

$$\hat{\mathcal{H}} = -J \sum_i \frac{1}{2} \left((T_i^\dagger T_{i+1}^- + T_i^- T_{i+1}^\dagger) + (1+\alpha) \underbrace{T_i^z T_{i+1}^z} \right)$$

$$\begin{aligned} A &= T_i^\dagger T_{i+1}^- \\ &= D_i C_{i+1}^\dagger D_{i+1} \\ &= C_i D_i D_{i+1}^\dagger C_{i+1}^\dagger \\ &\quad \underbrace{1 - 2C_i^\dagger C_i} \end{aligned}$$

$$\begin{aligned} &= C_i (1 - 2C_i^\dagger C_i) C_{i+1}^\dagger \\ &= (C_i - 2C_i C_i^\dagger C_i) C_{i+1}^\dagger \\ &= C_i C_{i+1}^\dagger - 2 \underbrace{C_i C_i^\dagger C_i}_{\{C_i, C_i^\dagger\}=1} C_{i+1}^\dagger \\ &\quad + C_i C_i^\dagger + C_i^\dagger C_i = 1 \end{aligned}$$

$$\begin{aligned} &= C_i C_{i+1}^\dagger - 2 C_i (1 - C_i C_i^\dagger) C_{i+1}^\dagger \\ &= C_i C_{i+1}^\dagger - \underbrace{(2C_i - 2C_i C_i C_i^\dagger)}_{=0} C_{i+1}^\dagger \\ &\quad - C_i C_i^\dagger C_i C_{i+1}^\dagger \end{aligned}$$

$$\begin{aligned} A &= -C_i C_{i+1}^\dagger \\ \{C_i, C_{i+1}^\dagger\} &= 0 \\ C_i C_{i+1}^\dagger &= -C_{i+1} \end{aligned}$$

$$B = C_i^\dagger C_{i+1}$$

$$\begin{aligned} C &= T_i^z T_{i+1}^z \\ &= \left(\frac{1}{2} - C_i^\dagger C_i \right) \left(\frac{1}{2} - C_{i+1}^\dagger C_{i+1} \right) \\ &= \frac{1}{4} - \frac{1}{2} C_{i+1}^\dagger C_{i+1} - \frac{1}{2} C_i^\dagger C_i + C_i^\dagger C_i C_{i+1}^\dagger C_{i+1} \\ &= \frac{1}{4} - \frac{1}{2} (C_{i+1}^\dagger C_{i+1} + C_i^\dagger C_i) \\ &\quad + \underbrace{C_i^\dagger C_i C_{i+1}^\dagger C_{i+1}}_{-C_{i+1}^\dagger C_i} \\ &= \frac{1}{4} - \frac{1}{2} (C_{i+1}^\dagger C_{i+1} + C_i^\dagger C_i) - C_i^\dagger C_{i+1}^\dagger C_i C_{i+1} \end{aligned}$$

$$\mathcal{H} = -J \sum_i \left(A + B + 2(1+\alpha) C \right)$$

$$= -\frac{J}{2} \sum_i \left(-C_i C_{i+1}^\dagger + C_i^\dagger C_{i+1} + 2(1+\alpha) \left(\frac{1}{2} - C_i^\dagger C_i \right) \left(\frac{1}{2} - C_{i+1}^\dagger C_{i+1} \right) \right)$$

$$= -\frac{J}{2} \sum_i \left(C_{i+1}^\dagger C_i + C_i^\dagger C_{i+1} + 2(1+\alpha) \left(\frac{1}{2} - C_i^\dagger C_i \right) \left(\frac{1}{2} - C_{i+1}^\dagger C_{i+1} \right) \right)$$