), t')$ we obtain

by differentiality:

$$L = \frac{dS}{dt} = \frac{\partial S}{\partial q}\dot{q} + \frac{\partial S}{\partial t} = p\dot{q} + \frac{\partial S}{\partial t}$$

- Introducing Hamiltonian (energy)

$$H = p\dot{q} - L = -\frac{\partial S}{\partial t}$$

transformation from $L = L(q, \dot{q}, t)$ to $H = p\dot{q} - L = H(p, q, t)$

$$\frac{\partial S}{\partial t} = -H$$

$\Rightarrow$ Legendre Transformation

$$H = H(p, \dot{q}, t)$$

- writing $H = H(p, q, t)$ [eliminate $\dot{q}$ by using $p = \frac{\partial L}{\partial \dot{q}}$]
+ [using $p = \frac{\partial S}{\partial q}$]

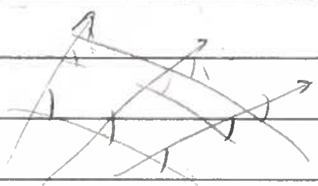
$\Rightarrow$ obtain a partial differential eq^n for S:

$$\boxed{\frac{\partial S}{\partial t} + H\left(\underline{q}, \frac{\partial S}{\partial \underline{q}}, t\right) = 0}$$

Hamiltonian - Jacobi Eq^n

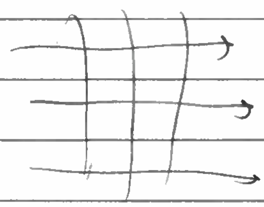
- Action waves:

$$\frac{\partial S}{\partial q} = p$$
$$= \nabla_q S$$



contour lines of S "wave fronts" $\Rightarrow$

We can go to infinity



beam of trajectories

one solution of the Hamiltonian-Jacobi Eq^n describes an entire family of trajectories & give an insight in the general behaviour.

Establishing connections to optics & quant mech.

- Example 1: one dimensional motion

$L = \frac{m}{2}\dot{x}^2 - U(x)$ (Lagrange)

$\cancel{p = \frac{\partial S}{\partial x}}$ $p = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$ (canonical momentum)

$H(x,p) = p\dot{x} - L$ (Hamiltonian)

$= p\frac{p}{m} - \left(\frac{m}{2}\left(\frac{p}{m}\right)^2 - U(x)\right)$

$= \frac{p^2}{2m} + U(x)$ - energy

$= \frac{1}{2m}\left(\frac{\partial S}{\partial x}\right)^2 + U(x)$ - energy as well

~~Hamiltonian~~ Hamilton-Jacobi Eq^n:

$\frac{\partial S}{\partial t} + H = 0$

$\frac{\partial S}{\partial t} + \frac{1}{2m}\left(\frac{\partial S}{\partial x}\right)^2 + U(x) = 0$

(U is not time-dependant => energy conserved)

$S(x,t) = W(x) - Et$ $W(x) = W_E(x)$ characteristic func => defining a probability distribution

$\frac{1}{2m}\left(\frac{dW}{dx}\right)^2 + U(x) = E$

$W(x) = \pm\int dx \sqrt{2m(E - U(x))}$

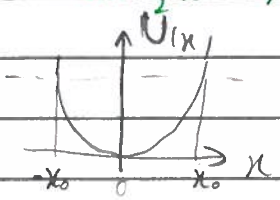
only allowed in the physically allowed region $\{x \mid U(x) \le E\}$

- Consider the harmonic oscillator:

$U(x) = \frac{k}{2}x^2$, $\omega = \sqrt{\frac{k}{m}}$

$\cos 2\theta = 2\cos^2\theta - 1$

$\frac{\cos 2\theta + 1}{2} = \cos^2\theta$



$E = \frac{k}{2}x_0^2$

$x_0 = \sqrt{\frac{2E}{k}}$

$W(x) = \pm\int dx \sqrt{2mE - mkx^2}$

$= \pm\sqrt{m}\int dx\sqrt{2E - kx^2} = \pm\sqrt{m}\sqrt{\frac{2E}{k}}\int\sqrt{2E - 2Ekx^2 / 2E}\, dz$

$z = \sqrt{\frac{k}{2E}}\,x$ $= \pm\frac{2E}{\omega}\int\sqrt{1 - z^2}\, dz = \pm\frac{2E}{\omega}\int \cos^2\theta \, d\theta$

$dz = \sqrt{\frac{k}{2E}}\, dx$ sub $z = \sin\theta$

$dz = \cos\theta\, d\theta$

$dx = \sqrt{\frac{2E}{k}}\, dz$ $= \pm\frac{E}{\omega}\int \cos 2\theta + 1 \, d\theta = \pm\frac{E}{\omega}\left(\frac{\sin 2\theta}{2} + \theta\right)$ $|z| \le 1$

$W(x) = \pm\frac{E}{\omega}\sin\theta\cos\theta + \theta = \pm\frac{E}{\omega}\left(z\sqrt{1 - z^2} + \arcsin z\right)$

Only for $|z| \le 1$ or $\frac{|x|}{x_0} \le 1$, $W(x)$ is $\boxed{\text{real}}$

complex $W(x)$ correspond to evanescent quantum waves.

7 Central force problem in polar coordinate:

$$L = \frac{m}{2}(\dot r^2 + r^2\dot\theta^2) - U(r)$$

canonical ~~moment~~ momentum: $p_r = m\dot r$

$p_\theta = mr^2\dot\theta = L_z$

Hamiltonian: $H = p\dot q - L = \sum_i p_i \dot q_i - L$

$$= p_r \dot r + p_\theta \dot\theta - L$$

$$= \frac{2p_r^2}{2m} + \frac{p_\theta^2}{mr^2} - \frac{m}{2}\frac{p_r^2}{m^2} - \frac{m}{2}r^2\left(\frac{p_\theta}{mr^2}\right)^2 + U$$

$$H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{mr^2} - \frac{m p_\theta^2}{2r^2 m} + U = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + U(r)$$

using the Hamilton-Jacobi eq^n (using $p_i = \frac{\partial S}{\partial q_i}$)

$$\hookrightarrow \quad \frac{\partial S}{\partial t} + \frac{1}{2m}\left[\left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial S}{\partial \phi}\right)^2\right] + U(r) = 0$$

$$\frac{1}{2m}\left[\left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial S}{\partial \theta}\right)^2\right] + U(r) = -\frac{\partial S}{\partial t}$$

$\rightsquigarrow$ partial diff eq^{ns} for $S(r, \theta, t)$

- separate time (as B4)

$S(r,\phi,t) = W(r,\theta) - Et$ $\Rightarrow$ $\frac{1}{2m}\left[\left(\frac{\partial W}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial W}{\partial \phi}\right)^2\right] + U(r) = E$

since $U(r)$ is independent of ϕ, the ϕ dependence of $W(r,\phi)$ is trivial

$\hookrightarrow$ we can use that $p_\phi = \frac{\partial S}{\partial \phi} = L_z = \text{const}$

$\frac{\partial W}{\partial \phi}$

$\hookrightarrow \Rightarrow W(r,\phi) = W_1(r) + L_z\phi$

$$U_{eff} = E - \frac{1}{2m}\left(\frac{dW_1}{dr}\right)^2$$

$$= U(r) + \frac{1}{2m}\left[\left(\frac{\partial W}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial W}{\partial \phi}\right)^2\right] - \frac{1}{2m}\left(\frac{\partial W}{\partial r}\right)^2$$

$$= U(r) + \frac{1}{2mr^2}\left(\frac{\partial W}{\partial \phi}\right)^2$$

$$= U(r) + \frac{p_\theta^2}{2mr^2} = U(r) + \frac{L_z^2}{2mr^2}$$

$$\boxed{U_{eff} = U(r) + \frac{L_z^2}{2mr^2}}$$

remaining diff eq^n of the problem:

$$\Rightarrow \frac{1}{2m}\left(\frac{dW_1}{dr}\right)^2 + U_{eff}(r) = E$$

Formal solution: $W_1(r) = \pm\int dr \sqrt{2m(E - U_{eff}(r))}$

~~$\frac{1}{2mr^2}\left(\frac{\partial W}{\partial \theta}\right)^2 = L_z\theta$~~

~~$L_z = mr^2\dot\theta$~~

~~$W_2(\theta) = L_z\theta$~~

~~$\frac{dW_2}{d\theta} = \frac{1}{\dot\theta}mr^2\dot\theta\dot\theta = mr^2\dot\theta = L_z$~~

~~$p_\theta = \frac{\partial S}{\partial \theta} = L_z$~~

$\frac{dF}{dr} = -\frac{\alpha}{r}$ $\quad \alpha = G(m_1+m_2)m = \mu m$ $\quad$??

example of this problem: Kepler problem, where $U(r) = -\frac{\alpha}{r}$, $\alpha > 0$

$$W_1(r) = \pm\int dr \sqrt{2m\left(E + \frac{\alpha}{r} - \frac{L_z^2}{2mr^2}\right)} = \pm \int dr \sqrt{\;}$$

$$\left(= \pm \int dr \sqrt{2m(E - U_{(x)})} \right)$$

make it into the form:

$$f(\rho) = \sqrt{a^2 - (g(\rho))^2}$$

$$U_{eff}(r) = U(r) + \frac{L_z^2}{2mr^2}$$

$$U_{eff}(r) = -\frac{\alpha}{r} + \frac{L_z^2}{2mr^2}$$

$r = p\rho$, $p = \frac{L_z^2}{\alpha m}$, $r = \frac{L_z^2}{\alpha m}\rho$

$x = \rho\cos\phi$, $y = \rho\sin\phi$, $r = r\hat{e}_x + r\hat{e}_y$, $r(x,y) \to r(\rho, \phi)$

$U(x) = \frac{dF}{dr} = -\frac{\alpha}{r}$ $\quad L_z = r\times(mv)$ $\quad p_\phi = \frac{\partial S}{\partial \phi} = L_z = \text{constant}$

$U(x) = -\frac{\alpha}{r}$ $\quad L_z = \vec{r}\times\vec{p}$ $\quad p = \frac{m^2 r^4 \dot{\rho}^2}{\alpha m}$

$r = r(x,y,z)$ $\quad mr^2\dot{\rho} = \vec{r}\times\vec{p}$ $\quad \alpha m p = L_z^2$ $\quad p = \frac{m r^4 \dot{\rho}^2}{\alpha}$

$\alpha m p = m^2 r^4 \dot{\rho}^2$

$p_\rho = mr^2\dot{\rho}$ $\quad \alpha r p = \frac{m r^2 \dot{\phi}^2}{\alpha m}$ $\quad \alpha p = r^4 \dot{\rho}^2$

$p_r = m\dot{r}$ $\quad \alpha r = \rho^2\dot{\phi}^2$

$p = p_r + p_\rho$ $\quad U(x)\cdot r = \rho^2\dot{\phi}^2$

$p = m\dot{r} + mr^2\dot{\phi}$ $\quad U(x).$ $\quad dr = p\,d\rho$ $\quad \rho = \frac{r}{p}$, $p = \frac{L_z^2}{\alpha m}$

if substitution comes first $\quad W(r) \to W(\rho)$ $\quad d\rho = \frac{1}{p}dr$, $d\rho = \frac{\alpha m}{L_z^2}dr$, $dr = \frac{L_z^2}{\alpha m}d\rho$

$\alpha = \mu m = GMm = G(m_1+m_2)m$, $p = \frac{L_z^2}{\alpha m}$

$$W_1(r) = \pm\int dr \sqrt{2m\left(E + \frac{\alpha}{r} - \frac{L_z^2}{2mr^2}\right)} = \pm \int dr \sqrt{2mE + \frac{2m\alpha}{r} - \frac{L_z^2}{2mr^2}}$$

$$\overset{\text{sub}}{\Rightarrow} = \pm\int dr\sqrt{2mE + \frac{2m\alpha}{p\rho} - \frac{L_z^2}{2m(p\rho)^2}} = \pm\int \frac{L_z^2}{\alpha m}\sqrt{2mE + \frac{2m\alpha(\alpha m)}{L_z^2\rho} - \frac{L_z^2(\alpha m)^2}{2mL_z^4\rho^2}}\,d\rho$$

$$= \pm\int \frac{L_z^2}{\alpha m}\sqrt{2mE + \frac{2(\alpha m)^2}{L_z^2\rho} - \frac{(\alpha m)^2}{2mL_z^2\rho^2}}\,d\rho = \pm\int\frac{L_z^2}{\alpha m}\sqrt{2mE + \frac{(\alpha m)^2}{L_z^2}\left(\frac{2}{\rho} - \frac{1}{2m\rho^2}\right)}\,d\rho$$

$$= \pm\int\frac{L_z^2}{\alpha m}\sqrt{2mE + \frac{(\alpha m)^2}{L_z^2}\left(-\left[\frac{1}{\rho^2} - \frac{2}{\rho} + 1 - 1\right]\right)}\,d\rho = \pm\int\frac{L_z^2}{\alpha m}\sqrt{2mE - \frac{(\alpha m)^2}{L_z^2}\left[\left(\frac{1}{\rho}-1\right)^2 - 1\right]}\,d\rho$$

$$= \pm\int\sqrt{\frac{2mEL_z^4}{\alpha^2 m^2} - \frac{(L_z)^2(\alpha m)^2}{(\alpha m)^2 L_z^2}\left[\left(\frac{1}{\rho}-1\right)^2 - 1\right]}\,d\rho$$

$$= \pm\int\sqrt{\frac{2EL_z^4}{\alpha^2 m} - L_z^2\left(\frac{1}{\rho}-1\right)^2 + L_z^2}\,d\rho = \pm L_z\int\sqrt{\frac{2EL_z^2}{\alpha^2 m} - \left(\frac{1}{\rho}-1\right)^2 + 1}\,d\rho$$

$$= \pm L_z\int\sqrt{1 + \frac{2EL_z^2}{\alpha^2 m} - \left(\frac{1}{\rho}-1\right)^2}\,d\rho = \pm L_z\int\sqrt{\epsilon^2 - \left(\frac{1}{\rho}-1\right)^2}\,d\rho = \pm L_z\int f(\rho)\,d\rho$$

where $\epsilon = \sqrt{1 + \frac{2EL_z^2}{\alpha^2 m}}$ $\quad f(\rho) = \sqrt{\epsilon^2 - \left(\frac{1}{\rho}-1\right)^2}$ which is in the form:

$$f(\rho) = \sqrt{\epsilon^2 - \left(\tfrac{1}{\rho}-1\right)^2}$$

$$\epsilon = \sqrt{1 + \frac{2E L_z^2}{\alpha^2 m}}$$

$$r = p\rho \qquad p = \frac{L_z^2}{\alpha m}$$

$$W_1(r) = W_1(\rho) = \pm L_z \int \sqrt{\epsilon^2 - \left(\tfrac{1}{\rho}-1\right)^2}\, d\rho$$

From the harmonic oscillator example:

$$W(x) = \pm \int dx \sqrt{2mE - mkx^2}$$

where $\xi = \left(\sqrt{\frac{k}{2E}}\right)x$

$$= \pm \frac{E}{w} \sin\theta\cos\theta + \theta = \pm \frac{E}{w}\left(\xi\sqrt{1-\xi^2} + \arcsin \xi\right)$$

Performing the integral:

$1 + \tan^2\theta = \sec^2\theta$

$\tan^2\theta = \sec^2\theta - 1$

$$W_1(\rho) = \pm L_z \int f(\rho)\, d\rho = \pm L_z \int \sqrt{\epsilon^2 - \left(\tfrac{1}{\rho}-1\right)^2}\, d\rho = \pm \int L_z \sqrt{\frac{\epsilon^2(\rho^{-1}-1)^2}{(\rho^{-1}-1)^2} - (\rho^{-1}-1)^2}\, d\rho$$

$$= \pm L_z \int (\rho^{-1}-1)\sqrt{\frac{\epsilon^2}{(\rho^{-1}-1)^2} - 1}\quad d\rho$$

try $u = \rho^{-1} - 1$

$\frac{du}{d\rho} = -\rho^{-2}$

$d\rho = du(-\rho^{2})$

$u + 1 = \rho^{-1}$

$(\rho)^2 = \left(\frac{1}{u+1}\right)^2$

$d\rho = \left(\frac{1}{1-u}\right)^2 du$

Wolfram alpha it

$$W_1(r) = \pm L_z \left[\rho f(\rho) + \arctan\left(\frac{\rho^{-1}-1}{f(\rho)}\right) - \frac{1}{\sqrt{1-\epsilon^2}} \arctan\left(\frac{\rho^{-1} - 1 + \epsilon^2}{f(\rho)\sqrt{1-\epsilon^2}}\right)\right.$$

for fun solve above ??

Roadmap for the Hamilton-Jacobi Theory:

Step 1: Establish Lagrangian of the system

$L = L(q, \dot{q}, t)$ where $q = (q_1, \cdots, q_n)$, $\dot{q} = (\dot{q}_1, \cdots, \dot{q}_n)$

* pick coordinates q that reflect the symmetries of the system

Step 2: Calculate Canonical Momentum:

$$p = \frac{\partial L}{\partial \dot{q}} = (p_1, \cdots, p_n) \qquad \left(p_i = \frac{\partial L}{\partial \dot{q}_i} \ (i = 1, \cdots, n)\right)$$

* if $\frac{\partial L}{\partial q_i} = 0$: p_i is conserved quantity! [follows from Euler-Lagrange eq^n]

Step 3: Obtain the Hamilton by Legendre Transformation

$$H = p\dot{q} - L = \sum_i p_i \dot{q}_i - L(q, \dot{q}, t) = H(q, p, t)$$

↑ express $\dot{q}_i$'s by coordinate q_i & momentum p_i using $p = \frac{\partial L}{\partial \dot{q}}$

Step 4: Hamilton-Jacobi ~~eq^n~~ diff eq^n for $S(q, t)$

$$\frac{\partial S}{\partial t} + H\left(q, \frac{\partial S}{\partial q}, t\right) = 0$$

$$\Rightarrow \frac{\partial S}{\partial t} + H\left(q_1, \cdots, q_n, \frac{\partial S}{\partial q_1}, \cdots, \frac{\partial S}{\partial q_n}, t\right) = 0$$

diff eq^n is: - first order
- partial
- non-linear (H contain p^2 terms $\left(\frac{\partial S}{\partial q}\right)^2$)

Step 5: Solve the Hamilton-Jacobi eq^n

↳ solution gives you $S = S(q_1, \cdots, q_n, t, \underbrace{\alpha_1, \cdots, \alpha_n}_{\text{integration constants}}) + \alpha_0$ ← trivial const. that can be dropped

Hint To get the ~~[illegible]~~ eq^n

- if L & H do not explicitly depend on time, Energy is conserved: $\alpha_1 = E$

$S(q, t) = -Et + W(q) = -Et + \underbrace{W(q_1, \cdots, q_n, \alpha_1 = E)}_{\text{characteristic func.}}$

If the problem is separable, there exists a set of coordinate $q = (q_1, \cdots, q_n)$ that $W(q_1, \cdots, q_n) = W_1(q_1) + \cdots + W_n(q_n)$

↳ n ordinary differential eq^n to solve much easier to solve

Step 6: Obtain dynamics $(q(t), p(t))$

from $\beta_i = \frac{\partial S}{\partial \alpha_i}$ ~~$(i = 1, \cdots, n)$~~ $(i = 1, \cdots, n)$ & $p_i = \frac{\partial S}{\partial q_i}$ $(i = 1, \cdots, n)$

Back to the central force problem!

After 1, 2, 3, 4 $\Rightarrow$ Hamilton-Jacobi eq^n for ~~$S(q,t)$~~ $S(r,\phi,t)$

$$\frac{\partial S}{\partial t}+\frac{1}{2m}\left(\left(\frac{\partial S}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial S}{\partial \phi}\right)^2\right)+U(r)=0$$

when does it NOT depend on time

$\Rightarrow$ Step 5: Hamilton & Lagrange do not explicitly depend on time $\Rightarrow \boxed{\alpha_1 = E}$ conserved.

$$\Rightarrow S(r,\phi,t) = -\underbrace{E}_{\text{Constant}}t + W(r,\phi,E) \quad \Rightarrow \quad \frac{1}{2m}\left[\left(\frac{\partial W}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial W}{\partial \phi}\right)^2\right]+U(r)=0$$

separation ansatz: $W(r,\phi)=W_1(r)+W_2(\phi)$

$$\frac{1}{2m}\left(\frac{dW_1}{dr}\right)^2+\frac{1}{2mr^2}\left(\frac{dW_2}{d\phi}\right)^2+U(r)=E$$

$$\frac{1}{2mr^2}\left(\frac{dW_2}{d\phi}\right)^2=E-U(r)-\frac{1}{2m}\left(\frac{dW_1}{dr}\right)^2$$

$$\left(\frac{dW_2}{d\phi}\right)^2=2mr^2(E-U(r))-r^2\left(\frac{dW_1}{dr}\right)^2$$

L.H.S depend on ϕ, R.H.S only on r

$\Rightarrow$ both sides equal to a constant $\left[\alpha_2^2=\left(\frac{dW_2}{d\phi}\right)^2=\left(\frac{\partial S}{\partial \phi}\right)^2=p_\phi^2=L_z^2\right]$

$\hookrightarrow \frac{dW_2}{d\phi}=L_z$

$\Rightarrow W_2(\phi)=L_z\phi$

$$\Rightarrow \frac{L_z^2}{2mr^2}=E-U(r)-\frac{1}{2m}\left(\frac{dW_1}{dr}\right)^2$$

~~$\Rightarrow \frac{dW_1(r)}{dr}=\pm\sqrt{2m}\sqrt{E-U_{eff}}$~~

$$\Rightarrow \frac{L_z^2}{2mr^2}+U(r)=E-\frac{1}{2m}\left(\frac{dW_1}{dr}\right)^2$$

$$U_{eff}(r)=E-\frac{1}{2m}\left(\frac{dW_1}{dr}\right)^2$$

$$\frac{dW_1}{dr}=\pm\sqrt{2m}\sqrt{E-U_{eff}(r)}$$

$$W_1(r)=\pm\sqrt{2m}\int dr\sqrt{E-U_{eff}(r)}$$

$$S(r,\phi,t,\alpha_1=E,\alpha_2=L_z)=-Et+L_z\phi\pm\sqrt{2m}\int dr\sqrt{E-U(r)-\frac{L_z^2}{2mr^2}}$$

$$S = -Et + L_z\phi \pm \sqrt{2m}\int dr \sqrt{E - U(r) - \frac{L_z^2}{2mr^2}}$$

Step 6: Obtaining the dynamics

$$\beta_1 = \frac{\partial S}{\partial \alpha_1} = \frac{\partial S}{\partial E} = -t + \sqrt{\frac{m}{2}}\int \frac{dr}{\sqrt{E - U_{eff}(r)}}$$

$\Leftrightarrow$ $\beta_1 = t_0$

$$t - t_0 = \pm\sqrt{\frac{m}{2}}\int \frac{dr}{\sqrt{E - U_{eff}(r)}}$$

$$\beta_2 = \frac{\partial S}{\partial \alpha_2} = \frac{\partial S}{\partial L_z} = \phi \pm \sqrt{\frac{L_z^2}{2m}}\int \frac{dr}{r^2\sqrt{E - U_{eff}(r)}}$$

consistent w/ previous results.

$\Leftrightarrow$ $\beta_2 = \phi_0$

$$\phi - \phi_0 = \pm\frac{L_z}{\sqrt{2m}}\int \frac{dr}{r^2\sqrt{E - U_{eff}(r)}}$$

Multivalued function (jump @ $\phi = \pi$)

$$\Rightarrow \Delta W = \oint dW = \oint \left(\frac{\partial W}{\partial q}dq + \underbrace{\frac{\partial W}{\partial t}}_{=0}dt\right) = \oint p\,dq \neq 0$$

quantum mechanical wavefunction can be written in the semiclassical limit

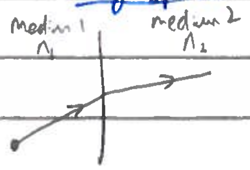
$$\psi = Ae^{iS/\hbar}$$

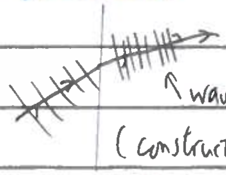
This must be single valued which ~~implied~~ implies:

$$\oint p\,dq = 2\pi\hbar \cdot n \Rightarrow n \in \mathbb{Z}^+$$

$\Rightarrow$ Bohr-Sommerfield quantization

For ray optics — wave optics





↑ wave front.
(constructed from Huygen's principle)

The light wave ~~may~~ may be written in the form:

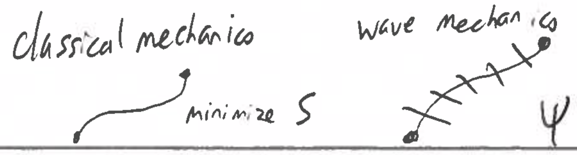
$u = a\cdot e^{i\phi}$, $\phi = \int(\underline{k}(r)d\underline{r} - \omega dt)$ $\Rightarrow$ action

for plane waves:
$\underline{k}, \omega$ constant
$\phi = \underline{k}\cdot\underline{r} - \omega t$

dispersion relation connects k & ω: $k = n(r)\frac{\omega}{c}$

Fermat's principle: Find optimum path ~~so~~ ~~ther~~ that $\delta\phi = 0$

- The semiclassical approximation



taking the hint from what we have just seen for optics

$\Rightarrow$ we anticipate that the semiclassical wavefunction can be written in the form:

$$\psi = A \cdot e^{iS/\hbar} \quad \text{]- semiclassical approximation}$$

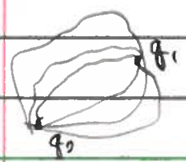
classical properties of S can be used to deduce a wave eq^n (P.D.E.) for ψ.

$\Rightarrow$ Schrödinger eq^n

- Broader context: Feynman path integral

~~Feynman~~ Feynman approached QM closer to ~~QA~~ classical mech, L & H

Wavefunction can be written as a sum over All Possible Paths from $q_0 \to q_1$, each weighted by a complex phase $e^{iS/\hbar}$



$$\psi(q) = \int \!\! \propto q\, e^{iS/\hbar}$$ "Superposition of waves"

summation over all paths from q_0 to q_1

in the classical limit $\Rightarrow$ only paths with constructive interference will contribute significantly, eg. only paths for which S is the same & optimal, $\delta S = 0$

but $\delta S = 0$ is just the least action principle

$$\Rightarrow \quad \psi = \int \!\! \propto q\, e^{iS/\hbar} \simeq A e^{iS/\hbar} \quad \text{w/ } \delta S = 0$$ for the classical path

- Deducing a wave equation for ψ: The Schrödinger eq^n

Principles ~~must~~ that the diff eq^n must satisfy

1. Superposition (ψ_1, ψ_2 sol^n $\Rightarrow$ $\psi_1 + \psi_2$ solution)

$\Rightarrow$ diff eq^n must be linear

2. ψ describes state (diff eq^n evolution in time)

$\Rightarrow$ 1st order in time

We can always write such an eq^n in the form:

$$i\hbar \frac{\partial \psi}{\partial t} = \hat{A}\,\psi$$

where does this come from

some linear operator whose form we must deduce

We use the ansatz $\psi = A e^{iS/\hbar}$ (semiclassical approximation)

+ Assume that A is not changing in time

$$\psi = A \cdot e^{iS/\hbar}$$

$$i\hbar \frac{\partial \psi}{\partial t} = i\hbar \frac{i}{\hbar}\frac{\partial S}{\partial t}\psi = -\frac{\partial S}{\partial t}\psi \underset{\substack{\text{Hamilton} \\ \text{Jacobi}}}{=} \frac{1}{2m}(\nabla S)^2 \psi + U\psi$$

$(\nabla S)^2$ can be related to $\nabla^2 \psi$:

$$\nabla^2 \psi = \nabla^2 \left(Ae^{iS/\hbar}\right) = \nabla\left((\nabla A)e^{iS/\hbar} + A \cdot \nabla e^{iS/\hbar}\right)$$

$$= (\nabla^2 A)e^{iS/\hbar} + 2(\nabla A)(\nabla e^{iS/\hbar}) + A \nabla^2 e^{iS/\hbar}$$

$$= (\nabla^2 A)e^{iS/\hbar} + 2(\nabla A)\cdot(\nabla S)e^{iS/\hbar} + A \cdot \nabla(\nabla e^{iS/\hbar})$$

$$= (\nabla^2 A)e^{iS/\hbar} + 2\frac{i}{\hbar}(\nabla A)\cdot(\nabla S)e^{iS/\hbar} + A\frac{i}{\hbar}\nabla\left((\nabla S)e^{iS/\hbar}\right)$$

$$= (\nabla^2 A)e^{iS/\hbar} + 2\frac{i}{\hbar}(\nabla A)\cdot(\nabla S)e^{iS/\hbar} + A\cdot\frac{i}{\hbar}\left[\nabla^2 S e^{iS/\hbar} + \nabla S \cdot \nabla e^{iS/\hbar}\right]$$

$$= (\nabla^2 A)e^{iS/\hbar} + 2\frac{i}{\hbar}(\nabla A)\cdot(\nabla S)e^{iS/\hbar} + A\frac{i}{\hbar}\left[\nabla^2 S e^{iS/\hbar} + \nabla S \cdot \nabla S \frac{i}{\hbar} e^{iS/\hbar}\right]$$

neglect derivatives of A and 2nd derivative of S

$$= \cancel{(\nabla^2 A)e^{iS/\hbar}} + \cancel{2\frac{i}{\hbar}(\nabla A)\cdot(\nabla S)e^{iS/\hbar}} + \cancel{\frac{i}{\hbar}A(\nabla^2 S)e^{iS/\hbar}} - \frac{1}{\hbar^2}A(\nabla S)^2 e^{iS/\hbar}$$

$$\simeq \underline{-\frac{1}{\hbar^2}(\nabla S)^2\psi} \simeq \nabla^2\psi$$

$$i\hbar\frac{\partial \psi}{\partial t} = \frac{1}{2m}(\nabla S)^2\psi + U\psi \qquad \nabla^2\psi = -\frac{1}{\hbar^2}(\nabla S)^2\psi$$

$$i\hbar\frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + U\psi \qquad -\hbar^2\nabla^2\psi = (\nabla S)^2\psi$$

$$i\hbar\left(\frac{\partial \psi}{\partial t}\right) = \left(-\frac{\hbar^2}{2m}\nabla^2 + U\right)\psi$$

$$\hat{H} = -\frac{\hbar^2}{2m}\nabla^2 + U \text{ — Hamilton operator}$$

⇒ next step: deduce how the probability density $|A|^2$ behaves

Assuming $|A|^2$ or A is real, w/o the loss of generality

Reducing the behaviour of probability density of $|A|^2$

We know:

$$i\hbar\left(\frac{\partial\psi}{\partial t}\right) = \left(-\frac{\hbar^2}{2m}\nabla^2 + U\right)\psi$$

$$\psi = Ae^{iS/\hbar}$$

Hamilton-Jacobi:

$$\frac{\partial S}{\partial t} + H\left(q, \frac{\partial S}{\partial q}, t\right) = 0$$

$$\frac{\partial S}{\partial q} = p$$

$$\frac{\partial S}{\partial t} = -H(q,p,t)$$

$$= -\left(\frac{p^2}{2m} + U(q,r)\right)$$

$$① = -\left[\frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2 + U(q,r)\right]$$

$$= -\left(\frac{1}{2m}(\nabla S)^2 + U(q,r)\right)$$

$$i\hbar\left(\frac{\partial\psi}{\partial t}\right) = \left(i\hbar\frac{\partial A}{\partial t} - A\frac{\partial S}{\partial t}\right)e^{iS/\hbar}$$

$$\left[\text{Hamilton-Jacobi}\right] = \left[\left(i\hbar\frac{\partial A}{\partial t}\right) + \frac{A}{2m}(\nabla S)^2 + AU\right]e^{iS/\hbar}$$

$$\|$$

$$-\frac{\hbar^2}{2m}\nabla^2\psi + U\psi = -\frac{\hbar^2}{2m}\Big(\underbrace{\nabla^2 A}_{\text{ignore}} + 2\tfrac{i}{\hbar}(\nabla A)(\nabla S) - \tfrac{1}{\hbar^2}A(\nabla S)^2 + \underbrace{\tfrac{i}{\hbar}A(\nabla^2 S)}_{\text{ignore}}\Big)e^{iS/\hbar} + AUe^{iS/\hbar}$$

$$= -\frac{\hbar^2}{2m}\left(2\frac{i}{\hbar}(\nabla A)(\nabla S) - \frac{1}{\hbar^2}A(\nabla S)^2\right)e^{iS/\hbar} + AUe^{iS/\hbar}$$

$$= \left[-i\frac{\hbar}{m}(\nabla A)(\nabla S) + \frac{1}{2m}A(\nabla S)^2\right]e^{iS/\hbar} + AUe^{iS/\hbar}$$

comparing w/ ①

$$= \underbrace{-i\frac{\hbar}{m}(\nabla A)(\nabla S)e^{iS/\hbar}}_{\left(i\hbar\frac{\partial A}{\partial t}\right)\psi} + \left[\frac{1}{2m}(\nabla S)^2 + U\right]\underbrace{Ae^{iS/\hbar}}_{\psi}$$

$$\Leftrightarrow \quad i\hbar\frac{\partial A}{\partial t} = -i\hbar\frac{1}{m}(\nabla A)\underbrace{(\nabla S)}_{p}$$

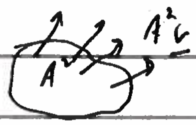
$$\frac{\partial A}{\partial t} = -\frac{p}{m}(\nabla A)$$

$$\frac{\partial A}{\partial t} = -\underline{v}(\nabla A) = -\frac{1}{2A}\nabla(A^2\underline{v})$$

$$(2A)\frac{\partial A}{\partial t} = -\nabla(A^2\underline{v})$$

describing a flow of probability

$$\frac{\partial}{\partial t}\underbrace{(A^2)}_{\text{probability density}} = -\nabla\underbrace{(A^2\underline{v})}_{\text{probability current}} \quad \text{continuity eq}^n$$



$$\nabla(A^2\underline{v}) = (\nabla A^2)\underline{v} + (\nabla\underline{v})A^2$$

$$= \nabla(A\cdot A)\underline{v} + \nabla\underline{v}A^2$$

$$= \left[(\nabla A)A + A(\nabla A)\right]\underline{v} + \nabla\underline{v}A^2$$

$$= 2A(\nabla A)\underline{v} + \underbrace{(\nabla\underline{v})A^2}_{\text{ignore this}}$$

why??

$$\left(\frac{\partial}{\partial x}v_x \quad \frac{\partial}{\partial y}v_y \quad \frac{\partial}{\partial z}v_z\right) = 0$$

Hamilton's Eq^n or canonical eq^{n}s

⇒ concerned w/ determining trajectories in phase space

$(q,p) = (\underbrace{q_1, \dots, q_n}_{\text{configuration space}}, \underbrace{p_1, \dots, p_n}_{\text{momentum space}})$ 2n-dimensional space

– Canonical momentum : $p = \frac{\partial L}{\partial \dot{q}}$

$$H = \dot{q}p - L = \dot{q}(q,p,t)\cdot p - L(q, \dot{q}(q,p,t), t) = H(q,p,t)$$

– The Hamiltonian is a function of phase-space coordinates (q,p)

– Equation of motion: Hamilton's eq^{n}s: $\boxed{\frac{\partial H}{\partial p} = \dot{q} \;;\; \frac{\partial H}{\partial q} = -\dot{p}}$

$$\frac{\partial H}{\partial p} = \frac{\partial}{\partial p}\left(\dot{q}p - L(q, \dot{q}(q,p,t), t)\right) = \dot{q} + \frac{\partial \dot{q}}{\partial p} p - \underbrace{\frac{\partial L}{\partial \dot{q}}}_{p} \frac{\partial \dot{q}}{\partial p} = \dot{q}$$

$$\frac{\partial H}{\partial q} = \frac{\partial}{\partial q}\left(\dot{q}p - L(q, \dot{q}(q,p,t), t)\right) = \frac{\partial \dot{q}}{\partial q} p + \cancel{\frac{\partial p}{\partial q}} \dot{q} - \frac{\partial L}{\partial q} - \frac{\partial L}{\partial \dot{q}} \frac{\partial \dot{q}}{\partial q}$$

$$= \cancel{\frac{\partial \dot{q}}{\partial q}} p - \frac{\partial L}{\partial q} - p \cancel{\frac{\partial \dot{q}}{\partial q}} = -\frac{\partial L}{\partial q} \underset{E.L}{=} -\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = -\frac{d}{dt}p = -\dot{p}$$

$$\left[\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q}\right]$$

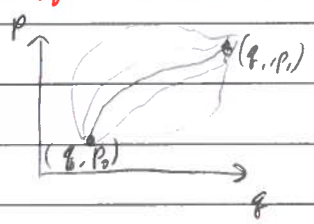
– Conservation of Energy:

$$\frac{\partial E}{\partial t} = \frac{\partial H}{\partial t} = \frac{\partial H}{\partial q}\frac{\partial q}{\partial t} + \frac{\partial H}{\partial p}\frac{\partial p}{\partial t} + \cancel{\frac{\partial H}{\partial t}\frac{\partial t}{\partial t}} = \frac{\partial H}{\partial q}\dot{q} + \frac{\partial H}{\partial p}\dot{p} + \frac{\partial H}{\partial t}$$

subbing Hamilton's eq^{n}s $= -\dot{p}\dot{q} + \dot{q}\dot{p} + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t}$

Energy is conserved ⇒ H does not depend explicitly on time

– Hamilton's Eq^n from principle of least Action



variation $(\delta q, \delta p)$
$\delta q(t_0) = \delta q(t_1) = 0$
$\delta p(t_0) = \delta p(t_1) = 0$

$$S = \int_{t_0}^{t_1} dt\, L(q,\dot{q},t) = \int_{t_0}^{t_1} dt\, (p\dot{q} - H(q,p,t))$$

$$\Rightarrow \delta S = \int_{t_0}^{t_1} dt \left[\underbrace{p\delta\dot{q}}_{\text{integrate by parts}} + \dot{q}\delta p - \left(\frac{\partial H}{\partial q}\delta q + \frac{\partial H}{\partial p}\delta p\right)\right] = \int_{t_0}^{t_1} dt\left[-\underbrace{\dot{p}}_{\frac{\partial H}{\partial q}}\delta q + \underbrace{\dot{q}}_{\frac{\partial H}{\partial p}}\delta p - \frac{\partial H}{\partial q}\delta q - \frac{\partial H}{\partial p}\delta p\right]$$

integrate by parts:
$u = p \quad v' = \delta\dot{q}$
$u' = \dot{p} \quad v = \delta q$
$\cancel{p\delta q\Big|_{t_0}^{t_1}} - \int_{t_0}^{t_1} \dot{p}\,\delta q$

$$= \int_{t_0}^{t_1} dt\left[\left(\frac{\partial H}{\partial p} - \frac{\partial H}{\partial p}\right)\delta p - \left(\frac{\partial H}{\partial q} + \frac{\partial H}{\partial q}\right)\delta q\right] = 0 \quad \forall\, \delta q, \delta p$$

Harmonic oscillator Eg.

$$H = \frac{p^2}{2m} + \frac{1}{2}kq^2 = \frac{p^2}{2m} + \frac{m\omega^2}{2}q^2 \qquad \omega = \sqrt{\frac{k}{m}}$$

$$\left.\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p} = \frac{p}{m} \\ \dot{p} &= -\frac{\partial H}{\partial q} = -m\omega^2 q \end{aligned}\right] \Rightarrow \frac{d}{dt}\begin{pmatrix} q(t) \\ p(t) \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{m} \\ -m\omega^2 & 0 \end{pmatrix}\begin{pmatrix} q(t) \\ p(t) \end{pmatrix}$$

$\Rightarrow$ 2 coupled 1^{st} order Diff eq^n

$\Rightarrow$ easily obtain decoupled 2nd order Diff eq^n for p & q:

$$\ddot{q} = \frac{1}{m}\dot{p} = -\omega^2 q$$

$$\ddot{p} = -m\omega^2\dot{q} = -\omega^2 p$$

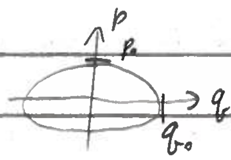
for phase space trajectories:

$q(0) = q_0 \qquad q(t) = q_0 \cos(\omega t)$

$p(0) = 0 \qquad p(t) = p_0 \sin(\omega t)$

Corresponds to an ellipse in phase space

$$\Rightarrow \frac{q^2}{q_0^2} + \frac{p^2}{p_0^2} = 1$$



- We can solve the problem by Canonical Transformations, ~~using~~ w/o using 2nd order eq^n's.

$\Rightarrow$ by diagonalizing / decoupling the problem

$$\frac{d}{dt}\begin{pmatrix} q(t) \\ p(t) \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{m} \\ -m\omega^2 & 0 \end{pmatrix}\begin{pmatrix} q(t) \\ p(t) \end{pmatrix}$$

\# transformation to new coordinate & momentum
$(Q) \quad (P)$

\# transformation has to be such that P is the canonical momentum of Q
+
form of Hamilton equations is preserved.

- Canonical Transformations

$$(q, p) \xrightarrow{\text{diffeomorphism}} (Q, P)$$

$$Q = Q(q, p, t)$$

$$P = P(q, p, t)$$

$H = H(q, p, t) \qquad \tilde{H} = \tilde{H}(Q, P, t)$

$\dot{q} = \frac{\partial H}{\partial p},\ \dot{p} = -\frac{\partial H}{\partial q} \qquad \dot{Q} = \frac{\partial \tilde{H}}{\partial P},\ \dot{P} = -\frac{\partial \tilde{H}}{\partial Q}$

How to construct said Transformations?

$H = H(q,p,t)$ $\dot{q} = \frac{\partial H}{\partial p}$; $\dot{p} = -\frac{\partial H}{\partial q}$

$\tilde{H} = \tilde{H}(Q,P,t)$ $\dot{Q} = \frac{\partial \tilde{H}}{\partial P}$; $\dot{P} = -\frac{\partial \tilde{H}}{\partial Q}$

- Hamilton's Eq^{ns} result from the principle of least action, $\delta S = 0$

$$S = \int dt \, (\underbrace{p\dot{q} - H(p,q,t)}_{L})$$

- Forms of eq^{ns} of motion remain invariant if Lagrangian $\tilde{L}$ in new coordinates is equal to the old ~~one~~ one, modulo a total time derivative

[invariance under multiplication w/ constants "for what tho!?" under adding total derivatives]

$$L = \tilde{L} + \frac{d}{dt} F_1(q,Q,t)$$

$\hookrightarrow$ $p\dot{q} - H(p,q,t) = P\dot{Q} - \tilde{H}(P,Q,t) + \frac{d}{dt} F_1(q,Q,t)$ [Eq^{n} $*$]

$\hookrightarrow$ Total time derivative of F_1

$$\frac{dF_1}{dt} = \frac{\partial F_1}{\partial q}\dot{q} + \frac{\partial F_1}{\partial Q}\dot{Q} + \frac{\partial F_1}{\partial t}$$

$$\Rightarrow \left(p - \frac{\partial F_1}{\partial q}\right)\dot{q} - H(p,q,t) = \left(P + \frac{\partial F_1}{\partial Q}\right)\dot{Q} - \tilde{H}(P,Q,t) + \frac{\partial F_1}{\partial t}$$

- We can guarantee the validity of $(*)$ & hence that the transformation is canonical if we postulate that:

$**$

$$p = \frac{\partial}{\partial q} F_1(q,Q,t)$$

$$P = -\frac{\partial}{\partial Q} F_1(q,Q,t)$$

and $\tilde{H}(P,Q,t) = H(p,q,t) + \frac{\partial}{\partial t} F_1(q,Q,t)$

every function $F_1(q,Q,t)$ generates a canonical transformation denoted above

Back to the harmonic oscillator example:

$$H = H(p,q) = \underbrace{\frac{p^2}{2m}}_{KE} + \underbrace{\frac{m}{2}\omega^2 q^2}_{PE} \qquad \omega = \sqrt{\frac{k}{m}}$$

Case 1: generating function $F_1(q,Q) = -\frac{Q}{q}$ → ??

Determining transformation $Q = Q(q,p)$, $P = P(q,p)$, Hamiltonian $\tilde{H} = \tilde{H}(Q,P)$

$$p = \frac{\partial}{\partial q} F_1 = \frac{Q}{q^2} \iff \left[Q = pq^2\right] \Rightarrow p = \frac{Q}{q^2} \qquad P = \frac{1}{q} \qquad q^2 = \frac{1}{P^2}$$

$$\boxed{p = QP^2}$$

$$\left[P = -\frac{\partial F_1}{\partial Q} = \frac{1}{q}\right]^*$$

$$\tilde{H}(P,Q) = H(p,q) + \underbrace{\frac{\partial}{\partial t} F_1}_{=0} = \frac{p^2}{2m} + \frac{m}{2}\omega^2 q^2$$

$$\tilde{H}(P,Q) = \frac{Q^2 P^4}{2m} + \frac{m}{2}\omega^2 P^{-2}$$

Hamilton's Eq^n's

$$\dot{Q} = \frac{\partial \tilde{H}}{\partial P} = \frac{2Q^2P^3}{m} - m\omega^2 P^{-3}$$

$$\dot{P} = -\frac{\partial \tilde{H}}{\partial Q} = -\frac{1}{m} Q P^4$$

⇒ The canonical transformation generated by $F_1 = -\frac{Q}{q}$ is not useful in this case since eq^n of motions are more complicated.

Case 2: $F_1(q,Q) = \frac{m}{2}\omega q^2 \cot Q$

and

$H = H(p,q)$ as before

$$p = \frac{\partial F_1}{\partial q} = m\omega q \cot Q \qquad p = m\omega \frac{\cos Q}{\sin Q}\sqrt{\frac{2P}{m\omega}} \qquad \cancel{\sin Q} = \sqrt{2Pm\omega}\cos Q \; *$$

$$P = \frac{m\omega q^2}{2\sin^2 Q} \Bigg] \Rightarrow q = \sqrt{\frac{2P\sin^2 Q}{m\omega}} = \sin Q \sqrt{\frac{2P}{m\omega}} \; *$$

$$\tilde{H}(P,Q) = H(p,q,t) + \underbrace{\frac{\partial}{\partial t} F_1(q,Q,t)}_{0} = \frac{p^2}{2m} + \frac{m}{2}\omega^2 q^2 = \frac{2Pm\omega}{2m}\cos^2 Q + \frac{m}{2}\omega^2 \sin^2 Q \frac{2P}{m\omega}$$

$*$

$$\tilde{H}(P,Q) = \omega P$$

Hamilton's Eq^n's:

$$\dot{Q} = \frac{\partial \tilde{H}}{\partial P} = \omega \Rightarrow Q(t) = \omega t + Q_0$$

$$\dot{P} = -\frac{\partial \tilde{H}}{\partial Q} = 0 \Rightarrow P(t) = P_0$$

$$\Rightarrow \quad q(t) = \sqrt{\frac{2P}{m\omega}} \sin(\omega t + Q_0)$$

$$p(t) = m\omega\sqrt{\frac{2P_0}{m\omega}} \cos(\omega t + Q_0) = m\dot{q}(t)$$

eq^n are much simpler, but finding generating function F_1 to achieve this is a difficult problem

Calculating generating functions for a given transformation

$$Q = \ln p, \quad P = -qp, \quad q,p>0$$

$$p = \frac{\partial}{\partial q} F_1(q,Q,t) \qquad P = -qp = -qe^Q = -\frac{\partial}{\partial Q} F_1(q,Q,t)$$

$$P = -\frac{\partial}{\partial Q} F_1(q,Q,t) \qquad \Rightarrow F_1(q,Q,t) = qe^Q + f(q) + g(t)$$

$$p = \frac{\partial}{\partial q} F_1 = e^Q + f'(q) = p + f'(q) \Rightarrow f'(q) = 0$$

or

$f(q) = c$ [can be dropped or absorbed in $g(t)$]

$$F_1(q,Q,t) = qe^Q + g(t)$$

4 Different types of generating functions of canonical transformations: $(q,p) \to (Q,P)$

$F_1(q,Q,t)$ $\quad F_2(q,P,t)$ $\quad F_3(p,Q,t)$ $\quad F_4(p,P,t)$

Not independent of each other & related by Legendre transformations

Taking $F_2(q,P,t)$ as an example:

$$p\dot{q} - H(p,q,t) = P\dot{Q} - \tilde{H}(P,Q,t) + \frac{d}{dt}F_1(q,Q,t) \qquad \cancel{F_2}$$

$$\Rightarrow dF_1(q,Q,t) = p\,dq - P\,dQ + (\tilde{H} - H)\,dt \qquad (*)$$

$$dF_2 = \frac{\partial F_2}{\partial q}dq + \frac{\partial F_2}{\partial P}dP + \frac{\partial F_2}{\partial t}dt \quad \checkmark$$

in $(*)$ rewrite $P\,dQ = d(PQ) - Q\,dP \Leftarrow d(PQ) = P\,dQ + Q\,dP$

$$\Rightarrow p\,dq - (d(PQ) - Q\,dP) + (\tilde{H} - H)\,dt$$

$$= p\,dq - d(PQ) + Q\,dP + (\tilde{H} - H)\,dt = dF_1$$

$$p\,dq + Q\,dP + (\tilde{H} - H)\,dt = d(F_1 + PQ) = dF_2$$

Legendre Transformation
same as between L & H
→ important in thermodynamics

$$F_2 = F_1 + PQ = F_1 + Q\frac{\partial F_1}{\partial Q}$$

$$p = \frac{\partial F_2}{\partial q}(q,P,t)$$

$$\tilde{H}(P,Q,t) = H(p,q,t) + \frac{\partial}{\partial t}F_2(q,P,t)$$

$$Q = \frac{\partial F_2}{\partial P}(q,P,t)$$

Back to Hamilton-Jacobi Theory!

$$\frac{\partial S}{\partial t} + H(q, \underbrace{\frac{\partial S}{\partial q}}_{p}, t) = 0$$

$$S = S(q, \alpha, t) = S(q_1, \ldots q_n, \underbrace{\alpha_1, \ldots \alpha_n}_{\text{integration constants}}, t)$$

α

integration constants $\Rightarrow$ View them as conserved canonical momentum $P = [P_1, \ldots, P_n]$ of some coordinates $Q = [Q_1, \ldots, Q_n]$

$\Rightarrow$ take $S(q,\alpha,t) = S(q,P,t)$ as the generating function $F_2(q,P,t)$ of a canonical transformation

$$\Rightarrow \tilde{H} = H + \frac{\partial F_2}{\partial t} = H + \frac{\partial S}{\partial t} = 0$$

sub in Hamilton-Jacobi

$$Q = \frac{\partial F_2}{\partial P}$$

$$= \frac{\partial}{\partial P} S(q,P,t) = \frac{\partial}{\partial \alpha}(q, \alpha, t)$$

$$Q = \beta = \text{const} \quad \because \dot{Q} = \frac{\partial \tilde{H}}{\partial P} = 0$$

This differentiation wrt integration constants we use as the last step in H.J theory to extract the dynamics.

Poisson Brackets:

Observable: $f(q,p,t)$, $q = (q_1, \ldots, q_n)$
$p = (p_1, \ldots, p_n)$

$$\frac{\partial H}{\partial p} = \dot{q}, \quad \frac{\partial H}{\partial q} = -\dot{p}$$

Hamilton's eq^n

Total time derivative

$$\frac{d}{dt} f = \frac{\partial f}{\partial q}\dot{q} + \frac{\partial f}{\partial p}\dot{p} + \frac{\partial f}{\partial t} = \left(\frac{\partial f}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial H}{\partial q}\right) + \frac{\partial f}{\partial t}$$

$$\Rightarrow \frac{d}{dt} f = [f, H] + \frac{\partial f}{\partial t}$$

— time evolution of observable is controlled by Hamiltonian

Poisson Bracket:

??

$$[f,g] = [f,g]_{q,p} = \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial q}$$

$$= \sum_i \left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}\right)$$

★ Some useful properties: [c_1, c_2, const]

(a) $[c_1 f + c_2 g, h] = c_1[f,h] + c_2[g,h]$ — linearity

(b) $[f,g] = -[g,f]$ — antisymmetry

(c) $[fg, h] = f[g,h] + [f,h]g$ — product rule

(d) $[f,[g,h]] + [h,[f,g]] + [g,[h,f]] = 0$ — Jacobi identity

~~Hamilton's eq^n can be written as:~~

~~$\dot{q}_i = \frac{\partial H}{\partial p_i} = [\ldots]$~~

Hamilton's eq^n can be written as: (w/ proof)

$\dot{q}_i = \frac{\partial H}{\partial p_i} = [q_i, H]$

$$[q_i, H] = \sum_j \Big(\underbrace{\frac{\partial q_i}{\partial q_j}}_{\delta_{ij}} \frac{\partial H}{\partial p_j} - \underbrace{\frac{\partial q_i}{\partial p_j}}_{=0} \frac{\partial H}{\partial q_j} \Big) = \sum_j \delta_{ij} \frac{\partial H}{\partial p_j} = \frac{\partial H}{\partial p_i} = \dot{q}_i$$

$\dot{p}_i = -\frac{\partial H}{\partial q_i} = [p_i, H]$

$$[p_i, H] = \sum_j \Big(\underbrace{\frac{\partial p_i}{\partial q_j}}_{=0} \frac{\partial H}{\partial p_j} - \underbrace{\frac{\partial p_i}{\partial p_j}}_{\delta_{ij}} \frac{\partial H}{\partial q_j} \Big) = -\frac{\partial H}{\partial q_i}$$

~~fundamental poisson bra~~ ~~Pos~~ Poisson Brackets,

$[q_i, q_j] = 0$
$$= \sum_k \Big(\frac{\partial q_i}{\partial q_k} \underbrace{\frac{\partial q_j}{\partial p_k}}_{=0} - \underbrace{\frac{\partial q_i}{\partial p_k}}_{=0} \frac{\partial q_j}{\partial q_k} \Big)$$

$[q_i, p_j] = \delta_{ij}$
$$= \sum_k \Big(\underbrace{\frac{\partial q_i}{\partial q_k}}_{\delta_{ik}} \underbrace{\frac{\partial p_j}{\partial p_k}}_{\delta_{jk}} - \underbrace{\frac{\partial q_i}{\partial p_k}}_{0} \underbrace{\frac{\partial p_j}{\partial q_k}}_{0} \Big)$$

$[p_i, p_j] = 0$ (similarly)

Harmonic oscillator example again:

$$\dot{x} = [x, H] = [x, \frac{p^2}{2m} + \frac{m\omega^2}{2} x^2] \underset{[x,x]=0}{=} [x, \frac{p^2}{2m}] \underset{(a)}{=} \frac{1}{2m}[x, p^2] \underset{(c)}{=} 2 \cdot \frac{1}{2m} p \underbrace{[x,p]}_{=1} = \frac{p}{m}$$

$$\dot{p} = [p, H] = [p, \frac{p^2}{2m} + \frac{m\omega^2}{2} x^2] \underset{[p,p]=0}{=} [p, \frac{m\omega^2}{2} x^2] \underset{(a)}{=} \frac{m\omega^2}{2} [p, x^2] \underset{(c)}{=} m\omega^2 x [p, x]$$

$$\underset{(b)}{=} -m\omega^2 x \underbrace{[x,p]}_{=1} = -m\omega^2 x$$

– Criterion for canonical transformation: [fundamental Poisson Brackets must be preserved]

$q_i \to Q_i(q,p,t)$
$p_i \to P_i(q,p,t)$ $(i = 1, \dots, n)$ [Canonical]

$\Leftrightarrow [Q_i, Q_j]_{q,p} = [P_i, P_j]_{q,p} = 0$
&
$[Q_i, P_j]_{q,p} = \delta_{ij}$
– conserved

Much easier to check than to find the generating function $F_1(q, Q, t)$

eg.: $Q = q^a \cos(bp)$, $P = q^a \sin(bp)$ } canonical?

$$[Q,P] = \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q} = aq^{a-1}\cos(bp)\, bq^a \cos(bp) + bq^a \sin(bp)\, aq^{a-1} \sin(bp) = abq^{2a-1}$$

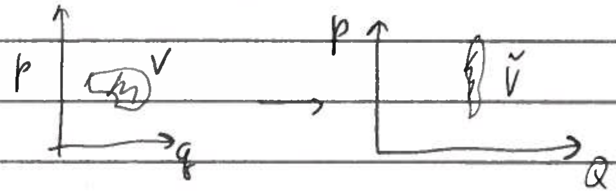
transformation is canonical $\Leftrightarrow a = \frac{1}{2}, b = 2$

Main idea?!

$\Rightarrow$ density of states in an ensemble of many identical states w/ diff init cond^{n} is const. along every trajectory in phase ~~sp~~ space

4.6 Liouville's Theorem

· Canonical invariance of phase-space volume



$$V = \iint dq\,dp \qquad \tilde{V} = \iint \underbrace{dQ\,dP}_{d\tilde{V}} = \iint \left| \begin{matrix} \frac{\partial Q}{\partial q} & \frac{\partial P}{\partial q} \\ \frac{\partial Q}{\partial p} & \frac{\partial P}{\partial p} \end{matrix} \right| \underbrace{dq\,dp}_{dV}$$

$$\left| \begin{matrix} \frac{\partial Q}{\partial q} & \frac{\partial P}{\partial q} \\ \frac{\partial Q}{\partial p} & \frac{\partial P}{\partial p} \end{matrix} \right| = \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial P}{\partial q}\frac{\partial Q}{\partial p} = [Q,P]_{q,p} = 1 \text{ for canonical transformation}$$

~~volume remains invariant~~

Phase-space ~~invariant~~ under canonical transformations, $V = V \to \tilde{V}$ //

– Time translation is canonical

$$Q(t) = q(t+\Delta t) = q(t) + \dot{q}(t)\Delta t + O(\Delta t^2) = q + \frac{\partial H}{\partial p}\Delta t + O(\Delta t^2)$$

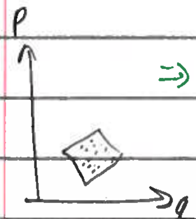
$$P(t) = p(t+\Delta t) = p(t) + \dot{p}(t)\Delta t + O(\Delta t^2) = p - \frac{\partial H}{\partial q}\Delta t + O(\Delta t^2)$$

$$[Q,P] = \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q} = 1 + O(\Delta t^2)$$

$\Rightarrow$ infinitesimal time translation is canonical

$\Rightarrow$ time translation ~~is~~ are canonical (by canonical transform grps)

– consider ensemble of G systems, each corresponds w/ a pt $(q,p) = (q_1 \cdots q_n, p_1 \cdots p_n)$ in Phase space P



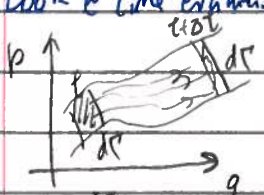
$\Rightarrow$ approximate by continuous distribution func $g(q,p,t)$

$\rightarrow$ g is positive definite

$\Rightarrow$ we can define a probability distribution

$$\int \cdots \int \underbrace{dq_1 \cdots dq_n\, dp_1 \cdots dp_n}_{\Gamma}\; g(q,p,t) = G \qquad \rho(q,p,t) = \frac{g(q,p,t)}{G}$$

– Look @ time evolution of g & ρ



$d\Gamma \Rightarrow$ small volume element in phase space Γ

– time evolution canonical $\Rightarrow d\Gamma$ does not change in time

– phase-space trajectories do not cross

$\Rightarrow dG$ must remain the same

$$\rightarrow 0 = \frac{dg}{dt} = [g,H] + \frac{\partial g}{\partial t}$$

$$g = \frac{dG}{d\Gamma} \qquad [g,H] + \frac{\partial g}{\partial t} = 0 \quad \Rightarrow \text{Liouville's theorem}$$

[number of phase space points in $d\Gamma$]

$\rangle r$