TWO BODIES PROBLEM. PLANETMOTION. LAWS OF KEPLER.

The Sun *S* (mass *M*) and the Planet P (mass *m*) are located at a distance *SP=r* of each other (fig.1). They attract each other according to the law of Newton with the force of gravitational attraction:

$$K = \gamma \frac{Mm}{r^2}$$

This invokes on *P* by *S* an acceleration a_1 in the direction *PS* with the size $a_1 = (\gamma M/r^2)$.
S undergoes by P an acceleration a_2 in the direction *SP*, equal to $a_2 = (\gamma m/r^2)$.
The relative acceleration of P with regard to S amounts to

$$a_r = a_1 + a_2 = \gamma \frac{(M+m)}{r^2}.$$

This is directed from *P* to *S*. If a rectangular coordinate system is chosen (in the plane of movement) with *S* as origin, and the direction of the *X*-ax arbitrary, then the components of the acceleration of *P* in *X*- en *Y*-direction are successively

$$a_x = -a_r \cos\phi = -\gamma \frac{(M+m)}{r^2} \cdot \frac{x}{r},$$

$$a_y = -a_r \sin\phi = -\gamma \frac{(M+m)}{r^2} \cdot \frac{y}{r},$$

Where *Φ* depicts the angle between the radius line *r=SP* and the positive *X*-ax.

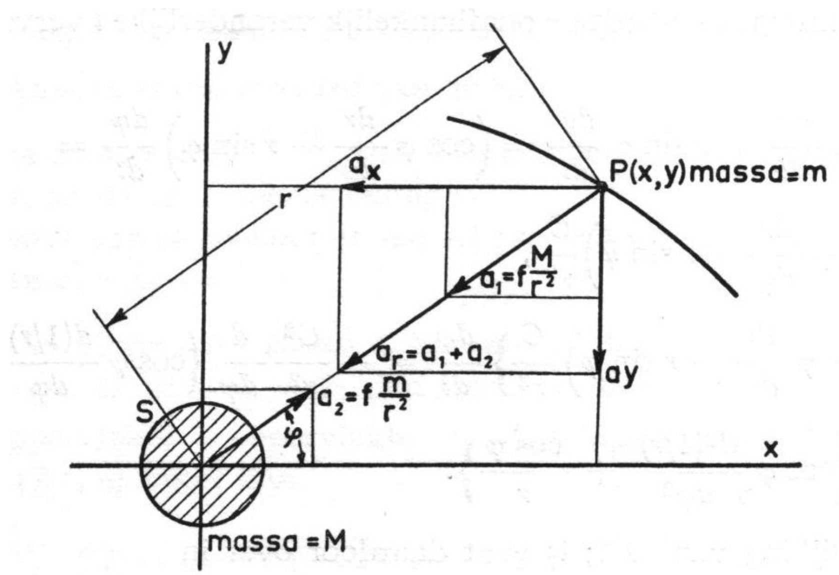

Fig.1

Hence the differential equations of the movement of P with respect to S are:

$$\left.\begin{aligned}\frac{d^2x}{dt^2}+\gamma\frac{(M+m)x}{r^3}=0,\\ \frac{d^2y}{dt^2}+\gamma\frac{(M+m)y}{r^3}=0.\end{aligned}\right\} \qquad (1)$$

From the equations (1):

$$x\frac{d^2y}{dt^2}-y\frac{d^2x}{dt^2}=\frac{d}{dt}\left(x\frac{dy}{dt}-y\frac{dx}{dt}\right)=0,$$

Thus after integration

$$x\frac{dy}{dt}-y\frac{dx}{dt}=C. \qquad (2)$$

The mathematical physical meaning of this integration constant C becomes already partially distinct by the introduction of polar coordinates

$$x=r\cos\varphi,\ y=r\sin\varphi$$

Thus

$$x\frac{dy}{dt}-y\frac{dx}{dt}=r^2\frac{d\phi}{dt}=C, \quad \text{of} \quad r^2d\phi=Cdt, \qquad (3)$$

$r^2d\phi=$ twice the plane-element passed by the radius in dt seconds. From (3) it follows $\int r^2d\phi=Ct$ = twice the sector (or the Area) passed by the radius line in t seconds. (4)

By (4) the famous *Area law= second law of Kepler* is expressed in the traditional saying:

The line from the Sun to the planet sweeps out equal areas in equal times.

The constant C from (3) and (4) has the name *area constant*. For the following integration the independent variable t replaced by ϕ, according (3)

$$\frac{dx}{dt}=\cos\phi\frac{dr}{dt}-r\sin\phi\frac{d\phi}{dt}=\left(\cos\phi\frac{dr}{d\phi}-r\sin\phi\right)\frac{d\phi}{dt}=\left(\cos\phi\frac{dr}{d\phi}-r\sin\phi\right)\frac{C}{r^2},$$

$$\frac{d^2x}{dt^2}=\frac{d}{d\phi}\left\{\left(\cos\phi\frac{dr}{d\phi}-r\sin\phi\right)\frac{C}{r^2}\right\}\frac{d\phi}{dt}=-\frac{C^2}{r^2}\frac{d}{d\phi}\left\{\left(\cos\phi\frac{d(1/r)}{d\phi}+\frac{\sin\phi}{r}\right)\right\}=-\frac{C^2}{r^2}\left(\cos\phi\frac{d^2(1/r)}{d\phi^2}+\frac{\cos\phi}{r}\right),$$

Therefore the first equation of (1) changes to

$$-\frac{C^2}{r^2}\cos\phi\left(\frac{d^2(1/r)}{d\phi^2}+\frac{1}{r}\right)+\frac{\gamma(M+m)\cos\phi}{r^2}=0,$$

or

$$\left(\frac{d^2(1/r)}{d\phi^2}+\frac{1}{r}\right)=\frac{\gamma(M+m)}{C^2}=\text{constant}, \tag{5}$$

This results in a common complete differential equation of the second order in $1/r$, with constant coefficients. (The second equation of (1) leads to the same result.) The integration of (5) results

$$\frac{1}{r}=\frac{\gamma(M+m)}{C^2}+A\cos\phi+\mathrm{B}\sin\phi=\frac{\gamma(M+m)}{C^2}+E\cos(\phi-\alpha), \tag{6}$$

Where *A*, *B*, *E* and *α* depict new integration constants. Instead of *C* and *E* two new constants *α* are *e* introduced, determined by

$$E=\frac{\gamma(M+m)}{C^2}.e \tag{7a}$$

$$\frac{\gamma(M+m)}{C^2}=\frac{1}{a(1-e^2)}. \tag{7b}$$

Hence the result of (6) becomes

$$r=\frac{a(1-e^2)}{1+e\cos(\phi-\alpha)}. \tag{8}$$

This is the *polar equation* of a cone cut, of which one focus coincides with the pole S. In the normal case is $e < 1$. In that case the cone cut becomes an ellipse. The significance of the parameters *a* and *e* is clear.

$$r_{\min}=\mathit{perihelium}\text{-distance}=\frac{a(1-e^2)}{1+e}=a(1-e)\text{ for }\phi=\alpha. \quad \text{(actually ABS(e))} \tag{8a}$$

$$r_{\max}=\mathit{aphelium}\text{-distance}=\frac{a(1-e^2)}{1-e}=a(1+e)\text{ for }\phi=\alpha+\pi.$$

(actually ABS(e)) (8b)

Thus

a is the length of the *half large* ax of the ellipse,

e is the *numeric eccentricity* of the ellipse.

The half small ax $b = a\sqrt{1-e^2}$. the result is expressed by the *first law of Kepler,* in the traditional saying:

The motion curve of the Planet is an ellipse, of which one of the foci is occupied by the Sun.

The significance of the *area constant C* now can be elucidated. If one calls the *revolution time* of the Planet =*T*, then in the time *T* an area is swept of which the surface = square of the ellipse = $\pi ab = \pi a^2\sqrt{1-e^2}$. From the area law (4) it follows then

$$CT = 2\pi a^2\sqrt{1-e^2},\ \text{thus } C = \frac{2\pi a^2\sqrt{1-e^2}}{T}. \tag{9}$$

Finally from (9) and (7b) it follows

$$\frac{a^3}{T^2} = \frac{\gamma(M+m)}{4\pi^2}. \tag{10}$$

This equation depicts, in *improved* form, the *third law of Kepler*. In general this law is, in elementary cosmography teaching books, formulated as follows:

For the different planets the squares of the revolution times relate as the third powers of the half large axes.

Thus when the half large axes of two planets are a_1 and a_2 and the subsequent revolution times T_1 en T_2, then it should be according to the third law of Kepler in this primitive form

$$\frac{a_1^3}{T_1^2} = \frac{a_2^3}{T_2^2} = \text{constant}$$

However it appears from (10) that this result is not constant from planet by planet, but depending on the individual mass of the planet. If the masses of the planets concerned are m_1 and m_2, then is

$$\frac{a_1^3}{T_1^2} = \frac{\gamma M\{1+(m_1/M)\}}{4\pi^2};\quad \frac{a_2^3}{T_2^2} = \frac{\gamma M\{1+(m_2/M)\}}{4\pi^2}.$$

However both results differ *very* little from the *constant* value $\gamma M/4\pi^2$, because in worst case is (Jupiter) *(m/M)* < 0.001 , so that the factor 1+*(m/M)* ∞ 1. Obviously here the very small disturbing mutual influences of both planets are neglected.

Determination of the motion curve of a planet when the position on the *X-ax* and *Y-ax* and the velocities *Vx* and *Vy* are known.

Thus the curve is based on equation (8)

$$r = \frac{a\left(1-e^2\right)}{1+e\cos\left(\varphi-\alpha\right)}.$$

The positions and the velocities on a certain point of time are called: *Xo*, *Yo*, *Vxo* en *Vyo*. Thus the unknowns *a*, *e*, *φ* and *α* shall be expressed in these starting values.

$$r_0 = \sqrt{x_0^2 + y_0^2}$$

$$\cos\varphi_0 = \frac{x_0}{r_0},\quad \sin\varphi_0 = \frac{y_0}{r_0},\quad tg\varphi_0 = \frac{y_0}{x_0}.$$ Thus

$$\varphi_0 = arctg\,\frac{y_0}{x_0}$$

(For convenience purposes, in the equations below, the indices (o) have been omitted. As it happens the formulae stay universally valid and the starting requirements can be filled in later.)

From $x = r\cos\varphi$ and $y = r\sin\varphi$ it can be derived:

$$\frac{dx}{dt} = V_x = \frac{dr}{dt}\cos\varphi - r\sin\varphi\,\frac{d\varphi}{dt} \qquad (11)$$

$$\frac{dy}{dt} = V_y = \frac{dr}{dt}\sin\varphi + r\cos\varphi\,\frac{d\varphi}{dt} \qquad (12)$$

Hence:

$$\frac{dr}{dt} = V_x x\cos\varphi + V_y\sin\varphi \qquad (13)$$

$$r\,\frac{d\varphi}{dt} = V_y \cos\varphi - V_x \sin\varphi$$

(14)

Differentiation of (8) results:

$$\frac{dr}{dt} = \frac{re\sin(\varphi-\alpha)}{1+e\cos(\phi-\alpha)}.\frac{d\varphi}{dt}$$

(15)

Further from (2):

$$C = xV_y - yV_x$$

(16)

From (7b)

$$a = \frac{C^2}{\gamma(M+m)(1-e^2)}$$

(17)

From (13), (15) and (15):

$$V_x x + V_y y = \frac{e\sin(\varphi-\alpha)}{1+e\cos(\varphi-\alpha)}.(V_y x - V_x y) = \frac{e\sin(\varphi-\alpha)}{1+e\cos(\varphi-\alpha)}C$$

$$\Rightarrow (xV_x + yV_y)(1+e\cos(\varphi-\alpha)) = eC\sin(\varphi-\alpha)$$

$$\Rightarrow (xV_x + yV_y) = eC\sin(\varphi-\alpha) - e\cos(\varphi-\alpha)(xV_x + yV_y)$$

$$\Rightarrow e = \frac{xV_x + yV_y}{C\sin(\varphi-\alpha) - (xV_x + yV_y)\cos(\varphi-\alpha)}$$

(18)

From (8) and (17):

$$r = \sqrt{x^2+y^2} = \frac{C^2}{\gamma(M+m)(1-e^2)}.\frac{(1-e^2)}{1+e\cos(\varphi-\alpha)} = \frac{C^2}{\gamma(M+m)(1+e\cos(\varphi-\alpha))}$$

$$\Rightarrow 1+e\cos(\varphi-\alpha) = \frac{C^2}{\gamma(M+m)\sqrt{x^2+y^2}}$$

$$\Rightarrow e = \frac{\dfrac{C^2}{\gamma(M+m)\sqrt{x^2+y^2}} - 1}{\cos(\varphi-\alpha)}$$

(19)

From (18) and (19):

$$\frac{xV_x + yV_y}{C\sin(\varphi-\alpha)-(xV_x + yV_y)\cos(\varphi-\alpha)}$$

$$= \frac{\dfrac{C^2}{\gamma(M+m)\sqrt{x^2+y^2}}-1}{\cos(\varphi-\alpha)}$$

$$\Rightarrow \frac{1}{\dfrac{C^2}{\gamma(M+m)\sqrt{x^2+y^2}}-1} = \frac{C\sin(\varphi-\alpha)-(xV_x + yV_y).\cos(\varphi-\alpha)}{(xV_x + yV_y).\cos(\varphi-\alpha)}$$

$$\Rightarrow \frac{1}{\dfrac{C^2}{\gamma(M+m)\sqrt{x^2+y^2}}-1} = \frac{C}{xV_x + yV_y}.tg(\varphi-\alpha)-1$$

$$\Rightarrow tg(\varphi-\alpha=)\frac{\dfrac{1}{\dfrac{C^2}{\gamma(M+m)\sqrt{x^2+y^2}}-1}+1}{\dfrac{C}{xV_x + yV_y}}$$

$$\Rightarrow tg(\varphi-\alpha=)\frac{\dfrac{\gamma(M+m)\sqrt{x^2+y^2}}{C^2-\gamma(M+m)\sqrt{x^2+y^2}}+1}{\dfrac{C}{xV_x + yV_y}} =$$

$$\Rightarrow= \frac{\gamma(M+m)\sqrt{x^2+y^2}+C^2-\lambda(M+m)\sqrt{x^2+y^2}}{C^2-\gamma(M+m)\sqrt{x^2+y^2}}.\frac{xV_x + yV_y}{C}$$

$$\Rightarrow tg(\varphi-\alpha)=\frac{C(xV_x + yV_y)}{C^2-\gamma(M+m)\sqrt{x^2+y^2}} = P = \frac{tg\varphi - tg\alpha}{1+tg\varphi.tg\alpha}$$

$$\Rightarrow tg\alpha = \frac{\dfrac{y}{x}-P}{\dfrac{y}{x}.P+1} = \frac{y-Px}{yp+x}$$

(20)

Thus herewith is stated:

$$P = \frac{C(xV_x + yV_y)}{C^2-\gamma(M+m)\sqrt{x^2+y^2}}$$

From (20):

$$\alpha = arctg\left(\frac{y - Px}{yp + x}\right)$$
(21)

Derivation to determine the velocity of the planet.

Differentiation of (8):

$$\frac{dr}{dt} = \frac{-r}{1 + e\cos(\varphi - \alpha)}.(-e\sin(\varphi - \alpha)).\frac{d\varphi}{dt}$$
(22)

From (11) and (22):

$$\Rightarrow V_x = \frac{dr}{dt}.\cos\varphi - r.\sin\varphi.\frac{d\varphi}{dt} = \frac{re\sin(\varphi - \alpha)}{1 + e\cos(\varphi - \alpha)}.\frac{d\varphi}{dt}.\cos\varphi - r\sin\varphi.\frac{d\varphi}{dt}$$

$$\Rightarrow V_x = \frac{re\sin(\varphi - \alpha)}{1 + e\cos(\varphi - \alpha)}.\frac{d\varphi}{dt}.\cos\varphi - r\sin\varphi.\frac{d\varphi}{dt} = r\frac{d\varphi}{dt}.\left[\frac{e\sin(\varphi - \alpha)}{1 + e\cos(\varphi - \alpha)}.\cos\varphi - \sin\varphi\right]$$

From (12) and (22):

$$\Rightarrow V_y = \frac{re\sin(\varphi - \alpha)}{1 + e\cos(\varphi - \alpha)}.\frac{d\varphi}{dt}.\sin\varphi + r\cos\varphi.\frac{d\varphi}{dt} = r\frac{d\varphi}{dt}.\left[\frac{e\sin(\varphi - \alpha)}{1 + e\cos(\varphi - \alpha)}.\sin\varphi + \cos\varphi\right]$$

$$\Rightarrow V_x = V_y.\frac{\left[\frac{e\sin(\varphi - \alpha)\cos\varphi}{1 + e\cos(\varphi - \alpha)} - \sin\varphi\right]}{\left[\frac{e\sin(\varphi - \alpha).\sin\varphi}{1 + e\cos(\varphi - \alpha)} + \cos\varphi\right]}$$

$$\Rightarrow V_x = V_y.\frac{e\sin(\varphi - \alpha).\cos\varphi - \sin\varphi - e\cos(\varphi - \alpha).\sin\varphi}{e\sin(\varphi - \alpha).\sin\varphi + \cos\varphi + e\cos(\varphi - \alpha).\cos\varphi} = K.V_y$$
(23)

From (16) and (23):

$$C = xV_y - yV_x \Rightarrow V_y = \frac{C + yV_x}{x} \Rightarrow V_x = K.\frac{C + yV_x}{x}$$

$$\Rightarrow xV_x = K.C + K.yV_x \Rightarrow V_x.(x - Ky) = K.C$$

$$\Rightarrow V_x = \frac{K.C}{x - y.K} \text{ en } V_y = \frac{C}{x - y.K}$$

Relate ellipse to the input data: longest distance; Rmax, and the shortest distance; Rmin (or optionally to the revolution time T)

From (8a) and (8b):

$$\frac{1+e}{1-e}=\frac{R\max}{R\min}\Rightarrow R\min+eR\min=R\max-eR\max\Rightarrow$$

$$\Rightarrow e=\frac{R\max-R\min}{R\max+R\min} \quad \text{en} \quad a=\frac{1}{2}\left(R\max+R\min\right) \qquad (23)$$

From (7b):

$$C=\sqrt{\gamma\left(M+m\right)a(1-e^2)} \quad \text{en} \quad T=\frac{2\pi a^2\sqrt{1-e^2}}{C}$$

With this the motion curve can be described.

Now when the distance (Rmax) as well as the revolution time is known.

From (10) follows:

$$\frac{a^3}{T^2}=\frac{\gamma\left(M+m\right)}{4\pi^2}. \qquad \Rightarrow a^3=\frac{T^2\gamma\left(M+m\right)}{4\pi^2}.$$

$$\Rightarrow a=\sqrt[3]{\frac{T^2\gamma\left(M+m\right)}{4\pi^2}}.$$

From (23) follows:

$$a=\frac{1}{2}\left(R\max+R\min\right)\Rightarrow R\min=2a-R\max$$

$$\Rightarrow R\min=2\sqrt[3]{\frac{T^2\gamma\left(M+m\right)}{4\pi^2}}-R\max.$$

$$\Rightarrow e=\frac{2R\max}{2\sqrt[3]{\frac{T^2\gamma\left(M+m\right)}{4\pi^2}}}-1$$

Back