Master in Foundations of Data Science — 2018-2019

NUMERICAL LINEAR ALGEBRA

Reevaluation exam, June 7th, 2019, from 15:00h till 19:00h, at room B1.

1. Given a matrix $A \in \mathbb{R}^{n\times n}$, Gaussian elimination with partial pivoting (GEPP) computes a factorization

$$A = P \cdot L \cdot U$$

where P is a permutation matrix, L is lower triangular, and U is upper triangular.

(1) Write down the GEPP algorithm in pseudocode notation and describe how it works.
(2) Prove that the complexity of this algorithm in terms of floating point operations *(flops)* is bounded by $\frac{2}{3}n^3 + O(n^2)$.
(3) Describe how to solve the linear equation $A \cdot x = b$ using the PLU factorization of the matrix A.

2. Let $\|\cdot\|$ be a matrix norm on $\mathbb{R}^{n\times n}$.

(1) Define the condition number $\kappa_{\|\cdot\|}(A)$ of a matrix $A \in \mathbb{R}^{n\times n}$. Knowing the condition number of a matrix, which useful information does it give you?
(2) Let $\|\cdot\|_\infty$ be the vector norm given by $\|(x_1,\dots,x_n\|_\infty = \max_i |x_i|$ and consider the matrix

$$A = \begin{pmatrix} \varepsilon & 1 \\ 1 & 1 \end{pmatrix}$$

for a small parameter ε. Compute the condition number of A and of the matrices in its LU factorization, with respect to the matrix norm associated to $\|\cdot\|_\infty$.
(3) Use the example in (2) to show that Gaussian elimination without pivoting can be a numerically unstable algorithm for solving a system of linear equations.

3. Consider the *singular value decomposition (SVD)* given with three significant digits as

$$A = \begin{pmatrix} 1 & 1 \\ -1 & 0 \\ 1 & 2 \end{pmatrix} \simeq \begin{pmatrix} -0.522 & 0.248 \\ 0.218 & -0.886 \\ -0.825 & -0.391 \end{pmatrix} \cdot \begin{pmatrix} 2.676 & 0 \\ 0 & 0.915 \end{pmatrix} \cdot \begin{pmatrix} -0.585 & -0.811 \\ 0.881 & -0.585 \end{pmatrix} = U \cdot S \cdot V^T$$

(1) Compute the best rank 1 approximation of A with respect to the 2-norm, and determine its distance to A.
(2) Compute the vector $x_{\min} \in \mathbb{R}^2$ that solves the least squares problem

$$\|Ax_{\min} - b\|_2 = \min_{x\in\mathbb{R}^2} \|Ax - b\|_2$$

for the vector $b = (1, 2, 0) \in \mathbb{R}^3$.

4. The *real Schur form* of an $n \times n$-matrix A is a factorization

$$A = Q \cdot R \cdot Q^T$$

where Q is an orthogonal matrix and R is block upper triangular, with blocks of size of 1×1 and 2×2.

(1) How can you compute the eigenvalues and the eigenvectors of A using its real Schur form?
(2) The *Hessenberg form* of A is a factorization $A = Q_0 \cdot H \cdot Q_0^T$ where Q_0 is an orthogonal matrix and H is upper Hessenberg. How can you compute it in the case when A is a 3×3 matrix?
(3) Given a matrix A, which algorithm would you apply to compute its real Schur form?

5. Consider the matrix and the vector

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

(1) For this matrix and vector, compute the corresponding *iterative scheme* for (a) the Jacobi method, (b) the Gauss-Seidel method, and (c) the SOR(ω) method, for an arbitrary parameter $\omega \in \mathbb{R}$.
(2) Check if the Jacobi and Gauss-Seidel iterative schemes for this system converge for any choice of a starting point $x_0 \in \mathbb{R}^2$.