Extra Note: (Summer 2020)

using Wolfram, from deriving Maxwell Boltzmann Distribution

Gaussian Integrals

$$I_n(a) = \int_{-\infty}^{\infty} e^{-ax^2} x^n \, dx$$

- $n=0 \Rightarrow I_0(a) = \sqrt{\frac{\pi}{a}}$
- $n>0$ (n is even) $\Rightarrow I_n(a) = \frac{(n-1)!!}{2^{n/2} a^{n/2}} \sqrt{\frac{\pi}{a}}$
- $n>0$ (n is odd) $\Rightarrow I_n(a) = \frac{\left(\frac{1}{2}[n-1]\right)!}{2a^{(n+1)/2}}$

Setting: $x = a^{-\frac{1}{2}} y$, $dx = a^{-\frac{1}{2}} dy$, $x^2 = \frac{y^2}{a}$

$$I_n(a) = \int_{-\infty}^{\infty} e^{-a\frac{y^2}{a}} (y a^{-\frac{1}{2}})^n a^{-\frac{1}{2}} dy$$

$$I_n(a) = a^{-\frac{n+1}{2}} \int_{-\infty}^{\infty} e^{-y^2} y^n dy = 2a^{-\frac{n+1}{2}} \int_0^{\infty} e^{-y^2} y^n dy$$

(by symmetry)

Solving $n=0$:

$$I_n(a) = a^{-\frac{n+1}{2}} \int_{-\infty}^{\infty} e^{-y^2} y^n dy$$

$$I_0(a) = \frac{1}{\sqrt{a}} \underbrace{\int_{-\infty}^{\infty} e^{-y^2} dy}_{I}$$

$$I^2 = \int_{-\infty}^{\infty} e^{-y^2} dy \int_{-\infty}^{\infty} e^{-x^2} dx$$

$$I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)} dx\,dy$$

$x = \rho\cos\phi$

$y = \rho\sin\phi$

$dA = dx\,dy$

$= \rho\, d\rho\, d\phi$

$\rho \in [0,\infty]$

$\phi \in [0, 2\pi]$

$$= \int_0^{2\pi}\int_0^{\infty} \rho e^{-\rho^2} d\rho\, d\phi$$

$$= 2\pi\left[-\frac{e^{-\rho^2}}{2}\right]_0^{\infty}$$

$$= \pi\left[-\underbrace{e^{-\infty^2}}_{0} + \underbrace{e^{0}}_{1}\right]$$

$$I^2 = \pi$$

$$I = \sqrt{\pi}$$

$$I_0(a) = \sqrt{\frac{\pi}{a}}$$

Solving $n=1$

$$I_1(a) = \frac{1}{a}\int_{-\infty}^{\infty} y e^{-y^2} dy$$

$$I_1(a) = \frac{1}{a}\left[\frac{e^{-y^2}}{-2}\right]_{-\infty}^{\infty}$$

$$= \frac{\cancel{2}}{a}\left[\frac{-e^{-y^2}}{\cancel{2}}\right]_0^{\infty}$$

$$= \frac{1}{a}\left[\underbrace{e^{-\infty^2}}_{0} + \underbrace{e^{0}}_{1}\right]$$

$$I_1(a) = \frac{1}{a}$$

Solving for $n > 0$: when n is even

using $I_n(a) = \int_{-\infty}^{\infty} e^{-ax^2} x^n dx = 2 \underbrace{\int_0^{\infty} e^{-ax^2} x^n dx}_{L_n(a)}$

- aim to get ~~an~~ a recursive expression and has a pattern which n goes down to 0

* for a recursive relationship to occur ; ~~&~~ has a pattern which n goes down to zero

trying:
$$-\frac{\partial}{\partial a}\int_0^{\infty} e^{-ax^2} x^n dx = \int_0^{\infty} x^2 e^{-ax^2} x^n = \int_0^{\infty} e^{-ax^2} x^{n+2}$$

when $n \Rightarrow n-2$
$$-\frac{\partial}{\partial a}\int_0^{\infty} e^{-ax^2} x^{n-2} dx = \int_0^{\infty} e^{-ax^2} x^{n+2-2} = \int_0^{\infty} e^{-ax^2} x^n = L_n(a)$$

$$\therefore -\frac{\partial}{\partial a} L_{n-2}(a) = L_n(a)$$

Suppose $n = 2s$ $\therefore n > 1$ & n is even

$$L_n(a) = -\frac{\partial}{\partial a} L_{n-2}(a) = \left(-\frac{\partial}{\partial a}\right)^2 L_{n-4}(a) = \left(-\frac{\partial}{\partial a}\right)^3 L_{n-6}(a) \ldots$$

$$L_n(a) = \left(-\frac{\partial}{\partial a}\right)^{\frac{2s}{2}} L_0(a)$$

$I_0(a) = \sqrt{\frac{\pi}{a}}$

$L_0(a) = \frac{1}{2}\sqrt{\frac{\pi}{a}}$

$$L_n(a) = (-1)^s \underbrace{\left(\frac{\partial^s}{\partial a^s} a^{-\frac{1}{2}}\right)}_{f_s(a)} \frac{\sqrt{\pi}}{2}$$

$$f_s(a) = \frac{(2s-1)!!\,(-1)^s}{2^s\, a^{s+\frac{1}{2}}}$$

$$L_n(a) = (-1)^s \frac{(2s-1)!!\,(-1)^s}{2^s\, a^{s+\frac{1}{2}}} \frac{\sqrt{\pi}}{2}$$

$\because$ even

$$L_n(a) = \frac{\cancel{(-1)^{2s}}(2s-1)!!}{2^s\, a^s} \frac{1}{2}\sqrt{\frac{\pi}{a}}$$

Sub $n = 2s$

$s = \frac{n}{2}$

$$L_n(a) = \frac{(n-1)!!}{2^{\frac{n}{2}}\, a^{\frac{n}{2}}} \frac{1}{2}\sqrt{\frac{\pi}{a}}$$

$$2L_n(a) = I_n(a)$$

$\therefore$ for n is even & $n > 1$:

$$I_n(a) = \frac{(n-1)!!}{2^{\frac{n}{2}}\, a^{\frac{n}{2}}}\sqrt{\frac{\pi}{a}}$$

Regarding $f_s(a)$: s goes from 1 to s

$s=1 \Rightarrow \frac{\partial}{\partial a} a^{-\frac{1}{2}} = \left(-\frac{1}{2}\right)\left(a^{-\frac{3}{2}}\right)$

$s=2 \Rightarrow \frac{\partial^2}{\partial a^2}(a^{-\frac{1}{2}}) = \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right) a^{-\frac{5}{2}}$

$s=3 \Rightarrow \frac{\partial^3}{\partial a^3}(a^{-\frac{1}{2}}) = \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right) a^{-\frac{7}{2}}$

$s=4 \Rightarrow \frac{\partial^4}{\partial a^4}(a^{-\frac{1}{2}}) = \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{7}{2}\right) a^{-\frac{9}{2}}$

By observing the pattern:

Top part

$(2s-1)!!$ $\because$ $1\times3\times5$ and so on $\|$ $(-1)^s$ $\begin{cases} \text{odd} \Rightarrow -ve \\ \text{even} \Rightarrow +ve \end{cases}$

2^s $\because$ $2\times2\times2\times2\ldots$ for s times

bottom part

$a^{s+\frac{1}{2}}$ $\because$ indicies of a go up by intervals of 1 from $\frac{1}{2}$

Solving for $n \geq 0$ when n is odd:

Using the recursive relationship:

$L_n(a) = \left(-\frac{\partial}{\partial a}\right) I_{n-2}(a)$

$= \left(-\frac{\partial}{\partial a}\right)^2 I_{n-4}(a) = \left(-\frac{\partial}{\partial a}\right)^3 I_{n-6}(a)$

$\because n$ is odd $\Rightarrow n = 2s+1 \Leftrightarrow s = \frac{n-1}{2}$

& while accounting for $n=0$

s goes from $0 \to s$

$L_n(a) = \left(-\frac{\partial}{\partial a}\right)^s I_1(a)$ | $I_1(a) = \frac{1}{a}$, $L_1(a) = \frac{1}{2a}$

$= \left(-\frac{\partial}{\partial a}\frac{1}{a}\right)^s$

$= \frac{(-1)^s}{2}\underbrace{\left(\frac{\partial}{\partial a}a^{-1}\right)^s}_{g_s(a)}$

Regarding $g_s(a)$:

$s=1 \Rightarrow \frac{\partial}{\partial a}a^{-1} = (-1)\frac{\partial}{\partial a}a^{-2}$

$s=2 \Rightarrow \frac{\partial^2}{\partial a^2}a^{-1} = (-1)(-2)\frac{\partial}{\partial a}a^{-3}$

$s=3 \Rightarrow \frac{\partial^3}{\partial a^3}a^{-1} = (-1)(-2)(-3)\frac{\partial}{\partial a}a^{-4}$

$s=4 \Rightarrow \frac{\partial^4}{\partial a^4}a^{-1} = (-1)(-2)(-3)(-4)\frac{\partial}{\partial a}a^{-5}$

Which is a simpler case:

~~Ln~~ $g_s(a) = \frac{s!(-1)^s}{2a^{s+1}}$

Sub $s = \frac{n-1}{2}$

$L_n(a) = (-1)^s \frac{(-1)^s s!}{2a^{s+1}}$

$\because$ even

$= \frac{(-1)^{2s} s!}{2a^{s+1}} = \frac{\left(\frac{n-1}{2}\right)!}{2a^{\frac{n-1}{2}+\frac{3}{2}}}$

$2L_n(a) = I_n(a) \Rightarrow I_n(a) = \frac{\frac{1}{2}(n-1)!}{a^{\frac{n+1}{2}}}$

for $n > 0$ & n is odd

In the next section: (from quantum stuff blackbody radiation)

Questions to be asked:

① solving $(u-5)e^u = -5$

using Lambert function $W(x) = xe^x$

② Solving $I = \int \frac{2\pi}{c^2}\frac{hf^3}{e^{hf/kT}}df$

$\vdots$

$\hookrightarrow \int_0^\infty \frac{u^3}{e^u - 1}du$

using Pi function (Π) $\Pi(x) = \int_0^\infty t^x e^{-t}dt = n!$

Gamma function (Γ) $\Gamma(n) = \int_0^\infty e^{n-1}e^{-x}dx$

Fourier Series

③ Extra:

Zeta function (ζ) $\zeta(x) = \sum_{n=1}^\infty \frac{1}{n^x}$

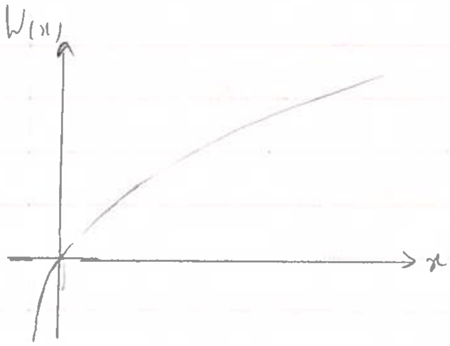
Eta function (η) $\eta(s) = \sum_{n=1}^\infty \frac{(-1)^{n-1}}{n^s}$

④ Deriving σ from $I = \sigma T^4$

Lambert function $W(x) = xe^x$

Solving for $(u-5)e^u = -5$

More on Wolfram



$(u-5)e^u = -5$

$u = W + 5$

$We^W \cdot e^5 = -5$

$\Rightarrow We^W = -5e^{-5} =$

$\Rightarrow W(x) = -5e^{-5}$

for the non-zero solution:

$W = W_0(-5e^{-5})$

$u = W_0(-5e^{-5}) + 5 = \text{productlog}(-5e^{-5}) + 5$

$= 4.965114$

$\ln(x) = g^{-1}(x) \Rightarrow g(x) = e^x$ $D: (-\infty, \infty)$

$D: (0, \infty)$ $R: (-\infty, \infty)$ $R: (0, \infty)$

$\gg \ln(g(x)) = \ln(e^x) = x$

$\gg g(\ln(x)) = e^{\ln x} = x$

$\rightarrow g^{-1}(g(x)) = g(g^{-1}(x)) = x$

$\rightarrow W(x) = f^{-1}(x) \Rightarrow f(x) = xe^x$

$D: [-\frac{1}{e}, \infty]$ $D: [-1, \infty]$

$R: [-1, \infty]$ $R: [-\frac{1}{e}, \infty]$

this does the job.

Now we have:

① $W[f(x)] = W(xe^x) = x$

② $f(W(x)) = W(x)e^{W(x)} = x$

$\rightarrow f(f^{-1}(x)) = x$

$= f^{-1}(f(x)) = x$

$W(x) = \text{product log}$

Pi function $\Pi(x) = \int_0^\infty t^x e^{-t}\,dt$

$\Pi(x) = n!$

Checking.

① $f(1) = 1$

$\Pi(1) = \int_0^\infty t e^{-t}\,dt$

	D	I
+	t	e^{-t}
−	1	$-e^{-t}$
+	0	e^{-t}

$$\Pi(1) = \left[-te^{-t} - e^{-t}\right]_{t=0}^{t=\infty}$$

using L.Hopital's rule:

$$\lim_{t\to\infty}\left[-\frac{t}{e^t}\right] = \lim_{t\to\infty}\left(-\frac{1}{e^t}\right) = 0$$

$$\Pi(1) = \left[0 - e^{-\infty}\right] - \left[-e^{-0}\right]$$

$\Pi(1) = 1$

② $f(n) = n \cdot f(n-1)$

$\Pi(n) = \int_0^\infty t^n e^{-t}\,dt$

	D	I
+	t^n	e^{-t}
−	nt^{n-1}	$-e^{-t}$

$$\Pi(n) = \left[-t^n e^{-t}\right]_{t=0}^{t=\infty} + \underbrace{\int_0^\infty n t^{n-1} e^{-t}\,dt}_{n f(n-1)}$$

using L.Hopital's rule:

$$\lim_{t\to\infty}\left[-\frac{t^n}{e^t}\right] \ldots \frac{d}{dt} \text{ infinitely many time}$$

$$= \lim_{t\to\infty}\left[\frac{n!}{e^t}\right] = 0$$

$\Pi(n) = n\,\Pi(n-1)$

Gamma function + Zeta function

$\Gamma(n) = (n-1)!$

$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x}\,dx$

$\Re(n) > 0$

for $n \in \mathbb{C}$

$\Re(n) > 0$

$\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s}$

for $s \in \mathbb{C}$

$t \in [0,\infty)$

$$\Gamma(x) = \int_{t=0}^{t=\infty} t^{x-1} e^{-t}\,dt$$

Let $t = n \cdot u$

$dt = n\,du$

$u \in [0,\infty)$

$$\Gamma(x) = \int_{u=0}^{u=\infty} (nu)^{x-1} e^{-nu}\, n\,du$$

$$= \int_0^\infty n^{x\cancel{-1}} u^{x-1} e^{-nu} \cancel{n}\,du$$

$$\Gamma(x) = n^x \int_0^\infty u^{x-1} e^{-nu}\,du$$

$$\sum_{n=1}^\infty \frac{\Gamma(x)}{n^x} = \sum_{n=1}^\infty \int_0^\infty u^{x-1} e^{-nu}\,du$$

$$\Gamma(x)\sum_{n=1}^\infty \frac{1}{n^x} = \int_0^\infty u^{x-1} \underbrace{\sum_{n=1}^\infty \left(e^{-u}\right)^n}_{\text{infinit geometric series!}} du$$

$$\Gamma(x)\sum_{n=1}^\infty \frac{1}{n^x} = \int_0^\infty u^{x-1} \frac{e^u\left(e^{-u}\right)}{e^u\left(1-e^{-u}\right)} du$$

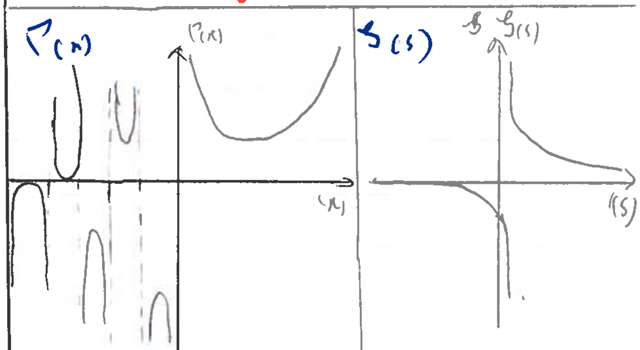
using $r = \frac{1}{e^u}$, $res = \frac{a_1}{1-r}$

$res = \frac{e^{-u}}{1-e^{-u}}$

$$\Gamma(x)\underbrace{\sum_{n=1}^\infty \frac{1}{n^x}}_{\zeta(x)} = \int_0^\infty \frac{u^{x-1}}{e^u - 1}\,du$$

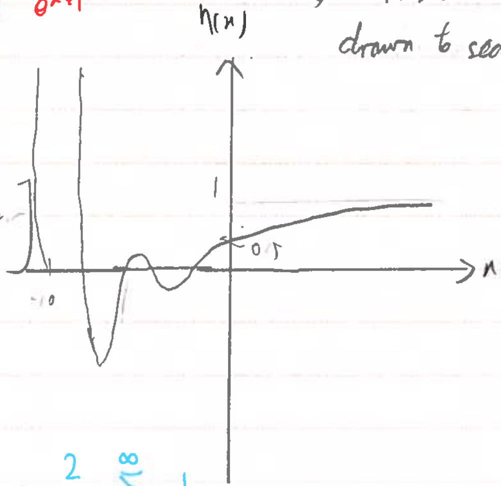
$x \to s$

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{u^{s-1}}{e^u - 1}\,du$$



Extra:

Eta function : $\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{x^{s-1}}{e^x+1}\,dx$



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in relation to the zeta func: $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$

$\zeta(s) - \eta(s) = \cancel{\frac{1}{1^s}} + \frac{1}{2^s} + \cancel{\frac{1}{3^s}} + \frac{1}{4^s} + \dots - \left[\cancel{\frac{1}{1^s}} - \frac{1}{2^s} + \cancel{\frac{1}{3^s}} - \frac{1}{4^s} + \dots\right]$

only even numbers stay:

$$= \frac{2}{2^s} + 2\frac{1}{4^s} + \frac{2}{6^s} + \frac{2}{8^s} + \dots$$

$$\zeta(s) - \eta(s) = \frac{2}{2^s \cdot 1^s} + \frac{2}{2^s \cdot 2^s} + \frac{2}{2^s 3^s} + \frac{2}{2^s 4^s} \dots = \frac{2}{2^s} \underbrace{\sum_{n=1}^{\infty} \frac{1}{n^s}}_{\zeta(s)}$$

$$\zeta(s) - \eta(s) = 2^{1-s}\zeta(s)$$

$$\eta(s) = \left[1 - 2^{1-s}\right]\zeta(s)$$

An integral representation $\eta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{x^{s-1}}{e^x+1}\,dx$

$$\int_0^{\infty} \frac{x^{s-1}}{e^x+1}\,dx = \int_0^{\infty} \frac{e^{-x} x^{s-1}}{1-(-e^{-x})}\,dx = \int_0^{\infty} e^{-x} x^{s-1} \sum_{n=0}^{\infty} (-e^{-x})^n\,dx = \int_0^{\infty} e^{-x} x^{s-1} \sum_{n=0}^{\infty} e^{-nx} (-1)^n\,dx$$

$$= \sum_{n=0}^{\infty} (-1)^n \int_0^{\infty} x^{s-1} e^{-x(n+1)}\,dx \quad \Bigg| \quad \begin{aligned} &\text{let } t = x(n+1) \implies x = \frac{t}{(n+1)} \\ &dt = (n+1)\,dx \end{aligned}$$

$$= \sum_{n=0}^{\infty} (-1)^n \int_0^{\infty} \left(\frac{t}{n+1}\right)^{s-1} e^{-t} \frac{dt}{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^s} \underbrace{\int_{0}^{\infty} t^{s-1} e^{-t}\,dt}_{\Gamma(s)}$$

$x =$

$$\underbrace{\frac{1}{\Gamma(s)} \int_0^{\infty} \frac{x^{s-1}}{e^x+1}\,dx}_{1} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} = \eta(s)$$

$n+1 \to n$

$n \to n-1$

$$\eta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{x^{s-1}}{\underbrace{e^x+1}}\,dx$$

Perseval's Theorem using Fourier Series (brief intro)

5 main integral identities, using Kronecker delta $\delta_{mn}\begin{cases} 0 & (n\neq m) \\ 1 & (n=m)\end{cases}$

① $\int_{-\pi}^{\pi} \sin(mx)\sin(nx)\,dx = \pi\delta_{mn}$

② $\int_{-\pi}^{\pi} \cos(mx)\cos(nx)\,dx = \pi\delta_{mn}$

③ $\int_{-\pi}^{\pi} \sin(mx)\cos(nx)\,dx = 0$

④ $\int_{-\pi}^{\pi} \sin(mx)\,dx = 0$

⑤ $\int_{-\pi}^{\pi} \cos(mx)\,dx = 0$

Generalized Fourier series:
↑
functions come from a complete orthogonal system

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(nx) + \sum_{n=1}^{\infty} b_n\sin(nx)$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$$

$$\int_{-\pi}^{\pi}(f(x))^2dx = \int_{-\pi}^{\pi}\left(\frac{a_0}{2}\right)^2dx + \underbrace{\int_{-\pi}^{\pi} a_0\sum_{n=1}^{\infty}(a_n\cos(nx)+b_n\sin(nx))\,dx}_{\text{using ④, ⑤, this term}=0} + \int_{-\pi}^{\pi}\sum_{n=1}^{\infty}(a_n\cos(nx)+b_n\sin(nx))\sum_{m=1}^{\infty}(a_m\cos(mx)+b_m\sin(mx))\,dx$$

$$= \frac{2\pi a_0^2}{4} + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\int_{-\pi}^{\pi}\underbrace{a_na_m\cos(nx)\cos(mx)}_{\text{using ②}} + \underbrace{a_nb_m\cos(nx)\sin(mx) + b_na_m\sin(nx)\cos(mx)}_{\text{using ③} \ \therefore\ \text{term}=0} + \underbrace{b_nb_m\sin(nx)\sin(mx)}_{\text{using ①}}$$

using ②: when $n=m$, $\delta_{nm}=1$

$$\Rightarrow \sum_{n=1}^{\infty}\int_{-\pi}^{\pi} a_n^2\,dx = \pi\sum_{n=1}^{\infty}(a_n)^2$$

$$\Rightarrow \sum_{n=1}^{\infty}\int_{-\pi}^{\pi}(b_n)^2dx = \pi\sum_{n=1}^{\infty}(b_n)^2$$

$$\int_{-\pi}^{\pi}(f(x))^2dx = \frac{\pi a_0^2}{2} + \pi\sum_{n=1}^{\infty}(a_n^2+b_n^2)$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi}(f(x))^2dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2+b_n^2)$$

Summarizing the stuff we know for derivation of the boltzmann constant (σ)

From thermodynamics:

$$dS = \frac{dQ}{T}$$

$$\Delta E = T \Delta S$$

$$\Delta S = k \ln W$$

Assuming change in energy cannot be smaller than a certain unit

$\Delta E = hf$ ($E = Nhf$)

determines the size of the energy packet

Available state of energy can only be in discrete chunks: $N+1 \to N$

$\therefore\ hf = k \Delta \ln W$

$\Rightarrow hf = kT \Delta \ln W$

$$\frac{hf}{kT} = \ln(N+1) - \ln N = \ln \frac{N+1}{N}$$

$$e^{\frac{hf}{kT}} = \frac{N+1}{N} \Rightarrow W = \frac{1}{e^{hf/kT} - 1}$$

$$\Rightarrow E(f) = \frac{hf}{e^{hf/kT} - 1}$$

Remembering:

$$I = \frac{c}{4} \int g(f)\, E(f)\, df$$

$g(f) = \frac{8\pi}{c^3} f^2$ (see density of states, cambridge)

$$I = \frac{c\, 8\pi}{4 c^3} \int \frac{f^2\, hf}{e^{hf/kT} - 1}\, df$$

Let $u = \frac{hf}{kT}$ $\quad h^3 = \frac{k^3 f^3}{k^3 T^3}$

$du = \frac{h}{kT} df$ $\quad f^3 = \frac{u^3 k^3 T^3}{h^3}$

$df = \frac{kT\, du}{h}$

$$\hookrightarrow I = \frac{2\pi h}{c^2} \frac{k^3 T^3}{h^3} \frac{kT}{h} \int \frac{h u^3}{e^u - 1} du$$

$$I = \frac{2\pi k^4 T^4}{c^2 h^3} \int \frac{u^3}{e^u - 1} du$$

Classical ~~way~~ way: using Gamma & Zeta function

Starting by using: $\int_0^\infty \frac{x^m e^x}{(e^x - 1)} dx$

$$= \int_0^\infty x^m e^{-x} \frac{1}{1 - e^{-x}} dx = \int_0^\infty x^m e^{-x} \sum_{n=1}^{\infty} (e^{-x})^n dx$$

$$= \sum_{n=1}^{\infty} \int_0^\infty \frac{x^m}{e^x} (e^{-x})^n dx = \sum_{n=1}^{\infty} \int_0^\infty x^m (e^{-x(n+1)}) dx$$

$$= \sum_{n=1}^{\infty} \int_0^\infty \left(\frac{u}{n+1}\right)^m e^{-u} \frac{1}{n+1} du = \sum_{n=1}^{\infty} \int_0^\infty u^m \frac{1}{(n+1)^{m+1}} e^{-u} du$$

$$= \sum_{n=1}^{\infty} \frac{1}{(n+1)^{m+1}} \int_0^\infty u^m e^{-u} du$$

$$= \Gamma(m+1) \times \zeta(m+1)$$

from infinite geometric series:

$$\frac{a}{1-r} = \sum_{k=0}^{\infty} a r^k$$

$$\hookrightarrow \sum_{n=1}^{\infty} (e^{-x})^n$$

Let $u = (n+1)x$

$du = (n+1)\, dx$

$dx = \frac{du}{n+1}$

When $m = 3$:

$$\Gamma(4) \times \zeta(4)$$

$$= (4-1)! \cdot \sum_{n=1}^{\infty} \frac{1}{n^4}$$

$$= 6 \cdot \sum_{n=1}^{\infty} \frac{1}{n^4}$$

Solving for $\sum_{n=1}^{\infty} \frac{1}{n^4}$ using Parseval's Theorem

$$\frac{1}{\pi} \int_{-\pi}^{\pi} (x^2)^2 dx = 2\left(\frac{\pi^2}{3}\right)^2 + \sum_{n=1}^{\infty} \left(\frac{4(-1)^n}{n^2}\right)^2$$

$$\frac{2\pi^5}{5\pi} = \frac{2\pi^4}{9} + \sum_{n=1}^{\infty} \frac{16}{n^4}$$

$$\sum_{n=1}^{\infty} \frac{16}{n^4} = \frac{9(2\pi^4)}{45} - \frac{(2\pi^4)5}{45}$$

$$\sum_{n=1}^{\infty} \frac{16}{n^4} = \frac{8\pi^4}{45} \times \frac{1}{16}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90} = \zeta(4)$$

$$\Gamma(4) \times \zeta(4) = \frac{6\pi^4}{90} = \frac{\pi^4}{15}$$

Set $f(x) = x^2$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} x^2 dx = \frac{\pi^2}{3}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cos(nx)\, dx$$

	D	I
+	x^2	$\cos(nx)$
−	$2x$	$\frac{\sin nx}{n}$
+	2	$-\frac{\cos nx}{n^2}$
−	0	$-\frac{\sin nx}{n^3}$

$$= \frac{1}{\pi}\left[\frac{x^2 \sin nx}{n} - \frac{x^2 \sin nx}{n^3} + \frac{2x\cos(nx)}{n^2}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{2\pi \cos n\pi}{n^2}\right] \times 2 = \frac{4}{n^2}\cos n\pi = \frac{4(-1)^n}{n^2}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \sin(nx)\, dx = 0$$

even × odd $\Rightarrow$ odd

integrating odd function

$b_n = 0$

$$\hookrightarrow \sigma = \frac{2\pi k^4}{c^2 h^3} \times \frac{\pi^4}{15}$$

$$= 5.67 \times 10^{-8}\ W m^{-2} K^{-4}$$

$$\underline{I = \sigma T^4}$$