Partial Differential Eq^{n}s

Zoology of Diff Eq^n

- Ordinary / Partial

Ordinary: function has only one variable

$$\frac{dy}{dx} = c \; ; \; y = xc + k \; ; y = y(x)$$

Partial: function has more than one variable!

$$\frac{\partial y}{\partial x} + \frac{\partial y}{\partial t} = c \; ; \; y = y(x,t)$$

↳ if variables are dependant

⇒ ~~need~~ need to specify which variables are held constant

$$\left.\frac{\partial y(x,t)}{\partial x}\right|_t = c(s,t)$$

- Order: order of its highest derivative

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}}{dx^{n-1}} + \dots + a_0(x)y = f(x)$$

- Linearity :⇒ written entirely as a linear function

(1) w/ no powers above the first power of the unknown function and its derivative ⇒ no products of the function (2) ⇒ function or its derivatives are inside another function (3)

eg. $x^2\frac{d^3y}{dx^2} - 6xy + 10y = \cos x^2$

non-linear: $\frac{d^2\theta}{dt^2} + \frac{5}{l}\sin\theta = 0$ (case 3) ⇒ linear: $\frac{d^2\theta}{dt^2} + \frac{g}{l}\theta = 0$

$\sin\theta \approx \theta$

general form: $a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}}{dx^{n-1}} + \dots a_0(x)y = b(x)$

- Homogeneous: if terms in the eq^n depend on the unknown function or its derivatives

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = G(x)$$

when $G(x)=0$

$\Rightarrow$ homogeneous

- Degree: - exponent of the highest order derivative involved

eg. $3y^2\left(\frac{dy}{dx}\right)^3 \cdot \frac{d^2y}{dx^2} = \sin(x^2)$

Order = 2, degree = 1

eg. $\sqrt{1+\left(\frac{dy}{dx}\right)^2} = y\frac{d^3y}{dx^3}$

$1+\left(\frac{dy}{dx}\right)^2 = y^2\left(\frac{d^3y}{dx^3}\right)^2$

order = 3, degree = 2

eg $\frac{d^2y}{dx^2} + \cos\frac{d^2y}{dx^2} = 5x$!

order = 2, degree $\Rightarrow$ undefined

- Solutions

$$f\left(x, y(x), \frac{dy}{dx}, \frac{d^2y}{dx^2}\right) = 0$$

there is some function $y = u(x)$ in the range $(a < x < b)$

for which the problem is defined.

$\Rightarrow$ question: Does $f\left(x, u(x), \frac{dy}{dx}, \ldots\right) = 0$?

$\Rightarrow$ check solution by substituting into the eq^n

or

↳ verifying using a posteriori

- Uniqueness: - a diff eq^n has more than one solution

↳ for an n^{th} order diff eq ⇒ usually n independent function where n boundary conditions are required to determine the constants.

- Existence: - there is no guarantee that ~~a~~ a diff eq^n will have the form $u(x)$

- Superposition: if $y_1(x)$ & $y_2(x)$ are solutions to a linear homogeneous Diff eq^n

↳ $y = Cy_1(x) + Dy_2(x)$ is also a solution

eg. $\frac{\partial^2 y}{\partial x^2} = -k^2 y$ (harmonic oscillator)

$y_1 = \cos kx$, $y_2 = \sin kx$

$y = A\cos kx + B\sin kx$

Separation of variables method:

Aim: transform a PDE in n variables into n separate ODEs

Consider:

$$a(x,y)\frac{\partial^2 u}{\partial x^2} + b(x,y)\frac{\partial^2 u}{\partial y^2} = 0$$

Assume $\underbrace{\text{hypothesis / ansatz}}_{\text{whats the diff}}$: $u(x,y) = X(x)Y(y)$

↳ $$a(x,y)Y(y)\frac{d^2X}{dx^2} + b(x,y)X(x)\frac{d^2Y}{dy^2} = 0$$

$$\frac{1}{XY}\left(a(x,y)Y(y)\frac{d^2X}{dx^2}\right) = \left(-b(x,y)X(x)\frac{d^2Y}{dy^2}\right)\frac{1}{XY}$$

$$\overbrace{a(x,y)\frac{1}{X}\frac{d^2X}{dx^2}}^{\mathrm{I}} = \overbrace{-b(x,y)\frac{1}{Y}\frac{d^2Y}{dy^2}}^{\mathrm{II}}$$

↳ Separable provided that:

I: written solely in terms of x ⟹ rearrange $a(x,y) \to A(x)$

II: " " " " " y $b(x,y) \to B(y)$

$$\frac{A(x)}{X}\frac{d^2X}{dx^2} = -\frac{B(y)}{Y}\frac{d^2Y}{dy^2}$$

or

$$f(x) = g(y)$$

But x & y are independant variables:

↳ each side must be equal to a constant:

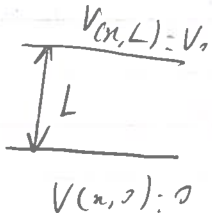
$$\frac{A(x)}{X}\frac{d^2X}{dx^2} = c$$

$$\frac{B(y)}{Y}\frac{d^2Y}{dy^2} = -c$$

* linear eq^n w/ constant variables can often ~~by~~ be solved by separation
* separability depends in general on the chosen system
* chose right ~~co~~ coordinate system

eg. In 2D:

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}=0$$



ansatz / hypothesis:

$$V(x,y)=X(x)\,Y(y)$$

$$\frac{1}{XY}:\quad Y(y)\frac{d^2X(x)}{dx^2}+X(x)\frac{d^2Y(y)}{dy^2}=0$$

$$\frac{d^2X(x)}{dx^2}\frac{1}{X}+\frac{d^2Y(y)}{dy^2}\frac{1}{Y}=0$$

$$c \quad - \quad c \quad = 0$$

$$\frac{d^2X}{dx^2}=cX\;;\quad \frac{d^2Y}{dy^2}=-cY$$

Possible solutions:

$$\underbrace{\cos(nx),\ \sin(nx)}_{e^{\pm inx}}\quad \text{or}\quad \underbrace{\cosh(mx);\ \sinh(mx)}_{e^{\pm mx}}$$

For $\frac{d^2X}{dx^2}=cX$

$$X_\lambda(x)=A_\lambda\cos(\lambda x)+B_\lambda\sin(\lambda x)$$

$$\frac{d^2X}{dx^2}=-\lambda^2X \quad \text{ie}\quad c=-\lambda^2$$

For $\frac{d^2Y}{dy^2}=-cY=\lambda^2Y$

$$Y_\lambda(y)=D_\lambda e^{\lambda y}+E_\lambda e^{-\lambda y}$$

Combine: $V(x,y)=X(x)\,Y(y)$

$$=(A_\lambda\cos(\lambda x)+B_\lambda\sin(\lambda x))(D_\lambda e^{\lambda y}+E_\lambda e^{-\lambda y})$$

A general solution:

$$V(x,y)=\sum_\lambda\left(A_\lambda\cos(\lambda x)+B_\lambda\sin(\lambda x)\right)\left(D_\lambda e^{\lambda y}+E_\lambda e^{-\lambda y}\right)$$

Special case when ~~c~~ $\lambda=0$ (c or λ)

$$\frac{d^2X}{dx^2}=0 \quad\text{and}\quad \frac{d^2Y}{dy^2}=0$$

$$X=A_0+B_0x \qquad Y=D_0+E_0y$$

$$\hookrightarrow V(x,y)=(A_0+B_0x)(D_0+E_0y)$$

$$+\sum_{\lambda\neq0}(A_\lambda\cos(\lambda x)+B_\lambda\sin(\lambda x))(D_\lambda e^{\lambda y}+E_\lambda e^{-\lambda y})$$

Solving w/ boundary conditions:

$$V(x,0)=0$$

$$V(x,0)=(A_0+B_0x)D_0+\sum_{\lambda\neq 0}(A_\lambda\cos\lambda x+B_\lambda\sin\lambda x)(D_\lambda e^{\lambda(0)}+E_\lambda e^{-\lambda(0)})=0$$

For $\lambda=0$: $(A_0+B_0x)D_0=0 \Rightarrow D_0=0$

For $\lambda\neq 0$: $(A_\lambda\cos(\lambda x)+B_\lambda\sin(\lambda x))(D_\lambda+E_\lambda)=0$

$$D_\lambda+E_\lambda=0$$
$$D_\lambda=-E_\lambda$$

$\Rightarrow$ ~~now~~ now we have:

$$V(x,y)=(A_0+B_0x)E_0y+\sum_{\lambda\neq 0}(A_\lambda\cos\lambda x+B_\lambda\sin\lambda x)(D_\lambda)(e^{\lambda y}-e^{-\lambda y})$$

another boundary condition:

$$V(x,L)=V_0$$

$$V(x,L)=(A_0+B_0x)E_0(L)+\sum_{\lambda\neq 0}(A_\lambda\cos\lambda x+B_\lambda\sin\lambda x)(D_\lambda)(e^{\lambda L}-e^{-\lambda L})=V_0$$

we cannot have any x dependence as V_0 is a constant

For $\lambda=0$: $B_0=0$

$\lambda\neq 0$: $\Rightarrow A_\lambda=B_\lambda=0$ and all $\lambda\neq 0$ terms go

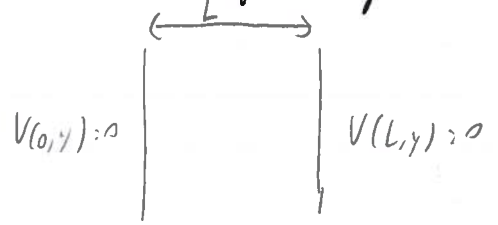
$$V(x,L)=A_0E_0L=V_0$$

$$A_0E_0=\frac{V_0}{L}$$

$$\Rightarrow V(x,y)=A_0E_0y$$

$$=\frac{V_0}{L}y$$

A different set of boundary conditions!



Start from general solution:

$$V(x,y) = (A_0 + B_0 x)(D_0 + E_0 y) + \sum_{\lambda \neq 0} (A_\lambda \cos(\lambda x) + B_\lambda \sin(\lambda x))(D_\lambda e^{\lambda y} + E_\lambda e^{-\lambda y})$$

3) Getting $\frac{d}{dy}$ isolating $\frac{\partial}{\partial y}$

from Ⓐ ④

$$r\sin\theta\frac{\partial}{\partial r} + \cos\theta\frac{\partial}{\partial\theta} = r\cos\phi\frac{\partial}{\partial x} + r\sin\phi\frac{\partial}{\partial y}$$

$\times \sin\theta\sin\phi$

$$r\sin^2\theta\sin\phi\frac{\partial}{\partial r} + \cos\theta\sin\theta\sin\phi\frac{\partial}{\partial\theta} = r\cos\phi\sin\theta\sin\phi\frac{\partial}{\partial x} + r\sin^2\phi\sin\theta\frac{\partial}{\partial y}$$

③ $\times \cos\phi$

$$\cos\phi\frac{\partial}{\partial\phi} = -r\sin\theta\sin\phi\cos\phi\frac{\partial}{\partial x} + r\sin\theta\cos^2\phi\frac{\partial}{\partial y}$$

Add:

$$r\sin^2\theta\sin\phi\frac{\partial}{\partial r} + \cos\theta\sin\theta\sin\phi\frac{\partial}{\partial\theta} + \cos\phi\frac{\partial}{\partial\phi} = r\sin\theta\frac{\partial}{\partial y}$$

$$\frac{\partial}{\partial y} = r\sin\theta\sin\phi\frac{\partial}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial}{\partial\theta} + \frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial\phi}$$

c) getting $\frac{d}{dz}$

$$\frac{\partial}{\partial r} = \sin\theta\cos\phi\frac{\partial}{\partial x} + \sin\theta\sin\phi\frac{\partial}{\partial y} + \cos\theta\frac{\partial}{\partial z}$$

$$\frac{\partial}{\partial\theta} = r\cos\theta\cos\phi\frac{\partial}{\partial x} + r\cos\theta\sin\phi\frac{\partial}{\partial y} - r\sin\theta\frac{\partial}{\partial z}$$

$$\frac{\partial}{\partial r} = \sin^2\theta\cos^2\phi\frac{\partial}{\partial r} + \frac{\sin\theta\cos\theta\cos^2\phi}{r}\frac{\partial}{\partial\theta} - \cancel{\frac{\sin\theta\cos\phi\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi}}$$
$$+ \sin^2\theta\sin^2\phi\frac{\partial}{\partial r} + \frac{\sin\theta\sin^2\phi\cos\theta}{r}\frac{\partial}{\partial\theta} + \cancel{\frac{\sin\theta\sin\phi\cos\phi}{r\sin\theta}\frac{\partial}{\partial\phi}}$$
$$+ \cos\theta\frac{\partial}{\partial z}$$

$$\frac{\partial}{\partial r} = \sin^2\theta\frac{\partial}{\partial r} + \left(\frac{\sin\theta\cos\theta}{r}\right)\frac{\partial}{\partial\theta} + \cos\theta\frac{\partial}{\partial z}$$

$$\cos^2\theta\frac{\partial}{\partial r} = \frac{\sin\theta\cos\theta}{r}\frac{\partial}{\partial\theta} + \cos\theta\frac{\partial}{\partial z}$$

$$\frac{\partial}{\partial z} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}$$

$$\nabla^2 = \left(\frac{\partial}{\partial x}\right)^2 + \left(\frac{\partial}{\partial y}\right)^2 + \left(\frac{\partial}{\partial z}\right)^2$$

$$= \left(\sin\theta\cos\phi\frac{\partial}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right)^2$$

$$+ \left(\sin\theta\sin\phi\frac{\partial}{\partial r} + \frac{\sin\phi\cos\theta}{r}\frac{\partial}{\partial\theta} - \frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right)^2$$

$$+ \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)^2$$

long-ass expansions

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\cancel{\psi}}{\partial r}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\cancel{\psi}}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\left(\frac{\partial^2}{\partial\phi^2}\right)$$

Laplace's Expression Eq^n in spherical polar coordinates

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\left(\frac{\partial^2 V}{\partial\phi^2}\right)=0$$

$$\nabla^2 V = 0$$

$x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta$

wrt: r $\frac{\partial x}{\partial r}=\sin\theta\cos\phi$ $\frac{\partial y}{\partial r}=\sin\theta\sin\phi$ $\frac{\partial z}{\partial r}=\cos\theta$

wrt: θ $\frac{\partial x}{\partial\theta}=r\cos\theta\cos\phi$ $\frac{\partial y}{\partial\theta}=r\cos\theta\sin\phi$ $\frac{\partial z}{\partial\theta}=-r\sin\theta$

wrt: ϕ $\frac{\partial x}{\partial\phi}=-r\sin\theta\sin\phi$ $\frac{\partial y}{\partial\phi}=r\sin\theta\cos\phi$ $\frac{\partial z}{\partial\phi}=0$

$\Downarrow$ using chain rules

$$\frac{\partial}{\partial r}=\sin\theta\cos\phi\frac{\partial}{\partial x}+\sin\theta\sin\phi\frac{\partial}{\partial y}+\cos\theta\frac{\partial}{\partial z} \quad -①$$

$$\frac{\partial}{\partial\theta}=r\cos\theta\cos\phi\frac{\partial}{\partial x}+r\cos\theta\sin\phi\frac{\partial}{\partial y}-r\sin\theta\frac{\partial}{\partial z} \quad -②$$

$$\frac{\partial}{\partial\phi}=-r\sin\theta\sin\phi\frac{\partial}{\partial x}+r\sin\theta\cos\phi\frac{\partial}{\partial y} \quad -③$$

$\Rightarrow$ Rearrange getting $\frac{\partial}{\partial x}$ (A)

① $\times r\sin\theta$

$$r\sin\theta\frac{\partial}{\partial r}=r\sin^2\theta\cos\phi\frac{\partial}{\partial x}+r\sin^2\theta\sin\phi\frac{\partial}{\partial y}+r\sin\theta\cos\theta\frac{\partial}{\partial z}$$

② $\times\cos\theta$

$$\cos\theta\frac{\partial}{\partial\theta}=r\cos^2\theta\cos\phi\frac{\partial}{\partial x}+r\cos^2\theta\sin\phi\frac{\partial}{\partial y}-r\sin\theta\cos\theta\frac{\partial}{\partial z}$$

add

$$\Rightarrow r\sin\theta\frac{\partial}{\partial r}+\cos\theta\frac{\partial}{\partial\theta}=r\cos\phi\frac{\partial}{\partial x}+r\sin\phi\frac{\partial}{\partial y} \quad -④$$

$\times\sin\theta\cos\phi$

$$r\sin^2\theta\cos\phi\frac{\partial}{\partial r}+\cos\theta\sin\theta\cos\phi\frac{\partial}{\partial\theta}=r\sin\theta\cos^2\phi\frac{\partial}{\partial x}+r\sin\theta\cos\phi\sin\phi\frac{\partial}{\partial y}$$

③ $\times\sin\phi$

$$\sin\phi\frac{\partial}{\partial\phi}=-r\sin\theta\sin^2\phi\frac{\partial}{\partial x}+r\sin\theta\cos\phi\sin\phi\frac{\partial}{\partial y}$$

Subtract:

$$r\sin^2\theta\cos\phi\frac{\partial}{\partial r}+\cos\theta\sin\theta\cos\phi\frac{\partial}{\partial\theta}-\sin\phi\frac{\partial}{\partial\phi}=r\sin\theta\frac{\partial}{\partial x}$$

$$\Rightarrow\frac{\partial}{\partial x}=\sin\theta\cos\phi\frac{\partial}{\partial r}+\frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial\theta}-\frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi}$$

Laplace eq^n in spherical polar coordinates

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial\phi^2} = 0$$

eg. $V = 2x^2 - y^2 - z^2$

$$\frac{\partial^2 V}{\partial x^2} = 4 \;;\; \frac{\partial^2 V}{\partial y^2} = -2 \;;\; \frac{\partial^2 V}{\partial z^2} = -2$$

In polar coordinates:

$$V = r^2\left(2\sin^2\theta\cos^2\phi - \sin^2\theta\sin^2\phi - \cos^2\theta\right)$$

check that $\nabla^2 V = 0$

Sp. Separation of variables:

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2} = 0$$

Ansatz: $V(r,\theta,\phi) = R(r)\Theta(\theta)\Phi(\phi)$

$$\Theta\Phi\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + R\Phi\frac{1}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + R\Theta\frac{1}{r^2\sin^2\theta}\left(\frac{d^2\Phi}{d\phi^2}\right) = 0$$

$\Rightarrow$ Now $\times \dfrac{r^2\sin^2\theta}{R\times\Theta\times\Phi}$

$$\Rightarrow \frac{\sin^2\theta}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = 0$$

$\Rightarrow \dfrac{1}{\Phi}\dfrac{d^2\Phi}{d\phi^2} = -m^2$

has solution $Ae^{\pm im\Phi}$

$\Rightarrow$ m is an integer and

$\Phi(\phi)$ is a periodic func.

$$\frac{\sin^2\theta}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) - m^2 = 0$$

$\Rightarrow$ Now divide by $\sin^2\theta$ & rearrange

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = \frac{m^2}{\sin^2\theta} - \frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)$$

L.H.S is a function of (r) only

R.H.S is function of (θ) only

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = \lambda R \quad \text{- (Radial eq}^n\text{)}$$

$$\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \left(\lambda\sin\theta - \frac{m^2}{\sin\theta}\right)\Theta = 0 \quad \text{- (polar eq}^n\text{)}$$

For the radial eq^n

$$r^2\frac{d^2R}{dr^2} + 2r\frac{dR}{dr} - \lambda R = 0$$

$\Rightarrow$ ansatz: $R(r) \sim r^\beta$

$$r^2\beta(\beta-1)r^{\beta-2} + 2r\beta r^{\beta-1} - \lambda r^\beta = 0$$

$$\beta(\beta-1)r^\beta + 2\beta r^\beta - \lambda r^\beta = 0$$

$$\beta^2 - \beta + 2\beta - \lambda = 0$$

cancelling r^β

$$\beta^2 + \beta - \lambda = 0$$

$$\beta = \frac{-1 \pm \sqrt{1+4\lambda}}{2}$$

Define: $\lambda = \ell(\ell+1)$

$$\beta = \frac{-1 \pm \sqrt{1+4\ell(\ell+1)}}{2}$$

$\beta = \ell$ or $-\ell-1$

$$R(r) = Ar^\ell + \frac{B}{r^{\ell+1}}$$

For Polar eqⁿ:

$$\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\left(l(l+1)\sin\theta-\frac{m^2}{\sin\theta}\right)\Theta=0$$

$\mu=\cos\theta \Rightarrow \frac{d\mu}{d\theta}=-\sin\theta$

$\frac{d}{d\theta}=-\sin\theta\frac{d}{d\mu}=-\sqrt{1-\mu^2}\frac{d}{d\mu}$

$$\Rightarrow -\sqrt{1-\mu^2}\frac{d}{d\mu}\left(\sqrt{1-\mu^2}\cdot-\sqrt{1-\mu^2}\frac{d\Theta}{d\mu}\right)+\left(l(l+1)\sqrt{1-\mu^2}-\frac{m^2}{\sqrt{1-\mu^2}}\right)\Theta=0$$

$$\frac{d}{d\mu}\left[(1-\mu^2)\frac{d\Theta}{d\mu}\right]+\left[l(l+1)-\frac{m^2}{1-\mu^2}\right]\Theta=0$$

[Legendre equation] to Legendre Polynomials

Eg: One-dimension wave eqⁿ

eg. guitar string is clamped @ $x=0$ & $x=L$

The displacement obeys:

$$\frac{\partial^2 y}{\partial x^2}-\frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}=0$$

$t \Rightarrow$ time

$v \Rightarrow$ speed of wave propagation

Ansatz: $y(x,t)=X(x)T(t)$

$$\Rightarrow T\frac{d^2X}{dx^2}-\frac{1}{v^2}X\frac{d^2T}{dt^2}=0$$

divide $y=XT$

$$\Rightarrow \frac{1}{X}\frac{d^2X}{dx^2}=\frac{1}{v^2T}\frac{d^2T}{dt^2}$$

[both sides are equal to a constant, $-k^2$

$$\frac{d^2X}{dx^2}+k^2X=0$$

$$\frac{d^2T}{dt^2}+k^2v^2T=0$$

Solution:

$X(x)=C\cos(kx)+D\sin(kx)$

Boundary conditions:

$y(0,t)=0$

$y(L,t)=0$

At $x=0$: $X(0)=C\cos 0+D\sin 0=0$

$\Rightarrow C=0$

At $x=L$: $X(L)=D\sin(kL)=0$

$\Rightarrow k=\frac{n\pi}{L}$ $n=1,2,3,\dots$

$T(t)$: $\frac{d^2T}{dt^2}+\left(\frac{n\pi}{L}v\right)^2T=0$

$T=A\cos\left(\frac{n\pi v}{L}t\right)+B\sin\left(\frac{n\pi v}{L}t\right)$

Complete solution:

$$y(x,t)=D\sin\left(\frac{n\pi}{L}x\right)\left[A\cos\left(\frac{n\pi vt}{L}\right)+B\sin\left(\frac{n\pi vt}{2}\right)\right]$$

using the superposition principle:

$$y(n,t) = \sum_{n=1}^{n} \sin\left(\frac{n\pi x}{L}\right)\left[A_n \cos\left(\frac{n\pi v}{L} t\right) + B_n \cos\left(\frac{n\pi v}{L} t\right)\right]$$

↳ General solution

Series Solutions of Differential Equations

2.1 The Harmonic oscillator

$$\frac{d^2 y}{dx^2} + y = 0$$

- We know that $y = A\cos x + B\sin x$ is a sol^n
  - ↳ trying a series sol^n

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$\frac{dy}{dx} = \sum_{n=0}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=0}^{\infty} a_n n(n-1) x^{n-2}$$

→ Inserting into H.O. eq^n

$$\sum_{n=0}^{\infty} a_n n(n-1) x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$\because = 0 \Rightarrow$ each correspondence of x must cancel each other

→ Replacing indicies in the first term

↳ i.e. $n = n' + 2 \Rightarrow n' = n - 2$

$$\sum_{n'=0}^{\infty} a_{n'+2} (n'+2)(n'+1) x^{n'} + \sum_{n=0}^{\infty} a_n x^n = 0$$

As n' is a dummy variable, we can call it anything, eg. n

$$\sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1) x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\left[\sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1) + a_n\right] x^n = 0$$

→ implying

$$a_{n+2}(n+2)(n+1) + a_n = 0$$

$\Rightarrow$ ~~[illegible]~~

$$a_{n+2} = -\frac{a_n}{(n+2)(n+1)}$$

$n = 0, 1, 2, 3$

☆ Recurrence relation

Dummy Index

$$\sum_{n=0}^{\infty} a_n (n(n-1)) x^{n-2} = 0$$

$$= 0 + 0 + 2a_2 + 6a_3 x + \dots$$

$$\sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n = 2a_2 + 6a_3 x + \dots$$

Terms from the recurrence relation :

$n=0$: $a_2 = \frac{-a_0}{2!}$ $\qquad$ $n=1$: $a_3 = \frac{-a_1}{3\times 2} = -\frac{a_1}{3!}$

$n=2$: $a_4 = \frac{-a_2}{(4)(3)} = \frac{+a_0}{(4!)}$ $\qquad$ $n=3$: $a_5 = \frac{-a_3}{(5)(4)} = \frac{+a_1}{5!}$

$n=4$: $a_6 = \frac{-a_4}{(6)(5)} = \frac{-a_0}{6!}$ $\qquad$ $n=5$: $a_7 = \frac{-a_5}{(7)(6)} = \frac{-a_1}{7!}$

General form:

$$a_{2m} = \frac{(-1)^m}{(2m)!} a_0 \quad [n \text{ is even}]$$

$$a_{2m+1} = \frac{(-1)^m}{(2m+1)!} a_1 \quad [n \text{ is odd}]$$

$$\Rightarrow y(x) = a_0\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots\right] \Rightarrow \cos(x)$$

$$+ a_1\left[x - \frac{-x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots\right] \Rightarrow \sin(x)$$

Taylor ~~expa~~ expansion

$$y(x) = a_0 \cos(x) + a_1 \sin(x)$$

But how do ~~we~~ we know these are linear independant sol^n

$\Rightarrow$ By using Wronskian test $\quad \begin{bmatrix} y_1 = \cos x & y_2 = \sin x \\ y_1' = -\sin x & y_2' = \cos x \end{bmatrix}$

$$\Rightarrow \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0 \Rightarrow y_1 y_2' - y_2 y_1' \neq 0$$

$$\cos^2 x + \sin^2 x = 1 \neq 0 \Rightarrow \text{linearly independant}$$

~~2.2 Fro~~

2.2 Frobenius's Method

For a general 2nd ODE:

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = 0$$

or

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0$$ [rewriting only]

$$p(x) = \frac{Q(x)}{P(x)}$$

$$q(x) = \frac{R(x)}{P(x)}$$

Method consists of:

1/ Choose a pt x_0, where series is centred

2/ Assume a sol^n of the form: $y(x) = \sum_{j=0}^{\infty} a_j (x-x_0)^{k+j}$ $(a_0 \neq 0)$

3./ Derive a condition => Indicial Eq^n for parameter k
=> solve it to determine possible values of k

4/ Insert expression for $y(x)$ & its differentials into original diff eq^n

5./ For each value of k, derive conditions on the coefficients a_j, and solve to completely determine the sol^n, $y(x)$

$a_0 \neq 0$ assumed ∵ can be imposed by defining parameter k

- sol^n in series form do not always exist
  ↳ even yes, may not converge for all values of x

- even if have sol^n in series form
↳ not always possible to analytically determine coefficients a_j

Indicial Eq^n

General form of the D.E.

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + q(x)y = 0$$

$$y(x) = \sum_{j=0}^{\infty} a_j (x-x_0)^{k+j}$$

For simplicity $x_0 = 0$

$$y(x) = \sum_{j=0}^{\infty} a_j x^{k+j}$$

$$y' = \sum_{j=0}^{\infty} a_j (k+j) x^{k+j-1}$$

$$y'' = \sum_{j=0}^{\infty} a_j (k+j)(k+j-1) x^{k+j-2}$$

$$\Rightarrow \sum_{j=0}^{\infty} a_j (k+j)(k+j-1) x^{k+j-2} + p(x)\sum_{j=0}^{\infty} a_j (k+j) x^{k+j-1} + q(x)\sum_{j=0}^{\infty} a_j x^{k+j} = 0$$

→ Multiply by: x^{2-k}:

$$\Rightarrow \sum_{j=0}^{\infty} a_j (k+j)(k+j-1) x^{j} + x\,p(x)\sum_{j=0}^{\infty} a_j (k+j) x^{j} + x^2 q(x)\sum_{j=0}^{\infty} a_j x^{j} = 0$$

→ For $j=0$, $x^j = 1$, $a_0 \neq 0$ ~~no a_0~~ → assumed $a_0 \neq 0$

$$k(k-1) + \underbrace{p_0}_{①} k + \underbrace{q_0}_{②} = 0$$ // indicial eq^n

[Quadratic eq^n in k, terms p_0 & q_0 are defined as:

$p_0 = \lim_{x\to 0} x\,p(x)$ – ①

$q_0 = \lim_{x\to 0} x^2 q(x)$ – ②

sol^n = values of k]

Special points & Fuch's Theorem

- Given the general 2nd order, linear, homogeneous diff eq^n

↳ we ~~have~~ can distinguish 3 kinds of points

1. Ordinary / Analytic points
   - where $p(x)$ & $q(x)$ are analytic
     ↳ $p(x)$ & $q(x)$ are finite
     ↳ expressed as:
     $p(x) = \sum_{j=0}^{\infty} p_j (x-x_0)^j$, $q(x) = \sum_{j=0}^{\infty} q_j (x-x_0)^j$

   ⇒ x_0 is an ordinary point

   IF both p & q diverge @ $x = x_0$
   ⇒ it is a singular pt

   eg. $y = \frac{1}{x^3}$ $(x=0)$; $y = \frac{1}{x-1}$ $(x=1)$; $y = \tan$ $(x = \frac{\pi}{2})$

2. Regular singular points
   where at least one of $p(x)$ or $q(x)$ is NOT analytic, BUT
   $\lim_{x\to x_0} (x-x_0)p(x)$ & $\lim_{x\to x_0} (x-x_0)^2 q(x)$ are both analytic

3. Essential singular points,
   at least one of the functions, $(x-x_0)p(x)$ & $(x-x_0)^2 q(x)$ is not analytic

Fuch's Theorem: given a linear, homogeneous diff eq^n of 2nd order, at least one series sol^n exist,

IF the expansion point is
- an ordinary point

or

- a regular singular point

2.4/ Example w/ singular point

$$\frac{d^2y}{dx^2} + \frac{1}{2x}\frac{dy}{dx} + \frac{1}{4x}y = 0$$

1/ $\Rightarrow p(x) = \frac{1}{2x}$; $q(x) = \frac{1}{4x}$

$\rightarrow$ Both are not analytic @ $x=0$

$\Rightarrow$ Rather than $y(x) = \sum_{n=0}^{\infty} a_n x^n$

2./ $\hookrightarrow$ we choose $y(x) = x^k \sum_{n=0}^{\infty} a_n x^n$ OR generally :

$$y(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^{n+k}$$

$$p_0 = \lim_{x \to x_0} p(x)(x-x_0) = \lim_{x\to 0} \frac{1}{2x}(x-0) = \frac{1}{2}$$

$$q_0 = \lim_{x\to x_0} q(x)(x-x_0)^2 = \lim_{x\to 0} \frac{1}{4x}(x^2) = 0$$

} Both analytic ✓

3./ Finding k using the indicial eq^n

$$\underline{k(k-1)+p_0k+q_0 = 0}$$

$$k^2 - k + p_0k + q_0 = 0$$

$$k^2 - k + \frac{1}{2}k + 0 = 0$$

$$k(k-\tfrac{1}{2}) = 0$$

$k = 0$ or $k = \frac{1}{2}$

4./ put $y = \sum_{n=0}^{\infty} a_n x^{n+k}$ into diff eq^n

$$\sum_{n=0}^{\infty} a_n(n+k)(n+k-1)x^{n+k-2} + \frac{1}{2x}\sum_{n=0}^{\infty} a_n(n+k)x^{n+k-1} + \frac{1}{4x}\sum_{n=0}^{\infty} a_n x^{n+k} = 0$$

$\times \frac{4x}{x^k}$

$$\Rightarrow \sum_{n=0}^{\infty}\left[4a_n(n+k)(n+k-1) + 2a_n(n+k)\right]x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

Changing dummy variable

$n' = n-1$, $n = n'+1$, dummy variable $n' \to n$

$$\sum_{n=-1}^{\infty}\left[4a_{n+1}(n+k+1)(n+k) + 2a_{n+1}(n+k+1)\right]x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$\because$ by comparing powers of x to R.H.S, x^{-1} has to go

$$\Rightarrow \sum_{n=0}^{\infty}\left[\overbrace{4a_{n+1}(n+k+1)(n+k) + 2a_{n+1}(n+k+1) + a_n}^{=0}\right]x^n = 0$$

$$a_{n+1} = \frac{-a_n}{2(n+k+1)(2(n+k)+1)}$$

$k_1 = 0$: $a_{n+1} = \dfrac{-a_n}{(2n+2)(2n+1)}$

$k_2 = \frac{1}{2}$: $a_{n+1} = \dfrac{-a_n}{(2n+3)(2n+2)}$

For $k=0$,

$$a_1 = \frac{-a_0}{2!}$$

$$a_2 = \frac{-a_1}{(4)(3)} = \frac{a_0}{4!}$$

$$a_3 = \frac{-a_3}{(6)(5)} = \frac{-a_0}{6!}$$

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+0}$$

$$= a_0 - \frac{a_0}{2!}x + \frac{a_0}{4!}x^2 - \frac{a_0}{6!}x^3 + \dots$$

$$= a_0 - \frac{a_0(\sqrt{x})^2}{2!} + \frac{a_0}{4}(\sqrt{x})^4 - \frac{a_0(\sqrt{x})^6}{6!} + \dots$$

$$= a_0 \cos(\sqrt{x})$$

For $k = \frac{1}{2}$

$$a_1 = \frac{-a_0}{3(2)} = -\frac{a_0}{3!}$$

$$a_2 = \frac{a_0}{5!}$$

$$a_3 = \frac{-a_0}{7!}$$

$$y(x) = \sqrt{x}\sum_{n=0}^{\infty} a_n x^n$$

$$y(x) = a_0\sqrt{x} - \frac{\sqrt{x}^3 a_0}{3!} + \frac{x^{\frac{5}{2}}a_0}{5!} - \frac{x^{\frac{7}{2}}a_0}{7!} + \dots$$

$$= a_0\left[\sqrt{x} - \frac{(\sqrt{x})^3}{3!} + \frac{(\sqrt{x})^5}{5!} - \frac{(\sqrt{x})^7}{7!} + \dots\right]$$

$$= a_0 \sin(\sqrt{x})$$

General sol^n: $y(x) = A\cos\sqrt{x} + B\sin\sqrt{x}$

(Put back & see if the sol^n works)

$\Rightarrow$ linearly independent

2.5 Another example:

$$x(x-1)\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + y = 0$$

$$\Rightarrow \frac{d^2y}{dx^2} + \frac{3}{x-1}\frac{dy}{dx} + \frac{1}{x(x-1)}y = 0$$

$$p(x) = \frac{3}{x-1} \qquad q(x) = \frac{1}{x(x-1)}$$

$p(x)$ is singular at $x=1$,
$q(x)$ is singular at $x=1$ or $x=0$

For $x=0$: $\quad p_0 = \lim_{x\to 0} \frac{3}{3x-1}\cdot x = 0$

$$q_0 = \lim_{x\to 0}\frac{1}{\cancel{x}(x-1)}\cdot x^{\cancel{2}} = \frac{x}{x-1} = 0$$

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+k}$$

$$y'(x) = \sum_{n=0}^{\infty} a_n(n+k)x^{n+k-1}$$

$$y''(x) = \sum_{n=0}^{\infty} a_n(n+k)(n+k-1)x^{n+k-2}$$

$$\Rightarrow \sum_{n=0}^{\infty} a_n(n+k)(n+k-1)x^{n+k-2} + \frac{3}{x-1}\sum_{n=0}^{\infty} a_n(n+k)x^{n+k-1}$$

$$+ \frac{1}{x(x-1)}\sum_{n=0}^{\infty} a_n x^{n+k} = 0 \quad — \quad (A)$$

$\Rightarrow \times\ x^{2-k}$

$$\hookrightarrow \sum_{n=0}^{\infty} a_n(n+k)(n+k-1)x^n + \frac{3x}{x-1}\sum_{n=0}^{\infty} a_n(n+k)x^n$$

$$+ \frac{x^2}{x(x-1)}\sum_{n=0}^{\infty} a_n x^n = 0$$

$\times (x-1)$

$$\Rightarrow \sum_{n=0}^{\infty} a_n x^n\left[(x-1)(n+k)(n+k-1) + 3x(n+k) + x\right] = 0$$

So set $x=0$, $\Rightarrow$ all terms with the exponent of $x>0$ vanishes

for $x=0$, $n=0$,

$$\left.\begin{aligned}&(-1)(k)(k-1) = 0\\ &\hookrightarrow k=0 \text{ or } k=1\end{aligned}\right\} - \text{indicial eq}^n$$

Finding recurrence relationship, $(A) \times [x(x-1)]$

$$\sum_{n=0}^{\infty} a_n(n+k)(n+k-1)(x^{n+k} - x^{n+k-1})$$

$$+ \sum_{n=0}^{\infty} 3a_n(n+k)x^{n+k} + \sum_{n=0}^{\infty} a_n x^{n+k} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} a_n x^{n+k}\left[(n+k)(n+k-1) + 3(n+k) + 1\right]$$

$$- \sum_{n=0}^{\infty} a_n(n+k)(n+k-1)(x^{n+k-1}) = 0$$

Changing dummy variables

~~$n' = n+1 \Rightarrow n'$~~

$n' = n-1 \Rightarrow n' \to n$

$$\Rightarrow \sum_{n=0}^{\infty} a_n x^{n+k}\left[(n+k)(n+k-1) + 3(n+k) + 1\right]$$

$$= \sum_{n=-1}^{\infty} a_{n+1}x^{n+k}(n+k+1)(n+k)$$

Equating powers of x:

$$\Rightarrow a_n(n+k+1)^2 = a_{n+1}(n+k+1)(n+k)$$

$$a_{n+1} = \frac{a_n(n+k+1)}{n+k}$$

For $k=1$:

$$a_{n+1} = a_n \frac{(n+2)}{n+1}$$

$$a_1 = a_0 \frac{2}{1} = 2a_0$$

$$a_2 = a_1 \cdot \frac{3}{2} = 3a_0$$

$$a_3 = a_2 \cdot \frac{4}{3} = 4a_0$$

$$\therefore y(x) = \sum_{n=0}^{\infty} a_n x^{n+k}$$

$$= a_0 x + 2a_0 x^2 + 3a_0 x^3 + 4a_0 x^4 + \dots$$

Putting $a_0 = 1$

$$\Rightarrow y(x) = x(1 + 2x + 3x^2 + 4x^3 + \dots)$$

$$y(x) = \frac{x}{(1-x)^2}$$

For $k=0$

$$a_{n+1} = a_n \left(\frac{n+1}{n}\right)$$

in this particular case, doesn't ~~work~~ work $\because \frac{n+1}{0}$ ✗

$$c_{n+1} = a_{n+1} x^{n+2} \; ; \; c_n = a_n x^{n+1}$$

$$= a_n \left(\frac{n+2}{n+1}\right) x^{n+2}$$

$$\Rightarrow S = \lim_{n\to\infty} \left| \frac{\cancel{a_n}(n+2) \cdot x}{\cancel{a_n}(n+1)} \right|$$

$$= \lim_{n\to\infty} |x|$$

$\Rightarrow$ convergence for $|x| < 1$

$\Rightarrow$ divergence for $|x| \geq 1$

D'Alembert Ratio Test:

↳ Given an infinite series of the form: $\sum_{j=0}^{\infty} c_j$

let s be defined as:

$$S = \lim_{j\to\infty} \left| \frac{c_{j+1}}{c_j} \right|$$

- $S < 1$: converging series
- $S = 1$: inconclusive
- $S > 1$: divergence

2.6/ Quantum Harmonic Oscillator

for a particle in a potential well, $V = \frac{1}{2}kx^2$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + \frac{1}{2}kx^2\psi(x) = E\psi(x)$$

- Substitutions. $y = \left(\frac{mk}{\hbar^2}\right)^{\frac{1}{4}} x$; $w = \sqrt{\frac{k}{m}}$; $\varepsilon = \frac{2E}{\hbar w}$

$$\Rightarrow \frac{d^2\psi(y)}{dy^2} - y^2\psi(y) = -\varepsilon\psi(y)$$

$\Rightarrow$ 1st solve $\frac{d^2\psi}{dy^2} - y^2\psi = 0$ //

sol^n: $\psi(y) = Ae^{-\frac{y^2}{2}} + \cancel{Be^{\frac{y^2}{2}}}$

$y \to \infty \Rightarrow \psi(y) \to 0$ $B = 0$

$\psi(y) = Ae^{-\frac{y^2}{2}}$

Assume full sol^n is

$\hookrightarrow \psi(y) = H(y)\exp\left(-\frac{y^2}{2}\right)$

$$\psi' = H'(y)\exp\left(-\frac{y^2}{2}\right) - y\psi$$

$$\psi'' = H''\exp\left(-\frac{y^2}{2}\right) - yH'\exp\left(-\frac{y^2}{2}\right) - \psi - yH'\exp\left(-\frac{y^2}{2}\right) + y^2\psi$$

$$\Rightarrow \frac{d^2\psi}{dx^2} - y^2\psi = \exp\left(-\frac{y^2}{2}\right)\left[\frac{d^2H}{dy^2} - 2y\frac{dH}{dy} - H\right] = -\varepsilon\psi$$

$\Rightarrow \frac{d^2H}{dy^2} - 2y\frac{dH}{dy} - H = -\varepsilon H$ (xH on both sides of)

$$\frac{d^2H}{dy^2} - 2y\frac{dH}{dy} + (\varepsilon - 1)H = 0$$

$\rightarrow$ no singular points, i.e. can obtain series sol^n expanding about $y = 0$

Assuming $H(y)=\sum_{n=0}^{\infty} a_n y^n$

$$\frac{dH}{dy}=\sum_{n=0}^{\infty} a_n n y^{n-1}$$

$$\frac{d^2H}{dy^2}=\sum_{n=0}^{\infty} a_n n(n-1) y^{n-2}$$

→ insert into diff eq^n

$$\sum_{n=0}^{\infty} a_n n(n-1) y^{n-2} - 2\sum_{n=0}^{\infty} a_n n y^n + (\varepsilon-1)\sum_{n=0}^{\infty} a_n y^n = 0$$

→ Combining terms & changing dummy variables in the first term

$$\sum_{n=0}^{\infty} a_{n+2} y^n (n+1)(n+2) + \sum_{n=0}^{\infty} (\varepsilon-1-2n) a_n y^n = 0$$

$$\Rightarrow a_{n+2}(n+1)(n+2) = -(\varepsilon-1-2n)a_n$$

$$a_{n+2} = \frac{2n+1-\varepsilon}{(n+1)(n+2)} a_n$$

Testing convergence:

$$\lim_{n\to\infty}\frac{a_{n+2}}{a_n} = \frac{2\alpha}{n^{\alpha}} = 0$$

Taking $e^{x^2}=\sum_{n=0}^{\infty}\frac{1}{n!}x^{2n}$

↳ set $2n=j$

$$\Rightarrow e^{x^2} = \sum_{j=0,2,4}^{\infty} \underbrace{\frac{1}{(j/2)!}x^j}_{c_j x^j}$$

$$\lim_{j\to\infty}\frac{c_{j+2}}{c_j} = \frac{(j/2)!}{\left(\frac{j+2}{2}\right)!} = \frac{1}{\frac{j+2}{2}} = \frac{2}{j+2} = \frac{1}{j/2+1} = \frac{2}{j}$$

which means our function goes like e^{y^2} and even w/ a damping term, $e^{-y^2/2}$, we still have $e^{y^2/2}$ and the series is not converging

缩短

We ~~must~~ need to truncate the series

↳ We turn series into a polynomial

+ require @ some point, a term is zero,

↳ ∴ series terminates → @ some pt, it stops

I.e. $a_{n+2} = \frac{(2n+1-\varepsilon)a_n}{(n+1)(n+2)} a_n = 0$

BUT $a_n \neq 0$

And $H(y)=\sum_{n=0}^{j} a_n y^n$

$\Rightarrow 2j+1-\varepsilon=0$

$\varepsilon = 2j+1$

~~Rec~~ Recalling $\varepsilon = \frac{2E}{\hbar\omega}$

$E=\hbar\omega(j+\frac{1}{2})$, $j=0,1,2,3\ldots$

Now writing down the polynomials

- Hermite polynomial

$$a_{n+2}=\frac{(2n+1-\varepsilon)a_n}{(n+1)(n+2)} = \frac{2(n-j)a_n}{(n+1)(n+2)}$$

$\varepsilon = 2j+1$

∵ a_{n+2} ∴ we can only equate even & odd parts separately for n & j

1. $j=0, n=0 \Rightarrow a_2 = \frac{2(0-0)}{2}a_0 = 0$

$a_4 = a_2 = 0$ etc

2. $j=1$, $n=1$ $\Rightarrow a_3 = \frac{2(0)}{(2)(3)}a_1 = 0$

$a_5 = 0$, etc

⇒

$$a_{n+2} = \frac{2(n-j)\,a_n}{(n+1)(n+2)}$$

3/ $j=2$, $n=0$

$$a_2 = \frac{2(0-2)a_0}{(1)(2)} = -2a_0$$

$a_4 = 0$, etc...

4/ $j=3, n=1 \Rightarrow a_3 = \frac{2(1-3)a_1}{(2)(3)} = -\frac{2}{3}a_1$

$a_5 = 0$, etc..

For convention on normalisation is that the highest term in a_j has a value 2^j.

$H_0(y)$: $a_0 = \underline{1}$ $[a_0 = 2^0 = 1]$

$H_1(y)$: $a_1 y' = 2y$ $[a_1 = 2^1 = 2]$

$H_2(y) = a_0 - 2a_0 y^2 =$

$\hookrightarrow y^2 \Rightarrow -2a_0 = 4$

$a_0 = -2$

$= -2 + 4y^2$

$H_3(y) = a_1 \boxed{-\frac{2}{3}a_1} y^3$

$2^3 \Rightarrow -\frac{2}{3}a_1 = 8$

$a_1 = -12$

$= -12y + 8y^3$

Back to the Schrödinger eq^n

$$\psi(y) = H(y)e^{-y^2/2}$$

$$\psi_0(x) = A.\,1.\,e^{-x^2/2\left(\frac{mk}{\hbar^2}\right)^{\frac{1}{2}}}$$

$$= Ae^{-x^2/2\left(\frac{m^2w^2}{\hbar^2}\right)^{\frac{1}{2}}}$$

$$= Ae^{-x^2/2\,(mw/\hbar)}$$

For $\psi_0(x)$

Use $H_0(y) = 1$

For normalisation:

$$\int_{-\infty}^{\infty} |\psi_0|^2 dx = 1$$

$$\int_{-\infty}^{\infty} e^{-\frac{mw}{\hbar}x^2} dx = 1 \quad \Leftarrow \text{gaussian}$$

$$A^2\sqrt{\frac{\pi\hbar}{mw}} = 1$$

$$\underline{A = \left(\frac{mw}{\pi\hbar}\right)^{\frac{1}{4}}}$$

$$\Rightarrow \psi_0(x) = \left(\frac{mw}{\pi\hbar}\right)^{\frac{1}{4}} e^{-mwx^2/2\hbar}$$

→ Corresponds to energy level

$E_0 = \frac{1}{2}\hbar w$

2.6/ The radial part of Laplace's eq^n

We had: $\frac{\sin^2\theta}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2}=0$

And $\frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2}=-m^2$

$\Rightarrow \underbrace{\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)}_{=\lambda}=\underbrace{\frac{m^2}{\sin^2\theta}-\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)}_{=\lambda}$

$\Rightarrow r^2\frac{d^2R}{dr^2}+2r\frac{dR}{dr}-\lambda R=0$

$\div r^2$

$\Rightarrow \frac{d^2R}{dr^2}+\frac{2}{r}\frac{dR}{dr}-\frac{\lambda R}{r^2}=0$

$p(r)=\frac{2}{r}$; $q(r)=-\frac{\lambda}{r^2}$

Singularity around $r=0$

Checking if can use Frobenius's method

$p_0=\lim_{r\to0}\frac{2}{r}\cdot r=2$

$q_0=\lim_{r\to0}\frac{-\lambda}{r^2}\cdot r^2=-\lambda$

$\Rightarrow$ regular singular pt. $r_0=0$ is a regular singular pt

Assuming $R(r)=\sum_{n=0}^{\infty}a_n r^{n+k}$

Indicial eq^n

$k(k-1)+p_0k+q_0=0$

$k^2+k-\lambda=0$

$k=\frac{-1\pm\sqrt{1+4\lambda}}{2}$

As before: $\underline{\underline{\lambda\equiv \ell(\ell+1)}}$

$k_1=\ell$, $k_2=-\ell-1$

$\frac{dR}{dr}=\sum_{n=0}^{\infty}a_n(n+k)r^{n+k-1}$

$\frac{d^2R}{dr^2}=\sum_{n=0}^{\infty}(n+k)(n+k-1)r^{n+k-2}$

Put into:

$r^2\frac{d^2R}{dr^2}+2r\frac{dR}{dr}-\lambda R=0$

$\rightarrow$ gives

$\sum_{n=0}^{\infty}a_n(n+k)(n+k-1)r^{n+k}+2\sum_{n=0}^{\infty}a_n(n+k)r^{n+k}-\lambda\sum_{n=0}^{\infty}a_nr^{n+k}=0$

$\Rightarrow\sum_{n=0}^{\infty}a_nr^{n+k}\left[(n+k)(n+k-1)+2(n+k)-\lambda\right]=0$

$a_0\neq0$

For $n=0$:

$a_0\underbrace{[k(k-1)+2k-\lambda]}_{\text{indicial eq}^n}=0$

For all other n:

$a_n[(n+k)(n+k-1)+(n+k)-\lambda]=0$

$\Rightarrow a_n=0$ $(n=1,2,3\ldots)$

$\Rightarrow$

Given that $a_0 \neq 0$, $a_n = 0$

$k_1 = \ell$, $k_2 = -\ell - 1$

$$R(r) = \underbrace{C_\ell r^\ell}_{a_0 \text{ term}} + \underbrace{D/r^{\ell+1}}_{a_0 \text{ term}}$$

a_0 term

$r^{n+k} = r^\ell$

a_0 term

$r^{0-\ell-1}$

Polar Eq^n:

$$\frac{1}{\Theta \sin\theta} \frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right) - \frac{m^2}{\sin^2\theta} = -\lambda$$

[Legendre equation] $= -\ell(\ell+1)$

need use Legendre polynomial

coming next.

One other example on series sol^n

$$2x^2 y'' - x y' + (1+x) y = 0$$

$$y'' - \frac{1}{2x} y' + \frac{1+x}{2x^2} y = 0$$

$p_0 = -\frac{1}{2}$

$q_0 = \frac{1}{2}$

$$k(k-1) - \frac{1}{2}k + \frac{1}{2} = 0$$

$$k^2 - k - \frac{1}{2}k + \frac{1}{2} = 0$$

$$k^2 - \frac{3}{2}k + \frac{1}{2} = 0$$

$$2k^2 - 3k + 1 = 0$$

$$(2k-1)(k-1) = 0$$

$k = \frac{1}{2}$ or $k = 1$

testing for convergence

$$S = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \frac{x^{n+1}}{x^n} \right| =$$

$$= |x| \lim_{n\to\infty} \left| \frac{\cancel{3 \cdot 5 \cdot 7 \cdot (2n+1)} n!}{[\cancel{3\times5\times7\cdots(2n+1)}(2n+2+1)(n+1)!} \right|$$

$$= |x| \lim_{n\to\infty} \left| \frac{\cancel{n!}}{(2n+3)(n+1)\cancel{n!}} \right| = 0$$

∴ series converges for all x

∴ general sol^n is $a_0 = 1$

$$y_1(x) = x\left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{(3\times5\times7\cdots(2n+1))n!}\right]$$

for the series sol^n

$$2\sum_{n=0}^{\infty} a_n (n+k-1)(n+k) x^{n+k} - \sum_{n=0}^{\infty} a_n x^{n+k}(n+k) + \sum_{n=0}^{\infty} a_n x^{n+k} + \underbrace{\sum_{n=0}^{\infty} a_n x^{n+k+1}}_{} = 0$$

$$\sum_{n=0}^{\infty} a_{n-1} x^{n+k}$$

$$a_n\left[2(n+k-1)(n+k) - (n+k) + 1\right] + a_{n-1} = 0 \qquad n \geq 1$$

$k = 1$

$$a_n = \frac{-a_{n-1}}{2(n+1)(n) - n \cancel{-1} + 1}$$

$$a_n = \frac{-a_{n-1}}{n(2(n+1) - 1)} = \frac{-a_{n-1}}{n(2n+2-1)}$$

$$a_n = \frac{-a_{n-1}}{n(2n+1)}$$

$$a_1 = \frac{-a_0}{3}$$

$$a_2 = \frac{-a_1}{2(5)} = \frac{a_0}{5\times3\times2}$$

$$a_3 = \frac{-a_2}{3(7)} = -\frac{a_0}{7\times5\times3\times3\times2\times1} \Rightarrow a_n = \frac{(-1)^n}{(3\cdot5\cdot7\cdot(2n+1))n!} =$$

for $k = \frac{1}{2}$

$$a_n = \frac{-a_{n-1}}{2(n+\frac{1}{2})(n-\frac{1}{2}) - n + \frac{1}{2}}$$

$$= \frac{-a_{n-1}}{2(n^2-\frac{1}{4}) - n + \frac{1}{2}}$$

$$= \frac{-a_{n-1}}{2n^2 - \frac{1}{2} - n + \frac{1}{2}} = \frac{-a_{n-1}}{n(2n-1)}$$

$$a_n = \frac{-a_{n-1}}{n(2n-1)}$$

$$a_1 = \frac{-a_0}{1(1)}$$

$$a_2 = \frac{-a_1}{2(3)} = \frac{a_0}{3\times2\times1}$$

$$a_3 = \frac{-a_2}{3(5)} = \frac{-a_0}{5\times3\times3\times2\times1\times1}$$

$$a_4 = \frac{-a_3}{4(7)} = \frac{a_0}{7\times5\times3\times1\times4\times3\times2\times1}$$

$$a_n = \frac{(-1)^n a_0}{n!\,(3\times5\times7\cdots(2n-1))} \qquad n \geq 1$$

$$y_2(x) = x^{\frac{1}{2}}\left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{[3\times5\times7\cdots(2n-1)]\,n!}\right]$$

3. Legendre polynomials

3.1 Laplace's Eq^n in spherical Coordinates
3.2 E-field from a sphere of radius R
3.3 Generating function
3.4 Orthogonality
3.5 Expansion of a function in terms of Legendre polynomials
3.6 Steric Spherical Harmonics
3.7. More Egs.

3.1 Laplace's Eq^n in spherical Coordinates

$$\frac{d^2\Phi}{d\phi^2} = -m^2\phi \quad [\text{Azimuthal Eq}^n]$$

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = \lambda \quad [\text{Radial Eq}^n]$$

$$\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) - \frac{m^2}{\sin^2\theta} = -\lambda \quad [\text{Polar eq}^n]$$
$$= -\ell(\ell+1)$$

Rewrite in singular form

$x = \cos\theta$; $\Theta(\theta) = y(x)$

$\sin^2\theta = 1 - \cos^2\theta = 1 - x^2$

$$\frac{dx}{d\theta} = -\sin\theta = -\sqrt{1-x^2}$$

$$\frac{d\Theta}{d\theta} = \frac{d\Theta}{dx}\frac{dx}{d\theta} = -\sin\theta\frac{d\Theta}{dx} = -\sqrt{1-x^2}\frac{dy}{dx}$$

$$\hookrightarrow \frac{d}{d\theta} = -\sqrt{1-x^2}\frac{d}{dx}$$

$$\frac{d^2\Theta}{d\theta^2} = \frac{d}{d\theta}\left[\frac{d\Theta}{d\theta}\right] = -\sqrt{1-x^2}\frac{d}{dx}\left(-\sqrt{1-x^2}\frac{dy}{dx}\right)$$
$$= -\sqrt{1-x^2}\left(-\frac{-2x}{2\sqrt{1-x^2}}\frac{dy}{dx} - \sqrt{1-x^2}\frac{d^2y}{dx^2}\right)$$
$$= -x\frac{dy}{dx} + (1-x^2)\frac{d^2y}{dx^2}$$

---

using the polar eq^n ($\times\Theta\sin\theta$)

$$\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \left[\ell(\ell+1)\sin\theta - \frac{m^2}{\sin\theta}\right]\Theta = 0$$

$$\cos\theta\frac{d\Theta}{d\theta} + \sin\theta\frac{d^2\Theta}{d\theta^2} + \left[\ell(\ell+1)\sin\theta - \frac{m^2}{\sin\theta}\right)\Theta = 0$$

Replacing $\cos\theta$ & $\sin\theta$

$$x(-\sqrt{1-x^2})\frac{dy}{dx} + (\sqrt{1-x^2})\frac{d^2\Theta}{d\theta^2}\left[\ell(\ell+1)\sqrt{1-x^2} - \frac{m^2}{\sqrt{1-x^2}}\right]\Theta = 0$$

replacing Θ, Θ', Θ'' :

$$-x\sqrt{1-x^2}\,y' + \sqrt{1-x^2}\left(-xy' + (1-x^2)y''\right)\left[\ell(\ell+1)\sqrt{1-x^2} - \frac{m^2}{\sqrt{1-x^2}}\right]y = 0$$

$\times \frac{1}{\sqrt{1-x^2}}$:

$$-xy' + (-xy' + (1-x^2)y'') + \left(l(l+1) - \frac{m^2}{1-x^2}\right)y = 0$$

$$\underbrace{(1-x^2)y'' - 2xy' + \left(l(l+1) - \frac{m^2}{1-x^2}\right)y = 0}$$

* first solving for $m=0$, i.e. no ϕ dependence $\Rightarrow$ having cylindrical symmetry

$$(1-x^2)y'' - 2xy' + (l(l+1))y = 0 \qquad \Big| \; l(l+1) = \lambda$$

$$y'' - \frac{2xy'}{1-x^2} + \frac{\lambda}{1-x^2}y = 0 \qquad \Big| \; \div(1-x^2)$$

$$p(x) = \frac{-2x}{1-x^2} \qquad q(x) = \frac{\lambda}{1-x^2}$$

$x = \pm 1$ is a singular pt

$x_0 = +1$

$$p_0(x) = \lim_{x\to 1} \frac{-2x}{(1-x)(1+x)}(x-1) = \lim_{x\to 1}\frac{2x}{1+x} = \frac{2}{2} = 1$$

$$q_0(x) = \lim_{x\to 1}\frac{\lambda}{(1-x)(x+1)}(x-1)^2 = \lim_{x\to 1}\frac{-\lambda(x-1)}{1+x} = 0$$

$x_0 = -1$

$$p_0(x) = \lim_{x\to -1} -\frac{2x}{(1-x)(1+x)}(1+x) = \lim_{x\to -1}\frac{-2x}{(1-x)} = 1$$

$$q_0(x) = \lim_{x\to -1} - \frac{\lambda(1+x)^{\cancel{2}}}{(1-x)\cancel{(1+x)}} = 0$$

$\to$ We can find a series sol^n

We will expand around $x=0$ as this is an ordinary pt between the singular points

* quick check

$$p_0 = \lim_{x\to 0}\frac{-2x}{1-x^2}\cdot x = 0$$

$$q_0 = \lim_{x\to 0}\frac{\lambda}{1-x^2}x^2 = 0$$

$\hookrightarrow$ Indicial Eq^n

$$k(k-1) = 0$$

$$k = 0 \text{ or } k = 1$$

* Assume $y(x) = \sum_{n=0}^{\infty} a_n x^n$

$$y' = \sum_{n=0}^{\infty} a_n n x^{n-1}$$

$$y'' = \sum_{n=0}^{\infty} a_n n(n-1)x^{n-2}$$

$$\hookrightarrow \sum_{n=0}^{\infty} a_n(n)(n-1)x^{n-2} - \sum_{n=0}^{\infty} a_n(n)(n-1)x^n - 2\sum_{n=0}^{\infty} a_n(n)x^n + l(l+1)\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^{n-1} = \sum_{n=0}^{\infty} a_n x^n\left[(n)(n+1) + 2n - l(l+1)\right]$$

$$a_{n+2} = \frac{n(n+1) - \ell(\ell+1)}{(n+1)(n+2)} a_n$$

Eg. $a_0 = 1$

$$\Rightarrow a_2 = \frac{-\ell(\ell+1)}{2}$$

$$a_4 = \frac{6-\ell(\ell+1)}{(3)(4)} \cdot \frac{-\ell(\ell+1)}{2} = \frac{\ell(\ell+1)(\ell-2)(\ell+3)}{4!}$$

$$a_3 = \frac{2-\ell(\ell+1)}{3\times 2} a_1 = \frac{-(\ell+1)(\ell+2)}{3!}$$

$$a_5 = \frac{12-\ell(\ell+1)}{5\times 4} \cdot \frac{-(\ell+2)(\ell+1)}{3!} = \frac{(\ell-1)(\ell+2)(\ell-3)(\ell+4)}{5!}$$

$$y_1(x) = 1 - \frac{\ell(\ell+1)}{2!}x^2 + \frac{\ell(\ell+1)(\ell-2)(\ell+3)}{4!}x^4$$

$$y_2(x) = x - \frac{(\ell+2)(\ell+1)}{3!}x^3 + x^5\frac{(\ell-1)(\ell+2)(\ell-3)(\ell+4)}{5!} + \dots$$

Convergence:

$$S = \lim_{n\to\infty}\left|\frac{a_{n+2}}{a_n}\frac{x^{n+2}}{x^n}\right| = \lim_{n\to\infty}\left|\frac{n(n+1)-\ell(\ell+1)}{(n+2)(n+1)}x^2\right| = |x^2| \Rightarrow \text{converges for } |x|<1$$

But series diverges for $|x|>1$

Recalling $x=\cos\theta$

$|x|<1$ means $0<\theta<\pi$

But $x=1 \Rightarrow \theta = 0$ or π

→ series diverges for $x=1$, ∴ we have to terminate the series, i.e. for some n recurrence relation yields $a_{n+2}=0$

↳ that all higher terms also vanish

& we have a set of polynomials

$$a_{n+2} = \frac{(n(n+1)) - \ell(\ell+1)}{(n+1)(n+2)} a_n = 0$$

$a_n \neq 0$

$\Rightarrow n(n+1) = \ell(\ell+1)$

$n=\ell$

↳ $y^{(\ell)}(x) = \sum_{n=0}^{\ell} a_n^{(\ell)} x^n = P_\ell(x)$

First terms:

$\ell=0$, $n=0$: $a_1 \neq 0$, $a_2 = \frac{0-0}{2}a_0 = 0$

$\Rightarrow P_0(x) = a_0 = 1$ [just setting up for convention that $P_\ell(1)=1$

$\ell=1$, $n=1$: $a_1 \neq 0$, $a_3 = 0$

$P_1(x) = a_1(x) = x$

$\ell=2$, $n=0$: $a_2 = \frac{-2(3)}{1\cdot 2}a_0 = -3a_0$

$\ell=2$, $n=2$: $a_4 = 0$

∴ $P_2(x) = a_0(1-3x^2)$

Normalisation $P_2(1) = a_0(1-3) = 1$

$a_0 = -\frac{1}{2}$

∴ $P_2(x) = \frac{1}{2}(3x^2-1)$

$\ell=3$, $n=1$: $a_3 = \frac{2-12}{2(3)}a_1 = -\frac{5}{3}a_1$

Normalisation $P_3(1) = a_1(1-\frac{5}{3}) = 1$

$a_1 = -\frac{3}{2}$

$n=3$: $a_5 = 0$

$P_3(x) = a_1(x - \frac{5}{3}x^3)$

$P_3(x) = \frac{1}{2}(5x^3 - 3x)$

3.2 Electric field from a sphere of radius R

=> looking for a sol^n for Laplace's eq^n, $\nabla^2 V = 0$, in 3D spherical coordinates

Consider the boundary condition

$$V(r=R,\theta,\phi) = V_0\cos^2\theta$$

$$V(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{\infty}\left(C_l r^l + \frac{D_l}{r^{l+1}}\right)P_l(\cos\theta)A_m e^{-im\phi}$$

$P_0(x)=1$	$P_0(\theta)=1$	$x=\cos\theta$
$P_1(x)=x$	$P_1(\theta)=\cos\theta$	
$P_2(x)=\frac{1}{2}(3x^2-1)$	$P_2(\theta)=\frac{1}{2}(3\cos^2\theta-1)$	

* We have cylindrical symmetry, i.e. no ϕ dependence
* $V \to 0$ as $r\to\infty$ => $C_l r^l$ vanishes

↳ $$\boxed{V(r,\theta) = \sum_{l=0}^{\infty}\frac{D_l}{r^{l+1}}P_l(\cos\theta)}$$ – Multipole expansion

But At @ $r=R$

$$V(R,\theta) = \frac{D_0}{R}P_0\cos\theta + \frac{D_1}{R^2}P_1\cos\theta + \frac{D_2}{R^3}P_2\cos\theta + \dots$$

$$= V_0\cos^2\theta$$

Equating terms: + putting in L. Polynomials

$$\frac{D_0}{R} + \frac{D_1}{R^2}\cos\theta + \frac{D_2}{R^3}\frac{1}{2}(3\cos^2\theta - 1) = V_0\cos^2\theta$$

1. $\frac{D_0}{R} - \frac{D_2}{R^3}\cdot\frac{1}{2} = 0$
2. $\frac{D_1}{R^2} = 0 \Rightarrow D_1 = 0$
3. $\frac{D_2}{R^3}\frac{3}{2} = V_0$

$\Rightarrow D_2 = \frac{2}{3}V_0R^3$

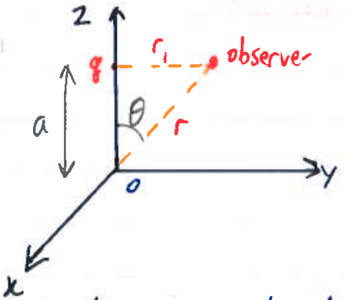
$\Rightarrow D_0 = \frac{D_2}{2R^2} = \frac{1}{3}V_0R$

$$V(r,\theta) = \frac{D_0}{r}P_0 + \frac{D_2}{r^3}P_2$$

$$= \frac{V_0R}{3r}(1) + \frac{\frac{2}{3}V_0R^3}{r^3}\cdot\frac{1}{2}(3\cos^2\theta - 1)$$

$$= \frac{V_0R}{3r}\left[1 + \frac{R^2}{r^2}(3\cos^2\theta - 1)\right]$$

3.3 Generating function



- considering a single charge q located on the z-axis @ a distance from O

Electrostatic potential, V, @ a distance r_1 is $V = \frac{q}{4\pi\varepsilon_0}\frac{1}{r_1}$

re-write in terms of r, θ

$$r_1^2 = r^2 + a^2 - 2ra\cos\theta$$

↳ $$V(r,\theta) = \frac{q}{4\pi\varepsilon_0}\frac{1}{\sqrt{r^2+a^2-2ra\cos\theta}}$$

Comparing to the multipole expansion:

$$V(r,\theta) = \frac{q}{4\pi\varepsilon_0}\frac{1}{\sqrt{r^2+a^2-2ra\cos\theta}} = \sum_{l=0}^{\infty}\frac{D_l}{r^{l+1}}\cdot P_l(\cos\theta)$$

Holds for all r & θ, so take $\theta = 0$

$$V(r,0) = \sum_{l=0}^{\infty}\frac{D_l}{r^{l+1}}P_l(1) = \sum_{l=0}^{\infty}\frac{1}{r}\frac{D_l}{r^l}$$

$$\text{Also } V(r,0) = \frac{q}{4\pi\varepsilon}\cdot\frac{1}{\sqrt{r^2+a^2-2ra}} = \frac{q}{4\pi\varepsilon_0}\frac{1}{|r-a|}$$

Assume $r \gg a$

$$V(r,0) = \frac{q}{4\pi\varepsilon}\frac{1}{r}\cdot\frac{1}{|1-\frac{a}{r}|} = \frac{q}{4\pi\varepsilon}\frac{1}{r}\sum_{l=0}^{\infty}\left(\frac{a}{r}\right)^l$$

Recalling Taylor expansion:

$$\frac{1}{1-x} = \sum_{l=0}^{\infty}x^l \quad \text{for } |x|<1$$

Now we have: $\frac{1}{r}\sum_{l=0}^{\infty}\frac{D_l}{r^l} = \frac{q}{4\pi\varepsilon_0}\frac{1}{r}\sum_{l=0}^{\infty}\left(\frac{a}{r}\right)^l$ for $r>a$

$$D_l = \frac{q}{4\pi\varepsilon_0}a^l$$

Putting $D_\ell = \frac{q}{4\pi\varepsilon_0} a^\ell$ into our original expression

$$V(r,\theta) = \frac{q}{4\pi\varepsilon_0} \underbrace{\frac{1}{\sqrt{r^2+a^2-2ra\cos\theta}}} = \sum_{\ell=0}^{\infty} \frac{q}{4\pi\varepsilon_0} \underbrace{\frac{a^\ell}{r^{\ell+1}} P_\ell \cos\theta}$$

$$\hookrightarrow \frac{1}{\sqrt{r^2+a^2-2ra\cos\theta}} = \frac{a^\ell}{r^{\ell+1}} P_\ell \cos\theta$$

$$= \frac{1}{r\sqrt{1+\frac{a^2}{r^2}-\frac{2}{r}a\cos\theta}} = \frac{1}{r}\sum_{\ell=0}^{\infty}\left(\frac{a}{r}\right)^\ell P_\ell \cos\theta$$

Set $\mu = \cos\theta$; $t = \frac{a}{r}$

$$\frac{1}{\sqrt{1+t^2-2t\mu}} = \sum_{\ell=0}^{\infty} (t)^\ell P_\ell(\mu) \quad |t|<1$$

$\therefore$ the function

$$g(t,\mu) = \frac{1}{\sqrt{1+t^2-2t\mu}}$$

is the generating function of the Legendre polynomials

Using the general form of the binomial expression

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots$$

$$n = -\tfrac{1}{2} \,;\; x = t^2 - 2t\mu$$

$$g(t) = 1 - \tfrac{1}{2}(t^2-2t\mu) + \frac{(-\frac{1}{2})(-\frac{3}{2})(t^2-2t\mu)^2}{2!} + \dots$$

$$= 1 - \frac{t^2}{2} + t\mu + \frac{3}{8}(t^4 + 4t^2\mu^2 - 4t^3\mu)$$

$$= \underbrace{1}_{P_0(\mu)=1} + \underbrace{\mu}_{P_1(\mu)=\mu} t + \underbrace{\tfrac{1}{2}(3\mu^2-1)}_{P_2(\mu)=\frac{1}{2}(3\mu^2-1)} t^2 + O(t^3)$$

$$g(t,\mu) = \sum_{\ell=0}^{\infty} P_\ell(\mu) t^\ell$$

Electric Multipoles:

$$V(r,\theta) = \frac{q}{4\pi\varepsilon_0 r}\sum_{\ell=0}^{\infty}\left(\frac{a}{r}\right)^\ell P_\ell \cos\theta$$

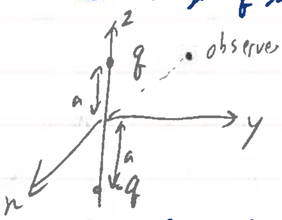
$\Rightarrow$ leading term: $\frac{q}{4\pi\varepsilon_0 r}$

which can also be seen from

$$V(r,\theta) = \frac{q}{4\pi\varepsilon_0}\frac{1}{\sqrt{r^2+a^2-2ar\cos\theta}}$$

as $r \to \infty$: $V(r,\theta) = \frac{q}{4\pi\varepsilon_0 r}$

Consider charge $-q$ at $z=-a$ also:



$$V = \frac{+q}{4\pi\varepsilon_0 r}\sum_{\ell=0}^{\infty}\left(\frac{a}{r}\right)^\ell P_\ell\cos\theta + \frac{-q}{4\pi\varepsilon_0 r}\sum_{\ell=0}^{\infty}\left(\frac{-a}{r}\right)^\ell P_\ell\cos\theta$$

$$= \frac{2q}{4\pi\varepsilon_0 r}\left[0 + P_1\cos\theta\,\frac{a}{r} + 0 + P_3\cos\theta\left(\frac{a}{r}\right)^3 + \dots\right]$$

$$= \frac{2q}{4\pi\varepsilon_0 r}\cdot\left[\cos\theta\cdot\frac{a}{r}\right] \rightarrow \text{the leading term}$$

$$V = \frac{2qa}{4\pi\varepsilon_0 r^2}\cos\theta \quad [\text{dipole term}]$$

3.4 Orthogonality & normalizations

We now have the Legendre eqⁿ:

$$(1-x^2)y'' - 2xy' + \lambda y = 0$$

Polynomial solutions are of: $y(x) = P_\ell(x)$

$$(1-x^2)P''_\ell(x) - 2xP'_\ell(x) + \ell(\ell+1)P_\ell(x) = 0 \quad ①$$

Rewriting another label: $\ell \to m$

$$(1-x^2)P''_m(x) - 2xP'_m(x) + m(m+1)P_m(x) = 0 \quad ②$$

① × P_m & integrate

$$\int_{-1}^{1}(1-x^2)P''_\ell P_m\,dx - \int_{-1}^{1} 2xP'_\ell P_m\,dx + \ell(\ell+1)\int_{-1}^{1} P_\ell P_m\,dx = 0$$

② × P_ℓ & integrate

$$\int_{-1}^{1}(1-x^2)P''_m P_\ell\,dx - \int_{-1}^{1} 2xP'_m P_\ell\,dx + m(m+1)\int_{-1}^{1} P_m P_\ell\,dx = 0$$

Integrating first term by parts

$u = (1-x^2)P_m$ $\quad v' = P''_\ell$

$u' = (1-x^2)P'_m - 2xP_m$ $\quad v = P'_\ell$

$$\int_{-1}^{1}(1-x^2)P''_\ell P_m\,dx = \cancel{\left[(1-x^2)P_mP'_\ell\right]_{-1}^{1}} + \int_{-1}^{1} 2xP_mP'_\ell\,dx - \int_{-1}^{1}(1-x^2)P'_\ell P'_m\,dx$$

* 2nd Term cancels w/ the 2nd term of the original expression from the full eqⁿ

$$\Rightarrow -\int_{-1}^{1}(1-x^2)P'_\ell P'_m\,dx + \ell(\ell+1)\int_{-1}^{1} P_\ell P_m\,dx = 0$$

and likewise

$$-\int_{-1}^{1}(1-x^2)P'_m P'_\ell\,dx + m(m+1)\int_{-1}^{1} P_m P_\ell\,dx = 0$$

Subtracting:

$$[\ell(\ell+1) - m(m+1)]\int_{-1}^{1} P_\ell(x)P_m(x)\,dx = 0$$

If $\ell = m$, L.H.S $= 0$

If $\ell \neq m$:

$$\int_{-1}^{1} P_\ell(x)P_m(x)\,dx = 0 \qquad = \delta_{\ell m}$$

Definition of orthogonality

For $\ell = m$ case:

$$P_\ell(x)\cdot P_\ell(x) = \int_{-1}^{1} P_\ell(x)P_\ell(x)\,dx$$

$$= \frac{2}{2\ell+1} \quad \Rightarrow 点嚟?$$

Using the generating function

$$\sum_{n=0}^{\infty} P_n(x)t^n = \frac{1}{\sqrt{1-2xt+t^2}}$$

$$\sum_{m=0}^{\infty} P_m(x)t^m = \frac{1}{\sqrt{1-2xt+t^2}}$$

Multiply:

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} P_n(x)P_m(x)t^{n+m} = \frac{1}{1-2xt+t^2}$$

Integrate:

$$\int_{-1}^{1}\frac{dx}{1-2xt+t^2} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\int_{-1}^{1} P_n(x)P_m(x)t^{n+m}\,dx$$

$$\text{L.H.S.} = -\frac{1}{2t}\left[\ln(1-2xt+t^2)\right]_{-1}^{1}$$

$$= -\frac{1}{2t}\left[\ln\left[(1-t)^2\right] - \ln\left[(1+t)^2\right]\right]$$

$$= \frac{1}{t}\ln\left(\frac{1+t}{1-t}\right)$$

Using Taylor expansions:

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$$

$$\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}$$

$$\Rightarrow \ln(1+x) - \ln(1-x) = x - \cancel{\frac{x^2}{2}} + \frac{x^3}{3} - \cancel{\frac{x^4}{4}} + \dots$$
$$+x + \cancel{\frac{x^2}{2}} + \frac{x^3}{3} + \cancel{\frac{x^4}{4}} + \dots$$
$$= 2\sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1}$$

$$\Rightarrow L.H.S = \frac{2}{t}\sum_{n=0}^{\infty} \frac{t^{2n+1}}{2n+1} = 2\sum_{n=0}^{\infty} \frac{t^{2n}}{2n+1}$$

$\Rightarrow R.H.S \Rightarrow$ integral is zero if $n \neq m$
$\hookrightarrow$ non zero if $n = m$

$$\cancel{\frac{1}{t}} \sum_{n=0}^{\infty} t^{2n} \int_{-1} P_n(x) P_n(x)\,dx$$

L.H.S = R.H.S

$$2\sum_{n=0}^{\infty} \frac{t^{2n}}{2n+1} = \sum_{n=0}^{\infty} t^{2n} \int_{-1}^{1} P_n(x) P_n(x)\,dx$$

Comparing coefficients t^{2n}

$$\int_{-1}^{1} P_n(x) P_n(x)\,dx = \frac{2}{2n+1}$$

$\Rightarrow$ We can write:

$$\int_{-1}^{1} P_\ell(x) P_n(x)\,dx = \frac{2}{2\ell+1} \cdot \delta_{\ell n}$$

$$\delta_{\ell n} = \begin{cases} 1 & \text{for } \ell = n \\ 0 & \text{for } \ell \neq n \end{cases}$$

$$\frac{5}{2}x^3 - \frac{3}{2}x$$

$$\frac{5}{2}\left(x^3 \cancel{+} - \frac{3}{5}x\right)$$

$$\frac{1}{2}\left(5x^3 - 3x\right)$$

3.5 Expansion of a function in terms of Legendre Polynomials

For any reasonable, continuous function $f(x)$ in the interval $-1<x<1$

$$f(x) = \sum_{\ell=0}^{\infty} C_\ell P_\ell(x)$$

To find C_ℓ:

~~Mult~~ Multiply by $P_m(x)$ & integrate:

$$\int_{-1}^{1} f(x) P_m(x)\,dx = \sum_{\ell=0}^{\infty} C_\ell \int_{-1}^{1} P_\ell(x) P_m(x)\,dx$$

$$= \sum_{\ell=0}^{\infty} C_\ell \, \delta_{\ell m} \frac{2}{2m+1} \qquad \delta_{\ell m} \left[\begin{array}{l} \ell = m \Rightarrow 1 \\ \ell \neq m \Rightarrow 0 \end{array} \right.$$

[can only hold for $\ell = m$

$$= C_m \cdot \frac{2}{2m+1}$$

$$\Rightarrow \boxed{C_m = \frac{2m+1}{2} \int_{-1}^{1} f(x) P_m(x)\,dx}$$

eg. $e^{\alpha x}$

$$C_0 = \frac{1}{2}\int_{-1}^{1} e^{\alpha x} P_0(x)\,dx$$
$$= \frac{1}{2}\int_{-1}^{1} e^{\alpha x}\,dx$$
$$= \frac{1}{2}\left[\frac{e^{\alpha x}}{\alpha}\right]_{-1}^{1}$$
$$= \frac{1}{2\alpha}\left(e^{\alpha} - e^{-\alpha}\right)$$
$$= \frac{1}{\alpha}\sinh\alpha$$

$$C_1 = \frac{3}{2}\int_{-1}^{1} e^{\alpha x} P_1(x)\,dx$$

$u = x \qquad v' = e^{\alpha x}$
$u' = 1 \qquad v = \frac{e^{\alpha x}}{\alpha}$

$$= \frac{3}{2}\int_{-1}^{1} e^{\alpha x}\, x\,dx$$
$$= \frac{3}{2}\left[\frac{x}{\alpha}e^{\alpha x}\right]_{-1}^{1} - \frac{3}{2}\int_{-1}^{1} \frac{1}{\alpha}e^{\alpha x}\,dx$$
$$= \frac{3}{2}\left[e^{\alpha} + e^{-\alpha}\right] - \frac{3}{2\alpha}\left[\frac{e^{\alpha x}}{\alpha}\right]_{-1}^{1}$$
$$= \frac{3}{\alpha}\left(\cosh\alpha - \frac{\sinh\alpha}{\alpha}\right)$$

3.6 / Spherical harmonics

Well-behaved ("physical") sol^n of Legendre's equation are possible if:

- l is a non-negative integer
- m is an integer w/ $-l \le m \le +l$

for $m>0$, P_l^m can be derived from P_l using:

$$P_l^m(x) = (1-x^2)^{m/2} \frac{d^m}{dx^m} P_l(x)$$

The orthogonality relation:

$$\int_{-1}^{1} P_l^m(x) P_n^m(x)\, dx = \frac{2}{2l+1} \frac{(l+m)!}{(l-m)!} \delta_{ln}$$

Relationship between $P_n^m(x)$ & $P_n^{-m}(x)$ is:

$$P_n^{-m}(x) = (-1)^m \frac{(n-m)!}{(n+m)!} P_n^m(x)$$

The orthogonality relation when m is different:

$$\int_{-1}^{1} P_n^m(x) P_n^k(x) \frac{dx}{1-x^2} = \frac{(n+m)!}{m(n-m)!} \delta_{mk}$$

First terms:

$l=1$, $m=1$: $P_1^1(x) = (1-x^2)^{\frac{1}{2}} \frac{d}{dx} x$

$= (1-x^2)^{\frac{1}{2}}$ | $x = \cos\theta$

$= \sin\theta$

$l=2$, $m=1$: $P_2^1(x) = (1-x^2)^{\frac{1}{2}} \frac{d}{dx}\left[\frac{1}{2}(3x^2-1)\right]$

$= (1-x^2)^{\frac{1}{2}} \cdot 3x$

$= 3\cos\theta\sin\theta$

$l=2$, $m=2$: $P_2^2(x) = (1-x^2)^1 \frac{d^2}{dx^2}\left[\frac{1}{2}(3x^2-1)\right]$

$= (1-x^2)\cdot 3 = 3\sin^2\theta$

Checking orthogonality

* $\int_{-1}^{1} P_2^1(x) P_1^1(x)\, dx = \int_{-1}^{1} 3x(1-x^2)^{\frac{1}{2}}(1-x^2)^{\frac{1}{2}} dx$

$= [3x - 3x^3]_{-1}^{1} dx$

$= 0$ => should get 0 ∴ not the same

* $\int_{-1}^{1} P_2^1(x)\cdot P_2^1(x)\, dx = \int_{-1}^{1} 9x^2(1-x^2)\, dx$

$= \left[3x^3 - \frac{9}{5}x^5\right]_{-1}^{1}$

$= 2\left[3 - \frac{9}{5}\right]$

$= \frac{12}{5}$

agrees

* $\int_{-1}^{1} P_l^m(x) P_n^m(x)\, dx = \frac{2}{2l+1} \frac{(l+m)!}{(l-m)!} \delta_{ln}$

=> $\int_{-1}^{1} P_2^1(x) P_2^1(x)\, dx = \frac{2}{5} \cdot \frac{3!}{1!} \cdot 1$ ($l=2$, $m=1$)

$= \frac{12}{5}$

In quantum mechanics, we write spherical harmonics $Y_l^m(\theta,\phi)$ as

$$Y_l^m(\theta,\phi) = c_{l,m} P_l^m(\cos\theta)\, e^{im\phi}$$

↳ Θ is a sol^n of the Legendre eq^n

By convention: Y_l^m is normalized as:

$$\int_{-1}^{1} d(\cos\theta) \int_0^{2\pi} d\phi\, Y_l^{m*}(\theta,\phi) Y_{l'}^{m'}(\theta,\phi) = \delta_{ll'}\delta_{mm'}$$

$$\therefore c_{l,m} = (-1)^m \sqrt{\frac{(l-m)!\,(2l+1)}{4\pi\,(l+m)!}}$$

$$\therefore Y_0^0(\theta,\phi) = \frac{1}{\sqrt{4\pi}}$$

$$Y_1^1(\theta,\phi) = -\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\phi}$$

$$Y_1^0(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^{-1}(\theta,\phi) = \sqrt{\frac{3}{8\pi}}\sin\theta e^{-i\phi}$$

Eg 1

A function $f(x) = 1 + 3x^2 + 4x^4$ is defined in the range $x = -1$ to 1

We can write the function as

$$f(x) = \sum_{n=0}^{\infty} C_n P_n(x)$$

Q Obtain the values for the coefficients C_n with $n \leq 3$

Eg 2 Spherical Harmonics are given by:

$$Y_\ell^m(\theta,\phi) = C_{\ell,m} P_\ell^m(\cos\theta) e^{im\phi}$$

where $C_{\ell,m} = (-1)^m \sqrt{\frac{(\ell-m)!\,(2\ell+1)}{4\pi\,(\ell+m)!}}$

& the associated legendre function, P_ℓ^m, can be derived from the legendre Polynomials using:

$$P_\ell^m(x) = (1-x^2)^{m/2} \frac{d^m}{dx^m} P_\ell(x)$$

Q Derive $Y_2^1(\theta,\phi)$ & $Y_2^0(\theta,\phi)$ using the above

4/ Lagrangian & Hamiltonian mechanics

4.1 The Euler-Lagrange eq^n
4.2 Lagrangian Mechanics
4.3 Hamiltonian mechanics
4.4 Extra examples.

4.1/ Euler-Lagrange eq^n

Consider the integral:

$$J = \int_{x_1}^{x_2} f(y, y', x)\, dx$$

But
→ dependence of y on x is not fixed.
→ Chose a path through (x_1, y_1) & (x_2, y_2) to minimize J

Assume: existence of a stationary path & look for arbitrary deformations around it.
→ described by $\eta(x)$ & a scale factor α to give the magnitude of variation

Impose: at $\eta(x_1) = \eta(x_2) = 0$

↳ $y(x,\alpha) = y(x,0) + \alpha\eta(x)$

$\delta y = y(x,\alpha) - y(x,0) = \alpha\eta(x)$

→ Choose $y(x,\alpha=0)$ as the unknown path that will minimize J

$$J(\alpha) = \int_{x_1}^{x_2} f[y(x,\alpha), y'(x,\alpha), x]\, dx$$

$$\left.\frac{\partial J}{\partial \alpha}\right|_{\alpha=0} = 0$$

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y}\frac{\partial y}{\partial \alpha} + \frac{\partial f}{\partial y'}\frac{\partial y'}{\partial \alpha}\right] dx$$

$$\frac{\partial y}{\partial \alpha} = \eta(x) \qquad \frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} \frac{\partial f}{\partial y}\eta(x) + \frac{\partial f}{\partial y'}\eta'(x)\, dx$$

$$\frac{\partial y'}{\partial \alpha} = \eta'(x)$$

Integrating 2nd ~~half~~ term by parts:

$$\int_{x_1}^{x_2} \frac{\partial f}{\partial y'}\frac{d\eta}{dx}\, dx = \left[\eta(x)\frac{\partial f}{\partial y'}\right]_{x_1}^{x_2} - \int_{x_1}^{x_2} \eta(x)\frac{d}{dx}\frac{\partial f}{\partial y'}\, dx$$

First term $= 0$ ∵ $\eta(x_1) = \eta(x_2) = 0$

For this to be true for arbitrary $\eta(x)$

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'}\right]\eta(x)\, dx = 0$$

$$\frac{\partial f}{\partial y} = \frac{d}{dx}\frac{\partial f}{\partial y'}$$] - Euler-Lagrange eq^n

* when $\frac{\partial f}{\partial y} = 0$, $\frac{\partial f}{\partial y'} = \text{constant}$

* $\frac{\partial f}{\partial x} - \frac{d}{dx}\left(f - y'\frac{\partial f}{\partial y'}\right) = 0$

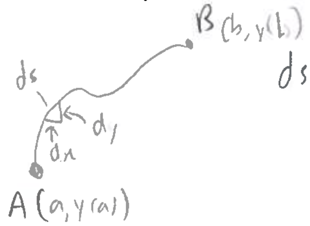
if $f = f(y, y')$

↳ $\frac{d}{dx}\left(f - y'\frac{\partial f}{\partial y'}\right) = 0$ - ~~Baltram~~

$f - y'\frac{\partial f}{\partial y'} = \text{constant}$

↳ Baltrami identity

Eg A straight line: determine the shortest distance between two pts in the x-y plane



$$ds = \sqrt{dx^2 + (dy)^2} = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}} dx$$

The total path length along the curve:

$$L=\int_{a,y(a)}^{b,y(b)} ds=\int_a^b (1+y'^2)^{\frac{1}{2}}dx$$

Using the Euler-Lagrange Eqn

$$\frac{\partial f}{\partial y}-\frac{d}{dx}\frac{\partial f}{\partial y'}=0$$

where $f(y,y',x)=(1+y'^2)^{\frac{1}{2}}$

$$\frac{\partial f}{\partial y}=0 \quad \frac{\partial f}{\partial y'}=\frac{y'}{(1+y'^2)^{\frac{1}{2}}}$$

$$\Rightarrow \frac{d}{dx}\frac{\partial f}{\partial y'}=\frac{d}{dx}\left(\frac{y'}{(1+y'^2)^{\frac{1}{2}}}\right)=0$$

$$\frac{y'}{(1+y'^2)^{\frac{1}{2}}}=C$$

$$\frac{y'^2}{1+y'^2}=C^2$$

$$y'=\sqrt{\frac{C^2}{1-C^2}}$$

$\Rightarrow y'(x)$ is a constant, say m

Integrate:

$$y(x)=mx+b$$

4.2 Lagrangian mechanics

Several dependent variables:

Original integral eqn is modified to:

$$J=\int_{x_1}^{x_2} f(y_1,y_1',y_2,y_2',\cdots,y_n,y_n',x)\,dx$$

where each of y_n, y_n' depend on x.

→ leads to a set of E.L. eqn

$$\frac{\partial f}{\partial y_i}-\frac{d}{dx}\left(\frac{\partial f}{\partial y_i'}\right)=0 \quad i=1,2,3,\dots n$$

Several independent variables

↳ dependency on x, y, z, rather than just x for n independent variables

$$J=\int\int\cdots\int f\left(y,\frac{\partial y}{\partial x_1},\frac{\partial y}{\partial x_2},\dots\frac{\partial y}{\partial x_n},x_1,x_2,\dots,x_n\right) dx_1 dx_2 \dots dx_n$$

$$\frac{\partial f}{\partial y}=\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial y_{x_i}}\right)$$

where $y_{x_i}=\frac{\partial y}{\partial x_i}$

∴ for 3-D, we have $u(x,y,z)$

$$\frac{\partial f}{\partial u}-\frac{\partial}{\partial x}\frac{\partial f}{\partial u_x}-\frac{\partial}{\partial y}\frac{\partial f}{\partial u_y}-\frac{\partial}{\partial z}\frac{\partial f}{\partial u_z}=0$$

To generalize further to more than 1 dependent & more than one independent variable:

Eg.

$$f=f(p(x,y,z),p_x,p_y,p_z,q(x,y,z),q_x,q_y,q_z,r(x,y,z),r_x,r_y,r_z,x,y,z)$$

We then have:

$$\frac{\partial f}{\partial p}-\frac{\partial}{\partial x}\frac{\partial f}{\partial p_x}-\frac{\partial}{\partial y}\frac{\partial f}{\partial p_y}-\frac{\partial}{\partial z}\frac{\partial f}{\partial p_z}=0$$

& similarly for q & r.

The **Lagrangian** is defined to be the difference between the K.E.s & P.Es

$$T \qquad V$$

$$L \equiv T - V$$

The physical system is defined in terms of
- coordinates $x_i(t)$
- velocity $\dot{x}_i(t) = \frac{dx_i}{dt}$

for a particle i as a function of time (t).

Hamilton's principle states that

in moving from one configuration @ time t_0 to another @ t_1,

the motion of such a system is a such to make

$$\mathcal{L} = \int_{t_0}^{t_1} L(x_1, x_2, \ldots, x_n, \dot{x}_1, \dot{x}_2, \ldots, \dot{x}_n, t)\, dt$$

E.L. eq^n: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i} = 0$

Application: A moving particle & Newton's 2nd Law

Lagrangian: $L = T - V$

moving particle has $T = \frac{1}{2} m\dot{x}^2$ → no x dependence

P.E., $V(x)$ as usual, the force is given by the -ve gradient of the potential

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0 \qquad F(x) = \frac{-dV(x)}{dx}$$

$$\frac{d}{dt}(m\dot{x}) - \frac{\partial}{\partial x}(T - V) = 0$$

$$m\ddot{x} - F(x) = 0$$

$F = ma$ ⇒ Newton's 2nd Law

4.3 Hamiltonian Mechanics

Hamiltonian formulation describes a system in terms of generalized coordinates (q_i)
- generalized momentum (p_i)

$L = L(q, \dot{q})$ - L w/ spatial & velocity parts

↳ $dL = \sum_i \frac{\partial L}{\partial q_i} dq_i + \sum_i \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i$

Defining generating momentum

$$p_i = \frac{\partial L}{\partial \dot{q}_i}$$

from E.L

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i}$$

$$\frac{d}{dt} p_i = \frac{\partial L}{\partial q_i}$$

$$\dot{p}_i = \frac{\partial L}{\partial q_i}$$

↳ ∴ $dL = \sum_i \dot{p}_i dq_i + \sum_i p_i d\dot{q}_i$

Note that

$$d(p_i \dot{q}_i) = \sum_i \dot{q}_i dp_i + \sum_i p_i d\dot{q}_i$$

⇒ $dL = \sum_i \dot{p}_i dq_i + d(p_i\dot{q}_i) - \sum_i \dot{q}_i dp_i$

Rearranging:

$$d(p_i\dot{q}_i - L) = -\sum_i \dot{p}_i dq_i + \sum_i \dot{q}_i dp_i$$

Hamiltonian, defined as:

$$H = \sum_i p_i \dot{q}_i - L$$

$$dH = -\sum_i \dot{p}_i dq_i + \sum_i \dot{q}_i dp_i$$

~~dH/dt~~

$$H = \sum_i p_i \dot{q}_i - L$$

$$dH = -\sum_i \dot{p}_i dq_i + \sum_i \dot{q}_i dp_i$$

$$dH = \sum_i \frac{\partial H}{\partial q_i} dq_i + \sum_i \frac{\partial H}{\partial p_i} dp_i$$

$$\Rightarrow -\dot{p}_i = \frac{\partial H}{\partial q_i} dq_i \;\&\; \dot{q}_i = \frac{\partial H}{\partial p_i}$$

Also $\frac{\partial H}{\partial t} = \frac{-\partial L}{\partial t}$ Hamilton's eq^n of motion

We can also write H in terms of the KE & PE

- $T = \sum_i \frac{1}{2} p_i \dot{q}_i$

- $H = \sum_i p_i \dot{q}_i - L$

$= \sum_i p_i \dot{q}_i - (T - V)$

$= \sum_i p_i \dot{q}_i - \frac{1}{2} \sum p_i \dot{q}_i + V$

$= \frac{1}{2} \sum_i p_i \dot{q} + V = T + V$

$H = T + V$

Example w/ the Harmonic Oscillator

eg a mass oscillating back & forth in 1-D on a spring

$KE = T = \frac{1}{2} mv^2 = \frac{1}{2} m\dot{x}^2$

$P.E. = V = \frac{1}{2} kx^2$ | k is the spring constant

$L = T - V$

$= \frac{1}{2} m\dot{x}^2 - \frac{1}{2} kx^2$

Using the E.L. ~~Lagrangian~~ eq^n

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

$$\frac{d}{dt}(m\dot{x}) + kx = 0$$

$$m\ddot{x} + kx = 0$$

$$\ddot{x} = -\frac{k}{m} x = -w^2 x \qquad \Big|\; w = \sqrt{\frac{k}{m}}$$

Or using the Hamiltonian

$$H = \frac{p^2}{2m} + \frac{1}{2} kx^2$$

using Hamilton's eq^n of motion

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \Rightarrow \dot{x} = \frac{p}{m}$$

$$\frac{d\dot{x}}{dt} = \frac{\dot{p}}{m} \quad (1)$$

$$-\dot{p}_i = \frac{\partial H}{\partial q_i} \Rightarrow -\dot{p} = \frac{\partial H}{\partial x} = kx$$

$$\dot{p} = -kx \quad (2)$$

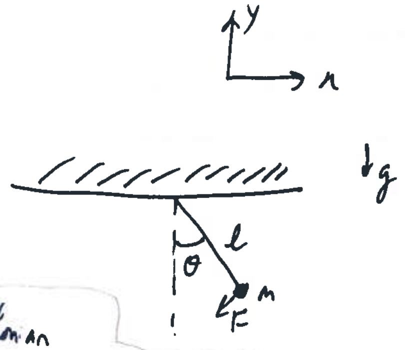
Combining (1) & (2)

$$\ddot{x} = \frac{\dot{p}}{m} = \frac{-kx}{m}$$

$$\ddot{x} = -\frac{k}{m} x = -w^2 x \qquad w = \sqrt{\frac{k}{m}}$$

4.4/ Extra Examples:

Eg 1. Simple Pendulum



Newtonian

$ma = -mg\sin\theta$

$a = -g\sin\theta$

The displacement from the vertical is an arc length s, & so the acceleration is $\ddot{s}$

$\ddot{s} = -g\sin\theta$

BUT $s = l\theta$

$\ddot{s} = l\ddot{\theta}$

$\Rightarrow l\ddot{\theta} = -g\sin\theta$

$\ddot{\theta} = -\frac{g}{l}\sin\theta$

Lagrangian approach:

Position $x = l\sin\theta$, $y = -l\cos\theta$

Velocity $\dot{x} = l\dot{\theta}\cos\theta$, $\dot{y} = l\dot{\theta}\sin\theta$

$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$

$= \frac{1}{2}ml^2\dot{\theta}^2$

→ Defining the potential energy: gravitational P.E, 0 when $\theta = 0$

$V = mgl(1-\cos\theta)$

The Lagrangian:

$L = T - V$

$= \frac{1}{2}ml^2\dot{\theta}^2 - mgl(1-\cos\theta)$

$\frac{\partial L}{\partial \theta} = -mgl\sin\theta$; $\frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta}$

$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = ml^2\ddot{\theta}$

Using E.L.

$\Rightarrow ml^2\ddot{\theta} + mgl\sin\theta = 0$

$\ddot{\theta} + \frac{g}{l}\sin\theta = 0$

Hamiltonian Approach

$H = T + V$

$p = \frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta} \Rightarrow \dot{\theta} = \frac{p}{ml^2}$

$\ddot{\theta} = \frac{\dot{p}}{ml^2}$ — (A)

$H = \frac{1}{2}ml^2\dot{\theta}^2 + mgl(1-\cos\theta)$

$= \frac{1}{2}ml^2\frac{p^2}{m^2l^4} + mgl(1-\cos\theta)$

$= \frac{1}{2}\frac{p^2}{ml^2} + mgl(1-\cos\theta)$

$-\dot{p} = \frac{\partial H}{\partial \theta} = mgl\sin\theta$

$\dot{p} = -mgl\sin\theta$ — (B)

Combining (A) & (B)

$\ddot{\theta} = \frac{\dot{p}}{ml^2} = \frac{-mgl\sin\theta}{ml^2}$

$\ddot{\theta} = -\frac{g}{l}\sin\theta$

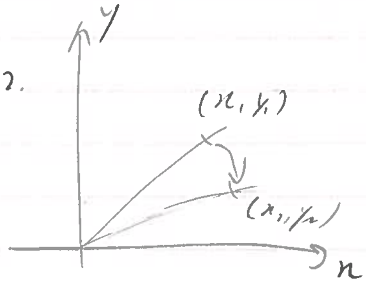
SAME RESULT

Eg 2 : Soap film

Consider a surface of revolution generated by revolving a curve $y(x)$ about x axis.
The curve passes through fixed end pts (x_1, y_1) & (x_2, y_2).

Find the curve such that the area of the surface is minimum.

let the curve be ~~$y(x)$~~ $x_i(t_i)$, $y_i(t_i)$ $i = 1, 2$.
$x_1(t_1)$



$$dS_x = 2\pi y(x)\, ds$$

$$ds = \sqrt{(dx)^2 + (dy)^2}$$

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$dS_x = 2\pi\, y(x)\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$x \in [x(t_1), x(t_2)]$
$y \in [y(t_1), y(t_2)]$
$t \in [t_1, t_2]$

$$S_x = 2\pi \int_{x_1, y_1}^{x_2, y_2} \underbrace{y(x)\sqrt{1 + y'^2}}\, dx$$

$$\delta S_x = 0 \qquad f = f(y, y', x) = y\sqrt{1 + y'^2}$$

E.L. $$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0$$

$$\frac{\partial f}{\partial y} = \sqrt{1 + y'^2}$$

$$\frac{\partial f}{\partial y'} = \frac{\partial}{\partial y'}\left(y(x)\,(1 + y'^2)^{\frac{1}{2}}\right)$$

$$= \cancel{\frac{\partial}{\partial y}}\, y(x)\, y'(1 + y'^2)^{-\frac{1}{2}}$$

$$\frac{d}{dx}\left[y(x)\, y'(1 + y'^2)^{-\frac{1}{2}}\right] = 0$$

$$\sqrt{1 + y'^2} - \frac{d}{dx}\left[\frac{y\, y'}{\sqrt{1 + y'^2}}\right] = 0$$

$$\frac{d}{dx}\left(\frac{yy'}{\sqrt{1 + y'^2}}\right) = \sqrt{1 + y'^2}$$

$$\frac{d}{dx}\left[f - y'\frac{\partial f}{\partial y'}\right] = 0$$

$$y\sqrt{1 + y'^2} - y'\frac{\cancel{y}y'}{\sqrt{1 + y'^2}} = a$$

$$y(1 + y'^2 - y'^2) = a\sqrt{1 + y'^2}$$

$$\frac{y^2}{a^2} = 1 + y'^2$$

$$y'^2 = \frac{y^2}{a^2} - 1$$

$$y' = \sqrt{\frac{y^2 - a^2}{a^2}}$$

$$\int \frac{dx}{dy} = \int \frac{a\, dy}{\sqrt{y^2 - a^2}}$$

$$x = a \operatorname{arccosh}\left(\frac{y}{a}\right) + C$$

$$\frac{x - C}{a} = \operatorname{arccosh}\frac{y}{a}$$

$$y = a \cdot \cosh\left(\frac{x - C}{a}\right)$$

a & c can be determined by:

$$y_1 = a\cosh\left(\frac{x_1 - c}{a}\right)$$

$$y_2 = a\cosh\left(\frac{x_2 - c}{a}\right)$$

↳ cannot be solved analytically

$\cancel{y = x^{\beta}}$

$\cancel{y' = \beta x^{\beta-1}}$

$\cancel{y'' = \beta(\beta-1)x^{\beta-2}}$

$\cancel{y = \frac{1}{x}}$

$\cancel{y' = x\ln(x+1)}$

$\cancel{y'' = \frac{x}{x+1}}$

$y = \ln$

$y = \ln(x+1)$

$y' = \frac{x}{x+1}$

$y'' = -\frac{x}{(x+1)^2}$

$$\int_{0,1}^{1,1} \left(\frac{c_1}{1+x}\right)^2 (1+x)\, dx$$

$=$

Fourier Series

Orthogonality of sine & cosine

Period of $2L$, $L = \pi$

Consider:

$\cos(n\pi)$, $\sin(n\pi)$ $\quad n \in \mathbb{N}$

$$f(x) \cdot g(x) = \int_{-\pi}^{\pi} f(x)\, g(x)\, dx$$

$$\sin(\alpha)\sin(\beta) = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)] \quad - (1)$$

$$\cos(\alpha)\cos(\beta) = \frac{1}{2}[\cos(\alpha-\beta) + \cos(\alpha+\beta)] \quad - (2)$$

$$\sin(\alpha)\cos(\beta) = \frac{1}{2}[\sin(\alpha-\beta) + \sin(\alpha+\beta)] \quad - (3)$$

Evaluating the scalar product between two functions

using (1) $\quad m \neq n$

$$\int_{-\pi}^{\pi} \sin(m)\sin(n) = \frac{1}{2}\int_{-\pi}^{\pi} \cos(m-n)x - \cos(m+n)x \, dx$$

$$= \frac{1}{2}\left[\frac{\sin(m-n)x}{m-n} - \frac{\sin(m+n)x}{m+n}\right]_{-\pi}^{\pi} = 0$$

for $m = n$

$$\Rightarrow \frac{1}{2}\int_{-\pi}^{\pi} 1 - \cos(2nx)\, dx$$

$$= \frac{1}{2}\Big[x - \underbrace{\frac{\sin(2nx)}{2n}}_{=0}\Big]_{-\pi}^{\pi}$$

$$= \frac{1}{2} 2\pi$$

$$= \underline{\pi}$$

for $\cos 0x$ i.e. $(n = m = 0)$

$$\int_{-\pi}^{\pi} \cos^2(0x)\, dx = \int_{-\pi}^{\pi} dx = 2\pi$$

$$\int_{-\pi}^{\pi} \sin(mx)\sin(nx)\, dx = \pi\,\delta_{mn} \quad \text{for } \max(m,n) > 0$$

$$\int_{-\pi}^{\pi} \sin(mx)\sin(nx)\, dx = 0 \quad m = n = 0$$

$$\int_{-\pi}^{\pi} \cos(mx)\cos(nx)\, dx = \pi\,\delta_{mn} \quad \text{for } \max(m,n) > 0$$

$$\int_{-\pi}^{\pi} \cos(mx)\cos(nx)\, dx = 2\pi \quad \text{for } m = n = 0$$

$$\int_{-\pi}^{\pi} \sin(mx)\cos(nx)\, dx = 0$$

Orthogonality of the set of periodic functions $\sin(nx)$ and $\cos(nx)$

Fourier series expansion (Period 2π)

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos(nx) + b_n\sin(nx)\right]$$

with coefficient:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\cos(nx)\, f(x)\, dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\sin(nx)\, f(x)\, dx$$

Multiply definition of $a_n \times \frac{1}{\pi}\cos(mx)$

⇒ integrate both sides $-\pi \to +\pi$

$$\frac{1}{\pi}\int_{-\pi}^{\pi}\cos(mx) f(x)\, dx = \frac{1}{\pi}\int_{-\pi}^{\pi}\cos(mx)\frac{a_0}{2}\, dx$$

$$+ \frac{1}{\pi}\sum_{n=1}^{\infty}\left[\int_{-\pi}^{\pi} a_n\cos(mx)\cos(nx)\, dx + b_n\int_{-\pi}^{\pi}\cos(mx)\sin(nx)\, dx\right]$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi}\cos(mx) f(x)\, dx = \delta_{m0}\, a_0 + \sum_{n=1}^{\infty}\delta_{mn}\, a_n = a_m$$

similar for def of b_n

$$\frac{1}{\pi}\int_{-\pi}^{\pi}\sin(mx) f(x)\, dx = b_m$$

Generic period, $2L$

$$x \to \frac{xL}{\pi}, \quad f\left(\frac{xL}{\pi}\right)$$

$$f\left[\frac{(x+2\pi)L}{\pi}\right] = f\left(\frac{xL}{\pi}\right)$$

$$f\left(\frac{xL}{\pi}\right) = \frac{a_0}{2} + \sum_{n=1}\left[a_n\cos(nx) + b_n\sin(nx)\right]$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\cos(nx)\, f(x)\, dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\sin(nx)\, f\left(\frac{xL}{\pi}\right) dx$$

multiply x by $\frac{\pi}{L}$

→ definition of Fourier series w/ period $2L$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos\left(n\frac{\pi}{L}x\right) + b_n\sin\left(n\frac{\pi}{L}x\right)\right]$$

$$a_n = \frac{1}{L}\int_{-L}^{L}\cos\left(n\frac{\pi}{L}x\right) f(x)\, dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L}\sin\left(n\frac{\pi}{L}x\right) f(x)\, dx$$

Scalar product $\int_{-L}^{L} f(x)\, g(x)\, dx$

w/ orthogonality set $\sin\left(n\frac{\pi}{L}x\right)$, $\cos\left(n\frac{\pi}{L}x\right)$

Writing again the orthogonality relationships:

$$\int_{-L}^{L}\sin\left(m\frac{\pi}{L}x\right)\sin\left(n\frac{\pi}{L}x\right) dx = L\delta_{mn} \qquad \max(m,n) > 0$$

$$\int_{-L}^{L}\sin\left(m\frac{\pi}{L}x\right)\sin\left(n\frac{\pi}{L}x\right) dx = 0 \qquad m = n = 0$$

$$\int_{-L}^{L}\cos\left(m\frac{\pi}{L}x\right)\cos\left(n\frac{\pi}{L}x\right) dx = L\delta_{mn} \qquad \max(m,n) > 0$$

$$\int_{-L}^{L}\cos\left(m\frac{\pi}{L}x\right)\cos\left(n\frac{\pi}{L}x\right) dx = 2L \qquad m = n = 0$$

$$\int_{-L}^{L}\sin\left(m\frac{\pi}{L}x\right)\cos\left(n\frac{\pi}{L}x\right) dx = 0$$

even function $f(x) = f(-x)$ } b_n vanish ∵ they would multiply $\sin(nx)$

odd function $f(x) = -f(x)$ } a_n vanish ∵ they would multiply $\cos(nx)$

Parseval's Identity

$\underline{u} = u_1 \underline{i} + u_2 \underline{j}$ $\underline{i}\cdot\underline{i} = 1$

$\underline{i}\cdot\underline{j} = 0$

$|\underline{u}|^2 = u_1^2 + u_2^2$ $\underline{j}\cdot\underline{j} = 1$

Parseval's Identity will be analogous to the above case

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(n\frac{\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right)\right]$$

$$\Rightarrow \frac{1}{2L}\int_{-L}^{L}[f(x)]^2\,dx = \left(\frac{a_0}{2}\right)^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

Proof

$$= \frac{1}{2L}\int_{-L}^{L}\left(\frac{a_0}{2} + \sum_{m=1}^{\infty}\left[a_m \cos\left(\frac{m\pi}{L}x\right) + b_m \sin\left(\frac{m\pi}{L}x\right)\right]\right)$$

$$\left(\frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(n\frac{\pi}{L}x\right)\right)dx\right]$$

$\downarrow$ using orthogonality relationship.

$$= \left(\frac{a_0}{2}\right)^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

[see extra notes]

$\cancel{f(x) = \left(\frac{a_0}{2}\right)^2 + \ldots}$

$$\langle f^2(x)\rangle = \frac{1}{2L}\int_{-L}^{+L}[f(x)]^2\,dx$$

$$= \left(\frac{a_0}{2}\right)^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

Complex Fourier series

using Euler's Formula:

$$\cos(\alpha) = \frac{e^{i\alpha} + e^{-i\alpha}}{2}$$

$$\sin(\alpha) = \frac{e^{i\alpha} - e^{-i\alpha}}{2}$$

$$c_n := \frac{(a_n - ib_n)}{2}$$

Complex Fourier Series

$$f(x) = \sum_{n=-\infty}^{+\infty} c_n e^{in\frac{\pi}{L}x}$$

with c_n given by:

$$c_n = \frac{1}{2L}\int_{-L}^{L} e^{-in\frac{\pi}{L}x} f(x)\,dx$$

$$f(x)\cdot g(x) = \int_{-L}^{L} f^*(x)\, g(x)\,dx$$

"Hilbert-Schmidt" scalar product

$$\left.\int_{-L}^{L} e^{im\frac{\pi}{L}x} e^{in\frac{\pi}{L}x}\,dx = 2L\,\delta_{mn}\right] - \text{orthogonality}$$

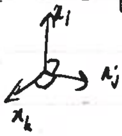
Parseval's Identity:

$$\frac{1}{2L}\int_{-L}^{L}|f(x)|^2\,dx = \sum_{n=-\infty}^{+\infty}|c_n|^2$$

* forming orthonormal set for identical function

orthonormal / orthogonal set

(can be function, vector)



$(x_i, x_j) = \delta_{ij}$ → orthogonal set

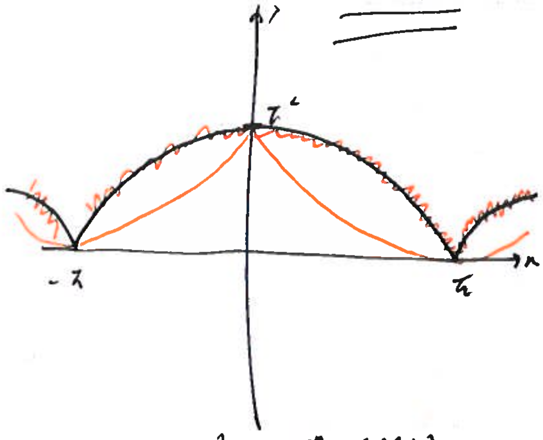
(orthonormal)

→ always 1

$m = n \Rightarrow$ orthonormal

Sketching

$-\pi < x < \pi$, $f(x) = (|x|-\pi)^2$



$$(|x|-\pi)^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{\cos(nx)}{n^2}$$

1. Take $x=0$

$$\pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

2. Take $x = \pi$

$$0 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = +\frac{\pi^2}{12}$$

$\sum_{n=1}^{\infty} \frac{1}{n^4}$ Parseval's identity state that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} [f(x)]^2 dx = \left(\frac{a_0}{2}\right)^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2+b_n^2)$$

$$\frac{1}{2\pi}\int_{-\pi}^{0} (-x-\pi)^4 dx + \frac{1}{2\pi}\int_{0}^{\pi} (x-\pi)^4 dx$$

$$= \frac{1}{2\pi}\left[\frac{1}{5}(x+\pi)^5\right]_{-\pi}^{0} + \frac{1}{2\pi}\left(\frac{1}{5}(x-\pi)^5\right]_{0}^{\pi}$$

$= \frac{\pi^4}{5}$ Persevals identity then gives

$$\frac{\pi^4}{5} = \left(\frac{\pi^2}{3}\right)^2 + \frac{16}{2}\sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}$$

Fourier Transforms

$$f(x) = \lim_{L\to\infty} \sum_{-\infty}^{\infty} c_n e^{in\frac{\pi}{L}x}$$

inserting our expression for c_n

$$f(x) = \lim_{L\to\infty} \sum_{-\infty}^{\infty} \frac{1}{2L}\int_{-L}^{L} e^{-in\frac{\pi}{L}y} f(y)\,dy\; e^{in\frac{\pi}{L}x}$$

$$= \frac{1}{2\pi}\lim_{L\to\infty}\sum_{-\infty}^{\infty}\frac{\pi}{L}\int_{-\infty}^{\infty} e^{in\frac{\pi}{L}(x-y)} f(y)\,dy$$

for any integrable function ~~fgk~~ $g(k)$

$$\lim_{L\to\infty}\sum_{-\infty}^{\infty}\frac{\pi}{L}\, g\!\left(n\frac{\pi}{L}\right) = \int_{-\infty}^{\infty} g(k)\,dk$$

$$\therefore f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\underbrace{\int_{-\infty}^{\infty} e^{ik(x-y)} f(y)\,dy}\,dk$$

Assuming this is integrable

IF $\int_{-\infty}^{\infty} |f(y)|^2\,dy$ is finite

square integrable

Defining the Fourier transform $\tilde{f}(k)$ of $f(x)$

$$\tilde{f}(k) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx} f(x)\,dx$$

Any complex function $f(x)$ such that $\int_{-\infty}^{\infty} |f(x)|^2\,dx$ is finite, can be expressed as:

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}\tilde{f}(k)\,dk$$

where Fourier transform $\tilde{f}(k)$ is defined as:

$$\tilde{f}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-ikx} f(x)\,dx$$

eg. QM

$$\begin{cases} \Psi(x) \to \text{position} \\ \tilde{\Psi}(k) \to \text{momentum} \end{cases}$$

$$\begin{cases} \tilde{\Psi}(t) \to \text{time} \\ \tilde{\Psi}(\omega) \to \text{frequency} \end{cases}$$

Trickery, Lies & Deceit

→ variable when integrating over when taking a Fourier transform is a dummy variable

↳ 幫佢改名 → keep diff integration variables distinct

$$\star\; f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}\tilde{f}(k)\,dk$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{izx}\tilde{f}(z)\,dz$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{i(y-w)x}\tilde{f}((y-w))\,d(y-w)$$

$$\star\; \tilde{f}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx} f(x)\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} (\cos(kx) - i\sin(kx))\, f(x)\,dx$$

if $f(x)$ is odd $\Rightarrow$ $f(x) = -f(-x)$ ~~applied~~

$$\tilde{f}(k) = \frac{i}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \sin(kx)\, f(x)\,dx$$

if $f(x)$ is even $\Rightarrow$ $f(x) = f(-x)$ ~~sin(ka)~~

$$\tilde{f}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \cos(kx)\, f(x)\,dx$$

differentiating:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx}\, \tilde{f}(k)\, dx$$

$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx}\, \tilde{f}(k)\, dx$

↳ $f'(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \underline{ik}\, e^{ikx}\, \tilde{f}'(k)\, dx$

$$\Rightarrow \tilde{f}'(k) = ik\, \tilde{f}(k)$$

Also: if $x \to x+a$

$$f(x+a) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ika}\, e^{ikx}\, \hat{f}(k)\, dk$$

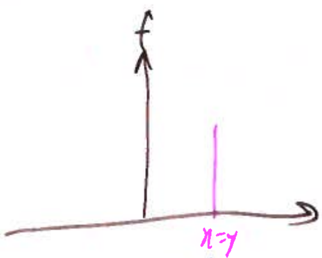
↳ F.T. of the shifted function $f(x+a)$, is given by $e^{ika}\tilde{f}(k)$ if $\tilde{f}(k)$ is the F.T. of $f(x)$

The Dirac Delta function

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{ik(x-y)} f(y)\, dy\, dk$$

$$= \int_{-\infty}^{\infty} \delta(x-y)\, f(y)\, dy$$

$$\delta(x-y) := \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(x-y)}\, dk$$



particular

↳ picking the value when $x=y$ while integrating over whole of the axis

Similar to the Kronecker delta:

$\delta_{mn} = 1 \quad m=n$
or
$0 \quad m \neq n$

↳ discrete version of the Dirac Delta function

$$\delta(x-y) := \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(x-y)}\, dk$$

$\Rightarrow$ only really has defined meaning inside an integral

⚠ more of a distribution than a function

$$c_n = \sum_{m=-\infty}^{\infty} \delta_{mn}\, c_m$$

Parseval's Identity for F.T.

$$\int_{-\infty}^{\infty} |f(x)|^2\, dx = \int_{-\infty}^{\infty} |\tilde{f}(k)|^2\, dk$$

$$f^*(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-izx}\, \hat{f}^*(z)\, dz$$

$$\int_{-\infty}^{\infty} |f(x)|^2\, dx = \frac{1}{2\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \underbrace{e^{ix(k-z)}}_{\delta}\, \hat{f}(k)\, \tilde{f}^*(z)\, dk\, dz\, dx$$

use Dirac Delta func

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \underbrace{\delta(k-z)}_{1 \text{ if } k=z}\, \tilde{f}^*(z)\, \tilde{f}(k)\, dk\, dz$$

picking out the value if $z=k$

$$= \int_{-\infty}^{\infty} \tilde{f}^*(k)\, \tilde{f}(k)\, dk$$

$$= \int_{-\infty}^{\infty} |\tilde{f}(k)|^2\, dk = \int_{-\infty}^{\infty} |f(x)|^2\, dx$$

Convolution Theorem

when measuring a physical quantity $f(x)$

↳ apparatus w/ resolution function $g(y)$

$$h(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, f(x)\, g(z-x)$$

$h(z)$ will be what actually is measured

⇒ convolution of the signal w/ resolving power of the apparatus

* each true value x is smeared by some resolution function g

ideally ⇒ $g(z-x) = \sqrt{2\pi}\, \delta(z-x)$

so that $h(z) = f(z)$

$$h = f * g = f \otimes g$$

which says that:

$$FT[h] = FT[f] \times FT[g]$$

eg. Defining :

$$F(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{iwx}\, dx$$

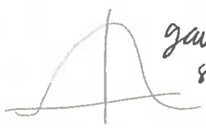
$$G(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x)\, e^{iwx}\, dx$$

$$H(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(x)\, e^{iwx}\, dx$$

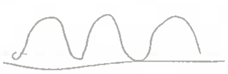
⇒ using $h = f * g$

$$H(w) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iwx}\, dx \int_{-\infty}^{\infty} dz\, f(z)\, g(x-z)$$

evaluating a function



gaussian smearing



evaluating the smearing function around ||

$$\Rightarrow H(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dz\, f(z)\, \underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iwz}\, dx\, g(x-z)}_{FT[G]}$$

* z is held constant ⇒ $d(x-z) = dx$

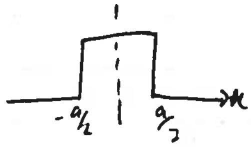
$$H(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dz\, f(z)\, e^{iwz}\, G(w)$$

$$H(w) = F(w)\, G(w)$$

Examples :

Evaluate the FT of "Tophat" or bo

$$F(x) = \begin{cases} 1 & |x| < a/2 \\ 0 & |x| > a/2 \end{cases}$$



Even function

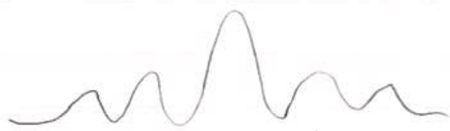
$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(x) \cos(kx)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a/2}^{a/2} \cos(kx)\, dx$$

$$= \frac{2 \sin(ak/2)}{\sqrt{2\pi}\, k} = \frac{4a}{\sqrt{2\pi}}\, \underbrace{\mathrm{sinc}\left(\frac{ak}{2}\right)}_{\frac{2 \sin(ak/2)}{ak/2}}$$

diffraction pattern



$$\left(\frac{\sin x}{x}\right)^2$$

~~use~~ Using double slit using convolution theorem

$$G(x) = \begin{cases} 1 & \cancel{\alpha} \ 0 \leq x \leq a \\ 0 & \text{otherwise} \end{cases}$$

$$G(x) = F(x - a/2$$

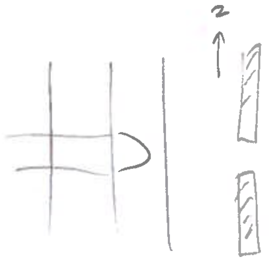
$$\tilde{G}(k) = e^{-\frac{iak}{2}} \tilde{F}(k)$$

FT of the dirac delta

$$\tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \delta(x - x_0) e^{-ikx} dk$$

$$= \frac{1}{\sqrt{2\pi}} e^{-ikx_0} \quad \leftarrow \text{if } x = x_0$$

if $k = 0$ $\quad \hat{f}(k) = \frac{1}{\sqrt{2\pi}}$



k_z : wavenumber in z direction

$= 0$

$k_z = 0$

z component of position of the wave is constant

(all z)

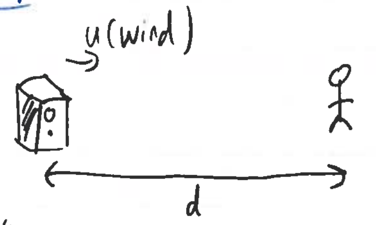
$\frac{\sin kx}{k}$ waveform

Special Relativity

~~Abs~~ "Absolute, true, & mathematical time of itself, & by its own true nature, flows uniformly on, w/o regard to anything external." ~ Newton's Principia

"Luminiferous aether" - how to detect?
↳ kind of "disproved" by Michelson-Morley Expt.

Example sound in air



time:
$t = \frac{d}{v_0}$
↳ speed of sound in air

Now, wind speed u.
⇒ speed $v_0 + u$

$t = \frac{d}{v_0 + u}$

frequency of sound = f

observer hears f

$\lambda = \frac{\text{speed}}{\text{freq}}$

$= \frac{v_0 + u}{f}$ $u \rightarrow$ (speaker to observer)

$\lambda = \frac{v_0 - u}{f}$ $\leftarrow u$ (observer to speaker)

number of ~~λ~~ wavelengths fitting into the distance d is

$\frac{d}{\lambda} = \frac{fd}{v_0 + u}$

or

$\frac{d}{\lambda} = \frac{fd}{v_0 - u}$

"There are observers for whom all isolated bodies move w/ a uniform velocity"

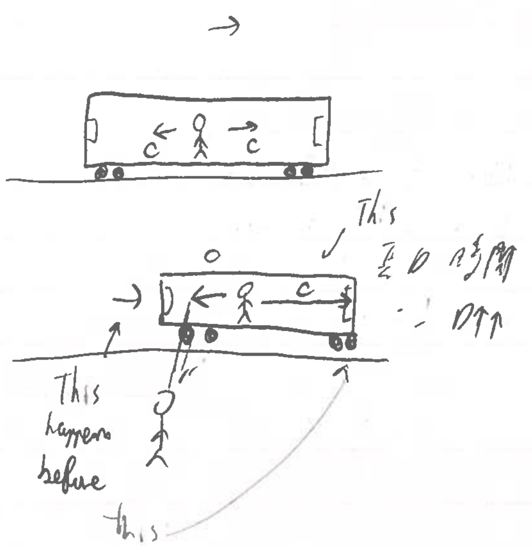
⇒ ~~Iner~~ Inertial observers

An inertial coordinate system
↳ a system of coordinates such that all isolated bodies move w/ uniform velocity in the coordinate

Einstein's Postulates:
1. The laws of physics have the same form in all inertial systems
↳ from Maxwell's eq^n
2. The velocity of light in empty space is a universal constant, the same for all observers.

implies

The relativity of simultaneity



Events & transformations

Event

$S \Rightarrow (t, x, y, z)$

$S' \Rightarrow (t', x', y', z')$

relationship of S & S'?? i.e. relationship between frames

$S \to S'$

$t, x, y, z \to t', x', y', z'$

The Galilean transform

S, S'

S' moves w/ velocity v w.r.t. S, along the z direction

Set them up so that:

Clock in S : $t=0$
Clock in S' : $t'=0$
origins: $x, y, z = 0$ $= x', y', z'$

Event in S (t, x, y, z)

Same event S' (t', x', y', z')

After time t : the origin of S' is at $z = vt$ in frame S.

$t' = t$

$x' = x$

$y' = y$

$z' = z - vt$

Galilean Transformation

$$\Rightarrow \begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -v & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$$

$z' = z - vt$

Lorentz Transformation

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & -\frac{\gamma v}{c^2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma v & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$$

$$\gamma \equiv \frac{1}{\sqrt{1 - v^2/c^2}} \qquad \beta \equiv \frac{v}{c}$$

$$t' = \frac{t - vz/c^2}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$x' = x$

$y' = y$

$$z' = \frac{z - vt}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Derivation:

$x = y = z = 0$, $t = 0$ $\Rightarrow S$
$x' = y' = z' = 0$, $t' = 0$ $\Rightarrow S'$
light emitted

Arrives @ z_0 on z axis in S

S: $t = \frac{z_0}{c}$ $\quad \left(\frac{z_0}{c}, 0, 0, z_0\right)$

using Galilean transformation:

S': $\left(\frac{z_0}{c}, 0, 0, z_0 - vt\right)$

Speed: $\frac{z_0 - v\, z_0 - vt}{z_0/c} = c - v\left(\frac{ct}{z_0}\right)$ $\quad z_0 = ct$

$= c - v$ ← contradict Einstein's postulate

$(t', x', y', z') \Leftrightarrow (t, x, y, z)$
$S' \qquad S$

$z' = Ct + D_2 + \cancel{Ax + Bx}$

$t' = At + Bz$

$$\Rightarrow \begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} A & 0 & 0 & B \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ C & 0 & 0 & D \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$$

Finding A, B, C, D

1. Event at origin of S' ($z'=0$)

z, t are given by

$$\frac{z}{t} = v$$

$$0 = z' = Ct + Dz \Rightarrow C = -Dv \checkmark$$

2. S' sees origin of S move w/ $-v$ along z axis

$$\Rightarrow \frac{z'}{t'} = -v \qquad z' = Ct + \underbrace{Dz}_{=0}$$

$$\div \quad t' = At + \underbrace{Bz}_{=0}$$

$$\Rightarrow \frac{z'}{t'} = \frac{Ct}{At} \Rightarrow \frac{C}{A} = -v \ \& \ C = -Dv$$

$$\therefore A = D$$

3. Light pulse emitted at $t = t' = 0$ from the origin

Event when the pulse arrives @ z_1, t_1 (in S)

z_1', t_1' (in S')

$$\frac{z'}{t'} = c = \frac{z}{t} = \frac{Ct + Dz}{At + Bz}$$

$$= \frac{C + Dc}{A + Bc}$$

($C = -Dv$)

$$\Rightarrow c = \frac{-Dv + Dc}{D + Bc} = \frac{c - v}{1 + cB/D}$$

solving for B/D ;

$$B = -\frac{v}{c^2} D$$ then, finding D then

4. light pulse emitted @ $t = t' = 0$ along y axis

y_1 on y-axis $\Rightarrow z = 0$

$$z' = Ct, \quad y' = y, \quad t' = At$$

Distance travelled by the pulse in S' is:

$$= \sqrt{(z')^2 + (y')^2}$$

$$\frac{y_1}{t} = c$$

$$\Rightarrow \frac{(z')^2 + (y')^2}{(t')^2} = c^2 = \frac{C^2 t^2 + y^2}{A^2 t^2} = \left(\frac{C}{A}\right)^2 + \left(\frac{c}{A}\right)^2$$

$$\Rightarrow C = -Dv, \quad A = D$$

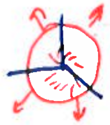
$$\Rightarrow c^2 = v^2 + \frac{c^2}{D^2}$$

$$D = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \qquad A = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$B = \frac{-v/c^2}{\sqrt{1 - \frac{v^2}{c^2}}} \qquad C = \frac{-v}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\left.\begin{aligned} t' &= \frac{t - vz/c^2}{\sqrt{1 - \frac{v^2}{c^2}}} \\ x' &= x \\ y' &= y \\ z' &= \frac{z - vt}{\sqrt{1 - \frac{v^2}{c^2}}} \end{aligned}\right\} \text{Lorentz transformation}$$

Invariance of the velocity of light



S: $c^2t^2 = x^2 + y^2 + z^2$

$c^2t^2 - x^2 - y^2 - z^2 = 0$

Sphere of light: ∵ c is spreading in all directions

S': $0 = c^2(t')^2 - (x')^2 - (y')^2 - (z')^2$

showing $\underline{S = S'}$

using Lorentz transform:

$$c^2(t')^2 - (x')^2 - (y')^2 - (z')^2 = \gamma^2 c^2\left(t^2 - \frac{2vzt}{c^2} + \frac{v^2z^2}{c^2}\right)$$

$$- \gamma^2(z^2 - 2zvt + v^2t^2)$$

$$- y^2 - x^2)$$

$$\Rightarrow \gamma^2\left(c^2t^2 - z^2 + \frac{v^2}{c^2}(z^2 - c^2t^2)\right) - y^2 - x^2$$

$$= \gamma^2(c^2t^2 - z^2)\cancel{\left(1 - \frac{v^2}{c^2}\right)} - y^2 - x^2$$

$$= c^2t^2 - z^2 - y^2 - x^2$$

$\boxed{\therefore \text{ invariant}}$

↳ 好似 3D vector 裡 嘅 length invariant w/ rotations/translations

Four-vectors

$\begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$ Four vectors are objects w/ transform according to the Lorentz transformation when boosted between inertial

4D not 3D

↳ -ve sign in scalar products

Metric $G = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

$(v \cdot w) = v^T G w$

$$(v_0 \; v_1 v_2 v_3)\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} w_0 \\ w_1 \\ w_2 \\ w_3 \end{pmatrix}$$

$= W_0W_0 - V_1W_1 - V_2W_2 - V_3W_3$

We are in <u>Clausius space</u>

Metric ⇒ encoding the structure of spacetime in the metric

↳ distinguishing the timeline coordinate from the 3 spatial coordinate

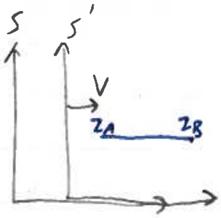
$\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$ = 4 vector specifying an event

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

symmetric b/w

Length Contraction

having a stick of length L_0 in rest frame @ rest in S'



$$\begin{pmatrix} ct \\ 0 \\ 0 \\ z_A \end{pmatrix}, \begin{pmatrix} ct \\ 0 \\ 0 \\ z_B \end{pmatrix} \quad \text{in frame } S$$

$$\begin{pmatrix} ct'_A \\ 0 \\ 0 \\ z_A' \end{pmatrix}, \begin{pmatrix} ct'_B \\ 0 \\ 0 \\ z_B' \end{pmatrix} \quad \text{In frame } S'$$

going from one frame to another:

$$\begin{pmatrix} ct'_A \\ 0 \\ 0 \\ z_A' \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & -\frac{\gamma v}{c} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{\gamma v}{c} & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} ct \\ 0 \\ 0 \\ z_A \end{pmatrix}$$

$$\Rightarrow z_A' = -\frac{\gamma v}{c} ct + \gamma z_A$$

$$= -\gamma v t + \gamma z_A = \gamma (z_A - vt)$$

Similarly for B:

$$\Rightarrow z_B' = -\gamma(z_B - vt)$$

$$L_0 = z_B' - z_A' = \gamma (z_B - z_A)$$

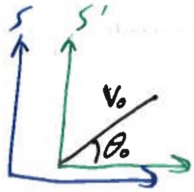
$$\underbrace{L_0}_{S'} = \gamma \underbrace{L}_{S}$$

$$L = \underbrace{L_0}_{\text{rest length}} \sqrt{1 - \frac{v^2}{c^2}}$$

length in moving frame is smaller

$\therefore$ = length contraction

Orientation of stick



- considering the positions of the ends of the stick in S'

$y'_A = 0$, $z_A' = 0$

$y'_B = L_0 \sin\theta_0$; $z_B' = L_0 \cos\theta_0$

$$\begin{pmatrix} ct'_A \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & -\frac{\gamma v}{c} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{\gamma v}{c} & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ y_A \\ z_A \end{pmatrix}$$

AND

$$\begin{pmatrix} ct'_B \\ 0 \\ L_0 \sin\theta_0 \\ L_0 \cos\theta_0 \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & -\frac{\gamma v}{c} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{\gamma v}{c} & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ y_B \\ z_B \end{pmatrix}$$

$z_A = 0$; $y_A = 0$

$\cancel{z_B' = L_0 \cos\theta_0 = \gamma z_B}$

$\cancel{y_B' = L_0 \sin\theta_0 = y_B}$

$\cancel{L = (z_B^2 + y_B^2)^{\frac{1}{2}} = \left[L_0^2 \cos^2\theta_0 \left(1 - \frac{v^2}{c^2}\right) \right] +}$

$z_A = 0$ $\quad z_B' = L_0 \cos\theta_0 = \gamma z_B$

$y_A = 0$ $\quad y_B' = L_0 \sin\theta_0 = y_B$

$$L = (z_B^2 + y_B^2)^{\frac{1}{2}} = \left[L_0^2 \cos^2\theta_0 \left(1 - \frac{v^2}{c^2}\right) + L_0^2 \sin^2\theta_0\right]^{\frac{1}{2}} = L_0 \left[1 - \frac{v^2}{c^2}\cos^2\theta_0\right]^{\frac{1}{2}}$$

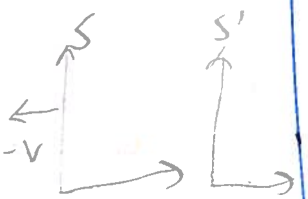
measuring the hypotenuse:

$$\theta = \tan^{-1}\left(\frac{y_B}{z_B}\right)$$

$$\theta = \tan^{-1}(\gamma \tan\theta_0) \quad \Big\| \quad \tan\theta_0 = \frac{y_B'}{z_B'}$$

$$= \tan^{-1}\left(\frac{\gamma\, y_B'}{z_B'}\right)$$

Time ~~ca~~ dilation



- helpful to have

~~S=Y~~

S in terms of S', rather than the other way around.

i.e. ⇒ inverse eqn

S moves at $-v$ relative to S'

∴ the transformation factor will just be the same, but with v switching signs

Inverse Lorentz transformation:

$$\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & \frac{\gamma v}{c} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{\gamma v}{c} & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix}$$

- clocks occur at: t_A' & t_B' [everything @ origin]

$ct_A = \gamma c t_A'$

$ct_B = \gamma c t_B'$

$$\Delta t = t_B - t_A = \gamma(t_B' - t_A') = \gamma \Delta t'$$

$$\Delta t = \frac{\Delta t'}{\sqrt{1 - \frac{v^2}{c^2}}} \qquad \gamma > 1 \qquad \Delta t > \Delta t'$$

Relativity of simultaneity

⇒ quantifying the question of simultaneity

$$\begin{pmatrix} ct_A \\ x_A \\ y_A \\ z_A \end{pmatrix} \& \begin{pmatrix} ct_B \\ x_B \\ y_B \\ z_B \end{pmatrix} \Rightarrow \text{specified in } S$$

$$\begin{pmatrix} ct_A' \\ x_A' \\ y_A' \\ z_A' \end{pmatrix} \& \begin{pmatrix} ct_B' \\ x_B' \\ y_B' \\ z_B' \end{pmatrix} \Rightarrow \text{same events specified in } S'$$

$$t_B - t_A \quad \Big\| \quad c(\cancel{t_B - t_A}) = \gamma\left(ct_B' - ct_A' + (z_B' - z_A')\frac{v}{c}\right)$$

Suppose: events are simultaneous in frame S ⇒ $t_B - t_A = 0$

$$t_B' - t_A' = -\frac{(z_B'-z_A)v}{c^2}$$

$$\underbrace{\gamma}_{\neq 0}\Big(\underbrace{ct_B' - ct_A' + (z_B'-z_A')\frac{v}{c}}_{=0}\Big) = 0$$

$$(t_B'-t_A')c = -\frac{(z_B'-z_A')v}{c}$$

$$t_B' - t_A' = \frac{-(z_B'-z_A')v}{c^2}$$

if $t_B - t_A = 0$ i.e. events are simultaneous in S

=> they are only simultaneous in S' if $z_B' = z_A'$

$$\begin{pmatrix} ct_A \\ x_A \\ y_A \\ z_A \end{pmatrix} - \begin{pmatrix} ct_B \\ x_B \\ y_B \\ z_B \end{pmatrix} = \begin{pmatrix} c\Delta t \\ \Delta x \\ \Delta y \\ \Delta z \end{pmatrix} = q$$

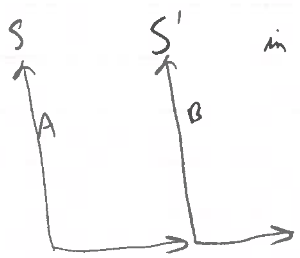
↑ also a 4 vector

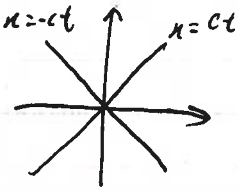
→ magnitude of 4-vector is invariant under Lorentz transformation

$$q^2 = c^2(\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2$$

- squared separation in space between A, B is $S^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$

$$\Rightarrow q^2 = \underbrace{(c\Delta t)^2}_{\text{max distance light can travel in } \Delta t} - \underbrace{S^2}_{\text{distance between events}}$$





if $q^2 > 0$:

$s^2 < (c\Delta t)^2$

- distance between events < light travelled in Δt between events

↳ particle can travel from one event to another less than c

↳ one event can plausibly cause another

or: can find ~~find~~ an intertial frame of both events happening @ the same time pt A&B

→ in that frame, distance will be zero

→ inertial frame can move from A to B

or: can find an inertial system that both events occur at the same position but @ diff times

* A can cause B
* A always happens before B
* There is a frame where A happens before B @ the same pt in space

if $q^2 = 0$: $s^2 = (c\Delta t)^2$

=> only light can move between events

=> only a light signal can go from one event to the other [light-like]

if $q^2 < 0$: $s^2 > (c\Delta t)^2$

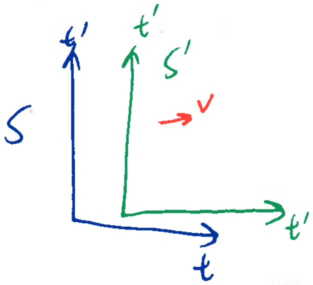
=> nothing can get between ~~A&B~~ A&B in Δt

=> nothing can travel in the spatial coordinate A to the spatial coordinate B

↳ no causal relation possible

↳ there will be an inertial system where the events are simultaneous, but in different places (when $\Delta t = 0$)

Addition of velocities



$u_x = \frac{x_B - x_A}{t_B - t_A}$, $u_y = \frac{y_B - y_A}{t_B - t_A}$, $u_z = \frac{z_B - z_A}{t_B - t_A}$

↓ primed version

$$u_x' = \frac{x_B' - x_A'}{t_B' - t_A'}$$

↓ Convert

Converting pr unprimed to primed using Lorentz transform

$x_B' - x_A' = x_B - x_A$

$z_B' - z_A' = \gamma\left(z_B - z_A - v(t_B - t_A)\right)$

$t_B' - t_A' = \gamma\left(t_B - t_A - \frac{v}{c^2}(z_B - z_A)\right)$

$$\Rightarrow u_z' = \frac{z_B' - z_A'}{t_B' - t_A'} = \frac{z_B - z_A - v(t_B - t_A)}{t_B - t_A - \frac{v}{c^2}(z_B - z_A)} \Big/ \frac{(t_B - t_A)}{(t_B - t_A)}$$

$$u_z' = \frac{u_z - v}{1 - u_z\frac{v}{c^2}}$$ //

y changes as well

$$u_y' = \frac{y_B' - y_A'}{\underbrace{t_B' - t_A'}_{\text{Time changes}}} = \frac{y_B - y_A}{\gamma\left(t_B - t_A - \frac{v}{c^2}(z_B - z_A)\right)}$$

$$u_y' = \frac{1}{\gamma}\frac{u_y}{1 - u_z v/c^2}$$

$$u_x' = \frac{1}{\gamma}\frac{u_x}{1 - u_z v/c^2}$$

$$u_x' = \frac{1}{\gamma}\frac{u_x}{1 - u_z\frac{v}{c^2}}$$

$$u_y' = \frac{1}{\gamma}\frac{u_y}{1 - u_z\frac{v}{c^2}}$$

$$u_z' = \frac{u_z - v}{1 - u_z\frac{v}{c^2}}$$

inverse version, swap sign of v

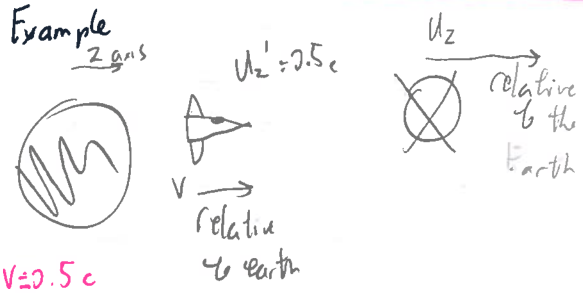
$$u_x = \frac{u_x'}{\gamma(1 + u_z'\frac{v}{c^2})}$$

$$u_y = \frac{u_y'}{\gamma(1 + u_z'\frac{v}{c^2})}$$

$$u_z = \frac{u_z' + v}{1 + u_z'\frac{v}{c^2}}$$

Example

z axis



$v = 0.5c$

$$u_z = \frac{u_z' + v}{1 + u_z' v/c^2} = \frac{0.5c + 0.5c}{1 + 0.5c\frac{0.5c}{c^2}} = \frac{c}{1 + \frac{1}{4}}$$

$u_z = 0.8c$

what if spaceship fire laser

$u_z' = c$
$v = 0.5c$

$$u_z = \frac{u_z' + v}{1 + \frac{u_z' v}{c^2}}$$

↑ from earth

$$= \frac{c + 0.5c}{1 + \frac{c(0.5c)}{c^2}} = \frac{1.5c}{1.5} = c$$

↑ still in c

Relativistic Doppler effect

T_0 = time between emission of pulses in rest frame of emitter

Emitter recedes w/ speed v
(from the observer)

T_1 = time between arrival of pulses @ the observer

using Time dilation

T = Interval between emission as seen by observer

$T = \gamma T_0$

Take $\frac{d}{c}$ for a pulse to travel distance d between emitter & observer
↳ measured in observer's rest frame

In time T between emission
↳ source travels: vT

$$T_1 = T + \underbrace{\frac{d+vT}{c}}_{\text{how far 2nd pulse travels}} - \underbrace{\frac{d}{c}}_{\text{how far first pulse travels}}$$

$$T_1 = T\left(1+\frac{v}{c}\right)$$

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{1}{\sqrt{(1+\frac{v}{c})(1-\frac{v}{c})}}$$

$$T_1 = \gamma T_0\left(1+\frac{v}{c}\right) = \frac{T_0\left(1+\frac{v}{c}\right)}{\sqrt{(1+\frac{v}{c})(1-\frac{v}{c})}}$$

$$T_1 = T_0\sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}}$$

getting frequency:

$$f = \frac{1}{T_1} = \frac{1}{T_0}\sqrt{\frac{1-v/c}{1+\frac{v}{c}}}$$

$$f = f_0\sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}}$$

∴ dropped.

$$\lambda = \lambda_0\sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}}$$

$\lambda\uparrow$ red
$\lambda\downarrow$ blue

* time between pulses is still dilated, even for transverse motion

$$f = \frac{f_0}{\gamma} \quad // \quad T = \gamma T_0$$

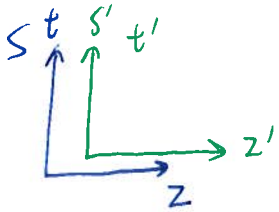
a source moving $\perp$ to the line between it & the observer

Mass & Energy

momentum

$$\underline{p} = m\underline{u}$$

$$p_x = m\frac{dx}{dt}, \quad p_y = m\frac{dy}{dt}, \quad p_z = m\frac{dz}{dt}$$



We want both observers to agree on p_x & p_y if boost is along z

∴ We need to use proper time to

time defined in an object's rest frame

$$p_z = m\frac{dz}{dt_0} = m\frac{dz}{dt}\frac{dt}{dt_0}$$

$$t = t_0\gamma$$

$$\frac{dt}{dt_0} = \gamma$$

$$\boxed{\underline{p} = \gamma m \underline{u}}$$

for $v \ll c$: $\gamma \approx 1$

$$\underline{p} = \gamma m \underline{v}$$

Energy:

force does work on a body

↳ causes acceleration

⇒ ↑ KE

Work: $F\delta x = \delta K$

if body moves with speed v,

$$\delta x = v\,\delta t$$

$$\frac{dK}{dt} = F\frac{dx}{dt} = F\cdot v$$

$$F = \frac{dp}{dt} = m\frac{dv}{dt}$$

$$\frac{dK}{dt} = v\frac{dp}{dt} = mv\frac{dv}{dt} = \frac{d}{dt}\left(\frac{1}{2}mv^2\right)$$

$$\therefore KE = \frac{1}{2}mv^2 \quad \Big\| \quad K=0 \text{ when } v=0$$

Consider now relativistic effects:

$$\frac{dK}{dt} = v\frac{dp}{dt} = mv\frac{d}{dt}(\gamma v)$$

$$\frac{d(\gamma v)}{dt} = \frac{d}{dt}\left(\frac{v}{\sqrt{1-\frac{v^2}{c^2}}}\right)$$

$$= \frac{\sqrt{1-\frac{v^2}{c^2}} + \frac{v^2/c^2}{\sqrt{1-\frac{v^2}{c^2}}}}{1-\frac{v^2}{c^2}}$$

$$= \left[\left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}} + \frac{v^2}{c^2}\left(1-\frac{v^2}{c^2}\right)^{-\frac{3}{2}}\right]\frac{dv}{dt}$$

$$= \left(1-\frac{v^2}{c^2}\right)^{-\frac{3}{2}}\frac{dv}{dt}$$

$$\therefore \frac{dK}{dt} = mv\left(1-\frac{v^2}{c^2}\right)^{-\frac{3}{2}}\frac{dv}{dt}$$

$$\int dK = m\int v\left(1-\frac{v^2}{c^2}\right)^{-\frac{3}{2}}dt$$

$$c^2\left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}}$$

$$\downarrow d$$

$$c^2\frac{v}{c^2}\left(1-\frac{v^2}{c^2}\right)^{-\frac{3}{2}}$$

$$\Rightarrow K = \gamma mc^2 + A$$

imposing $KE = 0$ when $v = 0$

$$K = \gamma mc^2 - mc^2$$

$$\boxed{KE = \gamma mc^2 - mc^2}$$

for $v \ll c$

$$KE = mc^2\left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}} - mc^2$$

$$= mc^2 + \frac{1}{2}mv^2 - mc^2 + O\left(\frac{v^2}{c^2}\right)^2$$

$$= \frac{1}{2}mv^2$$

increasing in energy due to a body moving w/ speed $\underline{v}$

$KE = \gamma mc^2 - mc^2$

"Total energy"

$E = \text{rest energy} + KE$

$E = \gamma mc^2 - mc^2 + mc^2$

$$\boxed{E = \gamma mc^2}$$

At rest $v = 0$; $\gamma = 1$

$\underline{\underline{E = mc^2}}$

Boojiboo

a particle @ rest still has energy

Energy in terms of momentum

$$E^2 = \underbrace{\frac{m^2c^4}{1 - v^2/c^2}}\ - m^2c^4 + m^2c^4$$

$$E^2 = \frac{m^2c^4 - m^2c^4 + m^2v^2c^2}{1 - \frac{v^2}{c^2}} + m^2c^4$$

$$= m^2c^4 + \frac{m^2v^2c^2}{1 - v^2/c^2} = m^2c^4 + \underbrace{\gamma^2 m^2 v^2}_{p^2} c^2$$

rest frame,

$$E^2 = \boxed{m^2c^4} + p^2c^2$$

$$E = \pm\sqrt{m^2c^4 + p^2c^2}$$

-ve sol^n important in QM

m is Lorentz invariant

$$\underbrace{E^2}_{\text{scalar relat to time}} - \underbrace{p^2c^2}_{\text{3 space-like components}} = m^2c^4$$

$\Rightarrow$ 4 momentum

Scalar product

Four momentum:

$$p = \begin{pmatrix} E/c \\ p_x \\ p_y \\ p_z \end{pmatrix}$$

In the rest frame S:

$$p = \begin{pmatrix} E/c \\ p_x \\ p_y \\ p_z \end{pmatrix} = \begin{pmatrix} mc \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

In frame S':

$~~p' = \begin{pmatrix} E'/c \\ p_x \end{pmatrix}~~$

$$p' = \begin{pmatrix} E'/c \\ p_x' \\ p_y' \\ p_z' \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & -\frac{v}{c}\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{v}{c}\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} mc \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \gamma mc \\ 0 \\ 0 \\ -\gamma mv \end{pmatrix}$$

transforming like a 4-vector

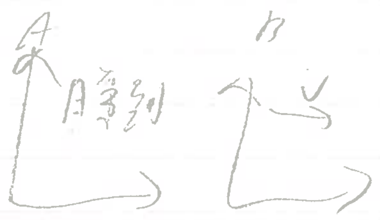
$E = \gamma mc^2$,

$p = -\gamma m\underline{v}$

magnitude of a 4-vector p^2

$$p^2 = \frac{E^2}{c^2} - p_x^2 - p_y^2 - p_z^2$$

$$= \frac{E^2}{c^2} - p^2 = m^2c^2$$



$$\Lambda B = \begin{pmatrix} 0 & 0 & 0 & -\frac{v}{c}\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{v}{c}\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} A \end{pmatrix}$$

Massless particle:

$$E = \sqrt{m^2c^4 + p^2c^2}$$

$m \to 0$

$$E = \pm\sqrt{p^2c^2}$$

$$E = |p|c$$

$$E = \gamma mc^2 = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}}$$

$\left.\begin{array}{l} m \to 0 \\ v \to \infty \end{array}\right]$ to have finite energy

✲ A particle can only travel at c if it is massless (otherwise $\gamma \to \infty$, $E \to \infty$)

- Particle A MUST travel @ c if it is massless otherwise $E \to 0$

In QM

$$E = hf = \frac{hc}{\lambda} \qquad h = 4.14 \times 10^{-24}$$

$$|p| = \frac{h}{\lambda}$$

4-momentum of photon:

$$\begin{pmatrix} E/c \\ p_x \\ p_y \\ p_z \end{pmatrix} = \begin{pmatrix} \frac{h}{\lambda} \\ 0 \\ 0 \\ \frac{hc}{\lambda} \end{pmatrix}$$

Magnitude of 4-momentum is zero for a photon $\because$ massless

$$p^2 = \frac{E^2}{c^2}$$

In any given inertial frame

✲ Four momentum is conserved component by component

$\begin{pmatrix} E/c \\ p_x \\ p_y \\ p_z \end{pmatrix}$ Energy & Momentum are conserved

$E = \gamma mc^2$, $p = \gamma mv$

Under Lorentz transform, magnitude of 4-momentum is invariant

---

$m = \gamma m_0$

$E = \gamma mc^2$

Tricks

$$\gamma = \frac{\gamma mc^2}{mc^2} = \frac{E}{mc^2}$$

$$\frac{\gamma mvc}{\gamma mc^2} = \frac{|p|c}{\gamma mc^2} = \frac{|p|c}{E} = \frac{v}{c}$$

photons boost in Lorentz frames