7/6/2021

# CMMP intern notes

- Classical $\Longrightarrow$ quantum

$$\underline{S} = \begin{pmatrix} S_x \\ S_y \\ S_z \end{pmatrix} \Rightarrow \hat{\underline{S}} = \begin{pmatrix} \hat{S}_x \\ \hat{S}_y \\ \hat{S}_z \end{pmatrix}$$

- acting on Hilbert space

$|i\rangle$ (ket)
$\langle j|$ (bra)

- dot product
  $\langle j | i \rangle \in \mathbb{C}$

- $\langle i | = (|i\rangle)^\dagger$ transpose + complex conjugate

- Matrix element : $A_{ij} = \langle j | \hat{A} | i \rangle \in \mathbb{C}$

- Commutator Relationships $\Rightarrow$ $[\hat{A}, \hat{B}] = 0$ (commute)

$$[\hat{A}, \hat{B}] = -[\hat{B}, \hat{A}]$$
$$[\hat{A}, \hat{A}] = 0$$
$$[\hat{A}, \hat{B} + \hat{C}] = [\hat{A}, \hat{B}] + [\hat{A}, \hat{C}] \quad \text{(linearity)}$$
$$[\hat{A}, \hat{B}\hat{C}] = [\hat{A}, \hat{B}]\hat{C} + \hat{B}[\hat{A}, \hat{C}]$$

- spin commutator relations (cyclic permutations)

$$[\hat{S}_x, \hat{S}_y] = i\hbar \hat{S}_z$$
$$[\hat{S}_z, \hat{S}_x] = i\hbar \hat{S}_y$$
$$[\hat{S}_y, \hat{S}_z] = i\hbar S_x$$

$$\Big\rangle \; [\hat{S}_\alpha, \hat{S}_\beta] = i\hbar \underbrace{\epsilon_{\alpha\beta\gamma} \hat{S}_\gamma}$$

↳ Einstein sum coeff

↳ $\epsilon_{\alpha\beta\gamma}$ $\Rightarrow$ non-zero only if no two indicies are the same

e.g. $\epsilon_{xxy} = 0$

$\epsilon_{xyz} = \epsilon_{zxy} = \epsilon_{yzx} = 1$

$\epsilon_{yxz} = \epsilon_{zyx} = \epsilon_{xzy} = -1$

$$[\hat{S}_z, \hat{S}^2] = 0$$

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$$\hat{S}^2 | s, m \rangle = \hbar^2 s(s+1) | s, m \rangle$$
$$\hat{S}_z | s, m \rangle = \hbar m | s, m \rangle$$

## Focusing on $S = \frac{1}{2}$

$|S, m\rangle$

$:= |\frac{1}{2}, \frac{1}{2}\rangle = |\uparrow\rangle$

$|\frac{1}{2}, -\frac{1}{2}\rangle = |\downarrow\rangle$

(Hilbert space spanned by two states [both : 2 dim space])

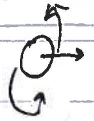
Pauli spin matricies:

What is the significance of $\hat{S}_x$ & $\hat{S}_y$ components?

$\hat{S}_x |\uparrow\rangle = \frac{\hbar}{2} |\downarrow\rangle$

$\hat{S}_x |\downarrow\rangle = \frac{\hbar}{2} |\uparrow\rangle$

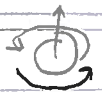
$\hat{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$



$\hat{S}_y |\uparrow\rangle = -i\frac{\hbar}{2} |\downarrow\rangle$

$\hat{S}_y |\downarrow\rangle = i\frac{\hbar}{2} |\uparrow\rangle$

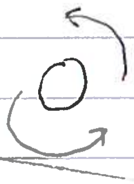
$\hat{S}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$



$\hat{S}_z |\uparrow\rangle = \frac{\hbar}{2} |\uparrow\rangle$

$\hat{S}_z |\downarrow\rangle = -\frac{\hbar}{2} |\downarrow\rangle$

$\hat{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$



defining: what does this mean??

$\hat{S}_+ := \hat{S}_x + i\hat{S}_y$

$\hat{S}_- := \hat{S}_x - i\hat{S}_y$

$S_+ = \hbar\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$S_- = \hbar\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

$\hat{S}_+ |\downarrow\rangle = \hbar |\uparrow\rangle$

$\hat{S}_- |\downarrow\rangle = 0$

$\hat{S}_+ |\uparrow\rangle = \hat{S}_x |\uparrow\rangle + i\hat{S}_y |\uparrow\rangle = \frac{\hbar}{2}|\downarrow\rangle + i\left(-i\frac{\hbar}{2}|\downarrow\rangle\right) = \hbar|\downarrow\rangle$

$\hat{S}_- |\uparrow\rangle = \hat{S}_x |\downarrow\rangle - i\hat{S}_y |\downarrow\rangle = 0$

Two-site problem:

$\underset{1}{\uparrow}$ $\underset{2}{\downarrow}$ => 2 arrangements

classically

AFM: ↑↓↑↓↑↓↑↓

FM: ↑↑↑↑↑↑↑

for two-interacting $s = \frac{1}{2}$ spins

$$\hat{H} = J\, \hat{\underline{S}}_1 \cdot \hat{\underline{S}}_2 \qquad J > 0 : \text{AFM coupling}$$

Hilbert space is spanned by:

$|\uparrow,\uparrow\rangle$, $|\uparrow,\downarrow\rangle$, $|\downarrow,\uparrow\rangle$, $|\downarrow,\downarrow\rangle$

$|1\rangle$, $|2\rangle$, $|3\rangle$, $|4\rangle$

4D Hilbert space

Hilbert space grows exponentially w/ # of lattice sites, N

↳ $2^N$ for spin $\frac{1}{2}$

$$\hat{H} = J\left(\frac{1}{2}\hat{S}_1^+ \hat{S}_2^- + \frac{1}{2}\hat{S}_1^- \hat{S}_2^+ + S_1^z S_2^z\right)$$

$$\hat{H}|1\rangle = \hat{H}|\uparrow,\uparrow\rangle = \frac{\hbar^2}{2} J |\uparrow,\uparrow\rangle$$

$$\hat{H}|4\rangle = \hat{H}|\downarrow,\downarrow\rangle = \frac{\hbar^2}{4} J |\downarrow,\downarrow\rangle$$

$$\hat{H}|2\rangle = \hat{H}|\uparrow,\downarrow\rangle = \frac{\hbar^2}{2} J |\downarrow,\uparrow\rangle - \frac{\hbar^2}{4} J |\uparrow,\downarrow\rangle$$

$$\hat{H}|3\rangle = \hat{H}|\downarrow,\uparrow\rangle = \frac{\hbar^2}{2} J |\uparrow,\downarrow\rangle - \frac{\hbar^2}{4} J |\downarrow,\uparrow\rangle$$

$$\underline{H} = \frac{\hbar^2}{4} J \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

haven't used for a while,
took some time to
familiarize

Diagonalizing : $A = \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix}$

$$\det \begin{pmatrix} -1-\lambda & 2 \\ 2 & -1-\lambda \end{pmatrix} = 0$$

$$(1+\lambda)^2 - 4 = 0$$

$$(1+\lambda)^2 = 4$$
$$1+\lambda = \pm 2$$
$$\lambda = \pm 2 - 1$$

$$\lambda_1 = 1 \qquad D = \begin{pmatrix} 1 & 0 \\ 0 & -3 \end{pmatrix}$$
$$\lambda_2 = -3$$

$AK = \lambda K$

$\lambda_1 = 1$

$$\begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\left.\begin{matrix} -a + 2b = a \\ 2a - b = b \end{matrix}\right] \quad a = b = 1 \qquad v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$L = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$\lambda_2 = -3$

$$L^{-1} = -\frac{1}{2}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$\left.\begin{matrix} -a + 2b = -3a \\ 2a - b = -3b \end{matrix}\right] \quad \begin{matrix} 2b = -2a \\ 2a = -2b \end{matrix}$$

$a = 1$

$b = -1$

$v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$LDL^{-1}$$
$$= \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -3 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

$$= \Big( 1$$

$$AL = \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} -1+2 & -1-2 \\ 2-1 & 2+1 \end{pmatrix} = \begin{pmatrix} 1 & -3 \\ 1 & 3 \end{pmatrix}$$

$$LD = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -3 \\ 1 & 3 \end{pmatrix}$$

$AL = LD \Rightarrow A = LDL^{-1}$ //

Resulting Hamiltonian for the two site, AFM coupling problem:

$$\underline{H} = \frac{\hbar^2}{4} J \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -3 & \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

shouldn't

$\underline{H}|2\rangle = \underline{H}|3\rangle$ ???

$\epsilon_1 \; \hat{H}|1\rangle = \frac{\hbar^2}{4} J |\uparrow,\uparrow\rangle$

$\epsilon_{24} \; \hat{H}|4\rangle = \frac{\hbar^2}{4} J |\downarrow,\downarrow\rangle$

$\epsilon_2 : \hat{H}|2\rangle = \frac{\hbar^2}{4} J |\uparrow,\downarrow\rangle$

$\epsilon_3 \; \hat{H}|3\rangle = -\frac{3\hbar^2}{4} J |\downarrow,\uparrow\rangle$

$$\cancel{\frac{\hbar^2}{4}} J \overset{|2\rangle}{\cancel{|\uparrow,\downarrow\rangle}} = 2\cancel{\frac{\hbar^2}{4}} J \overset{|3\rangle}{\cancel{|\downarrow,\uparrow\rangle}} - \cancel{\frac{\hbar^2}{4}} J \overset{|2\rangle}{\cancel{|\uparrow,\downarrow\rangle}}$$

$|2\rangle = 2|3\rangle - |2\rangle$

$2|2\rangle = 2|3\rangle$

$|2\rangle = |3\rangle$ ⟹ so need normalization?

+ i think not, coz assuming eigenbasis all normalized.

or

$-\frac{3}{4}|3\rangle = \frac{1}{2}|2\rangle - \frac{1}{4}|3\rangle$

$-2|3\rangle = 2|2\rangle$

$|2\rangle = -|3\rangle$

$$\underline{\underline{|2\rangle = \pm|3\rangle}}$$

PhD

ground state of some non-integrable model

need field theory

theory

eigenvectors:

$$\psi_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \psi_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

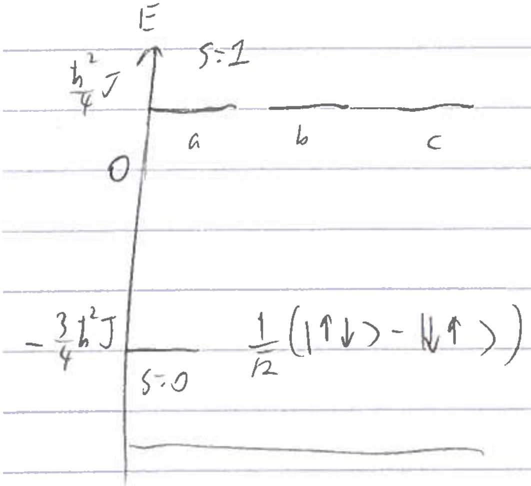
$$\psi_2 : \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} \quad \psi_3 = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$$

$\hookrightarrow \frac{1}{\sqrt{2}} \left( |\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle \right)$ $\quad \hookrightarrow \frac{1}{\sqrt{2}} \left( |\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle \right)$

1. Qm oscillator $a^+, a$
2. 2nd Quantisation

Zacherni, Kancka?

Resulting diagonalized Hamiltonian given as:



, $|S, m_s\rangle$

a: $|\uparrow\uparrow\rangle$ , $|1,1\rangle$

b: $\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$, $|1,0\rangle$

c: $|\downarrow\downarrow\rangle$ , $|1,-1\rangle$

, $|0,0\rangle$

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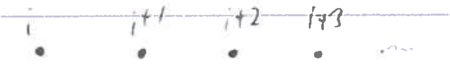
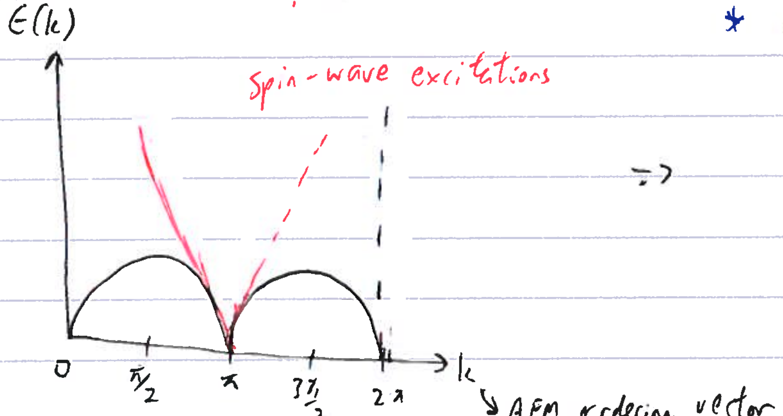
Heisenberg Antiferromagnetic Chain

AFM : $J>0$



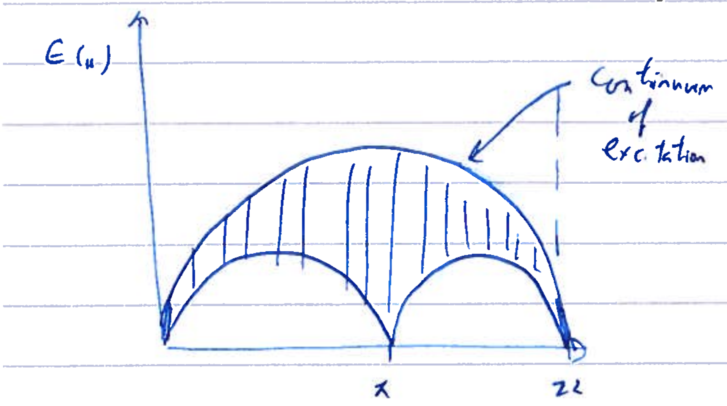
$$\hat{\mathcal{H}} = J \sum_i \hat{\underline{S}}_i \cdot \hat{\underline{S}}_{i+1}$$



$$e^{iqr} = e^{iqn} = e^{i\pi n} = (-1)^n$$

↳ Néel order (AFM order)

↑↓ ↑↓ ↑↓ ↑↓



Haldane Conjecture:

$S = \frac{1}{2}, \frac{3}{2}, \frac{5}{2} \dots$ ⇒ excitation is gapless

$S = 1, 2, 3, \dots$ => excitation spectrum is gapped

=>

* Spin $\frac{1}{2}$ is special => no sharp spin-wave excitations

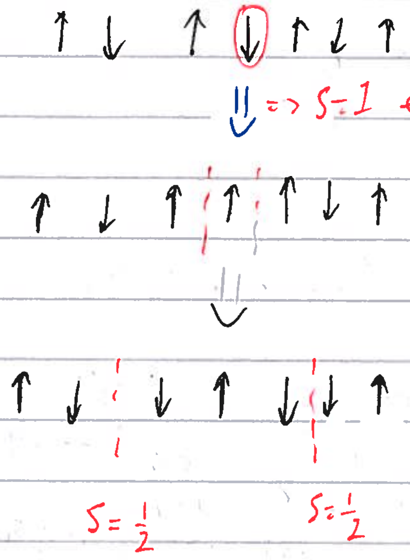
there is an exact spectrum

↳ Bethe Ansatz (see other sheets)

* Spin wave excitations

↳ quasiparticles (magnons)

spin-wave excitations break up into pair of "spinons" (domain walls)

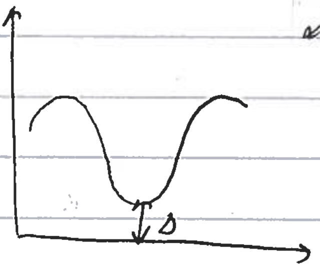


⇓ => S=1 excitation, $\Delta E = 4J$

domain walls are like fractionalised quasiparticles that can propagate freely (in a continuum.)

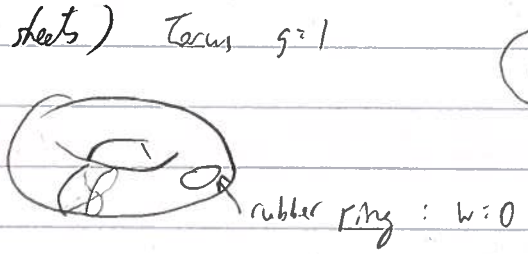
For the S=1 case:

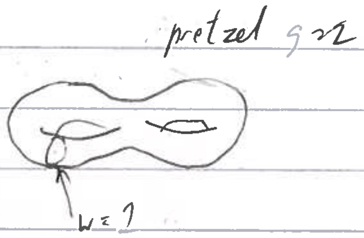
determined expts numerically & other methods



Δ: excitation => caused by Topological order
↳ topological gap.

Topological indicies: (see other sheets)





winding number (w)
genus of a manifold (g)

topological quantum state are potentially useful for quantum computing
↳ ∵ topological protection
↳ small, smooth perturbations cannot change the topology

# Generalisation of the Haldane model :

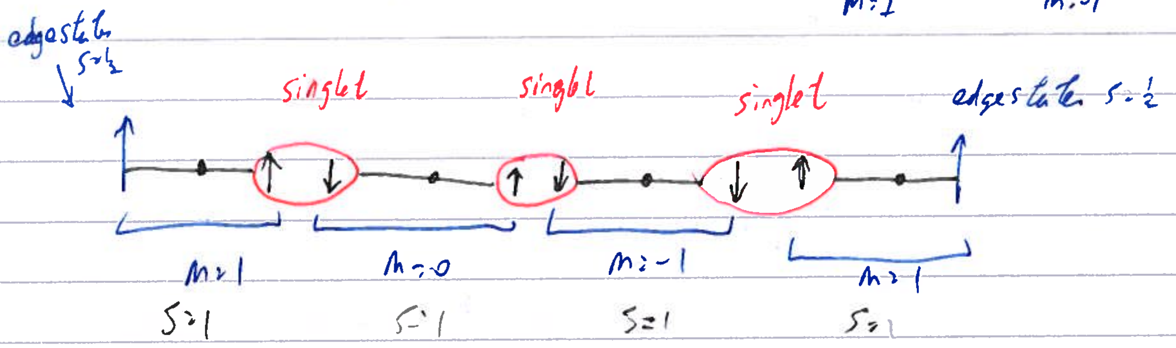
$$\hat{\mathcal{H}} = J \sum_i \hat{\underline{S}}_i \cdot \hat{\underline{S}}_{i+1} + \beta J \underbrace{\sum_i (\hat{\underline{S}}_i \cdot \hat{\underline{S}}_{i+1})^2}_{\text{biquadratic coupling}}$$

↳ biquadratic coupling

$\beta = 0$ : Haldane model is non-integrable

$\beta = \frac{1}{3}$ : AKLT model is integrable, w/ exact known ground state.

now we can write $S=1$ in terms of triplet state of two spin $\frac{1}{2}$ : $\overset{s=\frac{1}{2}}{\uparrow}\!-\!\overset{s=\frac{1}{2}}{\uparrow}$ ($m=1$), $\downarrow\downarrow$ ($m=-1$), $\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$ ($m=0$)



possible states (as a rule of thumb)

$+ - 0\,0 + - 0 + -$

$+ 0\,0 - 0\,0 + - + - 0\,0\,0 +$

↳ Néel state $+ - + -$

w/ any number of "0" insertions ⇒ hidden order

↳ can be linked to a string order parameter (see other notes)

↳ (product of spin operators) non-local operator ↳ infinite product

↳ can be used to map out the spins

✻ ground state is non-magnetic (product of singlets)

✻ magnetic excitations are gapped.

# EXTRA!

## The Harmonic Oscillator => Annihilation & Creation Operators

Modern Qm by Prof J.J. Sakurai

Pg 89.

$$\mathcal{H} = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2} \qquad \omega = \sqrt{\frac{k}{\mu}}$$

Def: two non-Hermitian operators: x & p are operators

$$a = \sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{ip}{m\omega}\right) \quad ; \quad a^{\dagger} = \sqrt{\frac{m\omega}{2\hbar}}\left(x - \frac{ip}{m\omega}\right)$$

(annihilation operator) (creation operator)

$$[a, a^{\dagger}] = aa^{\dagger} - a^{\dagger}a = \frac{m\omega}{2\hbar}\left[\left(x + \frac{ip}{m\omega}\right)\left(x - \frac{ip}{m\omega}\right) - \left(x - \frac{ip}{m\omega}\right)\left(x + \frac{ip}{m\omega}\right)\right]$$

$$= \frac{\cancel{m\omega}}{2\hbar}\left[\left(\cancel{x^2} - \frac{xip}{\cancel{m\omega}} + \frac{ipx}{\cancel{m\omega}}\right) - \left(\cancel{x^2} + \frac{xip}{\cancel{m\omega}} - \frac{ipx}{\cancel{m\omega}}\right)\right]$$

$$= \frac{\cancel{m\omega}}{2\hbar}\left[-xip + ipx - xip + ipx\right]$$

$$= \frac{i}{\hbar}\left[px - xp\right] = \frac{i}{\hbar}[p, x] = -\frac{i(i\hbar)}{\hbar} = 1$$

$$\boxed{[a, a^{\dagger}] = 1}$$

Number operator $N = a^{\dagger}a$

$$a^{\dagger}a = \frac{m\omega}{2\hbar}\left[\left(x^2 + \frac{p^2}{m^2\omega^2}\right) + \frac{xip}{m\omega} - \frac{ipx}{m\omega}\right]$$

$$= \frac{m\omega^2 x^2}{2\hbar\omega} + \frac{p^2}{2\hbar m\omega} + \frac{i}{2\hbar}(xp - px]$$

$$= \frac{1}{\hbar\omega}\underbrace{\left(\frac{m\omega^2 x^2}{2} + \frac{p^2}{2m}\right)}_{H} + \frac{i}{2\hbar}\underbrace{[x, p]}_{i\hbar}$$

$$N = a^{\dagger}a = \frac{H}{\hbar\omega} - \frac{1}{2} \Rightarrow H = \hbar\omega\left(N + \frac{1}{2}\right)$$

Some commutative relationship aside:

$$[N, a] = [a^{\dagger}a, a] = a^{\dagger}\underbrace{[a, a]}_{0} + \underbrace{[a^{\dagger}, a]}_{-1}a$$

$$[N, a] = -a$$

$$[N, a^{\dagger}] = a^{\dagger}$$

for some eigenstate $|n\rangle$ w/ eigenvalue $n$

$$N|n\rangle = n|n\rangle$$

$$H|n\rangle = (n+\tfrac{1}{2})\hbar\omega|n\rangle$$

☆ $$Na^\dagger|n\rangle = (\underbrace{[N,a^\dagger]}_{a^\dagger}+a^\dagger N)|n\rangle$$
$$= a^\dagger(1+N)|n\rangle = (n+1)a^\dagger|n\rangle$$

$\Rightarrow$ $a^\dagger|n\rangle$ is an eigenket of $N$ w/ eigenvalue ↑↑ by 1 [Creation]

∴ ↑↑ one unit of $\hbar\omega$

☆ $$Na|n\rangle = ([N,a]+aN)|n\rangle = (n-1)a|n\rangle$$

$\Rightarrow$ $a|n\rangle$ is an eigenket of $N$ w/ eigenvalue ↓↓ by 1

∴ ↓↓ one unit of $\hbar\omega$ [annihilation)

↳ implies $a|n\rangle$ & $|n-1\rangle$ are the same, up to a multiplicative constant, $c$.

$$a|n\rangle = c|n-1\rangle$$

for both $|n\rangle$ & $|n-1\rangle$ to be normalized. $\mathbb{C}$

$$\langle n|a^\dagger a|n\rangle = |c|^2$$

$$n = |c|^2 \Rightarrow$$

for $c>0$ and $c\in\mathbb{R}$ (by convention)

$$c = \sqrt{n}$$

$$\boxed{\begin{aligned} a|n\rangle &= \sqrt{n}\,|n-1\rangle \\ \hookrightarrow a^\dagger|n\rangle &= \sqrt{n+1}\,|n+1\rangle \end{aligned}}$$

in the condition that

☆ $$n = \langle n|N|n\rangle = (\langle n|a^\dagger).(a|n\rangle) \geq 0$$

EXTRA:

for $n=0$

$E_0 = \frac{\hbar\omega}{2} \Rightarrow H|n\rangle = E_0|n\rangle = E|0\rangle$

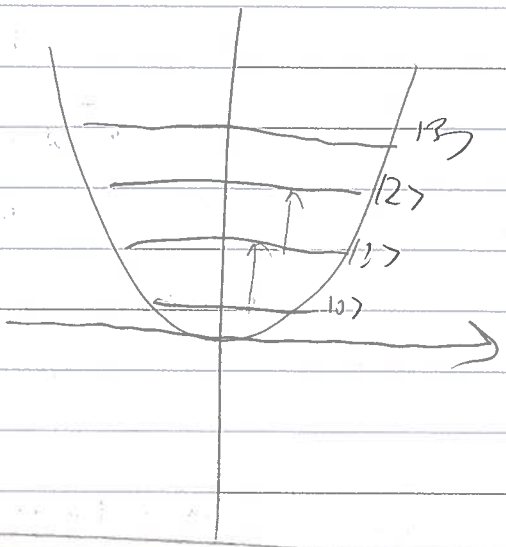
applying the creation operator to $|0\rangle$

$|1\rangle = a^\dagger|0\rangle$

$|2\rangle = \left(\frac{a^\dagger}{\sqrt{2}}\right)|1\rangle = \left(\frac{(a^\dagger)^2}{\sqrt{2}}\right)|0\rangle$

$\vdots$

$|n\rangle = \left[\frac{(a^\dagger)^n}{\sqrt{n!}}\right]|0\rangle$



2nd Quantisation (by Frank's [illegible] book: Condensed Matter Field Theory; A Altland & B. Simons, 2010)

- for many body systems
- for interpreting creation of quanta of energy by ladder operators ($a^\dagger$, $a$)

★ indistinguishability of fermions & bosons:

under particle exchange w.r.t. position $x_1$ & $x_2$ $(\lambda_1, \lambda_2)$

$$|\lambda_1,\lambda_2(x_1, x_2)\rangle = |\lambda_1,\lambda_2\rangle_{F(B)} \equiv \frac{1}{\sqrt{2}}\left(\langle x_1|\lambda_1\rangle\langle x_2|\lambda_2\rangle + \zeta\langle x_1|\lambda_2\rangle\langle x_2|\lambda_1\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(|\lambda_1\rangle \otimes |\lambda_2\rangle + \zeta|\lambda_2\rangle \otimes |\lambda_1\rangle\right)$$

★ Permutation operator

$\zeta = +1$ for bosons

$\zeta = -1$ for fermions

$\otimes$ : antisymmetrized / symmeterised product

$$\hat{P}_{ij}\,\psi(x_1, \dots x_i \dots x_j \dots x_n) = \zeta\psi(x_1, \dots, x_i, \dots, x_j \dots x_n)$$

$$\hat{P}_{ij}\,\hat{P}_{ji} = 1$$

Basis state for many particle state:

$$\psi(x_1,\dots,x_N) = N \sum_P (\pm)^{\overset{\text{order of permutation}}{P}} \underbrace{\psi_{\alpha_1}(x_1)\dots\psi_{\alpha_N}(x_N)}_{\text{Orthonormal basis states of a single particle}}$$

$N = \frac{1}{\sqrt{N!}}$ for normalization

- Occupational No.

generally: $|n_1, n_2 \dots\rangle$

Fermions: $0, 1$

Bosons: $0, 1, 2, \dots, \infty$

- Fock space: $\mathcal{F}$ => set of states w/ All possible combinations of occupation numbers

Hilbert direct sum of tensor product of copies of a single-particle Hilbert space $H$

$$\mathcal{F} \equiv \underbrace{\bigoplus_{N=0}^{\infty}}_{\text{direct sum}} \mathcal{F}^N$$

- some important commutation (anticommutation) relationships for fermions (bosons)

from: $\hat{c}_i^\dagger \hat{c}_j^\dagger |0\rangle = \pm c_j^\dagger c_i^\dagger |0\rangle$

$[c_i, c_j^\dagger] = \delta_{ij}$ $\{c_i, c_j^\dagger\} = \delta_{ij}$

$[c_i, c_j] = 0$ $\{c_i, c_j\} = 0$

$[c_i^\dagger, c_j^\dagger] = 0$ $\{c_i^\dagger, c_j^\dagger\} = 0$

- number operator: $n_\eta = c_\eta^\dagger c_\eta$

## Signature of topology order :

finite system : surface states which are topologically protected

=> for the Haldane chain : unpaired $s=\frac{1}{2}$ at the ends of the chain (edge states)
↳ topological edge states are protected by the gap, $\Delta$, of the bulk

✡ topological order cannot be destroyed by perturbation that do not close the gap

@ topological phase ~~the~~ transition : $\Delta=0$

Important: going from $\beta=\frac{1}{3}$ $(\Delta>0)$ to $\beta=0$, (Haldane model)

$\Delta$ remains finite all the way
=> Haldane model is topological

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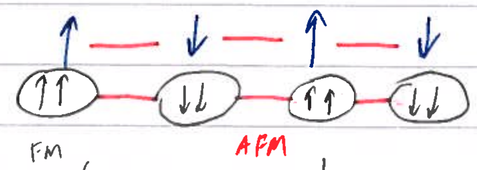
✡ In the project :

spin-$\frac{1}{2}$ model: $D=1$ using Jordan-Wigner Transformation to map onto a model of spinless ~~2~~ fermions
- contains a string operator [related to string operator which measures topological order]

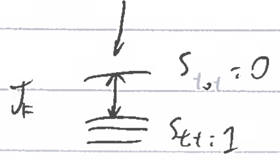
- for the AKLT ($\beta=\frac{1}{3}$, $S=1$) model, ground state is written in $S=\frac{1}{2}$ degrees of freedom
↳ why not start w/ $S=\frac{1}{2}$ model?

=> work up a $S=\frac{1}{2}$ model, that in a certain regime, behaves as the Haldane model $(s=1)$

For $J_K >> J_A$: (behave like a spin-1 object)



Haldane spin chain physics
✡ topological?



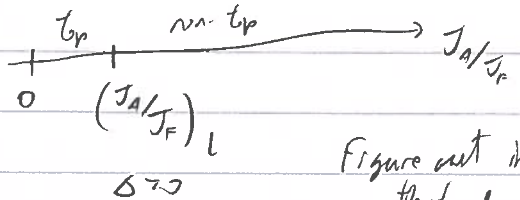


Figure out in which regime the topological order holds.

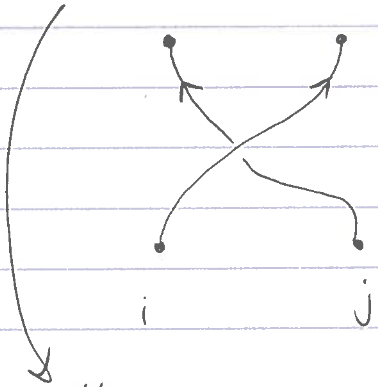
First quantised : => S.E.

=> fixed particle no.

→ N-particle wavefunctions that are eigenstate w/ energies

$\epsilon_1 \leq \epsilon_1 \leq \epsilon_2 \leq \epsilon_3 \dots \epsilon_N$

$$|\psi(r_i, \dots, r_i, \dots r_j)|^2 = |\psi(\dots, r_j, \dots r_i)|^2$$



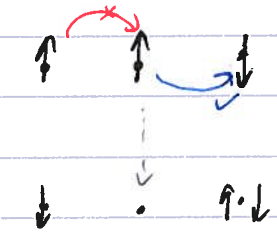
$$\psi(\dots, r_i \dots r_j \dots) = \overset{\text{bosons}}{\underset{\text{fermions}}{\pm}} \psi(\dots, r_j \dots, r_i \dots)$$

Pauli principle :

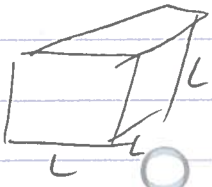
$$\psi(\dots, r_i, \dots r_i, \dots) = 0$$

✳ No two fermions can occupy the same quantum state (as from chem)

For fermions on a lattice $(\underset{\text{position}}{\underset{\uparrow}{r}}, \underset{\text{spin}}{\underset{\uparrow}{\sigma = \uparrow, \downarrow}})$



$$\psi(\underline{r} + L\hat{\underline{e}}_i) = \psi(\underline{r})$$



$V = L^3$

w/ periodic boundary cond$^{t}$:

$e^{i\underline{k}\underline{r}} e^{iLk_i}$

$= e^{i\underline{k}\underline{r}}$

$e^{iLk_i} = 1$

$k_i = \frac{2\pi}{L} n \quad n \in \mathbb{Z}$

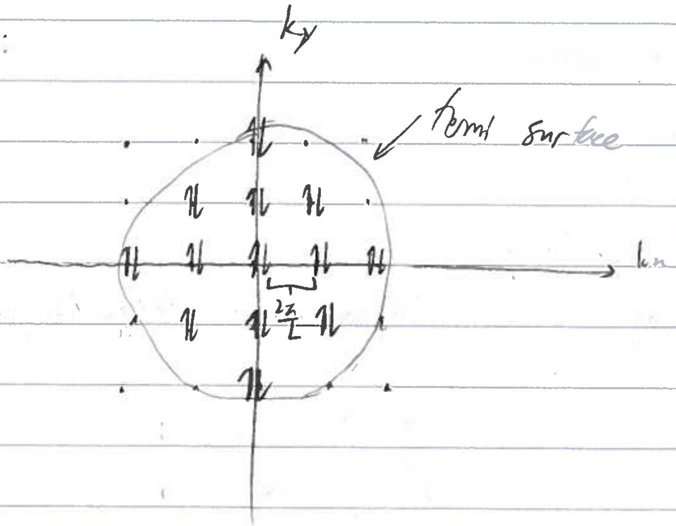
in momentum space : $(\underline{k}, \sigma = \uparrow, \downarrow)$

$$-\frac{\hbar^2}{2m} \nabla^2 \psi(r) = \epsilon \psi(r)$$

$\psi(r) \propto e^{i\underline{k}\cdot\underline{r}}$ (plane wave state)

$$\epsilon = \frac{\hbar^2 k^2}{2m}$$

$d=2$:



$L \to \infty$

density fixed

∆ ∵ Pauli principle ⇒ ground state of $N$ fermions is ∴ obtained by filling momentum states of ↑↑ energies (filling the fermi sea)

As $L \to \infty$, density is fixed

density is given by:

$V = L^3$

$\frac{2\pi}{L} = k$

$$\rho = \frac{N}{V} = \frac{N_\uparrow + N_\downarrow}{V} = \frac{2}{V}\sum_{\underline{k}}^{|k|\le k_F} = \frac{2}{(2\pi)^3}\sum_{\underline{k}}^{|\underline{k}|\le k_F} \Delta k^3$$

$k^3 = \left(\frac{2\pi}{L}\right)^3$

$L^3 = \frac{2\pi}{k^3}$

$$\xrightarrow[L\to\infty]{} \frac{2}{(2\pi)^3}\int_{|\underline{k}|\le k_F} d^3k = \frac{2}{(2\pi)^3}\,\frac{4}{3}\pi k_F^3 = \frac{1}{3\pi^2}k_F^3$$

Spherical coor

$k_F = (3\pi^2 \rho)^{\frac{1}{3}}$

$$\epsilon_F = \frac{\hbar^2 k^2}{2m} = (3\pi^2)^{\frac{2}{3}}\frac{\hbar^2}{2m}\rho^{\frac{2}{3}}$$

---

2nd Quantisation

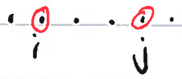
wavefunctions ⟶ operators

∆ Hilbert space for all possible particle no.

⇒ harmonic oscillator (0-dim) $\hat{H}|n\rangle = \hbar\omega(n+\frac{1}{2})|n\rangle$

$|0\rangle, |1\rangle, |2\rangle, \dots,$

$= \hbar\omega(b^\dagger b + \frac{1}{2})|n\rangle$ $n = b^\dagger b$



for the vacuum state:

$|0\rangle = |0\rangle_1 \otimes |0\rangle_2 \otimes |0\rangle_3 \dots$

(Fock space ??)

$[b, b^\dagger] = 1 \iff bb^\dagger - b^\dagger b = 1$

$[b, b] = 0, [b^\dagger, b^\dagger] = 0$

bosons

$$b_j^\dagger b_i^\dagger |0\rangle = \pm\, b_i^\dagger b_j^\dagger |0\rangle$$

fermions

$\Downarrow$ i & j exchange

$$b_i^\dagger b_j^\dagger \cancel{|0\rangle} = \pm\, b_j^\dagger b_i^\dagger \cancel{|0\rangle}$$ translating to

Commutator relationships for **bosons** $\longleftrightarrow$ Anticommutator relationship for **fermions**

$[b_i^\dagger, b_j^\dagger] = 0$ ~~for all~~ $i, j$

$[b_i, b_j] = 0$

$[b_i, b_j^\dagger] = \delta_{ij}$

$\{c_i^\dagger, c_j^\dagger\} = 0$

$\{c_i, c_j\} = 0$

$\{c_i, c_j^\dagger\} = \delta_{ij}$

eg. $c_i^\dagger c_i^\dagger |0\rangle = -c_i^\dagger c_i^\dagger |0\rangle = 0$

(Pauli)

## Hamiltonian in 2nd Quantisation

( $d=1$ , fermions) $\sigma = \uparrow, \downarrow$ ; $r = i$

$i \in \mathbb{Z}$ (lattice site)

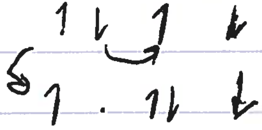
$$\hat{H} = \pm \sum_\sigma \sum_i \left( c_{i+1,\sigma}^\dagger c_{i\sigma} + c_{i\sigma}^\dagger c_{i+1,\sigma} \right)$$

$+ h.c.$ hermitian conjugate

interactions :

$$\hat{H}_{int} = U \sum_i \hat{n}_{i\uparrow} \hat{n}_{i\downarrow} = U \sum_i c_{i\uparrow}^\dagger c_{i\uparrow} c_{i\downarrow}^\dagger c_{i\downarrow} \Rightarrow$$

Hubbard Repulsion



+U

potential :

$$\hat{H} = \sum_{i\sigma} V_i \hat{n}_{i\sigma}$$

$$= \sum_{i\sigma} V_i c_{i\sigma}^\dagger c_{i\sigma}$$

- Bases of transformation:

通常 $x$ & $p$

position & momentum:

---

## Jordan-Wigner Transformation: (fermionize)

↳ for pairs of spins of diff sites.

- spin operators @ diff site commute
- fermion operators @ diff site anticommute

✱ main idea: writing spin operators in terms of number, creation & annihilation operators

$$\hat{S}^{+}_{m} = \hat{c}^{\dagger}_{m} \prod_{l<m} (1-2\hat{n}_{l}) \quad ; \quad \hat{S}^{+}_{m} = \hat{c}^{\dagger}_{m} e^{i\pi\sum_{l<m}\hat{n}_{l}} \quad ; \quad f^{\dagger}$$

$$\hat{S}^{-}_{m} = \prod_{l<m} (1-2\hat{n}_{l}) \hat{c}_{m} \quad ; \quad \hat{S}^{-}_{m} = e^{-i\pi\sum_{l<m}\hat{n}_{l}} \hat{c}_{m} \quad ; \quad f$$

$$\hat{S}^{z}_{m} = \hat{n}_{m} - \frac{1}{2} = c^{\dagger}_{m} c_{m} - \frac{1}{2} = f^{\dagger} f - \frac{1}{2}$$

$$\hat{c}^{\dagger}_{m}(1-2\hat{n}_{m}) = -(1-2\hat{n}_{m})\hat{c}^{\dagger}_{m}$$

$$\hat{c}_{m}(1-2\hat{n}_{m}) = -(1-2\hat{n}_{m})\hat{c}_{m}$$

---

## 2nd Quantisation ⇒ the Quantum Antiferromagnet (Pg. 79 Condensed Matter Field Theory)

$$\hat{H} = J \sum_{\langle mn \rangle} \hat{S}_{m} \cdot \hat{S}_{n}$$

$$J > 0$$

*

$$[\hat{S}^{+}, \hat{S}^{-}] = \hat{S}^{+}\hat{S}^{-} - \hat{S}^{-}\hat{S}^{+} = \hat{c}^{\dagger}_{m} \prod_{l<m} (1-2\hat{n}_{l}) \prod_{l<m} (1-2\hat{n}_{l}) \hat{c}_{m}$$

$$- \prod_{l<m} (1-2\hat{n}_{l}) \hat{c}_{m} \hat{c}^{\dagger}_{m} \prod_{l<m} (1-2\hat{n}_{l}) \quad \Rightarrow$$

→

$$[\hat{S}^+_m, \hat{S}^-_m] = \underbrace{\overbrace{c^\dagger_m \prod_{l<m}(1-2\hat{n}_l)}^{f^\dagger} \overbrace{\prod_{l<m}(1-2\hat{n}_l)\hat{c}_m}^{f}} - \underbrace{\prod_{l<m}(1-2\hat{n}_l)\hat{c}_m c^\dagger_m \prod_{l<m}(1-2\hat{n}_l)}$$

$$= \left[ - c^\dagger_m \prod_{l<m}(1-2\hat{n}_l)\, \hat{c}_m \prod_{l<m}(1-2\hat{n}_l) \right] - \left[ - \hat{c}_m \prod_{l<m}(1-2\hat{n}_l)\, \hat{c}^\dagger_m \prod_{l<m}(1-2\hat{n}_l) \right]$$

$$= f^\dagger f - f f^\dagger \quad = \quad \cancel{2f^\dagger f} \qquad f^\dagger f - 1 + f^\dagger f = 2f^\dagger f = 2\hat{n} - 1 = 2\hat{S}^z$$

$$\{f_i, f_i^\dagger\} = \delta_{ii} = 1$$

$$\Rightarrow f_i f_i^\dagger - f^\dagger f = 1$$

$$\cancel{-f^\dagger f = 1} \Rightarrow f f^\dagger$$

$$f_i f^\dagger = 1 + f^\dagger f$$

$$[\hat{S}^+_m, \hat{S}^-_m] = 2\hat{S}^z$$

(b) $\hat{S}^+_m \hat{S}^-_{m+1} = f^\dagger_m f_{m+1}$

$$\hat{S}^+_m \hat{S}^-_{m+1} = f^\dagger_m e^{i\pi \sum_{j<m} \hat{n}_m} e^{-i\pi \sum_{j<m+1} \hat{n}_{m+1}} f_{m+1} \qquad = 1 \ (\text{sum till } m)$$

~~aware of the different~~ indicies

$$= f^\dagger_m e^{i\pi \sum_{j<m} \hat{n}_l} e^{-i\pi \sum_{l<m+1} \hat{n}_l} f_{m+1} = f^\dagger_m e^{i\pi \sum_{j<m} \hat{n}_j} e^{-i\pi \sum_{l<m} \hat{n}_l} e^{-i\pi \hat{n}_m} f_{m+1}$$

$$[\hat{n}_j, \hat{n}_l] = 0$$

only $e^{-i\pi\hat{n}_m}$ remains

$$= f^\dagger_m e^{-i\pi\hat{n}_m} f_{m+1} = f^\dagger_m f_{m+1}$$

$\hat{n}_m \in \mathbb{Z}$

$$e^{-i\pi\hat{n}_m} \qquad 0$$

$$= \underbrace{\cos(\pi\hat{n}_m)}_{=1,-1} - i\overbrace{\sin(\pi\hat{n}_m)}$$

$\hat{n}_m = \{0, 1\}$

c)

Fermionic hopping & mapping

~~$\hat{H} \cdot |\theta| \hat{H} =$~~ $\qquad\qquad$ ~~$S^{n}$~~ $\underline{S} = \begin{pmatrix} S_x \\ S_y \\ S_z \end{pmatrix}$

~~Assuming the following form~~

~~$\hat{H} =$~~

~~$$\hat{H} = J \sum_{\langle m \rangle} \underline{\hat{S}}_m \cdot \hat{S}_{m+1} + \beta J \sum_{m} (\underline{\hat{S}}_m \cdot \underline{\hat{S}}_{m+1})^2$$~~

~~[biquadratic coupling using 1st order perturbation]~~

~~$$= \sum_m J [\hat{S}^x_m \hat{S}^x_{m+1} + \hat{S}^y_m \hat{S}^y_{m+1} + \hat{S}^z_m \hat{S}^z_{m+1}]$$~~

~~$$+ \beta J [\hat{S}^x_m \hat{S}^x_{n+} \hat{S}^y_m \hat{S}^y_{m+1} + \hat{S}^z_m$$~~

$S^{+}_m = \hat{S}^x + i \hat{S}^y$

$S^{-}_m = \hat{S}^x - i \hat{S}^y$

$$\hat{H} = \sum_n \left[ J_z \hat{S}^z_n \hat{S}^z_{n+1} + \frac{J_\perp}{2} (\hat{S}^+_n \hat{S}^-_{n+1} + \hat{S}^-_n \hat{S}^+_{n+1}) \right]$$

$\hat{S}^+ \hat{S}^- - \hat{S}^- \hat{S}^+ = 2\hat{S}^z$

using:

$$\Rightarrow \left| \begin{array}{l} [\hat{S}^+, \hat{S}] = 2\hat{S}^z \\ \hat{S}^+ S^-_{m+1} = f^+_n f_{m+1} \end{array} \right|$$

~~& Jordan-Wigner transformation~~

$\{f_i, f^\dagger_{i+1}\} = 0$

$f_i f^\dagger_{i+1} = -f^\dagger_i f_{i+1}$

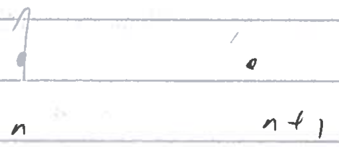
$$\hat{H} = \sum_n \left[ \left[ J_z (f^\dagger_n f_{n+1} - \tfrac{1}{2})(f^\dagger_n f_{n+1} - \tfrac{1}{2}) \right] + \left[ \frac{J_\perp}{2} (f^\dagger_n f_{n+1} + \underbrace{f_n f^\dagger_{n+1}}_{h.c.}) \right] \right]$$

$$\hat{H} = \sum_n J_z (f^\dagger_n f_{n+1} f^\dagger_n f_{n+1} - f^\dagger_n f_{n+1} + \tfrac{1}{4})$$

$$+ \frac{J_\perp}{2} (f^\dagger_n f_{n+1} + \overset{f_n f^\dagger_{n+1}}{h.c.})$$

$\Downarrow$ But how to map?

for XY-model, non-interacting tight-binding



$$\hat{H} = \sum_n \frac{J_\perp}{2} (f^\dagger_n f_{n+1} + f_n f^\dagger_{n+1})$$

$\hat{H} =$ ~~$\hat{H}$~~ Diagonalising using ~~discrete Fourier Transform~~.

finding eigen$\overset{\text{value}}{\text{vects}}$ in Fourier k space

$$\hat{H} = \frac{J}{2}\sum_n \left(f_n^\dagger f_{n+1} + f_n f_{n+1}^\dagger\right)$$

~~$\hat{H} = \frac{J}{2}\sum_n \left(f_n^\dagger \; f_{n+1}^\dagger\right)\begin{pmatrix} 0 & \epsilon(k) \\ \epsilon(k) & 0 \end{pmatrix}\begin{pmatrix} f_n \\ f_{n+1} \end{pmatrix}$~~ ??

$$d_k = \frac{1}{\sqrt{N}}\sum_{n=1}^{N} e^{-ik_n} f_n \quad \Rightarrow \quad f_n = \frac{1}{\sqrt{N}}\sum_{k=1}^{N} e^{ik_n} d_k$$

$$d_k^\dagger = \frac{1}{\sqrt{N}}\sum_{n=1}^{N} e^{-ik_n} f_n^\dagger \quad \text{reverse transform} \quad f_n^\dagger = \frac{1}{\sqrt{N}}\sum_{n=1}^{N} e^{-ik_n} d_n^\dagger$$

$$\sum_j e^{i(k_1-k_2)j} = N\delta_{k_1,k_2}$$

~~$\hat{H} = \frac{J}{2}\sum_n ($~~

rewriting each term of the Hamiltonian

$$\sum_n f_n^\dagger f_{n+1} = \sum_k e^{ik} d_k^\dagger d_k$$

$$\sum_n f_n f_{n+1}^\dagger = \sum_k e^{-ik} d_k^\dagger d_k$$

$$\epsilon(k) = J\cos(k\,\underbrace{d_k^\dagger d_k}_{a}) = J\cos(ka)$$

OK

$$\hat{H} = \sum_{ij} H_{ij}\,\hat{a}_i^\dagger \hat{a}_j = \sum_{ij} \underbrace{H_{ij}}\, a_i^\dagger a_j$$

↖Hermitian matrix ⇒ is real & can be diagonalised by a unitary transformation to a new basis

⇒

$$\hat{\alpha}_i^\dagger = U_{ij}\hat{a}_j^\dagger \Rightarrow \text{creation operator on a basis: } \hat{a}_j^\dagger = U_{ji}^\dagger \alpha_i^\dagger$$

(23)

$$\hat{H} = \sum_{ij} H_{ij} a_i^\dagger a_j$$

$$= \sum_{ijkl} H_{ij} U_{ik}\alpha_k^\dagger \alpha_l U_{lj}^\dagger = \sum_{kl} \underbrace{\sum_{ij} H_{ij} U_{ik} U_{lj}^\dagger}_{\Rightarrow (U^\dagger H U)_{kl}} \alpha_k^\dagger \alpha_l \overset{k=l}{=} \sum_l \epsilon_l \hat{\alpha}_l^\dagger \hat{\alpha}_l$$

finding a Unitary rotation that ~~diago~~ is diagonal

diagonalising $\hat{H}$

Diagonalise it:

$$\hat{\mathcal{H}} = \frac{J}{2} \sum_n ( f_n^\dagger f_{n+1} + f_n f_{n+1}^\dagger )$$

$$\Rightarrow \hat{\mathcal{H}} = \frac{J}{2} \sum_n \begin{pmatrix} f_n^\dagger & f_{n+1}^\dagger \end{pmatrix} \begin{pmatrix} 0 & \epsilon(k) \\ \epsilon(k) & 0 \end{pmatrix} \begin{pmatrix} f_n \\ f_{n+1} \end{pmatrix}$$ ✓ ??

try: $\det \begin{vmatrix} 0 & \epsilon(k) \\ \epsilon(k) & 0 \end{vmatrix} = 0$

$$-\epsilon^2(k) = 0$$

$$\epsilon^2(k) = 0$$

Trivially $\epsilon(k) = 0$

non-trivial solution:

$$\epsilon(k) = \cancel{k}(e^{i\cancel{kk}\,ka}) = 0$$

$$= \overset{k=}{\cancel{k}}\left(\cos(\overset{ka}{\cancel{kk}}) + i\sin(\tfrac{\pi}{2}k)\right)$$

$$k = \frac{\cancel{\pi} n}{2} \qquad n = 1, 2, 3$$

$$\epsilon(k) = \cos(ka)$$

$$ka = \frac{\pi}{2} n$$

$$k = \frac{\pi n}{2a}$$

∓

side proof

$$\cancel{\begin{pmatrix} f_n^\dagger & f_{n+1}^\dagger \end{pmatrix} \begin{pmatrix} \epsilon_k f_{n+1} \\ \epsilon_k f_n \end{pmatrix}}$$

$$\cancel{\Rightarrow f_n^\dagger f_{n+1} + f_{n+1}^\dagger f_n}$$

$$\begin{pmatrix} f_{n+1}^\dagger \epsilon(k) & f_n^\dagger \epsilon(k) \end{pmatrix} \begin{pmatrix} f_n \\ f_{n+1} \end{pmatrix}$$

$$= ( f_{n+1}^\dagger f_n + f_n^\dagger f_{n+1} )\, \epsilon(k)$$

kind of like a dot product

# Jordan - Wigner Transformation

1D, spin $\frac{1}{2}$

Taking the toy Hamiltonian

$$\mathcal{H} = J \sum_i \left( \underline{\hat{S}}_i \cdot \underline{\hat{S}}_{i+1} + \alpha \hat{\underline{S}}_i^z \hat{\underline{S}}_{i+1}^z \right)$$

$J > 0$ AFM

$\alpha \Rightarrow$ anisotropy term

$\alpha = 0$ isotropic case

$\Downarrow$

ground state: Néel AFM ↑↓↑↓

XXZ chain

$J_x = J_y = J \qquad J_z = (1+\alpha) J$

$\underline{\alpha > 0}$ : spins along z-axis ↑↓↑↓

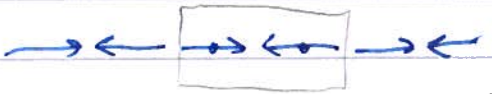
(easy - axis anisotropy)

$\underline{\alpha < 0}$ : spins in the xy plane →← →←

(eg. x-axis)

(easy-plane anisotropy)

unit cell contains two spins



we want to "flip"/"rotate" every other 2nd spin

(1 site per unit cell)

Using transformations of operators, which also satisfies with the commutator relationships

$$T_i^x = \overbrace{(-1)^i}^{e^{i\pi n} \quad n \in \mathbb{Z}} \hat{S}_i^x$$

$$T_i^y = \hat{S}_y^y$$

$$T_i^z = (-1)^i \hat{S}_z^z$$

$\Rightarrow$ $T^\alpha$ are proper spin $\frac{1}{2}$ operators

$\ast [\hat{S}_i^x, \hat{S}_i^y] = i\hbar \hat{S}_i^z$

$$\Rightarrow [T_i^x, T_i^y] = (-1)^i [\hat{S}_i^x, \hat{S}_i^y] = (-1)^i i\hbar \hat{S}_i^z$$

$$= i\hbar \hat{T}_i^z$$

$$\left.\begin{array}{lcl} 1, \; \hat{T}^+ = \hat{T}^x + i \hat{T}^y & \Leftrightarrow & \hat{T}^x = \frac{1}{2}(\hat{T}^+ + \hat{T}^-) \\ \hat{T}^- = \hat{T}^x - i \hat{T}^y & & \hat{T}^y = \frac{1}{2i}(\hat{T}^+ - \hat{T}^-) \end{array}\right] (\ast\ast\ast)$$

## Spin $\frac{1}{2}$ operator $\Longleftrightarrow$ Spinless fermions

1U

$\hat{T}^z |\frac{1}{2}, m\rangle = m |\frac{1}{2}, m\rangle$

$m = \pm\frac{1}{2}$ $(\hbar = 1)$

$\hat{T}^z |\uparrow\rangle = \frac{1}{2} |\uparrow\rangle$

$\hat{T}^z |\downarrow\rangle = -\frac{1}{2} |\downarrow\rangle$

State of spinless fermions

$|0\rangle$ : no fermion

$|1\rangle$: one fermion

nothing else -! Pauli principle

$\hat{T}^+ |\uparrow\rangle = 0$

$\hat{T}^+ |\downarrow\rangle = |\uparrow\rangle$

$\hat{T}^- |\downarrow\rangle = 0$

$\hat{T}^- |\uparrow\rangle = |\downarrow\rangle$

$\hat{c}^+ | 1\rangle = 0$

$\hat{c}^+ |0\rangle = |1\rangle$

$\hat{c}^- |0\rangle = 0$

$\hat{c}^- |1\rangle = |0\rangle$

$n = \hat{c}^+\hat{c}$ (occupation number operator)

identifying: $|\uparrow\rangle \hat{=} |0\rangle$

$|\downarrow\rangle \hat{=} |1\rangle$

$\hat{T}^z = \frac{1}{2} - \hat{c}^+\hat{c}$

$\hat{T}^+ = \hat{c}$

$\hat{T}^- = \hat{c}^+$

works on single site:

$\Rightarrow$ & assuming $c, c^\dagger$ satisfy:

$\{c, c\} = \{c^\dagger, c^\dagger\} = 0$ , $\{c_i, c_j^\dagger\} = \delta_{ij}$

$\{c, c^\dagger\} = 1$

single site @ the moment

$$[\hat{T}^x, \hat{T}^y] = \frac{1}{4i}[\hat{T}^+ + \hat{T}^-, \hat{T}^+ - \hat{T}^-] \overset{\text{linearity}}{\underset{\text{operators commute with itself}}{=}} \frac{1}{4i}[-\boxed{[\hat{T}^+, \hat{T}^-]} + [\hat{T}^-, \hat{T}^+]]$$

$$= \frac{1}{2i}[\hat{T}^-, \hat{T}^+]$$

$$= \frac{1}{2i}[\hat{T}^-\hat{T}^+ - \hat{T}^+\hat{T}^-] = \frac{1}{2i}[\hat{c}^+\hat{c} - \underbrace{\hat{c}c^+}_{1-\hat{c}^+\hat{c}}] = \frac{1}{2i}(2c^{+}c - 1) \times \frac{i}{i}$$

$$= i\left(\frac{1}{2} - \hat{c}^+\hat{c}\right) = i\,\hat{T}^z$$

✓

✡ spin operators on different site commute
fermion operators anti commute.

(**) $$\left[\begin{array}{l}[\hat{T}_i^\alpha, \hat{T}_j^\beta] = i\delta_{ij}\epsilon_{\alpha\beta\gamma}\hat{T}_i^\gamma \\ [\hat{T}_i^+, \hat{T}_j^-] = [\hat{T}_i^x + i\hat{T}_i^y, \hat{T}_j^x - i\hat{T}_j^y] = -i[\hat{T}_i^x, \hat{T}_j^y] + i[\hat{T}_i^y, \hat{T}_j^x] \\ \quad = -i[\hat{T}_i^x, \hat{T}_j^y] - i[\hat{T}^x{}_i, \hat{T}_j^y] = 2\delta_{ij}\hat{T}_i^z\end{array}\right.$$

(*) $$\left[\begin{array}{l}\{c_i, c_j\} = \{c_i^+, c_j^+\} = 0 \\ \{c_i, c_j^+\} = \delta_{ij}\end{array}\right.$$

for different sites:

$$[\hat{T}_i^\alpha, \hat{T}_j^\beta] = 0 \qquad [\hat{T}_i^+, \hat{T}_j^-] = 0$$

$$\{c_i, c_j\} = \{c^+{}_i, c_j^+\} = \{c_i, c_j^+\} = 0$$

repairing w/ string operator: (JW transformation)

$$\hat{T}_i^z = \frac{1}{2} - \hat{c}_i^+\hat{c}_i$$

$$\hat{T}_i^+ = \prod_{l<i}(1 - 2\hat{c}_l^+\hat{c}_l)\hat{c}_i$$

$$T_i^- = \hat{c}_i^+\prod_{l<i}(1 - 2\hat{c}_l^+\hat{c}_l)$$

String operator

$$\hat{D}_i = \prod_{l<i}(1 - 2\hat{c}_l^+\hat{c}_l)$$

same site:

$$[\hat{c}_i, \hat{D}_i] = \hat{c}_i\hat{D}_i - \hat{D}_i\hat{c}_i = \hat{c}_i\prod_{l<i}(1 - 2c_l^+c_l) - \hat{D}_i\hat{c}_i$$

$$\underset{(*)}{=} \underbrace{\prod_{l<i}(1 - 2\hat{c}_l^+\hat{c}_l)}_{D_i}c_i - \hat{D}_i\hat{c}_i = 0$$

$$[\hat{c}_i, \hat{D}_i] = 0 \quad ; [c_i^+, \hat{D}_i] = 0 \ ; \hat{D}_i^\dagger = \hat{D}_i$$

$$\hat{c}_i\prod_{l<i}(1 - 2\hat{c}_l^+\hat{c}_l) = \prod_{l<i}(c_i - 2\hat{c}_i\underbrace{\hat{c}_l^+}_{=\delta_{il} - \hat{c}_l^+\hat{c}_i}\hat{c}_l)$$

$$= \prod_{l<i}\hat{c}_i + 2\hat{c}_l^+\underbrace{\hat{c}_i\hat{c}_l}_{-\hat{c}_l\hat{c}_i} = \prod_{l<i}\hat{c}_i - 2\hat{c}_l^+\hat{c}_l\hat{c}_i$$

$$= \prod_{l<i}(1 - 2\hat{c}_l^+\hat{c}_l)c_i$$

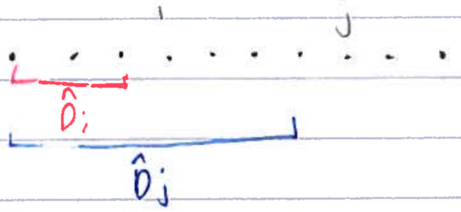
$$[\hat{D}_i^2 = \prod_{l<i}(1 - 2\hat{c}_l^+\hat{c}_l)\prod_{l<i}(1 - 2\hat{c}_l^+\underbrace{\hat{c}_l)}_{1 - \hat{c}_l^+\hat{c}_l}$$

$$= \prod_{l<i}(1 - 4c_l^+c_l + 4\hat{c}_l^+\overbrace{\hat{c}_l c_l^+}\hat{c}_l)$$

$$= \prod_{l<i}(1 - 4\cancel{c_l^+}c_l + 4\cancel{\hat{c}_l^+}c_l - 4\underbrace{\cancel{\hat{c}_l^+\hat{c}_l^+}c_lc_l}_{=0})$$

$$= 1 \qquad \boxed{\hat{D}_i^2 = 1}$$

$$\hat{D}_i\hat{D}_j = \prod_{l=i}^{j}(1 - 2\hat{c}_l^+\hat{c}_l)$$



For neighboring lattice sites:

$$\hat{D}_i\hat{D}_{i+1} = 1 - 2\hat{c}_i^+c_i$$

✡ if $\hat{c}_i, c_i^+$ satisfy fermion anti-commutator relations (*), then the operators $\hat{T}_i^\alpha$ defined by JW => satisfies the spin commutator relationships (**)

7/7/2021 : Things to do:

① For i<j, check that $[\hat{T}_i^+, \hat{T}_j^-] = 0$

② Rewrite Hamiltonian in terms of $c_i^+, c_i$, & simplify as much as possible.

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# Rewriting Hamiltonian :

$$\hat{\mathcal{H}} = J\sum_i \left( \underline{\hat{S}_i} \cdot \underline{\hat{S}_{i+1}} + \alpha \hat{S}_i^z \cdot \hat{S}_{i+1}^z \right)$$

$$= J\sum_i \left( (-1)^i \hat{T}_i^x (-1)^{i+1} \hat{T}_i^x + \hat{T}_i^y \hat{T}_{i+1}^y - (1+\alpha)\hat{T}_i^z \hat{T}_{i+1}^z \right)$$

$$= -J\sum_i \left( \hat{T}_i^x \hat{T}_{i+1}^x - T_i^y T_{i+1} + (1+\alpha) \hat{T}_i^z \hat{T}_{i+1}^z \right)$$

$$(***) \Longrightarrow -J\sum_i \left\{ \frac{1}{2}\left( \hat{T}_i^+ \hat{T}_{i+1}^- + \underbrace{\hat{T}_i^- \hat{T}_{i+1}^+}_{h.c\ ?} \right) + (1+\alpha)\hat{T}_i^z \hat{T}_{i+1}^z \right\}$$

$$= -J\sum_i \left\{ \frac{1}{2}\left( \underbrace{\hat{D}_i \hat{c}_i \hat{c}_{i+1}^\dagger \hat{D}_{i+1}}_{A} + \underbrace{\hat{c}_{i+1}^\dagger \hat{D}_{i+1} \hat{D}_i \hat{c}_i}_{B} \right) + (1+\alpha) \underbrace{\overset{\hat{T}_i^z}{\left(\frac{1}{2} - \hat{c}_i^\dagger \hat{c}_i\right)} \overset{\hat{T}_{i+1}^z}{\left(\frac{1}{2} - \hat{c}_{i+1}^\dagger \hat{c}_{i+1}\right)}}_{C}\right.$$

$$[\hat{T}_i^+, \hat{T}_{i+1}^-] = 0$$

$$\hat{T}_i^+ \hat{T}_{i+1}^- = \hat{T}_{i+1} \hat{T}_i \quad \Big\| \quad \hat{T}_i^- \hat{T}_{i+1}^+ = \hat{T}_{i+1} \hat{T}_i^-$$

$$C = \frac{1}{4} - \frac{1}{2}\left( \hat{c}_i^\dagger \hat{c}_i + \hat{c}_{i+1}^\dagger \hat{c}_{i+1} \right)$$
$$+ c_i^\dagger \underbrace{c_i c_{i+1}^\dagger}_{-c_{i+1}^\dagger c_i} c_{i+1}$$

$$= \frac{1}{4} - \frac{1}{2}\left( c_i^\dagger c_i + c_{i+1}^\dagger c_{i+1} \right)$$

$$- c_i^\dagger c_{i+1}^\dagger c_i c_{i+1}$$

$$A = \hat{D}_i \hat{c}_i \hat{c}_{i+1}^\dagger \hat{D}_{i+1} -$$

$$= c_i \underbrace{D_i D_{i+1}}_{1-2c_i^\dagger c_i} c_{i+1}^\dagger$$

$$\cancel{c_i, c_i^\dagger}$$

$$c_i c_i^\dagger + c_i^\dagger c_i = \cancel{0}1$$

$$= c_i (1 - 2 c_i^\dagger c_i) c_{i+1}^\dagger$$

$$= (c_i - 2 c_i c_i^\dagger c_i) c_{i+1}^\dagger$$

$$= c_i c_{i+1}^\dagger - 2 \underbrace{c_i c_i^\dagger}_{1-c_i^\dagger c_i} c_i c_{i+1}^\dagger = c_i c_{i+1}^\dagger - 2(1 - c_i^\dagger c_i)(c_i c_{i+1}^\dagger)$$

$$= c_i c_{i+1}^\dagger - 2 + 2 c_i^\dagger \underbrace{c_i c_i}_{=0} c_{i+1}^\dagger$$

$$A = c_i c_{i+1}^\dagger - 2$$

$$c_i c_i^\dagger + c_i^\dagger c_i = 1$$

$$B = c_i^\dagger \underbrace{D_i D_{i+1}}_{(1-2c_i^\dagger c_i)} c_{i+1}$$

$$= \left( c_i^\dagger - 2 c_i^\dagger c_i^\dagger c_i \right) c_{i+1}$$

$$= c_i^\dagger c_{i+1} - 2 \underbrace{c_i^\dagger c_i^\dagger}_{=0} c_i c_{i+1}$$

$$B = c_i^\dagger c_{i+1} \quad \checkmark$$

$$C = \frac{1}{4} - \frac{1}{2} \underbrace{c_i^\dagger c_i} - \frac{1}{2} c_{i+1}^\dagger c_{i+1} - c_i^\dagger c_{i+1}^\dagger c_i c_{i+1}$$

$$= \frac{1}{4} - \frac{1}{2}\left( c_i c_i^\dagger - 1 \right) - \frac{1}{2}\left( c_{i+1} c_{i+1}^\dagger - 1 \right) - c_i^\dagger c_{i+1}^\dagger c_i c_{i+1}$$

$$= \frac{1}{4} - \frac{1}{2} c_i c_i^\dagger + \frac{1}{2} - \frac{1}{2} c_{i+1} c_{i+1}^\dagger + \frac{1}{2} - c_i^\dagger \underbrace{c_{i+1}^\dagger c_i}_{1} c_{i+1}$$

$$= \frac{1}{4} + 1 - \frac{1}{2}\left( c_i c_i^\dagger + c_{i+1} c_{i+1}^\dagger \right) -$$

Checking $[T_i^{+}, T_j^{-}] : 0$ $\quad i<j$

$$[D_i c_i , D_j c_j^{\dagger}] = D_i \underbrace{c_i D_j}\, c_j^{\dagger} - \underbrace{D_j c_j^{\dagger}} D_i c_i$$

$$\hookrightarrow (1-2c_i^{\dagger}c_i)c_j^{\dagger} = (c_j^{\dagger} - 2c_i^{\dagger}c_i c_j^{\dagger})$$

$$c_i(\overset{\neq}{1-2c_i^{\dagger}c_i}) = c_i - 2c_i + \underbrace{2c_i^{\dagger}c_i c_i}_{=0} = -c_i$$

if $i \neq j \Rightarrow c_i(1-2c_j^{\dagger}c_j) = (1-2c_j^{\dagger}c_j)c_i$

~~$[c_i, c_j]$~~ $= c_i$ $\qquad = -D_i c_i c_j^{\dagger} - (c_j^{\dagger} - 2c_i^{\dagger}c_i c_j^{\dagger}) D_i c_i$

$\{c_i, c_j^{\dagger}\} = c_i c_j^{\dagger} + c$ $\qquad = -D_i c_i c_j^{\dagger} - (c_j^{\dagger} D_i c_i - 2c_i^{\dagger} c_i c_j^{\dagger} D_i c_i)$

$$= -D_i c_i c_j^{\dagger} - (+D_i \underbrace{c_j^{\dagger} c_i} + 2c_i^{\dagger} \underbrace{c_i c_j^{\dagger}} c_i D_i)$$

$\underline{+}$

$$\underset{=0}{=} -D_i c_i c_j^{\dagger} - (\cancel{+} D_i c_i c_j^{\dagger} \underbrace{-2c_i^{\dagger} c_j^{\dagger} c_i c_i D_i}_{=0})$$

$$\frac{1}{4} + \underbrace{\sum_i}_{\text{replace}} (\hat{n}_i \hat{n}_{i+1} - \frac{1}{2}(\hat{n}_{i+1} + \hat{n}_i))$$

Should get degen bands for new Hamiltonian

$$\sum_i \hat{n}_i \hat{n}_{i+1} = n_1 n_2 + n_2 n_3 \cdots n_i n_{i+1}$$

$$\sum_i^N \hat{n}_{i+1} = n_2 + n_3 + n_4 + \cdots + n_{N+1} = \sum_i^N n_1 + n_2 + n_3 + n_4 + n_5 \cdots n_{N+1} - n_1$$

$$\sum_i$$

$$= \sum_{k=1}$$

$$= \sum_i n$$

$i+1 \to k$

$k = i+1$

$i = k-1$

$$\sum_i^N \hat{n}_i + \hat{n}_{N+1}$$

$$\frac{N}{4} + \sum_i \hat{n}_i (\hat{n}_i + \hat{n}_{N+1}) - \frac{1}{2}\hat{n}_i - \frac{1}{2}\hat{n}_{N+1} - \frac{\hat{n}_1}{2}$$

~~$A = c_i c_{i+1}^\dagger - 2 \underbrace{c_i c_i^\dagger c_i}_{} c_{i+1}^\dagger = c_i c_{i+1}^\dagger - 2 c_i (1 - c_i c_i^\dagger) c_{i+1}^\dagger = c_i c_{i+1}^\dagger - (c_i + 2 c_i c_i^\dagger) c_{i+1}^\dagger$~~

~~$c_i c_i^\dagger + c_i^\dagger c_i = 1$~~

~~$= 1 - c_i c_i^\dagger$~~

~~$= c_i c_{i+1}^\dagger - 2 c_i c_{i+1}^\dagger - 2 c_i c_i^\dagger c_{i+1}^\dagger$~~

~~$= - c_i c_{i+1}^\dagger - 2 c_i c_i^\dagger c_{i+1}^\dagger$~~

Final Hamiltonian:

$$\hat{H} = -\frac{J}{2} \sum_i \overbrace{\left( c_{i+1}^\dagger c_i + c_i^\dagger c_{i+1} \right)}^{\text{fermionic hopping}}$$

$$+ (1+\alpha) J \sum_i \underbrace{\left(\frac{1}{2} - \hat{n}_i\right)\left(\frac{1}{2} - \hat{n}_{i+1}\right)}_{\text{contains } \hat{n}_i \hat{n}_{i+1}}$$

non-interacting terms:

- hopping terms: $\hat{c}_i^\dagger \hat{c}_{i+1} + h.c.$

- density terms: $\hat{a}^\dagger \hat{c}_i = \hat{n}_i$
- anomalous terms: $c_i^\dagger c_{i+1}^\dagger + c_{i+1} c_i$ (don't conserve particle number)

interactions:

$\hat{n}_i \hat{n}_{i+1}$ : Cannot solve the problem analytically :

→ variational method

→ perturbation theory

→ mean-field theory

$\alpha = -1$ : non-interacting system only x & y component of spin : XY spins

$$\Rightarrow \hat{H} = -\frac{J}{2} \sum_i \left( c_{i+1}^\dagger c_i + c_i^\dagger c_{i+1} \right)$$ [tight binding Hamiltonian]

$\checkmark$

$$\hat{H} = -\frac{J}{2}\sum_i (c_{i+1}^{\dagger} c_i + c_i^{\dagger} c_{i+1})$$

$\cdot \quad \cdot \quad \underbrace{\cdot \quad \cdot}_{a}$ in this case

F.T $c_x^{\dagger}$ & $c_x$ $\qquad$ $a=1$ for simplicity

to find energy eigenval. by diagonalisation

° changing from position space to momentum space, $k$ & $k'$

$$\hat{c}_x = \underbrace{\int_{-\pi/a}^{\pi/a}}_{\int_k} \frac{dk}{2\pi} \hat{c}_k^{\dagger} e^{ikx} \quad \bar{4}$$

$$c_x = \int_{k'} \hat{c}_{k'} e^{-ik'x}$$

$$\hat{H} = -\frac{J}{2}\sum_x (c_{x+a}^{\dagger} c_x + c_x^{\dagger} c_{x+a}) = J_{/2} \sum_x \int_k \int_{k'} \{ \hat{c}_k^{*} e^{ik(x+a)} \hat{c}_{k'} e^{-ik'x} + h.c. \}$$

Combining exp

$$= -\frac{J}{2}\sum_x \int_k \int_{k'} [ \hat{c}_k^{\dagger} \hat{c}_{k'} \underbrace{e^{i(k-k')x}}_{\substack{=1 \text{ if } k=k' \\ =0 \text{ if } k \neq k'}} e^{ika} + h.c. ]$$

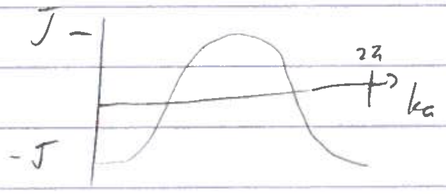
$$\sum_x e^{i(k-k')x} = \int_{-\infty}^{\infty} e^{i(k-k')x} dx = \delta_{kk'}$$

$$= -\frac{J}{2} \int_k \int_{k'} \delta_{kk'} [ \hat{c}_k^{\dagger} \hat{c}_{k'} e^{ika} + h.c. ]$$

$$\overset{k=k'}{=} -\frac{J}{2} \int_k (e^{ika} + e^{ika}) (\hat{c}_k c_k)$$

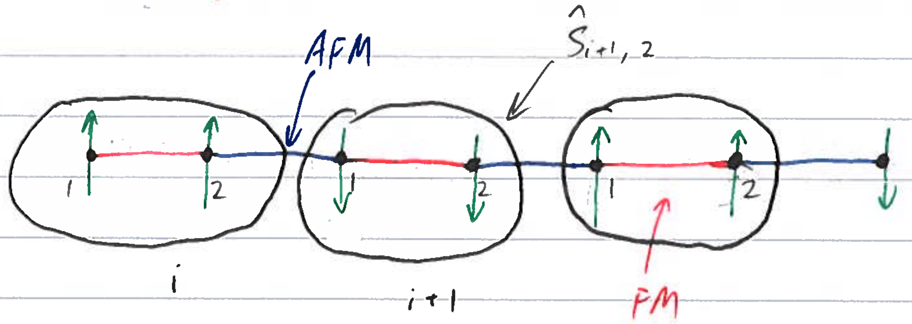
$$= -J \int_k \cos(ka) c_k^{\dagger} c_k \overset{\text{finite chain}}{=} -J \sum_k \cos(ka) c_k^{\dagger} c_k$$

Energy dispersion $\quad -J\cos(ka)$



Man

# The Model:



$$\hat{\mathcal{H}} = -J_F \sum_i \left( \underline{\hat{S}}_{i,1} \cdot \underline{\hat{S}}_{i,2} + \alpha \hat{S}^z_{i,1} \hat{S}^z_{i,2} \right)$$

$$+ J_A \sum_i \left( \underline{\hat{S}}_{i,2} \cdot \underline{\hat{S}}_{i+1,1} + \alpha \hat{S}^z_{i,2} \hat{S}^z_{i+1,1} \right)$$

$\alpha = 0$ Isotropic model (Heisenberg model)

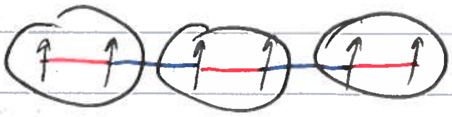
$\alpha = -1$ $xy$ model

$\therefore S^z$ does not enter

$$= -J_F \sum_i \left[ S^x_{i,1} S^x_{i,2} + S^y_{i,1} S^y_{i,2} + (1+\alpha)\left(S^z_{i,1} S^z_{i,2}\right) \right]$$

$$+ J_A \sum_i \left[ S^x_{i,2} S^x_{i+1,1} + S^y_{i,2} S^y_{i+1,1} + (1+\alpha)\left(S^z_{i,2} S^z_{i+1,1}\right) \right]$$

spin flip:



$$\left.\begin{aligned} \hat{T}^x_{i,m} &= (-1)^i \hat{S}^x_{i,m} \\ \hat{T}^y_{i,m} &= \hat{S}^y_{i,m} \\ \hat{T}^z_{i,m} &= (-1)^i \hat{S}^z_{i,m} \end{aligned}\right] \Rightarrow$$

$$H = -J_F \sum_i \Big[ \overbrace{T^x_{i,1} T^x_{i,2}}^{A} + \overbrace{T^y_{i,1} T^y_{i,2}}^{B} + (1+\alpha) \overbrace{T^z_{i,1} T^z_{i,2}}^{C}$$

$$- J_A \sum_i \Big[ \overbrace{T^x_{i,2} T^x_{i+1,1}}^{D} - \overbrace{T^y_{i,2} T^y_{i+1,1}}^{E} + (1+\alpha) \overbrace{T^z_{i,2} T^z_{i+1,1}}^{F} \Big]$$

$\Downarrow$ 1) $T^x, T^y \longrightarrow T^+, T^-$

2) JW ~~tranf~~ trans

$i \to i,m$

JW:

$$\hat{T}^z_{i,m} = \frac{1}{2} - c^\dagger_{i,m} c_{i,m}$$

$$\hat{T}^+_{i,m} = \hat{c}_{i,m} \hat{D}_{i,m}$$

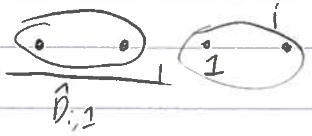
$$\hat{T}^-_{i,m} = \hat{c}^\dagger_{i,m} \hat{D}_{i,m}$$

$$\hat{D}^2_{i,m} = 1$$

$$T^x = \frac{1}{2}(T^+ + T^-)$$

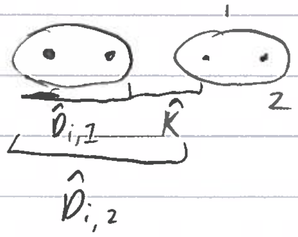
$$T^y = \frac{1}{2}(T^+ - T^-)$$

$$\hat{D}_{i,1} = \prod_{\ell < i} (1 - 2c^\dagger_{\ell_1} c_{\ell_1})(1 - 2c^\dagger_{\ell_2} c_{\ell_2})$$



$$\hat{D}_{i,2} = \prod_{\ell < i} (1 - 2c^\dagger_{\ell_1} c_{\ell_1})(1 - 2c^\dagger_{\ell_2} c_{\ell_2}) \overbrace{(1 - 2c^\dagger_{i_1} c_{i_1})}^{\hat{K}}$$

$$= \hat{D}_{i,1} (1 - 2c^\dagger_{i_1} c_{i_1})$$



Double checking the calculation

[illegible]

$$A+B: \; \underline{T^x_{1,1}T^x_{i,2} + T^y_{1,1}T^y_{i,2}} = \frac{1}{2}\left(T^+_{i,1}T^-_{i,2} + T^-_{i,1}T^+_{i,2}\right)$$

$$= \frac{1}{2}\left(c_{i,1}D_{i,1}\underline{D_{i,2}c^\dagger_{i,2}} + D_{i,1}c^\dagger_{i,1}D_{i,2}c_{i,2}\right)$$

$$= \frac{1}{2}\left(c_{i,1}\underbrace{D_{i,1}D_{i,1}}_{1}(1-2c^\dagger_{i,1}c_{i,1})c^\dagger_{i,2} + c^\dagger_{i,1}\underbrace{D_{i,1}D_{i,1}}_{1}(1-2c^\dagger_{i,1}c_{i,1})c_{i,2}\right)$$

$$= \frac{1}{2}\left(c_{i,1}c^\dagger_{i,2} - 2\underbrace{c_{i,1}c^\dagger_{i,1}}_{1-c^\dagger_{i,1}c_{i,1}}c_{i,1}c^\dagger_{i,2} + c^\dagger_{i,1}c_{i,2} - \underbrace{2c^\dagger_{i,1}c^\dagger_{i,1}c_{i,1}c_{i,2}}\right)$$

$$= \frac{1}{2}\left(c_{i,1}c^\dagger_{i,2} - 2c_{i,1}c^\dagger_{i,2} + 2c^\dagger_{i,1}c_{i,1}c_{i,1}c^\dagger_{i,2} + c^\dagger_{i,1}c_{i,2}\right)$$

$$= \frac{1}{2}\left(c^\dagger_{i,1}c_{i,2} + c^\dagger_{i,2}c_{i,1}\right)$$

$$H_F = -\frac{J_F}{2}\sum_i\left[c^\dagger_{i,1}c_{i,2} + c^\dagger_{i,2}c_{i,1} + 2(1+\alpha)\left(\frac{1}{2} - c^\dagger_{i,1}c_{i,1}\right)\left(\frac{1}{2} - c^\dagger_{i,2}c_{i,2}\right)\right]$$

D − E :

$$T^x_{i,2}T^x_{i+1,1} - T^y_{i,2}T^y_{i+1,1} = \frac{1}{4}\left(T^+_{i,2} + T^-_{i,2}\right)\left(T^+_{i+1,1} + T^-_{i+1,1}\right)$$
$$+\frac{1}{4}\left(T^+_{i,2} - T^-_{i,2}\right)\left(T^+_{i+1,1} - T^-_{i+1,1}\right)$$

$$\Rightarrow \frac{1}{4}\left(T^+_{i,2}T^+_{i+1,1} + \cancel{T^+_{i,2}T^-_{i+1,1}} + \cancel{T^-_{i,2}T^+_{i+1,1}} + T^-_{i,2}T^-_{i+1,1}\right)$$

$$+ \frac{1}{4}\left(T^+_{i,2}T^+_{i+1,1} - \cancel{T^+_{i,2}T^-_{i+1,1}} - \cancel{T^-_{i,2}T^+_{i+1,1}} + T^-_{i,2}T^-_{i+1,1}\right.$$

$$= \frac{1}{2}\left(T^+_{i,2}T^+_{i+1,1} + T^-_{i,2}T^-_{i+1,1}\right) = \frac{1}{2}\left(c_{i,2}D_{i,2}D_{i+1,1}c_{i+1,1} + c^\dagger_{i,2}D_{i,2}D_{i+1,1}c^\dagger_{i+1,1}\right)$$

$D-E = \frac{1}{2}(c_{i,2} D_{i,2} D_{i+1,1} c_{i+1,1} + c_{i,2}^\dagger D_{i,2} D_{i+1,1} c_{i+1,1}^\dagger)$

(*)

$= \frac{1}{2}(c_{i,2} \overbrace{D_{i,2} D_{i,2}}^{1} (1-2c_{i,2}^\dagger c_{i,2}) c_{i+1,1}$

$+ c_{i,2}^\dagger \overbrace{D_{i,2} D_{i,2}}^{1} (1-2c_{i,2}^\dagger c_{i,2}) c_{i+1,1}^\dagger)$

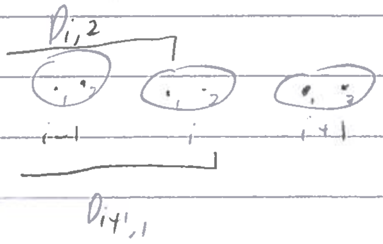
$= \frac{1}{2}(c_{i,2} c_{i+1,1} - 2 \overbrace{c_{i,2} c_{i,2}^\dagger}^{1-c_{i,2}^\dagger c_{i,2}} c_{i,2} c_{i+1,1}$

$+ c_{i,2}^\dagger c_{i+1,1}^\dagger - 2 \overbrace{c_{i,2}^\dagger c_{i,2}^\dagger}^{=0} c_{i,2} c_{i+1,1}^\dagger)$

$= \frac{1}{2}(c_{i,2} c_{i+1,1} - 2(1-c_{i,2}^\dagger c_{i,2}) c_{i2} c_{i+1,1}$

$+ c_{i,2}^\dagger c_{i+1,1}^\dagger)$

$D_{i+1,1} = D_{i,2}(1-2c_{i,2}^\dagger c_{i,2})$



(*) $D_{i+1,1} = D_{i,2}(1-2c_{i,2}^\dagger c_{i,2})$

$= \frac{1}{2}(c_{i,2} c_{i+1,1} - 2c_{i2} c_{i+1,1} + 2c_{i,2}^\dagger \cancel{c_{i2}} c_{i,2} c_{i+1,1} + c_{i,2}^\dagger c_{i+1,1}^\dagger)$

$= \frac{1}{2}(\overbrace{-c_{i,2} c_{i+1,1}}^{\{i,i\}=0} + c_{i,2}^\dagger c_{i+1,1}^\dagger) = \frac{1}{2}(c_{i+1,1} c_{i,2} + c_{i,2}^\dagger c_{i+1,1}^\dagger)$

|| using $\beta := \frac{J_A}{J_F}$

$\hat{H}_A = -\overbrace{\frac{J_A}{2}}^{\frac{J_F\beta}{2}} \sum_i [c_{i+1,1} c_{i,2} + c_{i,2}^\dagger c_{i+1,1}^\dagger + 2(1+\alpha)(\frac{1}{2} - c_{i,2}^\dagger c_{i,2})(\frac{1}{2} - c_{i+1,1}^\dagger c_{i+1,1})$

$\hat{H} = \hat{H}_F + \hat{H}_A$

$\hat{H} = -\frac{J_F}{2}\sum_i [c_{i,1}^\dagger c_{i,2} + c_{i,2}^\dagger c_{i,1} + \beta(c_{i+1,1} c_{i,2} + c_{i,2}^\dagger c_{i+1,1}^\dagger)]$

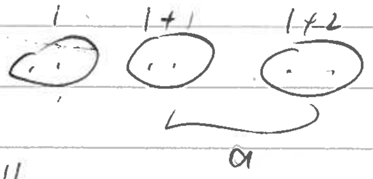
$- J_F(1+\alpha)\sum_i (\frac{1}{2} - c_{i,1}^\dagger c_{i,1})(\frac{1}{2} - c_{i,2}^\dagger c_{i,2})$

$- J_F\beta(1+\alpha)\sum_i (\frac{1}{2} - c_{i,2}^\dagger c_{i,2})(\frac{1}{2} - c_{i+1,1}^\dagger c_{i+1,1})$

Easy case: XY case $\Rightarrow \alpha = -1$

$\hat{H}_{XY} = -\frac{J_F}{2}\sum_i (c_{i,1}^\dagger c_{i,2} + c_{i,2}^\dagger c_{i,1} + \beta(c_{i+1,1} c_{i,2} + c_{i,2}^\dagger c_{i+1,1}^\dagger))$

FT to change to momentum space
& diagonalize



Hopping terms:

$$\sum_i (c^\dagger_{i,1} c_{i,2} + h.c.) \Rightarrow \sum_x (c_1^\dagger(x) c_2(x) + h.c.)$$

$$c_1^\dagger(x) = \int_k c_1^\dagger(k) e^{ikx}$$

$$c_1(x) = \int_k c_1(k) e^{-ikx}$$

$$= \sum_x \iint_{kk'} c_1^\dagger(k) e^{ikx} c_2(k') e^{-ik'x} + c_2^\dagger(k) e^{-ikx} c_1^{*}(k') e^{ikx}$$

$$= \sum_x \iint_{kk'} c_1^\dagger(k) c_2(k') e^{i(k-k')x} + c_2^\dagger(k) c_1^{*}(k') e^{-i(k-k')x}$$

$$\sum_x e^{i(k-k')x} = \delta_{kk'} \quad (**)$$

$$\overset{(**)(k=k')}{=} \int_k c_1^\dagger(k) c_2(k) + c_2^\dagger(k) c_1^{*}(k)$$

$$\overset{(***)}{=} \overset{??}{\frac{1}{2}} \int_0 \frac{dk}{2\pi}\Big[c_1^\dagger(k) c_2(k) + c_2^\dagger(k) c_1^{*}(k) - c_2(-k) c_1^\dagger(-k) - c_1(-k) c_2^\dagger(-k)\Big]$$

or

$$k \in [-\pi, \pi) \quad \int_{-\pi}^{\pi} \frac{dk}{2\pi} f(k)$$

$$\underset{\substack{\tilde{k}=-k \\ d\tilde{k}=-dk}}{\Rightarrow} = -\int_{\pi}^{-\pi} \frac{d\tilde{k}}{2\pi} f(-\tilde{k}) = \int_{-\pi}^{\pi} \frac{d\tilde{k}}{2\pi} f(-\tilde{k})$$

$$= \frac{1}{2} \int_{-\pi}^{\pi} \frac{dk}{2\pi} (f(k) + f(-k))$$

$$\overset{\text{even fcn}}{=} \int_0^{\pi} \frac{dk}{2\pi} (f(k) + f(-k))$$

or odd

$$= 0$$

$$\therefore \int_{-\pi}^{\pi} \frac{dk}{2\pi} f(k) = \int_0^{\pi} \frac{dk}{2\pi} (f(k) + f(-k)) \quad (***)$$

Anomalous terms: $\sum_i c^\dagger_{i,2} c^\dagger_{i+1,1} + c_{i+1,1} c_{i,2}$

$$= \sum_x c_2^\dagger(x) c_1^\dagger(x+a) + c_1(x+a) c_2(x)$$

$$\overset{FT}{=} \sum_x \iint_{kk'} c_2^\dagger(k) e^{ikx} e^{+ik'(x+a)} c_1^\dagger(k') + e^{-ik(x+a)} c_1(k) e^{-ik'x} c_2(k')$$

$$= \sum_x \iint_{kk'} c_2^\dagger(k) c_1^\dagger(k') e^{i(k+k')x} e^{ik'a} + c_1(k) c_2(k') e^{-i(k+k')x} e^{-ika}$$

$$= \sum_x \iint_{kk'} \delta_{k,k'} \Big[ e^{-ik'a} c_2^\dagger(k) c_1^\dagger(k') + e^{-ika} c_1(k) c_2(k') \Big) \Big]$$

2 Af

$$\overset{-k'=k}{=} \int_k e^{-ika} c_2^+(k) c_1^+(-k) + e^{-ika} c_1(k) c_2(-k))$$

$$= \int_{-\pi}^{\pi} \frac{dk}{2\pi} \left[ e^{-ika} c_2^+(k) c_1^+(-k) + e^{-ika} c_1(k) c_2(-k) \right]$$

$$\overset{(***)}{=} \int_0^{\pi} \frac{dk}{2\pi} \Big[ e^{-ika} c_2^+(k) c_1^+(-k) + e^{-ika} \overbrace{c_1(k) c_2(-k)}^{-c_2(-k)c_1(k))}$$

$$+ e^{ika} \underbrace{c_2^+(-k) c_1^+(k)}_{-c_1^+(k) c_2^+(-k)} + e^{ika} c_1(-k) c_2(k))$$

$$= \int_0^{\pi} \frac{dk}{2\pi} \Big[ e^{-ika} c_2^+(k) c_1^+(-k) - e^{-ika} c_2(-k) c_1(k)$$

$$- e^{ika} c_1^+(k) c_2^+(-k) + e^{ika} c_1(-k) c_2(k) \Big] //$$

$$\hat{H}_{xy} = -\frac{J_F}{2} \int_k (c_1^+(k), c_2^+(k), c_1(-k), c_2(-k)) \; \underline{\underline{M}} \begin{pmatrix} c_1(k) \\ c_2(k) \\ c_1^+(-k) \\ c_2^+(-k) \end{pmatrix} \Rightarrow$$

to incorporate anomalous & hopping terms

$$\underline{\underline{M}} = \left( \begin{array}{cc|cc} & \square & & \times \\ \square & & \times & \\ \hline & \times & & \square \\ \times & & \square & \end{array} \right)$$

$$\hat{H}_{xy} = -\frac{J_F}{2} \int_k c_1^+(k) c_2(k) + c_2^+(k) c_1(k) + e^{-ika} \beta \left( c_2^+(k) c_1^+(-k) + c_1(k) c_2(-k) \right) )$$

The Hamiltonian:

$$\hat{H} = -\frac{J_F}{2}\int_0^{\pi}\frac{dk}{2\pi}\Big[c_1^+(k)c_2(k) + c_2^+(k)c_1(k) - c_2(-k)c_1^+(-k) - c_1(-k)c_2^+(-k)$$

$$+ e^{-ika}\beta\, c_2^+(k)c_1^+(-k) - e^{-ika}\beta\, c_2(-k)c_1(k)$$

$$- e^{ika}\beta\, c_1^+(k)c_2^+(-k) + \beta e^{ika} c_1(-k)c_2(k)$$

By doubling dimensions:

$$\hat{H}_{xy} = -\frac{J_F}{2}\int_k \left(c_1^+(k), c_2^+(k), c_1(-k), c_2(-k)\right)\underline{\underline{M}}\begin{pmatrix} c_1(k) \\ c_2(k) \\ c_1^+(-k) \\ c_2^+(-k)\end{pmatrix}$$

$$\underline{\underline{M}} = \left(\begin{array}{cc|cc} 0 & 1 & 0 & -\beta e^{ika} \\ 1 & 0 & \beta e^{-ika} & 0 \\ \hline 0 & \beta e^{ika} & 0 & -1 \\ -\beta e^{ika} & 0 & -1 & 0 \end{array}\right) \checkmark$$

diagonalizing using Mathematica

$$\epsilon_1(k) = \pm\frac{J_F}{2}\sqrt{\beta e^{-ika} + (\beta^2+1)\beta e^{ika}} = \pm\frac{J_F}{2}\sqrt{\beta^2+1+2\beta\cos(ka)}$$

$$\epsilon_2(k) = \pm\frac{J_F}{2}\sqrt{(\beta^2+1)e^{2ika}e^{-2ika} - \beta e^{ika-2ika} + \beta e^{ika}}$$

$k \in [0, \pi]$

$\beta = J_A / J_F$

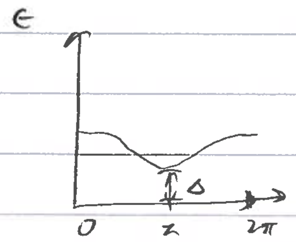
$$= \pm\frac{J_F}{2}\sqrt{\beta^2+1 - 2\beta\cos(ka)}$$

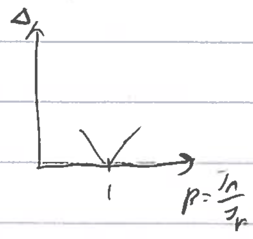
For $a=1$

$$\epsilon(k) = \frac{J_F}{2}\sqrt{\beta^2+1+2\beta\cos(k)}$$

Plot[ε(k), {k, 0, 2π}]   2π periodic

See Mathematica for plot





$\Delta = 0$ for $\beta = 1$ ⟺ $J_A = J_F$

⇒ topological phase transition

$$\Delta = \epsilon(\pi) = \frac{J_F}{2}\sqrt{\beta^2+1-2\beta} = \frac{J_F}{2}\sqrt{(\beta-1)^2}$$

$$= \frac{J_F}{2}|\beta - 1|$$

To do: do a easier 2x2 matrix

introducing new fermion operators: then do the Bogoliubov transformation

1. Check for:

$$d_1(k) = \frac{1}{\sqrt{2}}(c_1(k) + c_2(k))$$

$$d_2(k) = \frac{1}{\sqrt{2}}(c_1(k) - c_2(k)) \qquad k \in [-\pi, \pi]$$

$$\hat{H} = \int_k (d_1^{+}(k) d_1(-k)) \begin{pmatrix} & 2\times 2 & \end{pmatrix} \begin{pmatrix} d_1(k) \\ d_1^{+}(-k) \end{pmatrix} + \int_k (d_2^{+}(k) d_2(-k)) \underbrace{\begin{pmatrix} 2\times 2 \end{pmatrix}}_{\tilde{\underline{\underline{\Omega}}}} \begin{pmatrix} d_2(k) \\ d_2^{+}(-k) \end{pmatrix}$$

$$\hat{H} = \int_k ((d_1^{+}(k) d_1(-k)\ d_2^{+}(k) d_2(-k))) \tilde{\underline{\underline{\Omega}}} \begin{pmatrix} d_1(k) \\ d_1^{+}(-k) \\ d_2(k) \\ d_2^{+}(-k) \end{pmatrix}$$

$$\tilde{\underline{\underline{\Omega}}} = \left(\begin{array}{cc|cc} \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & 0 & 0 \\ \hline 0 & 0 & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot \end{array}\right) \qquad \pm\epsilon_1, \ \pm\epsilon_2$$

2. new "arrangement"

$$\hat{H} = \int_k (c_1^{+}(k), c_1(-k), c_2^{+}(k), c_2(-k)) \underline{\underline{\Omega}} \begin{pmatrix} c_1(k) \\ c_1^{+}(-k) \\ c_2(k) \\ c_2^{+}(-k) \end{pmatrix}$$

$k \in [0, \pi]$

3. See Bogoliubov transformation

# Mean field approx. MFA.

Let $\hat{A}, \hat{B}$ be operators

We have $\hat{A} \cdot \hat{B}$ in the Hamiltonian

for eg

$$\cancel{-\frac{J_F}{2}} - J_F \beta (1+\alpha) \sum_i \big(\underbrace{\tfrac{1}{2} - c^\dagger_{i2} c_{i2}}_{\hat{A}}\big)\big(\underbrace{\tfrac{1}{2} - c^\dagger_{i+1,1} c_{i+1,1}}_{\hat{B}}\big)$$ for

We write $\hat{A}\, 0\, \hat{B}$ as:

$\hat{A} = \langle \hat{A} \rangle + \delta \hat{A}$ $\quad$ $\langle \hat{O} \rangle$ are just numbers

$\hat{B} = \langle \hat{B} \rangle + \delta \hat{B}$

$$\hat{A} \cdot \hat{B} = \langle \hat{A} \rangle \langle \hat{B} \rangle + \langle \hat{A} \rangle \delta \hat{B} + \langle \hat{B} \rangle \delta \hat{A} + \cancel{\delta \hat{A} \cdot \delta \hat{B}}$$ ← neglect for mean-field approx

$$= \langle \hat{A} \rangle \langle \hat{B} \rangle + \langle \hat{A} \rangle (\hat{B} - \langle \hat{B} \rangle) + \langle \hat{B} \rangle (\hat{A} - \langle \hat{A} \rangle)$$

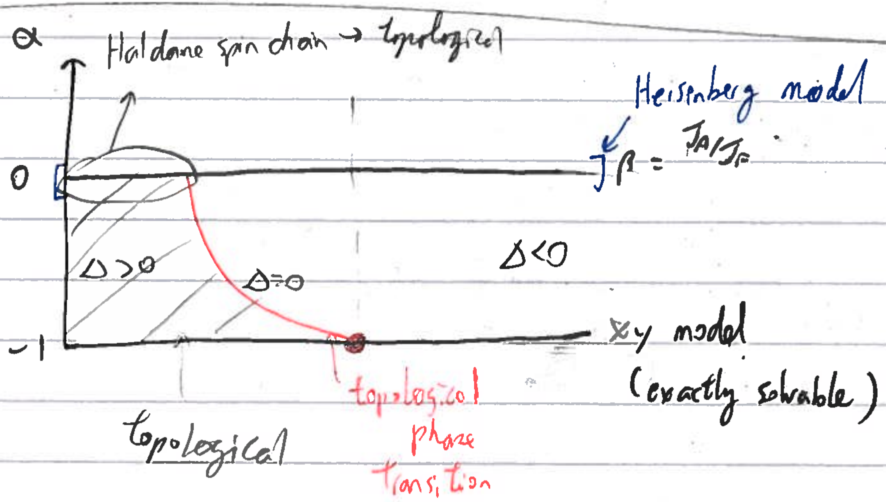
$$= \cancel{\langle \hat{A} \rangle \langle \hat{B} \rangle} + \langle \hat{A} \rangle \hat{B} - \cancel{\langle \hat{A} \rangle \langle \hat{B} \rangle} + \hat{A} \langle \hat{B} \rangle - \langle \hat{A} \rangle \langle \hat{B} \rangle$$

$$\hat{A} \cdot \hat{B} \approx \langle \hat{A} \rangle \hat{B} + \hat{A} \langle \hat{B} \rangle - \langle \hat{A} \rangle \langle \hat{B} \rangle$$

For the Hamiltonian:

do the same

Aim



$$\hat H = -\frac{J_F}{2}\sum_i\Big[\overbrace{c_{i1}^{\dagger}c_{i2}}^{\hat E} + \overbrace{c_{i2}^{\dagger}c_{i1}}^{\hat F} + \beta\big(\overbrace{c_{i+1,1}c_{i2}}^{\hat G} + \overbrace{c_{i2}^{\dagger}c_{i+1,1}^{\dagger}}^{\hat H}\big)\Big] \quad [I]$$

$$- J_F(1+\alpha)\sum_i\left(\tfrac12 - c_{i1}^{\dagger}c_{i1}\right)\left(\tfrac12 - c_{i2}^{\dagger}c_{i2}\right) \quad ]_{II}$$

$$- J_F\,\beta(1+\alpha)\sum_i\left(\tfrac12 - c_{i2}^{\dagger}c_{i2}\right)\left(\tfrac12 - c_{i+1,1}^{\dagger}c_{i+1,1}\right) \quad ]_{III}$$

Not quite right

Rewriting for mean-field approx to make interacting terms vanish for to make it exactly solvable & make it into non-interacting

$$\underbrace{\overbrace{c_{i1}^{\dagger}c_{i1}}^{\hat D}\,\overbrace{c_{i2}^{\dagger}c_{i2}}^{\cancel{\hat B}\,\hat A}}_{\hat E\quad \cancel{B}\quad F} \qquad c_{i1}^{\dagger}c_{i2}$$

$$\underbrace{\overbrace{c_{i2}^{\dagger}c_{i2}}^{\hat A}\,\overbrace{c_{i+1,1}^{\dagger}c_{i+1,1}}^{B}}_{\hat H\qquad\qquad \hat G}$$

$\hat A = c_{i2}^{\dagger}c_{i2}$ ; $\hat D = c_{i1}^{\dagger}c_{i1}$

$\hat B = c_{i+1,1}^{\dagger}c_{i+1,1}$ ; $\hat E = c_{i1}^{\dagger}c_{i2}$

$\hat F = c_{i2}^{\dagger}c_{i1}$ ; $\hat G = c_{i+1,1}c_{i2}$

$H = c_{i2}^{\dagger}c_{i+1,1}^{\dagger}$

$$\underline{III}\ \frac1N\sum_i \frac14 - \frac12 c_{i+1,1}^{\dagger}c_{i+1,1} - \frac12 c_{i2}^{\dagger}c_{i2} + c_{i2}^{\dagger}c_{i2}c_{i+1,1}^{\dagger}c_{i+1,1}$$

$$\overset{\text{MFA}}{\approx} \frac14 - \frac1N\sum_i\frac12\hat B + \frac12\hat A + \left(\hat A\langle\hat B\rangle + \langle\hat A\rangle\hat B - \langle\hat A\rangle\langle\hat B\rangle\right)$$

similarly ↓

$$\underline{II}:\ \frac14 - \frac12 c_{i2}^{\dagger}c_{i2} - \frac12 c_{i1}^{\dagger}c_{i1} + c_{i1}^{\dagger}c_{i1}c_{i2}^{\dagger}c_{i2}$$

$$\overset{\text{MFA}}{\approx} \frac14 - \frac12\hat D - \frac12\hat A + \left(\hat D\langle\hat A\rangle + \hat A\langle\hat D\rangle - \langle\hat D\rangle\langle\hat A\rangle\right)$$

$$\hat H = -\frac{J_F}{2}\sum_i\Big[\hat E + \hat F + \beta\hat G + \beta\hat H + 2(1+\alpha)\left(\frac14 - \frac12(\hat D + \hat A) + \hat D\langle\hat A\rangle + \hat A\langle\hat D\rangle - \langle\hat D\rangle\langle\hat A\rangle\right)$$
$$+ 2\beta(1+\alpha)\left(\frac14 - \frac12(\hat B + \hat A) + \hat A\langle\hat B\rangle + \langle\hat A\rangle\hat B - \langle\hat A\rangle\langle\hat B\rangle\right)\Big]$$

$$= -\frac{J_F}{2}\sum_i\Big[\hat E + \hat F + \beta\hat G + \beta\hat H + (2+2\alpha)\Big(\frac14 - \frac12(\hat A + \hat D) + \hat D\langle\hat A\rangle + \hat A\langle\hat D\rangle - \langle\hat D\rangle\langle\hat A\rangle$$
$$+ \frac{\beta}{4} - \frac{\beta}{2}(\hat B + \hat A) + \beta\hat A\langle\hat B\rangle + \beta\langle\hat A\rangle\hat B - \beta\langle\hat A\rangle\langle\hat B\rangle\Big)$$

MFA continued

A:

$$\hat{H} = -\frac{J_F}{2}\sum_i \Big[ \hat{E} + \hat{F} + \beta\hat{G} + \beta\hat{H} + \frac{1}{2} - \hat{A} - \hat{D} + 2\hat{D}\langle\hat{A}\rangle + 2\hat{A}\langle\hat{D}\rangle - 2\langle\hat{D}\rangle\langle\hat{A}\rangle$$

$$+ \frac{\beta}{2} - \beta(\hat{B} + \hat{A}) + 2\beta\hat{A}\langle\hat{B}\rangle + 2\beta\langle\hat{A}\rangle\hat{B} - 2\beta\langle\hat{A}\rangle\langle\hat{B}\rangle$$

$$+ \frac{\alpha}{2} - \alpha\hat{A} - \alpha\hat{D} + 2\alpha\hat{D}\langle\hat{A}\rangle + 2\alpha\hat{A}\langle\hat{D}\rangle - 2\alpha\langle\hat{D}\rangle\langle\hat{A}\rangle$$

$$+ \frac{\alpha\beta}{2} - \alpha\beta\hat{B} - \alpha\beta\hat{A} + 2\alpha\beta\hat{A}\langle\hat{B}\rangle + 2\alpha\beta\langle\hat{A}\rangle\hat{B} - 2\alpha\beta\langle\hat{A}\rangle\langle\hat{B}\rangle$$

$$= -\frac{J_F}{2}\sum_i \Big[ \hat{E} + \hat{F} + \frac{1}{2} - \hat{A} - \hat{D} + 2\hat{D}\langle\hat{A}\rangle + 2\hat{A}\langle\hat{D}\rangle - 2\langle\hat{D}\rangle\langle\hat{A}\rangle$$

$$+ \beta(\hat{G} + \hat{H}) + \beta\Big(\frac{1}{2} - \hat{B} - \hat{A} + 2\hat{A}\langle\hat{B}\rangle + 2\langle\hat{A}\rangle\hat{B} - 2\langle\hat{A}\rangle\langle\hat{B}\rangle\Big)$$

$$+ \alpha\Big(\frac{1}{2} - \hat{A} - \hat{D} + 2\hat{D}\langle\hat{A}\rangle + 2\langle\hat{A}\rangle\langle\hat{D}\rangle - 2\langle\hat{D}\rangle\langle\hat{A}\rangle$$

$$+ \alpha\beta\Big(\frac{1}{2} - \hat{B} - \hat{A} + 2\hat{A}\langle\hat{B}\rangle + 2\langle\hat{A}\rangle\hat{B} - 2\langle\hat{A}\rangle\langle\hat{B}\rangle \Big]$$

$$= -\frac{J_F}{2}\sum_i \Big[ \hat{E} + \hat{F} + (1+\alpha)\Big(\frac{1}{2} - \hat{A} - \hat{D} + 2\hat{D}\langle\hat{A}\rangle + 2\hat{A}\langle\hat{D}\rangle - 2\langle\hat{D}\rangle\langle\hat{A}\rangle\Big)$$

$$\beta(\hat{G} + \hat{H}) + \beta\Big(\frac{1}{2} - \hat{B} - \hat{A} + 2\hat{A}\langle\hat{B}\rangle + 2\langle\hat{A}\rangle\hat{B} - 2\langle\hat{A}\rangle\langle\hat{B}\rangle\Big)(1+\alpha)\Big]$$

$$\hat{H} = -\frac{J_F}{2}\sum_i \Big[ \hat{E} + \hat{F} + \beta(\hat{G} + \hat{H}) + (1+\alpha)\Big(\frac{1}{2} - \hat{A} - \hat{D} + 2\hat{D}\langle\hat{A}\rangle + 2\hat{A}\langle\hat{D}\rangle - 2\langle\hat{D}\rangle\langle\hat{A}\rangle$$

$$+ \beta\Big(\frac{1}{2} - \hat{B} - \hat{A} + 2\hat{A}\langle\hat{B}\rangle + 2\langle\hat{A}\rangle\hat{B} - 2\langle\hat{A}\rangle\langle\hat{B}\rangle\Big)\Big]$$

~~$\hat{H} = -\frac{J_F}{2}\sum_i [\hat{E} + \hat{F}$~~

OR

$$\frac{\partial H}{\partial \alpha} = 0 \quad ??$$

$$\frac{\partial H}{\partial \beta} = 0 \quad ??$$

for minimizing ?

~~$\hat{H} = \sum - \frac{J_F}{2} \sum_i$~~

$$\frac{\partial \hat{H}}{\partial \alpha} = -\frac{J_F}{2} \sum_i \Big( \frac{1}{2} - \hat{A}\ \hat{D} + 2\hat{D}\langle \hat{A} \rangle + 2\hat{A}\langle \hat{D} \rangle - 2\langle \hat{D} \rangle \langle \hat{A} \rangle$$
$$+ \beta\Big( \frac{1}{2} - \hat{B} - \hat{A} + 2\hat{A}\langle \hat{B} \rangle + 2\langle \hat{A} \rangle \hat{B} - 2\langle \hat{A} \rangle \langle \hat{B} \rangle \Big)\Big)$$

$$\frac{\partial \hat{H}}{\partial \beta} = -\frac{J_F}{2} \sum_i \hat{G} + \hat{H} + (1+\alpha)\Big(\frac{1}{2} - \hat{B} - \hat{A} + 2\hat{A}\langle \hat{B} \rangle\Big) + 2\langle \hat{A} \rangle \hat{B} - 2\langle \hat{A} \rangle \langle \hat{B} \rangle \Big)$$

$$\frac{\partial \hat{H}}{\partial \alpha} = \frac{\partial \hat{H}}{\partial \beta} = 0$$

$$\Rightarrow \frac{1}{2} - \hat{A} - \hat{D} + 2\hat{D}\langle \hat{A} \rangle + 2\hat{A}\langle \hat{D} \rangle - 2\langle \hat{D} \rangle \langle \hat{A} \rangle$$
$$+ \beta\Big( \frac{1}{2} - \hat{B} - \hat{A} - 2\hat{A}\langle \hat{B} \rangle + 2\langle \hat{A} \rangle \hat{B} - 2\langle \hat{A} \rangle \langle \hat{B} \rangle$$

~~$\hat{G} + \hat{H}$~~ $+ (1+\alpha)\Big(\frac{1}{2} - \hat{B} - \hat{A} + 2\hat{A}\langle \hat{B} \rangle + 2\langle \hat{A} \rangle \hat{B} - 2\langle \hat{A} \rangle \langle \hat{B} \rangle \Big)\Big)$

$$= \hat{G} + \hat{H}$$

$\langle \hat{A}$

$$\Rightarrow (\beta - 1 - \alpha)\Big(\frac{1}{2} - \hat{B} - \hat{A} + 2\hat{A}\langle \hat{B} \rangle + 2\langle \hat{A} \rangle \hat{B} - 2\langle \hat{A} \rangle \langle \hat{B} \rangle\Big)$$
$$= \hat{G} + \hat{H} - \frac{1}{2} + \hat{A} + \hat{D} - 2\hat{D}\langle \hat{A} \rangle + 2\hat{A}\langle \hat{D} \rangle - 2\langle \hat{D} \rangle \langle \hat{A} \rangle$$

$\langle \hat{A} \rangle : a \in \mathbb{R}$
$\langle \hat{D} \rangle : d \in \mathbb{R}$
$\langle \hat{B} \rangle = b \in \mathbb{R}$

$$\Rightarrow \quad (\beta - 1 - \alpha)\Big(\frac{1}{2} - \hat{B} - \hat{A} + 2b \cdot \hat{A} + 2a\hat{B} - 2ab\Big)$$
$$= \hat{G} + \hat{H} - \frac{1}{2} + \hat{A} + \hat{D} - 2a\hat{D} + 2d\hat{A} - 2ad$$

$$+ (\beta - \alpha)\Big(\frac{1}{2} - \hat{B} - \hat{A} + 2b\hat{A} + 2a\hat{B} - 2ab\Big) - \cancel{\frac{1}{2}} + \hat{B} + \cancel{\hat{A}} - 2b\hat{A} - 2a\hat{B} + 2ab$$

$$= \hat{G} + \hat{H} - \cancel{\frac{1}{2}} + \cancel{\hat{A}} + \hat{D} - 2a\hat{D} + 2d\hat{A} - 2ad$$

New arrangements

Introducing new fermion operators :

$$\begin{tikzcd} d_1(k) = \frac{1}{\sqrt{2}}\left(c_1(k) + c_2(k)\right) \arrow[d, leftrightarrow] \arrow[dr, leftrightarrow] & d_1^{+}(k) = \frac{1}{\sqrt{2}}\left(c_1^{+}(k) + c_2^{+}(k)\right) \arrow[d, leftrightarrow] \arrow[dl, leftrightarrow] \\ d_2(k) = \frac{1}{\sqrt{2}}\left(c_1(k) - c_2(k)\right) & d_2^{+}(k) = \frac{1}{\sqrt{2}}\left(c_1^{+}(k) - c_2^{+}(k)\right) \end{tikzcd}$$

checking the anti-commutator relationships

✓ $\{d_1(k), d_1^{+}k\} = d_1(k)d_1^{+}(k) + d_1^{+}(k)d_1(k)$

$$= \frac{1}{2}\left[\left(c_1(k)+c_2(k)\right)\left(c_1^{+}(k)+c_2^{+}(k)\right) + \left(c_1^{+}(k)+c_2^{+}(k)\right)\left(c_1(k)+c_2(k)\right)\right]$$

$$= \frac{1}{2}\Big[\overbrace{\{c_1(k), c_1^{+}(k)\}}^{1} + \overbrace{\{c_1(k), c_2^{+}(k)\}}^{0} \qquad = 1$$

$$+ \underbrace{\{c_2(k), c_1^{+}(k)\}}_{0} + \overbrace{\{c_2(k), c_2^{+}(k)\}}^{1}$$

✓ $\{d_2(k), d_2^{+}(k)\} = \frac{1}{2}\left[\left(c_1(k) - c_2(k)\right)\left(c_1^{+}(k) - c_2^{+}(k)\right) + \left(c_1^{+}(k) - c_2^{+}(k)\right)\left(c_1(k) - c_2(k)\right)\right]$

$$= \frac{1}{2}\Big[\{c_1(k), c_1^{+}(k)\} - \{c_1(k), c_2^{+}(k)\}$$

$$- \{c_2(k), c_1^{+}(k)\} + \{c_2(k), c_2^{+}(k)\}\Big] = 1$$

✓

$\{d_1(k), d_2^{+}(k)\} = \frac{1}{2}\left[\left(c_1(k) + c_2(k)\right)\left(c_1^{+}(k) - c_2^{+}(k)\right) + \left(c_1^{+}(k) - c_2^{+}(k)\right)\left(c_1(k) + c_2(k)\right)\right]$

$$= \frac{1}{2}\Big[\overbrace{\{c_1(k), c_1^{+}(k)\}}^{1} + \overbrace{\{c_2(k), c_1^{+}(k)\}}^{0} \qquad = 0$$

$$- \underbrace{\left(c_1(k)c_2^{+}(k) + c_2^{+}(k)c_1(k)\right)}_{0} - \underbrace{\{c_2(k), c_2^{+}(k)\}}_{1}\Big]$$

✓ $\{d_1^{+}(k), d_2(k)\} = \frac{1}{2}\left[\left(c_1^{+}(k) + c_2^{+}(k)\right)\left(c_1(k) - c_2(k)\right) + \left(c_1(k) - c_2(k)\right)\left(c_1^{+}(k) + c_2^{+}(k)\right)\right]$

$$= \frac{1}{2}\left[\{c_1^{+}(k), c_1(k)\} - \{c_1^{+}(k), c_2(k)\} + \{c_2^{+}(k), c_1(k)\} - \{c_2(k), c_2^{+}(k)\}\right]$$

$$= 0$$

~~[illegible] $d_1$~~ $d_1$, $d_1^{+}$, $d_2$, $d_2^{+}$ all satisfy the anticommutator relationship

Checking :

$$\hat{H} = \int_k \overbrace{\left( d_1^+(k) \; d_1(-k) \right) \underline{\underline{M}}_1 \begin{pmatrix} d_1(k) \\ d_1^+(-k) \end{pmatrix}}^{H_1}$$

$$+ \underbrace{\int_k \left( d_2^+(k) \; d_2(-k) \right) \underline{\underline{M}}_2 \begin{pmatrix} d_2(k) \\ d_2^+(-k) \end{pmatrix}}_{H_2}$$

1.

$$H_1 = \left( d_1^+(k) \; d_1(-k) \right) \underline{\underline{M}}_1 \begin{pmatrix} d_1(k) \\ d_1^+(-k) \end{pmatrix} = [illegible]$$

$$\left( d_1^+(k) \; d_1(-k) \right) \underline{\underline{M}}_1 = \frac{1}{\sqrt{2}} \left( \left( c_1^+(k) + c_2^+(k) \right), \left( c_1(-k) + c_2(-k) \right) \right) \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} \left[ m_1 \left( c_1^+(k) + c_2^+(k) \right) + m_3 \left( c_1(-k) + c_2(-k) \right) \right] \\ m_2 \left( c_1^+(k) + c_2^+(k) + m_7 \left( c_1(-k) + c_2(-k) \right) \right) \end{pmatrix}^T$$

OR

$$H_1 = \left( m_1 d_1^+(k) + m_3 d_1(-k) , \; m_2 d_1^+(k) + m_4 d_1(-k) \right) \begin{pmatrix} d_1(k) \\ d_1^+(-k) \end{pmatrix}$$

$$= m_1 d_1^+(k) d_1(k) + m_3 d_1(-k) d_1(k) + m_2 d_1^+(k) d_1^+(-k) + m_4 d_1(-k) d_1^+(-k)$$

similarly

$$H_2 = p_1 d_2^+(k) d_2(k) + p_3 d_2(-k) d_2(k) + p_2 d_2^+(k) d_2^+(-k) + p_4 d_2(-k) d_2^+(-k)$$

"doesn't really work"

$$\hat{H} = \int_k \left( d_1^+(k), \underline{d_1(-k)}, d_2^+(k), \underline{d_2(-k)} \right) \underline{\underline{\tilde{\Omega}}} \begin{pmatrix} d_1(k) \\ d_1^+(-k) \\ d_2(k) \\ d_1^+(-k) \end{pmatrix}$$

$$d_1^+(k)\, d_1(k) = \frac{1}{2}\left[ \overset{(0)}{c_1^+(k)\, c_1(k)} + \underline{\overset{(1)}{c_2^+(k)\, c_1(k)}} + \underline{\overset{(1)}{c_1^+(k)\, c_2(k)}} + \overset{(2)}{c_2^+(k)\, c_2(k)} \right)$$

$$d_2^+(k)\, d_2(k) = \frac{1}{2}\left[ c_1^+(k)\, c_1(k) - \underline{c_2^+(k)\, c_1(k)} - \underline{c_1^+(k)\, c_2(k)} + c_2^+(k)\, c_2(k) \right)$$

$$d_1(-k)\, d_1^+(-k) = \frac{1}{2}\left[ c_1(-k)\, c_1^+(-k) + \underline{c_1(-k)\, c_2^+(-k)} + \underline{c_2(-k)\, c_1^+(-k)} + c_2(-k)\, c_1^+(-k) \right]$$

$$d_2(-k)\, d_2^+(-k) = \frac{1}{2}\left[ c_1(-k)\, c_1^+(-k) - \underline{c_1(-k)\, c_2^+(-k)} - \underline{c_2(-k)\, c_1^+(-k)} + c_2(-k)\, c_2^+(-k) \right]$$

$$\left( d_1^+(k), d_1(-k), d_2^+(k), d_2(-k) \right) \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ m_1 & m_2 & m_3 & n_4 \\ n_1 & n_2 & n_3 & n_4 \end{pmatrix} \begin{pmatrix} d_1(k) \\ d_1^+(-k) \\ d_2(k) \\ d_2^+(-k) \end{pmatrix}$$

$$\begin{pmatrix} d_1^+(k)\, a_1 + d_1(-k)\, b_1 + d_2^+(k)\, m_1 + d_2(-k)\, n_1 \\ d_1^+(k)\, a_2 + d_1(-k)\, b_2 + d_2^+(k)\, m_2 + d_2(-k)\, n_2 \\ d_1^+(k)\, a_3 + d_1(-k)\, b_3 + d_2^+(k)\, m_3 + d_2(-k)\, n_3 \\ d_1^+(k)\, a_4 + d_1(-k)\, b_4 + d_2^+(k)\, m_4 + d_2(-k)\, n_4 \end{pmatrix}^T \begin{pmatrix} d_1(k) \\ d_1^+(-k) \\ d_2(k) \\ d_2^+(-k) \end{pmatrix}$$

$$\begin{aligned} = {} & a_1 d_1^+(k) d_1(k) + b_1 d_1(-k) \underline{d_1(k)} + m_1 d_2^+(k) d_1(k) + n_1 d_2(-k)\, d_1(k) && \text{JA} \\ & + a_2 d_1^+(k) \underline{d_1^+(-k)} + b_2 \underline{d_1(-k)} d_1^+(-k) + m_2 d_2^+(k) \underline{d_1^+(-k)} + n_2 \underline{d_2(-k)} d_1^+(-k) && \text{JB} \\ & + a_3 d_1^+(k) d_2(k) + b_3 d_1(-k) \underline{d_2(k)} + m_3 d_2^+(k) d_2(k) + m_3 d_2(-k) \underline{d_2(k)} && \text{JC} \\ & + a_4 d_1^+(k) d_2^+(-k) + b_4 d_1(-k)\, d_2^+(-k) + m_4 d_2^+(k) \underline{d_2^+(-k)} + m_4 d_2(-k) \underline{d_2^+(-k)} && \text{JD} \end{aligned}$$

$$(c_1^+(k)\; c_1(-k)\; c_2^+(k)\; \underline{c_2(-k)})^T\; \underline{\underline{\Omega}} \begin{pmatrix} c_1(k) \\ c_1^+(-k) \\ c_2(-k) \\ c_2^+(-k) \end{pmatrix}$$

$$\downarrow$$

$$\begin{pmatrix} 0 & 0 & \gamma & -\mu \\ 0 & 0 & \mu & -\gamma \\ \gamma & \mu^* & 0 & 0 \\ -\mu^* & -\gamma & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \gamma c_2^+(k) - \mu^* c_2(-k) \\ \mu^* c_2^+(k) - \gamma c_2(-k) \\ \gamma c_1^+(k) + \mu c_1(-k) \\ -\mu c_1^+(k) - \gamma c_1(-k) \end{pmatrix}^T \begin{pmatrix} c_1(k) \\ c_1^+(-k) \\ c_2(k) \\ c_2^+(-k) \end{pmatrix}$$

$$\mu = \tilde{\rho} e^{ika}$$

$$= \gamma c_2^+(k) c_1(k) - \mu^* c_2(-k) c_1(k)$$

$$+ \mu^* c_2^+(k) c_1^+(-k) - \gamma c_2(-k) c_1^+(-k)$$

$$+ \gamma c_1^+(k) c_2(k) + \mu c_1(-k) c_2(k)$$

$$- \mu c_1^+(k) c_2^+(-k) - \cancel{\gamma c_2(k)})\, \gamma c_1(-k)\, c_2^+(-k)$$

$$= \gamma \big( c_2^+(\cancel{-}k) c_1(k)_{\checkmark} - c_2(-k)\, c_1^+(-k)_{\checkmark} + c_1^+(\cancel{-}k)\, c_2(k)_{\checkmark} - c_1(-k) c_2^+(-k)_{\checkmark} \big)$$

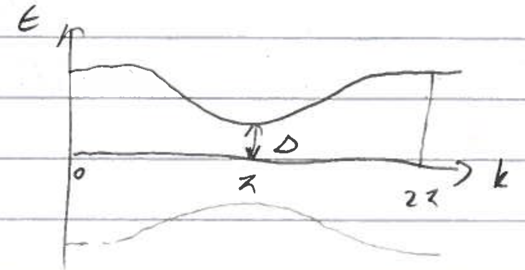
$$+ \mu^* \big( c_2^+(k)\, c_1^+(-k)_{\checkmark} - c_2(-k) c_1(k)_{\checkmark}$$

$$+ \mu \big( c_1(-k) c_2(k)_{\checkmark} - c_1^+(k)\; c_2^+(-k)_{\checkmark} \big)\big)$$

Can use the previous matrix, but switch the ones to $\gamma$, $\beta \to \tilde{\beta}$

$$\underline{\underline{M}}(k) = \left(\begin{array}{cc|cc} 0 & 0 & \gamma & -\mu \\ 0 & 0 & \mu & -\gamma \\ \hline \gamma & \mu^* & 0 & 0 \\ -\mu^* & -\gamma & 0 & 0 \end{array}\right) \qquad \mu = \tilde{\beta} e^{ika}$$

$$\epsilon(k) = \frac{J_F}{2}\sqrt{\tilde{\beta}^2 + \gamma^2 + 2\tilde{\beta}\gamma\cos(k)} \qquad k \in [0, 2\pi]$$



The energy density (energy per unit volume per Ferromagnetic coupling) [dimensionless]

$$\epsilon := \frac{E}{NJ_F} = -\frac{1}{2}\int_0^{2\pi}\frac{dk}{2\pi}\sqrt{\tilde{\beta}^2 + \gamma^2 + 2\tilde{\beta}\gamma\cos(k)}$$

$$+ (1+\alpha)\,(\beta\lambda^2 - t^2)$$

$$\boxed{\begin{aligned}\epsilon = -\frac{1}{4\pi}\int_0^{2\pi} dk \Big\{ & \beta^2[1+2\lambda(1+\alpha)]^2 + [1-2t(1+\alpha)]^2 \\ & + 2\beta[1+2\lambda(1+\alpha)][1-2t(1+\alpha)]\cos(k)\Big\}^{\frac{1}{2}} \\ & + (1+\alpha)(\beta\lambda^2 - t^2)\end{aligned}}$$

✡ $\alpha$ & $\beta$ are given parameters:

$\alpha \in [-1, 0]$
$\beta \in [0, 1]$ } need to be fixed first, then vary

✡ $\lambda, t$ need to be determined self-consistently:

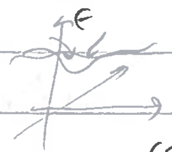
$\epsilon = \epsilon(\lambda, t)$

I. $\frac{\partial \epsilon}{\partial \lambda} = 0$

II $\frac{\partial \epsilon}{\partial t} = 0$

$$\Rightarrow \frac{\partial \epsilon}{\partial \lambda} - \frac{\partial \epsilon}{\partial t} = 0$$
$\phantom{\Rightarrow} = 0 \quad = 0$

✡ take random starting value $(\lambda_0, t_0)$



$$(\lambda_1, t_1) = (\lambda_0, t_0) - \overset{10^{-3}}{\overset{\parallel}{\delta}}\, \nabla\epsilon|_{\lambda_0, t_0}|$$

$\epsilon_1 < \epsilon_0$

$$\Rightarrow (\lambda_{n+1}, t_{n+1}) = (\lambda_n, t_n) - \delta\,\nabla\epsilon|_{\lambda_n, t_n})$$

where $\nabla\epsilon|_{\lambda_0, t_0} = \begin{pmatrix} \frac{\partial \epsilon}{\partial \lambda}\Big|_{\lambda_0, t_0} \\ \frac{\partial \epsilon}{\partial t}\Big|_{\lambda_0, t_0} \end{pmatrix}$ $\qquad \epsilon_{n+1} < \epsilon_n$

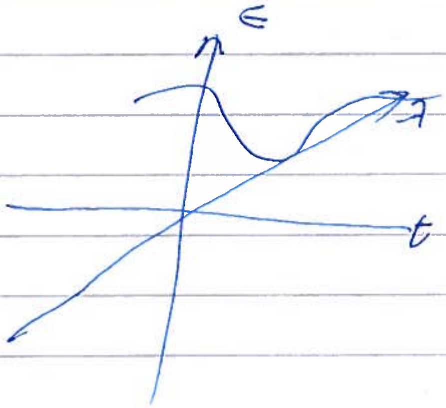
$$\hookrightarrow \frac{\partial \epsilon}{\partial \lambda}\Big|_{\lambda_0, t_0} = \lim_{a \to 0}\frac{\epsilon(\lambda_0 + a, t_0) - \epsilon(\lambda_0, t_0)}{a} \approx \frac{\epsilon(\lambda_0 + a, t_0) - \epsilon(\lambda_0, t_0)}{a}$$

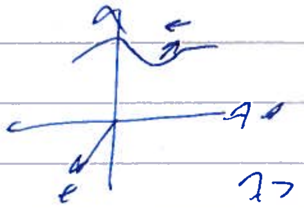
for ~~an~~ sufficiently small a

↳ need numerical integration as well. (elliptical integral)

~~[illegible] the int~~

Wig's theorem





$\lambda > 0$

shift left

else shift right

$\frac{\sqrt{r}}{2}\sqrt{\tilde{\beta}^2+\gamma^2+2\gamma\tilde{\beta}\cos(k)}$

$= \frac{\sqrt{r}}{2}\sqrt{\tilde{\beta}^2+\gamma(1+2\tilde{\beta}\cos(k))}$

~~$\cos^2(k)=1$~~

~~$\cos(2k) = 1 \Rightarrow 2\cos^2(k)-1$~~

$(1-2t(1+\alpha))^2 + 2\cos(k)(1-2t(1+\alpha))\beta(1+2(1+\alpha)\lambda) + \beta^2($[illegible]$(1+\alpha)\lambda^2)^2$ $\overset{\to 0}{\partial\lambda} > 0$

$\partial > 0$

non-convergence => stepsize

saddle point

$\frac{\partial E}{\partial k}$

calculate the gap

some line where the gap is zero

use new ∈ which has an elliptic function

$a > -0.1$ ; $b > 0.3$

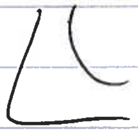
$\epsilon(a,b)$

phase boundary by bisection

$t_c = 0.13394$ //

$\lambda_c = 0.46476$ //

density a,b



work out where the slope is

Bisection

→ [illegible]

Notes for meeting ~~37~~

$$\hat{H}_{MF} = -\frac{J_F}{2} \sum_i \Big\{ \overbrace{(1-2t(1+\alpha))}^{\gamma} (c_{i2}^{\dagger} c_{i1} + c_{i1}^{\dagger} c_{i2})$$

$$+ \overbrace{\beta(1+2\lambda(1+\alpha))}^{} (c_{i+1,1} c_{i2} + c_{i2}^{\dagger} c_{i+1,1}^{\dagger}) \Big\}$$

$$+ N J_F (1+\alpha)(\beta \lambda^2 - t^2)$$

$$\gamma := (1-2t(1+\alpha))$$

$$\tilde{\beta} := \beta(1+2\lambda(1+\alpha))$$

$$\hat{H}_{MF} = -\frac{J_F}{2} \sum_i \gamma(c_{i2}^{\dagger} c_{i1} + c_{i1} c_{i2})$$
$$+ \tilde{\beta}\, c_{i1} + \ldots$$

hard core string order parameter

$\Delta = \Delta(\lambda, t)$

check gap.

integral numerically

determine $\lambda$ & $t$ ranges myself

$\lambda$ & $t$ < 1 maybe

try different $\lambda_0$ & $t_0$

to see if converge to local / global minimum

steepest gradient descent

elliptic integral

see optimized algorithms for numeric integration

I. fix $\alpha, \beta \overset{fix}{\Rightarrow}$ then loop

$\alpha \in [-1, 0]$

$\beta \in [0, 1)$

OR

$$\frac{\epsilon(\lambda_0 + a t_0) - \epsilon(\lambda_0, t_0)}{a}$$

for sufficiently small $a$.

II, correct $\lambda$ & $t$

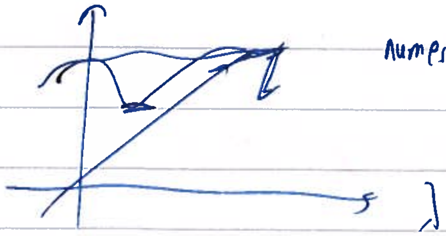
$$\Rightarrow \frac{\partial \epsilon}{\partial \lambda} = \frac{\partial \epsilon}{\partial t} = 0$$

$\approx 10^{-2} / 10^{-3}$

$\epsilon$

III take some starting value

$(\lambda_0, t_0)$



numerical 2 comp vector

IV numerically integrate the integral of cosh

converge to local / global minimum

hopefully

there will be recursion

$$\hat{H} = -\frac{J_F}{2}\sum_i \left[ c_{i1}^\dagger c_{i2} + c_{i2}^\dagger c_{i1} + \beta \left( c_{i+1,1} c_{i2} + c_{i2}^\dagger c_{i+1,1}^\dagger \right) \right]$$
$$- J_F (1+\alpha) \sum_i \left( \tfrac{1}{2} - c_{i1}^\dagger c_{i1} \right)\left( \tfrac{1}{2} - c_{i2}^\dagger c_{i2} \right) \quad \mathrm{I}$$
$$- J_F \beta (1+\alpha) \sum_i \left( \tfrac{1}{2} - c_{i2}^\dagger c_{i2} \right)\left( \tfrac{1}{2} - c_{i+1,1}^\dagger c_{i+1,1} \right) \quad \mathrm{II}$$

using all possible decoupling

$$c_{i1}^\dagger c_{i1} c_{i2}^\dagger c_{i2} = - c_{i1}^\dagger c_{i2} c_{i2}^\dagger c_{i1} \overset{MFA}{\approx} -\langle c_{i1}^\dagger c_{i2} \rangle c_{i2}^\dagger c_{i1} - c_{i1}^\dagger c_{i2} \langle c_{i2}^\dagger c_{i1} \rangle + \langle c_{i1}^\dagger c_{i2} \rangle \langle c_{i2}^\dagger c_{i1} \rangle$$
$$= -t\left( c_{i2}^\dagger c_{i1} + c_{i1}^\dagger c_{i2} \right) + t^2$$

$t = \langle c_{i1}^\dagger c_{i2} \rangle = \langle c_{i2}^\dagger c_{i1} \rangle \in \mathbb{R}$

constraint of $\rho$:

$$c_{i1}^\dagger c_{i1} c_{i2}^\dagger c_{i2} \simeq \rho\left( c_{i1}^\dagger c_{i1} + c_{i2}^\dagger c_{i2} \right) - \rho^2 - t\left( c_{i2}^\dagger c_{i1} + c_{i1}^\dagger c_{i2} \right) + t^2$$

$] \Rightarrow \rho = \langle c_{i1}^\dagger c_{i1} \rangle = \langle c_{i2}^\dagger c_{i2} \rangle = \frac{1}{2} \in \mathbb{R}$

$\hat{T}_{im}^{z} = \frac{1}{2} - c_{im}^\dagger c_{im} \quad m = 1, 2$

$\alpha < 2$ : XY anisotropy $\Rightarrow$ moments develop on XY plane

$\langle T_{in}^{z} \rangle = 0 \Leftrightarrow \langle c_{im}^\dagger c_{im} \rangle = \frac{1}{2} = \rho$

$$c_{i2}^\dagger c_{i2} c_{i+1,1}^\dagger c_{i+1,1} \simeq \rho\left( c_{i2}^\dagger c_{i2} + c_{i+1,1}^\dagger c_{i+1,1} \right) - \rho^2 + \lambda\left( c_{i2}^\dagger c_{i+1,1}^\dagger + c_{i+1,1} c_{i2} \right) - \lambda^2$$

$] \Rightarrow \lambda = \langle c_{i+1,1} c_{i2} \rangle = \langle c_{i2}^\dagger c_{i+1,1}^\dagger \rangle \in \mathbb{R}$

$$\mathrm{I}.\ \sum_i \left( \tfrac{1}{2} - c_{i1}^\dagger c_{i1} \right)\left( \tfrac{1}{2} - c_{i2}^\dagger c_{i2} \right) = \cancel{\frac{N}{4}} - \cancel{\frac{1}{2}\sum_i \left( c_{i1}^\dagger c_{i1} + c_{i2}^\dagger c_{i2} \right)} - t\left( c_{i2}^\dagger c_{i1} + c_{i1}^\dagger c_{i2} \right) + Nt^2$$
$$+ \cancel{\frac{1}{2}\sum_i \left( c_{i1}^\dagger c_{i1} + c_{i2}^\dagger c_{i2} \right)} - \cancel{\frac{N}{4}}$$
$$= Nt^2 - t\sum_i \left( c_{i2}^\dagger c_{i1} + c_{i1}^\dagger c_{i2} \right)$$

$$\mathrm{II}.\ \sum_i \left( \tfrac{1}{2} - c_{i2}^\dagger c_{i2} \right)\left( \tfrac{1}{2} - c_{i+1,1}^\dagger c_{i+1,1} \right) \Rightarrow \cancel{\frac{N}{4}} - \cancel{\frac{1}{2}\sum_i \left( c_{i2}^\dagger c_{i2} + c_{i+1,1}^\dagger c_{i+1,1} \right)} + \cancel{\frac{1}{2}\sum_i \left( c_{i2}^\dagger c_{i2} + c_{i+1,1}^\dagger c_{i+1,1} \right)} - \cancel{\frac{N}{4}}$$
$$+ \lambda \sum_i \left( c_{i2}^\dagger c_{i+1,1}^\dagger + c_{i+1,1} c_{i2} \right) - N\lambda^2$$

$\lambda = \langle c_{i+1,1} c_{i2} \rangle = \langle c_{i2}^\dagger c_{i+1,1}^\dagger \rangle \in \mathbb{R}$

$$= \lambda \sum_i \left( c_{i2}^\dagger c_{i+1}^\dagger + c_{i+1,1} c_{i2} \right) - N\lambda^2$$

$$\hat{H}_{Mf} = -\frac{J_F}{2}\sum_i \Big\{ \left[ 1 - 2t(1+\alpha) \right]\left( c_{i1}^\dagger c_{i2} + c_{i2}^\dagger c_{i1} \right) + \beta\left[ 1 + 2\lambda(1+\alpha) \right]\left[ c_{i+1,1} c_{i2} + c_{i2}^\dagger c_{i+1,1}^\dagger \right] \Big\} + N J_F (1+\alpha)\left( \beta\lambda^2 - t^2 \right)$$

Setting $\gamma := 1 - 2t(1+\alpha)$

$\tilde{\beta} := \beta\left[ 1 + 2\lambda(1+\alpha) \right]$

$$\hat{H}_{mf} = -\frac{J_F}{2}\int_k \left( c_1^\dagger(k)\ c_1(-k)\ c_2^\dagger(k)\ c_2(-k) \right) \underline{\underline{M}}(\underline{k}) \begin{pmatrix} c_1(k) \\ c_2^\dagger(-k) \\ c_2(k) \\ c_2^\dagger(-k) \end{pmatrix}$$
$$+ J_F (1+\alpha)\left( \beta\lambda^2 - t^2 \right)$$

$$\int_0^{2\pi} dk \sqrt{A^2+B^2+2AB\cos(k)}$$

mathematica

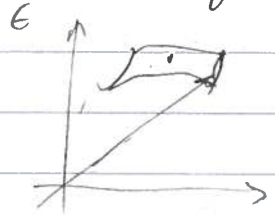
$$= 4|A+B| \text{ EllipticE}\left(\frac{4AB}{(A+B)^2}\right)$$

$A = \beta[1+2\lambda(1+\alpha)]$  $A = A(\beta,\lambda,\alpha)$

$B = 1-2t(1+\alpha)$  $B = B(t,\alpha)$

$$\Rightarrow \epsilon = -\frac{1}{\pi}|A+B| \text{ EllipticE}\left(\frac{4AB}{(A+B)^2}\right) + (1+\alpha)(\beta\lambda^2 - t^2)$$

* actually finding a saddle point



steepest grad descent for $\lambda$ $(-=)$

steepest grad ascent for $t$ $(+=)$

problems with the first code:

- energy's wrong for some reason
- need use a new $\epsilon$ which has an elliptic function
- if non-convergence over => use smaller steps

model answer for comparison:

for $a = -0.1$; $b: 0.8$

$t = 0.13394$

$\lambda: 0.464764$ //

Next step:

work out phase boundaries by bisection

Report ~~[illegible]~~ draft //

Method.

2nd Quantised form

Derivation :

Need a Picture

Start w/ Hamiltonian //

In 2nd Quantised form

w/ spin operators

↓

defining new operators

↓

JW transform

to map spinless fermions

↓ for XY case : α = -1

Fourier Transform to

change to momentum space

↓ Double dimension => (4x4) matrix

~~Boto~~ Bogoliubov transformation

to incorporate anomalous term

↓

obtaining matrix (4x4)

↳ diagonalise to get eigenvalue

and double checking by MATLAB (??)

& Mathematica

↓

Mean field approx to make

Interacting terms vanish,

by using all possible decouplings,

and introducing ~~[illegible]~~ 3 more

expectation values, with constraints

↳ similar to XY case, with

some extra substitutions

diagonalisation is exactly the

same as the α = -1 case

↳ obtaining energy density

equations

Computation:

$\alpha \in [0,1]$ ; $\beta \in [-1,0]$

~~[illegible]~~ $\lambda_0$ = randn(0,1)

$t_0$ = randn(0,1)

2 determine $\lambda$ & $t$ self-consistently

w/ $\frac{\partial \epsilon}{\partial \lambda} = 0$ ; $\frac{\partial \epsilon}{\partial t} = 0$

↳ description of code

Results

1. energy density plot

2. phase transition diagram

Appendix : GitHub link