Optimization Methods

1. Introduction.
2. Greedy algorithms for combinatorial optimization.
3. LS and neighborhood structures for combinatorial optimization.
4. Variable neighborhood search, neighborhood descent, SA, TS, EC.
5. Branch and bound algorithms, and subset selection algorithms.
6. Linear programming problem formulations and applications.
7. Linear programming algorithms.
8. Integer linear programming algorithms.
9. Unconstrained nonlinear optimization and gradient descent.
10. Newton's methods and Levenberg-Marquardt modification.
11. Quasi-Newton methods and conjugate direction methods.
12. Nonlinear optimization with equality constraints.
13. Nonlinear optimization with inequality constraints.
14. Problem formulation and concepts in multi-objective optimization.
15. Search for single final solution in multi-objective optimization.
16: Search for multiple solutions in multi-objective optimization.

Linear Programing Problems

$$\begin{aligned} \text{Maximize} \quad & 3x_1 + 4x_2 \\ \text{subject to} \quad & x_1 + x_2 = 10 \\ & x_1 \geq 0,\ x_2 \geq 0 \end{aligned}$$

$$\begin{aligned} \text{Maximize} \quad & 2x_1 + 3x_2 \\ \text{subject to} \quad & 2x_1 + \ \ x_2 \leq 20 \\ & x_1 + 3x_2 \leq 30 \\ & x_1 \geq 0,\ x_2 \geq 0 \end{aligned}$$

$$\text{Maximize} \quad (3,\ 4)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\text{subject to} \quad (1,\ 1)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 10$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\text{Maximize} \quad (2,\ 3)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\text{subject to} \quad \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq \begin{pmatrix} 20 \\ 30 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{aligned} \text{Maximize} \quad & \mathbf{c}^{\mathrm{T}}\mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

$$\begin{aligned} \text{Maximize} \quad & \mathbf{c}^{\mathrm{T}}\mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

Problem Formulation Exercise

We have two types of concrete. The first type contains 30% cement, 40% gravel, and 30% sand (all percentages of weight). The second type contains 10% cement, 20% gravel, and 70% sand. The first type of concrete costs \$5 per pound and the second type costs \$1 per pound. How many pounds of each type of concrete should you buy and mix together so that your cost is minimized but you get a concrete mixture that has at least a total of 5 pounds of cement, 3 pounds of gravel, and 4 pounds of sand?

(from E.K.P. Chong and S.H. Zak: An Introduction of Optimization)

$$\begin{array}{ll} \text{Minimize} & \mathbf{c}^{\mathrm{T}}\mathbf{x} \\ \text{subject to} & \mathbf{A}\mathbf{x} \geq \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{array}$$

$$\mathbf{c}^{\mathrm{T}} = (?, ?) \qquad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\mathbf{A} = \begin{pmatrix} ? & ? \\ ? & ? \\ ? & ? \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} ? \\ ? \\ ? \end{pmatrix}$$

Send your answer to TA

Problem Formulation Exercise

We have two types of concrete. The first type contains 30% cement, 40% gravel, and 30% sand (all percentages of weight). The second type contains 10% cement, 20% gravel, and 70% sand. The first type of concrete costs \$5 per pound and the second type costs \$1 per pound. How many pounds of each type of concrete should you buy and mix together so that your cost is minimized but you get a concrete mixture that has at least a total of 5 pounds of cement, 3 pounds of gravel, and 4 pounds of sand?

(from E.K.P. Chong and S.H. Zak: An Introduction of Optimization)

$$\begin{aligned} &\text{Minimize} && \mathbf{c}^{\mathrm{T}}\mathbf{x} \\ &\text{subject to} && \mathbf{A}\mathbf{x} \geq \mathbf{b} \\ & && \mathbf{x} \geq \mathbf{0} \end{aligned}$$

$$\mathbf{c}^{\mathrm{T}} = (5, 1) \qquad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\mathbf{A} = \begin{pmatrix} 0.3 & 0.1 \\ 0.4 & 0.2 \\ 0.3 & 0.7 \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} 5 \\ 3 \\ 4 \end{pmatrix}$$

Problem Solving Exercise

$$\begin{array}{ll} \text{Maximize} & z = x_1 + 2x_2 \\ \text{subject to} & x_1 + x_2 = 10 \\ & x_1 \geq 0,\ x_2 \geq 0 \end{array}$$

Your solution

$z^* = \underline{\ ?\ }$, $x_1^* = \underline{\ ?\ }$, $x_2^* = \underline{\ ?\ }$.

Send your answer to TA

Problem Solving Exercise

$$\begin{aligned} \text{Maximize} \quad & z = x_1 + 2x_2 \\ \text{subject to} \quad & x_1 + x_2 = 10 \\ & x_1 \geq 0,\ x_2 \geq 0 \end{aligned}$$

Your solution

$z^* = 20,\ x_1^* = 0,\ x_2^* = 10$

$$\begin{aligned} \text{Maximize} \quad & z = x_1 + x_2 \\ \text{subject to} \quad & x_1 + x_2 \leq 10 \\ & x_1 \geq 0,\ x_2 \geq 0 \end{aligned}$$

Your solution

$z^* = \underline{\ ?\ },\ x_1^* = \underline{\ ?\ },\ x_2^* = \underline{\ ?\ }.$

Send your answer to TA

Problem Solving Exercise

$$\begin{aligned} \text{Maximize} \quad & z = x_1 + 2x_2 \\ \text{subject to} \quad & x_1 + x_2 = 10 \\ & x_1 \geq 0,\ x_2 \geq 0 \end{aligned}$$

Your solution

$z^* = 20,\ x_1^* = 0,\ x_2^* = 10$

$$\begin{aligned} \text{Maximize} \quad & z = x_1 + x_2 \\ \text{subject to} \quad & x_1 + x_2 \leq 1 \\ & x_1 - x_2 \leq -2 \\ & x_1 \geq 0,\ x_2 \geq 0 \end{aligned}$$

Your solution

$z^* = \underline{\ ?\ },\ x_1^* = \underline{\ ?\ },\ x_2^* = \underline{\ ?\ }.$

Send your answer to TA

$$\begin{aligned} \text{Maximize} \quad & z = x_1 + x_2 \\ \text{subject to} \quad & x_1 + x_2 \leq 10 \\ & x_1 \geq 0,\ x_2 \geq 0 \end{aligned}$$

Your solution

$z^* = 10,\ x_1^* = t,\ x_2^* = 10 - t,$

$(0 \leq t \leq 10)$

Problem Solving Exercise

$$\begin{aligned} \text{Maximize} \quad & z = x_1 + 2x_2 \\ \text{subject to} \quad & x_1 + x_2 = 10 \\ & x_1 \geq 0, \; x_2 \geq 0 \end{aligned}$$

Your solution

$z^* = 20, \; x_1^* = 0, \; x_2^* = 10$

$$\begin{aligned} \text{Maximize} \quad & z = x_1 + x_2 \\ \text{subject to} \quad & x_1 + x_2 \leq 1 \\ & x_1 - x_2 \leq -2 \\ & x_1 \geq 0, \; x_2 \geq 0 \end{aligned}$$

Your solution

No feasible solution

$$\begin{aligned} \text{Maximize} \quad & z = x_1 + x_2 \\ \text{subject to} \quad & x_1 + x_2 \leq 10 \\ & x_1 \geq 0, \; x_2 \geq 0 \end{aligned}$$

Your solution

$z^* = 10, \; x_1^* = t, \; x_2^* = 10 - t,$

$(0 \leq t \leq 10)$

$$\begin{aligned} \text{Maximize} \quad & z = x_1 + x_2 \\ \text{subject to} \quad & -x_1 + x_2 \leq 1 \\ & x_1 - x_2 \leq 1 \\ & x_1 \geq 0, \; x_2 \geq 0 \end{aligned}$$

Your solution

$z^* = \underline{\;?\;}, \; x_1^* = \underline{\;?\;}, \; x_2^* = \underline{\;?\;}.$

Send your answer to TA

Problem Solving Exercise

$$\begin{aligned} &\text{Maximize} \quad z = x_1 + 2x_2 \\ &\text{subject to} \quad x_1 + x_2 = 10 \\ &\qquad\qquad\quad x_1 \geq 0,\ x_2 \geq 0 \end{aligned}$$

Your solution

$z^* = 20,\ x_1^* = 0,\ x_2^* = 10$

$$\begin{aligned} &\text{Maximize} \quad z = x_1 + x_2 \\ &\text{subject to} \quad x_1 + x_2 \leq 1 \\ &\qquad\qquad\quad x_1 - x_2 \leq -2 \\ &\qquad\qquad\quad x_1 \geq 0,\ x_2 \geq 0 \end{aligned}$$

Your solution

No feasible solution

$$\begin{aligned} &\text{Maximize} \quad z = x_1 + x_2 \\ &\text{subject to} \quad x_1 + x_2 \leq 10 \\ &\qquad\qquad\quad x_1 \geq 0,\ x_2 \geq 0 \end{aligned}$$

Your solution

$z^* = 10,\ x_1^* = t,\ x_2^* = 10 - t,$

$(0 \leq t \leq 10)$

$$\begin{aligned} &\text{Maximize} \quad z = x_1 + x_2 \\ &\text{subject to} \quad -x_1 + x_2 \leq 1 \\ &\qquad\qquad\quad x_1 - x_2 \leq 1 \\ &\qquad\qquad\quad x_1 \geq 0,\ x_2 \geq 0 \end{aligned}$$

Your solution

$z^* = \infty,\ x_1^* = \infty,\ x_2^* = \infty$

under the constraint conditions.

Linear Programming Problem

- Linear objective function & Linear constraint conditions

Four Different Cases of the Optimal Solution

(1) A Single Solution

$$\begin{aligned} &\text{Maximize} && z = x_1 + 2x_2 \\ &\text{subject to} && x_1 + x_2 = 10 \\ & && x_1 \geq 0,\ x_2 \geq 0 \end{aligned}$$

$$z^* = 20,\ x_1^* = 0,\ x_2^* = 10$$

(2) Multiple Solutions

$$\begin{aligned} &\text{Maximize} && z = x_1 + x_2 \\ &\text{subject to} && x_1 + x_2 \leq 10 \\ & && x_1 \geq 0,\ x_2 \geq 0 \end{aligned}$$

$$z^* = 10,\ x_1^* = t,\ x_2^* = 10 - t,\quad (0 \leq t \leq 10)$$

(3) No Feasible Solution

$$\begin{aligned} &\text{Maximize} && z = x_1 + x_2 \\ &\text{subject to} && x_1 + x_2 \leq 1 \\ & && x_1 - x_2 \leq -2 \\ & && x_1 \geq 0,\ x_2 \geq 0 \end{aligned}$$

No feasible solution

(4) Unbounded

$$\begin{aligned} &\text{Maximize} && z = x_1 + x_2 \\ &\text{subject to} && -x_1 + x_2 \leq 1 \\ & && x_1 - x_2 \leq 1 \\ & && x_1 \geq 0,\ x_2 \geq 0 \end{aligned}$$

$$z^* = \infty,\ x_1^* = \infty,\ x_2^* = \infty$$

Simplex Method for LP (1947)

by George Bernard Dantzig (1914-2005)

Linear programming is a mathematical method for determining a way to achieve the best outcome (such as maximum profit or lowest cost) in a given mathematical model for some list of requirements represented as linear relationships. Linear programming arose as a mathematical model developed during World War II to plan expenditures and returns in order to reduce costs to the army and increase losses to the enemy. It was kept secret until 1947. Postwar, many industries found its use in their daily planning.

The founders of this subject are Leonid Kantorovich, a Russian mathematician who developed linear programming problems in 1939, Dantzig, who published the simplex method in 1947, and John von Neumann, who developed the theory of the duality in the same year.

[From Wikipedia]

Simplex Method for LP (1947)

by George Bernard Dantzig (1914-2005)

An event in Dantzig's life became the origin of a famous story in 1939, while he was a graduate student at UC Berkeley. Near the beginning of a class for which Dantzig was late, professor Jerzy Neyman wrote two examples of famously unsolved statistics problems on the blackboard. When Dantzig arrived, he assumed that the two problems were a homework assignment and wrote them down. According to Dantzig, the problems "seemed to be a little harder than usual", but a few days later he handed in completed solutions for the two problems, still believing that they were an assignment that was overdue.[4][8]

[From Wikipedia]

Simplex Method for LP (1947)

by George Bernard Dantzig (1914-2005)

At a conference in Wisconsin in 1948, when Dantzig presented his algorithm, a senior academic objected: “But we all know the world is nonlinear.” Dantzig was nonplussed by this put-down, but an audience member rose to his defence: “The speaker titled his talk ‘Linear Programming’ and carefully stated his axioms. If you have an application that satisfies the axioms, then use it. If it does not, then don’t.” This respondent was none other than John von Neumann, the leading applied mathematician of the twentieth century.

nonplussed:
so surprised by something that you do not know what to say or do

George Bernard Dantzig (1914-2005)

- George Dantzig's talk on „Programming in a Linear Structure" (meeting of the Econometric Society, Univ. of Wisconsin in Madison), 1948.
- Harold Hotelling objected „...but we all know the world is nonlinear".
- John von Neumann defended the flustered young Dantzig, saying that „if one has an application that satiesfied the axioms of the model, then it can be used, otherwise not." (flustered: confused and nervous)
- Hotelling was right: The world **is** highly nonlinear.
- But: Systems of linear <u>in</u>equalities allow an approximation of most kinds of nonlinear relations encountered in practical applications.

Age:35

George Dantzig (1913-2005)

53

Harold Hotelling (1895-1973)

45

John von Neumann (1903-1957)

Advantages of Linear Programming (LP)

(i) Exact optimization methods: Solutions are always optimal.

(ii) Many software packages are available.

(iii) Very fast. The number of decision variables can be millions.

(iv) LP software packages need no parameter specifications.

You can easily use an available LP solver. Then, you will quickly obtain the optimal solution within a very short computation time.

Lab Session

Task 1: Find the optimal solution of each of the following problems.

Problem 1:

$$
\begin{aligned}
\text{Minimize} \quad & x_1 + x_2 \\
\text{subject to} \quad & x_1 + x_2 \geq 1 \\
& \mathbf{0} \leq \mathbf{x} \leq \mathbf{1}
\end{aligned}
$$

Problem 2:

$$
\begin{aligned}
\text{Minimize} \quad & x_1 + x_2 + x_3 + x_4 \\
\text{subject to} \quad & x_1 + x_2 = 1 \\
& x_2 + x_3 = 1 \\
& x_3 + x_4 = 1 \\
& x_1 + x_4 = 1 \\
& \mathbf{0} \leq \mathbf{x} \leq \mathbf{1}
\end{aligned}
$$

Task 2: Choose three LP solvers with different algorithms and/or different languages. Then solve the above two problems using each solver.

Integer Linear Programing Problems

$$\begin{aligned} \text{Maximize} \quad & 3x_1 + 4x_2 \\ \text{subject to} \quad & x_1 + x_2 = 10 \\ & x_1 \geq 0, \; x_2 \geq 0 \end{aligned}$$

x_1 and x_2 are integers.

$$\text{Maximize} \quad (3, 4)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\text{subject to} \quad (1, 1)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 10$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

x_1 and x_2 are integers.

$$\begin{aligned} \text{Maximize} \quad & \mathbf{c}^{\mathrm{T}}\mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

x_1 and x_2 are integers.

$$\begin{aligned} \text{Maximize} \quad & 2x_1 + 3x_2 \\ \text{subject to} \quad & 2x_1 + x_2 \leq 20 \\ & x_1 + 3x_2 \leq 30 \\ & x_1 \geq 0, \; x_2 \geq 0 \end{aligned}$$

x_1 and x_2 are integers.

$$\text{Maximize} \quad (2, 3)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\text{subject to} \quad \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq \begin{pmatrix} 20 \\ 30 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

x_1 and x_2 are integers.

$$\begin{aligned} \text{Maximize} \quad & \mathbf{c}^{\mathrm{T}}\mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

Problem Formulation Exercise

Assume the profit per chair is $20, whereas the profit per table is $30. To build a chair, a single unit of wood is required and three man-hours of labor. To build a table, six units of wood are required and one man-hour of labor. The production process has some restrictions: all the machines can only process 288 units of wood per day and there are only 99 man-hours of available labor each day. The question is, how many chairs and tables should the company build to maximize its profit?

$$\begin{aligned} &\text{Maximize} \quad z = \mathbf{c}^{\mathrm{T}}\mathbf{x} \\ &\text{subject to} \quad \mathbf{A}\mathbf{x} \le \mathbf{b} \\ &\qquad\qquad\qquad \mathbf{x} \ge \mathbf{0} \\ &\mathbf{x} \text{ is an integer vector.} \end{aligned}$$

$$\mathbf{c}^{\mathrm{T}} = (?, ?) \qquad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\mathbf{A} = \begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} ? \\ ? \end{pmatrix}$$

Send your answer to TA

Problem Formulation Exercise

Assume the profit per chair is $20, whereas the profit per table is $30. To build a chair, a single unit of wood is required and three man-hours of labor. To build a table, six units of wood are required and one man-hour of labor. The production process has some restrictions: all the machines can only process 288 units of wood per day and there are only 99 man-hours of available labor each day. The question is, how many chairs and tables should the company build to maximize its profit?

$$
\begin{aligned}
&\text{Maximize} \quad z = \mathbf{c}^{\mathrm{T}}\mathbf{x} \\
&\text{subject to} \quad \mathbf{A}\mathbf{x} \le \mathbf{b} \\
&\qquad\qquad\qquad \mathbf{x} \ge \mathbf{0}
\end{aligned}
$$

$\mathbf{x}$ is an integer vector.

$$\mathbf{c}^{\mathrm{T}} = (20,\ 30) \qquad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\mathbf{A} = \begin{pmatrix} 1 & 6 \\ 3 & 1 \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} 288 \\ 99 \end{pmatrix}$$

Problem Solving Exercise

Assume the profit per chair is $20, whereas the profit per table is $30. To build a chair, a single unit of wood is required and three man-hours of labor. To build a table, six units of wood are required and one man-hour of labor. The production process has some restrictions: all the machines can only process 288 units of wood per day and there are only 99 man-hours of available labor each day. The question is, how many chairs and tables should the company build to maximize its profit?

$$\begin{aligned} &\text{Maximize} \quad z = \mathbf{c}^{\mathrm{T}}\mathbf{x} \\ &\text{subject to} \quad \mathbf{A}\mathbf{x} \le \mathbf{b} \\ &\qquad\qquad\quad \mathbf{x} \ge \mathbf{0} \\ &\mathbf{x} \text{ is an integer vector.} \end{aligned}$$

$$\mathbf{c}^{\mathrm{T}} = (20, 30) \qquad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\mathbf{A} = \begin{pmatrix} 1 & 6 \\ 3 & 1 \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} 288 \\ 99 \end{pmatrix}$$

Optimal Solution:

Chairs: x1 = _?_ , Tables: x2 = _?_. **Send your answer to TA**

Advantages of Integer Linear Programming (ILP)

(i) Exact optimization methods: Solutions are always optimal.

(ii) Many software packages are available.

(iii) LP software packages need no parameter specifications.

Difficulties of Integer Linear Programming (ILP)

(i) Large computation time for a large-scale problem.

You can easily use an available ILP solver. Then, you will quickly obtain the optimal solution within a short computation time for a small-scale problem, and after a long computation time for a large-scale problem.

Use of LP as a heuristic method for integer programming problems

Knapsack Problem with Multiple Constraint Conditions

$$\text{Maximize } f(\boldsymbol{x}) = \sum_{i=1}^{n} v_i x_i$$

$$\text{Subject to } \sum_{i=1}^{n} w_{ij} x_i \le W_j,\ j = 1, 2, \ldots, m,$$

$$\text{and } x_i \in \{0, 1\},\ i = 1, 2, \ldots, n$$

LP-Relaxation Problem

$$\text{Maximize } f(\boldsymbol{x}) = \sum_{i=1}^{n} v_i x_i$$

$$\text{Subject to } \sum_{i=1}^{n} w_{ij} x_i \le W_j,\ j = 1, 2, \ldots, m,$$

$$\text{and } 0 \le x_i \le 1,\ i = 1, 2, \ldots, n$$

$$x_i = \begin{cases} 1, \text{ if } x_i^{\mathrm{LP}} = 1 \\ 0, \text{ if } x_i^{\mathrm{LP}} < 1 \end{cases}$$

Lab Session

Task 3: Compare the following four methods on your 100-item knapsack problem (used in the lab session on Simulated Annealing) with respect to the solution quality (i.e., the objective function value), and the total computation time.

(i) A heuristic method used to generate an initial solution for SA.

(ii) Simulated algorithm with your parameter setting

(iii) Use of an LP solver (i.e., use of the LP relaxation problem), and create a feasible solution from the LP solutions.

(iv) Use of an integer LP (ILP) solver

Task 4: You will receive 200-item and 400-item knapsack problems from our TA. Compare the above-mentioned four algorithms on each of those two problems in the same manner as in Task 3.