Science One Mathematics

*This exam has **10 questions** on **11 pages**, for a total of 74 points.*

Duration: 150 minutes

- Read all the questions carefully before starting to work.
- Give complete arguments and explanations for all your calculations; for changes of variables state how the variables are related. For integration by parts, state what the parts are. Answers without justifications will not be accepted.
- Continue on blank pages if you run out of space.
- This is a closed-book examination. **None of the following are allowed**: documents, cheat sheets or electronic devices of any kind (including calculators, cell phones, etc.)

First name: ______________________ Last name: ______________________

Student #: ______________________ Bamfield #: ______________

Signature: ____

Question:	1	2	3	4	5	6	7	8	9	10	Total
Points:	8	12	8	6	6	7	7	6	8	6	74
Score:											

8 marks

1. (a) Approximate $I = \int_0^1 \sin(\pi x^2)\, dx$ using a Riemann sum with $n = 2$ subintervals and left endpoints.

(b) Consider a function f continuous for all x and such that $\int_a^b f(x)dx = 3$. Which one of the following statements is true? Select all that apply. Justify your claim.

(A) **True/False**. $\int_a^1 f(x)dx + \int_1^b f(x)dx = 3$.

(B) **True/False**. $\dfrac{d}{dx}\displaystyle\int_a^b x f(y)dy = 3$

(C) **True/False**. $\int_a^b 2y f(y^2)dy = 3$.

(c) Let $G(x) = \int_{2x}^1 e^{(t^2)}dt$. Compute $G(1/2)$ and $G'(1/2)$.

$\boxed{\text{12 marks}}$ 2. Compute the following integrals.

(a) $\displaystyle\int_0^{\frac{\pi}{2}} e^{-\sin^2(y)} \sin(y) \cos(y)\, dy$

(b) $\displaystyle\int \sin(\sqrt{x})dx$

(c) $\displaystyle\int \frac{1}{t^3 - t}\, dt$

8 marks 3. Determine whether each series converges. Justify your answer, by stating which test you are using. You may use known facts about the convergence of geometric series and p-series.

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{1+e^{-k}}$

(b) $1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \frac{1}{81} \cdots$

(c) $\displaystyle\sum_{k=1}^{\infty} \frac{3^k}{2^k + 2^{2k}}$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n n!}{n^n}$

6 marks 4. Determine (with justification) whether each of the improper integrals converges or diverges, and if it converges find its value.

(a) $\displaystyle\int_2^\infty \frac{dx}{x(\ln(x))^2}$

(b) $\displaystyle\int_{-\infty}^\infty \frac{1}{x^2}\,dx$

(c) $\displaystyle\int_0^2 (x-1)^{-\frac{1}{3}}dx$

6 marks 5. Find the volume of the solid obtained by rotating the region $0 \leq y \leq 2 - e^x$, $0 \leq x \leq \ln(2)$ about the y-axis.

7 marks 6. Recall that the rate of change of the temperature of an object is proportional to the difference between the object's temperature and the temperature of the surroundings (Newton's law). This law applies to **both** cooling and warming processes.

You are served a grilled sandwich at $80°C$, and it begins cooling, in the $20°C$ room, according to Newton's law.

(a) Write down the ODE IVP for the temperature $T(t)$ (in $°C$) of your sandwich as a function of time t (in minutes).

(b) Solve the ODE IVP from part (a).

(c) After sitting for 10 minutes, your sandwich is $30°C$. How long should you put it in a $180°C$ oven to warm it back up to $80°C$ (still assuming Newton's law applies) ?

7 marks

7. Suppose the time T (in minutes) it takes to get your grilled sandwich at Ike's is a continuous random variable, and the probability that it takes longer than t minutes is

$$P(T > t) = \frac{1}{1+t^2}.$$

(a) What is the probability that you will get your sandwich within 2 minutes?

(b) Find the probability density function $f(t)$ for this random variable.

(c) Find the mean waiting time $\mu = \int_0^\infty t f(t) dt$.

6 marks 8. Your (2 dimensional) grilled sandwich has the shape of a triangle in the plane with vertices $(0, 0)$, $(0, 4)$, $(4, 0)$. Find the coordinates of the centroid of the region that remains after you eat the square with vertices $(0, 0)$, $(0, 1)$, $(1, 0)$, $(1, 1)$.

8 marks 9. (a) Find the first four non-zero terms of the Maclaurin series (Taylor series centred at 0) of the function $f(x) = \dfrac{\sin(2x)}{1+x^3}$.

(b) Consider the improper integral

$$\int_0^1 \frac{ax - f(x)}{x^2}\,dx$$

For which value of the constant a does the integral converge? Explain. (Do not evaluate the integral)

6 marks 10. Recall that a point charge produces an electric field whose strength is proportional to its charge, and inversely proportional to the square of the distance from it (Coulomb's law).

For the following arrangements of (**infinitely many**) charges on the real line, **determine whether the electric field strength at $x = 0$ is finite or infinite**:

(a) 1 electron at $x = 1$, 2 electrons at $x = 2$, 3 electrons at $x = 3$, ... etc. ;

(b) 1 electron at $x = 1$, 2 protons at $x = 2$, 3 electrons at $x = 3$, 4 protons at $x = 4$, ... etc.;

(c) 1 electron at $x = 2$, 2 electrons at $x = 4$, 3 electrons at $x = 8$, 4 electrons at $x = 16$, ..., n electrons at $x = 2^n$, ..., etc.;

(d) a dipole of polarization qd at $x = 1$, a dipole of polarization $2qd$ at $x = 2$, a dipole of polarization $3qd$ at $x = 3$, ... etc. Assume $d \ll 1$. Recall the magnitude of the electric field of a dipole is proportional to the dipole polarization qd, where q is the magnitude of the dipole charge, and inversely proportional to the cube of the distance from it.