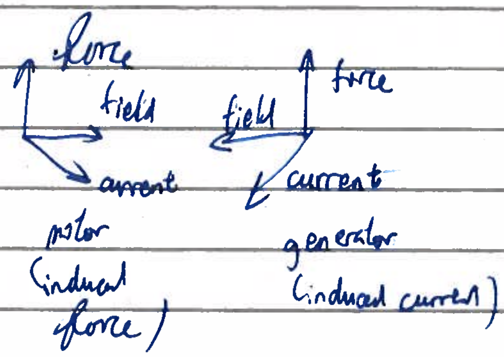
1. Force & energy

Electric charge => electrostatic force



w/ charge density:

$$\int_V \rho_{el}(\underline{r})\, d\tau = -e$$

$$\tau = \tau(x, y, z)$$

- Gauss Law

$$\int_S \underline{E} \cdot d\underline{s}$$

flux through a surface

or flow

$$\nabla \cdot \underline{E} = \frac{\rho}{\varepsilon_0}$$

flux

$$\int_V \nabla \cdot \underline{F}\, d\tau = \int_S \underline{F} \cdot d\underline{S}$$

divergence theorem

$$\int_S \underline{E} \cdot d\underline{S} = \frac{1}{\varepsilon_0} \int \rho\, d\tau$$

- Milestone in electromagnetism
  - Electrostatics
  - Conductors
  - Dielectrics
  - DC circuits
  - Magnetostatics
  - Electromagnetic induction
  - AC circuit
  - Maxwell's eqn

Faraday's Law

Differential form | Integral form

$$\nabla \times \underline{E} = -\frac{\partial B}{\partial t} \qquad \int_c \underline{E}\cdot d\underline{l} = -\int_A \left(\frac{\partial B}{\partial t}\right)\cdot d\underline{A} \qquad \text{Faraday's Law}$$

$EMF = -\frac{d\Phi}{dt}$ => Magnetic flux within a circuit produces an EMF

Φ across surface S

$$\Phi(t) = \int_S \underline{B}(t)\, d\underline{S}$$

EMF across the entire circuit

$$EMF_{total} = \oint d(EMF)$$

E-field comes ~~from~~ relates to force from electric charges

Definition of voltage:

$$V = \int \underline{E}\cdot d\underline{l}$$

$$\frac{dV}{d\underline{l}} = \underline{E}$$

voltage of 2 pts: sum of E field along the path i.e. line integral

By Stokes theorem => $\oint_{c=\partial S} \underline{E}\cdot d\underline{l} = \int_S \nabla\times E\cdot d\underline{S}$

~~We~~ We know $EMF = -\frac{d\Phi}{dt}$

$\therefore$ ~~$\oint \underline{E}\cdot d\underline{S} = \int_S \nabla\times E\cdot d\underline{S}$~~

$$\therefore \oint_{c=\partial S} \underline{E}\cdot d\underline{l} = \int_S \nabla\times \underline{E}\cdot d\underline{S} \quad (1)$$

$$= -\frac{d}{dt}\int_S \underline{B}(t)\cdot d\underline{S}$$

$$= \int_S -\frac{dB(t)}{dt}\cdot d\underline{S} \quad (2)$$

=> (1) & (2)

~~$\nabla\times E = d$~~

$$\nabla\times\underline{E} = -\frac{\partial B(t)}{dt}$$

- E-field gives rise to B-field
  B-field " " to E-field.

~~The static~~ /

- The rotational flow of E-field is caused by a magnetic field changing in time
  or
  the other way around
  ↳ works both ways.

Gauss's Law for E-fields

∯ electric charge acts as sources or sinks for E-fields

how E-field behaves around electric charge

Differential form

$\nabla \cdot \underline{D} = \rho_v$

$\nabla \cdot \underline{E} = \frac{\rho_v}{\varepsilon_0}$

Integral form

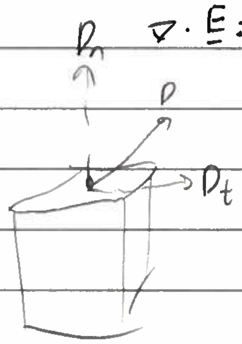
$\int_v (\nabla \cdot D)\, dv = \int_v \rho_v\, dv$

$\Rightarrow \oint_s \underline{D} \cdot d\underline{S} = Q_{enc} \Rightarrow \oint_s \underline{E} \cdot d\underline{S} = \frac{Q_{enc}}{\varepsilon_0}$

↘ summing up $D \cdot dS$ values at each pt along ∯ surface S

~~electric charge exiting~~

Electric flux exiting any volume is equal to the total charge inside



only considering D_n for D_n is the actual component leaving the volume

Behaviour:

div $\cdot\, \underline{D} = \rho_v$

异性相吸

同性相斥

1. if +ve ⇒ source ⇒ hence +ve div

   ← ⊕ →

   ← ⊕ →

   ← ⊕ →

2. if −ve ⇒ sink ⇒ hence −ve div

   → ⊖ ←

   → ⊖ ←

   → ⊖ ←

if 0 ⇒ no divergence ⇒ hence

→ → →

→ → →

→ → →

div.D = net amount of charge in that region

$|\underline{F}| = \frac{q_1 q_2}{4\pi\varepsilon_0 R^2}$

$|\underline{E}| = \frac{q_1}{4\pi\varepsilon_0 R^2}$

$|\underline{D}| = \frac{q_1}{4\pi R^2}$ ⇒ electric flux density

$\therefore \frac{|D|}{\varepsilon_0} = |E|$

~~Gauss's~~ Gauss' Law for B-fields

magnetic ~~ϕ~~ flux density

$$\nabla \cdot \underline{B} = 0$$

or

$$(\underline{B} = \mu \underline{H})$$

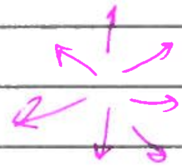
$$\nabla \cdot \underbrace{H}_{\text{magnetic field}} = 0$$

$\mu \Rightarrow$ permeability
↳ measure of how easily a magnetic field can pass through a medium

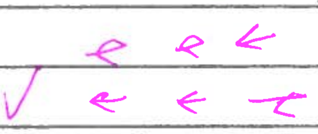
no "magnetic charge"
div B or div H is always zero through any volume.

$$\oint \vec{B} \cdot d\vec{A} = 0$$

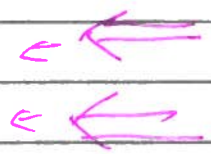
✓ no div
field don't flow in or out in any volume



✗ +ve div
∴ impossible



✓ field flow into & out of any volume equally



∵ uniform vector field
↳ bigger arrows → uniform in Y axis
But decreasing in X-axis
∴ div ~~B~~ ≠ 0

— [illegible] 入 [illegible] 另一[illegible]

↑ mean flux

Ampere's Law

$$\nabla \times H = \frac{\partial D}{\partial t} + J$$

$$\oint H \cdot dL = \underbrace{I_{enc}}_{\text{enclosed current}} = \int_S \underbrace{J}_{\text{current density}} \cdot d\underline{S} = \int_S (\nabla \times H) \cdot d\underline{S}$$

when you have a conductor (wire) carrying current I
⇒ current produces a $\underline{\text{B-field}}$ ∅ which circles the wire

$$\therefore \nabla \times \underline{H} = \underline{J}$$

using Stokes Theorem:

$$\oint H \cdot dL = \int_S (\nabla \times H) \cdot d\underline{S}$$

$$\nabla \times E = \frac{\partial B}{\partial t}$$

Symmetry:

$$\Rightarrow \frac{\partial D}{\partial t} = \nabla \times H$$

$$\therefore \nabla \times H = \frac{\partial D}{\partial t} + J \qquad \because \nabla \cdot J \neq 0$$

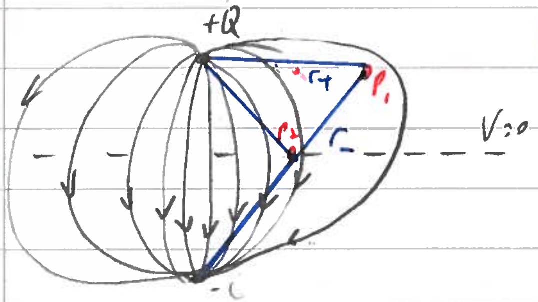
$$= \underbrace{J_d}_{\text{Displacement current density}} + J$$

∅ flowing current ($\underline{J}$) gives rise to a B-field ~~circle~~ circulating the current

∅ Electric flux density ($\underline{D}$) gives rise to a B-field that circles the $\underline{D}$ field.

More on method of images

Field lines on a dipole



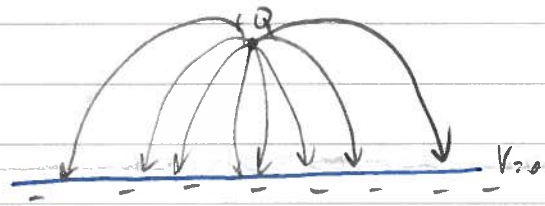
$$V = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r_+} - \frac{1}{r_-}\right)$$

$\underline{E} = -\nabla V$

Tells you how quickly the potential ~~change~~ changes
& in what direction

[it pts in the direction of maximum -ve change of electric potential]

Field lines between a charge & a conducting plane.



Method of images : procedure, where a difficult to solve configuration of charges & charge distribution is replaced by a simpler configuration

-need to satisfy

↳ Boundary conditions

↳ Poisson's Eq^n $\nabla^2 V = -\frac{\rho}{\varepsilon_0}$

it is possible to use Uniqueness Theorem

↳ there is only one sol^n which can satisfy the boundary condition & to the Poisson Eq^n

↳ ∴ No need to search for other sol^{n}s

solution only applies to the region where the original configuration & & the original images are identical

X apply to region replaced by a unique charge.

Charge & Coulomb's Law

- $e = 1.602 \times 10^{-19}\,C$

  ~~ef~~
  electric charge is quantized
  ↳ can only carry a charge q multiple of the elementary charge

$$q = Ne$$

- Conservation of charge:
  → charge only transferred, not made / transferred
  ∵ 冇另外
  e.g. $\gamma \to e^- + e^+$
  pair production

- Static electricity

- Repulsion in a nucleus

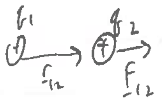
$$|F_{21}| = \frac{1}{4\pi\varepsilon_0} \frac{e^2}{r_{21}^2} = 14\,N$$

  Why ~~not protons~~ proton don't explode apart
  ⇒ due to strong nuclear force

- Coulomb's Law

$$F_{12} = \frac{1}{4\pi\varepsilon_0 r_{12}^2} q_1 q_2 \, \hat{r}_{12}$$

↳ unit vector (from 1 to 2)



in free

$\varepsilon_0 = 8.85 \times 10^{-12}\,C^2 N^{-1} m^{-2}$

- Materials in between charges
  ↳ decreases F

[Permittivity]
↳ how "permitted" the material is to absorb energy

$$\varepsilon = \varepsilon_r \varepsilon_0$$

ε_r: Relative permittivity

Vacuum	1
Paper	4
Methanol	30
Acid	87

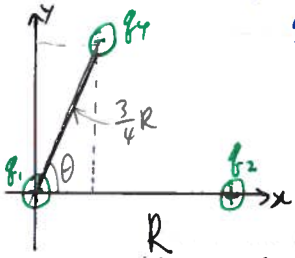
$\varepsilon_r = 1 + \chi$ [Susceptibility to polarization]

Vacuum	0
Paper	3
Methanol	29
Acid	79

* If it is easy for medium to absorb energy ⇒ High relative permittivity & susceptibility
↳ uses up some of the energy & results in a smaller field / force

Eg.

Force from multiple charges



given: $q_1 = 1C$
$q_2 = 2C$
$q_4 = -4C$

What is the net force on q_1 ~~to~~ due to q_2 & q_4

Force of 2&4 on 1

$$\underline{F}_{(2+4),1} = \underline{F}_{21} + \underline{F}_{41}$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{R^2}\hat{r}_{21} + \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_4}{(\frac{3}{4}R)^2}\hat{r}_{41}$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{q_1}{R^2}\left(q_2\hat{r}_{21} + \frac{16 q_4}{9}\hat{r}_{41}\right)$$

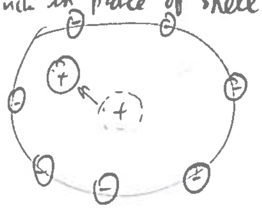
$$= \frac{1}{4\pi\varepsilon_0}\frac{1}{R^2}\left(2(-\hat{\underline{i}}) + \frac{16}{9}(-4)[\cos\theta(-\hat{\underline{i}}) + \sin\theta(-\hat{\underline{j}})]\right)$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{1}{R^2}\left(-2\hat{\underline{i}} - \frac{64}{9}\left[[(-\underline{i})\cos\theta] - \hat{\underline{j}}\sin\theta\right]\right)$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{1}{R^2}\left[\left(-2 + \frac{64}{9}\cos\theta\right)\hat{\underline{i}} + \frac{64}{9}\sin\theta\,\hat{\underline{j}}\right]$$

Forces inside an insulating shell

[e⁻'s stuck in place of shell]



Force inside ~~on a~~ a conducting shell?

e⁻'s move out of their symmetrical arrangement in response to the +ve charge.
↳ +ve charge feels a net force then move

Force felt by +ve charge :) $F = 0$ ∴ no movement of + by itself → same

Force felt by -ve charge :) $F \neq 0$ but charges stuck → same BUT charges move
↳ no movement of charges
↳ +ve charge feels a force and moves by itself

How can +ve charge ~~feel~~ feels no force, but e⁻'s do?
∵ + charge feels an $F \neq 0$ w/ each individual e⁻
↳ NET force on the +ve charge is zero

~~Electric field~~

Electric field :

electric force per +ve unit charge @ a pt ~~in space~~ or
electric force acting on a point charge in space divided by the magnitude of the charge

q_1

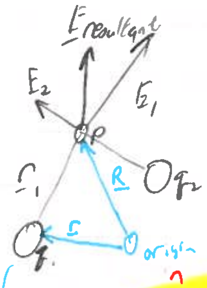
$$\underline{E} = \frac{1}{4\pi\varepsilon_0}\frac{q_1}{r^2}\hat{\underline{r}}$$

force a 2nd charge would feel exerted by q_1 if we placed it there

eg. another charge q_2 is introduced

$$\underline{F}_{12} = q_2 \underline{E}$$

For multiple pt charges :



Total E-field due to a group of ~~those~~ charges = vector sum of the E-fields of all charges

$$\underline{E} = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\frac{q_i}{r_i^2}\hat{\underline{r}}_i$$

$[\underline{r}_i = \underline{R} - \underline{r}]$

$\hat{\underline{r}} = \frac{\underline{R}-\underline{r}}{|\underline{R}-\underline{r}|}$

→ for a continuous charge distribution

Volume V has charge density ϱ

$\varrho = \varrho(x,y,z)$

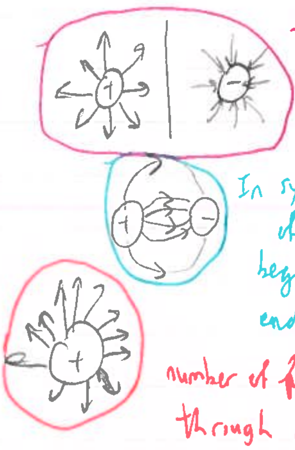
Charge in dV

$\Rightarrow dq = \varrho\, dV$

$$d\underline{E}(\underline{R}) = \frac{1}{4\pi\varepsilon_0}\frac{\underline{R}-\underline{r}}{|\underline{R}-\underline{r}|^3}\overbrace{\varrho(r)\,dV}^{dq}$$

$$\underline{E}(\underline{R}) = \int_V \frac{1}{4\pi\varepsilon_0}\frac{\underline{R}-\underline{r}}{|\underline{R}-\underline{r}|^3}\varrho(r)\,dV$$

Field lines :



field lines pt in a dir a +ve charge would go

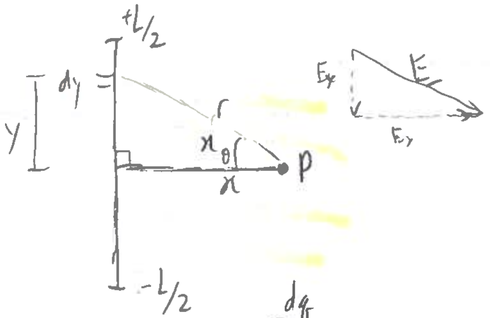
In systems w/ net charge of 0, all ~~field~~ field lines begin on a +ve charge & end on a -ve charge

number of field lines per unit area through a surface ⊥ to the field lines ~~is~~ ∝ magnitude of the field in that region

Eg.

Determine the electric field created by a segment of length L, carrying a linear charge density λ, @ a pt P located on the median plane of the wire.

1. 畫圖，線
2. 寫 $dq = \rho dV$
3. Figure out components
4. Expression for summing these chunks
5. Calc + simplify.



* finding E_x but integrating over dy
* x is a constant
* wire has same charge per unit length

$$|d\vec{E}| = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^2}\lambda\, dy\left(\cos\theta\,\hat{x} - \cancel{\sin\theta\,\hat{y}}\right) \quad (E_y = 0 \because \text{cancel out})$$

$$\Rightarrow E = \frac{\lambda}{4\pi\varepsilon_0}\int_{-\frac{L}{2}}^{+\frac{L}{2}} \frac{\cos\theta}{r^2}\, dy$$

$$= \frac{\lambda}{4\pi\varepsilon_0}\int_{y=-\frac{L}{2}}^{y=\frac{L}{2}} \frac{\cos\theta\,\cancel{\sec^2\theta}}{x^2\,\cancel{\sec^2\theta}}\, d\theta$$

$$= \frac{\lambda}{4\pi\varepsilon_0 x}\int_{y=-\frac{L}{2}}^{y=\frac{L}{2}} \cos\theta\, d\theta$$

$$= \frac{\lambda}{4\pi\varepsilon_0 x}\left[\sin\theta\right]_{y=-\frac{L}{2}}^{y=\frac{L}{2}}$$

$$= \frac{\lambda}{4\pi\varepsilon_0 x}\left(\frac{L}{\sqrt{x^2+\left(\frac{L}{2}\right)^2}}\right)$$

$\frac{y}{x} = \tan\theta$

$dy = \sec^2\theta\, d\theta$

$r^2 = x^2 + y^2$

$= x^2 + x^2\tan^2\theta$

$r^2 = x^2\sec^2\theta$

$\sin\theta = \frac{y}{r}$

$y = r\sin\theta$

and $r = \sqrt{x^2+\left(\frac{L}{2}\right)^2}$

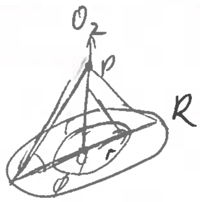
$\frac{L}{2} = \sqrt{x^2+\left(\frac{L}{2}\right)^2}\,\sin\theta$

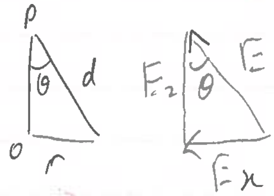
↑ L: $\sin\theta = \frac{L}{2\sqrt{x^2+\left(\frac{L}{2}\right)^2}}$

↓ $-L$: $\sin\theta = -\frac{L}{2\sqrt{x^2+\left(\frac{L}{2}\right)^2}}$

Eg 2. Charge Q is uniformly distributed over a disk of radius R and axis Oz

Determine the E-field @ a pt P on the z axis





E_y cancel out $\qquad E_z = E\cos\theta$

- charge on disk : $\sigma = \frac{Q}{\pi R^2}$

- charge on one ring : $dq = \sigma 2\pi r\, dr$

- distance of ring from P: $d = \sqrt{r^2+z^2}$

$$\Rightarrow \cos\theta = \frac{z}{d} = \frac{z}{\sqrt{r^2+z^2}}$$

E-field @ point P due to one very thin ring

$$dE_z = \frac{\sigma \cancel{2\pi} r\, dr}{2\cancel{4\pi}\varepsilon_0 (r^2+z^2)} \frac{z}{\sqrt{r^2+z^2}} = \cancel{\sigma z\, r\, dr}$$

$$dE_z = \frac{\sigma z\, r\, dr}{2\varepsilon_0 (z^2+r^2)^{\frac{3}{2}}}$$

$$E_z = \int_0^R \frac{\sigma z}{2\varepsilon_0} \frac{r\,dr}{(r^2+z^2)^{\frac{3}{2}}} = \frac{\sigma z}{2\varepsilon_0}\left[-(z^2+r^2)^{-\frac{1}{2}}\right]_0^R = \frac{\sigma z}{2\varepsilon_0}\left(-\left[(z^2)^{-\frac{1}{2}} - (z^2+R^2)^{-\frac{1}{2}}\right]\right).$$

$$= \frac{\sigma z}{2\varepsilon_0}\left(-\frac{1}{z} - \frac{1}{\sqrt{z^2+R^2}}\right)$$

$$= \frac{Qz}{2\pi\varepsilon_0 R^2}\left(\frac{1}{z} - \frac{1}{\sqrt{z^2+R^2}}\right)$$

$$\int \frac{r\,dr}{(r^2+z^2)^{\frac{3}{2}}}$$

$$\frac{1}{(r^2+z^2)^{\frac{3}{2}}}$$

$$= 2\frac{\cancel{2}r\left(\frac{1}{\cancel{2}}\right)}{(r^2+z^2)^{\frac{3}{2}}}$$

some interesting limits

i) $z >> R \Rightarrow E_z = \frac{Qz}{4\pi\varepsilon_0 r^2}$

$$\left[\frac{1}{\sqrt{z^2+R^2}} = \frac{1}{z\sqrt{1+\frac{R^2}{z^2}}} = \frac{1}{z}\left(1+\frac{R^2}{z^2}\right)^{-\frac{1}{2}} \approx \left(\frac{1}{z} - \frac{1}{2}\frac{R^2}{z^2}\right)\right.$$

$$E_z = \frac{Qz}{2\pi\varepsilon_0 R^2}\left(\cancel{\frac{1}{z}} - \cancel{\frac{1}{z}} + \frac{R^2}{2z^2}\right) = \frac{Qz}{4\pi\varepsilon_0 R^2}$$ same as point charge

ii) $R >> z \Rightarrow$

$$E_z = \frac{Qz}{2\pi\varepsilon_0 R^2}\left(\frac{1}{z} - \cancel{\frac{1}{\sqrt{z^2+R^2}}}\right)$$

$$= \frac{\sigma \pi R^2}{2\pi\varepsilon_0 R^2}$$

$$= \frac{\sigma}{2\varepsilon_0}$$ same as a charged plane

Electric Flux

Remembering point charge:

$$\underline{E} = \frac{1}{4\pi\varepsilon_0}\frac{q_i}{r^2}\hat{r}$$

=> Electric flux

$$\underline{\Phi} = \frac{\sum q_{enclosed}}{\varepsilon_0}$$

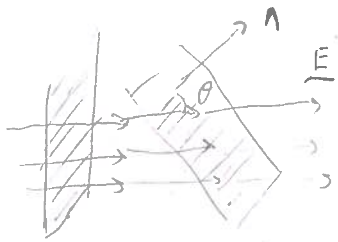
↳ Corresponds to the total number of field lines penetrating a surface

* $E \propto$ no. of field lines per unit area
* $\underline{\Phi} \propto$ no of field lines

— flux thru both surfaces is the same

=> for a surface ⊥ to $\underline{E}$

$$\underline{\Phi} = \underline{E}\, a$$



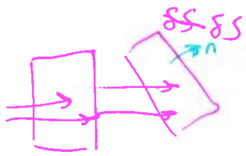
↳ $\underline{\Phi} = \underline{E}\cdot\hat{n}a$

[reducing $\underline{E}$ to give accurate flux]

defining $d\underline{a} = \hat{n}\,da$

→ $\underline{\Phi} = \int_S \underline{E}\cdot d\underline{a} = \int_S \underline{E}\cdot d\underline{a}$

OR



for a small area δa

$$d\underline{\Phi} = \underline{E}\cdot\hat{n}\,\delta a = \underline{E}\cdot\delta\underline{a}$$

summing up small chunks of $d\underline{\Phi}$

↳ $\underline{\Phi}_E = \int_S E\cdot\hat{n}\,da$

eg spherical enclosed charge

$$\oint \underline{E}\cdot d\underline{a} = \int \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r^2}\hat{r}\right)\cdot r^2 \sin\theta\, d\theta\, d\phi\, \hat{r} = \frac{q}{\varepsilon_0}$$

$$\underline{E} = \sum_{i=1}^{n} \underline{E}_i$$

flux through surface enclose them all

$$\oint \underline{E}\cdot d\underline{a} = \sum_{i=1}^{n}\left(\oint \underline{E}_i\cdot d\underline{a}\right) = \sum_{i=1}^{n}\left(\frac{1}{\varepsilon_0}\right)q_i$$

$$\oint \underline{E}\cdot d\underline{a} = \frac{Q_{enclosed}}{\varepsilon_0}$$

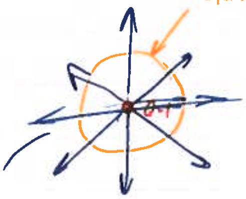
Gauss's law $\Phi_{net} = \oint \underline{E} \cdot d\underline{a} = \frac{Q_{internal}}{\varepsilon_0}$ $\nabla \cdot \underline{E} = \frac{\rho}{\varepsilon_0}$ $\nabla \times \underline{E} = 0$

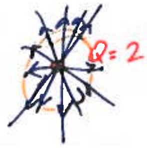
$= \int_V \nabla \cdot \underline{G}(r)\, dV$

Basic Idea:

Gaussian surface (imaginary surface)

only net charge contributes to net flux on the surface





2× net charge enclosed => 2× flux thru Gassian surface => 2× $\underline{E}$-field on surface

→ ∴ if find charge in the Gaussian surface
↳ we can find E-field at points on the surface

Let's say

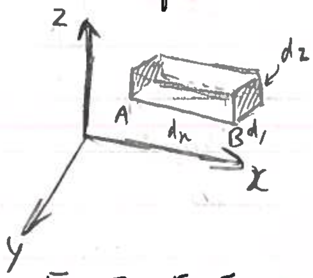
By comparison: going in → -ve
half of that of Q=1 → 0.5
↳ net charge enclosed = -0.5

=>

$\Phi_{net} = \oint \underline{E} \cdot d\underline{a} = \frac{Q_{internal}}{\varepsilon_0}$

→ valid for all surfaces
→ valid for all charge distributions
(唔理 charge 係邊)
→ independent of distance from charge & surface shape

Differential form



1. $\underline{E} = \underline{E}_x + \underline{E}_y + \underline{E}_z$ @ A

↳ @ B (same for dy & dz by symmetry)

$E_x + dE_x = E_x + \left(\frac{\delta E_x}{\delta x}\right) dx$

2. Net flux: change in flux between A & B → 相減

$$\left(E_x + \frac{\delta E_x}{\delta x} dx\right) dy dz - E_x dx dy dz = \frac{\delta E_x}{\delta x} dx dy dz$$

3. Total net flux (including the 2 other pairs of surfaces)

$$\left(\frac{\delta E_x}{\delta x} + \frac{\delta E_y}{\delta y} + \frac{\delta E_z}{\delta z}\right) dx dy dz = \underbrace{\frac{Q_{int}}{\varepsilon_0}}_{\text{R.H.S of Gauss's law}} = \frac{\rho\, dx dy dz}{\varepsilon_0}$$

$$\begin{pmatrix} \partial x \\ \partial y \\ \partial z \end{pmatrix} \cdot \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = \frac{\rho}{\varepsilon_0}$$

$$\boxed{\nabla \cdot \underline{E} = \frac{\rho}{\varepsilon_0}}$$

net flux per unit volume

~~net flux per unit volume centred @ a certain pt is equal to the charge~~

$\cancel{\nabla \nabla}$

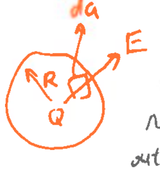
$\nabla^2 V = \frac{\rho}{\varepsilon_0}$

Deriving from Coulomb's law to Gauss's law

极柱

→ showing $\Phi = \frac{\sum Q_{enclosed}}{\varepsilon_0}$

- starting pt: we have a sphere
  ↳ symmetrical surface w/ charge @ centre



$\Phi_{net} = \int_S \underline{E} \cdot \underline{da} = \int_S E\,da = E \underbrace{\int da}_{S}$

net flux out of surface, S

$\underline{E}$ is // to $\underline{da}$ at all points
∵ spherical symmetry
↳ dot product will just be = the magnitude

$\underline{E}$ is constant throughout all points of the surface
∴ can be taken out of integration

$\Phi_{net} = E \int da = \frac{1}{4\pi\varepsilon_0}\frac{Q}{R^2}\int da = \frac{1}{4\pi\varepsilon_0}\frac{Q}{R^2} 4\pi R^2$

$= \frac{Q}{\varepsilon_0}$

∴ independent of radius & shape of the surface

- for a non-symmetrical surface

→ singled charge

we can draw an imaginary symmetrical surface for the charge

→ same amount of flux

✱ doesn't matter if we draw the imaginary surface inside or outside

→ multiple charges

- applying principle of superposition to electric field

↳ $\underline{E} = \sum_1^n \underline{E}_i$ (vector sum)

$\int_S \underline{E} \cdot \underline{da} = \sum \int_S \underline{E}_i \cdot \underline{da}$

$= \sum_i \frac{q_i}{4\pi\varepsilon_0} \int_S da_i$

$= \sum_i \frac{Q_i}{\varepsilon_0}$

$\Phi_{tot} = \sum_i^n \frac{Q_i}{\varepsilon_0}$ [total net flux parallel to the surface, add up total net charge within surface]

Consequences of Gauss's Law

- no electric field inside a conductor

  ↳ e/s all push away from each other to the edges ⇒ no charge ⇒ no field

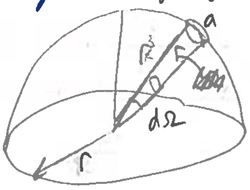
- no E-field inside a hollow conductor eg. Faraday cage

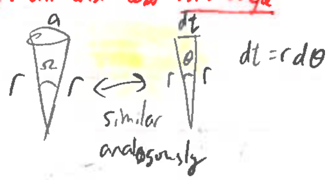
  ↳ in a hollow space inside a charged material / conductor

  ↳ no charge → ∴ no electric field (Faraday cage)

Solid angle (angle for 3D, using steradians) → can be used for any shape

Ω : dimensionless solid angle





Finding the total solid angle Ω in sphere:

$$da = r^2 d\Omega$$
$$d\Omega = \frac{da}{r^2}$$

if area is projected is r^2 away

$$\Omega = \frac{a}{r^2}$$

$$\Omega = \int_{sphere} d\Omega = \frac{1}{r^2}\int_{sphere} da = \frac{1}{r^2} 4\pi r^2 = 4\pi \text{ (steradians)}$$

輔修

Electrostatics in simple geometries

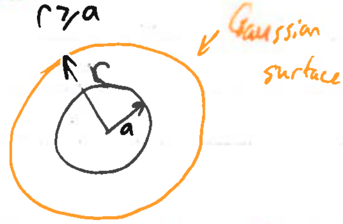
1. uniformly charged solid sphere
   w/ charge density ϱ and total charge:

$Q = V\varrho$

$Q = \frac{4}{3}\pi a^3 \varrho$

case 1:

$r \geq a$



$$\Phi_{net} = \int_S \underline{E} \cdot d\underline{a} = E \int_S da = E 4\pi r^2 = \frac{Q}{\varepsilon_0}$$

$\hookrightarrow E = \frac{1}{4\pi\varepsilon_0} \frac{Q}{r^2} \quad (r \geq a)$ → charge inside Gaussian surface for $r > a$ is always total charge, Q

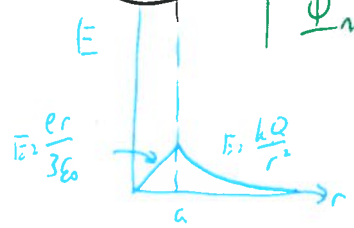
case 2:

$r < a$

$$\Phi_{net} = E 4\pi r^2 = \int_V \frac{\varrho}{\varepsilon_0} dV = \frac{\varrho}{\varepsilon_0} \frac{4}{3}\pi r^3$$

$E \cancel{4\pi} r^{\cancel{2}} = \frac{4}{3} \cancel{\pi} r^{\cancel{3}} \frac{\varrho}{\varepsilon_0}$

$\boxed{E = \frac{\varrho r}{3\varepsilon_0}} \quad (r < a)$ → charge inside Gaussian surface changes w/ r



2. Hollow sphere
   - negligible thickness
   - radius a
   - constant charge per unit area, σ

   ~~$Q = \frac{4}{3}\pi a^3 \varrho$~~

Case 1:

$r \geq a$

$Q = 4\pi a^2 \sigma$

$\Phi_{net} = E 4\pi r^2$

$\boxed{E = \frac{1}{4\pi\varepsilon_0} \frac{Q}{r^2}} \quad r \geq a$ Same for a pt charge or solid sphere

case 2:

$r < a$

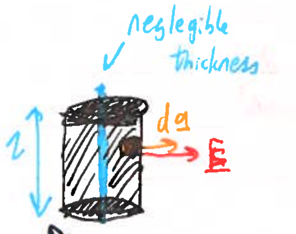
$E = 0 \quad (r < a)$

$\because Q_{inside} = 0$

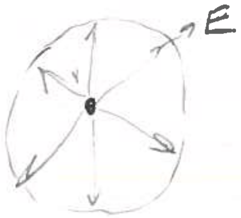
↳ R.H.S of Gauss's law is 0

$\therefore$ not net flux through Gaussian surface

3. Charged line



constant charge per unit length λ



$$\Phi_{net} = \int_S \underline{E} \cdot d\underline{a} = E \int_S da$$

for cylinder ends:

$E \cdot da = 0$

$\therefore E\cos(90)d\underline{a} = 0$

$\times$ 3 ends of cylinder

R.H.S

$$\frac{Q}{\varepsilon_0} = \frac{\lambda l}{\varepsilon_0}$$

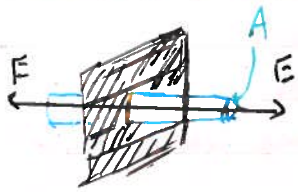
L.H.S

$$E 2\pi r l$$

$$\boxed{E = \frac{\lambda}{2\pi\varepsilon_0 r}}$$

- not affected by length, (l) we have chosen for the Gaussian surface
- E-fields falls off $\because$ $\frac{1}{r}$ from wire

4. Charged Plane → planar geometry



- infinite plane
- neglegible thickness
- constant charge per unit area σ

For only ends of the cylinder:

$$\Phi_{net} = \int_S \underline{E} \cdot d\underline{a} = E \int_S da = E 2\pi r^2$$

2 ends of the cylinder

[surface area of the cylinder]

R.H.S

$$\frac{Q}{\varepsilon_0} = \frac{\sigma A}{\varepsilon_0} = \frac{\sigma \pi r^2}{\varepsilon_0}$$

L.H.S

$$E 2\pi r^2$$

No need $\times 2$ $\because$ talking about area of the plane which is enclosed by the cylinder

$$\boxed{E = \frac{\sigma}{2\varepsilon_0}}$$

⇒ field strength is ~~ear~~ constant
↳ does not depend on the distance from the infinite sheet

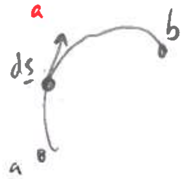
2.7/ Electric Potential.

- Electric potential energy (U)
  → Joules of energy a charge has due to the other charges present

$$\boxed{U = \frac{q_1 q_2}{4\pi\varepsilon_0 r}}$$

Work done! Force exerted by the E-field against the force pushing the charge closer

$$W = -\int_a^b \underline{E} \cdot d\underline{s} \quad (J)$$



- Electric Potential (V) [scalar]
  - Potential energy per Coulomb of a charge

$$V = \frac{1}{4\pi\varepsilon_0}\frac{q}{r} \quad J/C \text{ or } V$$

$$U = qV$$

※ force doing the pushing is in equal & opposite direction (-ve sign)

$$W = -\underbrace{q_2}_{}\int_a^b \underline{E} \cdot d\underline{s}$$

(charge) one creating the field

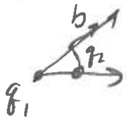
q_2: charge being pushed through the electric field

ΔV : potential difference [voltage]
- difference in the electric potential ΔV between two points
- difference in the energy of charge per unit charge between two pts.

$$W = -\frac{q_1 q_2}{4\pi\varepsilon_0}\int_a^b \frac{1}{r^2}\underbrace{\hat{r} \cdot d\underline{s}}_{}$$

$$E = \frac{q_1}{4\pi\varepsilon_0}\frac{1}{r^2}\hat{r}$$

∴ only interested when $\hat{r}$ & ds overlap



Rewriting in terms of V

$$\Rightarrow -q_2\int_a^b \underline{E} \cdot d\underline{s} = q_2[V(b) - V(a)]$$

$$\therefore W = -\frac{q_1 q_2}{4\pi\varepsilon_0}\int_a^b \frac{dr}{r^2} = -\frac{q_1 q_2}{4\pi\varepsilon_0}\left(\frac{1}{r_a} - \frac{1}{r_b}\right)$$

If we bring a charge from ∞ to b

$$\Rightarrow -\int_\infty^b \underline{E} \cdot d\underline{s} = V(b) - \underbrace{V(\infty)}_{0}$$

$$= V(b)$$

※ Path independent
only depend on start & end points
⇒ Principle of superposition

$$\boxed{V(b) = -\int_\infty^b \underline{E} \cdot d\underline{s}}$$

2.9

Electric field as grad of potential

eg from mechanics : $\underline{F} = -\underline{\nabla} W$ | $\underline{F} = q\underline{E}$ & $W = qV$

$q\underline{E} = -\underline{\nabla} qV$

$$\underline{E} = -\underline{\nabla} V$$

$$\underline{E} = -\left(\frac{\partial V}{\partial x}\underline{\hat{x}} + \frac{\partial V}{\partial y}\underline{\hat{y}} + \frac{\partial V}{\partial z}\underline{\hat{z}}\right)$$ [cartesian coordinates]

$$\underline{E} = -\left(\frac{\partial V}{\partial \rho}\hat{u}_\rho + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\hat{u}_\phi + \frac{\partial V}{\partial z}\hat{u}_z\right)$$ [cylindrical coordinates]

$x = \rho\cos\phi$
$y = \rho\sin\phi$
$z = z$

$$\underline{E} = -\left(\frac{\partial V}{\partial r}\hat{u}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{u}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\hat{u}_\phi\right)$$ [spherical coordinates]

$x = r\sin\theta\cos\phi$
$y = r\sin\theta\sin\phi$
$z = r\cos\theta$

Try to derive $\underline{\nabla}$ for cylindrical & spherical coordinates (Ex)

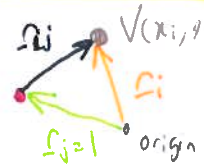
Q 2.9/ Potential of several point charges

2.10/ E-Potential for Discrete charge distribution

The potential @ pt (x_i, y_i, z_i) due to a set of charges, q_j @ position (x_j, y_j, z_j)

$$\boxed{V(x_i, y_i, z_i) = \frac{1}{4\pi\varepsilon_0}\sum_j \frac{q_j}{r_{ij}}}$$

Eg.



[in terms of vectors from the origin]

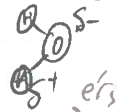
$$\underbrace{r_{ij}}_{\text{from charge to a point (vector)}} = |r_{ij}| = |r_i - r_j|$$

- why only summing over j? why not 加埋 i
  ∵ point ~~...~~ V doesn't change
  ∵ that's the point we are measuring at
  ∴ only summing the different charges in the different j positions

$$\boxed{E = -\nabla V = \frac{q}{4\pi\varepsilon_0}\frac{1}{r^2}\hat{r}}$$

2.11/ Electric dipole

eg. $\delta+$ More electronegative $\delta-$



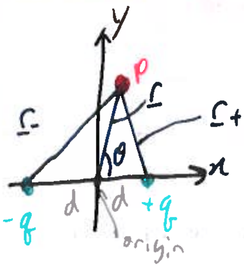
e⁻'s

e⁻/ dipole moment p

Electric dipole moment ∝ E ∝ p

$p = 2qd$

- Dipole: a system w/ 2 charges of equal magnitude & opposite sign separated by distance, 2d



Knowing: $V(p) = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r_+} - \frac{1}{r_-}\right)$ ← opposite charges

Using the cosine rule:

$a^2 = b^2 + c^2 - 2bc\cos A$

- $r_+^2 = r^2 + d^2 - 2rd\cos\theta$
- $r_-^2 = r^2 + d^2 - 2rd\cos(\pi - \theta)$
$\Rightarrow r_-^2 = r^2 + d^2 + 2rd\cos(\theta)$

$$V(P) = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{\sqrt{r^2 + d^2 - 2rd\cos\theta}} - \frac{1}{\sqrt{r^2 + d^2 + 2rd\cos\theta}}\right)$$

For the case where $r \gg d$

- First we have to rewrite to make it easier for binomial expansion

$$\frac{1}{r_\pm} = (r_\pm^2)^{-\frac{1}{2}} = \left[r^2\left(1 + \frac{d^2}{r^2} \mp \frac{2d\cos\theta}{r}\right)\right]^{-\frac{1}{2}}$$

$$= \frac{1}{r}\Big[1 + \underbrace{\frac{d^2}{r^2} \mp \frac{2d\cos\theta}{r}}_{x}\Big]^{-\frac{1}{2}}$$

For Binomial Expression:

$$(1+x)^{-\frac{1}{2}} \cong 1 - \frac{1}{2}x + \ldots$$

$$\frac{1}{r_\pm} = \frac{1}{r}\left(1 - \frac{d^2}{2r^2} \pm \frac{d\cos\theta}{r}\right) \Longleftarrow$$

$$\Rightarrow \frac{1}{r_+} - \frac{1}{r_-} = \frac{1\,2d\cos\theta}{r \cdot r}$$

putting Back to "V eq" ⇒

$$V_{dipole} = \frac{q}{4\pi\varepsilon_0}\frac{2d\cos\theta}{r^2}$$

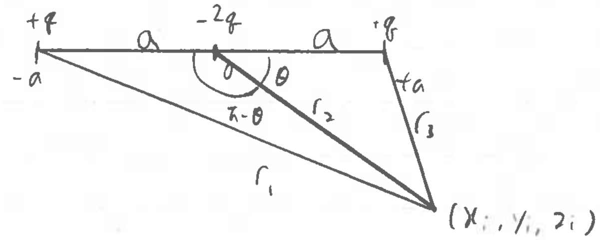
for $r \gg d$

2.11 / Ex 3: Consider the charge config fromed by a charge $-2q$ @ the origin & two charges $+q$ at the points $(\pm a, 0, 0)$

electric quadrupole

Show that the potential V at a distance r, which is large compared to a, $(r >> a)$ is approximately given by $V = -\frac{qa^2}{4\pi\varepsilon_0 r^3}$

where θ is the angle between r & the line through the charges.



$$V(x_i, y_i, z_i) = \frac{1}{4\pi\varepsilon_0} \sum_j \frac{q_i}{r_{ij}} \qquad r_{ij} = |r_i - r_j|$$

$$V(x_i, y_i, z_i) = \frac{1}{4\pi\varepsilon_0} \left[\frac{q}{r_1} - \frac{2q}{r_2} + \frac{q}{r_3} \right]$$

$$r_3^2 = a^2 + r_2^2 - 2ar_2\cos\theta$$

$$r_1^2 = a^2 + r_2^2 - 2ar_2\cos(\pi - \theta) = a^2 + r_2^2 + 2ar_2\cos\theta$$

$$\hookrightarrow V(x_i, y_i, z_i) = \frac{1}{4\pi\varepsilon_0} \left[\frac{q}{(a^2 + r_2^2 + 2ar_2\cos\theta)^{\frac{1}{2}}} + \frac{q}{(a^2 + r_2^2 - 2ar_2\cos\theta)^{\frac{1}{2}}} - \frac{2q}{r_2} \right]$$

$$= \frac{q}{4\pi\varepsilon_0 r_2} \left[\frac{1}{\left(1 + \frac{a^2}{r_2^2} + \frac{2a\cos\theta}{r_2}\right)^{\frac{1}{2}}} + \frac{1}{\left[1 + \frac{a^2}{r_2^2} - \frac{2a\cos\theta}{r_2}\right]^{\frac{1}{2}}} - 2 \right]$$

Using ~~ex~~ Binomial expansion:

$$(1+x)^n = 1 + nx \ldots \quad \text{when } x \text{ is small}$$

$$(1+x)^{-\frac{1}{2}} = 1 - \frac{1}{2}x$$

$$V(x_i, y_i, z_i) = \frac{q}{4\pi\varepsilon_0 r_2} \left[\cancel{1} - \frac{a^2}{2r_2^2} - \cancel{\frac{a\cos\theta}{r_2}} + \cancel{1} - \frac{a^2}{2r_2^2} + \cancel{\frac{a\cos\theta}{r_2}} \cancel{-2} \right]$$

$$= \frac{q}{4\pi\varepsilon_0 r_2} \left[-\frac{a^2}{r_2^2} \right] = -\frac{qa^2}{4\pi\varepsilon_0 r_2^3}$$

2.12 Potential for continuous charge distributions

→ splitting up ~~vol~~ volumes into tiny pieces of charge dq,

then sum:
$$V = \frac{1}{4\pi\varepsilon_0}\int \frac{dq}{r}$$

charge density

$$\Rightarrow V(x_i, y_i, z_i) = \frac{1}{4\pi\varepsilon_0}\int_{vol} \frac{\rho(x_j, y_j, z_j)}{r_{ij}}\, d\tau$$

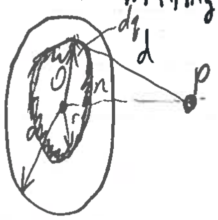
volume element

↳ $|r_{ij}| = |r_i - r_j|$

Eg. Charged disk, find V & E on axis

→ ~~An~~ A uniformly charged disk has a radius, a & surface charge density σ.

1. Find the electric potential along the perpendicular central axis of the disk
2. ~~Diffing the~~ Diff the potential → find the magnitude of the E field along the same axis



- charge on ring: $\sigma \cdot 2\pi r \cdot dr$

- Potential @ P due to the ring: $dV = \frac{\sigma \cdot 2\pi r \cdot dr}{4\pi\varepsilon_0\sqrt{x^2+r^2}}$

- Total potential: $V = \frac{\sigma}{2\varepsilon_0}\int_0^a \frac{r\,dr}{\sqrt{x^2+r^2}}$

$u = x^2 + r^2$

$\frac{du}{dr} = 2r$

$$\Rightarrow V = \frac{\sigma}{2\varepsilon_0}\int_{x^2}^{x^2+a^2} \frac{1}{\sqrt{u}}$$

$$= \frac{\sigma}{2\varepsilon_0}\left[\frac{1}{2}\cdot 2\sqrt{u}\right]_{x^2}^{x^2+a^2}$$

$$= \frac{\sigma}{2\varepsilon_0}\left[\left(\sqrt{x^2+a^2}\right) - x\right]$$

Electric Field

- By symmetry: $E_y = E_z = 0$

$$E_x = -\frac{dV}{dx}$$

$$= -\frac{\sigma}{2\varepsilon_0}\left[\frac{2x(\frac{1}{2})}{\sqrt{x^2+a^2}} - 1\right]$$

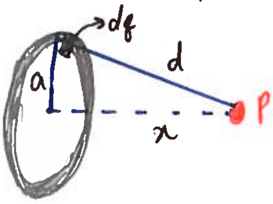
$$= \frac{\sigma}{2\varepsilon}\left[1 - \frac{x}{\sqrt{x^2+a^2}}\right]$$

$Q = \sigma\pi a^2$

$$= \frac{Q}{2\pi\varepsilon_0 a^2}\left(1 - \frac{x}{\sqrt{x^2+a^2}}\right)$$

Eg. Charged ring, find V & E on axis

- Find expression for V @ pt P located @ the perpendicular ~~distance~~ axis of a uniformly charged ring of radius a & total charge Q.
- Find an expression for magnitude of E-field @ P.



$$V = \int \frac{dq}{4\pi\varepsilon d}$$

∵ each section of the ring from pt P is the same distance

$$\therefore V = \int \frac{dq}{\underbrace{4\pi\varepsilon_0 d}_{\text{constant}}} = \frac{Q}{4\pi\varepsilon_0 d} = \frac{Q}{4\pi\varepsilon_0\sqrt{a^2+x^2}}$$

When $x \gg a$, the V goes to $V = \frac{Q}{4\pi\varepsilon_0 x}$ [same as pt charge]

Finding E: $E_y = E_z = 0$

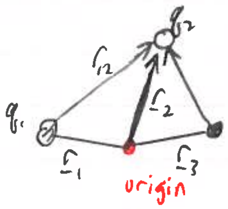
$$E_x = -\frac{dV}{dx}$$

$$= -\frac{d}{dx}\frac{1}{4\pi\varepsilon_0} Q(x^2+a^2)^{-\frac{1}{2}}$$

$$= \frac{1}{2}\frac{1}{4\pi\varepsilon_0} Q(x^2+a^2)^{-\frac{3}{2}}(2x)$$

$$= \frac{Qx}{4\pi\varepsilon_0(x^2+a^2)^{\frac{3}{2}}}$$

2/13 Electrostatic potential energy – discrete charges



★ Electrostatic potential Energy, U, of a system
= work, W, needed to bring the charges
from an infinite separation to their final positions

Potential due to q_1: $V_1(\underline{r}_2) = \frac{1}{4\pi\varepsilon_0}\frac{q_1}{r_{12}}$

– To bring q_2 from ∞ to $\underline{r}_2$ => need to do work against the field from q_1

$$W_2 = q_2 V_1 = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r_{12}}$$

→ same as bringing q_3, do work against q_1 & q_2:

$$W_3 = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_3 q_1}{r_{13}} + \frac{q_3 q_2}{r_{23}}\right)$$

→ Total potential energy:

$$U = W_2 + W_3 = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1 q_2}{r_{12}} + \frac{q_3 q_1}{r_{13}} + \frac{q_3 q_2}{r_{23}}\right)$$

In general:

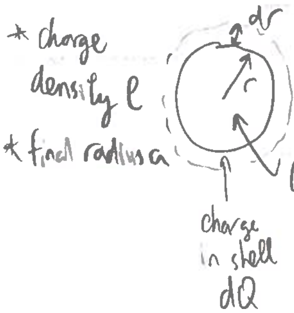
$$U = \frac{1}{4\pi\varepsilon_0}\sum_{\substack{\text{all}\\\text{pairs}}}\frac{q_i q_j}{r_{ij}} \quad \text{OR} \quad U = \frac{1}{4\pi\varepsilon_0}\sum_{\substack{i,j=1\\i\neq j}}\frac{q_i q_j}{r_{ij}}\left(\frac{1}{2}\right)$$

↖ avoid double counting
→ we want to have each pair once

$$U = \frac{1}{4\pi\varepsilon_0}\frac{1}{2}\sum_{\substack{i,j=1\\i\neq j}}\frac{q_i q_j}{r_{ij}}$$

2.14/ Electrostatic potential energy - continuous charge distributions

[Building up a charged sphere shell by adding up the total energy]



$$V_r = \frac{1}{4\pi\varepsilon_0}\frac{Q_r}{r}$$

- Work to bring next shell:

$$dU = dQ\,V_r = \frac{Q_r\,dQ}{4\pi\varepsilon_0 r}$$

where $Q_r = \frac{4}{3}\pi r^3 \rho$

$$dQ = \underbrace{4\pi r^2}_{\text{surface Area}}\rho\,\underbrace{dr}_{\text{width of shell}}$$

=> Subbing in

$$dU = \frac{1}{4\pi\varepsilon_0 r}\left(\frac{4}{3}\pi r^3\rho\right)(4\pi r^2\rho)\,dr$$

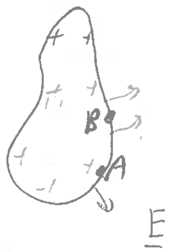
$$dU = \frac{4\pi\rho^2}{3\varepsilon_0}r^4\,dr \Rightarrow U = \int_0^a \frac{4\pi\rho^2}{3\varepsilon_0}r^4\,dr = \frac{3}{5}\frac{Q^2}{4\pi\varepsilon_0 a}$$

E & V in a conductor

Conductors are materials in which charges move freely

↳ if left isolated, charges will distribute themselves to achieve electrostatic eq^{m} → charges stationary

↳ $F = 0 \rightarrow E = 0$ inside

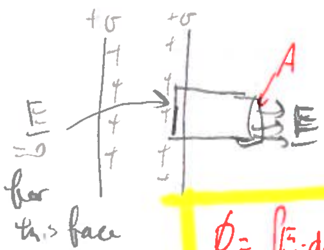


For electrostatic Eq^{m}

- E-field = 0 everywhere inside conductor
  ∴ $F = 0$, $E = 0$
- If isolated conductor carries ~~a~~ charge
  ↳ charge resides on surface
    ↳ By Gauss' Law, no charge contained in conductor if there is no field within it
- E-field just outside a charged conductor is ⊥ to surface of the conductor

∴ no sideways movement, tangential $E = 0$

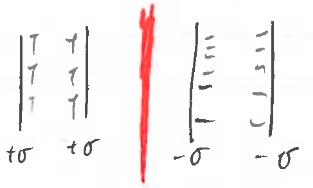
∴ All E ⊥ to surface



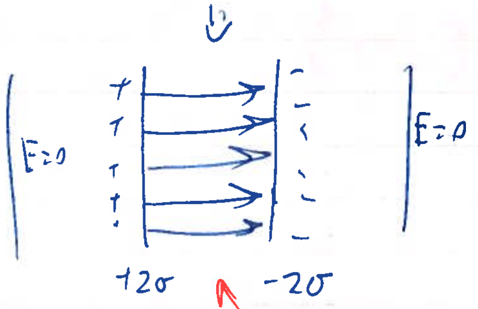
$$\boxed{\phi = \int E\cdot da = EA = \frac{\sigma A}{\varepsilon_0} \qquad E = \frac{\sigma}{\varepsilon_0}}$$

Applies to any shape of the conductor, immediately outside surface

3.3. $\underline{E}$ for two oppositely charged conducting plates:



$E = \frac{\sigma}{\varepsilon_0}$ for all surfaces



$E = \frac{2\sigma}{\varepsilon_0}$

* case for the mid-way situation (ppt pls)

$\nabla \cdot \underline{E} = \frac{\rho}{\varepsilon_0}$ | $\underline{E} = -\nabla V$

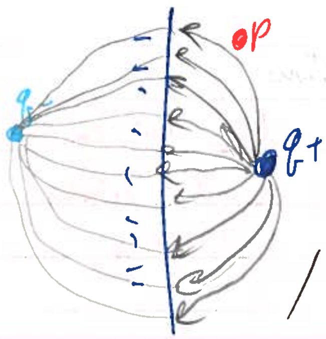
$\nabla \cdot \nabla V = -\frac{\rho}{\varepsilon_0}$

$\nabla^2 V = -\frac{\rho}{\varepsilon_0}$

Fields outside conductors

- using the Method of images:
  - ↳ replace some parts w/ imaginary charges
    - ↳ making sure the result have the same boundary conditions & satisfy Poisson's eq $\nabla^2 V = -\frac{\rho}{\varepsilon_0}$

* replacing the difficult plane to sth easier to calculate BUT giving the same result



- conductor connected to the ground has a potential of zero ∴ $V=0$ along the plate

∴ Replacing the plate w/ pt charge,

- Between q+ & q− ⇒ $V=0$
- $V=0$ @ ∞

Now becomes a 2 pt charge prob

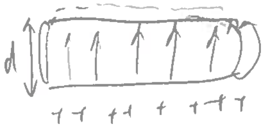
Same as dipole prob as 184

$$V(P) = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r_+} - \frac{1}{r_-}\right)$$

$$V(P) = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{\sqrt{r^2+d^2-2rd\cos\theta}} - \frac{1}{\sqrt{r^2+d^2+2rd\cos\theta}}\right)$$

3.5/ Vacuum capacitor : definition of capacitor

Capacitor: Two conductors w/ equal charges (Combination of) which have opposite signs



Between: $E = \frac{\sigma}{\varepsilon_0}$; $\sigma = \frac{q_1}{A}$ — charge on one plate

[ignoring edge effects]

↳ non-uniform E

Potential energy (U), to move charge q_2 from one plate to another : $U = F \cdot d = q_2 E d = \frac{q_1 q_2 d}{\varepsilon_0 A}$

↳ ∴ potential difference between two plates : $\Delta V = \frac{q_1 d}{\varepsilon_0 A} = \frac{U}{q_2}$

$V = V(x)$

$\Delta V = |V(0) - V(d)|$

$= \int_0^d E\,dx = E[x]_0^d = Ed$

or

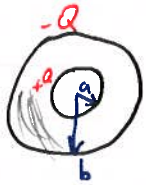
↳ $q_1 = \underbrace{\frac{\varepsilon_0 A}{d}}_{\text{capacitance}} \Delta V$ (A: area of one of the plates)

$q_1 = C \Delta V$

or

$C = \frac{q_1}{\Delta V}$ => charge that can be stored per volt of potential difference [farads, F]

3.7/ Spherical capacitor



[concentric shells]

Drawing Gaussian surface

$r > b \Rightarrow$ No net charge enclosed $\rightarrow E = 0$

$r < a \Rightarrow$ same as above

$a \le r \le b \Rightarrow$ same as if a pt charge $+Q$ enclosed

@ $r = b$: $V_b = \frac{1}{4\pi\varepsilon_0}\frac{Q}{b}$ @ $r = a$; $V_a = \frac{1}{4\pi\varepsilon_0}\frac{Q}{a}$

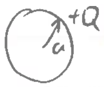
$$\Delta V = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right) = \frac{Q}{4\pi\varepsilon_0}\left(\frac{ab}{b-a}\right)$$

$\because$ $C = Q\Delta V$

$\Rightarrow$ $C_{sperical} = 4\pi\varepsilon_0 \frac{ab}{b-a}$

if $\rightarrow$ single sphere;

$b \rightarrow \infty$



$\lim_{b\to\infty} \frac{a}{(1 - a/b)}$

$C = 4\pi\varepsilon_0 a$

3.8/ Cylindrical capacitors

finding ΔV:

- neglecting end effects: field is radially outward & $\perp$ to the axis
- Gauss Law: charge on the outer cylinder do not contribute to the field inside
- field is due to the line of charge, for $r > a$ [outside of central conductor]

$a \le r \le b \rightarrow$ [If $l \gg b$]

↳ same as if infinite charge line enclosed

I.e. $E = \frac{\lambda}{2\pi\varepsilon_0 r}\hat{r}$ $\lambda = \frac{Q}{l}$ (charge density)

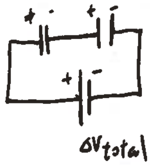
$$\Delta V = \int_a^b E \cdot dr = \frac{Q}{2\pi\varepsilon_0 l}\int_a^b \frac{dr}{r} = \frac{Q}{2\pi\varepsilon_0 l}\ln\left(\frac{b}{a}\right)$$

$$C = \frac{Q}{\Delta V}$$

$$C_{cylindrical} = \frac{2\pi\varepsilon_0 l}{\ln\left(\frac{b}{a}\right)}$$

cylindrical geometry $\rightarrow$ used in electricity & signal transmission [coaxial cable]

3.9/ Capacitors in series



- Same charge on each plate

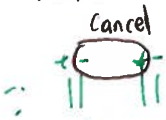
- adding voltage

$$\Delta V_{total} = \Delta V_1 + \Delta V_2 = \frac{Q}{C_1} + \frac{Q}{C_2}$$

$$= Q\left(\frac{1}{C_1} + \frac{1}{C_2}\right)$$

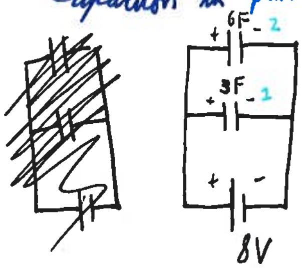
$$\frac{1}{C_{tot}} = \frac{1}{C_1} + \frac{1}{C_2}$$

→ using 2 (or more) capacitors, gives less capacitance than using one



↳ leaving the outside plates, ∴ one capacitor, but half the ΔV between the outside 2 plates ⇒ half the capacitance

Capacitors in parallel



[Same voltage drop]

→ 8 Volt drop in each of the ~~eq~~ capacitors
↳ charge splits in proportion to capacitances

$$\Delta V_{total} = \frac{Q_1}{C_1} = \frac{Q_2}{C_2}$$

⇒ storing the same total charge

$$Q_{tot} = Q_1 + Q_2$$

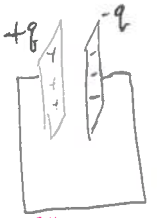
↳ ∴ same as merging both −ve & +ve plates

$$C_{//} = \frac{Q}{\Delta V} = \frac{Q_1 + Q_2}{\Delta V} = \frac{Q_1}{\Delta V} + \frac{Q_2}{\Delta V}$$

$$= \frac{C_1 \Delta V}{\Delta V} + \frac{C_2 \Delta V}{\Delta V}$$

↳ $C_{//} = C_1 + C_2$

3.10/ Energy of a capacitor



avg voltage drop is half of the initial voltage drop

Charge passed from one plate to another, against the E-field of the two plates

$dW = \Delta V\, dq$ | $\Delta V = \frac{q}{C}$ ⇒ total work done to move a unit charge

$$W = \int_0^Q \frac{q}{C}\, dq$$

$$= \frac{1}{2}\frac{Q^2}{C}$$

$C = \frac{Q}{\Delta V}$

$$U = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2} Q\Delta V = \frac{1}{2} C(\Delta V)^2$$

$\frac{Q}{C} = \Delta V$

* applies to any capacitor, independent of geometry
* applies in devices which store & rapidly deliver electrical energy

- eg. Defibrillator
  ↳ conductor is heart
- eg. Camera flash unit
  ↳ conductor is flash bulb

3.11 Energy density [energy stored per unit volume] (u)

U stored in capacitor can be seen as being stored in the electric field

- $E = \frac{\Delta V}{d}$
- $C = \frac{\varepsilon_0 A}{d}$

$\Rightarrow U = \frac{1}{2} C (\Delta V)^2 = \frac{1}{2} \frac{\varepsilon_0 A}{d} E^2 d^2$

$C = \frac{\varepsilon_0 A}{d}$

- $u = \frac{U}{Ad}$

$u = \frac{1}{2} \frac{\varepsilon_0 A E^2 d}{Ad}$

energy density

$\Rightarrow \boxed{u = \frac{1}{2}\varepsilon_0 E^2 = \frac{U}{Ad}}$ volume of capacitor

$\Rightarrow$ result applies generally to the energy density of a charged capacitor

---

4.3/ Polar molecules:

[have a permanent dipole moment, even w/o an external E-field]

↳ should consist at least of 2 different species of atoms w/ a difference in electronegativity [general chem stuff]

4.1/ Polarisation or Polarisation density

Definition: $\boxed{\underline{P} = \frac{\Delta \underline{p}}{\Delta V}}$ — electric dipole moment; volume

— electric dipole moment per unit volume

✱ measure how much the e's has been pushed over to one side in a certain material, which is isotropic

✱ ✱ opposite direction to E-field of dipoles

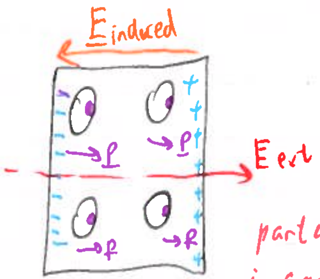
direction → -ve to +ve

Isotropic: polarization has the same direction as the E-field which causes it

in the limit of a weak applied E-field

↳ $\underline{P} = \varepsilon_0 \chi \underline{E}$

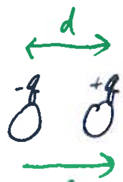
χ: electric susceptibility → relates to a dimensionless relative permittivity $\varepsilon_r = 1 + \chi$ or dielectric constant



part of the external field is cancelled by the E-field from the induced dipole

$E_{int} = E_{ext} + E_{induced}$

4.3/ Electric dipole in a field

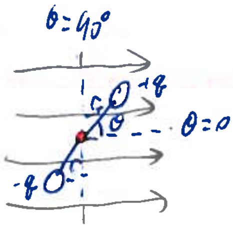


$\underline{p}$ electric dipole moment

$\underline{p} = q\underline{d}$

for Torque:

using $\underline{\tau} = \sum_i \underline{r}_i \times \underline{F}_i$



$\underline{\tau} = \underline{r}_+ (q\underline{E}) + \underline{r}_- (-q\underline{E})$

$= q \underbrace{(\underline{r}_+ - \underline{r}_-)}_{\underline{d}} \times \underline{E}$

$\boxed{\underline{\tau} = \underline{p} \times \underline{E}}$

Changes in potential energy:

$\Delta U = U_f - U_i$ ← work done by the field

$= \int_{\theta_i}^{\theta_f} |\underline{\tau}| \, d\theta$

$= pE \int_{\theta_i}^{\theta_f} \sin\theta \, d\theta$

$= -pE(\cos\theta_f - \cos\theta_i)$

choosing $U_i = 0$, $\theta_i = 90°$

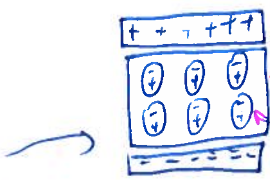
when $\theta = 90°$

$\Rightarrow U_f = -pE\cos\theta_f$

$\Rightarrow \boxed{U = -\underline{p} \cdot \underline{E}}$ — potential energy of a dipole in an external field, relative to the $\theta = 90°$ point

4.4/ Dielectrics in capacitors

Case 1: atoms have no natural polarization (non-polar)



dielectric = insulator [no free charges]

induced dipole

→ e's move to one side of the atom/molecule

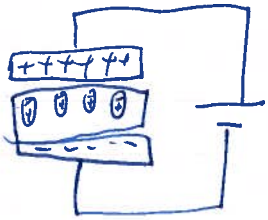
↳ acquire a dipole moment & becomes polarised

Case 2: atoms are already polarised (eg water) (polar)

← polar molecules already have a permanent dipole moment

↳ if they can rotate, they will align

w/ an external field to give the same result as above



Case a: charge capacitor then remove battery then insert dielectric

→ reduction in potential difference ∵ cancelled by charges in the dielectric

↳ By $C = \frac{Q}{V}$ $V\downarrow$, $C\uparrow\uparrow$ ⇒ capacitance ↑↑

Case b: leave battery in circuit → then insert dielectric

↳ battery will try to ↑↑ voltage until capacitor voltage = battery voltage

→ more charge can be stored
→ increased in stored charge → ↑↑ ~~the~~ capacitance

$C_{\text{with dielectric}} = \epsilon_r C$ ← dielectric ↑↑ value by factor ϵ_r

$U_{\text{with dielectric}} = \frac{1}{2}\epsilon_r \epsilon_0 E^2$] more susceptible the dielectric is to polarization
↳ stronger the polarization

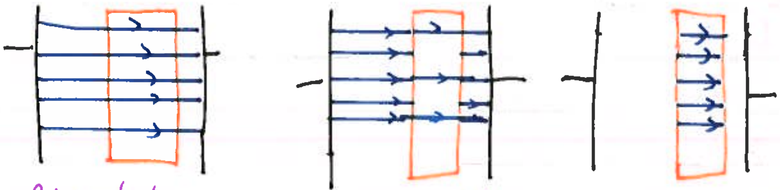
voltage across capacitor stays the same
each charge we add on to the plate ↑↑ V
↳ X just ↑↑ energy store by the plate, but also ↑↑ energy per charge
Makes sense ∵ W.D. to bring in the next charge keep ↑↑ ∵ need to move it against all the et stored charges

Electric Displacement, $\underline{D}$

$$\underline{D} = \epsilon_0 \underline{E} + \underline{P}$$

$\underline{P}$: electric dipoles per unit volume or [material's response to the $\underline{E}$-field]

$\underline{E}$: net $\underline{E}$-field due to external fields & internal field



$\underline{D}$: constant throughout the capacitor

$\underline{E}$: reduced inside the dielectric

$\underline{P}$: zero except for inside the dielectric

$\underline{D} = \epsilon_0 \underline{E} + \underline{P}$

& $\underline{P} = \epsilon_0 \chi \underline{E}$ sub. in

$\underline{D} = \epsilon_0 \underline{E} + \epsilon_0 \chi \underline{E}$

$= \underbrace{(1+\chi)}_{\epsilon_r} \epsilon_0 \underline{E}$

$= \underbrace{\epsilon_r \epsilon_0}_{\epsilon} \underline{E}$

$\boxed{\underline{D} = \epsilon \underline{E}}$

eg. for a point charge:

$\underline{E} = \frac{1}{4\pi\epsilon}\frac{q_1}{r^2}\hat{r}$

$\underline{D} = \frac{1}{4\pi}\frac{q_1}{r^2}\hat{r}$] independent of material

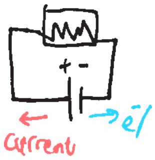
Split total charge density into free ρ_f & bound ρ_b

$\nabla \cdot \underline{D} = \epsilon_0 \nabla \cdot \underline{E} + \nabla \cdot \underline{P} = \rho - \rho_b = \rho_f$

$\oint \underline{E} \cdot d\underline{a} = \frac{Q_{\text{internal}}}{\epsilon}$

$\Rightarrow \oint \underline{D} \cdot d\underline{a} = Q_{\text{internal}}$

5.1 + 5.2 Current & resistance



* Steady state but not an electrostatic eq^m

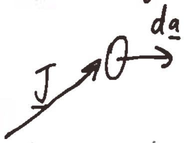
* Voltage doesn't ↑↑ w/ time
∵ batteries continuously separate +/- charge in a chemical Rx

In a resistor:
e/s undergo inelastic & lose energy
↳ electrical potential drops across resistor

Current: $I = \frac{dQ}{dt}$
[$C s^{-1}$; (A)]
[coulombs per second]

Ohm's law: $V = IR$
or
$I = VG \mid G = \frac{1}{R}$ [conductance]

Current density: net amount of charge crossing a unit area ⊥ to the drift velocity



- vector defined so that the charge crossing area $d\underline{a}$ per unit time !

$\underline{J} \cdot d\underline{a}$ $[\underline{J} = n q \underline{v}_d]$

number of charge carriers per unit volume

↳ charge

↳ charge crossing unit area ⊥ to J per unit time ($A m^{-2}$)

- Total current:

$$I = \int_A \underline{J} \cdot d\underline{a}$$

* charges (usually -ve electrons)
↳ have drift velocity v_d (non-relativistic)
due to acceleration of E-field between collisions

$\underline{J} = n q \underline{v}_d$

$v_d t$

- For steady currents, charge conservation gives:

$\nabla \cdot \underline{J} = 0$

or generally:

$\nabla \cdot \underline{J} = -\frac{\partial \rho}{\partial t}$

↳ total amount of charge leaving the little box
= reduction of charge which is currently inside the box

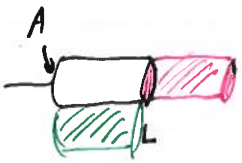
Resistor addition & subtractions → A-level stuff.

5.3 Resistivity & conductivity

$\left[R = \frac{V}{I}\right]$ ∀ in OHMIC materials

↳ R is constant, no matter what V or I there is → if change resistance

↳ change R need to change the resistor itself



↑↑ L, add another material / resistor in series

↳ $R \propto L$

↑↑ A, adding material / resistor in parallel

↳ $R \propto \frac{1}{A}$

$R = \rho \frac{L}{A}$

↳ Resistivity $[\Omega m]$ $\rho = \frac{1}{\sigma}$

↳ Conductivity $[(\Omega m)^{-1}]$

using Ohm's Law: $V = I\left(\rho \frac{L}{A}\right)$

$\frac{V}{L} = \rho \boxed{\frac{I}{A}}$

[if potential drop is uniform across a resistor.

$\underline{E} = -\frac{dV}{ds}$ (from $\underline{E} = -\underline{\nabla} V$)

↳ $\frac{V}{L} = \underline{E}$]

$\underline{E} = \rho \underline{J}$

↳ $\frac{1}{\sigma}$

:) $\underline{J} = \sigma \underline{E}$

current density & Electric field

$\frac{I}{A}$: current per unit area

↳ current density, $\underline{J}$

$\frac{\text{Charge}}{\text{Area}/s}$

$\underline{E}$-field ↑↑, P.D. ↑↑, e⁻/s speed up,

more charge going through an area per second

5.4 Electrical power

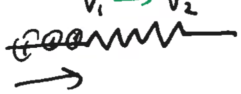
Resistors heat up due to inelastic collisions

∴ work is done by the E-field to drive the current

& how much energy is ~~dissa~~ dissipated.

$5JC^{-1}$ ΔV $2JC^{-1}$

$V_1 \rightarrow V_2$



use +ve ∵ easier w/o -ve signs

$\Delta V = 3JC^{-1}$

charge bump into atoms in resistor

↳ atoms vibrate more, i.e. hotter

↳ but how much hotter w/ time??

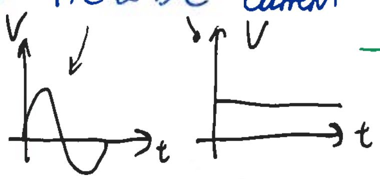
Power: $P = \frac{dU}{dt} = \frac{dQ\,\Delta V}{dt} = I\,\Delta V$

$$P = I\Delta V = I^2R = \frac{(\Delta V)^2}{R}$$

Watt (W) Js^{-1}

↳ how many Joules of energy are being converted in the resistor per second

5.5 AC & DC current



Tesla vs Edison

→ semiconductor Logic works w/ DC

↳ hence rectifier in laptop chargers

→ high V → lower current for same power

→ less losses

↳ cheaper + easier to step voltage up & down

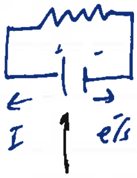
[more in transformers]

↳ BUT more losses due to interaction w/ outside the wire when changing direction [esp. in water]

& high voltage direct current generally used for international links (undersea)

General A-level stuff

5.6 EMF



batteries have to pump charge back to its original side
↳ requires work

$\boxed{\varepsilon = \frac{dW}{dq}}$ ⇒ work done per unit charge to move the charges against the E-field

↳ Electromotive force

↳ potential difference used to move charges from one side to the other [not a force]

5.7, 5.8

Kirchoff's Rules

1. In series 嘅 voltage

$dW = I^2 R\, dt$

$dW = \varepsilon\, dq$

$\boxed{\varepsilon = IR}$

IR → thermal energy per charge dissipated in the whole current

ε → chemical energy per charge gained in battery

Kirchoff's loop rule

$\sum_{\text{closed loop}} \Delta V = 0$

- including the battery EMF: there is no net voltage change in a closed loop

[conservation of energy]

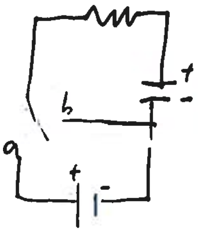
2. In parallel 嘅 current

Sum of current entering any junction must equal the sum of current leaving it:

$\sum I_{in} = \sum I_{out}$ } Kirchoff's junction rule

[conservation of charge

5.9 How ~~current~~ current varies as a capacitor charges up



Kirchoff's Loop Rule: $\mathcal{E} = IR + \frac{q}{C}$

↳ $\div R$, sub $I = \frac{dq}{dt}$

$$\frac{dq}{dt} + \left(\frac{1}{RC}\right) q = \frac{\mathcal{E}}{R}$$

$$\Rightarrow q(t) = C\mathcal{E}\left(1 - e^{-\frac{t}{RC}}\right)$$

$$\Rightarrow \boxed{I(t) = \frac{dq}{dt} = \frac{\mathcal{E}}{R} e^{-\frac{t}{RC}}}$$

↳ time constant [ΩF, s]

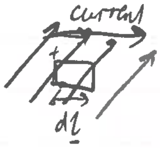
when $t = RC \Rightarrow$

$$\frac{I(t=RC)}{I(t=0)} = \frac{\frac{\mathcal{E}}{R}e^{-1}}{\frac{\mathcal{E}}{R}} = \frac{1}{e^1} = 0.368$$

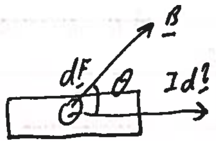
6. Magnetostatics

Moving charges cause magnetic fields :
- current flowing in a wire
- e/s in atomic-level current loops
- quantum spin

1. Magnetic Field [Magnetic Flux Density] $\underline{B}$



Tiny element of current : $I d\underline{l}$



if // → no magnetic ~~f~~ force felt

$F \propto B$, $F \propto \underline{I}$

$F \propto \sin\theta$ (∠ between B & l)

$F \propto l$ & $F \propto$ wire length

$d\underline{F} = I d\underline{l} \times \underline{B}$ using right hand rule

$\underline{F} = I \int_L d\underline{l} \times \underline{B}$

units for $\underline{B}$: $NA^{-1}m^{-1} = T$

2. Magnetic ~~fied~~ field lines :

Rules : 1. N → S
2. Tangent gives direction of $\underline{B}$
3. Density represents magnitude of $\underline{B}$
4. Can't cross
5. Continuous

3. Magnetic flux → $\underline{\underline{\phi_B}}$ (Wb)

$1 Wb = 1 T \cdot m^2$

$\underline{B} = \frac{\phi_B}{A}$ => Magnetic flux density as above

$\underline{\Phi_B} = \int_S \underline{B} \cdot d\underline{S}$

Φ_B → Scalar

$d\underline{S}$ → tiny bit of surface area in vector form

4. Motion of a charged particle in a $\underline{B}$-field

Force acting on a charged particle

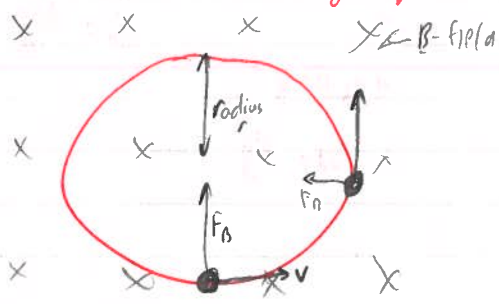
=> $\underline{F_B} = q \underline{v} \times \underline{B} = q v B \hat{\underline{r}}$

$q\underline{v}$ → charge & velocity of the particle

$\underline{B}$ → uniform magnetic field

※ $|\underline{v}|^2$ not changed by $\underline{F}$

※ direction of $\underline{v}$ is changed by $\underline{F}$



$\underline{v} = \underbrace{\underline{v}_\perp}_{\underline{v} \perp \underline{B}} + \underbrace{\underline{v}_{\parallel}}_{\underline{v} \parallel \text{ to } \underline{B}}$ => Helical motion

$\underline{v} \perp \underline{B}$ => circular motion

$\underline{v} \parallel$ to $\underline{B}$ => motion forwards in direction of $\underline{B}$

$\underline{B}$-force providing centripetal force $\frac{mv^2}{r}\hat{\underline{r}}$

=> $q \underline{v} B \hat{\underline{r}} = \frac{mv^2}{r} \hat{\underline{r}}$

$\frac{mv^2}{r} = qvB$

$r = \frac{mv}{qB}$

$\omega = \frac{2\pi}{T} = \frac{qB}{m}$ (cyclotron frequency)

$T = \frac{2\pi r}{v}$ (time period) $= \frac{2\pi m}{qB}$

Lorentz Force

for a moving charge

$$\underline{F} = q\underline{E} + q\underline{v}\times\underline{B}$$

$$= q(\underbrace{\underline{E}}_{\text{By E-field}} + \underbrace{\underline{v}\times\underline{B}}_{\text{By B-field}})$$

When $\underline{E} \perp \underline{B} \Rightarrow$ crossed fields

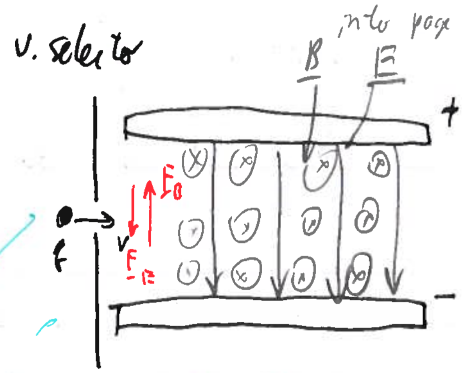
used in - velocity selector

- mass spectrometer
- Hall effect

for B-field detector

ATLAS dector @ CERN

v. selector

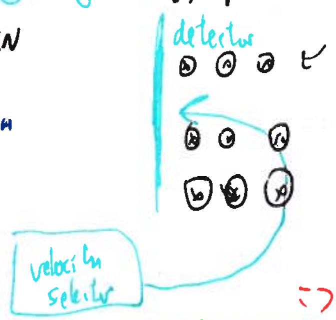


For particle to get through:

$$qE = qvB$$

$$\boxed{v = \frac{E}{B}}$$

mass spectrometer



w/ just B-field B_0

$$\underline{F}_B = q\underline{v}\times\underline{B}_0$$

$$= -qvB_0\hat{r}$$

$$= -\frac{mv^2}{r}\hat{r}$$

$$\Rightarrow r = \frac{mv}{qB_0}$$

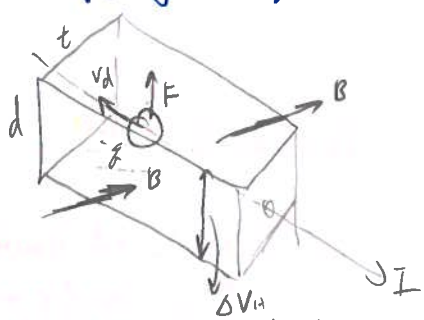
$$\frac{m}{q} = \frac{rB_0}{v} = \frac{rB_0B}{E}$$

from v. selector

∴ Can find q/m knowing B_0, B & E

Hall effect

- current flowing in $\underline{B}$ generates an $\underline{E}$ field E_H



ΔV_H Hall voltage

$$E_H = \frac{\Delta V_H}{d}$$

@ eq^m

$$qv_dB = qE_H$$

$$v_d = \frac{E_H}{B}$$

$$I = \underbrace{n}_{\text{no. of charge carriers}/m^3}\,\underbrace{q}_{\text{charge}}\,\underbrace{v_d}_{\text{drift vel.}}\,\underbrace{td}_{\text{cross-sectional S.A.}}$$

$$\Delta V_H = dE_H$$

$$= dv_dB$$

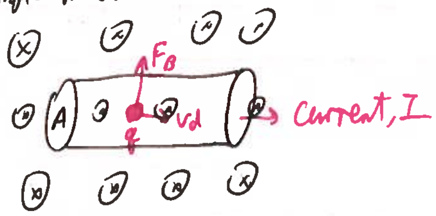
$$= \frac{IB}{nq\underbrace{t}_{\text{thickness}}} = \underbrace{R_H}_{\text{Hall coeff}}\frac{IB}{t}$$

$$\boxed{\Delta V_H = R_H\frac{IB}{t}}$$

$$R_H = \frac{1}{nq}$$

Magnetic force on a current-carrying conductor

Straight wire:



~~curr~~ on one charge:

$$\underline{F}_B = q\underline{v}_d \times \underline{B}$$

on whole segment

$$\underline{F}_B = \underbrace{\underbrace{n}_{\text{no. of charge carriers}}\, q\, \underbrace{AL}_{\text{volume}}}_{\text{total charge in the segment}} \underline{v}_d \times \underline{B}$$

$I = nqv_dA$

$$\underline{F}_B = I\underline{L} \times \underline{B}$$

if considering not straight

$$d\underline{F}_B = I\,d\underline{s} \times \underline{B}$$

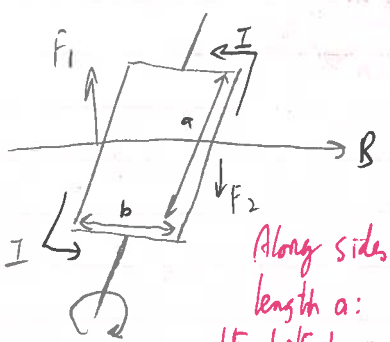
$$\underline{F}_B = I\int_a^b d\underline{s} \times \underline{B}$$

$$\underline{F}_B = I\left(\int_a^b d\underline{s}\right) \times \underline{B}$$

if closed loop: $\oint d\underline{s} = 0$ ↑ for uniform $\underline{B}$

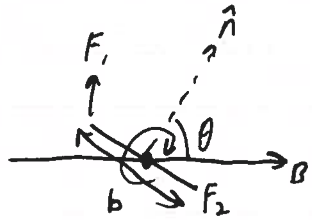
↳ $\underline{F}_B = 0$

Torque on Rectangular Current Loop



Along sides of length a:

$|\underline{F}_1| = |\underline{F}_2| = IaB$



Total torque:

$$\tau = F_1 \frac{b}{2}\sin\theta + F_2\frac{b}{2}\sin\theta$$

$$\tau = IabB\sin\theta = IAB\sin\theta$$

where $\underline{A} = A\hat{\underline{n}}$

$$\boxed{\underline{\tau} = I\underline{A} \times \underline{B}}$$

Magnetic Dipole

τ decreases θ

↳ U MΘ↑↑

As B4:

$$\tau = -\frac{dU}{d\theta} = -IAB\sin\theta$$

↳ ∠ between $\hat{\underline{n}}$ & $\underline{B}$

$$W = \int_{\theta_1}^{\theta_2} \tau\, d\theta'$$

↳ angle turned due to torque

$$= \int_{\theta_1}^{\theta_2} -\frac{dU}{d\theta}(-d\theta) \leftarrow$$

as $\theta' \uparrow$, $\theta \downarrow$

↳ small angle turned in θ', $\Delta\theta$ is same, but minus

↳ $d\theta' = -d\theta$

The integral of the energy gradient gives the total energy

$$\Rightarrow = \int_{\theta_1}^{\theta_2} IAB\sin\theta\, d\theta = IAB\int_{\theta_1}^{\theta_2}\sin d\theta$$

$$\Delta U = U_2 - U_1 = -IAB(\cos\theta_2 - \cos\theta_1)$$

Define $\theta_2 = \frac{\pi}{2}$, $U_2 = 0$

↳ $U = -IAB\cos\theta = -I\underline{A}\cdot\underline{B}$

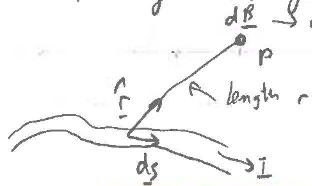
Defining magnetic dipole moment of current loop

$\underline{m} = I\underline{A}$ ∴ $\boxed{U = -\underline{m}\cdot\underline{B}}$

$$\underline{\tau} = \underline{m} \times \underline{B}$$

Biot-Savart law

↳ How currents generate magnetic fields



$\mu_0 = 4\pi\times10^{-7}$ $T\,m\,A^{-1}$ permeability of free space

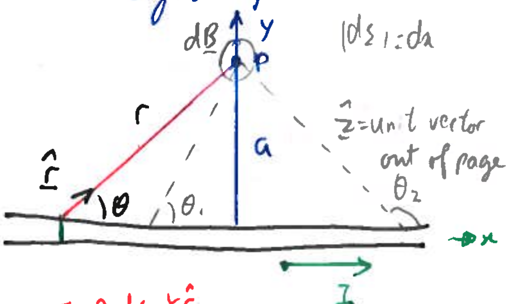
$$\underline{dB} = \frac{\mu_0 I}{4\pi}\frac{d\underline{s}\times\hat{r}}{r^2}$$

↳ B-field produced @ pt P due to current-length elements, $I d\underline{s}$

Total field @ P: due to whole wire

$$\underline{B} = \frac{\mu_0 I}{4\pi}\int\frac{d\underline{s}\times\hat{r}}{r^2}$$

Magnetic field due to a current in a long straight wire



using $\underline{B} = \frac{\mu_0 I}{4\pi}\int\frac{d\underline{s}\times\hat{r}}{r^2}$

$d\underline{s}\times\hat{r} = |d\underline{s}||\hat{r}|\sin\theta\,\hat{z} = dx\sin\theta\,\hat{z}$

$$\underline{dB} = \frac{\mu_0 I}{4\pi}\frac{dx\sin\theta}{r^2}\hat{z}$$

$dx \to d\theta$

$\sin\theta = \frac{a}{r} \to r = \frac{a}{\sin\theta}$

$\tan\theta = \frac{a}{-x} \to x = -\frac{a}{\tan\theta}$

(-ve) ∵ $d\underline{s}$ is at negative $\underline{x}$

↳ $\frac{dx}{d\theta} = \frac{a}{\sin^2\theta} \to dx = \frac{a\,d\theta}{\sin^2\theta}$

Element of B-field w/ new variables

$$\underline{dB} = \frac{\mu_0 I}{4\pi}\frac{a\,d\theta}{\sin^2\theta}\sin\theta\frac{\sin^2\theta}{a^2}$$

$$= \frac{\mu_0 I}{4\pi a}\sin\theta\,d\theta$$

doing the integration:

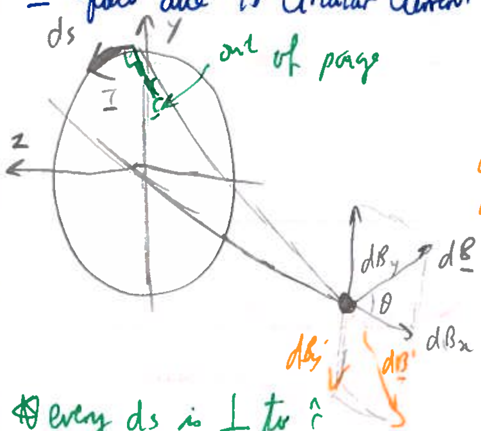
$$\underline{B} = \frac{\mu_0 I}{4\pi a}\int_{\theta_1}^{\theta_2}\sin\theta\,d\theta\,\hat{\underline{z}} = \frac{\mu_0 I}{4\pi a}(\cos\theta_1 - \cos\theta_2)\hat{\underline{z}}$$

if wire is v. long

$\theta_1 \to 0$, $\theta_2 \to \pi$

$$B_{v.\text{long wire}} = \frac{\mu_0 I}{2\pi a}$$

$\underline{B}$-field due to Circular Current Loop



$dB_x = dB\cos\theta$

$dB_y = dB\sin\theta$

every $d\underline{s}$ is ⊥ to $\hat{r}$

$|d\underline{s}\times\hat{r}| = |d\underline{s}||\hat{r}|\sin\frac{\pi}{2} = ds$

$r^2 = x^2 + R^2$

$$\Rightarrow dB = \frac{\mu_0 I}{4\pi}\frac{ds}{(x^2+R^2)}$$

$$B_x = \oint dB_x = \oint dB\cos\theta = \frac{\mu_0 I}{4\pi}\oint\frac{ds\cos\theta}{x^2+R^2} = \frac{\mu_0 I\overbrace{(\cos\theta)}^{\frac{R}{\sqrt{x^2+R^2}}}}{4\pi(x^2+R^2)}\underbrace{\oint ds}_{2\pi R}$$

$$B_x = \frac{\mu_0 I R^2}{2(x^2+R^2)^{\frac{3}{2}}}$$

Extreme cases:

1. $x = 0$

$$B_0 = \frac{\mu_0 I}{2R}$$

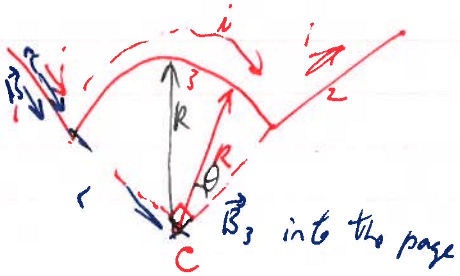
$x >> R$

$$B_x \approx \frac{\mu_0 I R^2}{2x^3}$$

From B4: $\underline{m} = I(\pi R^2)\hat{\underline{x}}$

$$\underline{B} = \frac{\mu_0}{4\pi}\frac{2m}{x^3}$$

Exercise:



Determine the direction of the B-field @ C.

Applying BS-law

$$d\underline{B} = \frac{\mu_0 I}{4\pi} \frac{d\underline{s} \times \hat{r}}{r^2}$$

$$d\underline{B} = \frac{\mu_0 I}{4\pi} \frac{ds \sin 0}{r^2} = 0$$

similarly $\underline{B}_1 = 0$

$\underline{B}_2 = 0$

$$\cancel{\underline{B}_3 = \frac{\mu_0 i}{4\pi} \frac{d\underline{s} \times \hat{r}}{r^2}}$$

$$\cancel{= \frac{\mu_0 i}{4\pi} \frac{R}{R^2}}$$

$$\cancel{= \frac{\mu_0 i}{4\pi R}}$$

$$\underline{B}_3 = \frac{\mu_0 i}{4\pi R} \frac{\pi}{2}$$

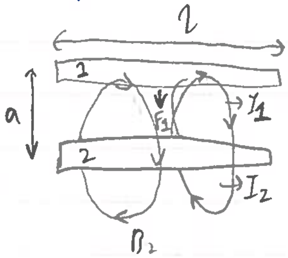
$d\underline{B} =$

$$\underline{B}_3 = \frac{\mu_0 i R}{4\pi R^2} \int_0^{\frac{\pi}{2}} d\phi$$

$$= \frac{\mu_0 i}{8R}$$

Exercise again

Magnetic force between 2 // conductors



wire 2 creates a field B_2

↳ exerts a force on wire 1:

$$F_1 = I_1 l \times B_2 = I_1 l B_2 \quad \because l \text{ is } \perp \text{ to } B_2$$

Assuming very long wires:

$$B_2 = \frac{\mu_0 I_2}{2\pi a}$$

// current => attract

$$F_1 = \frac{\mu_0 I_1 I_2}{2\pi a}$$

X // current => repel

Ampère's law:

Using Bio-Savart law:

$$dB = \frac{\mu_0 I}{4\pi}\frac{ds \times \hat{r}}{r^2}$$

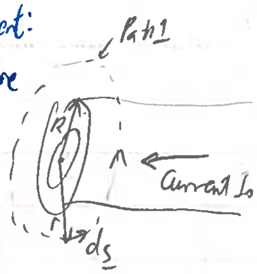
$\Downarrow$

Simpler way of applying Bio-Savart Law for cases w/ high symmetry

$$\oint B \cdot ds = \mu_0 I$$

↳ current passing through any closed path

B-field due to current: in a long straight wire



$$B(r) = \frac{\mu_0 I_0}{2\pi r}$$

using Ampere's Law:

L.H.S:

$$\oint B \cdot ds = B \oint ds = 2\pi r B$$

$B \parallel ds$

Path 1: $r > R$

$2\pi r B = \mu_0 I_0$

$$B = \frac{\mu_0 I_0}{2\pi r}$$

Part 2: $r < R$

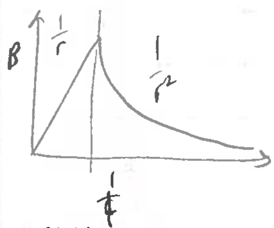
assuming uniform current density:

$$\frac{I}{I_0} = \frac{J\pi r^2}{J\pi R^2} = \frac{r^2}{R^2}$$

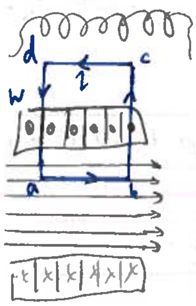
$$2\pi r B = \mu_0 I_0 \frac{r^2}{R^2}$$

$$B = \frac{\mu_0 I_0 r}{2\pi R^2}$$

$\Downarrow$



B-field due to a solenoid



$$\oint_{abcd} B \cdot ds = \mu_0 I_{enclosed}$$

$$\Rightarrow \oint_{ab} B\, ds + \cancel{\oint_{bc} B \cdot ds} + \cancel{\oint_{cd} B \cdot ds} + \cancel{\oint_{da} B \cdot da}$$

$$\Rightarrow \oint_{ab} B \cdot ds = Bl = \mu_0 I_{enclosed} = \mu_0 N I$$

↑ number of turns

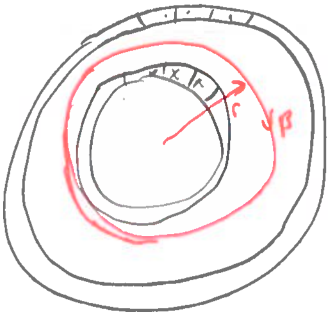
$$\boxed{B = \mu_0 n I}$$

$n = \frac{N}{l}$ } => no. of turns per unit length

Magnetic field due to a toroid

↳ donut shaped solenoid

↳ bent around so ends touch

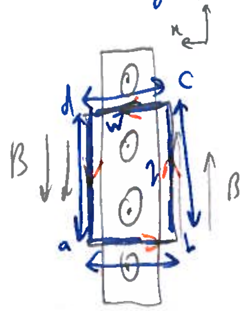


$$\oint \underline{B}\cdot d\underline{s} = B2\pi r = \mu_0 NI$$

$$\Rightarrow B = \frac{\mu_0 NI}{2\pi r} \quad \} \text{ inside toroid}$$

$B = 0$ outside toroid

B-field due to an infinite conducting sheet



$$\oint \underline{B}\cdot d\underline{s} = \oint_a^b \underline{B}\cdot d\underline{s} + \cancel{\oint_b^c \underline{B}\cdot d\underline{s}} + \cancel{\oint_c^d \underline{B}\cdot d\underline{s}} + \oint_d^a \underline{B}\cdot d\underline{s}$$

$\therefore$

$$\oint \underline{B}\cdot d\underline{s} = 2Bl = \mu_0 I_{encl} = \mu_0 J l$$

$$B = \frac{\mu_0 J}{2} \Leftrightarrow E = \frac{\sigma}{2\epsilon_0}$$

↳ independent of distance from the sheet, $\underline{B}$ is uniform

Magnetic Susceptibility → χ_m

$$B_m = \chi_m B_0$$

B_m: induced magnetic field in the material

B_0: External B-field

Total B-field: $\underline{B} = \underline{B_0} + \underline{B_m} = (1+\chi_m)B_0$

$= \mu_r$: relative permeability

$-1 \le \chi_m < 0$ diamagnetic — Ag, Au, Pb, Zn

$0 < \chi_m \le 1$ paramagnetic — Al, Cr, K, Mg

$1 \ll \chi_m$ ferromagnetic — eg. Fe, Ni, Co

Diamagnetism

$$\underline{B} = (1+\chi_m)B_0$$

$\Rightarrow \underline{B} < \underline{B_0} \Rightarrow$ density of field lines inside material is less than outside

OR some of the B-field is excluded from the material

* all materials show diamagnetism
  ↳ usually v. small [masked by other types of magnetism]

* see Meissner Effect in superconductors

Paramagnetic

atoms have individual permanent magnetic moments oriented randomly → B applied → atoms aligned w/ external field

$$\underline{B} = (1+\chi_m)B_0 > B_0$$
$$0 < \chi_m \le 1$$

$M = \frac{\mu}{V} \Rightarrow$ magnetisation = net magnetic moment per volume

$$\underline{M_{max}} = \frac{N\mu}{V}$$

N: no. of atoms; μ: magnetic moment of individual atom

Pierre Curie's Law:

$$\underline{M} = C\frac{B_0}{T}$$

C: Curie's const.

– only holds when $\frac{B}{T}$ is small

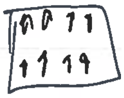
Ferromagnetism

↳ alignment of magnetic moment

→ Permanent magnetic moment due to aligned spins of unpaired e^-s

destroyed by heat / impact

Hard ferromagnetism (permanent magnet)



remove field → Hard ferromagnetism (permanent magnet)

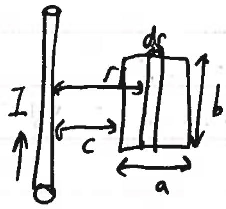
↘ soft ferromagnetism

Large external field

Curie Temp: T_c → all alignment breaks down ↳ no domains

Magnetic flux

$$\Phi_B = \int_S \underline{B} \cdot d\underline{S}$$



from B4: $B = \frac{\mu_0 I}{2\pi r}$

$$\Phi_B = \int_S |B|\,|dS|$$

$$= \int_S \frac{\mu_0 I}{2\pi r} \underbrace{dS}_{b\,dr}$$

$$= \int_S \frac{\mu_0 I}{2\pi r} b\,dr$$

$$= \int_c^{c+a} \frac{\mu_0 I b}{2\pi r} dr \Rightarrow \Phi_B = \frac{\mu_0 I b}{2\pi} \ln\left[1 + \frac{a}{c}\right]$$

$\underline{H}$, Magnetic field strength ['magnetic intensity']

From B4

$$\underline{D} = \epsilon_0 \underline{E} + \underline{P}$$

analogous

$$\underline{H} = \frac{1}{\mu_0}\underline{B} - \underline{M}$$

$\underline{B}$: total B-field inside the material (T)

$\underline{M}$: magnetisation of the material (Am^{-1})

$\underline{H}$: $A \cdot m^{-1}$

$$\underline{M} = \frac{1}{\mu_0}\underline{B_m} = \frac{1}{\mu_0}\chi_m \underline{B_0}$$

From B4: $\underline{B} = (1+\chi_m)\underline{B_0}$

$$\underline{H} = \frac{(1+\chi_m)\underline{B_0}}{\mu_0} - \frac{\chi_m \underline{B_0}}{\mu_0} = \frac{\underline{B_0}}{\mu_0}$$

$$= \frac{\underline{B}}{(1+\chi_m)\mu_0}$$

$$= \frac{\underline{B}}{\mu_r \mu_0}$$

$= \cancel{\underline{H}}\ \underline{B_0}$

Constitutive relations:

$$\underline{H} = (\mu_0 \mu_r)^{-1} \underline{B}$$

$$\underline{D} = \epsilon_0 \epsilon_r \underline{E}$$

Gauss's Law for Magnetism:

for electrostatics:

$$\underline{\nabla} \cdot \underline{E} = \frac{\rho}{\epsilon_0} \qquad \int_S \underline{E} \cdot d\underline{S} = \frac{Q}{\epsilon_0}$$

☆ Aim is to calculate $\underline{\nabla} \cdot \underline{B}$

using Biot-Savart law:

$$d\underline{B} = \frac{\mu_0 I}{4\pi} \frac{d\underline{S} \times \hat{\underline{r}}}{r^2}$$

using $\underline{\nabla} \cdot (\underline{P} \times \underline{Q}) = \underline{Q} \cdot (\underline{\nabla} \times \underline{P}) - \underline{P} \cdot (\underline{\nabla} \times \underline{Q})$

$\underline{P} = d\underline{S}$; $\underline{Q} = \hat{\underline{r}}$

$$\underline{\nabla} \cdot d\underline{B} = \frac{\mu_0 I}{4\pi r^2} \underline{\nabla} \cdot (d\underline{S} \times \hat{\underline{r}}) = \frac{\mu_0 I}{4\pi}\left[\underbrace{\hat{\underline{r}} \cdot (\underline{\nabla} \times d\underline{S})}_{=0} - d\underline{S} \cdot \left(\underline{\nabla} \times \frac{\hat{\underline{r}}}{r^2}\right)\right]$$

$$\frac{\hat{\underline{r}}}{r^2} = -\underline{\nabla}\left(\frac{1}{r}\right)$$

$$d\underline{B} = \frac{\mu_0 I}{4\pi}\left[d\underline{S} \cdot \underbrace{\left(\underline{\nabla} \times \underline{\nabla}\left(\frac{1}{r}\right)\right)}_{=0}\right]$$

↳ $\underline{\nabla} \cdot \underline{B} = 0$

Generally:

$$\int_S \underline{A}\cdot d\underline{S} = \int_V \nabla\cdot\underline{A}\,dV$$

integral of vector over closed surface

integral of divergence of vector over enclosed volume

$$\Rightarrow \quad \Phi_B = \int_S \underline{B}\cdot d\underline{S} = \int_V \nabla\cdot\underline{B}\,dV = 0$$

→ Same number of B-field lines leave the volume as entering it

→ ∄ No 'magnetic charge'

or

No 'magnetic monopoles'

→ all B-field are continuous

For B-fields, no net flux, no sources or sinks

Stoke's Theorem / Rotation Theorem

$$\int \underline{B}\cdot d\underline{s} = \iint \nabla\times\underline{B}\cdot d\underline{S}$$

integral around a closed path, s, of a vector quantity

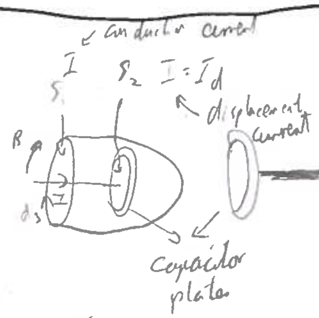
integral of the curl of that quantity, integrated over the surface S enclosed by the path

See MM3

Ampère-Maxwell Law

using Ampère's Law:

$$\int \underline{B}\cdot d\underline{s} = \mu_0 I$$



[no current between plates]

for S_1: R.H.S $= \mu_0 I$

S_2: R.H.S $= 0$

In the presence of time-varying E-field

→ Ampere's Law is incomplete

→ postulating a 'displacement current'

$$I_d = \epsilon_0 \frac{d\phi_E}{dt}$$ ↘ Electric flux

Ampère-Maxwell Law:

$$\oint \underline{B}\cdot d\underline{s} = \mu_0 I + \mu_0\epsilon_0 \frac{d\phi_E}{dt}$$

for the capacitor

$$\phi_E = EA = \left(\frac{Q}{\epsilon_0 A}\right)A = \frac{Q}{\epsilon_0}$$

$$I_d = \epsilon_0 \frac{d\phi_E}{dt} = \frac{dQ}{dt}$$

Magnetic fields can be produced

both by conduction currents
& by time-varying electric fields

$$\oint \underline{B}\cdot d\underline{s} = \mu_0 I + \mu_0 \epsilon_0 \frac{d\phi_E}{dt} \rightarrow \frac{dQ}{dt}$$

$$\oint \underline{B}\cdot d\underline{s} = \mu_0 \iint \underline{J}\cdot d\underline{S} + \mu_0 \epsilon_0 \frac{d}{dt}\iint \underline{E}\cdot d\underline{S}$$

$$= \iint \nabla\times\underline{B}\cdot d\underline{S}$$

stoke's therem

Differential form:

$$\nabla\times\underline{B} = \mu_0 \underline{J} + \mu_0\epsilon_0 \frac{d\underline{E}}{dt}$$

Faraday's Law

↳ a changing magnetic flux through a circuit induces a transient current

- when the magnetic flux through a circuit is changing
  ↳ EMF is induced
- magnitude of the EMF ∝ rate of change of flux

$$\mathcal{E} = -\frac{d\phi_B}{dt}$$

with $\phi_B = \int_S \underline{B}\cdot d\underline{a}$

↳ Magnetic flux through the circuit in which the EMF, $\mathcal{E}$ is induced

Lenz's Law

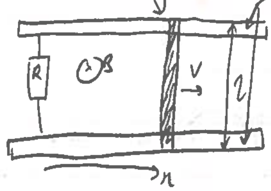
↳ direction of the EMF / induced current

* An induced current has a direction such that the B-field due to the current opposes the change in the magnetic flux that induces the current

---

Some examples:

1. Conducting bar on conducting rails



EMF in bar:

$$\Phi_B = \int_S \underline{B}\cdot d\underline{S} = Blx$$

$$\mathcal{E} = -\frac{d\phi_B}{dt}$$

$$= -Blv$$

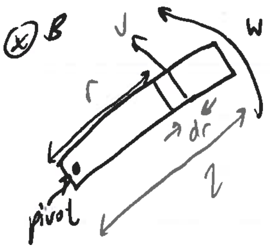
current:

$$I = \frac{|\mathcal{E}|}{R} = \frac{Blv}{R}$$

* circuit wants to reduce this extra magnetic flux going through it
  ∴ magnetic flux induced by the current will oppose the magnetic flux
  ∴ flux lines produced by the current will be ~~out~~ coming out of the page

→ anticlockwise for current
Force to the left

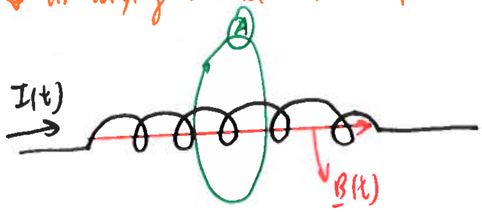
Rotating bar



$$\text{EMF} = -Bv\,dr$$

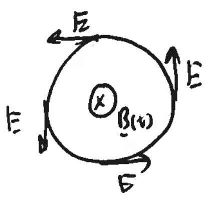
$$\varepsilon = -B\omega \int_0^l r\,dr = -\tfrac{1}{2}B\omega l^2$$

★ Time varying B-field → EMF



time varying current → time varying B inside → time varying ϕ_B inside → induces E-field → EMF

Applying Stoke's Theorem:



$$\varepsilon = \oint \underline{E}\cdot d\underline{s} = \iint \nabla\times\underline{E}\cdot d\underline{S}$$

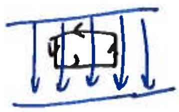
With:

$$\varepsilon = -\frac{d\phi_B}{dt} \qquad \Phi_B = \int_S \underline{B}\cdot d\underline{S}$$

$$\iint \nabla\times\underline{E}\cdot d\underline{S} = -\frac{d}{dt}\iint \underline{B}\cdot d\underline{S}$$

$$\Rightarrow \nabla\times E = -\frac{dB}{dt}$$

E-field from static charges:



$$\oint \underline{E}\cdot d\underline{s} = 0$$

E-field is conservative

E-field from magnetic induction

$$\oint \underline{E}\cdot d\underline{s} = \varepsilon \neq 0$$

Work is done by the E-field in moving a charge around a closed loop

→ E-field not conservative

no cancelling effect

$\nabla\times E \neq 0$

Self-Inductance / Back EMF

time varying current

↳ time varying B-field

↳ time varying ϕ_B

↳ EMF ε_L induced

[opposes source EMF]

$$\varepsilon_L = -\frac{d\phi_B}{dt}$$

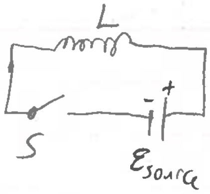
change of rate of flux is caused by rate of change of current

$$\hookrightarrow \frac{d\phi_B}{dt} \propto \frac{dB}{dt} \propto \frac{dI}{dt}$$

where I is the source current

$$\varepsilon_L = -\underbrace{L}_{\text{Inductance}}\frac{dI}{dt}$$

The inductor ~~of~~ in a circuit: ideal solenoid



B-field in solenoid:

$$\cancel{B = \mu_0 I} \quad B = \frac{\mu_0 N I}{l}$$

Through solenoid:

$$\phi_{B,1\,turn} = BA = \frac{\mu_0 N A \overset{\text{area}}{I}}{l}$$

$$\underbrace{\phi_{B,N\,turns}}_{\text{total}} = \frac{\mu_0 N^2 A I}{l} = N\phi_{B,1\,turn}$$

$$\mathcal{E}_L = -\frac{d\phi_B}{dt} = -\underbrace{\frac{\mu_0 N^2 A}{l}}_{L}\frac{dI}{dt}$$

$L \Rightarrow$ depends on geometry of the solenoid

$$\boxed{L = \frac{\mu_0 N^2 A}{l}}$$

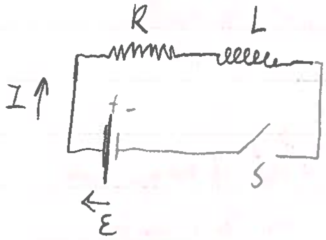
(H) or

1H = 1VsA^{-1} (henry)

RL circuits

resistor & inductor in series

→ the inductor opposes changes in the current through that circuit



using Kirchoff's rules:

$$\mathcal{E} - IR - L\frac{dI}{dt} = 0$$

$$\frac{L}{R}\frac{dI}{dt} = \frac{\mathcal{E}}{R} - I$$

$$\frac{R}{L}dt = \frac{dI}{\frac{\mathcal{E}}{R} - I} \Rightarrow \int dI \frac{1}{\frac{\mathcal{E}}{R} - I} = \int \frac{R}{L} dt$$

Boundary condition: $I_{t=0} = 0$

$$-\ln\left[\frac{\mathcal{E}}{R} - I(t)\right] + \ln\left[\frac{\mathcal{E}}{R}\right] = \frac{R}{L}t$$

$$\boxed{I(t) = \frac{\mathcal{E}}{R}\left(1 - e^{-\frac{t}{\tau}}\right) = \frac{\mathcal{E}}{R}\left(1 - e^{-\frac{R}{L}t}\right)}$$

where $\tau = \frac{L}{R}$

↳ time constant

Energy stored in magnetic field

from B4

$\varepsilon - IR = L\frac{dI}{dt}$

$\downarrow \times I$

$I\varepsilon = \underbrace{I^2R}_{\text{Rate of energy dissipated by resistor}} + \underbrace{IL\frac{dI}{dt}}_{\text{rate of energy stored in the inductor}}$

$I\varepsilon$: Rate of energy supplied by battery

$\hookrightarrow \frac{dU}{dt} = IL\frac{dI}{dt}$

$U = \int_0^U dU = L\int_0^I I\,dI$

$\Rightarrow \boxed{U = \frac{1}{2}I^2L}$ || $U = \frac{1}{2}\frac{Q^2}{C}$ comparing with $\downarrow$ for capacitor

for an ideal solenoid:

$L = \frac{\mu_0 N^2 A}{l}$ $V = Al$

$B = \frac{\mu_0 NI}{l}$

for the energy density of the field:

$u = \frac{U}{V} = \frac{\frac{1}{2}I^2\frac{\mu_0 N^2 A}{l}}{Al} = \frac{1}{2}\frac{\mu_0 N^2}{l^2}I^2$

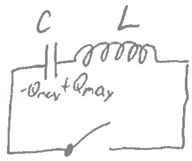
$B = \frac{\mu_0 NI}{l}$

$\frac{B^2}{\mu_0} = \frac{\mu_0 N^2 I^2}{l^2}$

$u = \frac{B^2}{2\mu_0}$ || $u = \frac{\varepsilon_0 E^2}{2}$ comparing w/ $\downarrow$ for capacitor

Oscillations in an LC circuit

inductor and capacitor in series



at $t>0 \Rightarrow$ switch is closed

E stored in capacitor $U_E = \frac{1}{2}\frac{Q^2}{C}$

E stored in inductor $U_B = \frac{1}{2}LI^2$

Total energy: $U = U_E + U_B$

$\frac{dU}{dt} = 0$ $\because$ no energy dissipation

$\frac{dU}{dt} = \frac{1}{2}\frac{d}{dt}\left(\frac{Q^2}{C} + LI^2\right)$

$= \frac{Q}{C}\underbrace{\frac{dQ}{dt}}_{I} + LI\underbrace{\frac{dI}{dt}}_{\frac{d^2Q}{dt^2}} = 0$

$\frac{Q}{C}\frac{dQ}{dt} + L\frac{dQ}{dt}\frac{d^2Q}{dt^2} = 0$

$\frac{dQ}{dt}\left(\frac{Q}{C} + L\frac{d^2Q}{dt^2}\right) = 0$

$\frac{dQ}{dt} = 0$

or

$\left.\frac{d^2Q}{dt^2} = -\frac{Q}{LC}\right] \rightarrow$ same as S.H.M

defining $\boxed{w = \frac{1}{\sqrt{LC}}}$

$\frac{d^2Q}{dt^2} = -w^2Q$

$\boxed{Q = A\cos(wt + \phi)}$

$Q = Q_{max}$ at $t = 0$

$Q(t) = Q_{max}\cos(wt)$ $I_{max} = -wQ_{max}$

$I(t) = -wQ_{max}\sin(wt) = I_{max}\sin(wt)$

See graph in notes

For the total energy of the circuit:

$$U(t) = U_E(t) + U_B(t)$$

$$= \frac{1}{2}\frac{Q^2}{C} + \frac{1}{2}LI^2$$

$$= \frac{Q^2_{max}}{2C}[\cos(\omega t)]^2 + \frac{LI^2_{max}}{2}[\sin(\omega t)]^2$$

using $I_{max} = -\omega Q_{max} = -\frac{Q_{max}}{\sqrt{LC}}$

$$U(t) = \frac{Q^2_{max}}{2C}[\cos^2\omega t + \sin^2\omega t]$$

$$= \frac{Q^2_{max}}{2C}$$

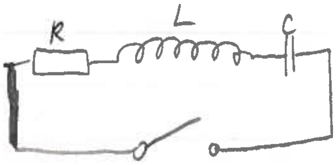
~~接去C, L 之~~

C stores $\xrightarrow{t_0}$ L stores

∞ loop ideally

$\xleftarrow{t_0}$

Damped oscillations in RLC circuit



damped ∵ of resistor

↳ energy lost/dissipated

↳ oscillations damped [reduce in amplitude]

Rate of Energy Loss in resistor : $\frac{dU}{dt} = -I^2R$

$$\frac{Q}{C}\frac{dQ}{dt} + LI\frac{dI}{dt} = -I^2R$$

$$LI\frac{dI}{dt} + I^2R + \frac{Q}{C}\frac{dQ}{dt} = 0$$

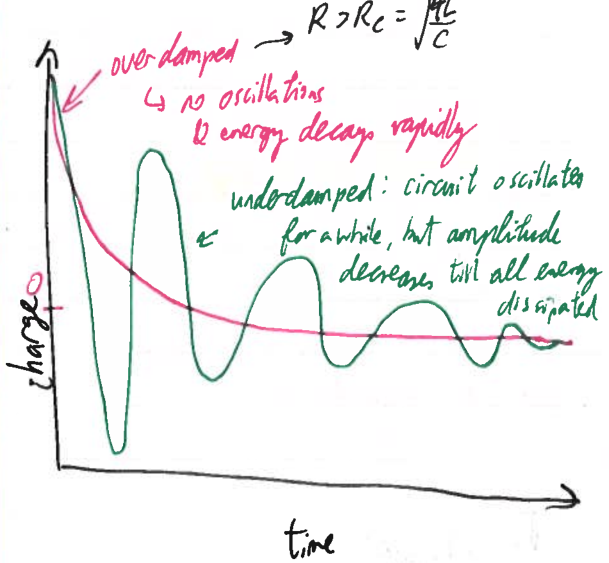
$$L\frac{dQ}{dt}\left(\frac{d^2Q}{dt^2}\right) + \left(\frac{dQ}{dt}\right)^2R + \frac{Q}{C}\frac{dQ}{dt} = 0$$

$$\Rightarrow L\frac{d^2Q}{dt^2} + \frac{dQ}{dt}R + \frac{Q}{C} = 0$$

Solution for the 2nd ODE:

$$Q(t) = Q_{max}e^{-\frac{Rt}{2L}}\cos(\omega_d t)$$

$$\omega_d = \sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}$$



Alternating current

external EMF:

$\varepsilon = \varepsilon_{max} \sin \omega t$

↳ 230V

$f = 50Hz$

$T = 0.02s$

EMF delivers an AC current of:

$I = I_{max} \sin(\omega t - \phi)$ → phase angle

- V_d of e/s = 4×10^{-5}
- reversing direction every 0.01s for current
- e/s move only about 4×10^{-7}m in a half cycle [100 atomic spacings]

But still useful although not travelling v. far.

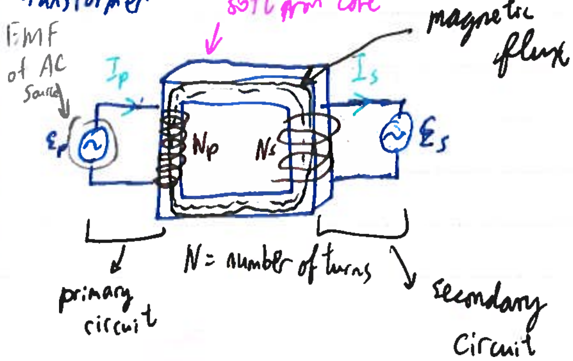
→ as long there is movement, the component the e/ is travelling through will only know e/s going past, not how far they will be travelling

↓ large current can still flow across a conductor if there are enough e/s in motion

↳ energy can still be dissipated through resistance

AC currents:
- easy to generate
- easy to transform
- easy to use

Transformer



From Faraday's law:

$$\varepsilon_p = -N_p \frac{d\phi_B}{dt}$$

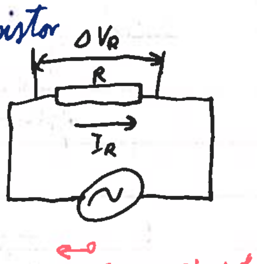
$$\varepsilon_s = -N_s \frac{d\phi_B}{dt}$$

$$\rightarrow \frac{\varepsilon_s}{\varepsilon_p} = \frac{N_s}{N_p}$$

if $N_s > N_p \Rightarrow \varepsilon_s > \varepsilon_p$

$\Rightarrow$ step up transformer

AC circuit w/ resistor



using Kirchoff's loop rule:

$\varepsilon = \Delta V_R$

$\hookrightarrow \Delta V_R = \varepsilon_{max} \sin \omega t$

$= \Delta V_{max} \sin \omega t$

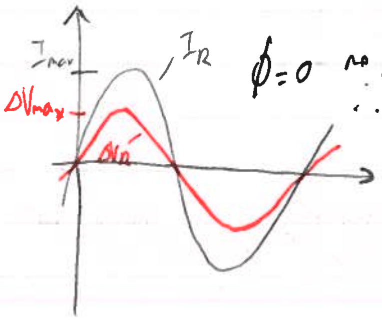
for the instantaneous current:

$V = IR \Rightarrow I = \dfrac{\Delta V_R}{R}$

$= \dfrac{\Delta V_{max}}{R} \sin \omega t$

$= I_{max} \sin \omega t$

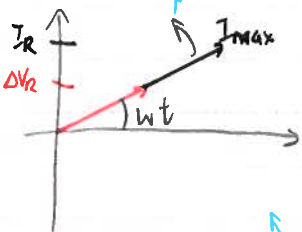
Phase diagrams



$\phi = 0$ no phase shift

$\therefore$ in phase w/ each other

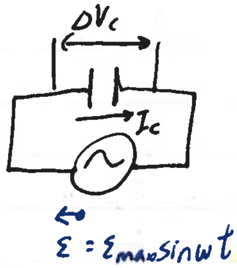
phasors rotate at ω



phasor diagram

$\Rightarrow$ representing sinusoidal variations

AC w/ capacitor



using Kirchoff's loop rule:

$\varepsilon = \Delta V_c$

$= \Delta V_{max} \sin \omega t$

instantaneous charge:

$Q = C \Delta V_c$

$Q = C \Delta V_{max} \sin \omega t$

$\Downarrow \dfrac{dQ}{dt} = I$

$I = \underbrace{\omega C \Delta V_{max}}_{= I_{max}} \cos \omega t$

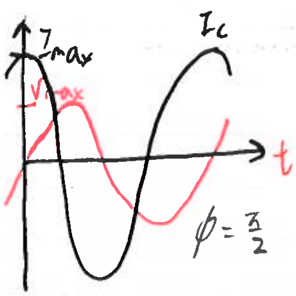
$= I_{max} \sin(\omega t + \frac{\pi}{2})$

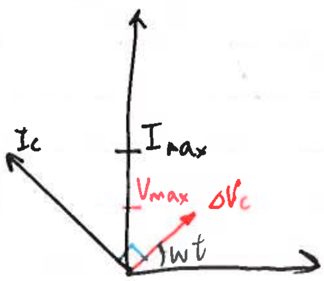
$I_{max} = \omega C \Delta V_{max}$

$= \dfrac{\Delta V_{max}}{X_c}$

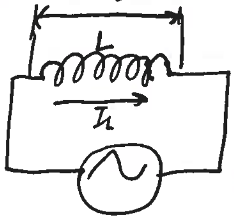
$|X_c| = \dfrac{1}{\omega C}$ $\Rightarrow$ capacitance reactance (Ω)

Phase diagrams:





AC w/ Inductor



Kirchoff's loop rule:

$\mathcal{E} = \Delta V_L$

$= \Delta V_{max} \sin \omega t$

$$\Delta V_L = L \frac{dI_L}{dt} = \Delta V_{max} \sin \omega t$$

$$\int dI_L = \int \frac{\Delta V_{max}}{L} \sin \omega t \, dt$$

Current $\Rightarrow$ $I_L = \frac{\Delta V_{max}}{L} \int \sin \omega t \, dt$

$$= -\frac{\Delta V_{max}}{\omega L} \cos \omega t$$

$$I_L = I_{max} \sin\left(\omega t - \frac{\pi}{2}\right)$$

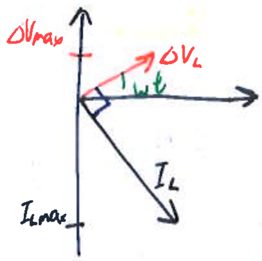
↳ current lags behind the potential

$$I_{max} = \frac{\Delta V_{max}}{X_L}$$

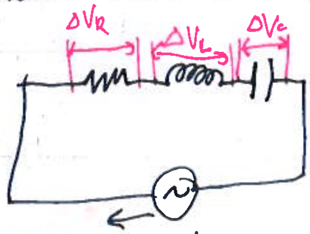
$$X_L = \omega L$$

↳ inductive reactance

$\phi = -\frac{\pi}{2}$



RLC circuit



$\therefore$ Same current throughout

$\Rightarrow I = I_{max} \cos(\omega t - \phi)$

$$e^{i\omega t} = \cos \omega t + i \sin \omega t$$

Source EMF: $\mathcal{E} = \mathcal{E}_{max} e^{i\omega t}$

$\mathcal{E} = \mathbb{R}(\mathcal{E})$

Current in the circuit

$I(t) = I_0 e^{i\omega t}$ $\Rightarrow$ if I_0 has an imaginary part, current & voltage will not be in phase ↳ why?

$I(t) = \mathbb{R}(I_0 e^{i\omega t})$

Using Kirchoff's Loop rule:

$$\frac{d}{dt}\, \mathcal{E} - RI - L\frac{dI}{dt} - \frac{Q}{C} = \frac{d}{dt}\, 0$$

$$\Rightarrow \frac{d\mathcal{E}}{dt} = R\frac{dI}{dt} - L\frac{d^2 I}{dt} + \frac{I}{C}$$

showing in complex form

$$\Rightarrow i\omega \mathcal{E}_{max} e^{i\omega t} = i\omega R I_0 e^{i\omega t} - \omega^2 L I_0 e^{i\omega t} + \frac{I_0}{C} e^{i\omega t}$$

$$i\omega \mathcal{E}_{max} e^{i\omega t} = i\omega R I_0 e^{i\omega t} + (i\omega)^2 L I_0 e^{i\omega t} + \frac{I_0}{i\omega C} e^{i\omega t}$$

$$\mathcal{E}_{max} e^{i\omega t} = I_0 e^{i\omega t}\left[R + i\omega L + \frac{1 \times i}{i\omega C \times i}\right]$$

$$= I_0 e^{i\omega t}\left[R + i\left(\omega L - \frac{1}{\omega C}\right)\right]$$

$$\Rightarrow \underbrace{\mathcal{E}_{max} e^{i\omega t}}_{\mathcal{E}} = \underbrace{I_0 e^{i\omega t}}_{I} \underbrace{\left[R + i\left(\omega L - \frac{1}{\omega C}\right)\right]}_{Z}$$

Resistance, inductive reactance, capacitive reactance

$$Z = R + i\left(\omega L - \frac{1}{\omega C}\right) = R + i(X_L - X_C)$$

↳ complex impedance of the circuit

↳ all things opposing the current in the circuit

Z depends on angular frequency ω.
→ faster current is switching
↳ $Z \uparrow\uparrow$

in complex form:

$$Z = Z e^{i\phi} = Z(\cos\phi + i\sin\phi)$$

$$|Z| = \sqrt{R^2 + (X_L - X_C)^2} \quad \text{(magnitude)}$$

phase angle ϕ:

$X_L - X_C = Z\sin\theta$

$R = Z\cos\theta$

$$\phi = \arctan\frac{X_L - X_C}{R}$$

for the current:

$$I = \frac{\mathcal{E}}{Z} = \frac{\mathcal{E}_{max} e^{i\omega t}}{Z e^{i\phi}} = \frac{\mathcal{E}_{max}}{Z} e^{i(\omega t - \phi)}$$

if $X_L > X_C \Rightarrow \phi > 0$ Voltage peaks then Current peaks ⇐ current lags

if $X_L < X_C \Rightarrow \phi < 0$ Current peaks then voltage peaks

⇒ voltage lags

@ frequency ω_0

$X_L = X_C \Rightarrow \phi = 0$ also impedance = resistance ($Z = R$)

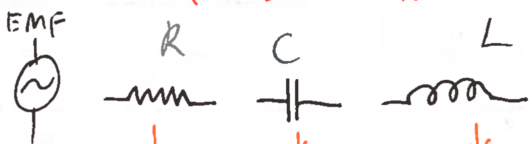
$\omega = \omega_0$

$$\omega_0 L = \frac{1}{\omega_0 C}$$

$$\omega_0 = \frac{1}{\sqrt{LC}} \Rightarrow \text{Resonant frequency}$$

for current: $I_{max} = \dfrac{\Delta V_{max}}{Z} = \dfrac{\Delta V_{max}}{R}$

Power in the RLC series circuit



EMF → energy source

R → dissipate energy through heat

C → energy stored in E-field

L → energy stored in B-field

∴ Energy is only dissipated through heat via resistor

Average energy stored is constant

$$P = I\Delta V_R = I_{max}\cos(\omega t - \phi)\,\Delta V_{max}\cos\omega t$$

$$= \Delta V_{max} I_{max}[(\cos\omega t\cos\phi + \sin\omega t\sin\phi)\cos\omega t]$$

$$= \Delta V_{max} I_{max}[\underbrace{\cos^2\omega t}_{=\frac{1}{2}}\cos\phi + \underbrace{\sin\omega t\cos\omega t}_{=0}\sin\phi]$$

Time average: average in time

$$\langle P \rangle_t = \frac{1}{2} I_{max}\Delta V_{max}\cos\phi$$

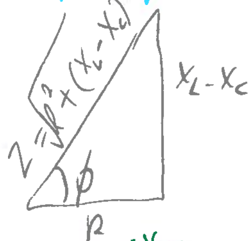
Average power in terms of rms current

$$I_{rms} = \sqrt{\langle I^2\rangle_t} = \sqrt{\langle I^2_{max}\sin^2(\omega t-\phi)\rangle_t}$$

$$I_{rms} = \frac{I_{max}}{\sqrt{2}} \quad ; \quad \Delta V_{rms} = \frac{V_{max}}{\sqrt{2}}$$

→ writing average power $\langle P\rangle_t$ in terms of I_{rms} & V_{rms}

$$\langle P\rangle_t = \underbrace{I_{rms}\Delta V_{rms}\cos\phi}_{\text{power factor}}$$



$$\cos\phi = \frac{R}{Z} = \frac{I_{max}R}{\Delta V_{max}} = \frac{I_{rms}\cancel{\sqrt{2}}R}{\cancel{\sqrt{2}}V_{rms}}$$

$$I_{max} = \frac{\Delta V_{max}}{Z}$$

$$\tan\phi = \frac{X_L - X_C}{R}$$

$$\langle P\rangle_t = I^2_{rms}R$$

→ in an ideal RLC circuit, power is only dissipated by the resistor

Resonance in the RLC circuit

resonance occurs when $I = I_{max}$

$$I_{rms} = \frac{\Delta V_{rms}}{Z} = \frac{\Delta V_{rms}}{\sqrt{R^2+(\underbrace{X_L-X_C}_{=0})^2}}$$

∴ resonance occurs when $X_L = X_C$, $Z = Z_{min} = R$

$\omega_0 = \frac{1}{\sqrt{LC}}$ ← happens when

$$\langle P\rangle_t = I^2_{rms}R = \frac{\Delta V^2_{rms}R}{Z^2} = \frac{(\Delta V_{rms})^2R\omega^2}{R^2\omega^2+L^2(\omega^2-\omega_0^2)^2}$$

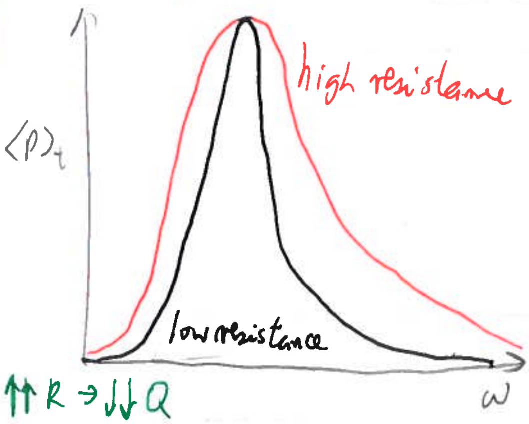
when $\omega = \omega_0$

$$\langle P\rangle_t = \frac{(\Delta V_{rms})^2}{R}$$ ⇒ maximum value for average power delivered

sharpness of curve $\langle P(\omega)\rangle_t$

↳ described by quality factor Q

$$Q = \frac{\omega_0}{\Delta\omega}$$



↑↑ R → ↓↓ Q

↳ can reach resonance w/ a broader range of frequencies

Maxwell's equation:

	differential form	integral form	
Gauss' Law in Electrostatic	$\nabla \cdot \underline{E} = \frac{\rho}{\varepsilon_0}$	$\int_S \underline{E} \cdot d\underline{S} = \frac{Q}{\varepsilon_0}$	
Gauss' Law in magnetism	$\nabla \cdot \underline{B} = 0$	$\int_S \underline{B} \cdot d\underline{S} = 0$	
	$\nabla \times \underline{E} = -\frac{\partial B}{\partial t}$	$\oint \underline{E} \cdot d\underline{s} = -\frac{d\Phi_B}{dt}$	Faraday's Law of Induction
	$\nabla \times \underline{B} = \mu_0 \underline{J} + \varepsilon_0 \mu_0 \frac{\partial E}{\partial t}$	$\oint \underline{B} \cdot d\underline{s} = \mu_0 I + \varepsilon_0 \mu_0 \frac{d\Phi_E}{dt}$	Ampère-Maxwell Law

Waves in free space

- no charges
- no currents
- no polarization charges
- no magnetisation

$\nabla \cdot \underline{E} = 0 \qquad \nabla \times \underline{E} = -\frac{\delta B}{\delta t}$

$\nabla \cdot \underline{B} = 0 \qquad \nabla \times \underline{B} = \varepsilon_0 \mu_0 \frac{\delta E}{\delta t}$

$$\nabla \times (\nabla \times \underline{E}) = -\frac{\delta(\nabla \times \underline{B})}{\delta t} = -\varepsilon_0 \mu_0 \frac{\delta^2 \underline{E}}{\delta t^2}$$

[using identity:

$$\nabla \times (\nabla \times \underline{E}) = \nabla(\underbrace{\nabla \cdot \underline{E}}_{=0}) - (\nabla \cdot \nabla)\underline{E}$$

$$\nabla \times (\nabla \times \underline{E}) = -\nabla^2 \underline{E}$$]

$$\therefore \nabla^2 \underline{E} = \varepsilon_0 \mu_0 \frac{\delta^2 \underline{E}}{\delta t^2}$$

similarly

$$\nabla^2 \underline{B} = \varepsilon_0 \mu_0 \frac{\delta^2 \underline{B}}{\delta t^2}$$

— 3D wave eq^n

In general, the 3D wave eq^n is:

$$\boxed{\nabla^2 \underline{A} = \frac{1}{v^2} \frac{\delta^2 \underline{A}}{\delta t^2}}$$

which has plane wave sol^{n}s

$$\underline{E} = \underline{E}_0 e^{i(\underline{k} \cdot \underline{r} - \omega t)}$$

travelling @ phase velocity:

$$\boxed{v = c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}}$$