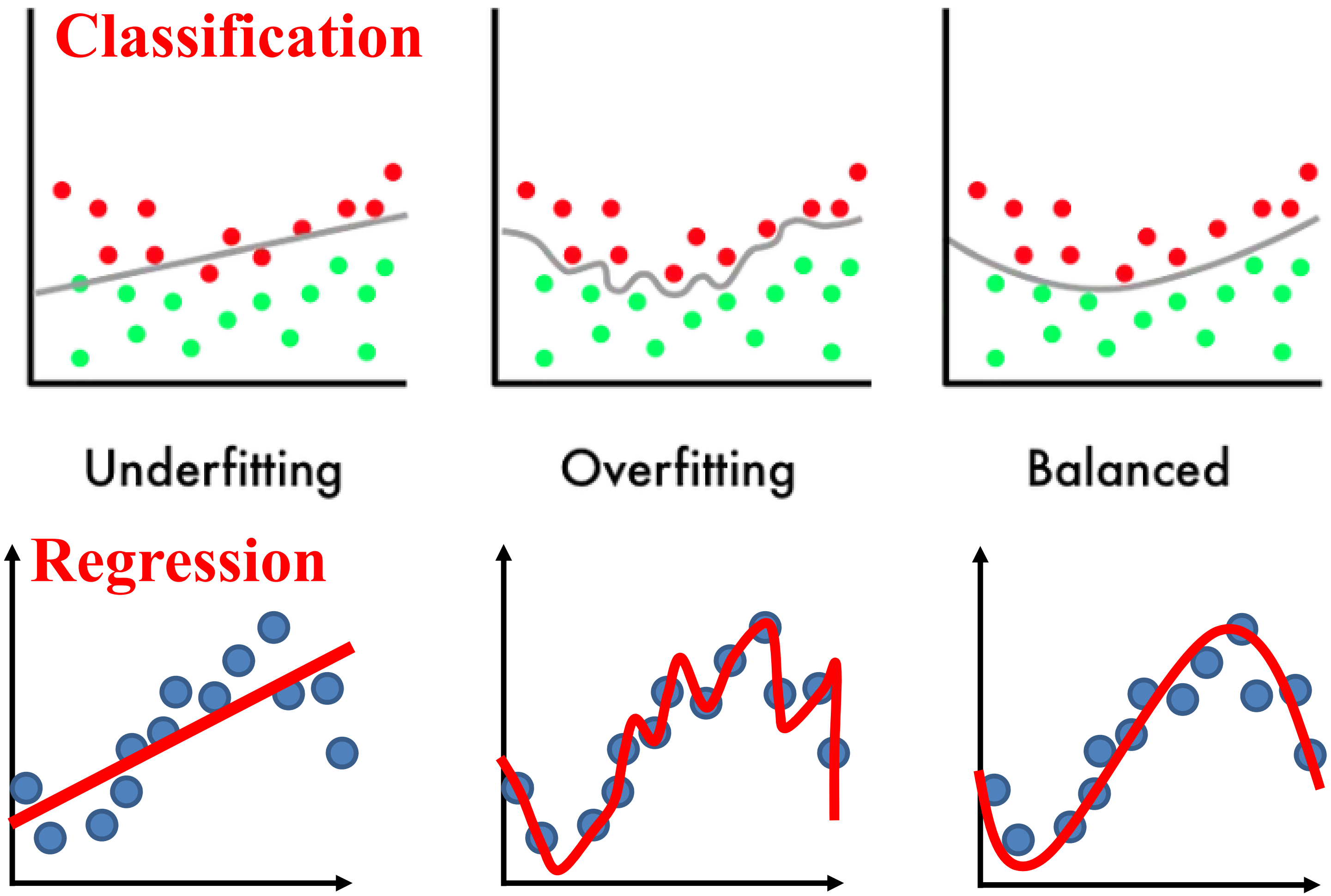
Optimization Methods

1. Introduction.
2. Greedy algorithms for combinatorial optimization.
3. LS and neighborhood structures for combinatorial optimization.
4. Variable neighborhood search, neighborhood descent, SA, TS, EC.
5. Branch and bound algorithms, and subset selection algorithms.
6. Linear programming problem formulations and applications.
7. Linear programming algorithms.
8. Integer linear programming algorithms.
9. Unconstrained nonlinear optimization and gradient descent.
10. Newton's methods and Levenberg-Marquardt modification.
11. Quasi-Newton methods and conjugate direction methods.
12. Nonlinear optimization with equality constraints.
13. Nonlinear optimization with inequality constraints.
14. Problem formulation and concepts in multi-objective optimization.
15. Search for single final solution in multi-objective optimization.
16: Search for multiple solutions in multi-objective optimization.

Other Modification: Regularization

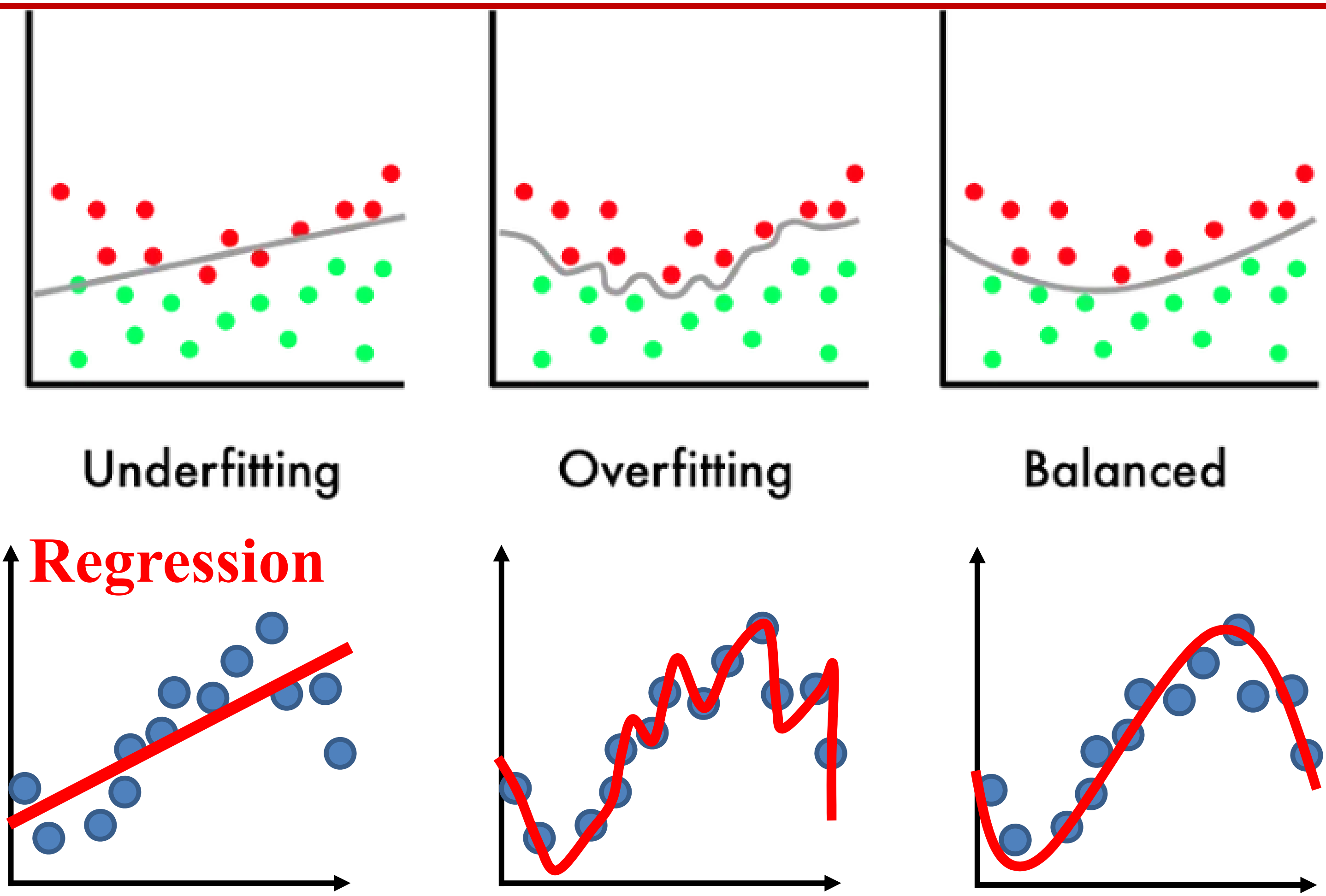
$$E(\mathbf{w}) \Rightarrow E(\mathbf{w}) + \lambda \|\mathbf{w}\|^2$$



Other Modification: Regularization

$$E(\mathbf{w}) \Rightarrow E(\mathbf{w}) + \lambda \|\mathbf{w}\|^2$$

Q. How to specify the value of λ ?   **Your answer**: ____________.



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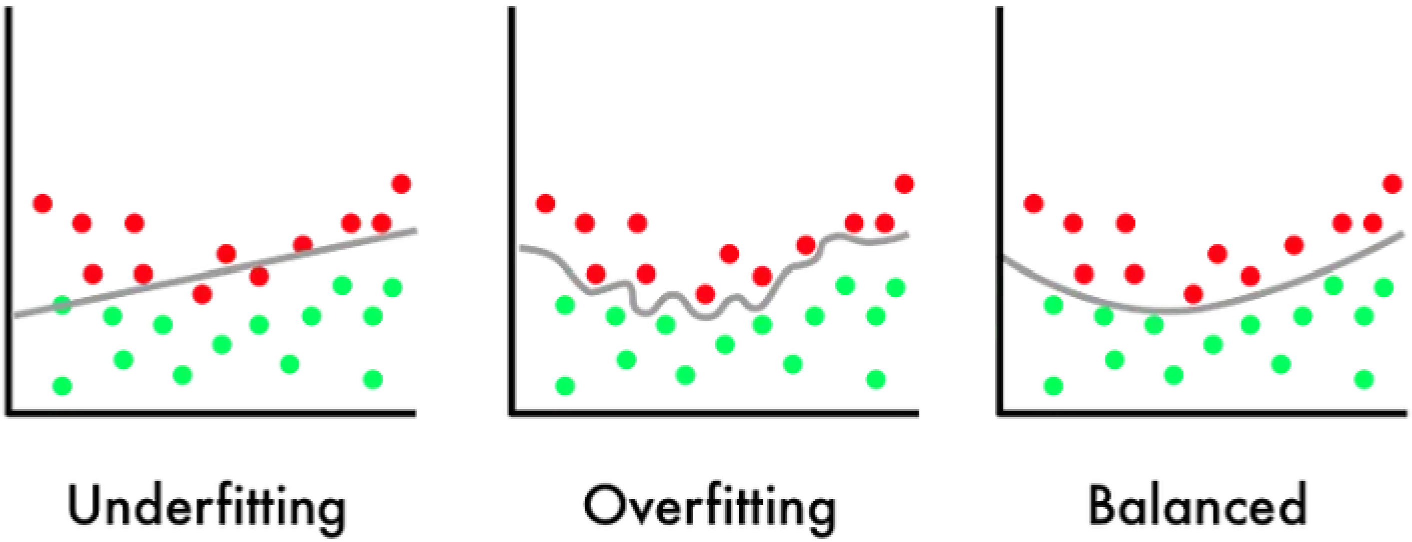


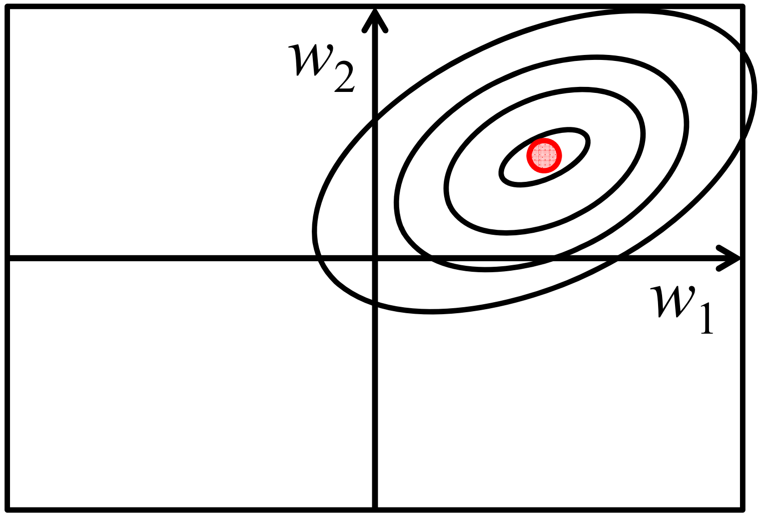
Single-Objective Problem

Minimization of $E(\mathbf{w}) + \lambda \|\mathbf{w}\|^2$

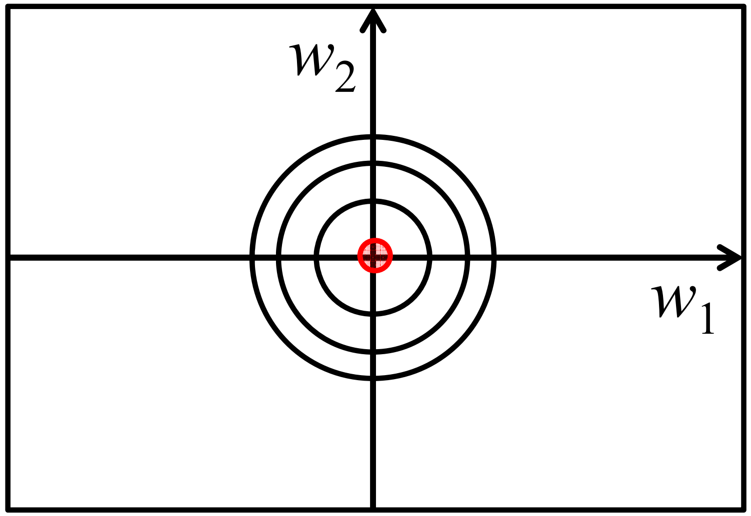
Multi-Objective Problem

Minimization of $E(\mathbf{w})$ and $\|\mathbf{w}\|^2$

Minimization of $E(\mathbf{w})$

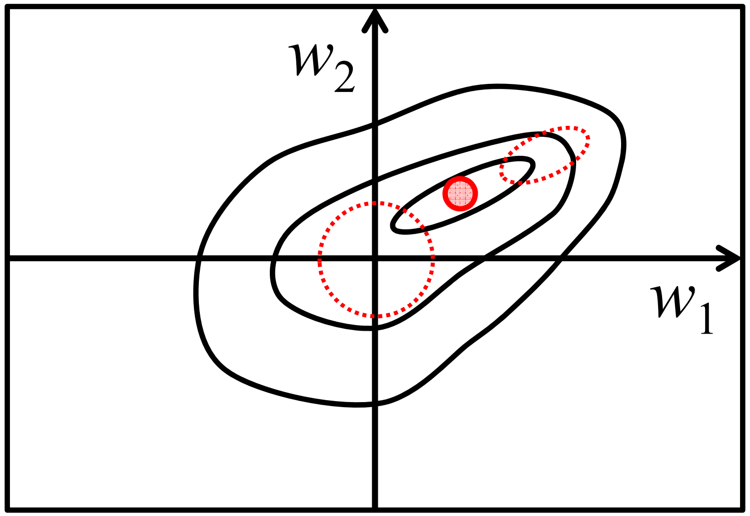


Minimization of $||\mathbf{w}||^2$

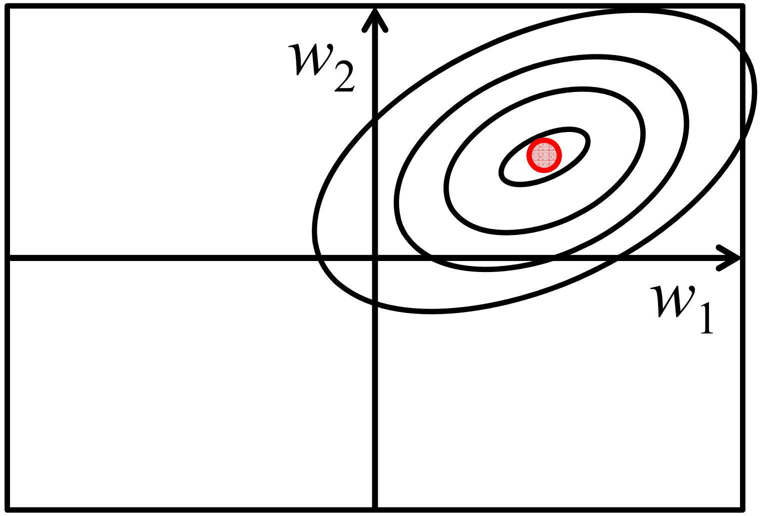


Minimization of $E(\mathbf{w}) + \lambda ||\mathbf{w}||^2$

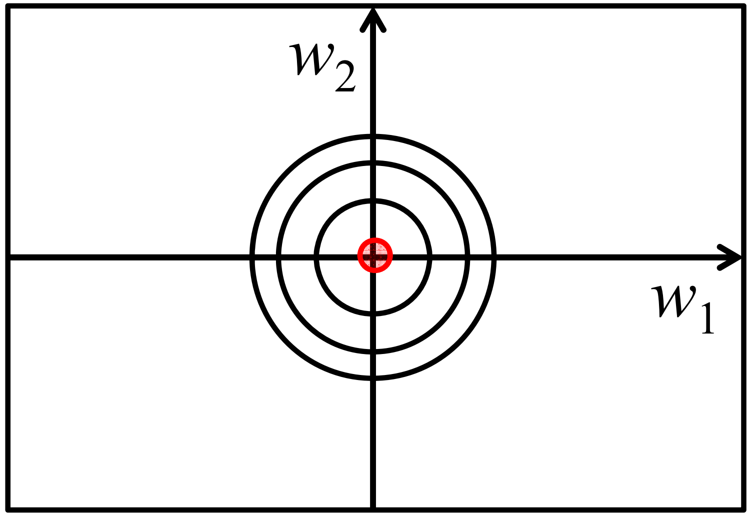
How to specify λ ?



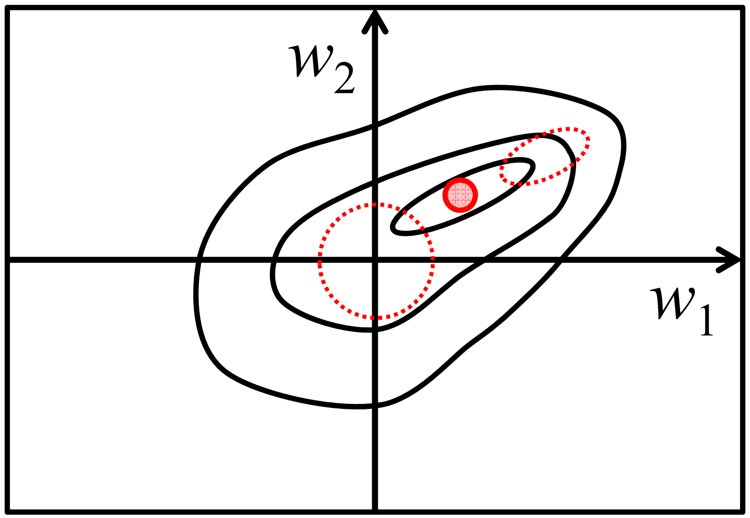
Minimization of $E(\mathbf{w})$



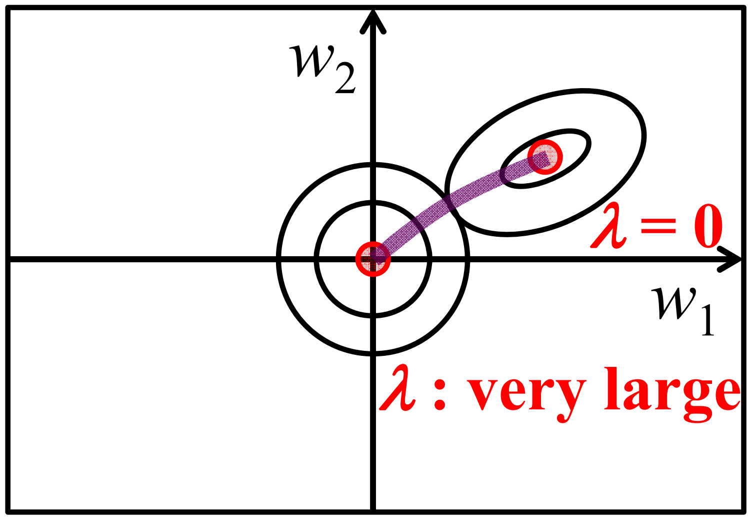
Minimization of $||\mathbf{w}||^2$



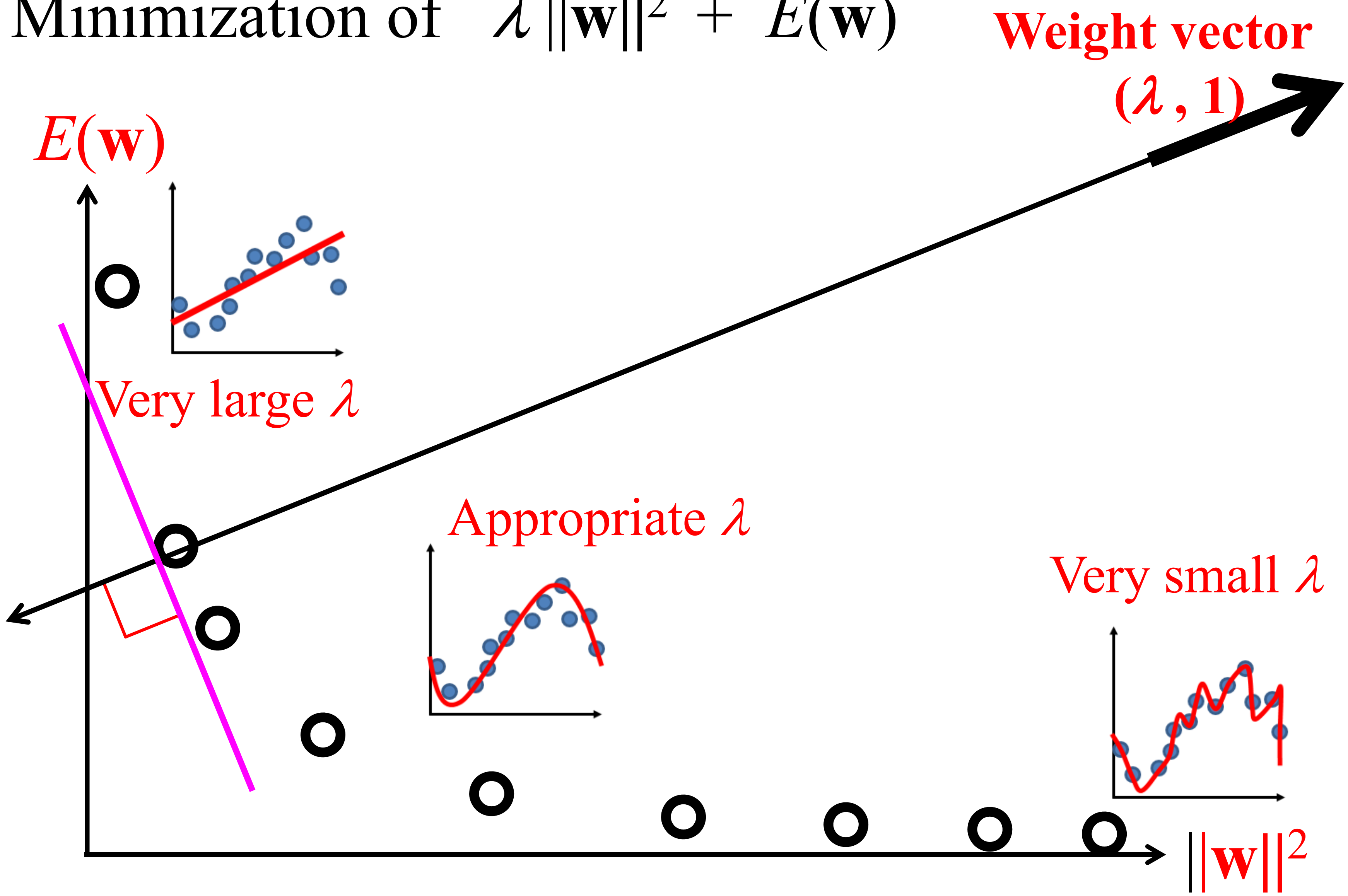
Minimization of $E(\mathbf{w}) + \lambda ||\mathbf{w}||^2$



How to specify λ ?

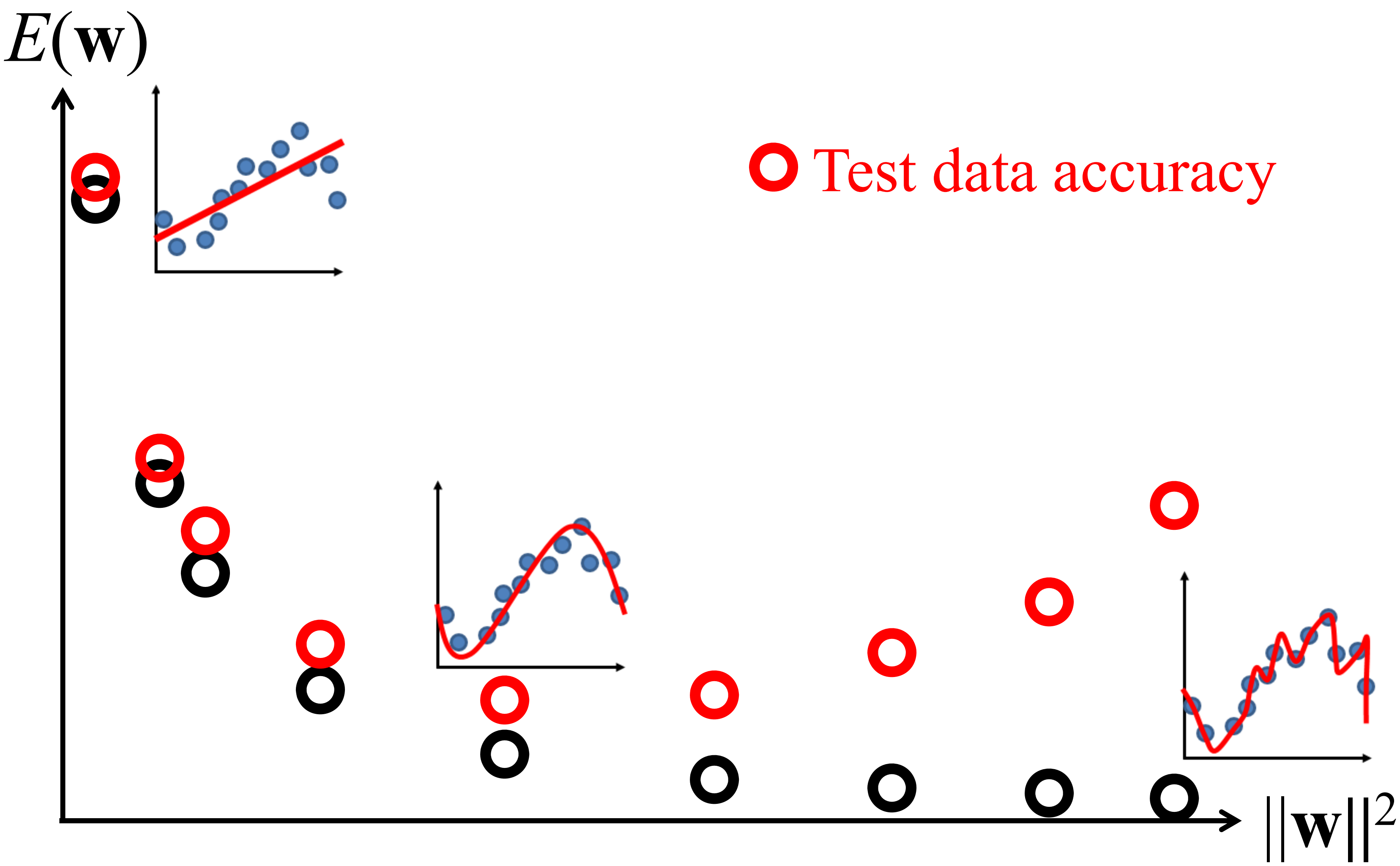


Minimization of $\lambda \, ||\mathbf{w}||^2 + E(\mathbf{w})$



Results of using various values of λ

Minimization of $\lambda \|\mathbf{w}\|^2 + E(\mathbf{w})$



Results of using various values of λ

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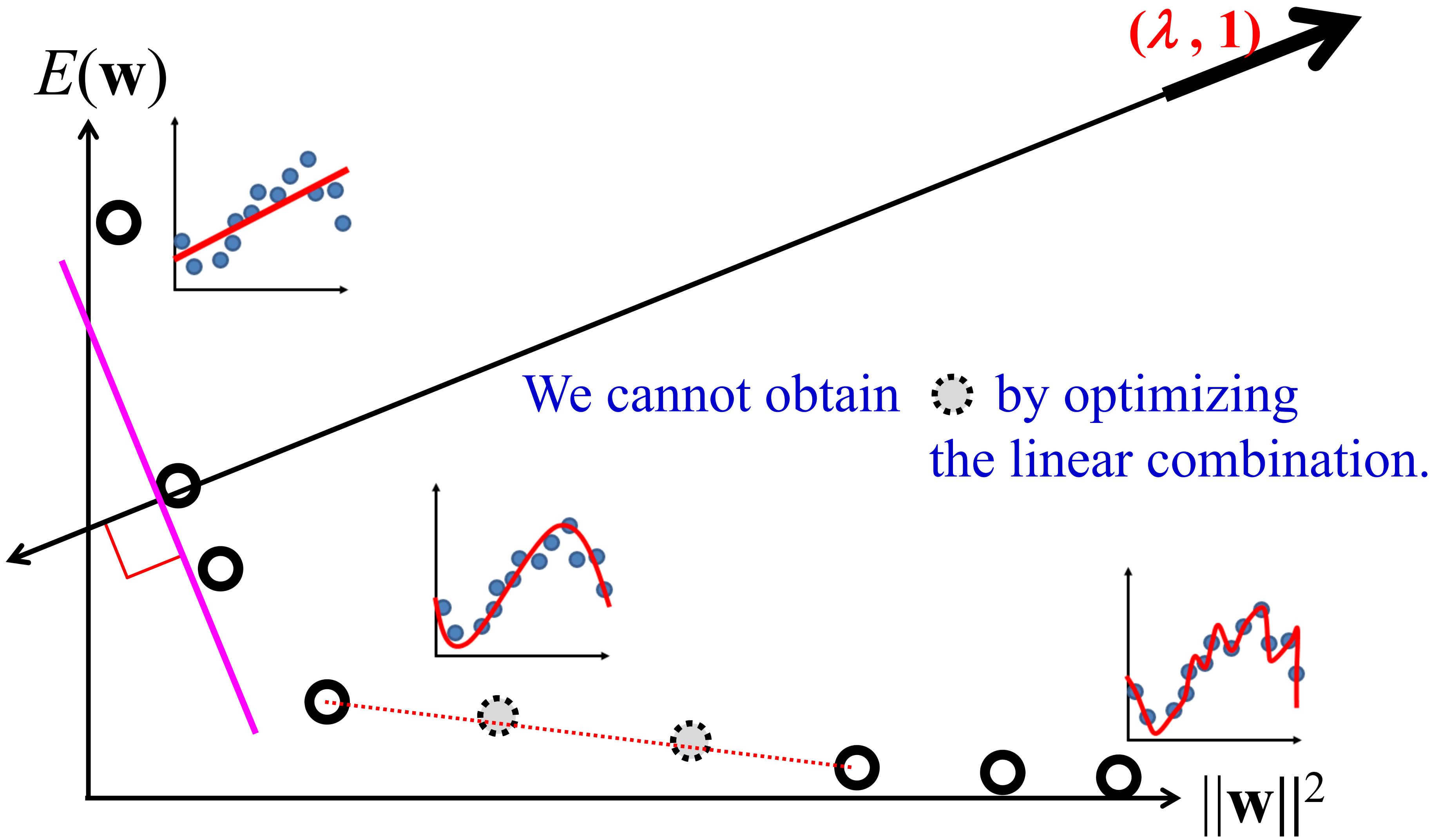
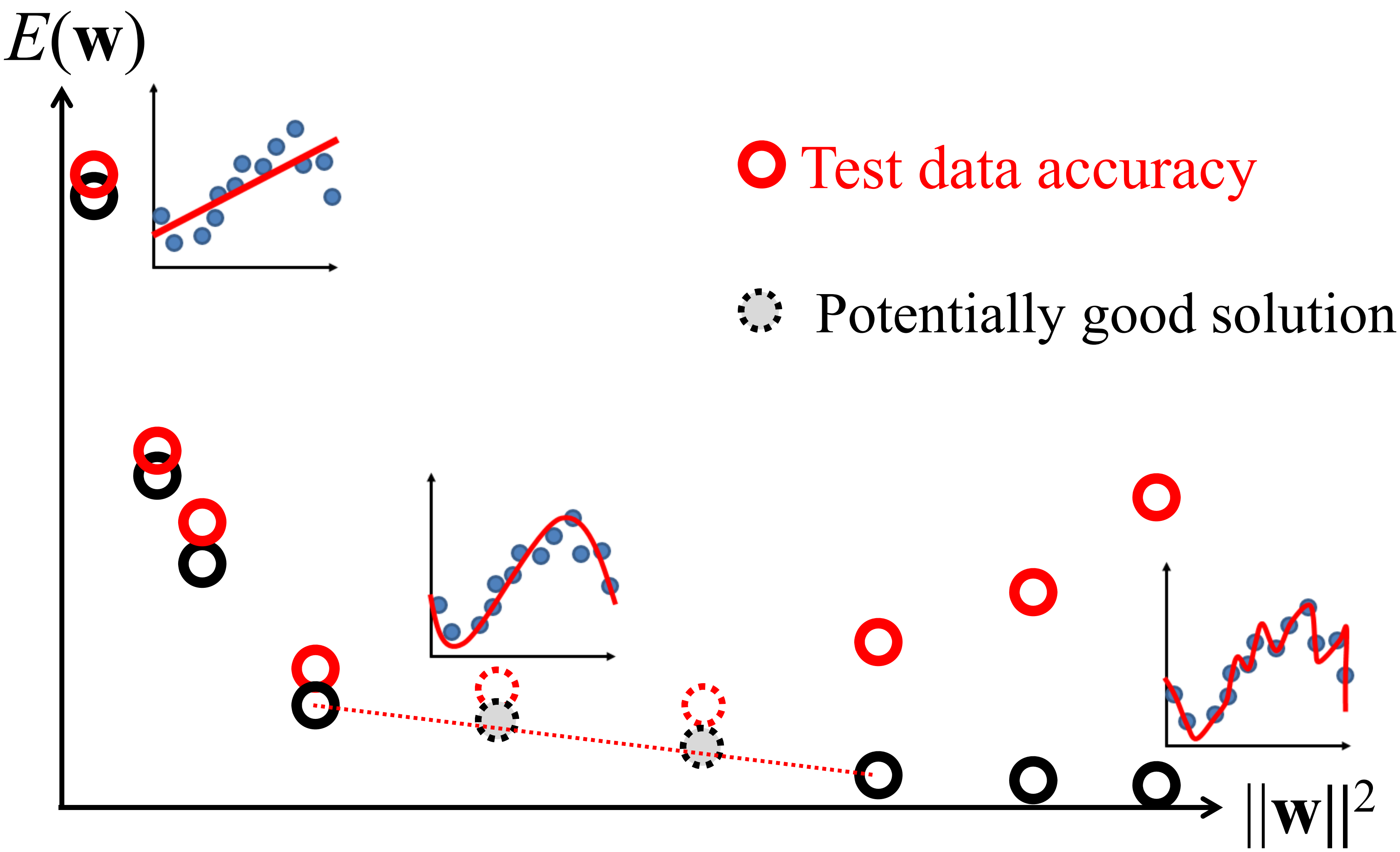


Results of using various values of λ

Minimization of $\lambda \|\mathbf{w}\|^2 + E(\mathbf{w})$



Results of using various values of λ

Classification of Optimization Problems

(with respect to the number of objectives)

Optimization Problems

= {Single-objective problems, multi-objective problems}

= {Single-objective problems,
{multi-objective problems, many-objective problems}}

A. Single-Objective Problems: Maximize $f(\boldsymbol{x})$

B. Multi-Objective Problems: Maximize $f_1(\boldsymbol{x}),\ f_2(\boldsymbol{x}),\ ...,\ f_m(\boldsymbol{x})$

B.1. Multi-Objective Problems:

Maximize $f_1(\boldsymbol{x}),\ f_2(\boldsymbol{x})$

Maximize $f_1(\boldsymbol{x}),\ f_2(\boldsymbol{x}),\ f_3(\boldsymbol{x})$

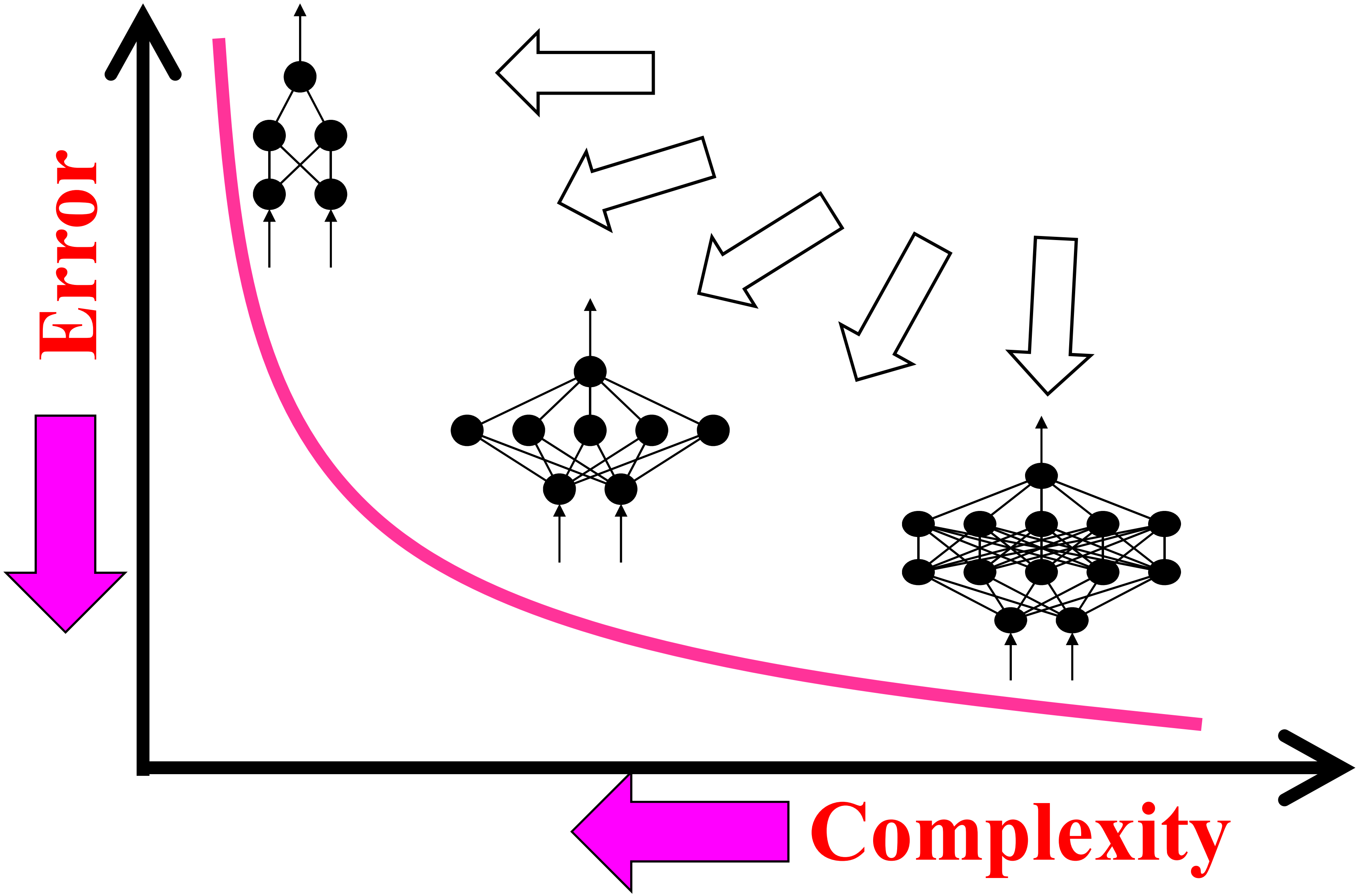
B2. Many-Objective Problems:

Maximize $f_1(\boldsymbol{x}),\ f_2(\boldsymbol{x}),\ f_3(\boldsymbol{x}),\ f_4(\boldsymbol{x})$

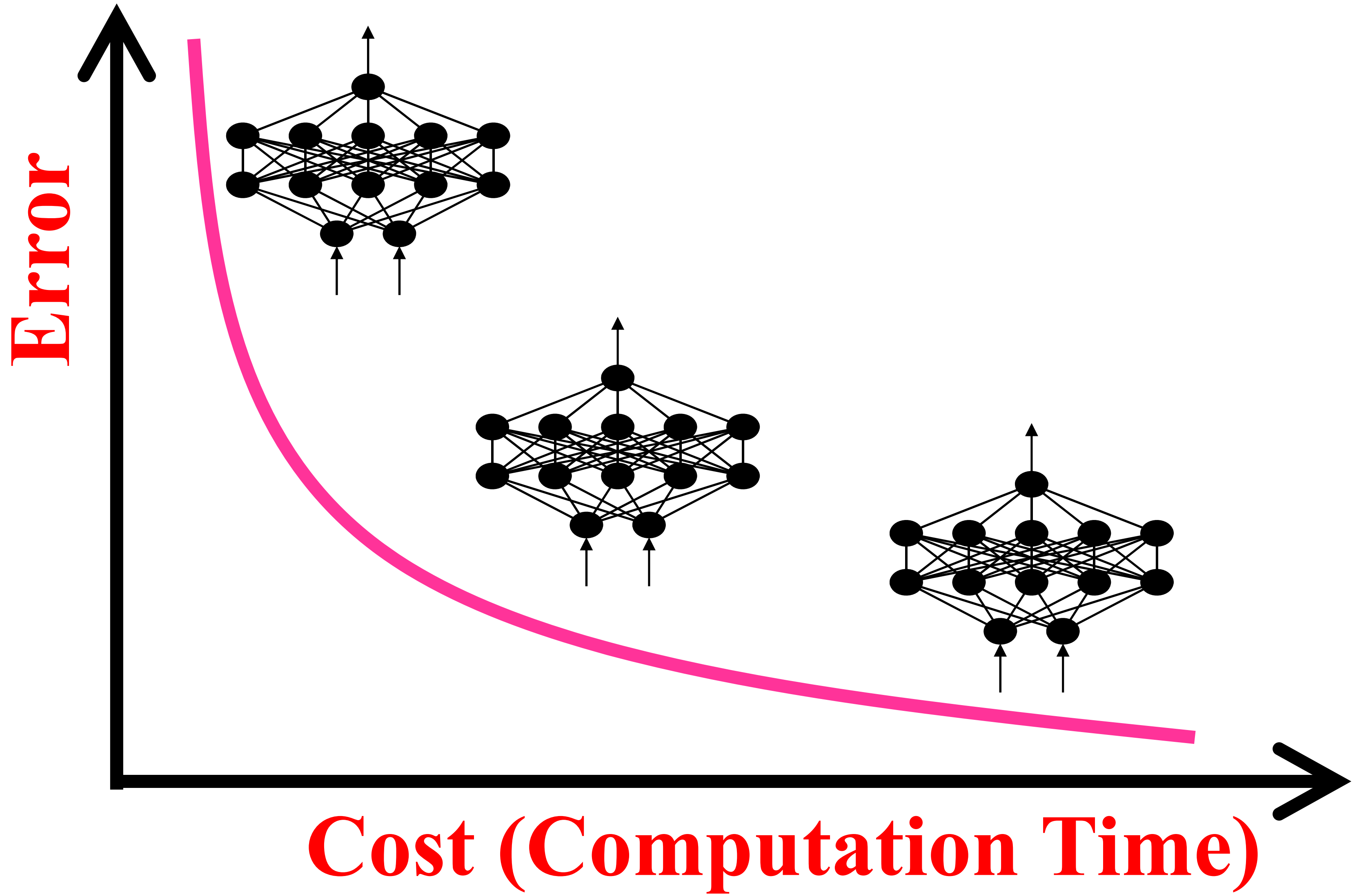
Maximize $f_1(\boldsymbol{x}),\ f_2(\boldsymbol{x}),\ f_3(\boldsymbol{x}),\ f_4(\boldsymbol{x}),\ f_5(\boldsymbol{x})$

Maximize $f_1(\boldsymbol{x}),\ f_2(\boldsymbol{x}),\ f_3(\boldsymbol{x}),\ f_4(\boldsymbol{x}),\ f_5(\boldsymbol{x}),\ f_6(\boldsymbol{x})$

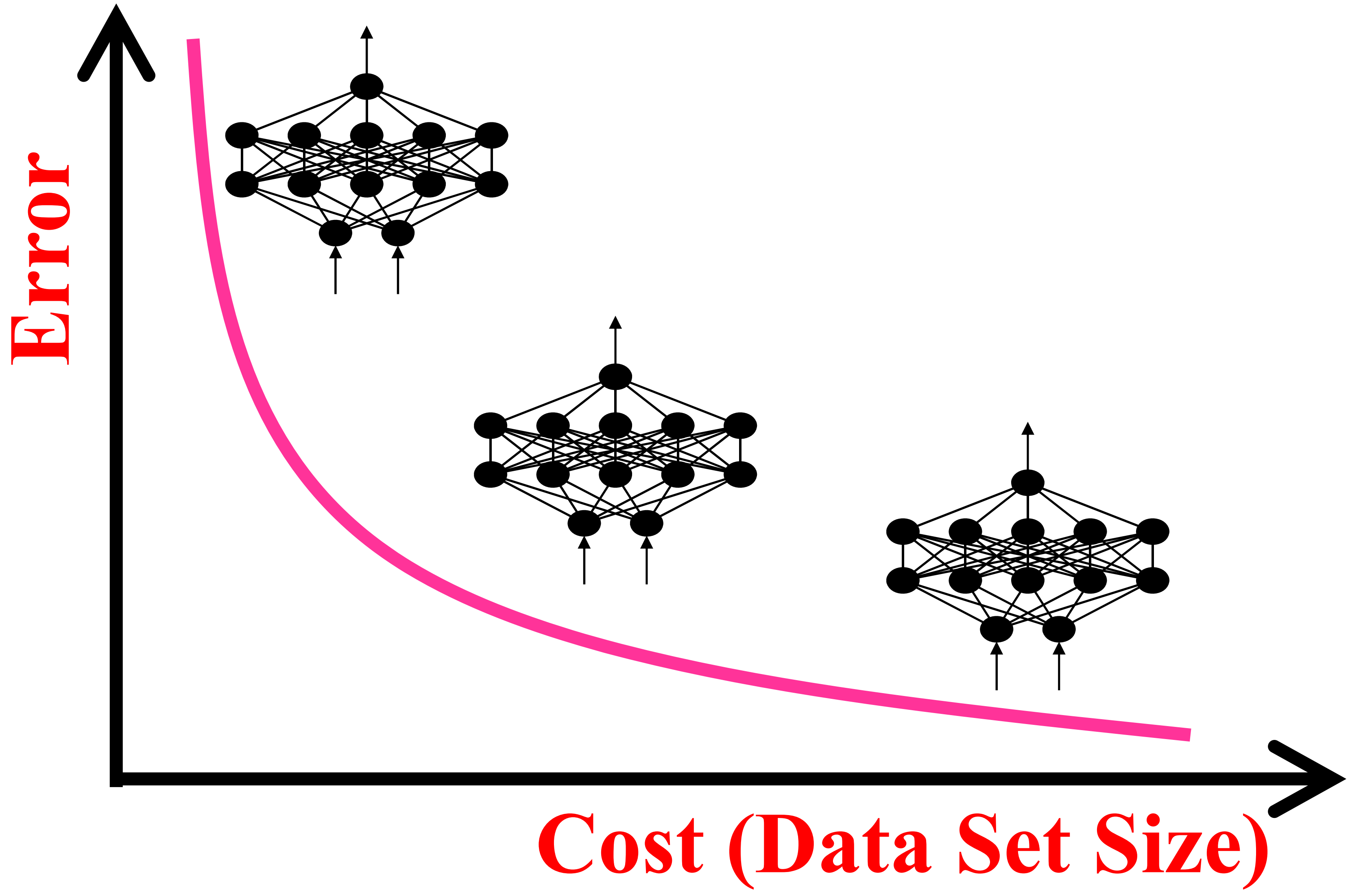
Almost all problems have multiple objectives



Almost all problems have multiple objectives



Almost all problems have multiple objectives



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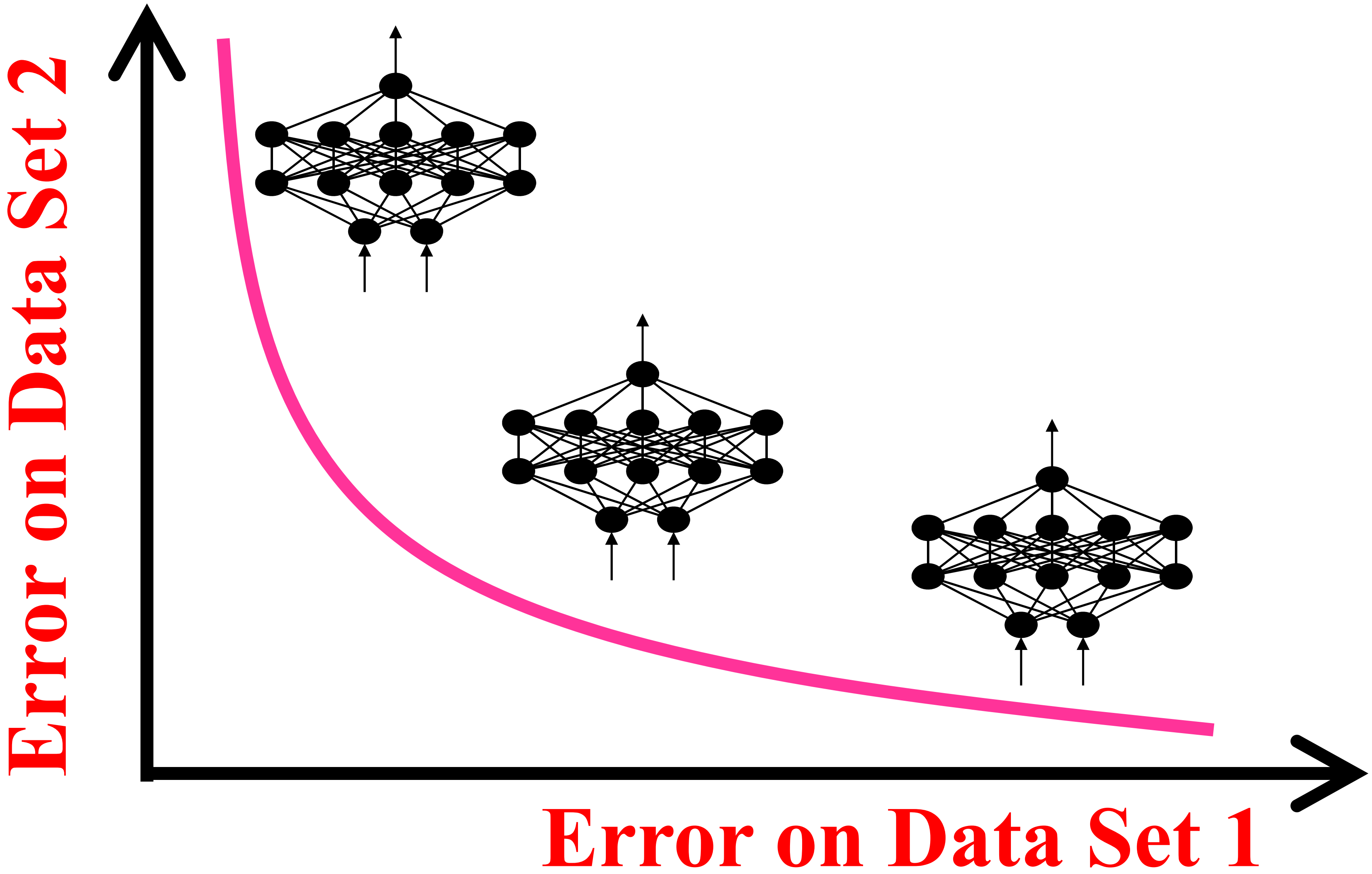
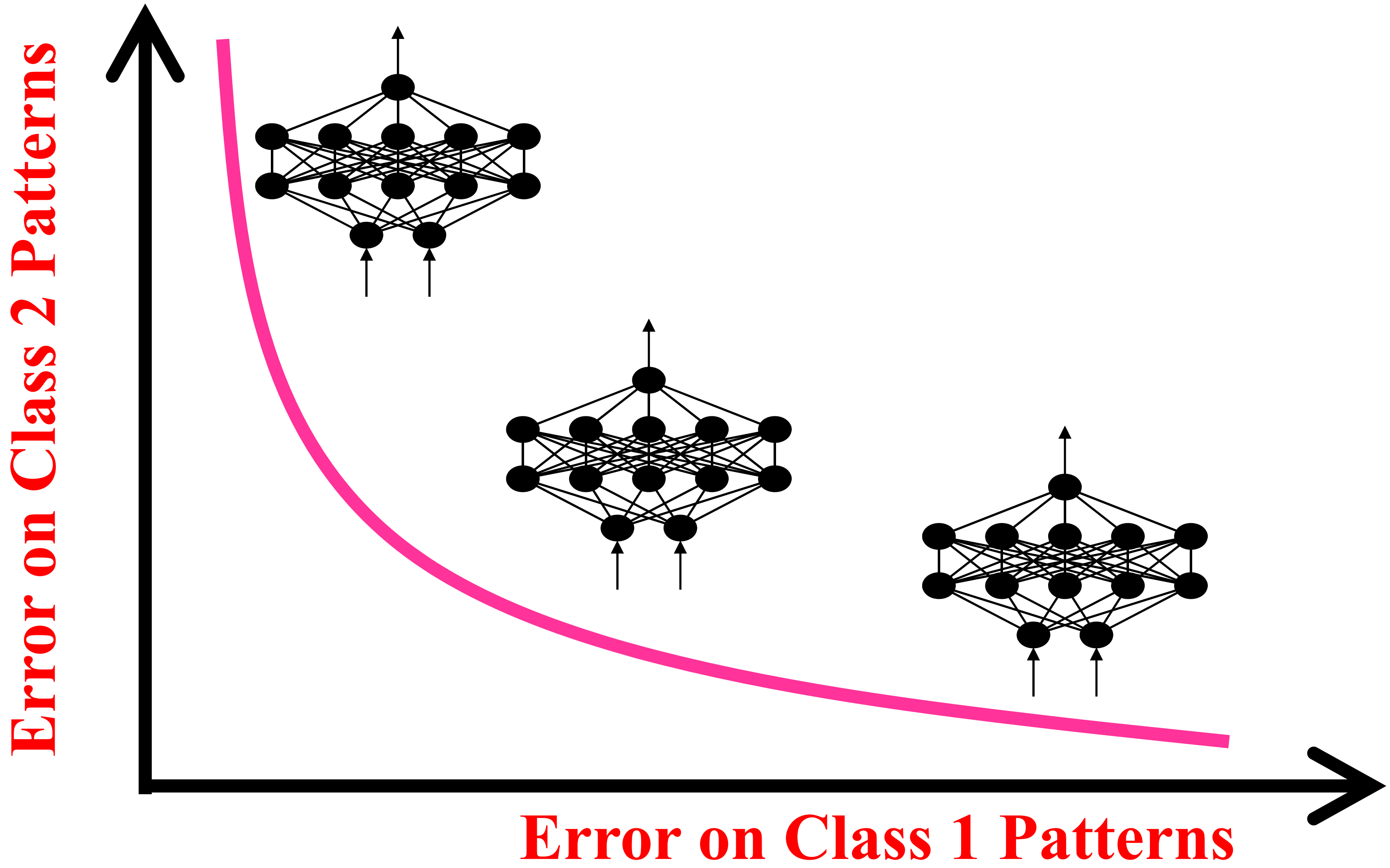
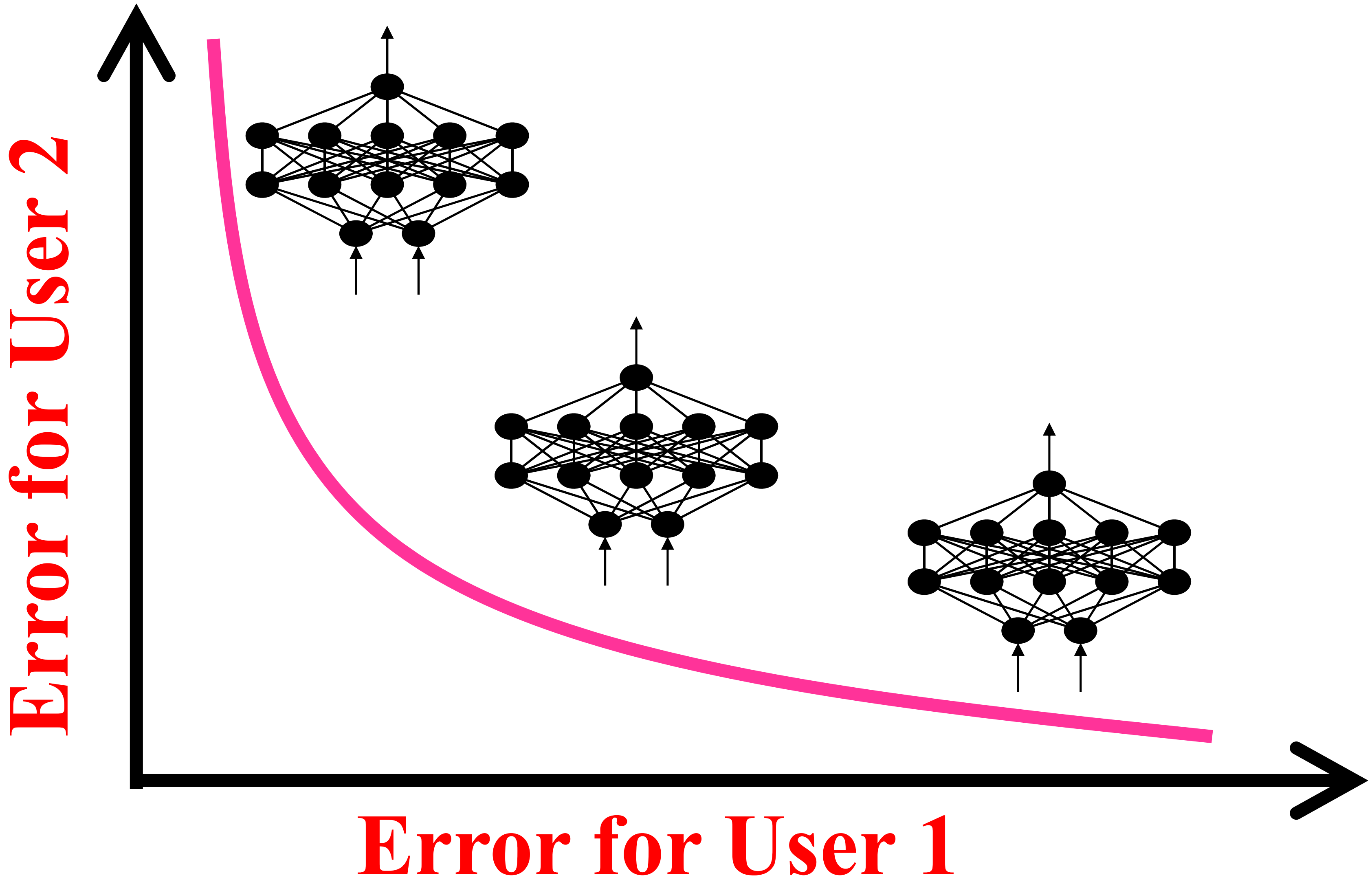


Almost all problems have multiple objectives



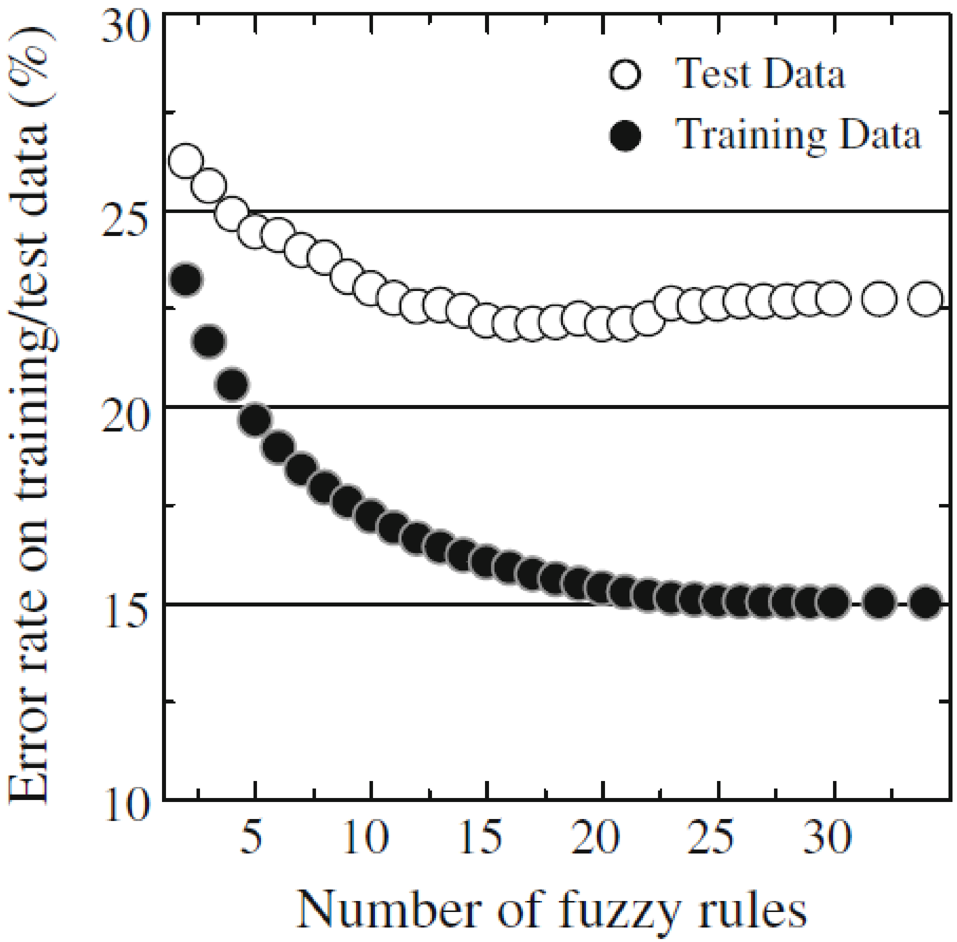
Almost all problems have multiple objectives



Some objectives are totally conflicting:

Example from my research: Fuzzy classifier design

- Minimization of the classification error on training data
- Minimization of the number of fuzzy if-then rules



Some objectives are related (and partially conflicting):

Maximization of an Interval Objective Function

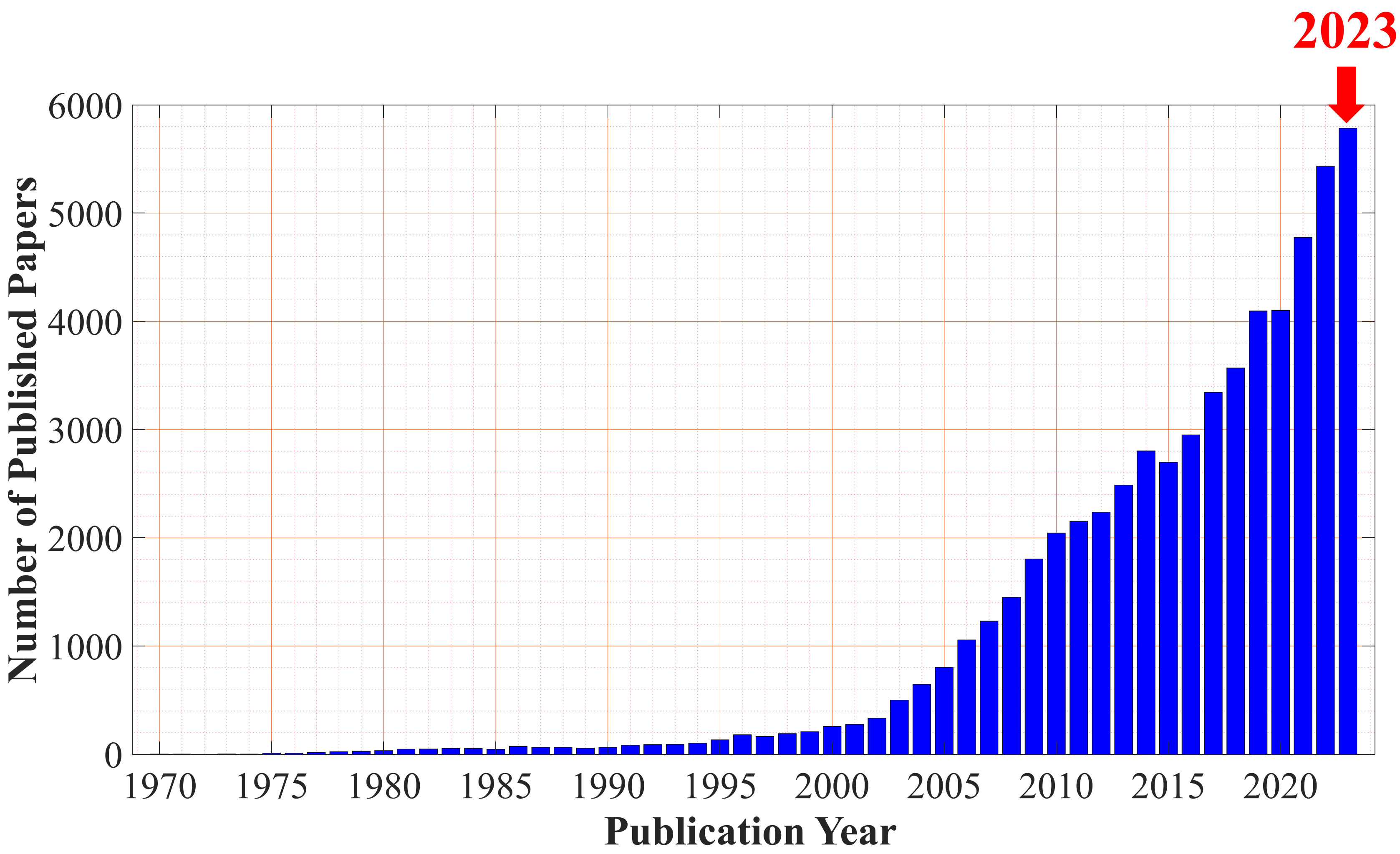
- Maximization of the lower limit (worst case)
- Maximization of the upper limit (best case)
- Maximization of the center (average case)
- Minimization of the width (uncertainty)

Scheduling Problems

- Minimization of the makespan
- Minimization of the maximum delay
- Minimization of the total delay

Popularity of Multi-Objective Optimization Research

The number of papers with “Multi-objective” or “Multiobjective” in the paper titles (Scopus Database: January 17, 2024)



Multiobjective Optimization

An Example from Newspaper (about Audi)

www.usatoday.com

USA TODAY

Money

SECTION B

Friday, June 12, 2009

Test Drive

By Jim Fets, Audi

Audi diesel has style but not much space

Good-looking, sure-footed crossover is less roomy than its rivals, 4B

Multiobjective Optimization

An Example from Newspaper (about Audi)

4B · FRIDAY, JUNE 12, 2009 · USA TODAY

Test Drive

Every Friday

Audi diesel has power, elegance

Too bad it doesn't have more space inside

The Audi Q7 TDI is a recently launched diesel-power variant of the brand's popular (by Audi's low-volume standards) big SUV.

TDI stands for turbocharging and direct injection of the fuel. Turbocharging lets an engine accelerate more quickly. Direct injection of the fuel improves power and cuts pollution. Programmed just so, as it apparently is in the Q7, direct injection can minimize a diesel's signature rocks-in-a-tin-can sound.

Diesels can get 25% to 40% better mileage than

What stands out

▸ **Powerful:** At low speed, where Americans like it.

2009 Audi Q7 TD

▸ **What?** Diesel-engine version of the brand's large, four-door, seven-passenger crossover SUV.

▸ **When?** The diesel version, called TDI, went on

truck-ba

erwise v

Q7 is

Multiobjective Optimization

Explanation using an Exaggerated Example

Multiobjective Optimization

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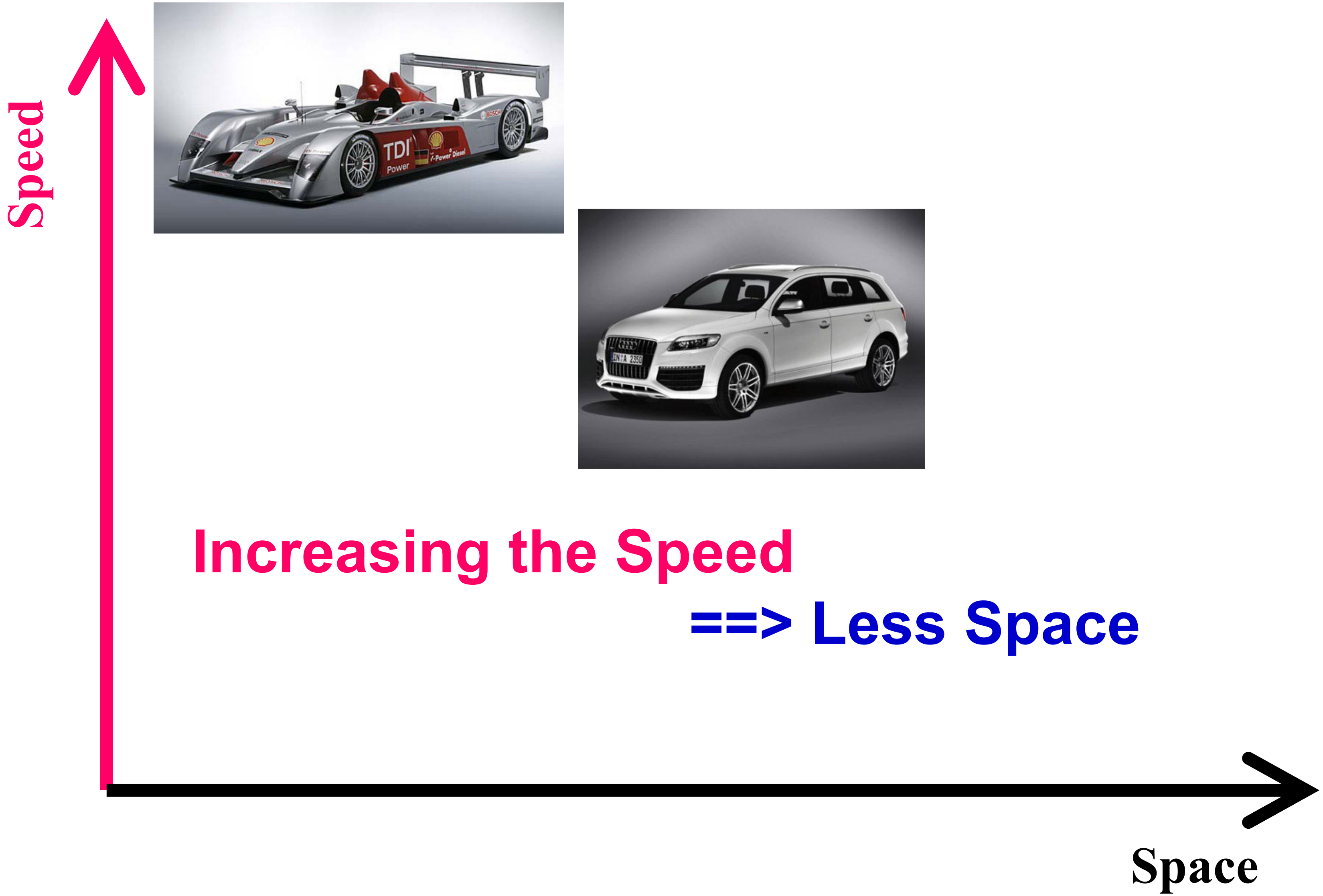
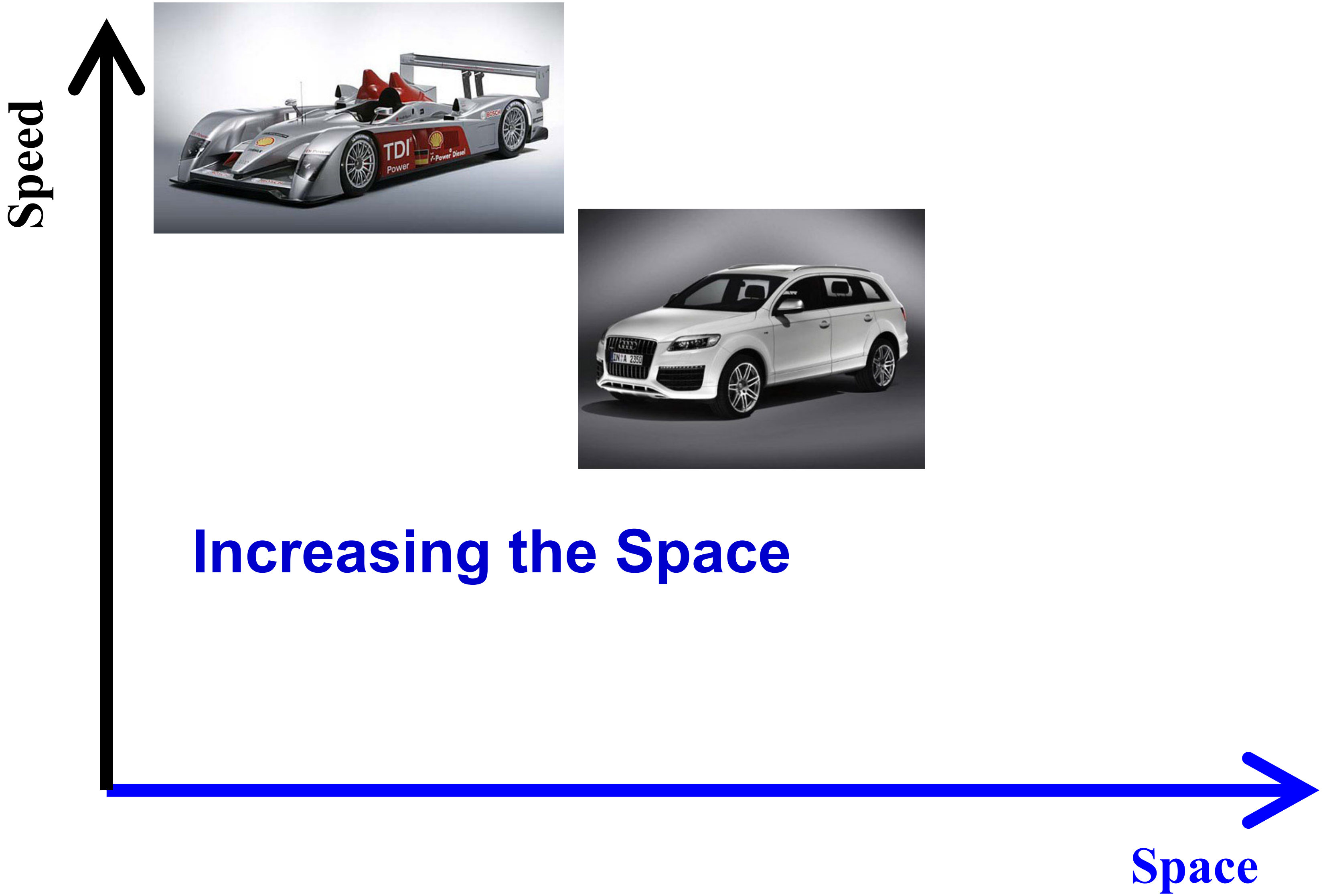
Multiobjective Optimization

Explanation using an Exaggerated Example



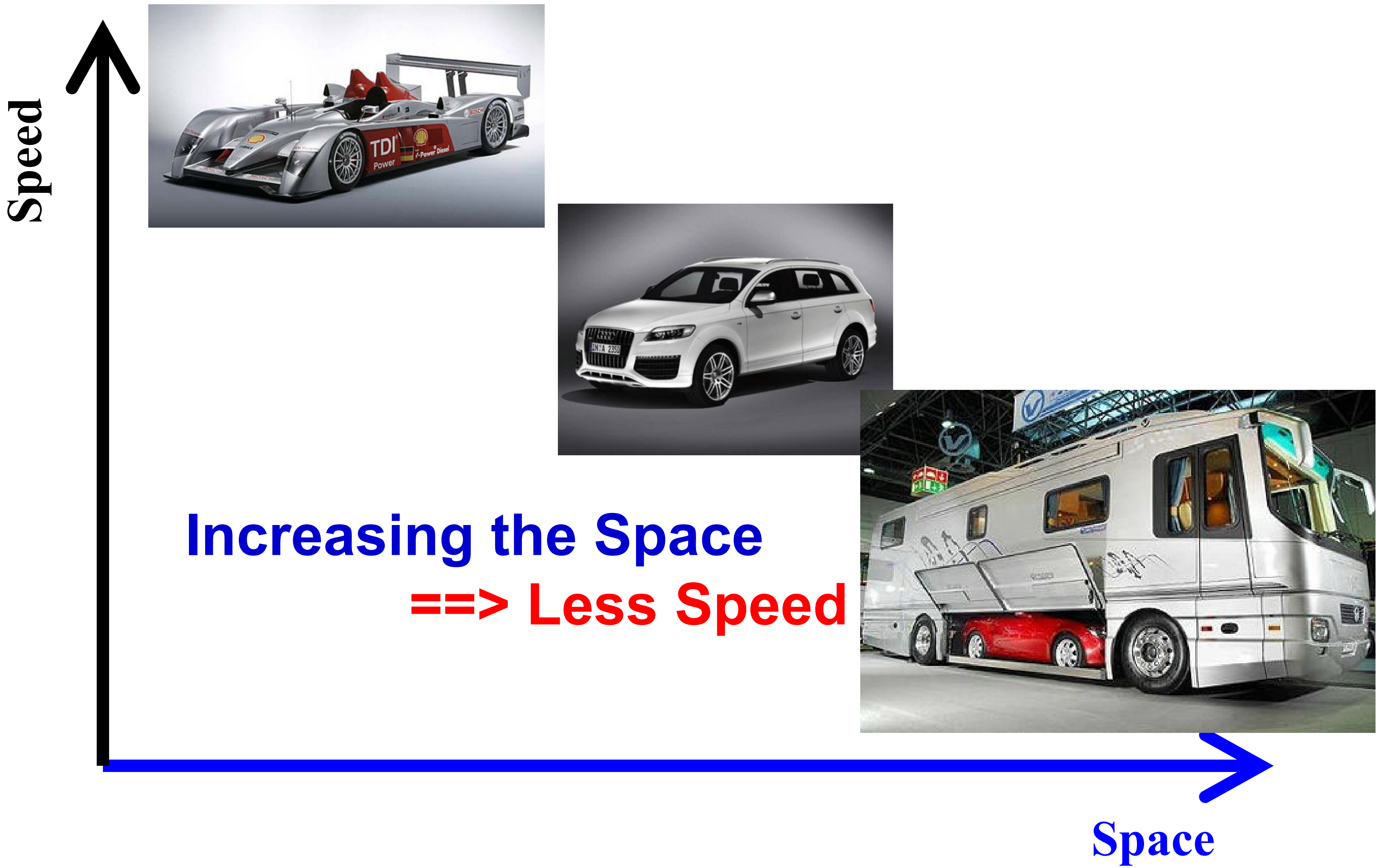
Multiobjective Optimization

Explanation using an Exaggerated Example



Multiobjective Optimization

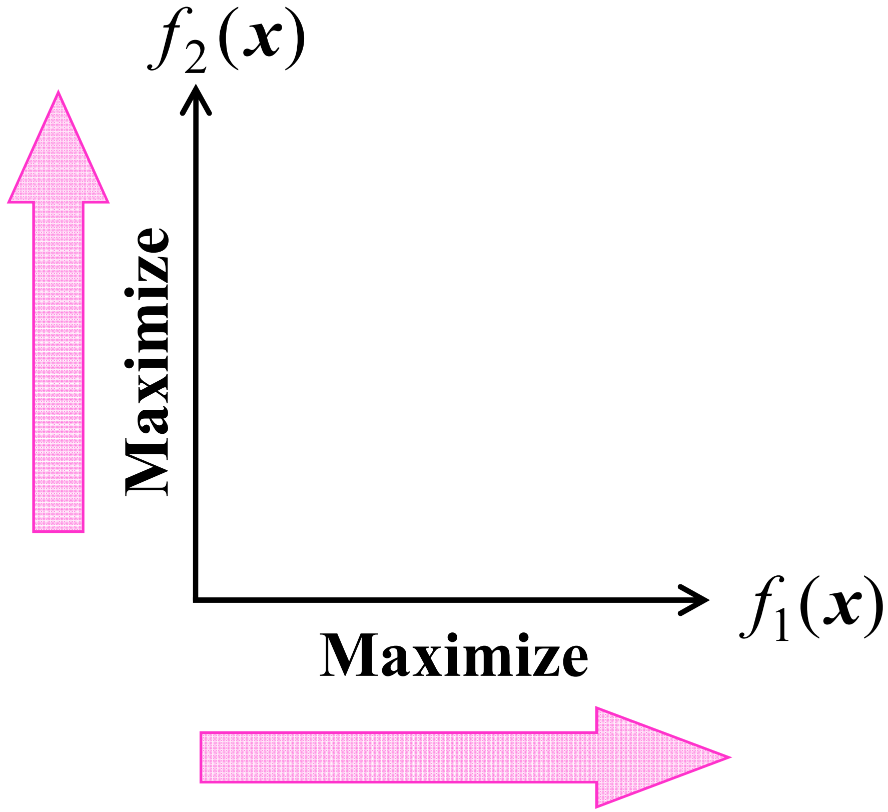
Explanation using an Exaggerated Example



Multiobjective Optimization

Two-Objective Maximization Problem:

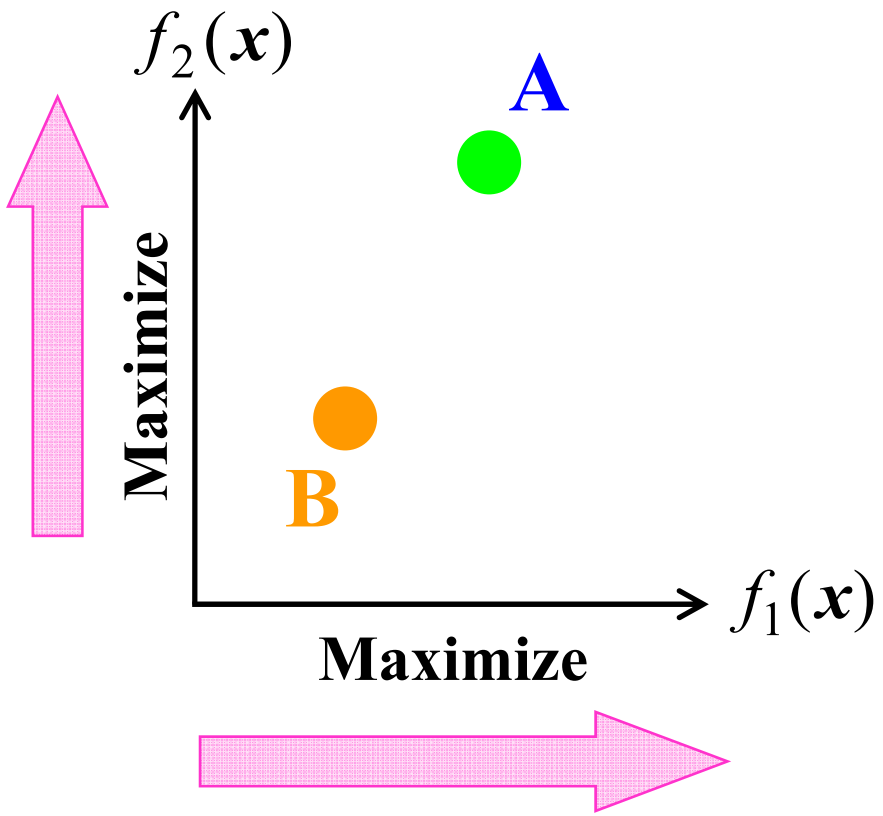
$$\text{Maximize} \quad \boldsymbol{f}(\boldsymbol{x}) = (f_1(\boldsymbol{x}),\ f_2(\boldsymbol{x}))$$



Comparison between Two Solutions

Pareto Dominance

$$\textbf{Maximize} \quad \boldsymbol{f}(\boldsymbol{x}) = (f_1(\boldsymbol{x}),\ f_2(\boldsymbol{x}))$$

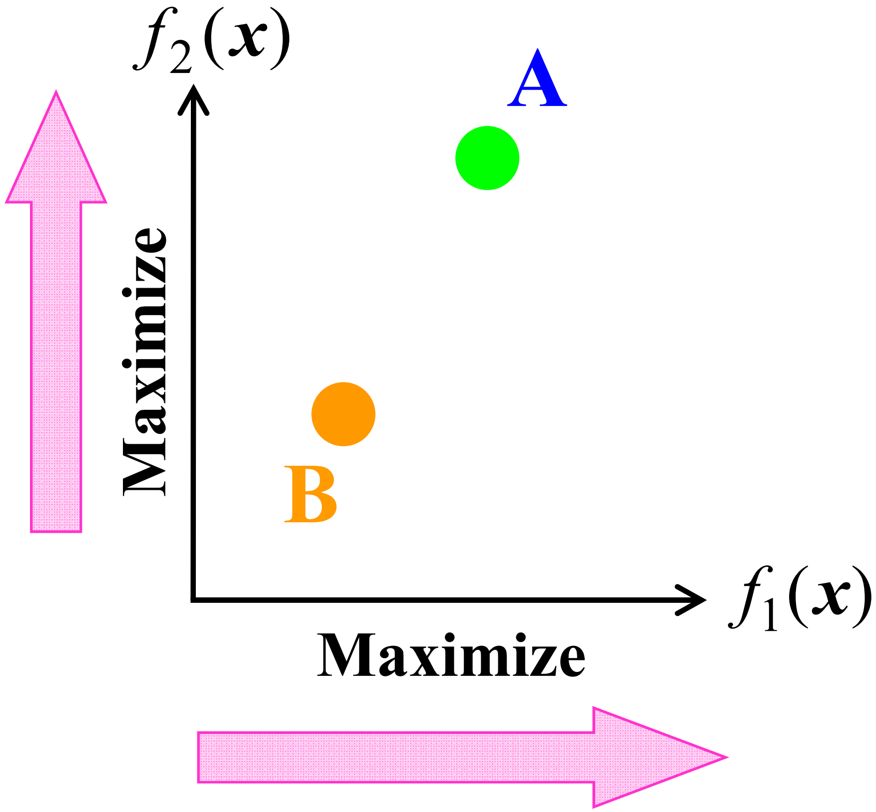


A is better than B

Comparison between Two Solutions

Pareto Dominance

Maximize $\boldsymbol{f}(\boldsymbol{x}) = (f_1(\boldsymbol{x}),\ f_2(\boldsymbol{x}))$



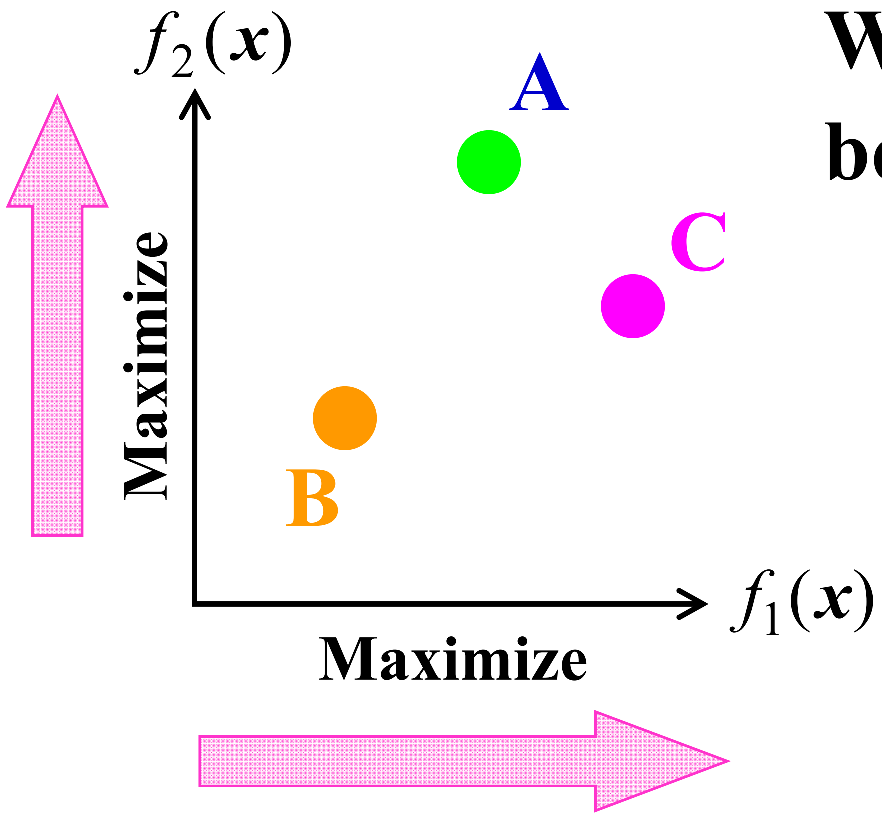
A is better than B

A dominates B

B is dominated by A

Comparison between Two Solutions
Pareto Dominance

$$\textbf{Maximize} \quad \boldsymbol{f}(\boldsymbol{x}) = (f_1(\boldsymbol{x}),\ f_2(\boldsymbol{x}))$$

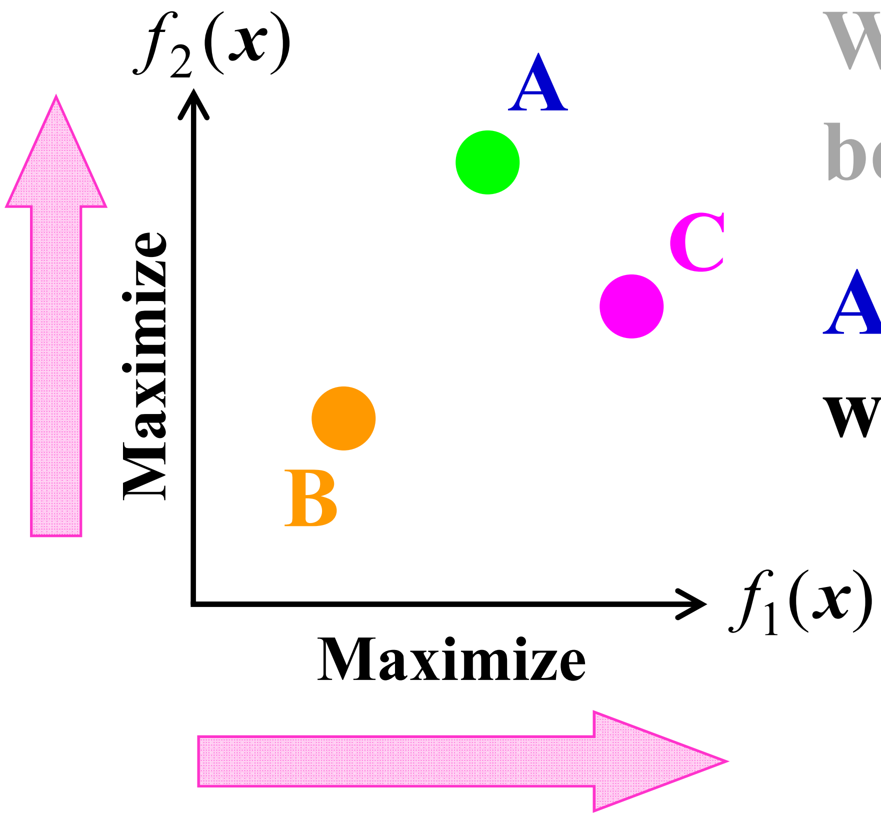


We cannot say which is better between A and C.

Comparison between Two Solutions

Pareto Dominance

Maximize $\boldsymbol{f}(\boldsymbol{x}) = (f_1(\boldsymbol{x}),\ f_2(\boldsymbol{x}))$



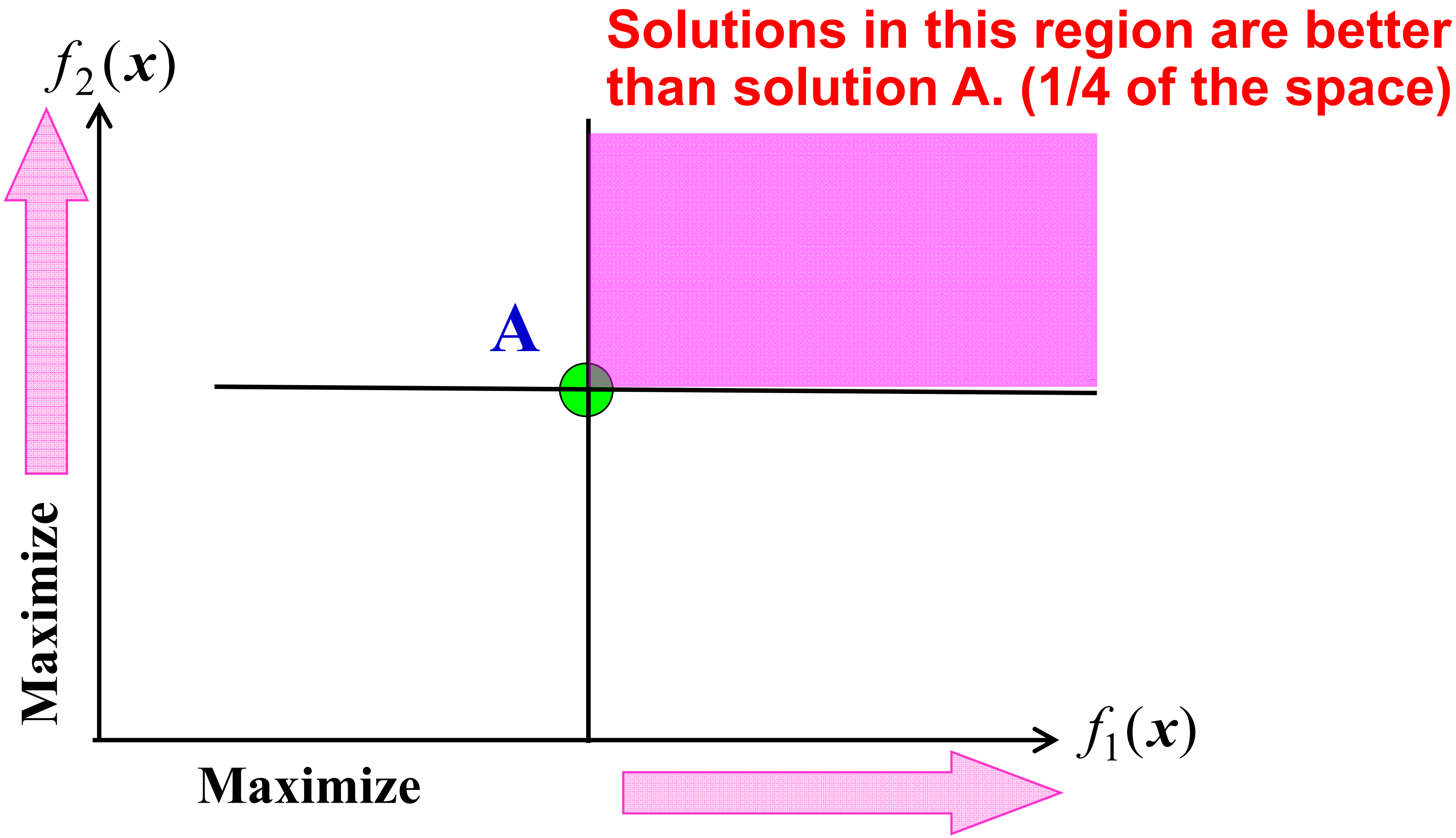
We cannot say which is better between A and C.

A and C are non-dominated with each other.

Comparison between Two Solutions

Pareto Dominance

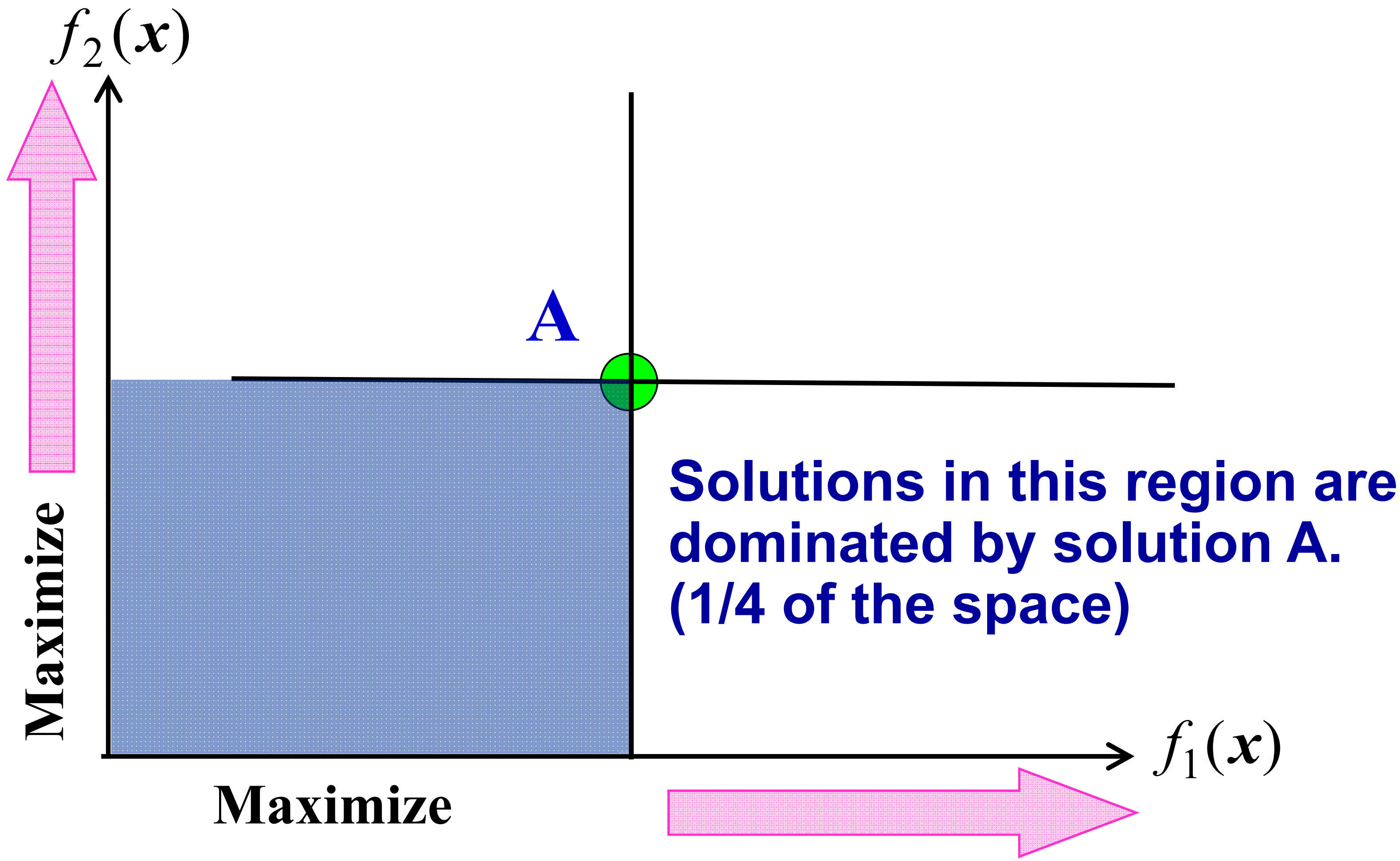
Maximize $\boldsymbol{f}(\boldsymbol{x}) = (f_1(\boldsymbol{x}),\ f_2(\boldsymbol{x}))$



Comparison between Two Solutions

Pareto Dominance

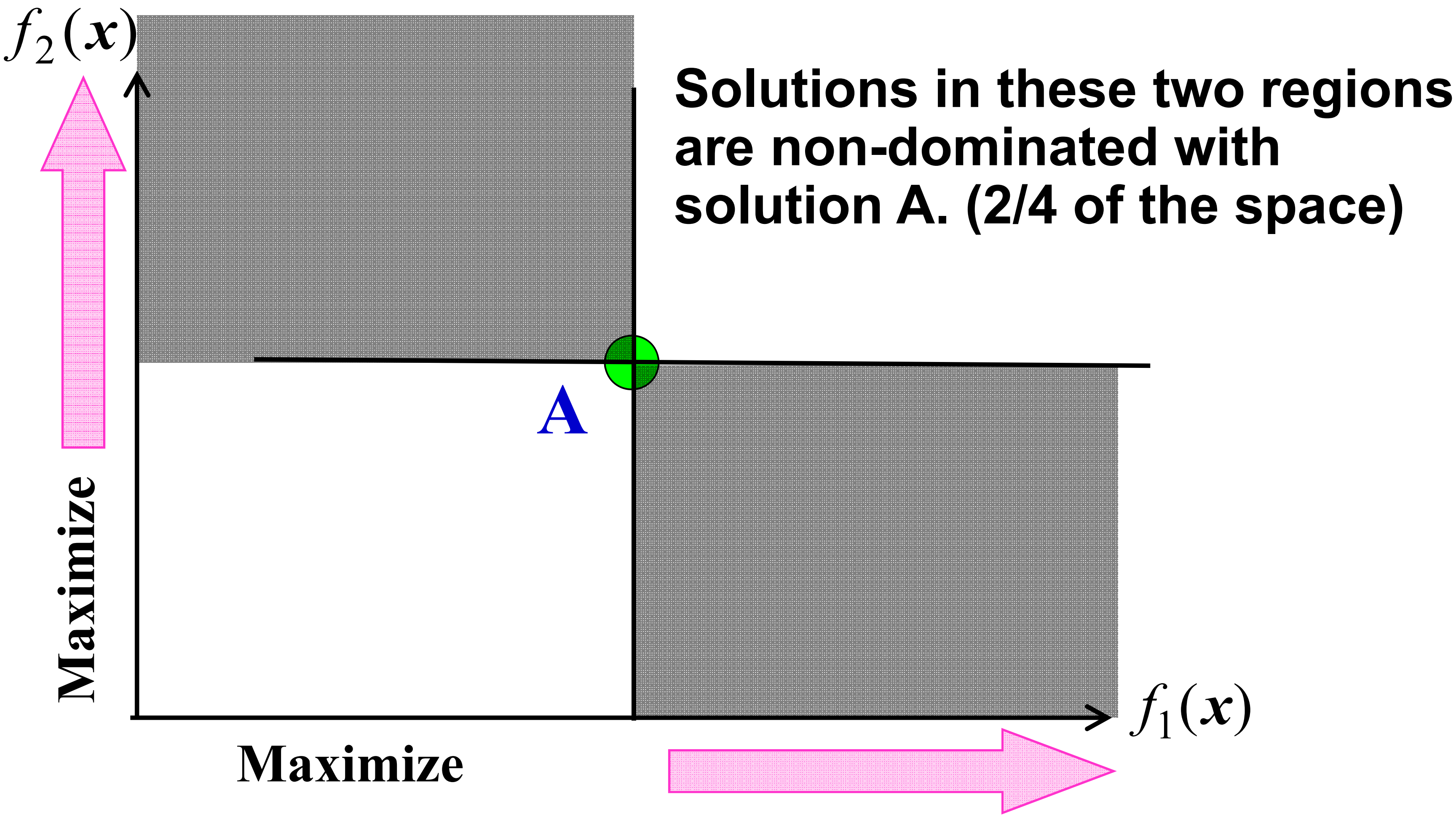
$$\text{Maximize} \quad \boldsymbol{f}(\boldsymbol{x}) = (f_1(\boldsymbol{x}),\ f_2(\boldsymbol{x}))$$



Comparison between Two Solutions

Pareto Dominance

Maximize $\boldsymbol{f}(\boldsymbol{x}) = (f_1(\boldsymbol{x}),\ f_2(\boldsymbol{x}))$



Pareto Dominance Relation

(Very Important Concept)

Maximize $\boldsymbol{f}(\boldsymbol{x}) = (f_1(\boldsymbol{x}),\ f_2(\boldsymbol{x}),\ \ldots,\ f_m(\boldsymbol{x}))$

Definition: $\boldsymbol{x}$ is dominated by $\boldsymbol{y}$

$$\Leftrightarrow \quad \forall i,\ f_i(\boldsymbol{x}) \le f_i(\boldsymbol{y}) \text{ and } \exists j,\ f_j(\boldsymbol{x}) < f_j(\boldsymbol{y}).$$

The same definition: $\boldsymbol{x}$ is dominated by $\boldsymbol{y}$

$$\Leftrightarrow \quad \forall i,\ f_i(\boldsymbol{x}) \le f_i(\boldsymbol{y}) \text{ and } \boldsymbol{f}(\boldsymbol{x}) \ne \boldsymbol{f}(\boldsymbol{y}).$$

Pareto Dominance Relation

(Very Important Concept)

Minimize $\boldsymbol{f}(\boldsymbol{x}) = (f_1(\boldsymbol{x}),\ f_2(\boldsymbol{x}),\ \ldots,\ f_m(\boldsymbol{x}))$

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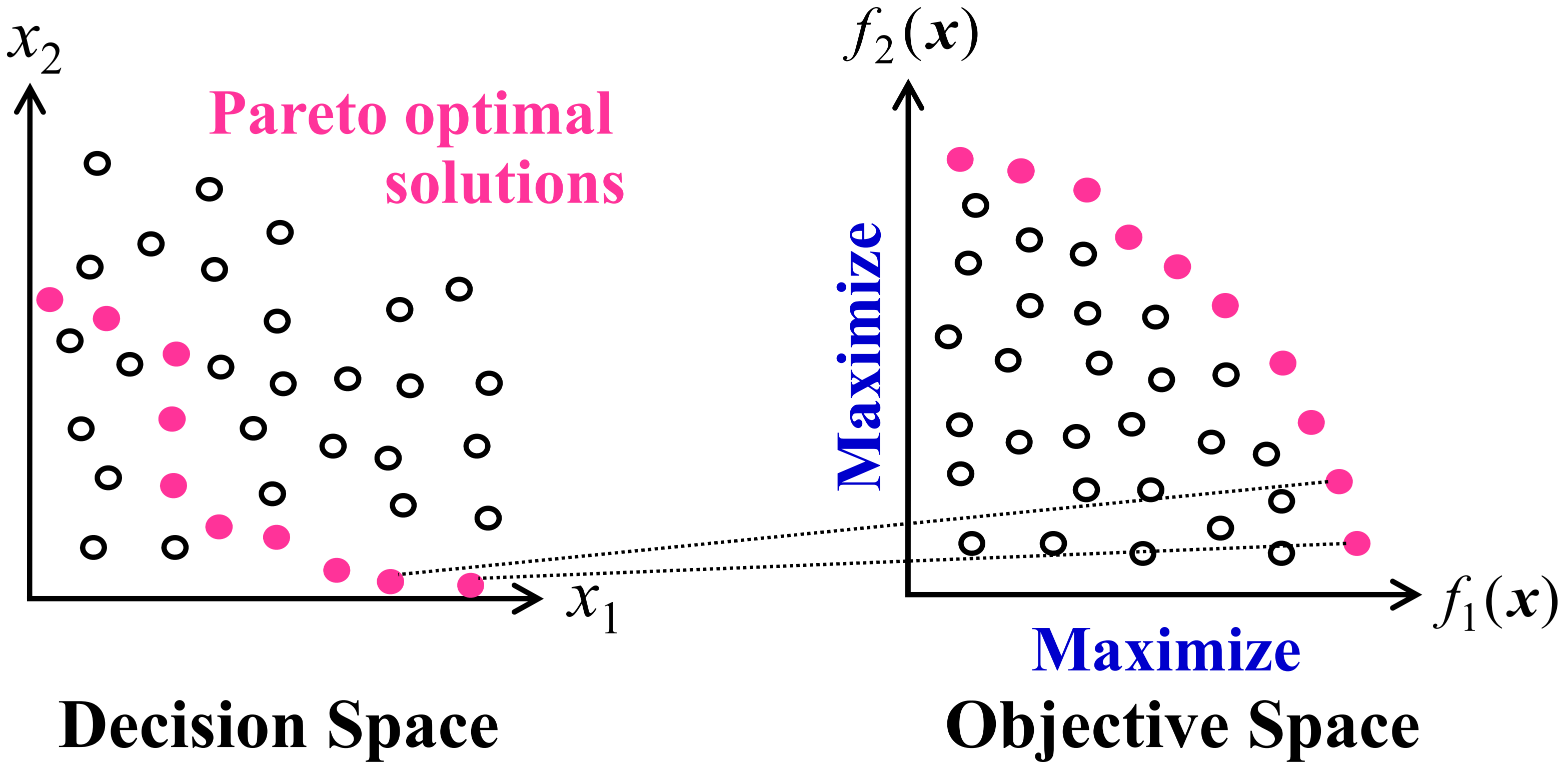
The same definition: $\boldsymbol{x}$ is dominated by $\boldsymbol{y}$

$$\Leftrightarrow \quad \forall i,\ f_i(\boldsymbol{x}) \geq f_i(\boldsymbol{y}) \ \text{ and } \ \boldsymbol{f}(\boldsymbol{x}) \neq \boldsymbol{f}(\boldsymbol{y}).$$

Pareto Optimal Solutions

Optimality in Multi-Objective Optimization

A Pareto optimal solution is a solution that is not dominated by any other solutions.



Pareto Optimal Solutions

Optimality in Multi-Objective Optimization

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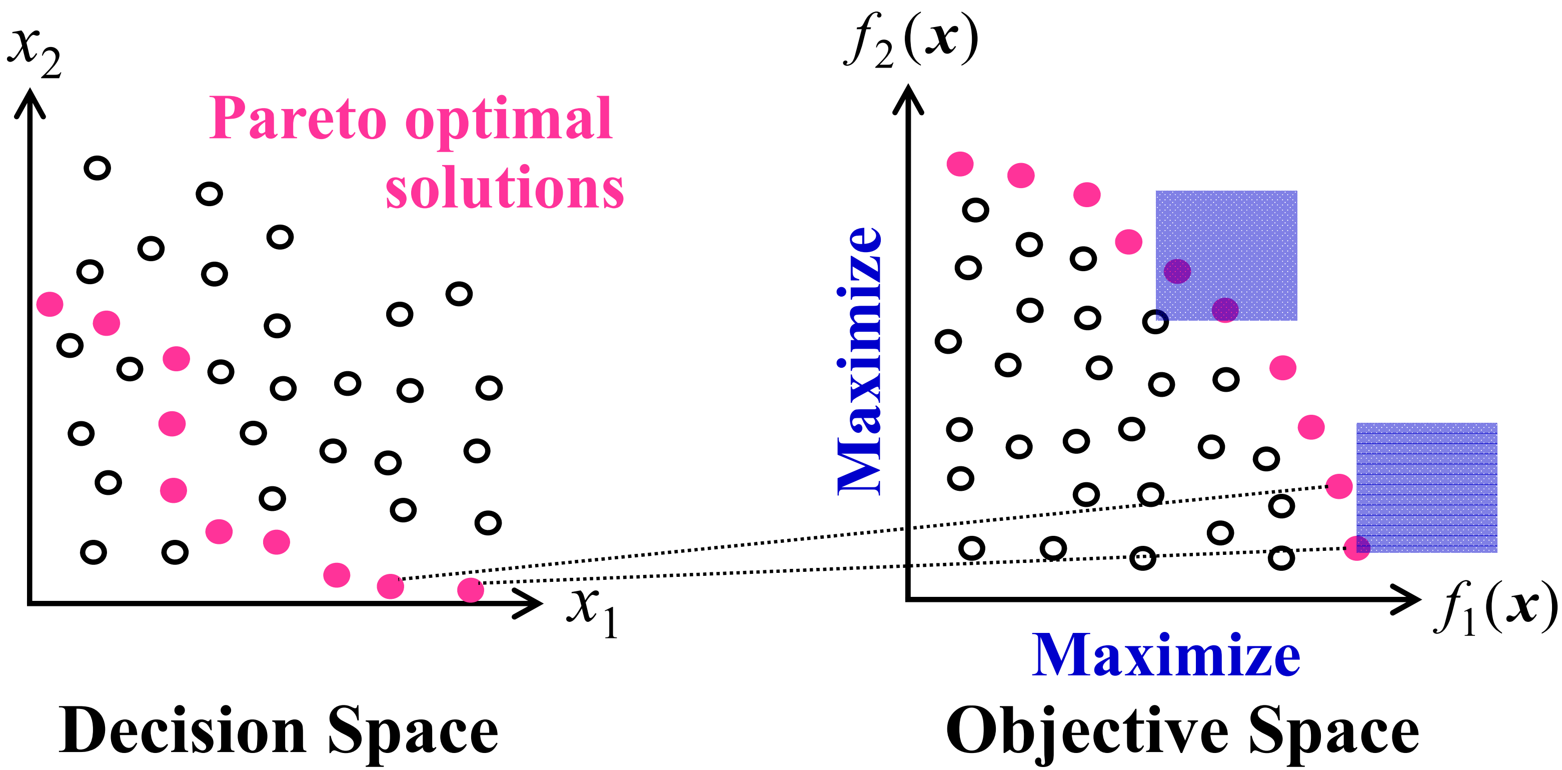
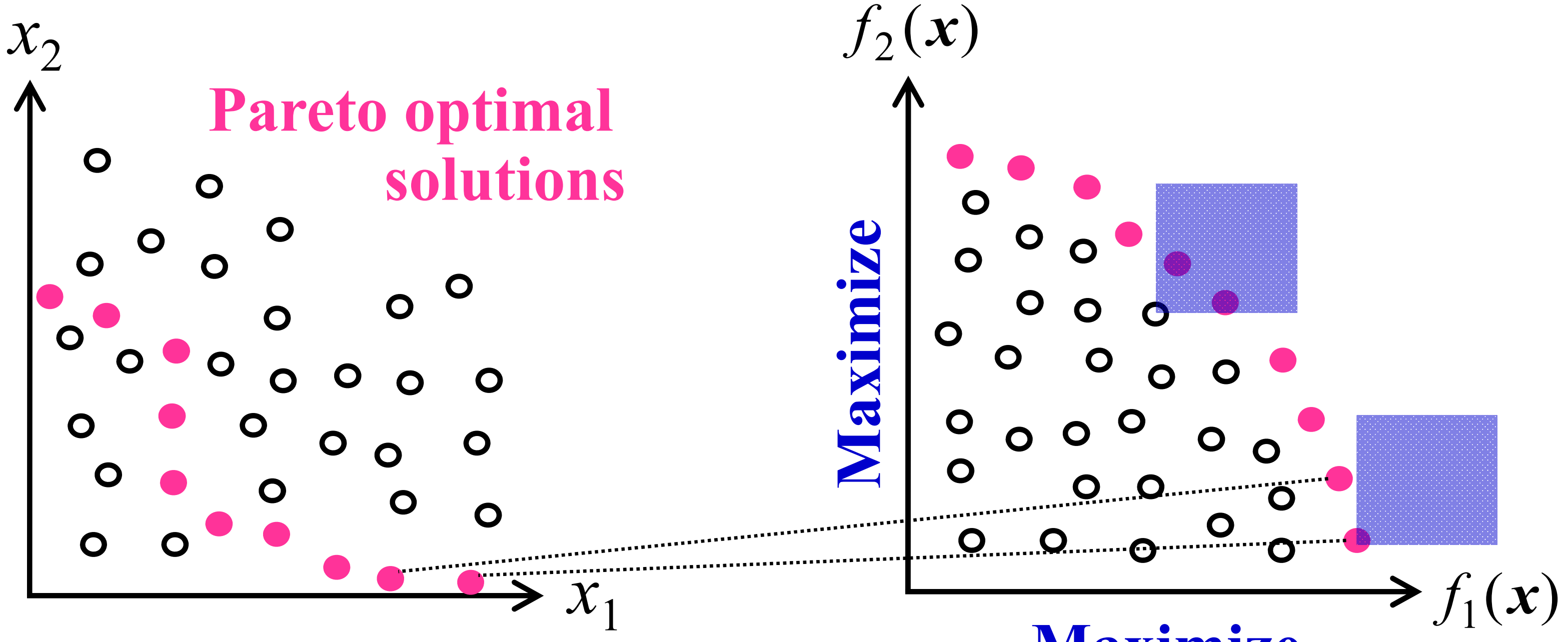


Pareto Optimal Solutions

Optimality in Multi-Objective Optimization

A Pareto optimal solution is a solution that is not dominated by any other solutions.

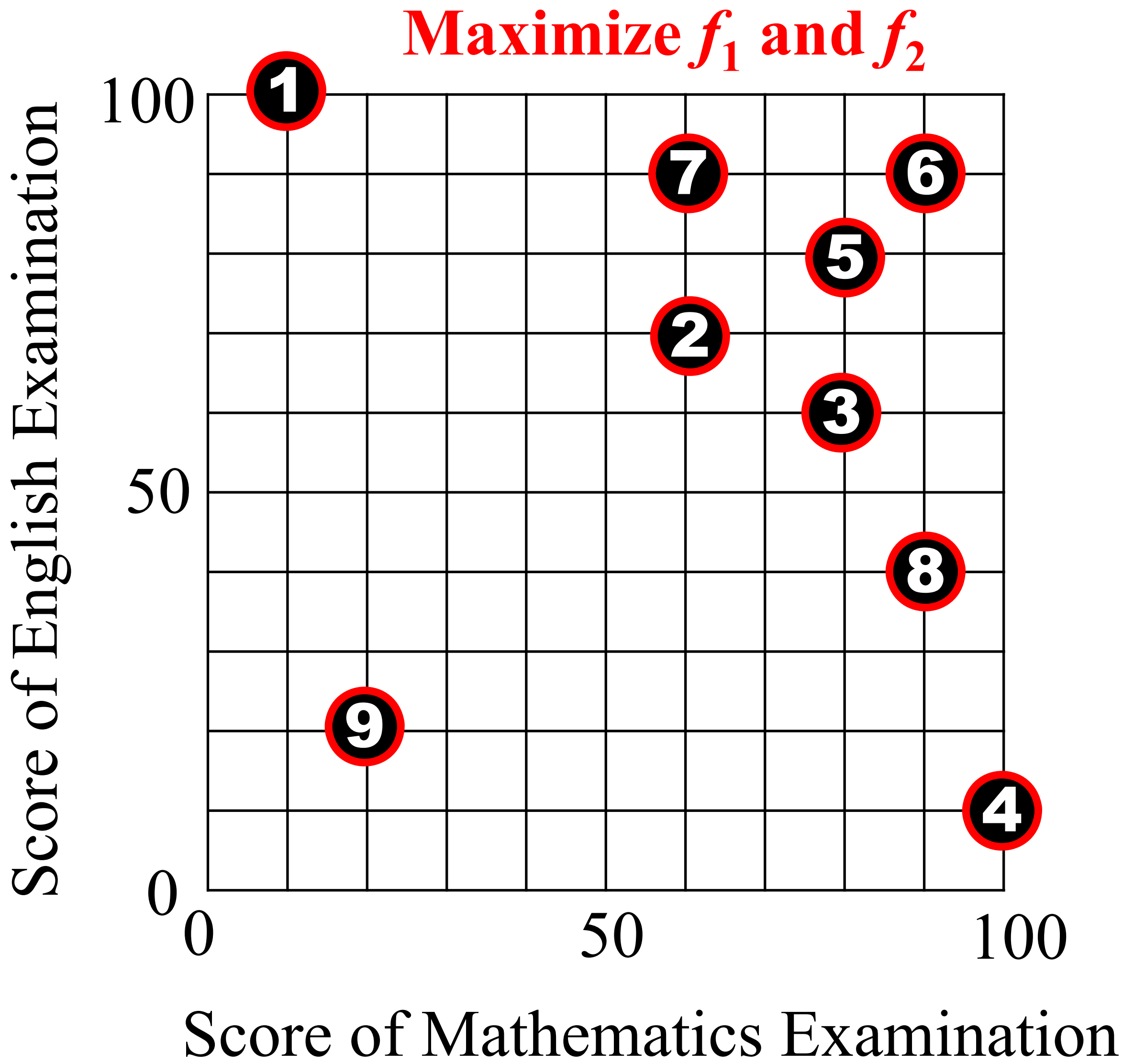
We cannot say which solution is the best solution among them.



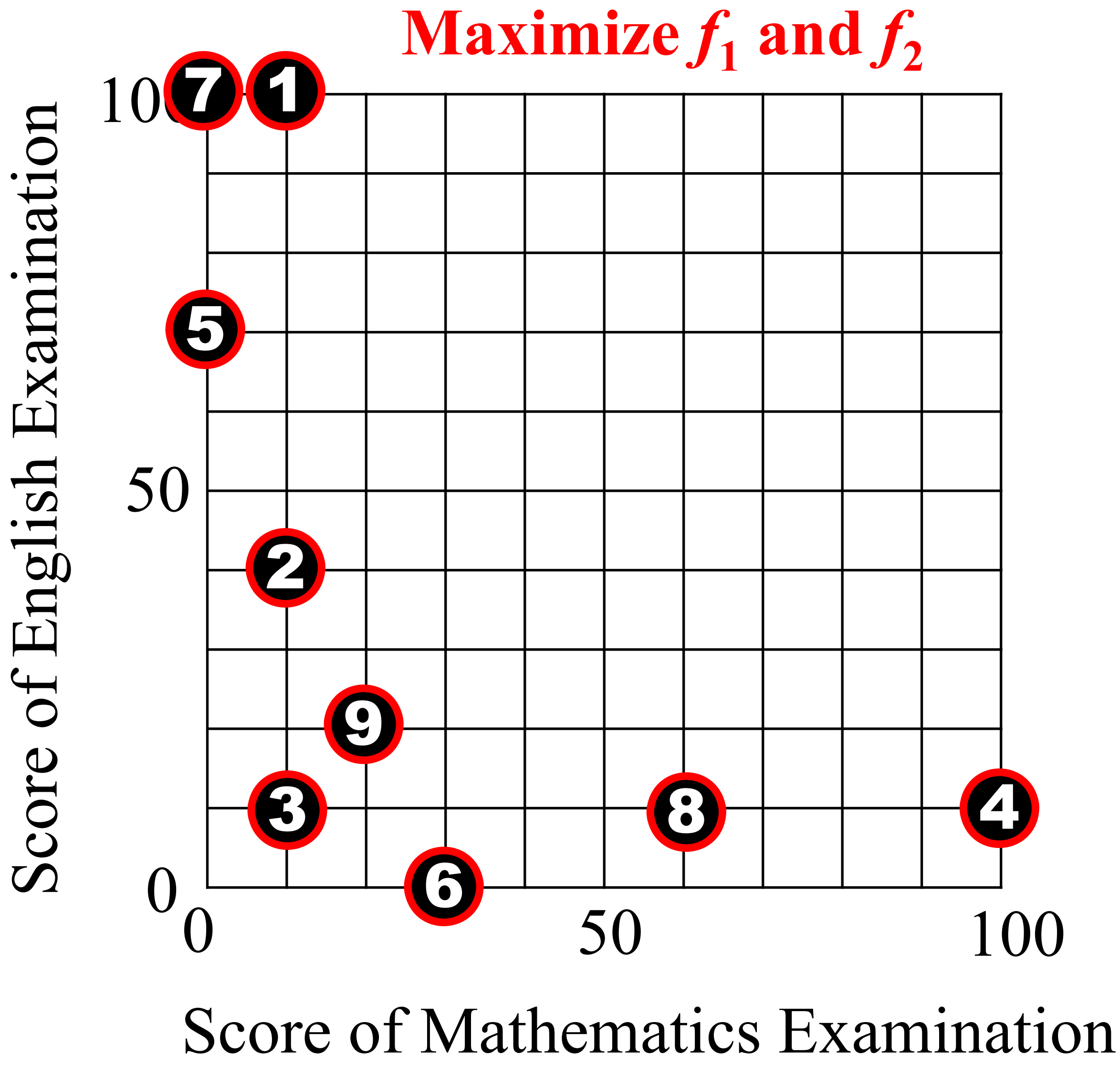
Decision Space

Objective Space

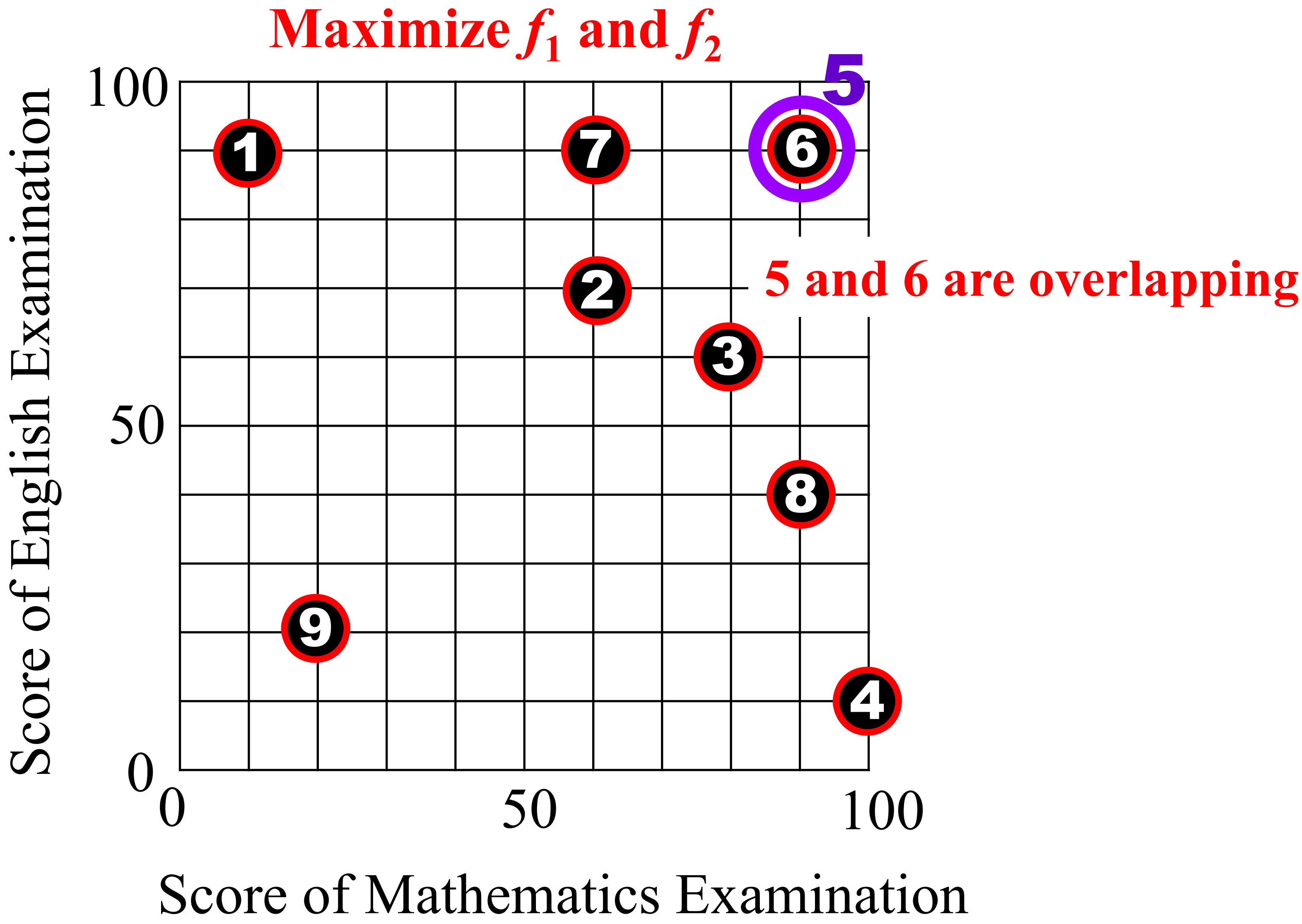
Q. Choose all Pareto optimal solutions



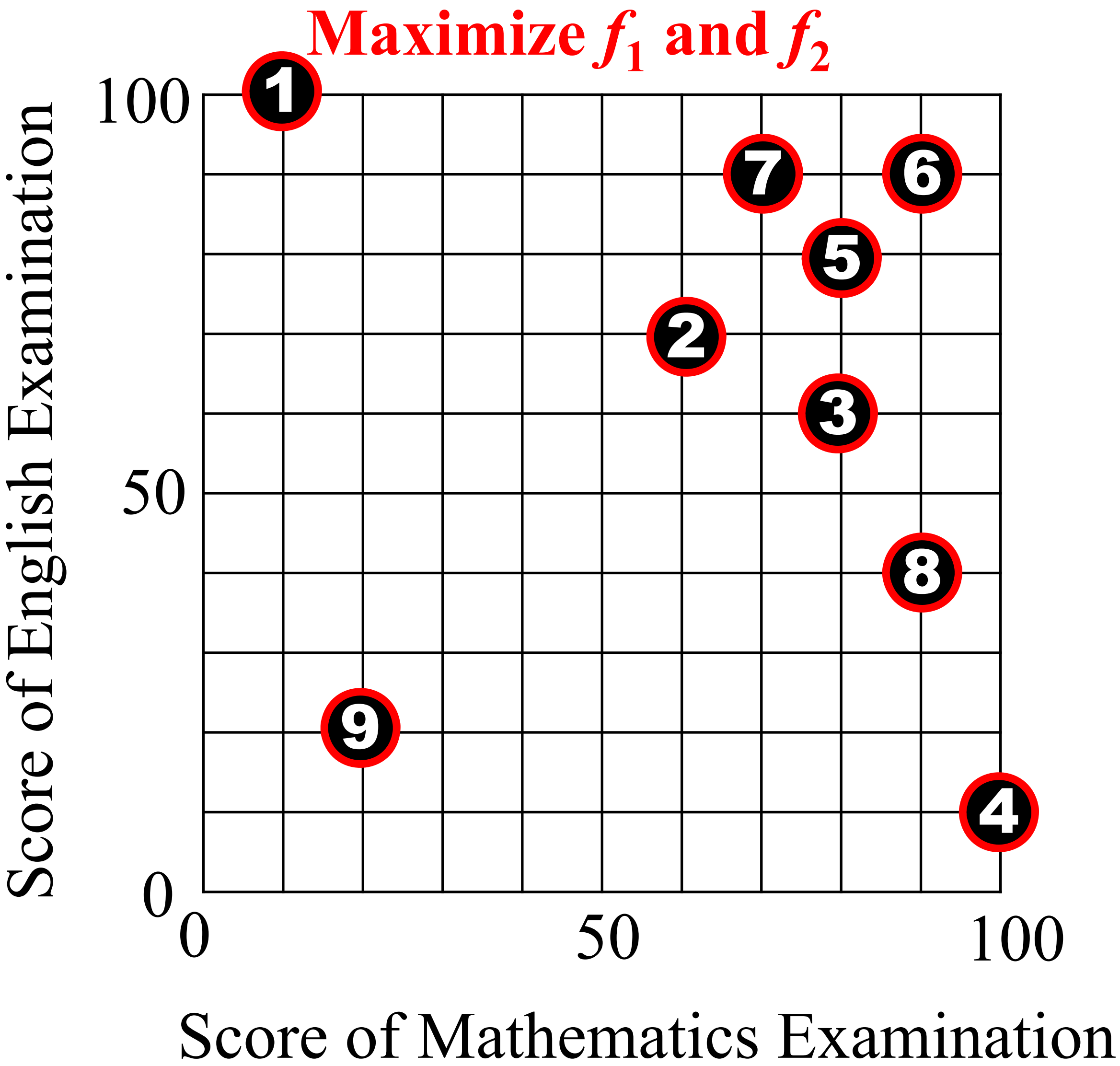
Q. Choose all Pareto optimal solutions



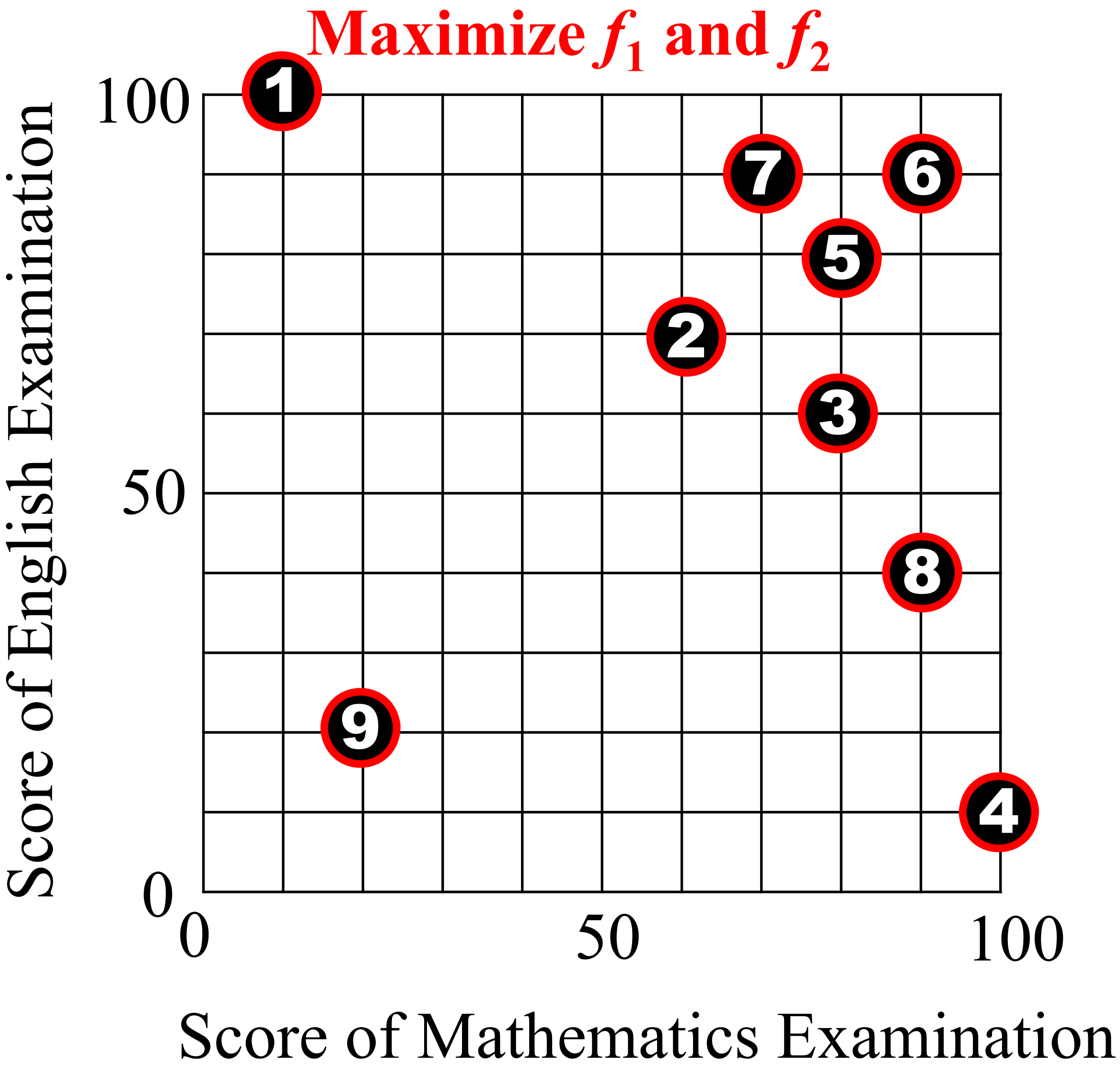
Q. Choose all Pareto optimal solutions



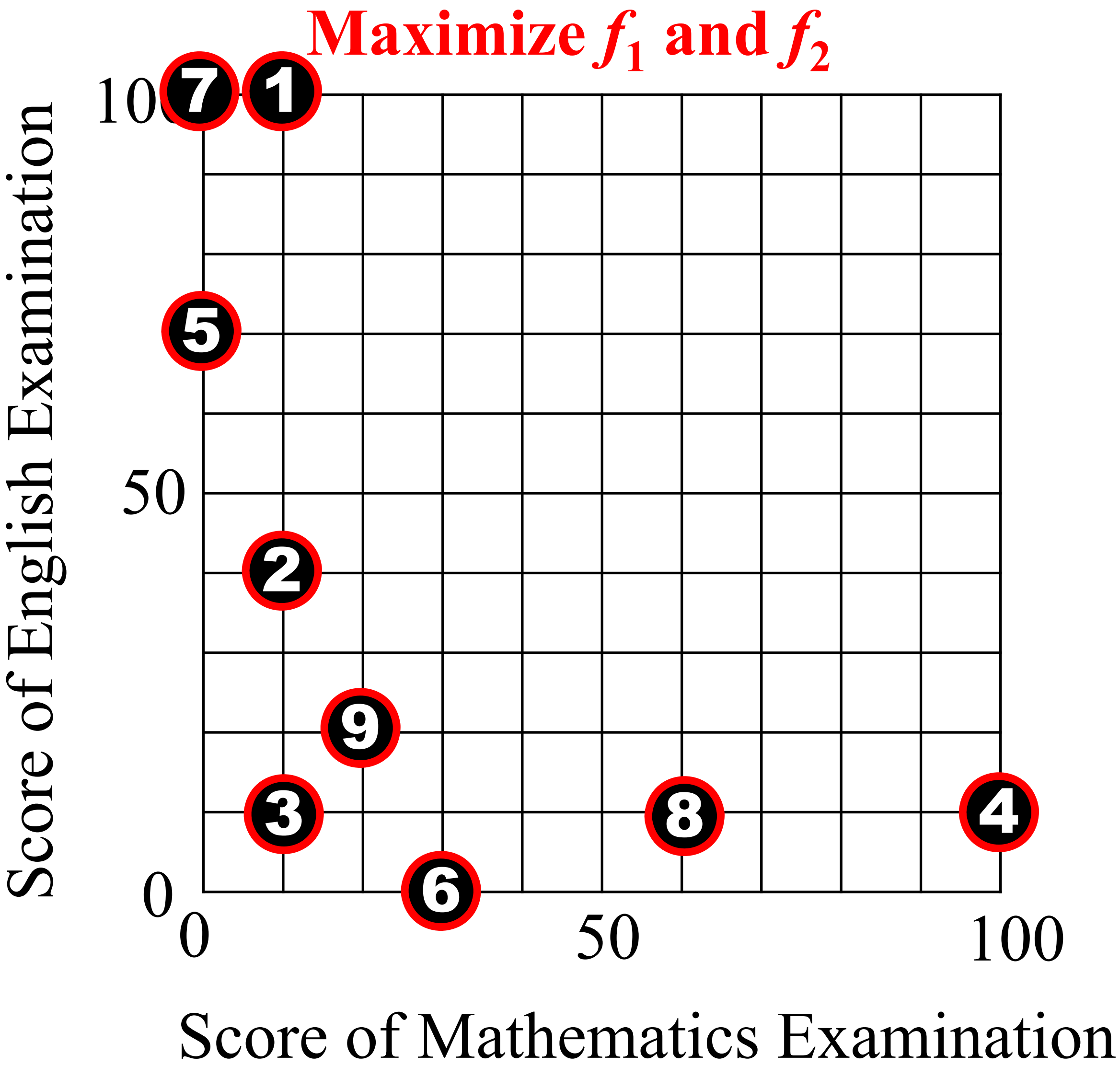
Q. Choose the best student (a single student)



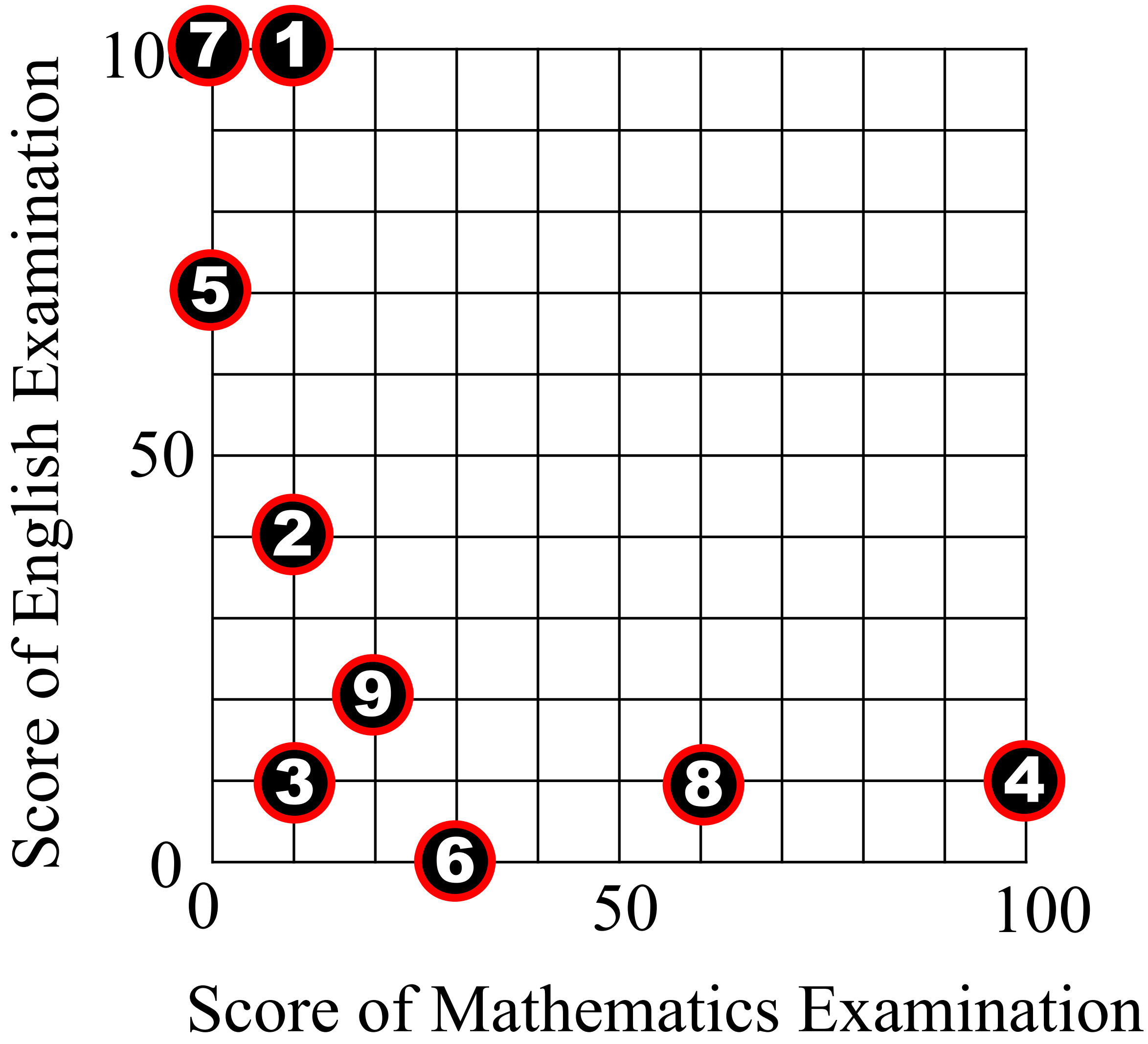
Q. Choose the best 3 students



Q. Choose the best student (a single student)



Q. Choose the best 3 students

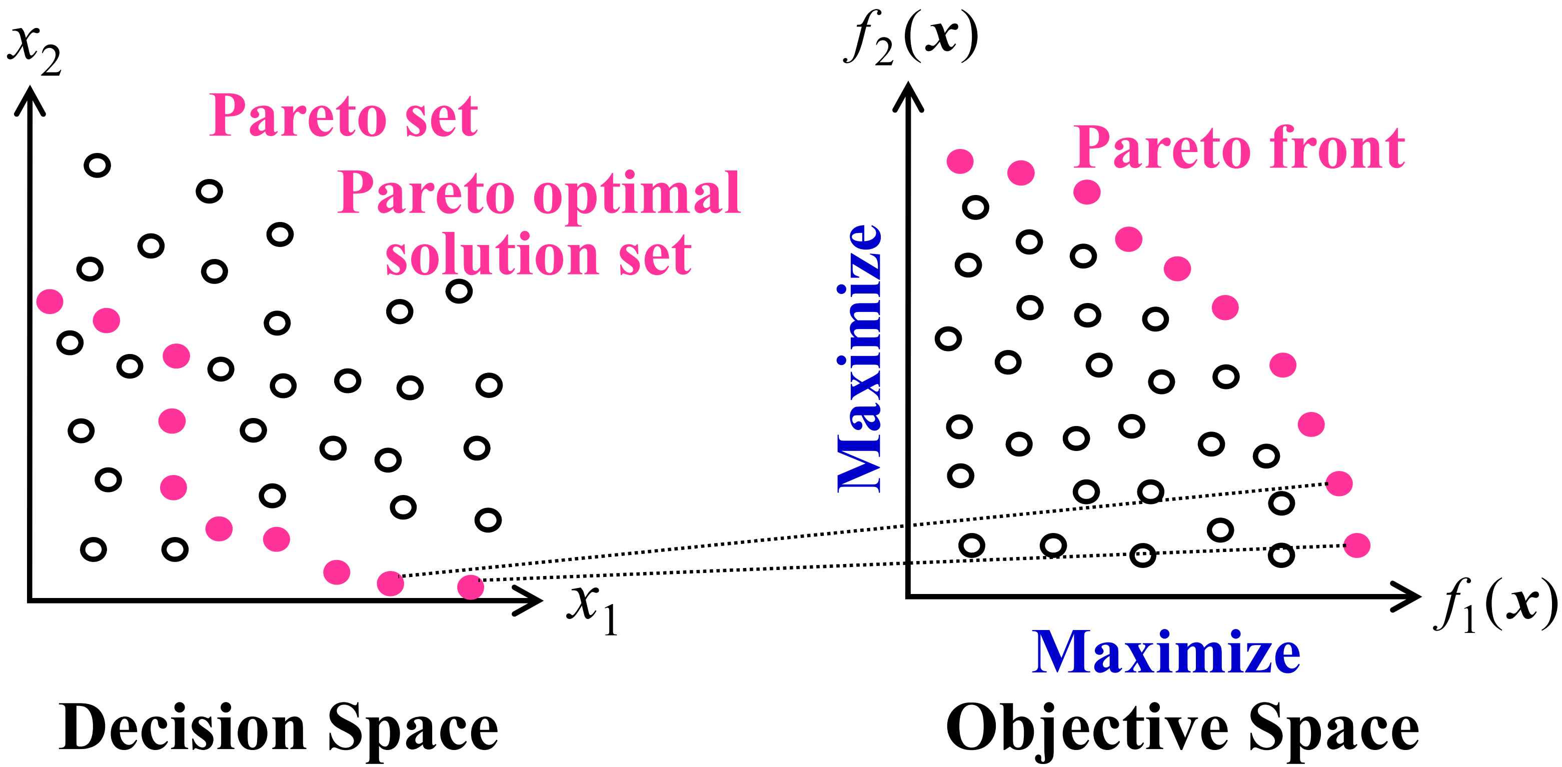


Pareto Set and Pareto Front

Optimality in Multi-Objective Optimization

The set of all Pareto optimal solutions in the decision space is called **the Pareto (optimal solution) set**.

The set of all Pareto optimal solutions in the objective space is called **the Pareto front**.

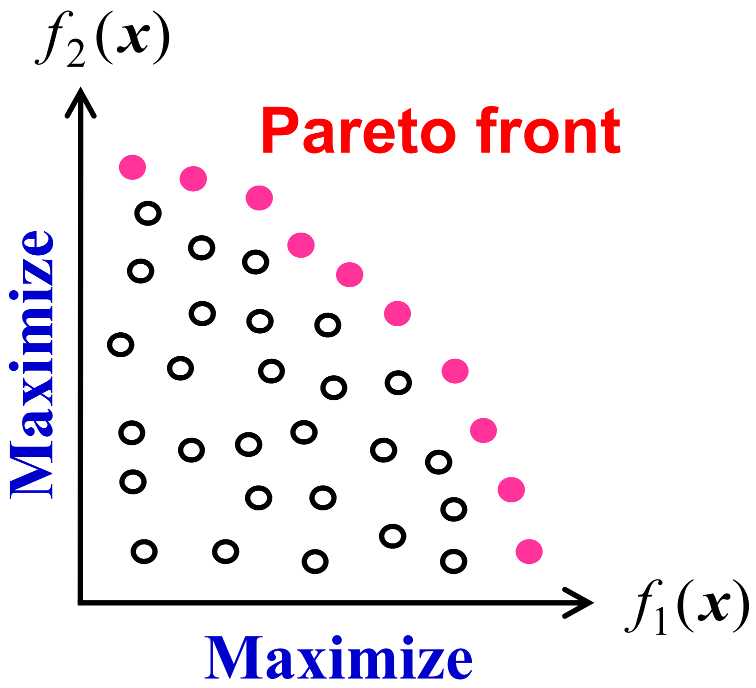


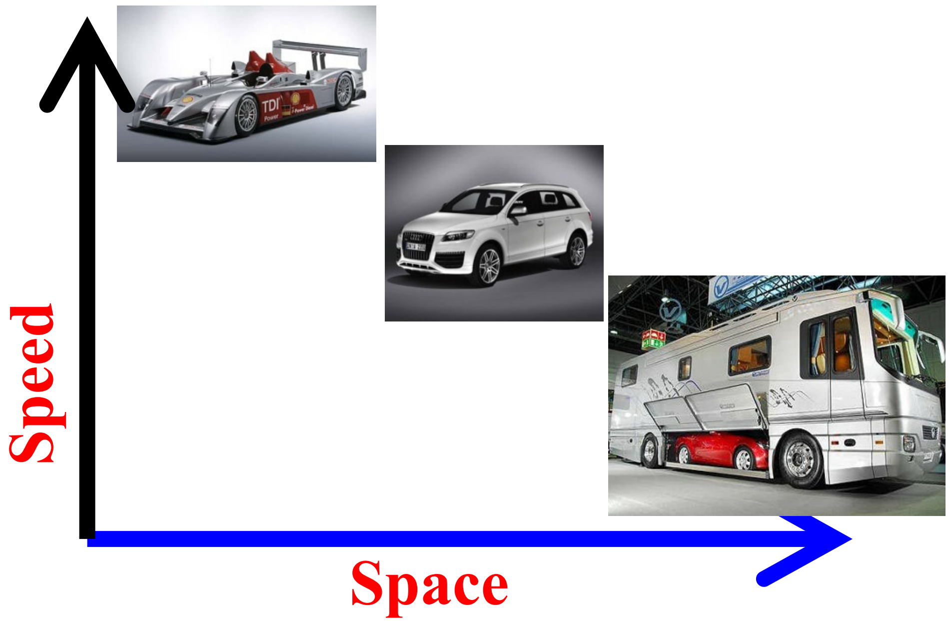
Pareto Front

Optimality in Multi-Objective Optimization

The set of all Pareto-optimal solutions in the objective space is called **the Pareto front**.

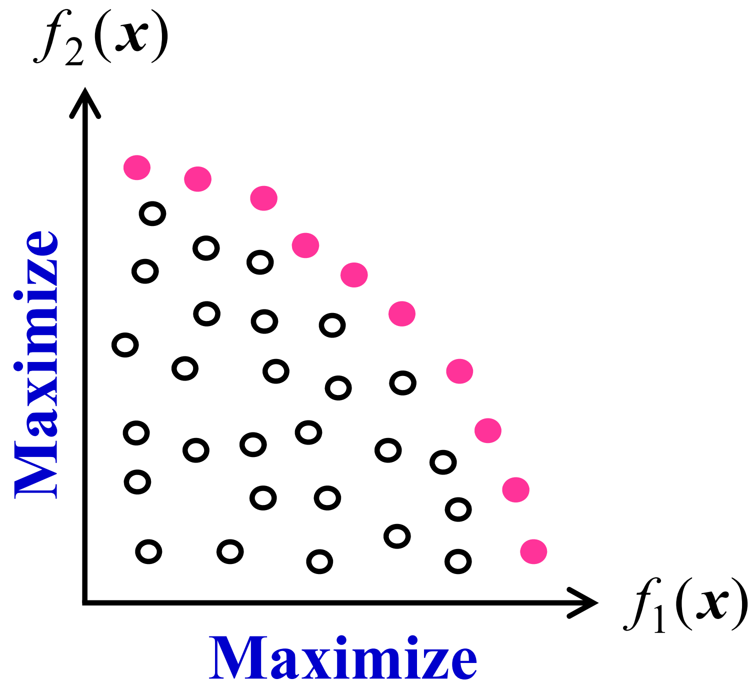
The Pareto front shows the trade-off relation.



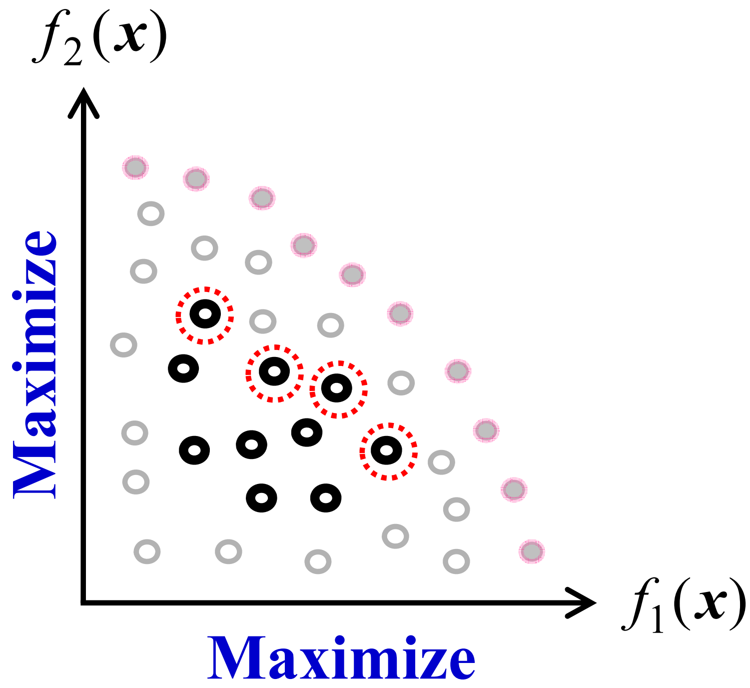


Non-Dominated Solutions

A non-dominated solution is a solution that is not dominated by any other solutions in a given solution set.



Pareto optimal solutions
(in all feasible solutions)

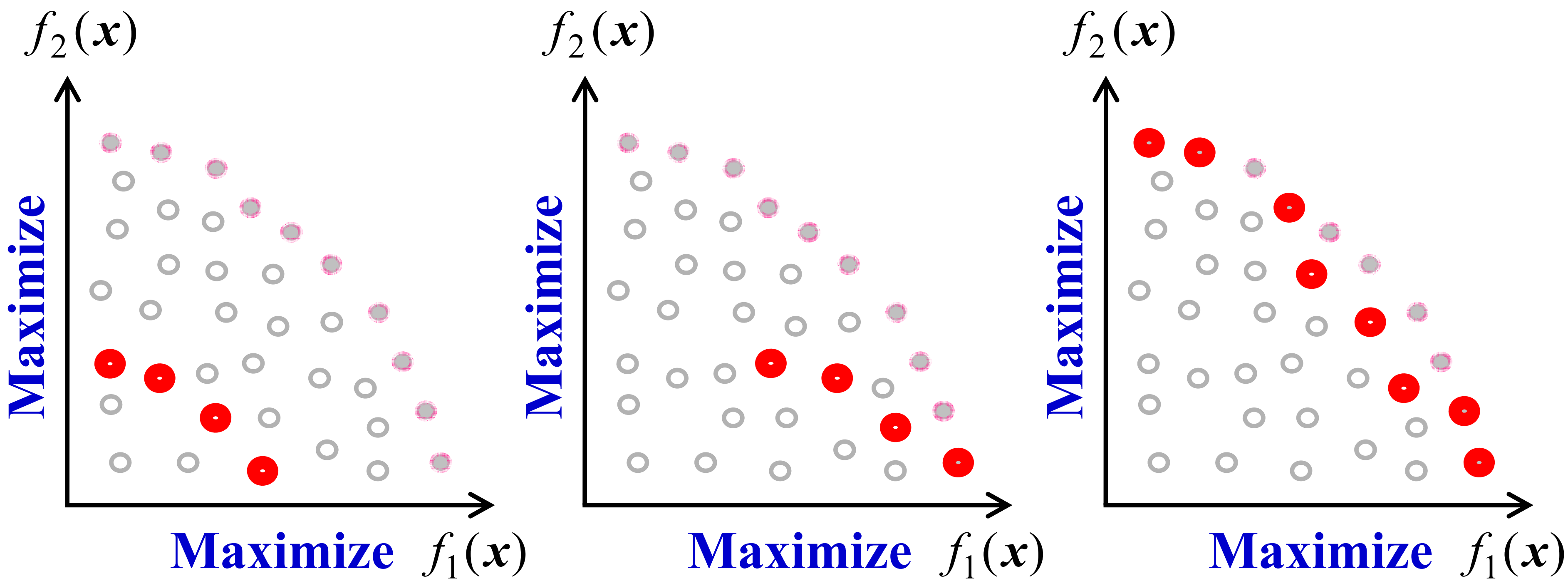


Non-dominated solutions
(in a given solution set)

Non-Dominated Solution Set

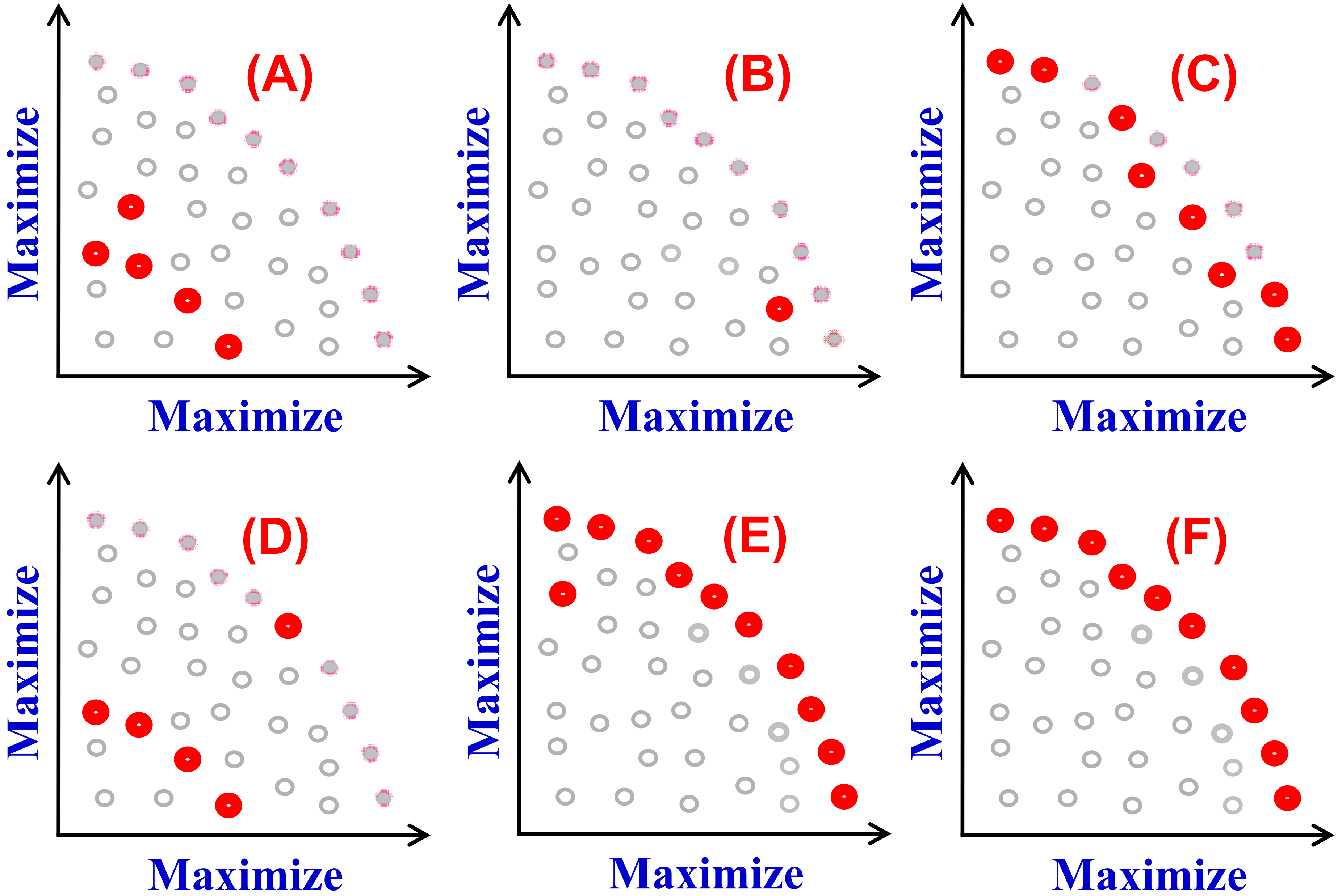
If all solutions in a solution set are non-dominated, the set is referred to as the non-dominated solution set.

Examples of non-dominated solution sets:



A non-dominated solution set can be a good solution set and a bad solution set. The Pareto set is also a non-dominated solution set.

Q. Choose all non-dominated solution sets



Lab Session Task:

For each of the following problems, (i) mathematically calculate the solution, and (ii) numerically estimate the solution through computer simulations with a large number of trials.

(1) We randomly generate two points A and B in $[0, 1]^2$. Calculate the probability of A being dominated by B.

(2) We randomly generate two points A and B in $[0, 1]^4$. Calculate the probability of A being dominated by B.

(3) We randomly generate two points A and B in $[0, 1]^{10}$. Calculate the probability of A being dominated by B.

(4) We randomly generate 200 points in $[0, 1]^2$. Calculate the expected number of non-dominated solutions among them.

(5) We randomly generate 2000 points in $[0, 1]^2$. Calculate the expected number of non-dominated solutions among them.

(6) We randomly generate 200 points in $[0, 1]^{10}$. Calculate the expected number of non-dominated solutions among them.

(7) We randomly generate 2000 points in $[0, 1]^{10}$. Calculate the expected number of non-dominated solutions among them.

Many algorithms and test problems are available through the Internet:

jMetal (for Java users)

J.J. Durillo, and A. J. Nebro, "jMetal: A Java framework for multi-objective optimization," Advances in Engineering Software (2011).

PlatEMO (for MATLAB users)

Y. Tian, R. Cheng, X. Zhang, and Y. Jin, "PlatEMO: A MATLAB platform for evolutionary multi-objective optimization," *IEEE Computational Intelligence Magazine* (2017)

Pymoo (for Python users)

J.Blank and K. Deb, "Pymoo: Multi-objective optimization in Python," *IEEE Access* (2020).