Gödel's Incompleteness Theorems

The way Gödel did it

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Formal Proof Systems

- A formal proof system provides a notion of well-formed formulas (WFFs), and a set of axioms along with rules of inference.
- A *proof* is a sequence of WFFs, each of which is either an axiom or follows from previous statements by a rule of inference.
- A *sentence* is a WFF with no free variables.
- A sentence is *provable* if there is a proof of it. For a sentence ϕ, we write

$$\vdash \phi$$

to denote that ϕ is provable. We say that ϕ is a *theorem*.

Theories and Theorems

- For a sentence ϕ, we write

$$\models \phi$$

to denote that ϕ is *true*.

Consistency and Completeness

- A formal proof system (taken with a theory) is *consistent* if every theorem is true.

$$\vdash \phi \implies \vDash \phi$$

- A formal proof system is *complete* if every true sentence is a theorem.

$$\vDash \phi \implies \vdash \phi$$

The Language of Arithmetic

The language of arithmetic consists of:

- the constants 0 and 1,
- the binary operations $+$ and $\cdot$,
- the relation $=$,
- symbols for variables, and
- symbols from first-order logic.

First Incompleteness Theorem

Theorem (1931)

There cannot exist a sound and complete proof system for arithmetic.