1. Differential Vector Operators

- partial derivatives $\frac{\partial \phi}{\partial x} = \partial_x \phi$

- $\hat{\underline{u}} = 1 = \sqrt{u_x^2 + u_y^2 + u_z^2}$

unit vector

grad ϕ:

$$\vec{\nabla}\phi = \hat{e}_x\left(\frac{\partial \phi}{\partial x}\right) + \hat{e}_y\left(\frac{\partial \phi}{\partial y}\right) + \hat{e}_z\left(\frac{\partial \phi}{\partial z}\right) = \begin{pmatrix} \partial_x \phi \\ \partial_y \phi \\ \partial_z \phi \end{pmatrix}$$

$\nabla\phi$ of a scalar field ϕ is always pointing towards the direction w/ maximum increase of ϕ || $\nabla\phi$ points along the maximum increase of ϕ

directional derivation of ϕ along $\hat{u}$: $g(\hat{u}) = \hat{\underline{u}} \cdot \nabla\phi = |\hat{u}|\,|\nabla\phi| \cos\alpha = |\nabla\phi| \cos\alpha$

- directional derivative vanishes if $\hat{u}$ is tangential to contour lines / surfaces given by $\phi(r) = C$: [directional derivative : $\hat{u} \cdot \nabla\phi$ $\cap d_1$ e_1)

$\nabla\phi$ is $\perp$ to hypersurfaces defined by $\phi(r) = C$

gradient of a rotationally symmetric field :

$\phi(\underline{r}) = f(r)$ with $r = |\underline{r}|$

given by : $f'(r)\,\hat{e}_r$ $\quad \hat{e}_r = \underline{r}/r$ which is the unit vector along direction of $\underline{r}$

- Total differential of fields

$$d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz$$

→ changing x by dx
↳ ϕ changes by $d\phi$

$= \nabla\phi \cdot dr$ where dr is vectorial element $\begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}$ // to $\partial_x \phi$

⇒ measures change $d\phi$ of ϕ as x, y, z are changed by x, y, z

- ~~Divergence & curls of vector fields~~

- Divergence & curl of vector field

$$A(r) = \begin{pmatrix} A_x(r) \\ A_y(r) \\ A_z(r) \end{pmatrix}$$

- $\mathrm{div}\, A = \nabla \cdot A = \partial_x A_x + \partial_y A_y + \partial_z A_z$

$$= \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} \cdot \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} = \begin{pmatrix} \partial_x A_x \\ \partial_y A_y \\ \partial_z A_z \end{pmatrix}$$

- $\nabla \cdot A \neq A \cdot \nabla$ - indicating sinks & source

- Scalar field

- $\mathrm{curl}\, \underline{A} = \nabla \times A = \cancel{\begin{vmatrix} e_x & e_y & e_z \\ \partial_x & \partial_y & \partial_z \\ A_x & A_y & A_z \end{vmatrix}} = \begin{pmatrix} \partial_y A_z - \partial_z A_y \\ \partial_z A_x - \partial_x A_z \\ \partial_x A_y - \partial_y A_x \end{pmatrix} = \begin{pmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ \partial_x & \partial_y & \partial_z \\ A_x & A_y & A_z \end{pmatrix}$ - indicating rotational flow

$$= \cancel{\begin{pmatrix} \partial_y & \partial_z \\ A_y & A_z \end{pmatrix} e_x - \begin{pmatrix} \partial_x & \partial_z \\ A_x & A_z \end{pmatrix} e_y + \begin{pmatrix} \partial_x & \partial_y \\ A_x & A_y \end{pmatrix} e_z}$$

- vector field

- Product Rule: $\frac{d}{dx} f(x) \cdot g(x) = f' \cdot g + f \cdot g'$

Setting $\phi(r), e(r) \Rightarrow$ scalar fields
$A(r), B(r) \Rightarrow$ vector fields

Proving 1. $\nabla(\phi e) = (\nabla \phi) e + (\nabla e) \phi$

2. $\nabla \cdot (\phi \underline{A}) = (\nabla \phi) \cdot A + \phi (\nabla \cdot A)$

(a): $\nabla(\phi e) = \begin{pmatrix} \partial_x \phi e \\ \partial_y \phi e \\ \partial_z \phi e \end{pmatrix} = \begin{pmatrix} (\partial_x \phi) e + (\partial_x e) \phi \\ (\partial_y \phi) e + (\partial_y e) \phi \\ (\partial_z \phi) e + (\partial_z \phi) e \end{pmatrix} = \begin{pmatrix} \partial_x \phi \\ \partial_y \phi \\ \partial_z \phi \end{pmatrix} e + \begin{pmatrix} \partial_x e \\ \partial_y e \\ \partial_z e \end{pmatrix} \phi$

$= (\nabla \phi) e + (\nabla e) \phi$

b. $\nabla \cdot (\phi \underline{A}) = \nabla \cdot \begin{pmatrix} \phi A_x \\ \phi A_y \\ \phi A_z \end{pmatrix} = \partial_x(\phi A_x) + \partial_y(\phi A_y) + \partial_z(\phi A_z)$

$= (\partial_x \phi) A_x + (\partial_x A_x) \phi + (\partial_y \phi) A_y + (\partial_y A_y) \phi + (\partial_z \phi) A_z + (\partial_z A_z) \phi$

$= \begin{pmatrix} \partial_x \phi \\ \partial_y \phi \\ \partial_z \phi \end{pmatrix} \cdot \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} + \phi (\partial_x A_x + \partial_y A_y + \partial_z A_z)$

$$= \underbrace{\nabla \phi \cdot A}_{\text{Scalar}} + \underbrace{\phi (\nabla \cdot A)}_{\text{Scalar}}$$

- 2nd Order Variations of fields, Laplace Operator:

consider $\phi(r)$ is a scalar field

$\nabla\phi(r)$ is a vector field

∇^2 Laplacian

$\Delta\phi := \nabla^2\phi = \text{div grad}\,\phi = \partial_x^2\phi + \partial_y^2\phi + \partial_z^2\phi$ — scalar field

↳ Laplacian of ϕ

$\Delta = \nabla^2 = \partial_x^2 + \partial_y^2 + \partial_z^2$ => called the laplacian operator

$\nabla\times\nabla\phi$ curl grad:

$$\nabla\times\nabla\phi = \nabla\times\begin{pmatrix}\partial_x\phi\\ \partial_y\phi\\ \partial_z\phi\end{pmatrix} = \begin{pmatrix}e_x & e_y & e_z\\ \partial_x & \partial_y & \partial_z\\ \partial_x\phi & \partial_y\phi & \partial_z\phi\end{pmatrix} = \begin{pmatrix}\partial_y\partial_z\phi - \partial_z\partial_y\phi\\ \partial_x\partial_z\phi - \partial_z\partial_x\phi\\ \partial_x\partial_y\phi \quad \partial_y\partial_x\phi\end{pmatrix} = 0$$

↳ partial derivatives are commute

$\nabla\cdot(\nabla\times A)$ div curl A

$\nabla\cdot(\nabla\times A) = 0$ => zero divergence

2. Multi-dimensional Integration

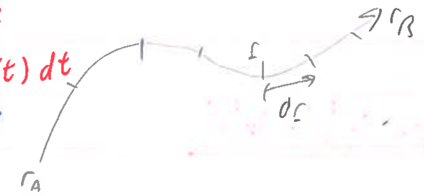
– Def Line integral

$$I = \int_C G(r) \cdot dr$$

* coordinate free definition

$$I = \int_{t_1}^{t_2} G(r(t)) \cdot r'(t)\, dt$$

$t \in [t_1, t_2]$



– breaking up the path into small displacements dr

– define integral as the sum over elementary contributions $G(r) \cdot dr$ in the limit $dr \to 0$

– result of a line integral is a scalar
↳ "Dot product" $G(r) \cdot dr$

– defines on which part is taken

Special case when $r_A = r_B$

$$\oint G(r) \cdot dr$$

loop integral

line element:

$$dr(t) = \begin{pmatrix} dx(t) \\ dy(t) \\ dz(t) \end{pmatrix} = \begin{pmatrix} x'(t)\,dt \\ y'(t)\,dt \\ z'(t)\,dt \end{pmatrix} = r'(t)\,dt$$

– Conservative vector fields
↳ if line integrals do not depend on which path is taken from r_A to r_B

checking whether field is conservative:

$G(r)$ is conservative when curl $G = \nabla \times G = 0$

$G = G(r(t))$

$$G(r) = \nabla \phi$$

↑ potential of $G(r)$

considering a path C parametrised by $r(t)$ where $t \in [a,b]$ $r(a) = r_A$; $r(b) = r_B$

$$I = \int_C G(r) \cdot dr = \int_a^b G(r(t)) \cdot r'(t)\,dt = \int_a^b \nabla\phi(r(t)) \cdot r'(t)\,dt$$

$d\phi = \nabla\phi \cdot dr$

$$= \int_a^b \frac{d}{dt}\phi(r(t))\,dt = [\phi(r(t))]_a^b = \phi(r_B - r_A)$$

∴ independent of the path taken!
just depends on initial & final positions

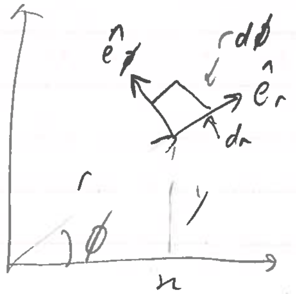
Solving $G(r) = \nabla\phi$
Solve simultaneous Eq^n

$G(r)$ is conservative when: scalar field $\phi(r)$: $G = \nabla\phi$
$\nabla \times G = 0$

Area integrals

$$A = \int_A \sigma(r)\, dA = \int_{x_1}^{x_2} dy \int_{x_1}^{x_2} x \quad \sigma(x,y)$$

polar coordinates



coordinate transformation

$x = r\cos\phi$

$y = r\sin\phi$

inverse transformation:

$r = \sqrt{x^2+y^2}$

$\tan\phi = y/x$

$\phi = \arctan(y/x)$

line element:

$$d\underline{r} = \hat{e}_r\, dr + \hat{e}_\phi\, d\theta$$

$$\hat{e}_r = \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix}$$

$$\hat{e}_\theta\, \cancel{d\theta} = \begin{pmatrix} -\sin\phi \\ \cos\phi \end{pmatrix}$$

orthogonal & have lengths of 1

$$\underline{r} = r\begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix} = r\begin{pmatrix} x \\ y \end{pmatrix}$$

$$d\underline{r} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}$$

using total differential

$x = x(r,\phi)$

$y = y(r,\phi)$

$$dr = \begin{pmatrix} \frac{\partial x}{\partial r} dr + \frac{\partial x}{\partial \phi} d\phi \\ \frac{\partial y}{\partial r} dr + \frac{\partial y}{\partial \phi} d\phi \end{pmatrix} = \begin{pmatrix} \cos\phi\, dr - r\sin\phi\, d\phi \\ \sin\phi\, dr + r\cos\phi\, d\phi \end{pmatrix} = \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix} dr + r\begin{pmatrix} -\sin\phi \\ \cos\phi \end{pmatrix} d\phi$$

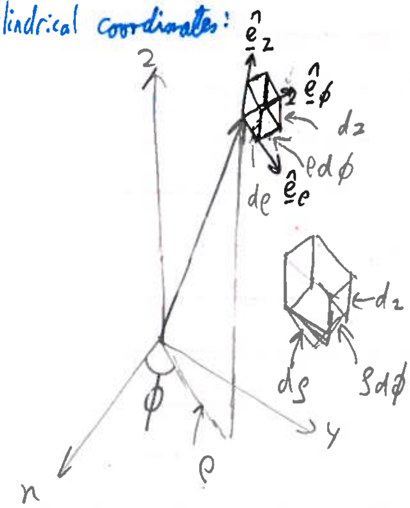
area element: $dA = dr \cdot r\, d\phi$

$= r\, dr\, d\phi$

Volume integrals:

$$I = \int_V f(\underline{r})\, dV = \iiint_V f(x,y,z)\, dx\, dy\, dz$$

Cylindrical coordinates:



$x = \rho\cos\phi$

$y = \rho\sin\phi$

$z = z$

~~area element = $\rho d\rho \cdot d\phi \cdot dz$~~

~~area element~~ =

volume element = $\rho\, d\rho\, d\phi\, dz$

Line element: $x = x(\rho,\phi)$, $y = y(\rho,\phi)$, $z = z$

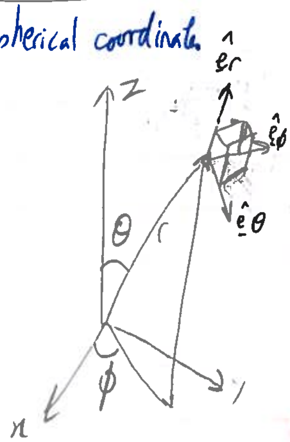
$$d\underline{r} = \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial \rho} d\rho + \frac{\partial x}{\partial \phi} d\phi \\ \frac{\partial y}{\partial \rho} d\rho + \frac{\partial y}{\partial \phi} d\phi \\ dz \end{pmatrix}$$

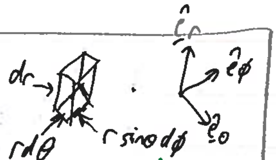
$$= \begin{pmatrix} \cos\phi\, d\rho - \rho\sin\phi\, d\phi \\ \sin\phi\, d\rho + \rho\cos\phi\, d\phi \\ dz \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} \cos\phi \\ \sin\phi \\ 0 \end{pmatrix}}_{\hat{e}_\rho d\rho} d\rho + \underbrace{\begin{pmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{pmatrix}}_{\hat{e}_\phi} \rho\, d\phi + \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{\hat{e}_z} dz$$

all orthogonal

spherical coordinates





$x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta$

volume element

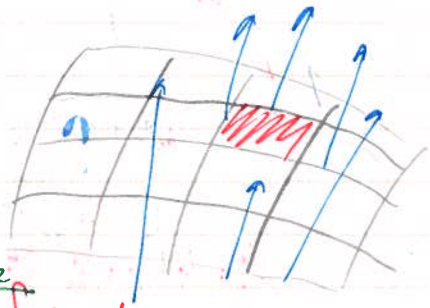
$r^2 dr \sin\theta\, d\theta\, d\phi$

$$d\underline{r} = \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi\, dr + r\cos\theta\cos\phi\, d\theta - r\sin\theta\sin\phi\, d\phi \\ \sin\theta\sin\phi\, dr + r\cos\theta\sin\phi\, d\theta + r\sin\theta\cos\phi\, d\phi \\ \cos\theta\, dr - r\sin\theta \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}}_{\hat{e}_r} dr + r \underbrace{\begin{pmatrix} \cos\theta\cos\phi \\ \cos\theta\sin\phi \\ -\sin\theta \end{pmatrix}}_{\hat{e}_\theta} d\theta + r\sin\theta \underbrace{\begin{pmatrix} \sin\phi \\ \cos\phi \\ 0 \end{pmatrix}}_{\hat{e}_\phi} d\phi$$

all orthogonal to each other

Surface integrals



1. Area of surface

$I_1 = \int_S f(\underline{r})\, dS$

- divide S into small area elements dS
  ↳ summing individual contributions $f(\underline{r})dS$ in the limit $dS \to 0$

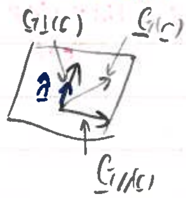
eg total charge on a curved plate

eg. total flux of blood cells through membrane

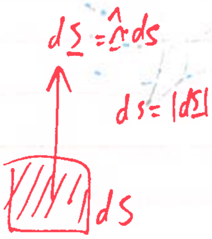
2. Total flux of a vector field $\underline{G}(\underline{r})$ through surface S

$I_2 = \int_S (\underline{G}\cdot\hat{\underline{n}})\, dS = \int_S \underline{G}\cdot d\underline{S}$

- decomposing $\underline{G}$ into components $//$ & $\perp$ to S



- only $G_\perp(\underline{r})$ contributes through S
  - assume we know normal vector $\hat{\underline{n}}$ $\perp$ to S at pt $\underline{r}$



$\underline{G}(\underline{r})\cdot\hat{\underline{n}} = \underline{G}_\perp(\underline{r})\cdot\hat{\underline{n}} + \underbrace{\underline{G}_{//}(\underline{r})\cdot\hat{\underline{n}}}_{0}$

$= \underline{G}_\perp(\underline{r})\cdot\hat{\underline{n}}$

$\boxed{\underline{G}(\underline{r})\cdot\hat{\underline{n}} = G_\perp}$

integrating $G_\perp(\underline{r})$ scalar gives type 1 surface area

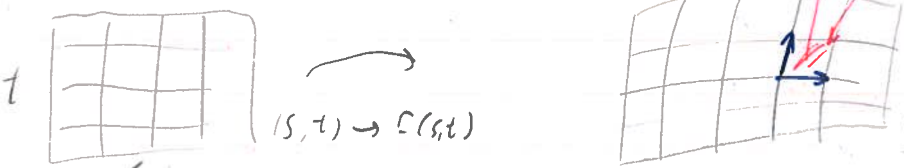
$I_2 = \int_S \underline{G}(\underline{r})\cdot\hat{\underline{n}}\, dS = \int_S \underline{G}(\underline{r})\cdot d\underline{S}$

Parametrising surfaces:

$|a \times b| = A$
$= ab \sin\theta$

$d\underline{S} = \left(\frac{\partial \underline{r}}{\partial s} \times \frac{\partial \underline{r}}{\partial t}\right) ds\, dt = d\underline{u} \times d\underline{v}$

ds



$du = \frac{\partial \underline{r}}{\partial t} dt$

$dv = \frac{\partial \underline{r}}{\partial s} ds$

$s \in [s_1, s_2]$, $t \in [t_1, t_2]$ ($s \in [s_{min}, s_{max}]$, $t \in [t_{min}, t_{max}]$)

$d\underline{S} = d\underline{u} \times d\underline{v} = \left(\frac{\partial \underline{r}}{\partial s} \times \frac{\partial \underline{r}}{\partial t}\right) ds\, dt$

$$I_1 = \int_{s_1}^{s_2} ds \int_{t_1}^{t_2} dt\, f(\underline{r}(s,t)) \left|\frac{\partial \underline{r}}{\partial s} \times \frac{\partial \underline{r}}{\partial t}\right|$$

$$I_2 = \int_{s_1}^{s_2} ds \int_{t_1}^{t_2} dt\, G(\underline{r}(s,t)) \cdot \left(\frac{\partial \underline{r}}{\partial s} \times \frac{\partial \underline{r}}{\partial t}\right)$$

$$d\underline{r} = d\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial s} ds + \frac{\partial x}{\partial t} dt \\ \partial_s y\, ds + \partial_t y\, dt \\ \partial_s z\, ds + \partial_t z\, dt \end{pmatrix}$$

$$= \underbrace{\frac{\partial \underline{r}}{\partial s} ds}_{} + \underbrace{\frac{\partial \underline{r}}{\partial t} dt}_{}$$

changing s by ds, r changes by $d\underline{u}$

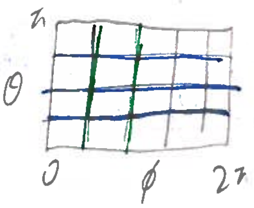
changing t by dt, r changes by $d\underline{v}$

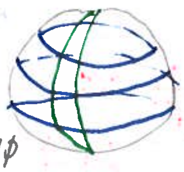
1 Spherical angles

By definition

$\theta \in [0, \pi]$
$\phi \in [0, 2\pi]$

$$\underline{r}(\theta, \phi) = R\begin{pmatrix} \sin\theta \cos\phi \\ \sin\theta \sin\phi \\ \cos\theta \end{pmatrix}$$





$$d\underline{S} = \left(\frac{\partial \underline{r}}{\partial \theta} \times \frac{\partial \underline{r}}{\partial \phi}\right) d\theta\, d\phi$$

$$= R^2 \begin{pmatrix} \cos\theta \cos\phi \\ \cos\theta \sin\phi \\ -\sin\theta \end{pmatrix} \times \begin{pmatrix} -\sin\theta \sin\phi \\ \sin\theta \cos\phi \\ 0 \end{pmatrix} d\theta\, d\phi$$

$$= R^2 \begin{pmatrix} \sin^2\theta \cos\phi \\ \sin^2\theta \sin\phi \\ \sin\theta \cos\theta \end{pmatrix} d\theta\, d\phi$$

$$= R^2 \sin\theta\, d\theta\, d\phi \underbrace{\begin{pmatrix} \sin\theta \cos\phi \\ \sin\theta \sin\phi \\ \cos\theta \end{pmatrix}}_{\hat{e}_r}$$

$$d\underline{S} = R^2 \sin\theta\, d\theta\, d\phi\, \hat{\underline{e}}_r$$

Surfaces defined by $z = g(x,y)$

[upper half sphere of $r = R$] $z = \sqrt{R^2 - x^2 - y^2}$

parametrise:

$$\underline{r} = \underline{r}(x,y) = \begin{pmatrix} x \\ y \\ g(x,y) \end{pmatrix}$$

vectorial surface element:

$$d\underline{S} = \left(\frac{\partial \underline{r}}{\partial x} \times \frac{\partial \underline{r}}{\partial y}\right) dx\, dy$$

$$= \begin{pmatrix} 1 \\ 0 \\ \partial_x g \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ \partial_y g \end{pmatrix} dx\, dy =$$

$$d\underline{S} = \begin{pmatrix} -\partial_x g \\ -\partial_y g \\ 1 \end{pmatrix} dx\, dy$$

$$dS = |d\underline{S}| = \sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2}\, dx\, dy$$

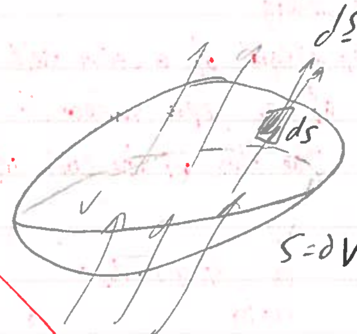
$\nabla\cdot\underline{G}(\underline{r})$ gives sources & sinks

pts outwards

Gauss's Divergence Theorem

$$\int_{S=\partial V} \underline{G}(\underline{r})\cdot d\underline{S} = \int_V \nabla\cdot\underline{G}(\underline{r})\,dV$$

surface integral for flux = volume integral

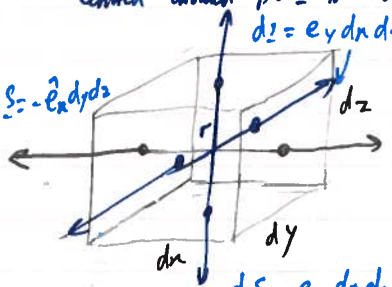


state that:
flux of a vector field $\underline{G}(\underline{r})$ through a closed surface $S=\partial V$ is equal to the integral of div $\underline{G}$ over enclosed volume of V.

Proving the theorem via sketch

Step 1:

prove the theorem for an infinitesimal cuboid of volume $dV = dx\,dy\,dz$ centred around pt $\underline{r}$ in volume V.



Step 2:

field where $\underline{r}$: $\vec{r} + \frac{dx}{2}\hat{e}_x$

x direction, 半個 dx

infinitesimal flux, dF through surface when $\underline{G}(\underline{r})$ is a vector field:

$$\begin{aligned} dF = {} & \vec{G}\left(\vec{r} + \frac{dx}{2}\hat{e}_x\right)\cdot\hat{e}_x\,dy\,dz + \vec{G}\left(\vec{r} - \frac{dx}{2}\hat{e}_x\right)\cdot(-\hat{e}_x)\,dy\,dz \\ & + \vec{G}\left(\vec{r} + \frac{dy}{2}\hat{e}_y\right)\cdot\hat{e}_y\,dx\,dz + \vec{G}\left(\vec{r} - \frac{dy}{2}\hat{e}_y\right)\cdot(-\hat{e}_y)\,dx\,dz \\ & + \vec{G}\left(\vec{r} + \frac{dz}{2}\hat{e}_z\right)\cdot\hat{e}_z\,dx\,dy + \vec{G}\left(\vec{r} - \frac{dz}{2}\hat{e}_z\right)\cdot(-\hat{e}_z)\,dx\,dy \end{aligned}$$

focusing on first line:

$$\begin{aligned} & \left[G_x\left(\vec{r} + \frac{dx}{2}\hat{e}_x\right) - G_x\left(\vec{r} - \frac{dx}{2}\hat{e}_x\right)\right]dy\,dz \\ & = \left[\left(\cancel{G_x(\underline{r})} + \partial_x G_x(\underline{r})\frac{dx}{2}\right) - \left(\cancel{G_x(\vec{r})} - \partial_x G_x(\underline{r})\frac{dx}{2}\right)\right]dy\,dz \\ & = \partial_x G_x(\underline{r})\,dx\,dy\,dz \end{aligned}$$

$$\therefore\ dF = (\partial_x G_x + \partial_y G_y + \partial_z G_z)\,dx\,dy\,dz = \nabla\cdot\underline{G}\,dV$$

Summing fluxes $dF = \underline{G}\cdot d\underline{S}$ through surfaces of all cuboids → only surface contributions survive, internal contributions cancel out

$$\int_V dF = \int_{S=\partial V} dF = \int_{S=\partial V} \underline{G}\cdot d\underline{S}$$

Step 3: consider integral over volume V with surface $S=\partial V$
→ defined as 'sum' over little cuboids dV

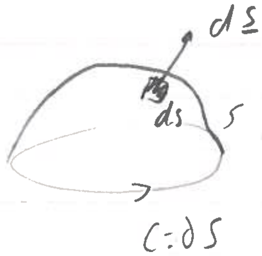
integrating: $\int_V \mathrm{div}\,\underline{G}\,dV$,

Stoke's Theorem :

~~loop integ~~

loop integral of a vector field $\underline{G}(\underline{r})$ around the boundary $C = \partial S$ of an <u>open</u> surface S is equal to the flux of the curl of the vector field, $\nabla \times G$ through the surface

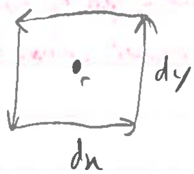
$$\oint_{C=\partial S} \underline{G}(\underline{r}) \cdot d\underline{r} = \int_S \nabla \times \underline{G} \cdot d\underline{S}$$



relative orientation of $d\underline{S}$ & C satisfies the right hand rule

Proving Stoke's theorem

1. start w/ infinitesimal surface area element dS
   - introduce local coordinate system (x, y) containing patch



$d\underline{S} = \hat{e}_z \, dx \, dy$

2. $\oint \underline{G}(\underline{r}) \cdot d\underline{r}$ given by the sum of contributions from 4 sides

$$dI = \underline{G}\left(\underline{r} - \frac{dy}{2}\hat{e}_y\right) \cdot \hat{e}_x \, dx + \underline{G}\left(\underline{r} + \frac{dx}{2}\hat{e}_x\right) \cdot \hat{e}_y \, dy + \underline{G}\left(\underline{r} + \frac{dy}{2}\hat{e}_y\right) \cdot (-\hat{e}_y) \, dx + \underline{G}\left(\underline{r} - \frac{dx}{2}\hat{e}_x\right) \cdot -\hat{e}_y \, dy$$

$$= \left[G_x\left(r - \frac{dy}{2}\hat{e}_y\right) - G_x\left(r + \frac{dy}{2}\hat{e}_y\right)\right] dx + \left[G_y\left(r + \frac{dx}{2}\hat{e}_x\right) - G_y\left(r - \frac{dx}{2}\hat{e}_x\right)\right] dy$$

$$= -\partial_y G_x(r) \, dx \, dy + \partial_x G_y(\underline{r}) \, dx \, dy = \nabla \times G \,|_z \, dx \, dy = (\nabla \times G) \cdot \hat{e}_z \, dx \, dy$$

$$dI = (\nabla \times \underline{G}) \cdot d\underline{S}$$

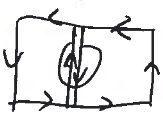
3. Summing up the individual loop integrals dI from all surface area elements

↳ integrate over surface S.

$$\text{R.H.S} = \int_S (\nabla \times \underline{G}) \cdot d\underline{S}$$

↳ internal contributions cancel

↳ only contributions from the boundary surf survive



$$\int_S dI = \int_{C=\partial S} \underline{G} \cdot d\underline{C}$$

ODEs

- seperable → A-level stuff

$$\frac{d}{dx}(f\cdot g) = f\cdot g' + g\cdot f'$$

$$\frac{d}{dx}(S\cdot y) = S\frac{dy}{dx} + y\,S'$$

- Linear 1st order ODEs (non-seperable)

Integrating factor method:

$$\frac{dy}{dx} + P(x)y = Q(x)$$

$$S(x)\left[\frac{dy}{dx} + P(x)y\right] = [Q(x)]S(x) \rightarrow S(x)\frac{dy}{dx} + P(x)S(x)y = S(x)Q(x)$$

$$S' = P(x)S(x) = \frac{dS}{dx} \Rightarrow \text{finding } S(x) \text{ that satisfies } P(x)S(x) = \frac{dS}{dx}$$

$$S(x) = e^{\int P(x)\,dx}$$

integrating factor

$$\Rightarrow S\frac{dy}{dx} + y\frac{dS}{dx} = S(x)Q(x) = \frac{d}{dx}(S\cdot y)$$

$$\frac{d(Sy)}{dx} = SQ$$

$$\boxed{y = \frac{1}{S(x)}\left(\int S(x)Q(x)\,dx + c\right) \qquad S(x) = e^{\int P(x)\,dx}}$$

Perfect differential method / Exact-differential method

$$Q(x,y)\frac{dy}{dx} + P(x,y) = 0$$

$$P(x,y)\,dx + Q(x,y)\,dy = 0$$

Assume: P & Q are partial diff wrt x & y of $f(x,y)$

I. $\frac{\partial f}{\partial x} = P(x,y)$ II. $\frac{\partial f}{\partial y} = Q(x,y)$

$$\therefore\ df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0$$

$$df = 0 \quad \therefore f(x,y) = C \quad C \in \mathbb{R}$$

I: $f(x,y) = \int \frac{\partial f}{\partial x}\,dx = f_I + g(y)$

II: $f(x,y) = \int \frac{\partial f}{\partial y}\,dy = f_{II} + h(x)$

determining $g(y)$ & $h(x)$ so that $f(x,y) = f_I + g(y) = f_{II} + h(x)$

so that $f(x,y) = \text{Const}$

Can use if

$$\frac{\partial P}{\partial y} = \frac{\partial^2 f}{\partial y\partial x} = \frac{\partial^2 f}{\partial x\partial y} = \frac{\partial Q}{\partial x}$$

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

Sufficient condition

$y = y_{cf} + y_{PI}$

2nd order:

y_{cf} — Homogeneous: $y'' + py' + qy = 0$

$y = e^{kx}$

$k_{1,2} = \dfrac{-p \pm \sqrt{p^2 - 4q}}{2}$

1. Real roots:

$\frac{p^2}{4} - q < 0$

$y = Ae^{k_1 x} + Be^{k_2 x}$

2. Complex roots:

$\frac{p^2}{4} - q < 0$

$k_{1,2} = \overset{Re}{\alpha} \pm \overset{Im}{i\beta}$

$\alpha = -\frac{p}{2}$, $\beta = \sqrt{q - p^2/4}$

$y = Ae^{k_1 x} + Be^{k_2 x}$

$= Ae^{(\alpha + i\beta)x} + Be^{(\alpha - i\beta)x}$

$= e^{\alpha x}(Ae^{i\beta x} + Be^{-i\beta x})$

$y = e^{\alpha x}[\underbrace{(A+B)}_{C}\cos(\beta x) + \underbrace{i(A-B)}_{D}\sin(\beta x)]$

$y = e^{\alpha x}[C\cos(\beta x) + D\sin(\beta x)]$

$y = e^{\alpha x}[(A+B)\cos(\beta x) + i(A-B)\sin(\beta x)]$

3. Degenerate roots

$\frac{p^2}{4} - q = 0$

$y = Ae^{kx} + Bxe^{kx}$

y_{PI} — Inhomogeneous

1. Polynomials

$f(x) = A_0 + A_1 x \cdots\cdots + A_n x^n$

$\hookrightarrow y_{PI} = \alpha_0 + \alpha_1 x + \cdots\cdots \alpha_n x^n$

2. exponentials

$f(x) = A_0 e^{wx}$

$y_{PI} = \alpha_0 e^{wx}$

$A_0 e^{wx} = \alpha_0 \quad y'' + py' + q$

$= \alpha_0 (w^2 + pw + q) e^{wx}$

$\alpha_0 = \dfrac{A_0}{w^2 + pw + q}$

if $w = k_{1,2}$ (from y_{11})

$y_{PI} = Bxe^{wx}$

3. sin U cos

$f(x) = A_0 \cos(wx) + A_1 \sin(wx)$

$y_{PI} = \alpha_0 \cos(wx) + \alpha_1 \sin(wx)$

comparing coefficient

Linear Algebra 芳 special relativity

Kronecker Delta $\delta_{ij} = \begin{cases} 1 & (//) \\ 0 & (\perp) \end{cases}$

$\hat{e}_x = \hat{e}_1$ $\qquad \hat{e}_1 \cdot \hat{e}_1 = \hat{e}_2 \cdot \hat{e}_2 = \hat{e}_3 \cdot \hat{e}_3 = 1$ (// to each other)

$\hat{e}_y = \hat{e}_2 \quad \Rightarrow$

$\hat{e}_z = \hat{e}_3$ $\qquad \hat{e}_1 \cdot \hat{e}_2 = \hat{e}_2 \cdot \hat{e}_3 = \hat{e}_3 \cdot \hat{e}_1 = 0$ ($\perp$ to each other)

↳ Summarising : $\hat{e}_i \cdot \hat{e}_j = \delta_{ij} = \begin{cases} 1 & (i=j) \to // \\ 0 & (i \neq j) \to \perp \end{cases}$

Kronecker delta

$S = 3$

$\underline{v} = v_1 \hat{e}_1 + v_2 \hat{e}_2 + v_3 \hat{e}_3$

Coefficient $v_i \Rightarrow v_i = \hat{e}_i \cdot \underline{v}$

$$= \hat{e}_i \cdot \begin{pmatrix} v_i \\ v_j \\ v_k \end{pmatrix} = \hat{e}_i \cdot v_i + 0 + 0 = \underline{\underline{v_i}}$$

Scalar product:

$$\underline{u} \cdot \underline{v} = (u_1 \hat{e}_1 + u_2 \hat{e}_2 + u_3 \hat{e}_3 \dots u_n \hat{e}_n) \cdot (v_1 \hat{e}_1 + v_2 \hat{e}_2 + v_3 \hat{e}_3 \dots v_n e_n)$$

$$\underline{u} \cdot \underline{v} = \sum_{i=1}^{n} u_i v_i$$

length of vector: $v = |\underline{v}| = \sqrt{v_1^2 + v_2^2 \dots v_n^2}$

$\underline{v} = 1 \to$ unit vector

$\underline{v} = 0 \to$ null vector

3D space

↳ defined as one where there are 3 (but no more) orthonormal linearly dependent vectors $\hat{e}_i$

Linear Dependence:

Set of vectors are linearly dependent when it is possible to find a set of scalar coefficients such that

$$c_1 \underline{x}_1 + c_2 \underline{x}_2 + \dots c_n \underline{x}_n = 0$$

Else ⇒ vectors are linearly independent

✱ Basis vectors are linearly independent

∴ no linear combination of $\hat{e}_i$ ~~which~~ which vanishes (unless α, β are all zero).

$$\hat{e}_3 \neq \alpha \hat{e}_1 + \beta \hat{e}_2$$

n- Dimensional Linear Vector Space $(a \in S, b \in S)$

Def:

$\underline{c} = \underline{a} + \underline{b} = \underline{b} + \underline{a}$ → Commutative

$(\underline{a} + \underline{b}) + \underline{c} = \underline{a} + (\underline{b} + \underline{c})$ → Associative

Exists a null vector $\underline{0} \in S$

↳ $\underline{a} + \underline{0} = \underline{a}$

$\underline{a} \in S \Rightarrow \lambda \underline{a} \in S \quad (\lambda \in \mathbb{C})$

$\lambda(\underline{a} + \underline{b}) = \lambda \underline{a} + \lambda \underline{b}$

$\lambda(\mu \underline{a}) = (\lambda\mu)\underline{a} \quad (\mu \in \mathbb{C})$

for every vector $\underline{a}$ → exist a unique vector $-\underline{a}$

↳ $\underline{a} + (-\underline{a}) = 0$

Basis vectors & components:

– not assumed that the basis vectors are unit vectors $\hat{e}_i$ or orthogonal to one another $\delta_{ij} \neq 0$

basis of a vector space is a set of linearly independent vectors that span the full space

$$\underline{v} = \sum_{i=1}^{n} v_i \underline{e}_i$$

Definition of scalar product in S_n
(only // components contribute)

$\underline{u} \cdot \underline{v} = u_1^* v_1 + u_2^* v_2 + \cdots + u_n^* v_n$

$\underline{w} \cdot (\alpha \underline{u} + \beta \underline{v}) = \alpha \underline{w} \cdot \underline{u} + \beta \underline{w} \cdot \underline{v}$

$\underline{u} \cdot \underline{v} = 0 \quad (\perp)$

$v_i = \hat{\underline{e}}_i \cdot \underline{v}$

$\underline{v} \cdot \underline{u} = v_1^* u_1 + v_2^* u_2 + \cdots + v_n^* u_n = (\underline{u} \cdot \underline{v})^*$

Matrices → operators allowing you to do linear transformation on vectors

Linear transformation

– expressing operation : $\hat{A}$ in terms of basis

$$\hat{\underline{e}}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \qquad \underline{a}_1 = \hat{A} \hat{\underline{e}}_1$$

$$\underline{a}_1 = \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \\ \vdots \\ a_{n1} \end{pmatrix}$$

acting on all basis vectors $\hat{\underline{e}}_j$ (generalized)

$$\underline{a}_j = \hat{A} \hat{\underline{e}}_j = \begin{pmatrix} a_{1j} \\ a_{2j} \\ a_{3j} \\ \vdots \\ a_{nj} \end{pmatrix}$$

$$\underline{a}_j = a_{1j} \hat{\underline{e}}_1 + a_{2j} \hat{\underline{e}}_2 + \cdots + a_{nj} \hat{\underline{e}}_n = \sum_{i=1}^{n} a_{ij} \cdot \hat{\underline{e}}_i$$

∵ $\hat{e}_i$ has 1 in i^{th} position & 0 in every where else

evaluating action of $\hat{A}$ on $\underline{v} = \sum_j v_j \hat{e}_j$

$$\underline{u} = \hat{A} \underline{v} = \sum_j v_j (\hat{A} \hat{\underline{e}}_j) = \sum_{ij} a_{ij} v_j \hat{\underline{e}}_i$$

also:

$$\underline{u} = \sum_i u_i \hat{\underline{e}}_i = \sum_{ij} a_{ij} v_j \hat{\underline{e}}_i$$

comparing coefficients

$$u_i = \sum_{j=1}^{n} a_{ij} v_j$$

→ vector undergone linear transformation

Set of no. a_{ij} represents $\hat{A}$

$$\underline{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & & a_{nn} \end{pmatrix}$$

where it is a square matrix

Y_{ij}

row i, column j

Matrix add & subtract.

– commutative (調) & associative (brackets)

$\cancel{M_{ij}+}$ $M_{ij} = a_{ij} + b_{ij}$

Matrix multiplication

(月第 term 1)

$\cdot\ \underline{M} = m_{ij} = (\underline{A}\underline{B})_{ij}$

$= \sum_{k=1}^{\hat{}} a_{ik} b_{kj} \equiv a_{ik} b_{kj}$

non commutative $\underline{A}\underline{B} \neq \underline{B}\underline{A}$

associative : $(\underline{A}\underline{B})\underline{C} = \underline{A}(\underline{B}\underline{C})$

associative proof :

$[(AB)C]_{il} = \sum_j (AB)_{ij} c_{jl}$

$= \sum_j \sum_k (a_{ik} b_{kj}) c_{jl} = \sum_j \sum_k a_{ik}(b_{kj} c_{jl})$

$= \sum_k a_{ik} (BC)_{kl} = [A(BC)]_{il}$

Considering

$\underline{w} = \hat{B}\underline{v}$

$\underline{u} = \hat{A}\underline{w}$

$\underline{u} = \hat{A}\hat{B}\underline{v} = \hat{C}\underline{v}$

[operation terminology]

finding matrix representation of $\hat{C}$:

$w_i = \sum_j b_{ij} v_j$

$u_k = \sum_i a_{ki} w_i = \sum_{i,j} a_{ki} b_{ij} v_j$

$= \sum_j C_{kj} v_j$

$C_{kj} = \sum_{i=1}^{\hat{}} a_{ki} b_{ij}$

Determinants

(月第 term 1)

$2\times2: \quad \det A = \begin{vmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{vmatrix}$

$= a_{11}a_{22} - a_{12}a_{21}$

3x3

$$\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

cofactor of matrix element: $C_{ij} = (-1)^{i+j} \det M_{ij}$

M_{ij} is the matrix remaining when row i & column j have been removed from the original matrix

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

4x4 → reduce to 3x3 → reduce to 2x2 ez

Properties of determinants:

1. If rows are written as columns, columns written as rows
↳ det is unchanged

$$\Delta' = \begin{vmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \Delta$$

2. Determinant vanishes if a row/column has all zeroes

3. If we multiply a row/column by a constant,
↳ det will also be multiplied by the constant

$$\Delta' = \begin{vmatrix} \alpha a_{11} & \alpha_{12} \\ a_{21} & a_{22} \end{vmatrix} = \alpha a_{11} a_{22} - \alpha_{1221} = \alpha \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

4. If 2 rows/columns are multiples of each other,
det = 0

if $a_{i2} = \alpha a_{i1}$ for $i = 1, 2$

$\Delta = \alpha a_{11} a_{21} - \alpha a_{11} a_{21} = 0$

can be used to test for linear independency

$\underline{u} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ $\underline{v} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$ $\underline{w} = \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}$

$$\begin{vmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{vmatrix} = 0$$

5. If a pair of row/column interchange
det changes sign

$$\Delta' = \begin{vmatrix} a_{12} & a_{11} \\ a_{22} & a_{21} \end{vmatrix} = a_{12}a_{21} - a_{11}a_{22}$$

$$\Delta' = -(a_{11}a_{22} - a_{12}a_{21}) = -\Delta$$

6. Adding a multiple of one row/column to another
↳ det doesn't change

(det remains unchanged if you add a multiple of a row to another row)

$$\Delta' = \begin{vmatrix} (a_{11} + \alpha a_{12}) & a_{12} \\ (a_{21} + \alpha a_{22}) & a_{22} \end{vmatrix} = (a_{11} + \alpha a_{12} a_{22}) - a_{12}(a_{21} + \alpha a_{22})$$

$$= a_{11}a_{22} + \cancel{\alpha a_{12} a_{22}} - a_{12}a_{21} - \cancel{\alpha a_{12} a_{22}}$$

$$= a_{11}a_{22} - a_{12}a_{21} = \Delta = 0$$

Determinant of a Matrix product

$|AB| = |A| \times |B|$ for any square matrices

determinant of AB product = product of A, B's determinants

$\therefore |AB| = |BA| = -1$

Equal Matrices:

$A = B$ if $a_{ij} = b_{ij}$ & all corresponding elements are equal, $n \times m$ are equal

Multiplication of Matrix by a scalar

$B = \lambda A \Rightarrow b_{ij} = \lambda a_{ij}$

Identity / unit matrix I

$IA = AI = A$ for square matrix A

$$I = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \vdots \\ 0 & 0 & 1 & \vdots \\ \cdots & & \cdots & 1 \end{pmatrix}$$

Component notation

$(IA)_{ij} = \sum_k \delta_{ik} a_{kj} = a_{ij}$

$(AI)_{ij} = \sum_k a_{ik} \delta_{kj} = a_{ij}$

$\therefore \delta_{ij} = 0$ for $i \neq j$

Transpose of a matrix

$$A^T = \begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}$$

$(A^T)_{ij} = a_{ji}$

Transpose of $n \times m$ = $m \times n$

Transpose of a transpose

$(A^T)^T = A$

if $A^T = A \Rightarrow A$ is symmetric

if $A^T = -A \Rightarrow A$ is antisymmetric

Transposing matrix products

transposing a product of matricies reverses order of multiplication

$(AB)^T = B^T A^T$

$(ABC)^T = C^T B^T A^T$

$C_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} \quad (C = AB)$

$C^T: \quad c_{ji} = \sum_{k=1}^{n} b_{jk} a_{ki}$

Orthogonal Matrices:

if $A^T A = I \Rightarrow A$ is orthogonal

$|A^T A| = |I| = 1$

$|A^T||A| = |I| = 1$ det of A^T = det of A

$|A^T| = |A|$

$\hookrightarrow |A||A| = |A^2| = |I| = 1$

$|A| = \pm 1$

Product of orthogonal matrices

suppose $C = AB$, A & B are orthogonal

C is also orthogonal

$C^T C = (AB)^T (AB)$

$C^T C = B^T \underbrace{A^T A}_{} B$

$\hookrightarrow C^T C = B^T B A^T A = I$

Complex conjugation

$(A^*)_{ij} = a^*_{ij}$

change signs in imaginary part

if $A = A^*$

$A \in \mathbb{R}$

Hermitian Conjugation

combining complex conjugation w/ transposition
(order does not matter)

$$A^{+} = (A^{T})^{*} = (A^{*})^{T}$$

$$(A^{+})^{+} = A$$

if $A^{+} = A \Rightarrow A$ is Hermitian

$A^{+} = -A \Rightarrow A$ is anti-Hermitian

✱ All real symmetric matrices are Hermitian

$$(AB)^{+} = B^{+}A^{+}$$

Proof:

Unitary Matrices

matrix U is unitary

if

$$U^{+}U = I$$

using $|AB| = |A| \times |B|$

$|U^{+}||U| = |I| = 1$

$|U^{*}|\ |U| = |I| = 1$ ($\because$ changing rows & columns in det does nth)

proof:

Trace of a matrix

↳ sum of the diagonal elements:

$$Tr(A) = a_{11} + a_{22} + a_{33} \dots a_{nn} = \sum a_{ii} = a_{ii}$$

using Einstein summation convention → repeated indicies in a term imply a sum

$$Tr(A) = Tr(A^{T})$$

Scaling of volume, Jacobian matrix

2D: $|A| = a \rightarrow$ any transformation scales the area by a factor 'a'

3D: volume is scaled by the det of the transformations (see notes for examples)

Q. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ w/ 2 vectors $\underline{u} = (a, c)$; $\underline{v} = (b, d)$ show that $|A|$ is the area of the //gram spanned by $\underline{u}$ & $\underline{v}$

Effect of a linear transformation on the volume / area element

we have 2 sets of coordinates: (x_1, x_2, x_3) & (x_1', x_2', x_3')

Change in prime coordinates using partial derivative

$$dx_1' = \frac{\partial x_1'}{\partial x_1} dx_1 + \frac{\partial x_1'}{\partial x_2} dx_2 + \frac{\partial x_1'}{\partial x_3} dx_3$$

$$dx_2' = \frac{\partial x_2'}{\partial x_1} dx_1 + \frac{\partial x_2'}{\partial x_2} dx_2 + \frac{\partial x_3'}{\partial x_3} dx_3$$

$$dx_3' = \frac{\partial x_3'}{\partial x_1} dx_1 + \frac{\partial x_3'}{\partial x_2} dx_2 + \frac{\partial x_3'}{\partial x_3} dx_3$$

example of spherical coordinates in notes

$\downarrow$ matrix form

$$\underbrace{\begin{pmatrix} dx_1' \\ dx_2' \\ dx_3' \end{pmatrix}}_{d\underline{r}'} = \underbrace{\begin{pmatrix} \frac{\partial x_1'}{\partial x_1} & \cdots & \\ \vdots & \ddots & \\ & & \frac{\partial x_3'}{\partial x_3} \end{pmatrix}}_{J} \underbrace{\begin{pmatrix} dx_1 \\ dx_2 \\ dx_3 \end{pmatrix}}_{d\underline{r}}$$

$d\underline{r}' = J d\underline{r}$

where J = Jacobian

$\det J$ = factor by which the volume element changes when we make the transformation

Multiplicative inverse of a matrix

$AA^{-1} = I$

$A = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}$

$A^{-1} = \frac{1}{|A|} \underbrace{C^T}$ adjoint matrix to A, A^{adj}

$C^T = \begin{pmatrix} \delta & -\gamma \\ -\beta & \alpha \end{pmatrix}$

$(A^{-1})_{ij} = \frac{1}{|A|} c_{ji}$

do prob to get meth

$C = \begin{pmatrix} +1(\ ) & -1(\ ) & 1(\ ) \\ -1(\ ) & 1(\ ) & -1(\ ) \\ 1(\ ) & -1(\ ) & 1(\ ) \end{pmatrix}$

$(\ )$ minor matrix det $\quad \Downarrow\ C^T$

PST 4 by Frank 3

$\det \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & & \vdots \\ \vdots & \vdots & \ddots & \\ 0 & 0 & 0 & a_{nn} \end{pmatrix} = a_{11} \det \begin{pmatrix} a_{22} & \cdots & a_{2n} \\ 0 & \ddots & \vdots \\ 0 & 0 & a_{nn} \end{pmatrix}$

$\det \begin{pmatrix} x & x & x & x \\ x & x & x & x \\ x & x & x & x \\ x & x & x & x \end{pmatrix} = \det \begin{pmatrix} x & x & x & x \\ 0 & x & x & x \\ 0 & x & x & x \\ 0 & x & x & x \end{pmatrix} = \det \begin{pmatrix} x & x & x & x \\ 0 & x & x & x \\ 0 & 0 & x & x \\ 0 & 0 & x & x \end{pmatrix}$

change to 0

target → make zeros below lower triangle

$= \det \begin{pmatrix} x & x & x & x \\ 0 & x & x & x \\ 0 & 0 & x & x \\ 0 & 0 & 0 & x \end{pmatrix} = \cancel{\det}$ → product of diagonal = det

Solutions of Linear Simultaneous Eq^n

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 \; a_{33}x_3 = b_3$$

$$\downarrow$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$\underline{A} \quad \underline{x} = \underline{b}$$

$$\underbrace{A^{-1}A}_{I=1} \quad \underline{x} = A^{-1}b$$

$$x = \underbrace{A^{-1}}_{\frac{1}{|A|}A^{adj}} b$$

$$\Rightarrow x_j = \Sigma (A^{adj})_{ji}\, b_i / |A|$$

★ Homogeneous eq^n

$$A\underline{x} = \underline{0}$$

Solutions involve $|A| = 0$

(eigenvalues)

Vanishing det

if $|A| = 0 \Rightarrow A$ is singular $\Rightarrow$ no inverse

at least one eq^n is not independent

↳ $n-1$ eq^n

→ solve by trial & error

Cramer's rule

$\Sigma (A^{adj})_{ji} b_i$ → det obtained

(proof in HW)

[by replacing the j^{th} column of A

by the column vector of $\underline{b}$

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

eg

⇓ using Cramer's rule

$$x_1 = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{23} & a_{33} \end{vmatrix} \div \Delta \qquad \Delta = |A|$$

$$x_2 = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix} \div \Delta$$

$$x_3 = \begin{vmatrix} a_{11} & a_{21} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix} \div \Delta$$

$3x_1 - 2x_2 - x_3 = 4$

$2x_1 + x_2 + 2x_3 = 10$

$x_1 + 3x_2 - 4x_3 = 5$

$$D = \begin{vmatrix} 3 & -2 & -1 \\ 2 & 1 & 2 \\ 1 & 3 & -4 \end{vmatrix} = \begin{vmatrix} 1 & -2 & -1 \\ 3 & 1 & 2 \\ 4 & 3 & -4 \end{vmatrix} = \begin{vmatrix} 0 & -2 & -1 \\ 5 & 1 & 2 \\ 0 & 3 & -4 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 5 & 0 \\ -2 & 1 & 3 \\ -1 & 2 & -4 \end{vmatrix} = 5 \, |8 - (-3)| = -55$$

$$D x_1 = \begin{vmatrix} 4 & -2 & -1 \\ 10 & 1 & 2 \\ 5 & 3 & -4 \end{vmatrix} = \begin{vmatrix} 4 & -4 & -1 \\ 10 & 5 & 2 \\ 5 & -5 & -4 \end{vmatrix} = \begin{vmatrix} 0 & -4 & 1 \\ 15 & 5 & 2 \\ 0 & -5 & 4 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 15 & 0 \\ -4 & 5 & -5 \\ 1 & 2 & 4 \end{vmatrix} = 15(-16+5) = 15(-11) = -165$$

$$x_1 = \frac{-165}{-55} = 3$$

$$D x_2 = \begin{vmatrix} 3 & 4 & -1 \\ 2 & 10 & 2 \\ 1 & 5 & -4 \end{vmatrix} = \begin{vmatrix} 3 & 4 & -1 \\ 3 & 15 & -2 \\ 1 & 5 & -4 \end{vmatrix} = \begin{vmatrix} 0 & -11 & 1 \\ 3 & 15 & -2 \\ 1 & 5 & -4 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & -11 & 1 \\ 0 & 0 & 10 \\ 1 & 5 & -4 \end{vmatrix}$$

$-2 - (\cancel{4/3})$ -12

$$= -\begin{vmatrix} 0 & 0 & 10 \\ 0 & -11 & 1 \\ 1 & 5 & -4 \end{vmatrix} = -10 \, |-(-11)| = -10(11) = -110$$

$$x_2 = \frac{-110}{-55} = 2$$

$$D x_3 = \begin{vmatrix} 3 & -2 & 4 \\ 2 & 1 & 10 \\ 1 & 3 & 5 \end{vmatrix} = \begin{vmatrix} 3 & -2 & 4 \\ 3 & 4 & 15 \\ 1 & 3 & 5 \end{vmatrix} = \begin{vmatrix} 0 & -6 & -11 \\ 3 & 4 & 15 \\ 1 & 3 & 5 \end{vmatrix} = \begin{vmatrix} 0 & -6 & -11 \\ 0 & -5 & 0 \\ 1 & 3 & 5 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & 1 \\ -6 & -5 & 3 \\ -11 & 0 & 5 \end{vmatrix} = -55$$

$\cancel{D x_3 = 1}$

$x_3 = 1$

Scalar product using Matrices

$\underline{v}$ & $\underline{w}$

$$\underline{V} \cdot \underline{W} = \underline{V}^T \underline{W} = \sum V_i V_i$$

— in cartesian space:

$$\underline{V} \cdot \underline{W} = (V_1\ V_2\ V_3) \begin{pmatrix} W_1 \\ W_2 \\ W_3 \end{pmatrix}$$

$$= \underline{V_1 W_1 + V_2 W_2 + V_3 W_3}$$

obvious

⇓ but generally:

$$\underline{V} \cdot \underline{W} = \underline{V}^T \underline{\underline{G}}\, \underline{w} = (V_1\ V_2\ V_3) \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix} \begin{pmatrix} W_1 \\ W_2 \\ W_3 \end{pmatrix}$$

for cartesian → G is the identity matrix

↳ $ds^2 = dx^2 + dy^2 + dz^2$ ∴ can ignore ∵ metric : $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$

surface elements for areas

$$ds^2 = dr^2 + r^2 d\theta^2 + dz^2$$

Cylindrical

metric : $\begin{pmatrix} 1 & 0 & 0 \\ 1 & r^2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$

Metric defines how diff coordinates combine to give length elements ⇒ eg. Jacobian as well

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Rev on Vector cross product [only ⊥ components contribute]

two linearly independent vectors

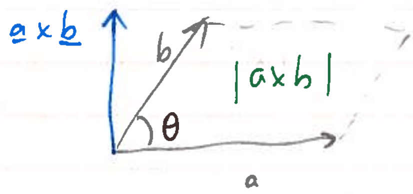
$\underline{a}$ & $\underline{b}$

$a \times b$ → vector ⊥ to both $\underline{a}$ and $\underline{b}$

resulting vector is normal to the plane

geometric meaning

$|\underline{a} \times \underline{b}|$ ⇒ magnitude of the cross product is the area of the //gram spanned by the $\underline{a}$ & $\underline{b}$ vectors



$$|a \times b| = |a||b| \sin\theta$$

Basis vectors:

$\hat{i} \times \hat{j} = \hat{k} \longleftrightarrow \hat{j} \times \hat{i} = -\hat{k}$ $\hat{i} \times \hat{i} = 0$

$\hat{j} \times \hat{k} = \hat{i} \longleftrightarrow \hat{k} \times \hat{j} = -\hat{i}$ $\hat{j} \times \hat{j} = 0$

$\hat{k} \times \hat{i} = \hat{j} \longleftrightarrow \hat{i} \times \hat{k} = -\hat{j}$ $\hat{k} \times \hat{k} = 0$

$\underline{a} = a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}$; $\underline{b} = b_1 \hat{i} + b_2 \hat{j} + b_3 \hat{k}$

$$\underline{a} \times \underline{b} = (a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}) \times (b_1 \hat{i} + b_2 \hat{j} + b_3 \hat{k})$$

$$= a_1 b_2 \hat{k} - a_1 b_3 \hat{j} - a_2 b_1 \hat{k} + a_2 b_3 \hat{i} + a_3 b_1 \hat{j} - a_3 b_2 \hat{i}$$

$$a \times b = (a_2 b_3 - a_3 b_2)\hat{i} + (a_3 b_1 - a_1 b_3)\hat{j} + (a_1 b_2 - a_2 b_1)\hat{k}$$

The Eigenvalues & eigenvectors of a linear operator scaling

characteristic eq^n

scalar → eigenvalues → factor of which the change of basis is squished or stretched

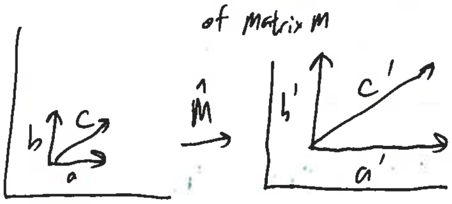
$$\underbrace{M}_{\text{transformation matrix}}\ \underbrace{\underline{v}}_{\text{column vector}} = \underbrace{\lambda}_{\text{eigenvalues of matrix } M}\ \underline{v} \rightarrow \text{eigenvectors}$$

linear transformation

↳ shows the effect of the operator $\hat{M}$ on $\underline{v}$

↳ new vector has the same direction of the original vector

eg.



↳ vector which remains in its own span after linear transformation

matrix-vector multiplication

$A\vec{v} = \underbrace{\lambda\vec{v}}_{\text{scalar multiplication}}$ / stay at its own span during transformation A

Eigenvalues can be determined by considering
$M = IM$

$\therefore\ IM\underline{v} = \lambda I\underline{v} \Rightarrow M\underline{v} = I\lambda\underline{v}$ Aim to solve for λ & $\vec{v}$

$\rightarrow (M-\lambda I)\underline{v} = \vec{0}$ (homogeneous eq^n) → why need? only way possible for a matrix to become zero w/ a non-zero vector

$\vec{0}$ ↳ zero vector

We know that a non zero $\underline{v}$ satisfying the eq^n exists only if $|M-\lambda I| = 0$

↳ transformation associated w/ a matrix squishes space into a lower dimension → dsquisification → det = 0

∴ we determined the condition for a scalar λ to be an eigenvalue of M

& λ must be a root of the characteristic eq^n

~~[illegible]~~ $M\underline{v} = \lambda\underline{v}$

& for square matrix only

$$\det(M-\lambda I) = 0$$
$$= \begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & \\ \vdots & & & \\ a_{n1} & \cdots & & a_{nn}-\lambda \end{vmatrix}$$

linearly independent

More than one eigenvector may correspond to the same eigenvalue λ_j

if we multiply eigenvector by another scalar
↳ yields another eigenvector associated to the same eigenvalue

(Proved @ next page)

there is a value of λ
And a non-zero vector $\vec{v}$ so that
$\underbrace{(M-\lambda I)}_{T}\vec{v} = 0$

T can squish space into a line & $T\vec{v} = \vec{0}$

$\vec{v}$ is an eigenvector of M
∵

$(M-\lambda I)\vec{v} = 0$
$M\vec{v} - \lambda\vec{v}I = 0$
$M\vec{v} = \lambda I\vec{v}$
$M\vec{v} = \lambda\vec{v}$

Let's assume $\underline{v}_j$ is an eigenvector of M with λ_j [$\therefore$ $\underline{M\underline{v}_j = \lambda_j \underline{v}_j}$]

AND consider action of M on the vector $c\underline{v}_j$

$$M(c\underline{v}_j) = c M\underline{v}_j = \underline{c\lambda_j \underline{v}_j} = \underline{\lambda_j c \underline{v}_j}$$

true

$\therefore M\underline{v} = \lambda \underline{v}$ => proving $c\underline{v}_j$ is also an eigenvector w/ eigenvalue λ_j

↳ eigenvectors can only be determined up to an arbitrary multiplicative factor

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Recipe to find eigenvalues & eigenvectors

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

1. Write down characteristic eq^n => solve it to find $\underline{\underline{\lambda}}$ (if no real solutions ↳ no eigenvectors eg $\lambda^2+1=0$)

$$\det(A - \lambda I) = 0 \qquad \lambda^2 - 1 = 0$$

$$\det\begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix} = 0 \qquad \lambda = \pm 1$$

2. For each λ there will be an eigenvector to give => find eigenvector $\underline{v}_1$ corresponding to λ_1

$$(M - \lambda_k I) v_k = 0 \qquad A\underline{v}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} b \\ a \end{pmatrix} = -\underline{v}_1 \quad (\lambda = -1)$$

$$v_1 = \begin{pmatrix} a \\ b \end{pmatrix} \qquad ↳ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} b \\ a \end{pmatrix} = \lambda_1 \underline{v}_1 = -\underline{v}_1 = \begin{pmatrix} -a \\ -b \end{pmatrix}$$

[solve eigensystem $A\underline{v}_1 = \lambda_1 \underline{v}_1$

$\boxed{b = -a}$; $a = -b$

3. Set one of the elements arbitrary => pick one element → make sure eigensystem does not imply $V_1 = 0$

↳ set $v_1 = 1$ & solve the eigensystem

Let's say $a = 1$ => $\underline{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

4. $\because$ no. of eigenvectors is $\underline{\text{indeterminate}}$ => determine normalized eigenvectors => eigenvectors w/ modulus of $\underline{1}$

↳ can have a number of eigenvectors for one eigenvalue

$$\underline{v} = \frac{1}{\|\underline{v}\|}\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$

5. $\therefore$ normalized eigenvector by using the length / $\underline{\text{euclidian}}$ length

↳ to get the basis vector just like how $\hat{\imath}, \hat{\jmath}$ & $\hat{k}$ are normal vector

$\underline{\hat{\imath}}, \underline{\hat{\jmath}}, \underline{\hat{k}}$ scale each unit = 1

$$|V| = \sqrt{V_1^2 + V_2^2 + \dots + V_n^2} = (\underline{V}^T \underline{V})^{\frac{1}{2}}$$

$$\underline{V}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \div \sqrt{1^2 + (-1)^2} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}\frac{1}{\sqrt{2}}$$

$\lambda_2 = +1 \rightarrow V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}\frac{1}{\sqrt{2}}$ [normalized] $\qquad \underline{V}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Real matrices can have non-real eigenvalues

$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

1. $\det(\lambda I - B) = 0$

$\det\begin{pmatrix} \lambda & -1 \\ 1 & \lambda \end{pmatrix} = \lambda^2 + 1 = (\lambda - i)(\lambda + i) = 0$

$\lambda_1 = -i$ or $\lambda_2 = +i$ (eigenvalues)

2. $\underline{v}_1 = \begin{pmatrix} a \\ b \end{pmatrix}$

solving $B\underline{v}_1 = \lambda_1 \underline{v}_1$ for a, b

$B\underline{v}_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} b \\ -a \end{pmatrix} = \lambda_1 v_1$

$= -i\underline{v}_1 = -i\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -ia \\ -ib \end{pmatrix} = \begin{pmatrix} b \\ -a \end{pmatrix}$

$b = -ia$

3. Set $a = 1 \Rightarrow b = -i$

$v_1 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$

4 Normalize:

$\underline{v}_1 = \begin{pmatrix} 1 \\ -i \end{pmatrix} / \sqrt{1^2 + i(i)} = \begin{pmatrix} 1 \\ -i \end{pmatrix} \frac{1}{\sqrt{2}}$

$v_2 = \begin{pmatrix} -i \\ 1 \end{pmatrix} \frac{1}{\sqrt{2}}$

Degenerate eigenvalues

a sheer

$M\underline{v} = \lambda \underline{v}$

- if two or more solutions this eq^n coincide
  - ↳ then degenerate roots → if eq^n is of order n
    - ↳ there will be less than n eigenvalues

eg.

$$B = \begin{pmatrix} 5 & 1 & 2 \\ 1 & 5 & -2 \\ 2 & -2 & 2 \end{pmatrix}$$

choices of coeffs are arbitrary
→ choices led to $\underline{v}_2$ & $\underline{v}_3$ being orthogonal

$\det(\lambda I - B) = 0$

$$\det \begin{pmatrix} \lambda-5 & -1 & -2 \\ -1 & \lambda-5 & 2 \\ -2 & 2 & \lambda-2 \end{pmatrix} = 0$$

$(\lambda-5)\left[(\lambda-5)(\lambda-2) - 4\right] -$
$-(-1)\left[(2-\lambda)+4\right]$ $\qquad = \lambda^3 - 12\lambda^2 + 36\lambda = \lambda(\lambda-6)^2 = 0$
$+(-2)\left[(-2)-(\lambda-5)\cdot 2\right]$

$\lambda_1 = 0$

$\lambda_2 = \lambda_3 = 6$

$\lambda_1 = 0$

Solving eigenvalue for $B\underline{v}_1 = \lambda_1 \underline{v}_1$

$$B\underline{v}_1 = \begin{pmatrix} 5 & 1 & 2 \\ 1 & 5 & -2 \\ 2 & -2 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 5a + b + 2c \\ a + 5b - 2c \\ 2a - 2b + 2c \end{pmatrix} = \lambda_1 \underline{v}_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$5a + b + 2c = 0$

$a + 5b - 2c = 0$

$2a - 2b + 2c = 0$

⇓

$6a = -6b$ $\qquad$ $4a + 2c = 0$

$a = -b$ $\qquad$ $c = -2a$

Setting $c = 1$ $\quad a = -\frac{1}{2}$ $\quad b = \frac{1}{2}$

$$\Rightarrow \quad v_1 = \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \end{pmatrix}$$

normalized $\underline{v}_1 = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} / \sqrt{6}$

$\lambda_2 = 6$ $\quad$ ⇒ setting $\underline{v}_2 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$B\underline{v}_2 = \lambda_2 \underline{v}_2$

$$B\underline{v}_2 = \begin{pmatrix} 5 & 1 & 2 \\ 1 & 5 & -2 \\ 2 & -2 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 5a + b + 2c \\ a + 5b - 2c \\ 2a - 2b + 2c \end{pmatrix} = \lambda_2 \underline{v}_2 = \begin{pmatrix} 6a \\ 6b \\ 6c \end{pmatrix}$$

⇒ $a - b - 2c = 0$
all vectors satisfying this relationship are eigen values for λ_2

Set arbitrary 2 elements $\quad a = 1, c = 1$ $\qquad$ $a = 1$ & $c = 0$

$\underline{v}_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ $\qquad$ normalized ⇒ $\underline{v}_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} / \sqrt{3}$ $\qquad$ $\underline{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ $\qquad$ normalized ⇒ $\underline{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} / \sqrt{2}$

$\underline{v}_2$ & $\underline{v}_3$ are both eigenvectors to the same eigenvalue λ_2

Diagonal Matrix: → all basis vectors are eigenvectors
→ diagonals of the matrix are their eigenvalues
↳ a square matrix w/ elements only along the diagonal

$$A = \begin{pmatrix} a_{11} & 0 & 0 & \cdots \\ 0 & a_{22} & 0 & \cdots \\ 0 & 0 & a_{33} & \cdots \\ \vdots & \vdots & & \ddots \end{pmatrix}$$

eigen basis
when both basis vectors are eigenvectors
↳ eg $\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$

$(A)_{ij} = a_{ij}\delta_{ij}$ using $\delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$

A & B are both ~~diag~~ diagonal

$$(AB)_{ij} = \sum_k a_{ik} b_{kj} = \sum a_i \delta_{ik} b_k \delta_{kj}$$
$$= \sum a_i \delta_{ik} \delta_{kj} b_k =$$
$$= \underline{\underline{a_i b_i}} \delta_{ij} \quad \Rightarrow AB \text{ is also a diagonal matrix}$$

~~AB~~ $AB = BA \Rightarrow$ commutative

Diagonalizing a matrix will simplify it, Not all matricies can be diagonalized

an eg. of a diagonalisable matrix

$A = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$

$\lambda_1 = 2 \quad \lambda_2 = 5$

normalized eigenvectors:

$\underline{v}_1 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}\sqrt{5} \qquad \underline{V}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}\sqrt{2}$

consider a new matrix B that has columns the normal eigenvectors of A

$$B = \begin{pmatrix} -\frac{1}{\sqrt{5}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$B^{-1} = \begin{pmatrix} -\frac{\sqrt{5}}{3} & \frac{\sqrt{5}}{3} \\ \frac{2\sqrt{2}}{3} & \frac{\sqrt{2}}{3} \end{pmatrix}$$

$$B^{-1}AB = \begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix} \Rightarrow A \text{ is diagonalizable.}$$

Generalizing the procedure

1. let $L \Rightarrow n \times n$ matrix with columns equal to the n vectors $\underline{v}_j$

$L = (v_1 \cdots v_j \cdots v_n)$

2. ∴ L must be invertible

L^{-1} must exist so that $L^{-1}L = I$

→ representing L^{-1}:

$$L^{-1} = \begin{pmatrix} \underline{w}_1 \\ \vdots \\ \underline{w}_j \\ \vdots \\ \underline{w}_n \end{pmatrix}$$

$$L^{-1}L = \begin{pmatrix} \underline{w}_1 \\ \vdots \\ \underline{w} \\ \vdots \\ \underline{w}_n \end{pmatrix} (v_1 \cdots v_j \cdots v_n) = \begin{pmatrix} w_1 v_1 & \cdots & w_1 v_n \\ w_k v_1 & & \\ w_n v_1 & \cdots & w_n v_n \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & & & & \\ 0 & & 1 & & & \\ \vdots & & & 1 & & \\ 0 & & & & & 1 \end{pmatrix}$$

~~[illegible]~~ OR

~~AA~~ $\underline{w}_k \underline{v}_j = \delta_{jk}$

~~made up of eigenvalues~~

$$^{*}D = L^{-1}ML \qquad \because M\underline{v} = \lambda_{\underline{v}}$$

$$D = L^{-1}M(v_1 \ldots v_j \ldots v_n) = L^{-1}(M\underline{v}_1 \ldots M\underline{v}_j \ldots M\underline{v}_n)$$

$$= L^{-1}(\lambda_1\underline{v}_1 \ldots \lambda_j\underline{v}_j \ldots \lambda_n\underline{v}_n) =$$

$$= \begin{pmatrix} w_1 \\ \vdots \\ w_k \\ \vdots \\ w_n \end{pmatrix}(\lambda_1\underline{v}_1 \ldots \lambda_j\underline{v}_j \ldots \lambda_n\underline{v}_n) = \begin{pmatrix} \lambda_1 w_1 v_1 & \cdots & \lambda_n \underline{w}_1 \underline{v}_n \\ \vdots & & \vdots \\ \lambda_1 \underline{w}_k \underline{v}_1 & & \\ \vdots & & \\ \lambda_n w_{n-1} v_1 & \cdots & \lambda_n \underline{w}_n v_n \end{pmatrix}$$

$$D = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \ddots & & \vdots \\ 0 & & \lambda_j & \\ 0 & & & \ddots \ \lambda_n \end{pmatrix}$$

$\therefore$ the expansion of the diagonalisable linear transformation M in the basis given by its eigenvectors is a diagonal matrix D w/ the eigenvalues on the main diagonal

* L diagonizes M & is constructed as the matrix whose columns are the normalized eigenvalues of M

eg.

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \underline{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \underline{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$L = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} / \sqrt{2}$$

$$L^{-1} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} / \sqrt{2}$$

$$D = L^{-1}AL = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

has to span whole space

$$= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

-1 & 1 are eigenvalues of matrix A

$\Rightarrow$ $\exists!$ eigenbasis

$D = L^{-1}ML \rightarrow$ normalized eigenvectors of M as columns

made up eigenvalues of M in the main diagonal $\Rightarrow$ eigenbasis

Eigenvalues are invariant under a change of basis of the vector space

diagonalizing vector $\Rightarrow$ change coordinate system so that eigenvectors are basis vectors $\Rightarrow$ using eigenvectors as basis

$L = [\,\square\;\square\,]$ — eigenvectors; change of basis matrix

$D = L^{-1}ML$ → result will be a matrix representing that same transformation but from the perspective of the new basis vectors coordinate system $\Rightarrow$ new matrix will be diagonal

Not all square matricies can be diagonalized

$$J = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow J\underline{v}_j = \lambda \underline{v}_j$$

$\Rightarrow \lambda^2 = 0 \quad \lambda_1 = 0$

$\lambda_1 = 0 \qquad v_1 = \begin{pmatrix} a \\ b \end{pmatrix}$

$J\underline{v}_1 = \lambda \underline{v}_1$

$$J\underline{v}_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} b \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$b = 0$

$\Rightarrow$ no other linearly independant eigenvector exists

$\therefore$ J is not diagonalizable

Jordan block $\Rightarrow$ $\lambda = 0$ led by $J\underline{v} = \lambda \underline{v}$

Introducing an invertible matrix N (Using $D = L^{-1}ML$)

$$M' = N^{-1}MN$$

$\downarrow$ 兩邊 $\times$ N & N^{-1}

$$NMN^{-1} = \underbrace{NN^{-1}}_{I} M \underbrace{NN^{-1}}_{I}$$

$\Rightarrow \quad M = NM'N^{-1}$

$\downarrow$ Sub $D = L^{-1}ML$

$$D = L^{-1}ML = L^{-1}\underbrace{NM'N^{-1}}_{M}L \quad (L' = N^{-1}L)$$

$\downarrow$ but there must be $\qquad NL' = L$

$D = L'^{-1}M'L' \because D = L^{-1}ML$

$$D = L'^{-1}M'L'$$

matrix of eigen vectors change from L into $N^{-1}L$

$\downarrow$

$\Rightarrow$ diagonal matrix of eigenvalues D stays the same

$\downarrow$

invariance

Two more invariant quantities:

$\det(AB) = \det(A)\det(B)$ $\rightarrow$ Binet's formula

1. Det of M

$\det(L^{-1}ML) = \det(L^{-1})\det(M)\det(L)$

using Binet's formula for det of products $\det(L^{-1}ML) = \frac{1}{\det L}\det(L)\det(M) = \det M$

2. Trace: $Tr(M)$ $\rightarrow$ sum of all elements in the diagonal $Tr(M) = \sum_{j=1}^{n} \lambda_j$

eg. $A = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$; $\det(A) = 10$; $tr(A) = 7$

$\lambda_1 = 2, \lambda_2 = 5$

$$D = \begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix} \Rightarrow \det(A) = 10 \; ; \; tr(A) = 7$$

Application of diagonalization (as long as matrix is diagonizable)

Powers of matrix:

if $D = L^{-1} M L$

isolating M: $M = L D L^{-1}$

- computing M^2: # remember matrix product is associative using the usual trick

$$M^2 = (LDL^{-1})^2 = (LDL^{-1})(LDL^{-1}) = LD(L^{-1}L)DL^{-1}$$

$$M^2 = L D^2 L^{-1}$$

For any power: $\underline{M^n = L D^n L^{-1}}$

Eigenvalues & eigenvalues of Hermitian matrices

$H = H^{\dagger}$ is diagonisable

Denoting λ_j & λ_k be 2 eigenvalues of H w/ the 2 corresponding eigenvectors $\underline{v}_j$ & $\underline{v}_k$

$$H\underline{v}_j = \lambda_j \underline{v}_j \quad - 4.73$$

$$H\underline{v}_k = \lambda_k \underline{v}_k \quad - 4.74$$

- multiplying 4.73 by $\underline{v}_k^{\dagger}$

$$\underline{v}_k^{\dagger} H \underline{v}_j = \lambda_j \underline{v}_k^{\dagger} \underline{v}_j$$

- consider eq^n. 4.75 & recall: $(AB)^{\dagger} = B^{\dagger}A^{\dagger}$

↳ $\underline{v}_k^{\dagger} H^{\dagger} = \underline{v}_k^{\dagger} H = \lambda_k^{*} \underline{v}_k^{\dagger}$ (using $H = H^{\dagger}$) [property]

$$\Rightarrow \underline{v}_k^{\dagger} H \underline{v}_j = \lambda_k^{*} \underline{v}_k^{\dagger} \underline{v}_j$$

$$\underline{v}_k^{\dagger} H \underline{v}_j = \lambda_j \underline{v}_k^{\dagger} \underline{v}_j$$

$$\rightarrow (\lambda_j - \lambda_k^{*}) \underline{v}_k^{\dagger} \underline{v}_j = 0$$

$(\lambda_j - \lambda_k^*)\, \underline{v}_k^\dagger \underline{v}_j = 0$

consequences:

- if $j = k$, then $\underline{v}_k^\dagger \underline{v}_j = \underline{v}_j^\dagger v_j = |v_j|^2 \neq 0$

  $\rightarrow \lambda_j = \lambda_j^*$

  => eigenvalues of hermitian matrices are always real

- if $j \neq k$ & $\lambda_j \neq \lambda_k$, then $\underline{v}_k^\dagger \underline{v}_j = 0$

  $\rightarrow$ eigenvectors of Hermitian operators associated to diff eigenvalues are orthogonal

- Diagonize H:

  $D = U^{-1} H U$

  $\rightarrow$ matrix w/ columns = eigenvectors of H

  orthonormal

  $U = (\underline{v}_1 \ldots \underline{v}_j \ldots \underline{v}_n)$

  w/ $\underline{v}_j^\dagger \underline{v}_k = \delta_{jk}$ [note that $\underline{v}_j^\dagger \underline{v}_k = \delta_{jk}$ is inner product]

  $$\Rightarrow U^{-1} = \begin{pmatrix} \underline{v}_1^\dagger \\ \vdots \\ \underline{v}_j^\dagger \\ \vdots \\ \underbrace{\underline{v}_d^\dagger}_{\text{where???}} \end{pmatrix} = U^\dagger$$

  $\rightarrow$ hud lun??

  => U is unitary

  any Hermitian matrix is diagonisable by a unitary matrix

Summary of special matrices

Recall:

- $M^T \rightarrow$ transpose of M [swapping rows]
  obtained by writing the rows as columns
  $M^T_{jk} = M_{kj}$
- $M^\dagger \rightarrow$ Hermitian conjugate
  obtained by complex conjugation of M^T
  $M^\dagger_{jk} = M^*_{kj}$

<u>Special matrices</u> :

- Square matrix M is normal
  if $M^\dagger M = M M^\dagger$
- square matrix H is Hermitian
  if $H^\dagger = H$
- Square matrix V is unitary
  if $U^\dagger U = U U^\dagger = I$
- square matrix O is orthorgonal
  if $O^T O = O O^T = I$
- square matrix S is symmetric
  if entries are real
  &
  $S^T = S$

Dependancies

1. Real orthorgonal matrices are unitary

$$O^\dagger O = O^T O = I$$

2. Real symmetric matrices are Hermitian

$$S^\dagger = S^T = S$$

3. Hermitian matrices are normal :

$$H^\dagger H = HH = HH^\dagger$$

4. normal matrices are diagonisable
   ↳ unitary, hermitian, orthorgonal & real symmetric matrices
   are all diagonisable ∵ they are normal

Real quadratic forms

quadratic forms → polynomials w/ n variables, all of degree two

Let x & y be two sets of variables (real)

$E = \sum_{i,j=1}^{n} q_{ij} x_i y_j$ → in matrix form: $E = \underline{x}^T Q \underline{y}$

Let's have 2 variables x_1 & x_2

∴ $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ compute the following expression $\underline{X}^T A \underline{X}$

eg

$$(x_1 \; x_2)\begin{pmatrix} 3 & -2 \\ -2 & 7 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (x_1 x_2)\begin{pmatrix} 3x_1 - 2x_2 \\ -2x_1 + 7x_2 \end{pmatrix}$$

$$= x_1(3x_1 - 2x_2) + x_2(-2x_1 + 7x_2)$$

$$= 3x_1^2 - 4x_1x_2 + 7x_2^2$$ → quadratic

recalling $A = \begin{pmatrix} 3 & -2 \\ -2 & 7 \end{pmatrix}$

eg Potential energy

$$PE = \frac{1}{2}k(x_1 - x_2)^2 = \frac{1}{2}k(x_1^2 + x_2^2 - 2x_1x_2)$$

$$= \frac{1}{2}kx_1^2 + \frac{1}{2}kx_2^2 - \frac{k}{2}2x_1x_2$$

$$= \frac{1}{2}kx_1^2 + \frac{1}{2}kx_2^2 - kx_1x_2$$

$\underline{x}^T K \underline{x}$

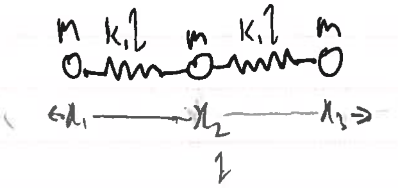
↓

$$K = \begin{pmatrix} \frac{1}{2}k & -\frac{k}{2} \\ -\frac{k}{2} & \frac{1}{2}k \end{pmatrix}$$

Example for Real & quadratic forms:

Normal modes of oscillation
(Looking for classical, non-quantum system)

Consider 3 particles of equal mass $\boxed{m}$
- joined by springs w/ elastic constant $\boxed{k}$
- & rest length $\boxed{l}$



equations of motion governing the positions x_1, x_2, x_3 of the 3 particles:

$$m\ddot{x}_1 = k(x_2 - x_1 - l) = -kx_1 + kx_2 - kl$$
$$m\ddot{x}_2 = -k(x_2 - x_1 - l) + k(x_3 - x_2 - l) = -kx_2 + kx_1 + \cancel{kl} + kx_3 - kx_2 - \cancel{kl}$$
$$m\ddot{x}_3 = -k(x_3 - x_2 - l) = kx_2 - kx_3 + kl \quad = kx_1 - 2kx_2 + kx_3$$

→ now, we define: $\underline{x} = (x_1, x_2, x_3)^T \Rightarrow \underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$\ddot{x} = (\ddot{x}_1, \ddot{x}_2, \ddot{x}_3)^T \Rightarrow \ddot{\underline{x}} = \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{pmatrix}$

$$m\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{pmatrix} = k\begin{pmatrix} -x_1 + x_2 + 0x_3 - l \\ x_1 - 2x_2 - x_3 - 0l \\ 0x_1 + x_2 - x_3 + l \end{pmatrix}$$

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{pmatrix} = \frac{k}{m}\underbrace{\begin{pmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{pmatrix}}_{A}\underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}_{\underline{x}} + \frac{k}{m}\underbrace{\begin{pmatrix} -l \\ 0 \\ l \end{pmatrix}}_{\underline{x}_0}$$

$$\ddot{\underline{x}} = \frac{k}{m}\underline{A}\,\underline{x} + \frac{k}{m}\underline{x}_0 \quad \text{— 4.93}$$

$$A = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{pmatrix} \quad \rightarrow \text{diagonalize}$$

diagonalize: $A = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{pmatrix}$

find ~~the~~ its eigenvalues & eigenvectors (normalized)

$(A - I\lambda)\underline{v} = 0$

$\det(A - I\lambda) = 0$

$$d\begin{vmatrix} -1-\lambda & 1 & 0 \\ 1 & -2-\lambda & 1 \\ 0 & 1 & -1-\lambda \end{vmatrix} = 0$$

$(-1-\lambda)\left[(-[2+\lambda])(-[1+\lambda]) - 1\right] - (1)(-1-\lambda) = 0$

$(-1-\lambda)\left((2+\lambda)(1+\lambda) - 1\right) - (-1-\lambda) = 0$

$(-1-\lambda)\left((2+\lambda)(1+\lambda) - 1 - 1\right) = 0$

$(1+\lambda)(\cancel{2} + 2\lambda + \lambda + \lambda^2 - \cancel{2}) = 0$

$(1+\lambda)(\lambda^2 + 3\lambda) = 0$

$\lambda(1+\lambda)(\lambda+3) = 0$

$\boxed{\begin{matrix}\lambda_1 = 0 \\ \lambda_2 = -1 \\ \lambda_3 = -3\end{matrix}}$

$\begin{pmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \lambda\begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$\underline{\lambda_1 = 0}$

$A\underline{v} = \lambda\underline{v}$

$\begin{pmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0\begin{pmatrix} a \\ b \\ c \end{pmatrix}\lambda_v)$

$-a + b = 0$

$a - 2b + c = 0$

$-b + c = 0$

$\cancel{a + b + c}$

$a = b = c$

Let $a = 1$

$\underline{V_1} = \frac{1}{\sqrt{1+1+1}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$\underline{V_1} = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$\underline{\lambda_2 = -1}$

$-a + b = -a$

$a - 2b + c = -b$

$-b + c = -c$

$a = 1$

$b = 0$

$1 - 0 + c = -0$

$c = -1$

$\underline{V_2} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

$\lambda_3 = -3$

$-a + b = -3a$

$a - 2b + c = -3b$

$-b + c = -3c$

$b = -2a$

$a + b + c = 0$

$1 - 2 + c = 0$

$-1 + c = 0$

$c = 1$

$a = 1$

$b = -2$

$c = 1$

$\underline{V_3} = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

$\lambda_1 = 0$; $\lambda_2 = -1$; $\lambda_3 = -3$

$$\underline{v}_1 = \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix} \quad \underline{v}_2 = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{pmatrix} \quad \underline{v}_3 = \begin{pmatrix} 1/\sqrt{6} \\ -2/\sqrt{6} \\ 1/\sqrt{6} \end{pmatrix}$$

$\because$ matrix is real & symmetric
↳ eigenvalues are real
eigenvectors are real & orthogonal

→ transformation R which diagonalises A is $\therefore$ the orthogonal transformation given by:

$$R = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{pmatrix}$$

$R^{-1} = R^T$ (orthogonality)

$$R^T A R = D \implies D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -3 \end{pmatrix}$$

$$A = RDR^T$$

now call $\underline{y} = R^T \underline{x}$ $\quad \ddot{\underline{y}} = R^T \ddot{\underline{x}}$
&
$\underline{y}_0 = R^T \underline{x}_0 = \begin{pmatrix} 0 \\ -\sqrt{2}\,l \\ 0 \end{pmatrix}$

recalling $x_0 = \begin{pmatrix} -l \\ 0 \\ l \end{pmatrix}$

$$\implies \ddot{\underline{x}} = \frac{k}{m} RDR^T \underline{x} + \frac{k}{m} \underline{x}_0$$

$\downarrow \times R^T$

$$R^T \ddot{\underline{x}} = \frac{k}{m} D \underbrace{R^T \underline{x}}_{y} + \frac{k}{m} \underbrace{R^T \underline{x}_0}_{y_0}$$

rmb $R^T R = I$
↳ orthogonal

4.p3: $\ddot{\underline{x}} = \frac{k}{m} A \underline{x} + \frac{k}{m} \underline{x}_0$

$$\cancel{R^T} R^T \ddot{x} = \frac{k}{m} D\underline{y} + \frac{k}{m} y_0 \implies \ddot{y} = \frac{k}{m} D\underline{y} + \frac{k}{m} y_0$$

$$\ddot{y} = \frac{k}{m} \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -3 \end{pmatrix} + \frac{k}{m} \begin{pmatrix} 0 \\ -\sqrt{2}\,l \\ 0 \end{pmatrix}$$

$\ddot{y}_1 = 0$

$\ddot{y}_2 = -\frac{k}{m} y_2 + \sqrt{2} \frac{kl}{m}$

$\ddot{y}_3 = -3 \frac{k}{m} y_3$

normal modes of motion:
- y_1 – no external force is acting on the 3 particles
- y_2 – breathing mode
- y_3 – Egyptian mode