Optimization

Màster de Fonaments de Ciència de Dades

Lecture II. Unconstrained and constrained optimization with equalities. Optimality conditions

Main issues in Optimization

1. Characterization of extrema (maxima/minima)
   - Necessary conditions
   - Sufficient conditions
   - Lagrange multiplier theory
   - The Karush-Kuhn-Tucker theory
2. Iterative algorithms for the computation of the extrema
   - Iterative descent
   - Approximation methods
   - Dual and primal-dual methods

Characterization of minima. Local and global minima

Let a function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$. A point $\boldsymbol{x}^* \in \mathbb{R}^n$ is a:

- LOCAL minimum of f if:

  there is an $\epsilon > 0$ such that $f(x^*) \leq f(\boldsymbol{x})$ for all $\boldsymbol{x} \in \mathbb{R}^n$ when $\|\boldsymbol{x} - \boldsymbol{x}^*\| \leq \epsilon$.

- STRICT LOCAL minimum of f if:

  there is an $\epsilon > 0$ such that $f(\boldsymbol{x}^*) < f(\boldsymbol{x})$ for all $\boldsymbol{x} \in \mathbb{R}^n \setminus \{\boldsymbol{x}^*\}$ when $\|\boldsymbol{x} - \boldsymbol{x}^*\| \leq \epsilon$.

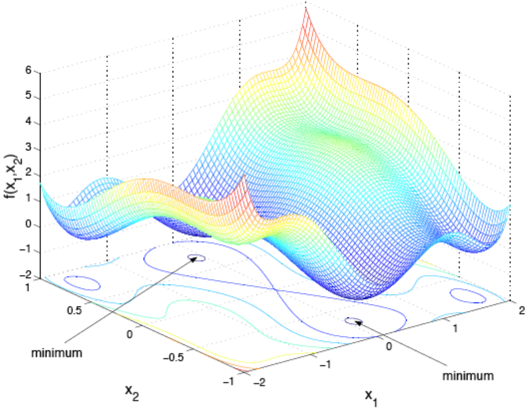
- GLOBAL minimum of f if:

  $f(\boldsymbol{x}^*) \leq f(\boldsymbol{x})$ for all $\boldsymbol{x} \in \mathbb{R}^n$.

- STRICT GLOBAL minimum of f if:

  $f(\boldsymbol{x}^*) < f(\boldsymbol{x})$ for all $\boldsymbol{x} \in \mathbb{R}^n \setminus \{\boldsymbol{x}^*\}$.

Local and global minima



The function $f(x_1, x_2) = x_1^2(4 - 2.1x_1^2 + \frac{1}{3}x_1^4) + x_1x_2 + x_2^2(-4 + 4x_2^2)$ *has two global minima, (0.089, –0.717) and (–0.0898, 0.717) and four local minima.*

Derivatives and notation

Recall that if $x \in \mathcal{C} \subset \mathbb{R}^n$ is a point where the real function

$$f : \mathcal{C} \longrightarrow \mathbb{R}$$

is differentiable then:

- We define the gradient of f at $\boldsymbol{x}$ as the vector $\nabla f(\boldsymbol{x})$ given by:

$$\nabla f(\boldsymbol{x}) = \left(\frac{\partial f(\boldsymbol{x})}{\partial x_1}, \ldots, \frac{\partial f(\boldsymbol{x})}{\partial x_n} \right)^T .$$

- If f is twice continuously differentiable at $\boldsymbol{x}$ we define the Hessian matrix of f at $\boldsymbol{x}$ as the $n \times n$ symmetric matrix $\nabla^2 f(x)$ given by:

$$\nabla^2 f(\boldsymbol{x}) = \left(\frac{\partial^2 f(\boldsymbol{x})}{\partial x_i \partial x_j} \right), \quad i, j = 1, \ldots, n.$$

- The directional derivative $D_{\boldsymbol{u}} f(\boldsymbol{x})$ of the function f, at point $\boldsymbol{x} \in \mathcal{C}$, in the direction $\boldsymbol{u} \in \mathbb{R}^n$ ($\|\boldsymbol{u}\| = 1$) is defined as

$$D_{\boldsymbol{u}} f(\boldsymbol{x}) = \lim_{\lambda \to 0} \frac{f(\boldsymbol{x} + \lambda \boldsymbol{u}) - f(\boldsymbol{x})}{\lambda} = \nabla f(\boldsymbol{x}) \cdot \boldsymbol{u} = \nabla f(\boldsymbol{x})^T \boldsymbol{u}$$

Necessary and sufficient conditions for extrema

One must be careful with necessary and sufficient conditions.

Theorem : Let $c \in (a; b)$ and f be a real valued **continuous** function at c. If for some $\delta > 0$, f is increasing on $(c - \delta, c)$ and decreasing on $(c, c + \delta)$, then f has a local maximum at c.

Proof: Choose any x_1 and x such that $c - \delta < x_1 < x < c$. Then $f(x_1) \leq f(x)$ and by the continuity of f at c we have:

$$f(x_1) \leq \lim_{x \to c^-} f(x) = f(c).$$

Similarly, if $c < x < x_2 < c + \delta$, then

$$f(x_2) \leq \lim_{x \to c^+} f(x) = f(c).$$

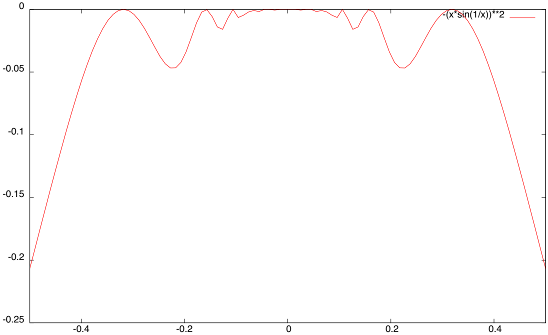
This proves the result.

Remark: This Theorem gives a sufficient condition of maximum but not a necessary one.

Necessary and sufficient conditions for extrema

Remark: The converse of the above Theorem is not true, i.e.: If f is continuous at c and f has a local maximum at c, then f need not be increasing on $(c-\delta, c)$ or decreasing on $(c, c+\delta)$ **for any** $\delta > 0$.
Take, for example, $c = 0$ and

$$f(x) = -\left(x \sin\frac{1}{x}\right)^2 \quad \text{if} \quad x \neq 0, \quad \text{and} \quad f(0) = 0.$$



Necessary and sufficient conditions for extrema

Theorem (Necessary condition)

Let $f : \mathcal{C} \to \mathbb{R}$ and $\boldsymbol{x}^*$ an interior point of $\mathcal{C}$ at which f has a local minimum (or a local maximum). If f is **differentiable** *at $\boldsymbol{x}^*$ then:*

$$\nabla f(\boldsymbol{x}^*) = 0.$$

Proof. As $\boldsymbol{x}^*$ is a local minimum, one has:

$$f(\boldsymbol{x}^*) \leq f(\boldsymbol{x}^* + \lambda \boldsymbol{s}) \quad \text{for all} \quad \boldsymbol{s} \in \mathbb{R}^n, \ \|\boldsymbol{s}\| = 1 \quad \text{and} \quad \lambda \in \mathbb{R} \quad \text{small enough.}$$

Fix $\boldsymbol{s}$ and define $F(\lambda) = f(\boldsymbol{x}^* + \lambda \boldsymbol{s})$, which is continuous at $\lambda = 0$. Then the above inequality becomes

$$F(0) \leq F(\lambda), \quad \forall\, |\lambda| < \delta.$$

From the Mean Value Theorem, we have

$$F(\lambda) = F(0) + F'(\theta\lambda)\lambda,$$

where $\theta \in [0, 1]$.

Now...

Necessary and sufficient conditions for extrema

Now...

– If $F'(0) > 0$, then, by the continuity assumptions, there exists an $\epsilon > 0$ such that:

$$F'(\theta\lambda) > 0, \quad \forall\theta \in [0,1], \text{ and } \quad \forall\lambda \quad s.t. \quad |\lambda| < \epsilon.$$

Hence, we can find a $\lambda < 0$, such that $|\lambda| < \delta$, and, since:

$$F(\lambda) = F(0) + F'(\theta\lambda)\lambda \quad \Rightarrow \quad F(0) > F(\lambda), \quad \text{contradiction!!}$$

– Assuming $F'(0) < 0$ would lead to a similar contradiction (taking $\lambda > 0$).

Thus

$$F'(0) = \nabla f(\boldsymbol{x}^*)^T \boldsymbol{s} = 0.$$

Since $\boldsymbol{s}$ is an arbitrary nonzero vector ($\|\boldsymbol{s}\| = 1$), we must have:

$$\nabla f(\boldsymbol{x}^*)^T = 0.$$

Necessary and sufficient conditions for extrema

Theorem (Sufficient conditions)

Let $\boldsymbol{x}^*$ be an interior point of $\mathcal{C}$ at which f is **twice continuously differentiable**. *If*

$$\nabla f(\boldsymbol{x}^*) = 0, \quad \boldsymbol{z}^T \nabla^2 f(\boldsymbol{x}^*)\boldsymbol{z} > 0, \;\; \forall \boldsymbol{z} \neq \boldsymbol{0},$$

then f has a local minimum at $\boldsymbol{x}^*$.
If

$$\nabla f(\boldsymbol{x}^*) = 0, \quad \boldsymbol{z}^T \nabla^2 f(\boldsymbol{x}^*)\boldsymbol{z} < 0, \;\; \forall \boldsymbol{z} \neq \boldsymbol{0},$$

then f has a local maximum at $\boldsymbol{x}^*$.
Moreover, the extrema are strict local extrema.

Proof. Use the Taylor expansion of f arounf $\boldsymbol{x}^*$.

Remark. Note that the converses of the above results are not true. For instance, $f(x) = -x^4$, has a maximum at $x^* = 0$, is twice differentiable at 0 but $\nabla f(0) = f''(0) = 0$, which is not strictly less than 0.

Remark. The condition $\boldsymbol{z}^T \nabla^2 f(\boldsymbol{x}^*)\boldsymbol{z} > 0, \;\; \forall \boldsymbol{z} \neq 0$ means that $\nabla^2 f(\boldsymbol{x}^*)$ is positive definite.

Necessary and sufficient conditions for extrema

Example. Let

$$f(x) = x^{2p}, \quad x \in \mathbb{R}, \quad p \in \mathbb{Z}_+,$$

and let $\mathcal{C}$ be the whole real line.

- The gradient of f is

  $$\nabla f(x) = f'(x) = 2px^{2p-1}.$$

  Clearly $\nabla f(0) = 0$, that is $x = 0$ satisties the necessary condition for a minimum or a maximum.
- The Hessian of f is

  $$\nabla^2 f(x) = f''(x) = (2p-1)2px^{2p-2}.$$

  For $p = 1$, $\nabla^2 f(0) = 2 > 0$, that is, the sufficient conditions for a strict local minimum are satisfied.
- If we take $p > 1$, then $\nabla^2 f(0) = 0$ and the sufficient conditions for a local minimum are not satisfied, yet f has a minimum at the origin.

  By taking any neighborhood of the origin, it can be verifiyed that all the conditions for a local minimum given in the next Theorem are satisfied for this example.

Necessary and sufficient conditions for extrema

Theorem

Let $\boldsymbol{x}^*$ *be an interior point of* $\mathcal{C}$ *and assume that* f *is* **twice continuously differentiable** *on* $\mathcal{C}$*, then:*

(a) *Necessary conditions for a local minimum of* f *at* $\boldsymbol{x}^*$ *are:*

$$\nabla f(\boldsymbol{x}^*) = 0, \quad \boldsymbol{z}^T \nabla^2 f(\boldsymbol{x}^*)\boldsymbol{z} \geq 0, \quad \forall \boldsymbol{z} \in \mathbb{R}^n.$$

(b) *Sufficient conditions for a local minimum are:*

$$\nabla f(\boldsymbol{x}^*) = 0,$$

and that **for every** $\boldsymbol{x}$ *in some neighborhood* $N_\epsilon(\boldsymbol{x}^*)$ *and* **for every** $\boldsymbol{z} \in \mathbb{R}^n$*, we have:*

$$\boldsymbol{z}^T \nabla^2 f(\boldsymbol{x})\boldsymbol{z} \geq 0.$$

(c) *If the sense of the inequalities is reversed, then the theorem applies to a local maximum.*

Proof of the Theorem

Proof

(a) The first order condition for local minimum

$$\boldsymbol{x}^* \quad \text{minimum} \quad \Rightarrow \quad \nabla f(\boldsymbol{x}^*) = 0,$$

has already been proved (pages 8 and 9).

Turning to the second-order conditions, we have by Taylor's theorem applied to the function $F(\theta) = f(\boldsymbol{x}^* + \theta \boldsymbol{s})$, with $\|\boldsymbol{s}\| = 1$ fixed,

$$F(\theta) = F(0) + \nabla F(0)\theta + \frac{1}{2}\nabla^2 F(\lambda\theta)\theta^2, \quad \lambda \in (0,1).$$

If $\nabla^2 F(0) < 0$, then, by continuity, there exists $\epsilon' > 0$ such that $\nabla^2 F(\lambda\theta) < 0$ for $\lambda \in (0,1)$ and $|\theta| < \epsilon'$. Since $\nabla F(0) = 0$, this inequality implies that for such a θ:

$$F(\theta) < F(0),$$

which is a contradiction. Consequently

$$\nabla^2 F(0) = \boldsymbol{s}^T \nabla^2 f(\boldsymbol{x}^*) \boldsymbol{s} \geq 0.$$

Since this inequality holds for all unitary vector $\boldsymbol{s}$, it must hold for all vector $\boldsymbol{z}$.

Proof of the Theorem (cont.)

(b) Assume that $\nabla f(\boldsymbol{x}^*) = 0$ and that $\boldsymbol{z}^T \nabla^2 f(\boldsymbol{x}^*) \boldsymbol{z} \geq 0$ for all $\boldsymbol{x} \in N_\delta(\boldsymbol{x}^*)$ and all $\boldsymbol{z} \in \mathbb{R}^n$, but that $\boldsymbol{x}^*$ is not a local minimum.

Then there exists a $\boldsymbol{w} \in N_\delta(\boldsymbol{x}^*)$ such that $f(\boldsymbol{w}) < f(\boldsymbol{x}^*)$.

Write $\boldsymbol{w} = \boldsymbol{x}^* + \theta \boldsymbol{y}$, with $\|\boldsymbol{y}\| = 1$ and $\theta > 0$. By Taylor's theorem:

$$f(\boldsymbol{w}) = f(\boldsymbol{x}^*) + \theta \, \nabla f(\boldsymbol{x}^*)^T \boldsymbol{y} + \frac{1}{2}\theta^2 \boldsymbol{y}^T \nabla^2 f(\boldsymbol{x}^* + \lambda\theta\boldsymbol{y}) y,$$

with $\lambda \in (0, 1)$. Our assumptions lead then to

$$\boldsymbol{y}^T \nabla^2 f(\boldsymbol{x}^* + \lambda\theta\boldsymbol{y}) \boldsymbol{y} < 0,$$

contradicting the hypothesis, since $\boldsymbol{x}^* + \lambda\theta\boldsymbol{y} \in N_\delta(\boldsymbol{x}^*)$.

Example

Let

$$f(x,y) = \frac{1}{2}x^2 + xy + 2y^2 - 4x - 4y - y^3.$$

Then

$$\nabla f(x,y) = \left(x + y - 4,\ x + 4y - 4 - 3y^2\right)^T, \quad H(x,y) = \begin{pmatrix} 1 & 1 \\ 1 & 4 - 6y^2 \end{pmatrix}.$$

$\nabla f(x,y) = 0$ has exactly two solutions, $\boldsymbol{x}_1 = (4,0)$, $\boldsymbol{x}_2 = (3,1)$, and

$$H(\boldsymbol{x}_1) = \begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix}, \qquad H(\boldsymbol{x}_2) = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}.$$

$$(x\ y)\begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = x^2+2xy+4y^2 = (x+y)^2+3y^2 > 0 \quad \text{if} \quad (x\ y) \neq (0,0).$$

$$(x\ y)\begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = x^2 + 2xy + 4y^2 = (x+y)^2 - 2y^2.$$

So $H(\boldsymbol{x}_1)$ is positive definite and $H(\boldsymbol{x}_2)$ is indefinite, therefore, the only extrema is the local minimum is $\boldsymbol{x}_1$.

Convexity

Convexity notions play an important role in nonlinear programming. Some reasons for that are:

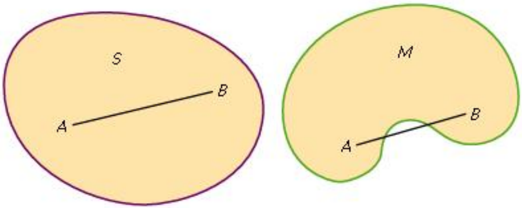
1. Convex optimization includes least-squares and linear programming problems, which can be solved numerically very efficiently.
2. When the cost function f is convex, every local maximum/minimum is also global.
3. We will see that if $f(\boldsymbol{x}) = \boldsymbol{x}^T Q \boldsymbol{x} + \boldsymbol{c}^T \boldsymbol{x} + d$ is such that Q is positive semidefinite, then f is convex.
4. The (fisrt order) necessary condition $\nabla f(\boldsymbol{x}^*) = 0$ is also sufficient for global optimality if f is convex.
5. The behavior of convex functions allows for very fast algorithms to optimize them.
6. Many optimization problems admit a convex (re)formulation.

Convex sets and convex functions

- Let two points $x_1, x_2 \in \mathbb{R}$, and $0 \leq \lambda \leq 1$ be given. Then, the point

$$x = \lambda x_1 + (1 - \lambda) x_2,$$

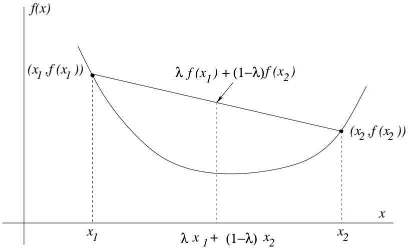
  is a convex combination of the two points x_1, x_2.
- The set $\mathcal{C} \subset \mathbb{R}^n$ is called convex, if all convex combinations of any two points $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathcal{C}$ are again in $\mathcal{C}$.



Convex sets and convex functions

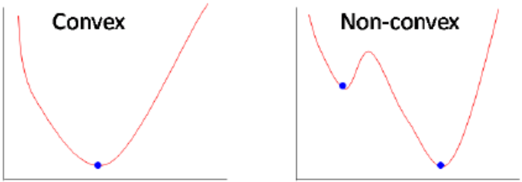
- A function $f : \mathcal{C} \longrightarrow \mathbb{R}$ defined on a convex set $\mathcal{C}$ is called convex if for all $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathcal{C}$ and $0 \leq \lambda \leq 1$ one has:

$$f(\lambda \boldsymbol{x}_1 + (1-\lambda)\boldsymbol{x}_2) \leq \lambda f(\boldsymbol{x}_1) + (1-\lambda) f(\boldsymbol{x}_2).$$



- For a convex function, the linear interpolation $\lambda f(\boldsymbol{x}_1) + (1-\lambda) f(\boldsymbol{x}_2)$ overestimates the function value $f(\lambda \boldsymbol{x}_1 + (1-\lambda)\boldsymbol{x}_2)$.

Convex sets and convex functions



- A function $f : \mathcal{C} \longrightarrow \mathbb{R}$ defined on a convex set $\mathcal{C}$ is called strictly convex if for all $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathcal{C}$ with $\boldsymbol{x}_1 \neq \boldsymbol{x}_2$ and $0 < \lambda < 1$ one has.

$$f(\lambda \boldsymbol{x}_1 + (1-\lambda)\boldsymbol{x}_2) < \lambda f(\boldsymbol{x}_1) + (1-\lambda) f(\boldsymbol{x}_2).$$

- A function $f : \mathcal{C} \longrightarrow \mathbb{R}$ defined on a convex set $\mathcal{C}$ is called concave if $-f$ is convex.
- **Remark.** Recall that the domain of a convex function must be a convex set.
- **Remark.** Later we will give a more general definition of convex function.

Examples of convex functions

Proposition

a) A linear function, $f(\lambda \boldsymbol{x} + \mu \boldsymbol{y}) = \lambda f(\boldsymbol{x}) + \mu f(\boldsymbol{y})$, is convex.

b) Any vector norm, $f(\boldsymbol{x}) = \|\boldsymbol{x}\|$, is a convex function.

c) The weighted sum of convex functions $(\sum \alpha_i f_i(\boldsymbol{x}))$ with positive weights $(\alpha_i > 0)$ is convex.

d) If I is an index set, $\mathcal{C} \subset \mathbb{R}^n$ is a convex set and $f_i : \mathcal{C} \to \mathbb{R}$ are convex functions for each $i \in I$, then the function:

$$\begin{aligned} h : \mathcal{C} &\longrightarrow (-\infty, \infty] \\ \boldsymbol{x} &\longrightarrow \sup_{i \in I} f_i(\boldsymbol{x}) \end{aligned}$$

is also convex (recall that the $\sup_{i \in I} f_i(\boldsymbol{x})$ is the least upper bound of $\{f_i(\boldsymbol{x});\ i \in I\}$).

Proof. a) and)c) are consequences of the definition of convexity.

b) Let $\|\cdot\|$ be a norm. Then, for any $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ and any $\alpha \in [0, 1]$ we have:

$$\|\alpha \boldsymbol{x} + (1-\alpha)\boldsymbol{y}\| \leq \|\alpha \boldsymbol{x}\| + \|(1-\alpha)\boldsymbol{y}\| = \alpha\|\boldsymbol{x}\| + (1-\alpha)\|\boldsymbol{y}\|.$$

d) For every $i \in I$ we have

$$f_i(\alpha \boldsymbol{x} + (1-\alpha)\boldsymbol{y}) \leq \alpha f_i(\boldsymbol{x}) + (1-\alpha) f_i(\boldsymbol{y}) \leq \alpha h(\boldsymbol{x}) + (1-\alpha) h(\boldsymbol{y}).$$

Taking the supremum over all $i \in I$ we conclude:

$$h(\alpha \boldsymbol{x} + (1-\alpha)\boldsymbol{y}) \leq \alpha h(\boldsymbol{x}) + (1-\alpha) h(\boldsymbol{y}).$$

Necessary and sufficient conditions for extrema for convex functions

Theorem (Necessary condition in the convex case)
Let $f : \mathcal{C} \to \mathbb{R}$ be a convex function over the convex set $\mathcal{C} \subset \mathbb{R}^n$.

a) *A local minimum of f over $\mathcal{C}$ is also a global minimum over $\mathcal{C}$.*

b) *If, in addition, f is strictly convex, then there exists at most one global minimum of f.*

c) *If f is convex, the set $\mathcal{C}$ is open, and f is differentiable at $\boldsymbol{x}^*$, then $\nabla f(\boldsymbol{x}^*) = 0$ is a necessary (see Theorem pg. 8) and sufficient condition for $\boldsymbol{x}^* \in \mathcal{C}$ to be a global minimum of f over $\mathcal{C}$.*

Proof

a) Suposse that $\boldsymbol{x}$ is a local minimum of f but not a global minimum. Then there exists some $\boldsymbol{y} \neq \boldsymbol{x}$ such that $f(\boldsymbol{y}) < f(\boldsymbol{x})$. Since f is convex:

$$f(\alpha\boldsymbol{x} + (1-\alpha)\boldsymbol{y}) \leq \alpha f(\boldsymbol{x}) + (1-\alpha) f(\boldsymbol{y}) < f(\boldsymbol{x}), \quad \forall \alpha \in [0,1).$$

This contradicts the assumption that $\boldsymbol{x}$ is a local minimum.

b) Suppose that two distinct global minima $\boldsymbol{x}$ and $\boldsymbol{y}$ exist ($f(\boldsymbol{x}) = f(\boldsymbol{y})$). Then $(\boldsymbol{x}+\boldsymbol{y})/2 \in \mathcal{C}$, since $\mathcal{C}$ is convex, and also:

$$f((1/2)\boldsymbol{x} + (1/2)\boldsymbol{y}) < f(\boldsymbol{x}), \qquad f((1/2)\boldsymbol{x} + (1/2)\boldsymbol{y}) < f(\boldsymbol{y}),$$

and since $\boldsymbol{x}$ and $\boldsymbol{y}$ are global minima, we obtain a contradiction.

Proof (cont.)

c) By the convexity of $\mathcal{C}$, and using the convexity of f, we have that for all $\boldsymbol{x} \in \mathcal{C}$ and $\alpha \in [0, 1]$:

$$\alpha \boldsymbol{x} + (1-\alpha)\boldsymbol{x}^* = \boldsymbol{x}^* + \alpha(\boldsymbol{x} - \boldsymbol{x}^*) \quad \Rightarrow$$
$$\Rightarrow f(\boldsymbol{x}^* + \alpha(\boldsymbol{x} - \boldsymbol{x}^*)) = f(\alpha \boldsymbol{x} + (1-\alpha)\boldsymbol{x}^*) \le \alpha f(\boldsymbol{x}) + (1-\alpha) f(\boldsymbol{x}^*),$$

$$f(\boldsymbol{x}^* + \alpha(\boldsymbol{x} - \boldsymbol{x}^*)) - f(\boldsymbol{x}^*) \le \alpha f(\boldsymbol{x}) + (1-\alpha) f(\boldsymbol{x}^*) - f(\boldsymbol{x}^*) = \alpha(f(x) - f(x^*)).$$

It follows that:

$$\frac{f(\boldsymbol{x}^* + \alpha(\boldsymbol{x} - \boldsymbol{x}^*)) - f(\boldsymbol{x}^*)}{\alpha} \le f(\boldsymbol{x}) - f(\boldsymbol{x}^*).$$

Furthermore

$$\lim_{\alpha \to 0} \frac{f(\boldsymbol{x}^* + \alpha(\boldsymbol{x} - \boldsymbol{x}^*)) - f(\boldsymbol{x}^*)}{\alpha} = \nabla f(\boldsymbol{x}^*)^T (\boldsymbol{x} - \boldsymbol{x}^*).$$

Taking the limit as $\alpha \to 0$ we obtain

$$\nabla f(\boldsymbol{x}^*)^T (\boldsymbol{x} - \boldsymbol{x}^*) \le f(\boldsymbol{x}) - f(\boldsymbol{x}^*), \quad \forall \boldsymbol{x} \in \mathcal{C}.$$

If $\nabla f(\boldsymbol{x}^*) = 0$, we obtain

$$0 \le f(\boldsymbol{x}) - f(\boldsymbol{x}^*) \quad \Rightarrow \quad f(\boldsymbol{x}) \ge f(\boldsymbol{x}^*) \quad \forall \boldsymbol{x} \in \mathcal{C},$$

so $\boldsymbol{x}^*$ is a global minimum.

Remark

The inequality of the above page:

$$\nabla f(\boldsymbol{x}^*)^T(\boldsymbol{x} - \boldsymbol{x}^*) \leq f(\boldsymbol{x}) - f(\boldsymbol{x}^*), \quad \forall \boldsymbol{x} \in \mathcal{C},$$

can be written as:

$$f(\boldsymbol{x}) \geq f(\boldsymbol{x}^*) + \nabla f(\boldsymbol{x}^*)^T(\boldsymbol{x} - \boldsymbol{x}^*), \quad \forall \boldsymbol{x} \in \mathcal{C}.$$

In fact, we have proven the more general one:

$$\boxed{f(\boldsymbol{x}) \geq f(\boldsymbol{y}) + \nabla f(\boldsymbol{y})^T(\boldsymbol{x} - \boldsymbol{y}), \quad \forall \boldsymbol{x}, \boldsymbol{y} \in \mathcal{C},} \tag{1}$$

since we have not used the condition $\nabla f(\boldsymbol{x}^*) = 0$.

The inequality is, in fact, a consequence of the following characterization of differentiable convex functions.

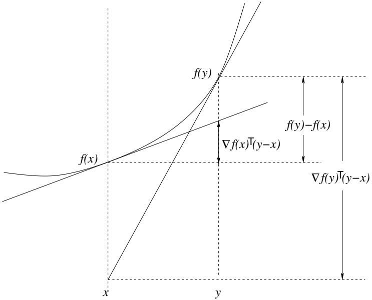
First characterization theorem of convex functions

Theorem

f is convex on $\mathcal{C}$ if and only if for any two points $\mathbf{x}, \mathbf{y} \in \mathcal{C}$ one has

$$\nabla f(\mathbf{x})^T(\mathbf{y}-\mathbf{x}) \leq f(\mathbf{y}) - f(\mathbf{x}) \leq \nabla f(\mathbf{y})^T(\mathbf{y}-\mathbf{x}). \tag{2}$$

If the inequalities are strict whenever $\mathbf{x} \neq \mathbf{y}$, then f is strictly convex over $\mathcal{C}$.



Remarks. As it follows from the proof, the two inequalities (2) in the Theorem can be substituted by (1) $\left(f(\mathbf{x}) \geq f(\mathbf{y}) + \nabla f(\mathbf{y})^T(\mathbf{x}-\mathbf{y})\right)$, since one inequality is a consequence of the other.
The proof for the strictly convex case is identical to the convex case.

Proof of the characterization theorem

Proof.

Assume that f is convex. The second inequality directly follows from (1). Interchanging the roles of $\boldsymbol{x}$ and $\boldsymbol{y}$ in (1), one gets for all $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{C}$ that

$$f(\boldsymbol{y}) \geq f(\boldsymbol{x}) + \nabla f(\boldsymbol{x})^T(\boldsymbol{y} - \boldsymbol{x}), \quad \Rightarrow f(\boldsymbol{y}) - f(\boldsymbol{x}) \geq \nabla f(\boldsymbol{x})^T(\boldsymbol{y} - \boldsymbol{x}),$$

which is the first inequality in (2).

To proof the converse, supose that (1) is true and we must proof that f is convex. We fix some $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{C}$ and some $\alpha \in [0, 1]$. Let $\boldsymbol{z} = \alpha\boldsymbol{x} + (1 - \alpha)\boldsymbol{y}$. Using the inequality twice, we get

$$\begin{aligned} f(\boldsymbol{x}) &\geq f(\boldsymbol{z}) + \nabla f(\boldsymbol{z})^T(\boldsymbol{x} - \boldsymbol{z}), \\ f(\boldsymbol{y}) &\geq f(\boldsymbol{z}) + \nabla f(\boldsymbol{z})^T(\boldsymbol{y} - \boldsymbol{z}). \end{aligned}$$

Multiplying the first inequality by α, the second by $(1 - \alpha)$ and adding, we obtain:

$$\alpha f(\boldsymbol{x}) + (1 - \alpha)f(\boldsymbol{y}) \geq f(\boldsymbol{z}) + \nabla f(\boldsymbol{z})^T(\alpha\boldsymbol{x} + (1 - \alpha)\boldsymbol{y} - \boldsymbol{z}) = f(\boldsymbol{z}),$$

which proves that f is convex.

Applications

- Many elementary (and and not so elementary) inequalities follow from the above Theorem.

  Example: the well know inequality[1]

  $$e^x \geq 1 + x,$$

  can also be proved by using the convexity of the function $f(x) = e^x$.

  Since $\nabla f(x) = f'(x) = e^x$, taking $y = 0$, so $\nabla f(y) = f'(0) = 1$, and using inequality (2):

  $$\nabla f(x)^T (y - x) \leq f(y) - f(x) \leq \nabla f(y)^T (y - x),$$

  we get:

  $$e^x(0 - x) \leq 1 - e^x \leq 0 - x \quad \Rightarrow \quad x \leq e^x - 1 \leq xe^x, \quad \forall x \in \mathbb{R},$$

  or

  $$e^x \geq 1 + x, \quad \text{and} \quad (1 - x)e^x \leq 1, \; \forall x \in \mathbb{R}.$$

[1] It can be easily be proved noting that $g(x) = e^x - (1 + x)$ has a minimum at $x = 0$.

Applications

Exercise 3. To be delivered before 20-X-2019 as: `Ex03-YourSurname.pdf`

Proof, whithout using the above theorem, that for any $a \in \mathbb{R}$, $f(x) = e^{ax}$ *is a convex function.*

Characterization of convexity for twice differentiable functions

Theorem.

Let $\mathcal{C} \subset \mathbb{R}^n$ be a convex set, let $f : \mathbb{R}^n \to \mathbb{R}$ be a twice continuously differentiable function over $\mathcal{C}$, and let Q be a real symmetric $n \times n$ matrix.

a) *If $\nabla^2 f(\boldsymbol{x})$ is positive semidefinite for all $\boldsymbol{x} \in \mathcal{C}$, then f is convex over $\mathcal{C}$.*

b) *If $\nabla^2 f(\boldsymbol{x})$ is positive definite for all $\boldsymbol{x} \in \mathcal{C}$, then f is strictly convex over $\mathcal{C}$.*

c) *If $\mathcal{C} = \mathbb{R}^n$ and f is convex, then $\nabla^2 f(\boldsymbol{x})$ is positive semidefinite for all $\boldsymbol{x} \in \mathcal{C}$.*

d) *The quadratic function $f(\boldsymbol{x}) = \boldsymbol{x}^T Q \boldsymbol{x}$, where Q is a symmetric matrix, is convex if and only if Q is positive semidefinite. Furthermore, f is strictly convex if and only if Q is positive definite.*

Proof.

a) According to Taylor's formula, for all $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{C}$

$$f(\boldsymbol{y}) = f(\boldsymbol{x}) + \nabla f(\boldsymbol{x})^T(\boldsymbol{y} - \boldsymbol{x}) + \frac{1}{2}(\boldsymbol{y} - \boldsymbol{x})^T \nabla^2 f(\boldsymbol{x} + \alpha(\boldsymbol{y} - \boldsymbol{x}))(\boldsymbol{y} - \boldsymbol{x}),$$

for some $\alpha \in [0, 1]$. Therefore, using the positive semi-definiteness of $\nabla^2 f(\boldsymbol{x})$, $(\boldsymbol{z}^T \nabla^2 f(\boldsymbol{x}) \boldsymbol{z} \geq 0, \quad \forall \boldsymbol{z})$, we obtain

$$f(\boldsymbol{y}) \geq f(\boldsymbol{x}) + \nabla f(\boldsymbol{x})^T(\boldsymbol{y} - \boldsymbol{x}), \quad \forall \boldsymbol{x}, \boldsymbol{y} \in \mathcal{C},$$

from which we can conclude that f is convex.

Proof of the Theorem (cont.)

b) Similar to the proof of part a).

c) Suposse that $f : \mathbb{R}^n \to \mathbb{R}$ is convex and that $\boldsymbol{x} \in \mathcal{C}$. For some small $\alpha > 0$ and any $\boldsymbol{y} \in \mathbb{R}^n$, we have that $\boldsymbol{x} + \alpha \boldsymbol{y} \in \mathcal{C}$. From Taylor's formula:

$$f(\boldsymbol{x} + \alpha \boldsymbol{y}) = f(\boldsymbol{x}) + \alpha \nabla f(\boldsymbol{x})^T \boldsymbol{y} + \frac{\alpha^2}{2} \boldsymbol{y}^T \nabla^2 f(\boldsymbol{x}) \boldsymbol{y} + o(\|\alpha \boldsymbol{y}\|^2).$$

Since f is convex, we know that for any $\boldsymbol{a}$ and $\boldsymbol{b}$:

$$f(\boldsymbol{a}) \geq f(\boldsymbol{b}) + \nabla f(\boldsymbol{b})^T (\boldsymbol{a} - \boldsymbol{b}),$$

so, taking $\boldsymbol{a} = \boldsymbol{x} + \alpha \boldsymbol{y}$ and $\boldsymbol{b} = \boldsymbol{x}$, we get

$$f(\boldsymbol{x} + \alpha \boldsymbol{y}) \geq f(\boldsymbol{x}) + \alpha \nabla f(\boldsymbol{x})^T \boldsymbol{y}.$$

Therefore, we have that for any $\boldsymbol{y} \in \mathbb{R}^n$

$$\frac{\alpha^2}{2} \boldsymbol{y}^T \nabla^2 f(\boldsymbol{x}) \boldsymbol{y} + o(\|\alpha \boldsymbol{y}\|^2) \geq 0.$$

Dividing by $\alpha^2/2$ and taking $\alpha \to 0$, we get

$$\boldsymbol{y}^T \nabla^2 f(\boldsymbol{x}) \boldsymbol{y} \geq 0, \quad \forall \boldsymbol{y} \in \mathbb{R}^n.$$

Proof of the Theorem (cont.)

d) If $f(\boldsymbol{x}) = \boldsymbol{x}^T Q \boldsymbol{x}$ then $\nabla^2 f(\boldsymbol{x}) = 2Q$. Hence, from a) and c) it follows that f is convex if and only if Q is positive semidefinite.

For the strict convexity, supose that f is strictly convex, then, according to c), Q is positive semidefinite and it remains to show that Q is positive definite.

It can be shown that Q is positive definite if and only if all its eigenvalues are posive.

Assume that zero is an eigenvalue, then there exists some $\boldsymbol{x} \neq 0$ such that $Q\boldsymbol{x} = 0$. It follows that

$$0 = f(0) = f\left(\frac{1}{2}\boldsymbol{x} + \frac{1}{2}(-\boldsymbol{x})\right) = \frac{1}{2}f(\boldsymbol{x}) + \frac{1}{2}f(-\boldsymbol{x}) = 0,$$

which contradicts the strict convexity of f.

Optimization with equality constraints. Lagrange multiplier theory

- Consider the problem of finding the minimum (or maximum) of a real-valued function f with domain $\mathcal{C} \subset \mathbb{R}^n$

  $$f : \mathcal{C} \longrightarrow \mathbb{R},$$

  subject only to the equality constraints

  $$g_i(\mathbf{x}) = 0, \quad i = 1, ..., m, \quad m < n, \tag{3}$$

  where each of the g_i is a real-valued function defined on $\mathcal{C}$. This is, the problem is to find an extremum of f in the region determined by the equations (3).

- The first and most intuitive method of solution of such a problem involves the elimination of m variables from the problem by using equations (3).
- The conditions for such an elimination are stated by the Implicit Function Theorem, that assumes differentiability of the functions g_i and that the $n \times m$ Jacobian matrix $(\partial g_i / \partial x_j)$ has rank m.
- The actual solution of the unconstraint equations for m variables in terms of the remaining $n - m$ can often be a difficult, if not impossible, task.

Optimization with equality constraints

Example *Find the area of the largest rectangle that can be inscribed in the ellipse*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Solution Suppose that the upper righthand corner of the rectangle is at the point (x, y), then the area of the rectangle is $S = 4xy$. We have:

$$\frac{2x}{a^2}dx + \frac{2y}{b^2}dy = 0 \quad \Rightarrow \quad \frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0 \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{b^2 x}{a^2 y},$$

so

$$\frac{dS}{dx} = 4y + 4x\frac{dy}{dx} = 4y - \frac{4b^2x^2}{a^2y}, \qquad \frac{dS}{dx} = 0 \quad \Rightarrow \quad y^2 = \frac{b^2x^2}{a^2}.$$

Since, according to the equation of the ellipse

$$y^2 = b^2 - \frac{b^2x^2}{a^2},$$

we get

$$y^2 = b^2 - y^2 \quad \Rightarrow \quad y = \frac{b}{\sqrt{2}} \quad \text{and} \quad x = \frac{a}{\sqrt{2}} \quad \Rightarrow \quad S_{max} = 2ab.$$

Lagrange multipliers

Another method for finding the minimum, also based on the idea of transforming a constrained problem into an unconstrained one, was proposed by Lagrange.

Before introducing this method, we present the following result:

Theorem

Let f and g_i, $i =, \dots, m$, be real-valued functions on $\mathcal{C} \subset \mathbb{R}^n$ and continuosly differentiable on a neighborhood $N_\epsilon(\boldsymbol{x}^*) \subset \mathcal{C}$. Suppose that $\boldsymbol{x}^*$ is a local minimum (or maximum) of f for all points $\boldsymbol{x} \in N_\epsilon(\boldsymbol{x}^*)$ that also satisfy:

$$g_i(\boldsymbol{x}) = 0, \quad i = 1, \dots, m.$$

Assume also that the Jacobian matrix $(\partial g_i / \partial x_j)$ at $\boldsymbol{x}^*$ has rank m. Under these hypothese, there exist real numbers λ_i^* such that

$$\nabla f(\boldsymbol{x}^*) = \sum_{i=1}^{m} \lambda_i^* \nabla g_i(\boldsymbol{x}^*).$$

Before the proof. Example

Consider the problem

$$\max f(x, y) = x\ y,$$

subject to the constraint

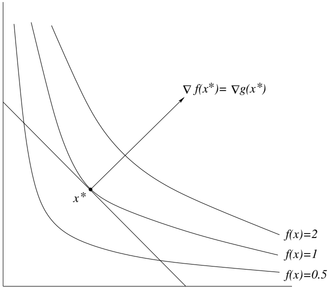
$$g(x, y) = x + y - 2 = 0.$$

The solution of this system problem is:

$$x^* = y^* = 1.$$

In this case:

$$\nabla f(\boldsymbol{x}^*) = \begin{pmatrix} y \\ x \end{pmatrix}_{(x,y)=(1,1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \nabla g(\boldsymbol{x}^*) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}_{(x,y)=(1,1)}.$$



Proof of the Theorem

Proof. By suitable rearrangement and relabeling of rows, we can always assume that the $m \times m$ matrix formed by taking the first m rows of the Jacobian $(\partial g_i(\boldsymbol{x}^*)/\partial x_j)$, is nonsingular since it has rank m.

What we want to proof is that there exist $\lambda_1^*, ..., \lambda_m^*$ such that:

$$\nabla f(\boldsymbol{x}^*) = \lambda_1^* \nabla g_1(\boldsymbol{x}^*) + \lambda_2^* \nabla g_2(\boldsymbol{x}^*) + ... + \lambda_m^* \nabla g_m(\boldsymbol{x}^*),$$

that can also be written as:

$$\left(\begin{array}{c} \frac{\partial f(\boldsymbol{x}^*)}{\partial x_1} \\ \vdots \\ \frac{\partial f(\boldsymbol{x}^*)}{\partial x_m} \\ \hline \vdots \\ \frac{\partial f(\boldsymbol{x}^*)}{\partial x_n} \end{array}\right) = \left(\begin{array}{ccc} \frac{\partial g_1(\boldsymbol{x}^*)}{\partial x_1} & \dots & \frac{\partial g_m(\boldsymbol{x}^*)}{\partial x_1} \\ \vdots & & \vdots \\ \frac{\partial g_1(\boldsymbol{x}^*)}{\partial x_m} & \dots & \frac{\partial g_m(\boldsymbol{x}^*)}{\partial x_m} \\ \hline \vdots & & \vdots \\ \frac{\partial g_1(\boldsymbol{x}^*)}{\partial x_n} & \dots & \frac{\partial g_m(\boldsymbol{x}^*)}{\partial x_n} \end{array}\right) \begin{pmatrix} \lambda_1^* \\ \vdots \\ \lambda_m^* \end{pmatrix}.$$

We will first proof that there exist $\lambda_1^*, ..., \lambda_m^*$ such that:

$$\begin{pmatrix} \frac{\partial f(\boldsymbol{x}^*)}{\partial x_1} \\ \vdots \\ \frac{\partial f(\boldsymbol{x}^*)}{\partial x_m} \end{pmatrix} = \begin{pmatrix} \frac{\partial g_1(\boldsymbol{x}^*)}{\partial x_1} & \dots & \frac{\partial g_m(\boldsymbol{x}^*)}{\partial x_1} \\ \vdots & & \vdots \\ \frac{\partial g_1(\boldsymbol{x}^*)}{\partial x_m} & \dots & \frac{\partial g_m(\boldsymbol{x}^*)}{\partial x_m} \end{pmatrix} \begin{pmatrix} \lambda_1^* \\ \vdots \\ \lambda_m^* \end{pmatrix}.$$

Proof of the Theorem (cont. 1)

Since the matrix of the above linear system is non-singular, the set of linear equations

$$\sum_{i=1}^{m} \frac{\partial g_i(\boldsymbol{x}^*)}{\partial x_j} \lambda_i = \frac{\partial f(\boldsymbol{x}^*)}{\partial x_j}, \quad j = 1, \ldots, m, \tag{4}$$

has a unique solution: λ_i^*, $i = 1, \ldots, m$. In this way we have seen that the first m components of the gradients verify the equality that we want to proof.

Let us see that the remaining $n - m$ components also fulfil the same equality. Let $\hat{\boldsymbol{x}} = (x_{m+1}, \ldots, x_n)$, then, applying the Implicit Function Theorem to the equations $g_i(\boldsymbol{x}^*) = 0$, it follows that there exist real functions

$$h_j(\hat{\boldsymbol{x}}) = h_j(x_{m+1}, \ldots, x_n), \quad j = 1, \ldots, m,$$

defined in an open set $\hat{D} \subset \mathbb{R}^{n-m}$ containing $\boldsymbol{x}^*$ such that

$$\begin{aligned} x_j^* &= h_j(\hat{\boldsymbol{x}}^*) = h_j(x_{m+1}^*, \ldots, x_n^*), \quad j = 1, \ldots, m, && (5) \\ f(\boldsymbol{x}^*) &= f(h_1(\hat{\boldsymbol{x}}^*), \ldots, h_m(\hat{\boldsymbol{x}}^*), x_{m+1}^*, \ldots, x_n^*). && (6) \end{aligned}$$

Using the same Theorem, we have also that for $j = m + 1, \ldots, n$

$$\sum_{k=1}^{m} \frac{\partial g_i(\boldsymbol{x}^*)}{\partial x_k} \frac{\partial h_k(\hat{\boldsymbol{x}}^*)}{\partial x_j} + \frac{\partial g_i(\boldsymbol{x}^*)}{\partial x_j} = 0, \quad i = 1, \ldots, m. \tag{7}$$

Proof of the Theorem (cont. 2)

If $\boldsymbol{x}^*$ is a minima of f its first partial derivatives with respect to $x_{m+1}, \dots, x_n$ must vanish at $\boldsymbol{x}^*$. Thus

$$\frac{\partial f(\boldsymbol{x}^*)}{\partial x_j} = \sum_{k=1}^{m} \frac{\partial f(\boldsymbol{x}^*)}{\partial x_k} \frac{\partial h_k(\hat{\boldsymbol{x}}^*)}{\partial x_j} + \frac{\partial f(\boldsymbol{x}^*)}{\partial x_j} = 0, \quad j = m+1, \dots, n. \tag{8}$$

Multiplying each of the equations in (7) by λ_i^* and adding up, we get:

$$\sum_{i=1}^{m} \left(\sum_{k=1}^{m} \lambda_i^* \frac{\partial g_i(\boldsymbol{x}^*)}{\partial x_k} \frac{\partial h_k(\hat{\boldsymbol{x}}^*)}{\partial x_j} + \lambda_i^* \frac{\partial g_i(\boldsymbol{x}^*)}{\partial x_j} \right) = 0, \quad j = m+1, \dots, n.$$

Substracting this equality from (8) we get:

$$\sum_{k=1}^{m} \left[\frac{\partial f(\boldsymbol{x}^*)}{\partial x_k} - \sum_{i=1}^{m} \lambda_i^* \frac{\partial g_i(\boldsymbol{x}^*)}{\partial x_j} \right] \frac{\partial h_k(\hat{\boldsymbol{x}}^*)}{\partial x_j} + \frac{\partial f(\boldsymbol{x}^*)}{\partial x_j} - \sum_{i=1}^{m} \lambda_i^* \frac{\partial g_i(\boldsymbol{x}^*)}{\partial x_j} = 0,$$

for $j = m+1, \dots, n$. Since, due to (4), the expression in the brackets is zero, we get the desired result

$$\frac{\partial f(\boldsymbol{x}^*)}{\partial x_j} - \sum_{i=1}^{m} \lambda_i^* \frac{\partial g_i(\boldsymbol{x}^*)}{\partial x_j} = 0, \quad j = m+1, \dots, n.$$

Lagrange's method

Lagrange's method consists of transforming an equality constrained extremum problem into a problem of finding a stationary point of the Lagrangian function

$$L(\mathbf{x}, \boldsymbol{\lambda}) = f(\mathbf{x}) - \sum_{i=1}^{m} \lambda_i g_i(\mathbf{x})$$

Theorem (Necessary conditions)
Suppose that f and g_i, $i = 1, \dots, m$, are real-valued functions that satisfy the hypoteses of the preceeding Theorem, this is:

- $$f : \mathcal{C} \subset \mathbb{R}^n \longrightarrow \mathbb{R}, \quad \text{and} \quad g_i : \mathcal{C} \subset \mathbb{R}^n \longrightarrow \mathbb{R}, \quad i = 1, \dots, m.$$
- *They are all continuosly differentiable on a neighborhood $N_\epsilon(\mathbf{x}^*) \subset \mathcal{C} \subset \mathbb{R}^n$.*
- *The Jacobian matrix $(\partial g_i(\mathbf{x}^*)/\partial x_j)$ has rank m.*
- *$\mathbf{x}^*$ is a local minimum (or maximum) of f in $N_\epsilon(\mathbf{x}^*)$.*
- *If $\mathbf{x} \in N_\epsilon(\mathbf{x}^*)$, then*
$$g_i(\mathbf{x}) = 0, \quad i = 1, \dots, m.$$

Then, there exists a vector of multipliers $\boldsymbol{\lambda}^* = (\lambda_1^*, \dots, \lambda_m^*)^T$ such that

$$\nabla L(\mathbf{x}^*, \boldsymbol{\lambda}^*) = 0.$$

Proof. Follows directly from the definition of L and the preceeding Theorem.

Lagrange's method

Theorem (Sufficient conditions).
Let f, g_1,...,g_m be twice continuously differentiable real-valued functions in $\mathbb{R}^n$.
If there exist vectors $\boldsymbol{x}^* \in \mathbb{R}^n$, $\boldsymbol{\lambda}^* \in \mathbb{R}^m$ such that

$$\nabla L(\boldsymbol{x}^*, \boldsymbol{\lambda}^*) = 0,$$

and for every $\boldsymbol{z} \in \mathbb{R}^n$, $\boldsymbol{z} \neq 0$ satisfying

$$\boldsymbol{z}^T \nabla g_i(\boldsymbol{x}^*) = 0, \quad i = 1, ..., m,$$

it follows that

$$\boldsymbol{z}^T \nabla_x^2 L(\boldsymbol{x}^*, \boldsymbol{\lambda}^*) \boldsymbol{z} > 0,$$

then, f has a strict local minimum at $\boldsymbol{x}^*$ subject to $g_i(\boldsymbol{x}) = 0$, $i = 1, ..., m$,
(similar for a maximum if $\boldsymbol{z}^T \nabla_x^2 L(\boldsymbol{x}^*, \boldsymbol{\lambda}^*) \boldsymbol{z} < 0$).

Proof of the Theorem

Proof.

Assume that $\boldsymbol{x}^*$ is not a strict local minimum. Then there exist a neighborhood $N_\delta(\boldsymbol{x}^*)$ and a sequence $\{\boldsymbol{z}^k\}_{k\in\mathbb{Z}}$, $\boldsymbol{z}^k \in N_\delta(\boldsymbol{x}^*)$, $\boldsymbol{z}^k \neq \boldsymbol{x}^*$, converging to $\boldsymbol{x}^*$ such that for every $\boldsymbol{z}^k$ in the sequence

$$g_i(\boldsymbol{z}^k) = 0, \quad i = 1, ..., m, \quad f(\boldsymbol{x}^*) \geq f(\boldsymbol{z}^k). \tag{9}$$

Let $\boldsymbol{z}^k = \boldsymbol{x}^* + \theta^k \boldsymbol{y}^k$, where $\theta^k > 0$ and $\|\boldsymbol{y}^k\| = 1$. The sequence $\{(\theta^k, \boldsymbol{y}^k)\}_{k\in\mathbb{Z}}$ has a subsequence that converges to $(0, \overline{\boldsymbol{y}})$, where $\|\overline{\boldsymbol{y}}\| = 1$. By the Mean Value Theorem, for each k in this subsequence

$$g_i(\boldsymbol{z}^k) - g_i(\boldsymbol{x}^*) = \theta^k\, (\boldsymbol{y}^k)^T \nabla g_i(\boldsymbol{x}^* + \eta_i^k \theta^k \boldsymbol{y}^k) = 0, \quad i = 1, ..., m. \tag{10}$$

with $0 < \eta_i^k < 1$ and

$$f(\boldsymbol{z}^k) - f(\boldsymbol{x}^*) = \theta^k\, (\boldsymbol{y}^k)^T \nabla f(\boldsymbol{x}^* + \xi^k \theta^k \boldsymbol{y}^k) \leq 0, \tag{11}$$

with $0 < \xi_i^k < 1$. Dividing (10) and (11) by θ^k and taking limits as $k \to \infty$, we get

$$\begin{aligned} \overline{\boldsymbol{y}}^T \nabla g_i(\boldsymbol{x}^*) &= 0, \quad i = 1, ...m \\ \overline{\boldsymbol{y}}^T \nabla f(\boldsymbol{x}^*) &\leq 0. \end{aligned}$$

Proof of the Theorem (cont)

From Taylor's theorem we have

$$\begin{aligned} L(\boldsymbol{z}^k, \boldsymbol{\lambda}^*) &= L(\boldsymbol{x}^*, \boldsymbol{\lambda}^*) + \theta^k (\boldsymbol{y}^k)^T \nabla_x L(\boldsymbol{x}^*, \boldsymbol{\lambda}^*) \\ &\quad + \frac{1}{2}(\theta^k)^2 (\boldsymbol{y}^k)^T \nabla_x^2 L(\boldsymbol{x}^* + \eta^k \theta^k \boldsymbol{y}^k, \boldsymbol{\lambda}^*) \boldsymbol{y}^k, \end{aligned} \tag{12}$$

with $0 < \eta^k < 1$. Dividing this equality by $(\theta^k)^2/2$, using the definition of L, the hypothesis $\nabla L(\boldsymbol{x}^*, \boldsymbol{\lambda}^*) = 0$ and the conditions (9), we get (exercise)

$$(\boldsymbol{y}^k)^T \nabla_x^2 L(\boldsymbol{x}^* + \eta^k \theta^k \boldsymbol{y}^k, \boldsymbol{\lambda}^*) \boldsymbol{y}^k \leq 0.$$

Letting $k \to \infty$, we obtain $\overline{\boldsymbol{y}}$ verifying

$$\overline{\boldsymbol{y}}^T \nabla_x^2 L(\boldsymbol{x}^*, \boldsymbol{\lambda}^*) \overline{\boldsymbol{y}} \leq 0.$$

This completes the proof, since $\overline{\boldsymbol{y}} \neq 0$ and satisfyes $\overline{\boldsymbol{y}}^T \nabla g_i(\boldsymbol{x}^*) = 0$ for $i = 1, \ldots m$.

Example

Consider again the problem

$$\max f(x, y) = x\ y,$$

subject to the constraint

$$g(x, y) = x + y - 2 = 0.$$

The Lagrangian is

$$L(\boldsymbol{x}, \lambda) = xy - \lambda(x + y - 2).$$

Setting $\nabla L(\boldsymbol{x}, \lambda) = 0$, we get:

$$\begin{aligned}
\frac{\partial L(\boldsymbol{x}, \lambda)}{\partial x} &= y - \lambda = 0, \\
\frac{\partial L(\boldsymbol{x}, \lambda)}{\partial y} &= x - \lambda = 0, \\
\frac{\partial L(\boldsymbol{x}, \lambda)}{\partial \lambda} &= -x - y + 2 = 0.
\end{aligned}$$

The solution of this system of equations is

$$x^* = y^* = \lambda^* = 1.$$

According to the Theorem on necessary conditions, the point $(\boldsymbol{x}^*, \lambda^*) = (1, 1, 1)$ satisfies the necessary conditions for a maximum.

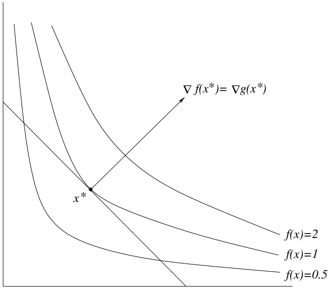
Example (cont.)

The linear dependence between ∇f and ∇g at the maxima, is clearly illustrated in the figure. In fact, in this case they concide, since

$$\nabla f(\boldsymbol{x}^*) = \left(\begin{array}{c} y \\ x \end{array} \right)_{(x,y)=(1,1)} = \left(\begin{array}{c} 1 \\ 1 \end{array} \right).$$

and

$$\nabla g(\boldsymbol{x}^*) = \left(\begin{array}{c} 1 \\ 1 \end{array} \right)_{(x,y)=(1,1)} = \left(\begin{array}{c} 1 \\ 1 \end{array} \right).$$



Example (cont.)

Turning to the sufficient conditions, we compute $\nabla_x^2 L(\boldsymbol{x}^*, \boldsymbol{\lambda}^*)$:

$$\frac{\partial^2 L(\boldsymbol{x}^*, \boldsymbol{\lambda}^*)}{\partial x \partial x} = 0, \quad \frac{\partial^2 L(\boldsymbol{x}^*, \boldsymbol{\lambda}^*)}{\partial x \partial y} = 1, \quad \frac{\partial^2 L(\boldsymbol{x}^*, \boldsymbol{\lambda}^*)}{\partial y \partial y} = 0.$$

Hence

$$\boldsymbol{z}^T \nabla_x^2 L(\boldsymbol{x}^*, \boldsymbol{\lambda}^*) \boldsymbol{z} = (z_1, z_2) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = 2 z_1 z_2,$$

According to the last Theorem, we must determine the sign of $2z_1z_2$ for all $\boldsymbol{z} \neq 0$ such that $\boldsymbol{z}^T \nabla g(\boldsymbol{x}^*) = 0$.
Since

$$\frac{\partial g(\boldsymbol{x}^*)}{\partial x} = \frac{\partial g(\boldsymbol{x}^*)}{\partial y} = 1,$$

the last condition $\boldsymbol{z}^T \nabla g(\boldsymbol{x}^*) = 0$ is equivalent to $z_1 + z_2 = 0$, from which we get

$$\boldsymbol{z}^T \nabla_x^2 L(\boldsymbol{x}^*, \boldsymbol{\lambda}^*) \boldsymbol{z} = -2z_1^2 < 0.$$

Thus, $(1, 1)$ is a strict local maximum.