- Sym break

- Topological order, topological phase, topological phase transition, $J_A - J_F$

- Alternating FM & AM Couplings

- Haldane spin chain

- determine phase boundary of the topological phase as a function of relative strength of the exchange coupling & the magnetic exchange anisotropy

- Jordan - Wigner transformation ✓
- non-local string operators

- mean-field approx
- coupled self-consistency eq^{n}s

- Simple ~~mode~~ model Hamiltonian ✓

- Fermion creation & annihilation operators ✓

- Strong anisotropy ($\gamma \to 1$) => Z-comp of spin frozen => Fermions non-interacting
  have topological phase transition @ $J_A = J_F$

- weaker anisotropy => mean-field approx => numerical solution of coupled self-consistency equations

- $\gamma \to 0$

1D spin chains:

⇒ Heisenberg chain (Bethe Ansatz)

⊕ → $\underline{\text{XY chain}}$ (Jordan-Wigner transformation) [Pg 82]

use

⊕ Periodic boundary conditions: Pg 28

Section 2.7 (An Intro to Quantum Spin systems)

~~end~~

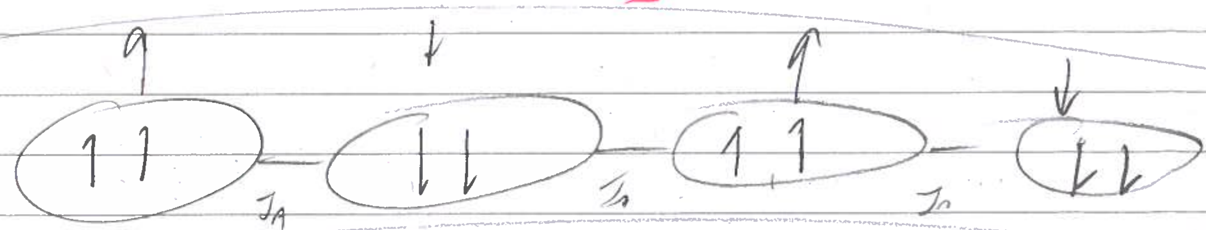
End of chain are joined

⇒ N^{th} site is a nearest-neighbour of the 1^{st}, as well as the $(N-1)^{th}$

taking $N \to \infty$

$$\mathcal{H} = J[\underline{S}_1 \cdot \underline{S}_2 + \underline{S}_2 \cdot \underline{S}_3 + \cdots + \underline{S}_N \cdot \underline{S}_1]$$

$$= J \sum_{i=1}^{N} S_i \cdot S_{i+1} \quad i+N = i$$



| A — B — C

A w/o B/C   A + w/C

Pg 82. γ: anisotropy → combination of Heisenberg & Ising

$= 0$ → XX model

$= 1$ → Ising case

⊕ correction factors for anisotropy ??

Hamiltonian ⇒

$$\mathcal{H} = J \sum_j [(1+\gamma) S_j^x S_{j+1}^x + (1-\gamma) S_j^y S_{j+1}^y]$$

Pg 82 ⇒ ~~$\mathcal{H} = J \sum \Delta S_j^z S_{j+1}^z + S$~~

$$= J \sum_j [\underbrace{\gamma(S_j^x S_{j+1}^x - S_j^y S_{j+1}^y)}_{\Delta S^z S^z} + (S_j^x S_{j+1}^x + S_j^y S_{j+1}^y)]$$

Spin operators to Fermion Operators I_5 62

Pauli Spin Matrices

$S_i^x = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$S_i^y = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $\hbar = 1$

$S_i^z = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

raising operator

$a_i^\dagger = S_i^x + i S_i^y$

$a_i = S_i^x - i S_i^y$

For a single site i $+ \equiv \uparrow$
$- \equiv \downarrow$

$$\hat{S}^+ = \hat{S}^x + i\hat{S}^y$$
$$\hat{S}^- = \hat{S}^x - i\hat{S}^y$$

$S^+|+\rangle = 0 \qquad S^-|-\rangle = 0$
$S^+|-\rangle = |+\rangle \qquad S^-|+\rangle = |-\rangle$ Page 60

$S_i^+|-\rangle = |+\rangle \qquad S_i^+|+\rangle = 0$

$S_i^-|-\rangle = 0 \qquad \cancel{S_x^-|+\rangle} = |\cancel{0}\rangle \; S_i^-|+\rangle = |-\rangle$

$S_i^z|-\rangle = -\frac{1}{2}|-\rangle \qquad S_i^z|+\rangle = \frac{1}{2}|+\rangle$

i) $S_i^- S_i^+ + S_i^+ S_i^- = 1$ — $(S_i^- S_i^+ + S_i^+ S_i^-)|+\rangle = |+\rangle$

ii) $S_i^{-2} = S_i^{+2} = 0$

Commutator relationship for a single sites 2, $i \neq j$

$$[S_i^-, S_j^+] \equiv S_i^- S_j^+ - S_j^+ S_i^- = 0$$

$$S_i^- S_j^+ + S_j^+ S_i^- = 2 S_i^- S_j^+$$
$$S_i^- S_j^- + S_j^- S_i^+ = 2 S_i^- S_j^-$$
$$S_i^+ S_j^+ + S_j^+ S_i^+ = 2 S_i^+ S_j^+$$

$\hookrightarrow$ Fe

anticommutator for fermions
$\Updownarrow$ analogous
commutator for bose particles

Introducing Fermion operators : c_i & $c_i^\dagger$

- Fermions => anticommute

$$\{c_i, c_j^\dagger\} \equiv c_i c_j^\dagger + c_j^\dagger c_i = \delta_{ij} \begin{cases} i=j \Rightarrow 1 \\ i\neq j \Rightarrow 0 \end{cases}$$

$$\{c_i, c_j\} = 0$$

$$\boxed{\{c_i^\dagger, c_j^\dagger\} = 0}$$ => can be simultaneously measured

For single site => can do $S_i^- = c_i$ $S_i^\dagger = c_i^\dagger$

BUT => For different sites

spin operator can be represented in terms of the fermions (c_i)

Transformation of to fermion operators

$$\Rightarrow \left[\begin{array}{ll} S_1^- = c_1 & S_1^\dagger = c_1^\dagger \\ (6.13) \Rightarrow S_2^- = [\exp(i\pi c_1^\dagger c_1)] c_2 & S_2^\dagger = c_2^\dagger [\exp(-i\pi c_1^\dagger c_1)] \\ S_i^- = Q_i c_i \quad i \geq 1 & S_i^\dagger = c_i^\dagger Q_i^\dagger \quad i \geq 1 \end{array}\right]$$

$$\boxed{Q_i = \exp\left[i\pi \sum_{l=1}^{i-1} c_l^\dagger c_l\right]}$$

=> Now, need to prove the spin operators from 6.13 commute when on different sites ($i \neq j$)

introduce new operators, ~~Ai and Ti~~ n_i, T_i, Q_i

Let $|+\rangle$ & $|-\rangle$ be the basis for the i^{th} state

Fermion operators acting on these basis functions

$$\boxed{\begin{array}{ll} c_i^\dagger|+\rangle = 0 & c_i^\dagger|-\rangle = |+\rangle \\ c_i|+\rangle = |-\rangle & c_i|-\rangle = 0 \end{array}}$$

Defining: $n_i =$ number operator

$n_i \equiv c_i^+ c_i$ ↳ 'counts' the value of the z-component of the ang momenta

$n_i^+ = n_i$ relative to the state $|-\rangle$

$n_i|+\rangle = c_i^+ c_i |+\rangle = c_i^+ |-\rangle = |+\rangle$

$n_i|-\rangle = c_i^+ c_i |-\rangle = 0$

$n_i|-\rangle = 0\,|-\rangle$ spin dn

$n_i^+|+\rangle = 1\,|+\rangle$ spin up

defining $\boxed{T_i = e^{i\pi c_i^+ c_i} = e^{i\pi n_i}}$

$\Rightarrow T_i|+\rangle e^{i\pi n_i}|+\rangle = e^{i\pi 1}|+\rangle = -|+\rangle$ $n_i = 1$ for spin up

$T_i|-\rangle = e^{i\pi n_i}|-\rangle = e^{i\pi(0)}|-\rangle = +|-\rangle$ $n_i = 0$ for spin down

$T_i^+ = e^{-i\pi n_i}$

$T_i^+|-\rangle = -|+\rangle$

$T_i|-\rangle = |-\rangle$

$\therefore$ effect of T_i' is the same as that of T_i on the two basis states

$\Rightarrow T_i^+ = T_i$

$\therefore T_i^2|+\rangle = |+\rangle$ & $T_i^2|-\rangle = |-\rangle$

↳ $T_i^2 = 1$

$n_i n_j = n_j n_i$

$\Rightarrow n_i n_j - n_j n_i = 0$ by commutator relationships

$\boxed{[n_i, n_j] = 0 \quad [T_i, T_j] = 0}$

for $\cancel{[S_i, S_j]} = 5^+$

$\{c_i, c_j^+\}$

$$Q_i = \exp\left[i\lambda \sum_{l=1}^{i-1} n_l \right] \quad \text{all } n_l \text{ commute.}$$

~~The proof~~

Provided that $[A,B]=0$

$$e^{A+B} = e^A e^B$$

$$Q_i = \prod_{l=1}^{i-1} e^{i\lambda n_l} = \cancel{Q} \prod_{l=1}^{i-1} T_l$$

$$Q_i^\dagger = \prod_{l=1}^{i-1} e^{-i\lambda n_l} = \prod_{l=1}^{i-1} T_l^\dagger = \prod_{l=1}^{i-1} T_l = Q_i$$

$$\boxed{Q_i^\dagger = Q_i}$$

$$[c_i, n_j] = c_i n_j - n_j n_i = c_i c_j^\dagger c_j - c_j^\dagger c_j c_i$$
$$= -c_j^\dagger c_i c_j + c_j^\dagger c_i c_j = 0$$

$[c_i, n_j] = 0$ $\quad$ $\{c_i, T_i\} = 0$ $\quad$ see 6.18 – 6.21

$[c_i, T_j] = 0$ $\quad$ $[c_i, T_i] \neq 0$

$[c_i^\dagger, T_j] = 0$ $\quad$ $[c_i, Q_i] = 0$ $\quad$ Pg. 65

$[c_i^\dagger, Q_i] = 0$

$[S_i^-, S_j^+] = 0$

using 6.1

$$S_j^x S_{j+1}^x = \frac{1}{2}(S_j^+ + S_j^-)\frac{1}{2}(S_{j+1}^+ + S_{j+1}^-)$$
$$= \frac{1}{4}(S_j^+ S_{j+1}^+ + S_j^- S_{j+1}^-) + \frac{1}{4}(S_j^- S_{j+1}^+ + S_j^+ S_{j+1}^-)$$

??

$$S_j^y S_{j+1}^y = \frac{1}{2i}(S_j^+ - S_j^-)\frac{1}{2i}(S_{j+1}^+ - S_{j+1}^-)$$
$$= -\frac{1}{4}(S_j^+ S_{j+1}^+ + S_j^- S_{j+1}^-) + \frac{1}{4}(S_i^- S_{j+1}^+ + S_j^+ S_{j+1}^-)$$

Hamiltonian of the XY model in terms of the spin operators

6.1

$$\mathcal{H} = J \sum_{j=1}^{N} [(1+\gamma) S_j^x S_{j+1}^x + (1-\gamma) S_j^y S_{j+1}^y]$$

$$= \frac{J}{2} \sum_{j=1}^{N} [(S_j^- S_{j+1}^+ + S_j^+ S_{j+1}^-) + \gamma (S_j^+ S_{j+1}^+ + S_j^- S_{j+1}^-)]$$

$\gamma = 0 \Rightarrow$ XY-model

$\Downarrow$ in terms of ~~spin~~ fermion operators

$$S_j^- S_{j+1}^+ = Q_j c_j c_{j+1}^+ Q_{j+1}$$

$$= c_j T_j c_{j+1}^+ \quad \text{[using commutator relationships]}$$

$c_j T_j |+\rangle = -|-\rangle$; comp ; $c_j |+\rangle = |-\rangle$

$c_j T_j |-\rangle = 0$; ; $c_j |-\rangle = 0$

$$c_j T_j = -c_j$$

$$S_j^- S_{j+1}^+ = -c_j c_{j+1}^+ = c_{j+1}^+ c_j$$

similarly

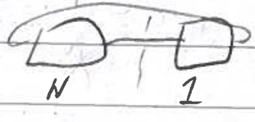
$$S_j^+ S_{j+1}^- = c_j^+ c_{j+1}$$

$$S_j^+ S_{j+1}^+ = c_j^+ c_{j+1}^+$$

$$S_j^- S_{j+1}^- = c_{j+1} c_j$$

For $j = N$, at the end of the series, for periodic boundary conditions:

$$S_N^+ S_1^- = Q_N c_N^+ c_1 \neq c_N^+ c_1 = Q_N c_N^+ c_1 \neq c_N^+ c_1$$



Subbing to the Hamiltonian

$$\mathcal{H} = \frac{J}{2} \sum_{j=1}^{N} [(c_{j+1}^+ c_j + c_j^+ c_{j+1}) + \gamma (c_j^+ c_{j+1}^+ + c_{j+1} c_j)]$$

$$- \frac{J}{2} [(c_1^+ c_N + c_N^+ c_1) + \gamma (c_N^+ c_1^+ + c_1 c_N)]$$

$$+ \frac{J}{2} Q_N [(c_N c_1^+ + c_N^+ c_1) + \gamma (c_N^+ c_1^+ + c_N c_1)]$$

can be neglected in the limit that $N \to \infty$

C-cyclic problem

in Ann Phy 16, 407 1961

it is in matrix form

In textbook An intro to Quantum Spin Systems

p 412 - spins can be arranged in a definite order,

- only neighboring spins interact
- z-components of spins cannot be entered.

$\cos \quad \frac{e^{ik}+e^{-ik}}{2} = \cos(k)$

Eq 6.27 :

$$\mathcal{H} = \frac{1}{2}\sum_{j=1}^{N}\left[(c_{j+1}^{\dagger}c_j + c_j^{\dagger}c_{j+1}) + \gamma(c_j^{\dagger}c_{j+1}^{\dagger} + c_{j+1}c_j)\right]$$

- ~~fmah~~ quadratic Hamiltonian involving only fermion operators

Diagonalise to make use of the translational invariance by introducing discrete Fourier transformed operators d_k & $d_k^{\dagger}$

→ diagonalize the Hamiltonian — find Eigenvalue

$$d_k = \frac{1}{\sqrt{N}}\sum_{j=1}^{N} e^{-ikj} c_j$$

$k \in [-\pi, \pi]$

~~kx~~

$k = \lambda \frac{2\pi}{N}$

$$d_k^{\dagger} = \frac{1}{\sqrt{N}}\sum_{j=1}^{N} e^{ikj} c_j^{\dagger}$$

$\lambda = (-\frac{N}{2}+1), \dots (\frac{N}{2})$

$d_{k_1}d_{k_2} = -d_{k_2}d_{k_1}$

$d_{k_1}d_{k_2} + d_{k_2}d_{k_1} = 0$

↓ reverse transform ↓

$\{d_{k_1}, d_{k_2}\} = \{d_{k_1}^{\dagger}, d_{k_2}^{\dagger}\} = 0$

$$c_j = \frac{1}{\sqrt{N}}\sum_k e^{ikj} d_k$$

using the properties of c_j & $c_j^{\dagger}$ → $\Rightarrow$ $\{d_{k_1}, d_{k_2}^{\dagger}\} = \delta_{k_1 k_2}$

$$c_j^{\dagger} = \frac{1}{\sqrt{N}}\sum_k e^{-ikj} d_k^{\dagger}$$

d's are fermions

~~using $\sum_j e^{ik_j}$~~ using $\sum_j e^{i(k_1-k_2)j} = N\delta_{k_1 k_2}$ [orthogonality relationships]

↓ rewriting each term of the Hamiltonian

$$\sum_j c_{j+1}^{\dagger} c_j = \sum_j \frac{1}{N}\sum_{k_1}\sum_{k_2} e^{-ik_1(j+1)} e^{ik_2 j} d_{k_1}^{\dagger} d_{k_2}$$

$$= \frac{1}{N}\sum_{k_1}\sum_{k_2} e^{-ik_1} N\delta_{k_1 k_2} d_{k_1}^{\dagger} d_{k_2}$$

↓ picking $k_1 = k_2 = k$

$$\Rightarrow \sum_k e^{-ik} d_k^{\dagger} d_k$$

Similarly: $\sum_j c_j^{\dagger} c_{j+1} = \sum_k e^{ik} d_k^{\dagger} d_k$ $\Rightarrow \sum_j c_{j+1} c_j^{\dagger} = \sum_k e^{-ik} d_k^{\dagger} d_k$

$$\sum_j c_j^{\dagger} c_{j+1}^{\dagger} = \sum_k e^{ik} d_k^{\dagger} d_{-k}^{\dagger}$$

$$\sum_j c_{j+1} c_j = \sum_k e^{ik} d_k d_{-k}$$

lin comb of fermion operators

↳ Bogoliubov transformation

⇓

quasi-particle operators

$$\longrightarrow \mathcal{H} = J \sum_k \left[\cos k \,(d_k^\dagger d_k) + \gamma \frac{e^{ik}}{2} \left(d_k^\dagger d_{-k}^\dagger + d_k d_{-k}\right)\right]$$

- no coupling between states w/ diff $|k|$
- still need to diagonalize the coupled k & $-k$ terms

[in momentum space] ??

instead of summing $k \in [-\pi, \pi]$

⇓ combine k & $-k$

↳ summing $k \in [0, \pi]$

$$\mathcal{H} = J \sum_{k=0}^{\pi} \Big[\cos(k)(d_k^\dagger d_k + d_{-k}^\dagger d_{-k}) + \gamma \Big[\frac{e^{ik}}{2}(d_k^\dagger d_{-k}^\dagger + d_k d_{-k}) + \frac{e^{-ik}}{2}(d_{-k}^\dagger d_k^\dagger + d_{-k} d_k) \Big]\Big]$$

using anticommutation relationships

$$\mathcal{H} = J \sum_{k=0}^{\pi} \left[\cos k \,(d_k^\dagger d_k + d_{-k}^\dagger d_{-k}) + \gamma [i \sin k \,(d_k^\dagger d_{-k}^\dagger + d_k d_{-k})]\right]$$

⇓ - only d_k and d_{-k}, $d_k^\dagger$ and $d_{-k}^\dagger$

diagonalize the Hamiltonian

↳ we need to look for two diff lin-comb η_k & η_{-k}

??

also fermion operators

$$\eta_k = A_k d_k + B_k d_{-k}^\dagger$$
$$\eta_{-k} = C_k d_{-k} + D_k d_k^\dagger$$

w/ ~~commutation~~ anticommutation relationships

$$\{\eta_k, \eta_k\} = \{\eta_k^\dagger, \eta_k^\dagger\} = 0$$
$$\{\eta_{-k}, \eta_{-k}\} = \{\eta_{-k}^\dagger, \eta_{-k}^\dagger\} = 0$$
$$\{\eta_k, \eta_{-k}^\dagger\} = \{\eta_{-k}, \eta_k^\dagger\} = 0$$
$$\{\eta_k, \eta_k^\dagger\} = \{\eta_{-k}, \eta_{-k}^\dagger\} = 1$$
$$\{\eta_k, \eta_{-k}\} = \{\eta_k^\dagger, \eta_{-k}^\dagger\} = 0$$

$$|A_k|^2 + |B_k|^2 = 1$$
$$|C_k|^2 + |D_k|^2 = 1$$

$$\mathcal{H} = J\sum_{k>0}^{k} \left[\cos k\,(d_k^\dagger d_k + d_{-k}^\dagger d_{-k}) + \gamma\,[\, i \sin k\,(d_k^\dagger d_{-k}^\dagger + d_k d_{-k})] \right]$$

tgt w/ lin comb of η_k, η_{-k} & d_k, d_{-k}

has the form below:

$$\mathcal{H} = \sum_{k>0}^{\pi} \left[\Lambda_{1k}\, \eta_k^\dagger \eta_k + \Lambda_{2k}\, \eta_{-k}^\dagger \eta_{-k} + \underbrace{X_k}_{\text{constant}} \right]$$