MM II cheat sheet

Multivariable Calculus & Differential operators

$$\text{grad}\,\phi : \vec{\nabla}\phi = \begin{pmatrix} \partial_x\phi \\ \partial_y\phi \\ \partial_z\phi \end{pmatrix}$$

:- points along the max ↑ of ϕ
- directional derivative of ϕ along $\hat{a} = |\nabla\phi|\cos\alpha$
- $\perp$ to hypersurfaces defined by $\phi(r) = C$

$|\hat{a}|\nabla\phi = 0$ when plane & surf $\perp$

Total differential of fields:

$$\partial\phi = \partial_x\phi\, dx + \partial_y\phi\, dy + \partial_z\phi\, dz$$

=> measures change of $d\phi$ of ϕ as x, y, z are changed

$$= \nabla\phi \cdot dr$$

$$\text{div}\, A = \nabla\cdot A = \partial_x A_x + \partial_y A_y + \partial_z A_z$$

:- Scalar
- indicates sinks & sources

$$\text{curl}\, A = \nabla\times A = \begin{pmatrix} \partial_y A_z - \partial_z A_y \\ \partial_z A_x - \partial_x A_z \\ \partial_x A_y - \partial_y A_x \end{pmatrix}$$

:- vector
- indicates rotational flow

$$\underline{\Delta\phi} = \nabla^2\phi = \text{div grad}\,\phi = \partial_x^2\phi + \partial_y^2\phi + \partial_z^2\phi$$

- laplacian of ϕ
- scalar field

$$\text{curl grad}\,\phi = \nabla\times\nabla\phi = \begin{pmatrix} \partial_y\partial_z\phi - \partial_z\partial_y\phi \\ \partial_z\partial_x\phi - \partial_x\partial_z\phi \\ \partial_x\partial_y\phi - \partial_y\partial_x\phi \end{pmatrix} = 0$$

=> zero divergence

Line integral:

$$I = \int_C G(r)\cdot dr = \int_a^b G(r(t))\cdot r'(t)\, dt = \int_a^b \nabla\phi(r(t))\cdot r'(t)\, dt$$

Area integrals:

$$A = \int_A \sigma(r)\, dA = \int_{y_1}^{y_2} dy \int_{x_1}^{x_2} dx\, \sigma(x,y)$$

Gauss's Divergence Theorem:

$$\int_{S=\partial V} G(r)\cdot d\underline{s} = \int_V \nabla\cdot G(r)\, dV$$

Stoke's Theorem:

$$\oint_{C=\partial S} G(r)\cdot d\underline{r} = \int_S \nabla\times G\cdot d\underline{S}$$

Volume integrals:

$$I = \int_V f(\underline{r})\, dV = \iiint_V f(x,y,z)\, dx\, dy\, dz$$

Surface integrals:

Area of surface
$$I_1 = \int_S f(r)\, ds = \int_{s_1}^{s_2} ds \int_{t_1}^{t_2} dt\, f(\underline{r}(s,t)) \left| \frac{\partial r}{\partial s}\times\frac{\partial r}{\partial t} \right|$$

$$\underline{r} = \begin{pmatrix} x \\ y \\ g(x,y) \end{pmatrix}$$

Total flux of vector field through S:
$$I_2 = \int_S (G\cdot\hat{n})\, ds = \int_S G\cdot d\underline{s} = \int_{s_1}^{s_2} ds \int_{t_1}^{t_2} dt\, G(\underline{r}(s,t))\cdot\left(\frac{\partial r}{\partial s}\times\frac{\partial r}{\partial t}\right)$$

$$d\underline{s} = d\underline{u}\times d\underline{v} = \left(\frac{\partial \underline{r}}{\partial s}\times\frac{\partial \underline{r}}{\partial t}\right) ds\, dt$$

$$ds = |d\underline{s}| = \sqrt{1+\left(\frac{\partial g}{\partial x}\right)^2+\left(\frac{\partial g}{\partial y}\right)^2}\, dx\, dy$$

Polar coordinates:

$x = r\cos\phi$

$y = r\sin\phi$

$dA = r\,dr\,d\phi$

Cylindrical coordinates:

$x = \rho\cos\phi$

$y = \rho\sin\phi$

$z = z$

$dV = \rho\,d\rho\,d\phi\,dz$

Sperical coordinates:

$x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta$

$dV = r^2 dr\,\sin\theta\,d\theta\,d\phi$

$d\underline{S} = R^2\sin\theta\,d\theta\,d\phi\,\hat{e}_r$

ODEs:

$\boxed{\text{Seperable} \quad -\text{☐}\,dy, -\text{☐}\,dx.}$

Non - seperable

integrating factor method

$S(x)\left[\frac{dy}{dx} + P(x)y\right] = [Q(x)]\,S(x)$

where $S(x) = e^{\int P(x)dx}$ find $P(x)S(x) = \frac{dS}{dx}$

$\hookrightarrow S \cdot Q = \frac{d}{dx}(S\cdot y)$

$\Rightarrow y = \frac{1}{S(x)}\left[\int S(x)Q(x)\,d(x) + C\right]$

perfect / exact differential method

condition: $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

$Q(x,y)\frac{dy}{dx} + P(x,y) = 0 \Rightarrow \underline{P(x,y)dx + Q(x,y)dy = 0}$

I. $f(x,y) = \int \frac{\partial f}{\partial x}dx$

$= f_I + g(y)$

II $f(x,y) = \int \frac{\partial f}{\partial y}dy$

$= f_{II} + h(x)$

determine $g(y)$ & $h(x)$

$\therefore \Rightarrow f(x,y) = f_I + g(y)$

$= f_{II} + h(x)$

$\Rightarrow$ so that $f(x,y) = \text{const.}$

2nd order ODEs:

(Y_{CF}) Homogeneous $y'' + py' + qy = 0$ guess: $e^{kx} = 0$

- Real roots : $y = Ae^{k_1 x} + Be^{k_2 x}$
- Complex roots : $y = e^{\alpha x}[(A+B)\cos(\beta x) + i(A-B)\sin(\beta x)]$
- Degenerate roots : $y = Ae^{kx} + Bxe^{kx}$

$Y = Y_{CF} + Y_{PI}$

(Y_{PI}) Inhomogeneous: $y'' + py' + qy = f(x)$

- polynomials: $f(x) = A_0 + A_1 x + \dots + A_n x^n$

  $Y_{PI} = \alpha_0 + \alpha_1 x \dots + \alpha_n x^n$

- exponentials: $f(x) = A_0 e^{wx}$

  $Y_{PI} = \alpha_0 e^{wx}$ $\quad \alpha_0 = \frac{A_0}{w^2 + pw + q}$

  when $w = k_{1,2}$ (from Y_{CF})

  $Y_{PI} = Bxe^{wx}$

- Trig func. $\sin x / \cos x \Rightarrow f(x) = A_0\cos(wx) + A_1\sin(wx)$

  $Y_{PI} = \alpha_0\cos(wx) + \alpha_1\sin(wx)$

Vector (Griffiths, EOM Pg 11)

A vector is any set of 3 components that transforms in the same manner as a displacement when you change the coordinates commutative => [illegible]

associative => bracket 亂擺

Linear Algebra:

Kronecker Delta: $\hat{e}_i \cdot \hat{e}_j = \delta_{ij} = \begin{cases} 1 & (i=j) \Rightarrow \parallel \\ 0 & (i \neq j) \Rightarrow \perp \end{cases}$

Basis vectors & components:

$\underline{v} = \sum_{i=1}^{n} v_i \hat{e}_i$

coefficient $v_n = \underline{v}\,\hat{e}_n$

scalar product: $\underline{u} \cdot \underline{v} = u_1 v_1 + u_2 v_2 + u_3 v_3 \cdots\cdots + u_n v_n$

[proof using δ_{ij}] $= \sum_{i=1}^{n} u_i v_i = \underline{u}^T \underline{w}$

length of vector: $v = |\underline{v}| = \sqrt{v_1^2 + v_2^2 \cdots\cdots + v_n^2}$

Special cases: $\underline{v} = 1 \Rightarrow$ unit vector

$\underline{v} = 0 \Rightarrow$ null vector

Linear dependence: scalar coefficients $C_1 \cdots C_n$

satisfy $\Rightarrow C_1 \underline{X}_1 + C_2 \underline{X}_2 + \cdots\cdots + C_n \underline{X}_n = 0$

Matrix: addition

$m_{ij} = a_{ij} + b_{ij}$

multiplication

$\underline{m} = m_{ij} = (\underline{A}\,\underline{B})_{ij}$

$= \sum_{k=1}^{n} a_{ik} b_{kj} \equiv a_{ik} b_{kj}$

[proof associativity]

Determinants:

2×2: $\det A = \begin{vmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{vmatrix}$

$= a_{11} a_{22} - a_{12} a_{21}$

3×3: $\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

$= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

Properties of determinants:

⭐ = hv changes

- ⭐ if rows are written as columns & columns are written as rows => det unchanged
- ⭐ for square matrices: $|AB| = |A||B|$ ⭐ $|AB| = |BA| = -1$
- ⭐ det vanishes if a row / column has all zeros
- ⭐ if we multiply a row / column by a constant, det will also be multiplied by the constant
- ⭐ if 2 rows / columns are multiples of each other det = 0
- ⭐ if a pair of row / column interchanges det changes sign
- ⭐ det does not change when adding a multiple of one row / column to another

for simultaneous eq^n

Multiplicative inverse of a matrix

$AA^{-1} = \underline{I}$ using to solve $\underline{A}\underline{x} = \underline{b}$

$\underline{x} = A^{-1} \underline{b}$

$A^{-1} = \frac{1}{|A|} \underline{C}^T$ $\left[(A^{-1})_{ij} = \frac{1}{|A|} C_{ji} \right]$

$\underline{C}^T \Rightarrow$ transpose of the cofactor

$\underline{C}$ cofactor of matrix element:

$C_{ij} = (-1)^{i+j} \det M_{ij}$ $\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$

Cramer's rule: [proof]

$\sum (A^{adj})_{ji} b_i$ => det obtained by replacing the j^{th} column of A by the column vector of $\underline{b}$

$x_1 = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{23} & a_{33} \end{vmatrix} \div |A|$

$x_2 = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix} \div |A|$

$x_3 = \begin{vmatrix} a_{11} & a_{21} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{23} & b_3 \end{vmatrix} \div |A|$

Special matricies:

Equal Matrices:

$A = B \quad a_{ij} = b_{ij}$

when all corresponding elements are equal

Identity / Unit matrix: I

$AI = IA = A$

$(AI)_{ij} = \sum_k a_{ik}\delta_{kj} = \sum_k \delta_{ik} a_{kj} = (IA)_{ij}$

$$I = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & \cdots & & 1 \end{pmatrix}$$

T

Transpose of a matrix: A^T

$$A^T = \begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix} \qquad [(A^T)^T = A$$

$(A^T)_{ij} = a_{ji}$

if $A^T = A \Rightarrow$ symmetric

if $A^T = -A \Rightarrow$ antisymmetric

Orthogonal Matricies: ⊥

$A^T A = I \Rightarrow$ orthogonal

$|A^T||A| = |I| = 1$

$|A||A| = |A|^2 = 1 \quad (|A^T| = |A|)$

$|A| = \pm 1$

Product of orthogonal matricies:

$C = AB \quad A$ & B are orthogonal

$\Rightarrow C$ is also orthogonal

testing for orthogonality $\Rightarrow$ $C^T C = (AB)^T (AB) = B^T \underbrace{A^T A}_{} B$

$C^T C = I$

* Complex conjugation

$(A^*)_{ij} = a_{ij}$

changes sign of the Im part

if $A = A^* \quad A \in \mathbb{R}$

Hermitian Conjugation † $T \to †$

$A^\dagger = (A^T)^* = (A^*)^T$

$(A^\dagger)^\dagger = A$

if $A^\dagger = A \Rightarrow$ Hermitian

$A^\dagger = -A \Rightarrow$ anti-Hermitian

* all REAL, SYMMETRIC Matricies are Hermitian

Transpose of matrix products:

$(AB)^T = B^T A^T$

$(ABC)^T = C^T B^T A^T$

$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$ using $C = AB$

$\hookrightarrow C^T \Rightarrow c_{ji} = \sum_{k=1}^{n} b_{jk} a_{ki}$

Unitary Matrices: U

$U^\dagger U = I \Rightarrow$ Unitary

$|U^\dagger||U| = |I| = 1$

$|U^*||U| = |I| = 1$

Tr

Trace of a matrix [sum of (\) elements]

$Tr(A) = a_{11} + a_{22} + a_{33} \cdots + a_{nn}$

$= \sum a_{ii} = a_{ii}$

$Tr(A) = Tr(A^T)$

Scalar products:

$$\underline{v} \cdot \underline{w} = \underline{v}^T \underline{G}\, \underline{w} = (v_1 \; v_2 \; v_3) \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$$

G: metric $\Rightarrow$ defining how different coordinates combine to give length elements

$$G_{cartesian}: \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

$ds^2 = dx^2 + dy^2 + dz^2$

$$G_{cylindrical} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$ds^2 = dr^2 + r^2 d\theta^2 + dz^2$

Scaling of Volume: Jacobian Matrix

* 2D det $\Rightarrow$ scaling of area by a lin. transform

3D det $\Rightarrow$ scaling of volume by a lin. transform

$d\underline{r}' = J d\underline{r} \Rightarrow J \Rightarrow$ Jacobian matrix

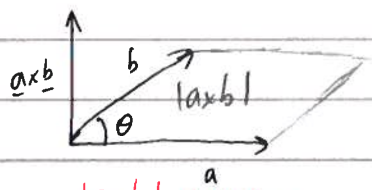
$\det J \Rightarrow$ factor by which the volume element changes when we make the transformation

vector cross products:

$$\underline{a} \times \underline{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = (a_2 b_3 - a_3 b_2)\hat{i} + (a_3 b_1 - a_1 b_3)\hat{j} + (a_1 b_2 - b_1 a_2)\hat{k}$$

[apply basis vectors multiplications]

$|a \times b| \Rightarrow$ area of //gram spanned by vectors a & b



$|a \times b| = |a||b| \sin\theta$

For 2 linearly independent & mutually orthogonal vectors

Eigenvalues & Eigenvectors of a linear operator

Characteristic eq^n:

$$\underbrace{M}_{\text{transformation matrix}} \underbrace{\underline{v}}_{\text{column vector}} = \underbrace{\lambda}_{\text{eigenvalue}} \overbrace{\underline{v}}^{\text{eigen vector}}$$

Steps:

1. Solve λ by characteristic eq^n
2. Solve eigenvector, $\underline{v}$, of corresponding λs
3. Set one of the elements arbitrary (eg. $v_1 = 1$)
   [make sure system does not imply $v_1 = 0$]
4. Determine normalized eigenvectors:
   (ie. eigenvectors with modulus of 1)

$$\underline{v} = \frac{1}{|\underline{v}|}\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$

* using euclidian length
↳ to get the basis vectors, like $\hat{i}, \hat{j}, \hat{k}$ [scale each unit by 1]

where

$$|\underline{v}| = \sqrt{v_1^2 + v_2^2 \cdots v_n^2} = (\underline{v}^T\underline{v})^{\frac{1}{2}}$$

Intuition:

Main idea: their exists a vector, $\underline{v}$ which remains at its own span after the effect of linear transform [M acting on $\underline{v}$]

λ(eigenvalue): factor of which the change of basis is squished or stretched.

* Real matricies can have non-real eigenvalues
* Degenerate eigenvalues corresponds to a sheer. There will be less eigenvalues

Solving λ & $\underline{v}$ by considering:

$$I M \underline{v} = \lambda I \underline{v}$$

$$M\underline{v} = I\lambda\underline{v} \quad (M = IM)$$

$$(M - I\lambda)\underline{v} = 0$$

Solving for $\det(M - I\lambda) = 0$

for square matrices:

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & & \\ \vdots & & \ddots & \\ a_{n1} & \cdots & & a_{nn}-\lambda \end{vmatrix} = 0 \Rightarrow$$

∵ the transformation associated w/ the matrix squishes/stretches space into a lower dimension
↳ Area/Volume ⇒ $\det(M - I\lambda) = 0$

Diagonal Matrix:
→ all basis vectors are eigenvectors
→ diagonals of the matrix are eigenvalues

def: square matrix w/ elements along the diagonal

(Proof) $(AB)_{ij} = a_i b_i \delta_{ij}$
↳ [AB is a diagonal matrix]

Main idea: diagonalising ⇒ change the coordinate system so that eigenvectors are basis vectors

Result for D ⇒ matrix representing that same transformation BUT from the perspective of the new basis vector's coordinate system

Generally:

Assuming we have an $n \times n$ matrix $\underline{M}$ which is diagonalisable w/ basis of eigenvectors $\underline{v}_j$, $j = 1, 2 \cdots n$

1. Let $M\underline{v}_j = \lambda_j \underline{v}_j$

$$L = (\underline{v}_1, \cdots \underline{v}_j, \cdots \underline{v}_n)$$

2. L must be invertible

so $L^{-1}L = I$

$$L^{-1} = \begin{pmatrix} \underline{w}_1 \\ \vdots \\ \underline{w}_k \\ \underline{w}_n \end{pmatrix}$$

3. $L^{-1}L = I$ can be expressed by $\begin{pmatrix} \underline{w}_1 \\ \vdots \\ \underline{w}_k \\ \vdots \\ \underline{w}_n \end{pmatrix}(\underline{v}_1 \cdots \underline{v}_j \cdots \underline{v}_n)$

$$\begin{pmatrix} \underline{w}_1\underline{v}_1 & \cdots & \underline{w}_1\underline{v}_j & \cdots & \underline{w}_1\underline{v}_n \\ \vdots & & \vdots & & \\ \underline{w}_k\underline{v}_1 & \cdots & \underline{w}_k\underline{v}_j & & \underline{w}_k\underline{v}_n \\ \vdots & & \vdots & & \\ \underline{w}_n\underline{v}_1 & & \underline{w}_n\underline{v}_j & & \underline{w}_n\underline{v}_n \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \vdots & 1 & & \vdots \\ & & \ddots & \\ 0 & \cdots & 0 & 1 \end{pmatrix} = I$$

$$= \underline{w}_k\underline{v}_j = \delta_{jk}$$

4. Evaluate $D = L^{-1}ML$

$$D = L^{-1}M(\underline{v}_1, \cdots \underline{v}_j \cdots \underline{v}_n) = L^{-1}(M\underline{v}_1, \cdots M\underline{v}_j \cdots \underline{v}_n)$$

using 1: $D = L^{-1}(\lambda_1\underline{v}_1, \cdots \lambda_j\underline{v}_j \cdots \lambda_n\underline{v}_n)$

$$= \begin{pmatrix} \underline{w}_1 \\ \vdots \\ \underline{w}_k \\ \vdots \\ \underline{w}_n \end{pmatrix}(\lambda_1\underline{v}_1, \cdots \lambda_j\underline{v}_j \cdots, \lambda_n\underline{v}_n)$$

similar to 3:

$$\begin{pmatrix} \lambda_1\underline{w}_1\underline{v}_1 & \cdots & \lambda_j\underline{w}_1\underline{v}_j & \cdots & \lambda_n\underline{w}_1\underline{v}_n \\ \lambda_1\underline{w}_k\underline{v}_1 & & \lambda_j\underline{w}_k\underline{v}_j & \cdots & \lambda_n\underline{w}_k\underline{v}_n \\ \lambda_1\underline{w}_n\underline{v}_1 & \cdots & \lambda_j\underline{w}_n\underline{v}_j & \cdots & \lambda_n\underline{w}_n\underline{v}_n \end{pmatrix} = \begin{pmatrix} \lambda_1 & \cdots & 0 & \cdots & 0 \\ \vdots & & \lambda_j & & \vdots \\ 0 & \cdots & & & \lambda_n \end{pmatrix}$$

* L which diagonalises M are constructed as a matrix which columns are eigenvectors of M

$$L = [\,\square\ \square\,]$$

(eigenvector)

Special case where matrix is non diagonalisable:

Jordan blocks

characteristic eq^n

$J\underline{v} = \lambda\underline{v}$

leads to $\lambda^2 = 0$

eg. $J = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

Invariants & eigenvalues:

using note e.g. to demonstrate:

$D = L^{-1} M L$ — ①

$M \to M' = N^{-1} M N$

$\hookrightarrow N M' N^{-1} = N N^{-1} M N N^{-1}$

$M = N M' N^{-1}$

Sub into ① & $L' = N^{-1} L$

$D = L^{-1} M L = L^{-1} \overbrace{N M' N^{-1}}^{M} L$

$L' = N^{-1} L \quad L'^{-1} N \quad L'$

$L' L'^{-1} = N^{-1} L L'^{-1}$

$N I = \cancel{N} N^{-1} L L'^{-1}$ $\quad D = L'^{-1} N L'$

$L^{-1} N I = L^{-1} L L'^{-1}$

$L'^{-1} = L^{-1} N$

轉 basis 後 value 唔轉

Invariance ① diagonal matrix of eigenvalue D stays the same under the change of basis (Eigenvalues)

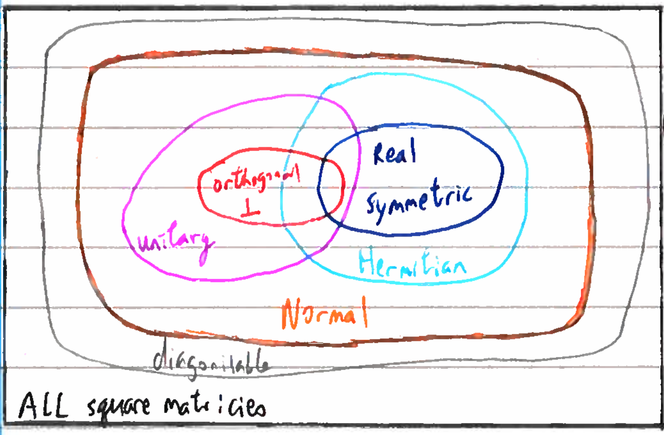
② Det of M (use Binet's formula for det)

③ Trace, $Tr(M)$ sum of all elements in the diagonal

An application of diagonalization → computing powers of a matrix

$D = L^{-1} M L \Rightarrow M^n = L D^n L^{-1}$

Summary of special matricies: (w/ diagonisability)



Normal: $M^+ M = M M^+$ ⟷ Hermitian: $H^+ = H$

Unitary: $U^+ U = U U^+ = I$

orthogonal: $O^T O = O O^T = I$

Symmetric: $S^T = S$ & Real

✱ ALL DIAGONISABLE ∵ ALL ARE NORMAL

Eigenvalue & Eigenvectors of Hermitian matricies

using: $H v_j = \lambda_j v_j$ — ①

$H v_k = \lambda_k v_k$ — ②

using ① $\quad \underbrace{v_k^+ H}_{} v_j = \lambda_j v_k^+ v_j$ — ③

$v_k^+ H^+ = (H v_k)^+ = \lambda_k^* v_k^+$

$\hookrightarrow v_k^+ H v_j = \lambda_k^* v_k^+ v_j$ — ④

$\hookrightarrow$ ③ & ④

$\Rightarrow \lambda_j v_k^+ v_j = \lambda_k^* v_k^+ v_j$

$(\lambda_j - \lambda_k^*) v_k^+ v_j = 0$

if $j = k$

$v_k^+ v_j = v_j^+ v_j = |v_j|^2 \neq 0$

$\Rightarrow \lambda_j = \lambda_j^*$

eigenvalues for Hermitian matricies are always real

if $j \neq k$

$\lambda_j \neq \lambda_k$

$\Rightarrow v_k^+ v_j = 0$

eigenvectors of Hermitian operators (+) associated w/ diff eigenvalues are orthogonal (⊥)

Diagonalising H:

eigenvectors of H → orthonormal

$D = U^{-1} H U \quad D = [\,]$

using transformation U, with columns equal to the eigenvectors of H, which are orthonormal

inner product (v_j, v_k)

$U = (v_1 \ldots v_j \ldots v_n)$ w/ $v_j^+ v_k = \delta_{jk}$

$\hookrightarrow$ using invariance

$U^{-1} = \begin{pmatrix} v_1^+ \\ \vdots \\ v_j^+ \\ \vdots \\ v_n \end{pmatrix} = U^+$

$U^+ U = U^{-1} U = I$

∴ UNITARY

✱ eigenvectors of a hermitian operator may always be chosen to form an orthonormal basis

✱ any hermitian matrix is diagonalisable AND can be diagonalised by a unitary transformation

✱ Real symmetric matricies can always be diagonalised by rotating the basis of the vector space