MM III cheat sheet

Series Solution to PDEs

Zoology

1. Ordinary → one variable
2. Partial → more than one variable
3. Order → highest derivative
4. Linearity → no powers above the first power of the unknown func & its derivative
   → function nor its derivative are inside another function

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}}{dx^{n-1}} + \dots a_0(x)y = b(x)$$

5. Homogeneous → terms of eq^n depend on unknown func or its derivative

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = 0$$

6. Degree → exponent of the highest order derivative involved
7. Solutions → some function for which the problem is defined
8. Uniqueness → 有多過一個答案
   ↳ n^{th} order diff eq^n
   → 要有 n independent functions
   & n boundary conditions
9. Existence → no guarantee have the form $u(x)$ for a diff eq
10. Superposition → 加埋 0 sol^ns

1. Separation of variables
   ↳ 拆 PDE → n separate ODEs w/ n variables
2. Frobenius's Method
   ↳ get indicial eq^n

& Testing for convergence:

$$S = \lim_{j\to\infty}\left|\frac{c_{j+1}}{c_j}\right|$$

S < 1 : converging

S = 1 : inconclusive

S > 1 : divergence

$$y''(x) + P(x)y' + q(x)y = 0$$

$$\text{sol}^n:\ y(x) = \sum_{j=0}^{\infty} a_j (x-x_0)^{k+j}$$

indicial eq^n: $k(k-1) + p_0 k + q_0 = 0$

$$p_0 = \lim_{x\to 0} x\,p(x)$$

$$q_0 = \lim_{x\to 0} x^2 q(x)$$

for finding values of k

3. Hermite polynomials
   ↳ 縮短 series → 將 series 轉 polynomial
   @ some pt, a term = 0, series terminates
4. Legendre Polynomials

Generating Function:

$$g(t,\mu) = \frac{1}{\sqrt{1+t^2-t\mu}} = \sum_{l=0}^{\infty} P_l(\mu)t^l = P_0(\mu) + P_1(\mu)t + P_2(\mu) + P_3(\mu) + O(t^4)$$

Spherical Harmonics

$$P_l^m(x) = (1-x^2)^{\frac{m}{2}}\frac{d^m}{dx^m}P_l(x)$$

$$P_l(x) = y^{(l)}x = \sum_{n=0}^{l} a_n^l x^n$$

Special Points:

1. Ordinary / Analytic Points → $p(x)$ & $q(x)$ are analytic & both diverge @ $x = x_0$
2. Regular Singular points → $p(x)$ or $q(x)$ X analytic BUT p_0 & q_0 are analytic
3. Essential Singular points → At least one of p_0 & q_0 ~~are~~ is not analytic

If expansion is 1 or 2 ⇒ at least 1 sol^n exists for a linear, homogeneous diff eq^n of 2nd order
↳ Fuch's Theorem

Lagrangian & Hamiltonian Mechanics

E.L. eq^n

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0$$

or

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i} = 0$$

Lagrangian: $L \equiv T - V$

Hamiltonian: $H \equiv T + V$

Beltrami identity:

for $f = f(y, y')$

$$\frac{d}{dx}\left(f - y'\frac{\partial f}{\partial y'}\right) = 0$$

Generating momentum:

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \; ; \; \dot{p}_i = \frac{\partial L}{\partial q_i}$$

$$P_x = \frac{\partial L}{\partial \dot{x}} \; , \; \dot{P}_x = \frac{\partial L}{\partial x}$$

eq^n of motion using H:

$$\frac{\partial H}{\partial p} = \dot{q} \; ; \; \frac{\partial H}{\partial q} = -\dot{p}$$

Hamiltonian:

$$H = \sum_i p_i \dot{q}_i - L$$

Hamilton's eq^n of motion

$$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$$

for generic period $2L$

Fourier Transform: $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(n\frac{\pi}{L}x\right) + b_n \sin\left(n\frac{\pi}{L}x\right)\right]$

Orthogonal relationships:

$$\int_{-L}^{L} \sin\left(m\frac{\pi}{L}x\right)\sin\left(n\frac{\pi}{L}x\right)dx = L\delta_{mn}$$

$\max(m,n) > 0$

$$\int_{-L}^{L} \sin\left(m\frac{\pi}{L}x\right)\sin\left(n\frac{\pi}{L}x\right)dx = 0$$

$m = n = 0$

$$\int_{-L}^{L} \cos\left(m\frac{\pi}{L}x\right)\cos\left(n\frac{\pi}{L}x\right) = L\delta_{mn}$$

$\max(m,n) > 0$

$$\int_{-L}^{L} \cos\left(m\frac{\pi}{L}x\right)\cos\left(n\frac{\pi}{L}x\right)dx = 2L$$

$m = n = 0$

$$\int_{-L}^{L} \sin\left(m\frac{\pi}{L}x\right)\cos\left(n\frac{\pi}{L}x\right)dx = 0$$

Even function: $f(x) = f(-x)$

Odd function: $f(x) = -f(x)$

Parseval's Identity

$$\frac{1}{2L}\int_{-L}^{L}[f(x)]^2 dx = \left(\frac{a_0}{2}\right)^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

$$\frac{1}{2L}\int_{-L}^{L}|f(x)|^2 dx = \sum_{n=-\infty}^{\infty}|c_n|^2$$

Complex Fourier Series

$$f(x) = \sum_{n=-\infty}^{+\infty} c_n e^{i\left(n\frac{\pi}{L}x\right)}$$

$$c_n = \frac{1}{2L}\int_{-L}^{L} e^{-in\frac{\pi}{L}x} f(x)\,dx$$

Hilbert-Schmidt Scalar Product

$$f(x)\cdot g(x) \equiv \int_{-L}^{L} f^*(x)\,g(x)\,dx$$

Fourier Transform:

$$\tilde{f}(k) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx} f(x)\,dx$$

$$\Rightarrow f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}\tilde{f}(k)\,dk$$

for any complex square integrable function

Differentiating:

$$\tilde{f}'(k) = ik\tilde{f}(k)$$

Dirac-Delta function

$$\delta(x-y) := \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(x-y)}\,dk$$

Parseval's Identity for Fourier Transform:

$$\int_{-\infty}^{\infty}|f(x)|^2 dx = \int_{-\infty}^{\infty}|\tilde{f}(k)|^2 dk$$

Convolution theorem

$$h(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\, f(x)\, g(z-x)$$

SP Special Relativity

Lorentz Transformation

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & -\frac{\gamma v}{c} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{\gamma v}{c} & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$$

$$t' = \frac{t - zV/c^2}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$x' = x$$

$$y' = y$$

$$z' = \frac{z - vt}{\sqrt{1-\frac{v^2}{c^2}}}$$

Four Vectors & scalar products:

$$(V \cdot W) = V^T G W = V_0W_0 - V_1W_1 - V_2W_2 - V_3W_3$$

Metric

$$G = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

[Clausius space]

Length contraction:

$$L = L_0\sqrt{1-\frac{v^2}{c^2}}$$

Orientation of stick:

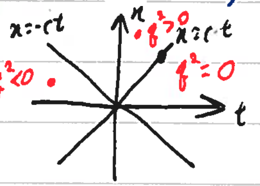
$$\theta = \tan^{-1}\left(\frac{\gamma y_B'}{z_B'}\right)$$

Time dilation:

$$\Delta t = \frac{\Delta t'}{\sqrt{1-\frac{v^2}{c^2}}}$$

Time cone for

$$q^2 = (c\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2$$



Addition of velocities:

$$u_x' = \frac{1}{\gamma}\frac{u_x}{1-u_z\frac{v}{c^2}} \qquad u_x = \frac{u_x'}{\gamma\left(1+u_z\frac{v}{c^2}\right)}$$

$$u_y' = \frac{1}{\gamma}\frac{u_y}{1-u_z\frac{v}{c^2}} \qquad u_y = \frac{u_y'}{\gamma\left(1+u_z\frac{v}{c^2}\right)}$$

$$u_z' = \frac{u_z - v}{1-u_z\frac{v}{c^2}} \qquad u_z = \frac{u_z' + v}{1+u_z\frac{v}{c^2}}$$

Relativistic Doppler effect

$$\lambda = \lambda_0\sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}}$$

$$f = \frac{f_0}{\gamma} \qquad T = \gamma T_0$$

Relativistic momentum:

$$p = \gamma m \underline{v}$$

Four momentum:

$$p = \begin{pmatrix} E/c \\ p_x \\ p_y \\ p_z \end{pmatrix} = \begin{pmatrix} mc \\ p_x \\ p_y \\ p_z \end{pmatrix}$$

$$p' = \begin{pmatrix} \gamma & 0 & 0 & -\frac{\gamma v}{c} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{\gamma v}{c} & 0 & 0 & \gamma \end{pmatrix}\begin{pmatrix} mc \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$p' = \begin{pmatrix} \gamma mc \\ 0 \\ 0 \\ -\gamma mv \end{pmatrix}$$

Four momentum of photon:

$$\begin{pmatrix} E/c \\ p_x \\ p_y \\ p_z \end{pmatrix} = \begin{pmatrix} h/\lambda \\ 0 \\ 0 \\ h/\lambda \end{pmatrix}$$

* $h = 4.14 \times 10^{-24}$ GeV·s

Relativistic Energy:

KE:

$$KE = \gamma mc^2 - mc^2$$

Total energy = rest energy + KE

$$E = \gamma mc^2$$

In terms of momentum:

$$E^2 = m^2c^4 + p^2c^2$$

Magnitude of 4-momentum:

$$p^2 = \frac{E^2}{c^2} - |p|^2 = m^2c^2$$