General Relativity

Bundling of

My derivations

And

deliberations.

By

Albert Prins

8 September 2018

1 Contents

1. The Laplace operator applied on the gravitational potential outside and inside a static sphere

Newton:

$$F = mg = \frac{mMG}{r^2} \text{—>} \; gravitational\ field{:}\; g = \frac{MG}{r^2} \text{—>} \; gravitational\ potential{:}\; \emptyset_{newton} = \frac{-MG}{r} \; where \;\; g = \frac{d\emptyset_{newton}}{dr}$$

Gravitational potential **outside** a sphere in General Relativity (GR)

$$\emptyset = 1 - \frac{2GM}{c^2 r} = 1 + \frac{2\emptyset_{newton}}{c^2}$$

$$\emptyset_{newton} = -\frac{GM}{r}$$

Gravitational potential **inside** a sphere (see derivation below)

$$\emptyset = 1 - \frac{3GM}{c^2 R} + \frac{GM}{c^2}\frac{r^2}{R^3} = 1 + \frac{2}{c^2}.\left(-\frac{3GM}{2R} + \frac{GM}{2}\frac{r^2}{R^3}\right)$$

$$\emptyset_{newton} = -\frac{3GM}{2R} + \frac{GM}{2}\frac{r^2}{R^3} \quad (see\ chapter\ 3\ formula\ 3)$$

Now application of the Laplace operator:

$$r^2 = x^2 + y^2 + z^2$$

1.1 Outside a sphere (Laplace):

$$\frac{\partial\emptyset_{newton}}{\partial x} = \frac{\partial\emptyset_{newton}}{\partial r}\frac{\partial r}{\partial x} = \frac{GM}{r^2}.\frac{x}{r} = \frac{GMx}{r^3}$$

$$\frac{\partial^2\emptyset_{newton}}{\partial x^2} = \frac{-3GMx}{r^4}.\frac{x}{r} + \frac{GM}{r^3} = \frac{-3GMx^2}{r^5} + \frac{GM}{r^3}$$

$$\Delta\emptyset_{newton} = \frac{\partial^2\emptyset_{newton}}{\partial x^2} + \frac{\partial^2\emptyset_{newton}}{\partial y^2} + \frac{\partial^2\emptyset_{newton}}{\partial z^2}$$

$$\Delta\emptyset_{newton} = \frac{-3GM}{r^3}.\frac{x^2 + y^2 + z^2}{r^2} + 3\frac{GM}{r^3} = \frac{-3GM}{r^3} + 3\frac{GM}{r^3} = 0$$

Thus

$$\boldsymbol{\Delta\emptyset_{newton} = 0}.$$

1.2 Inside a sphere (Poisson):

$$\frac{\partial\emptyset_{newton}}{\partial x} = \frac{\partial\emptyset_{newton}}{\partial r}\frac{\partial r}{\partial x} = \frac{2GM}{2}\frac{r}{R^3}\frac{x}{r} = \frac{GMx}{R^3}$$

$$\frac{\partial^2\emptyset_{newton}}{\partial x^2} = \frac{GM}{R^3}$$

$$\Delta\emptyset_{newton} = \frac{\partial^2\emptyset_{newton}}{\partial x^2} + \frac{\partial^2\emptyset_{newton}}{\partial y^2} + \frac{\partial^2\emptyset_{newton}}{\partial z^2} = \frac{3GM}{R^3} = \frac{3G.\frac{4}{3}\pi R^3\rho}{R^3} = 4\pi G\rho$$

Thus

$$\boldsymbol{\Delta\emptyset_{newton} = 4\pi G\rho}.$$

This is in agreement with the Poisson equation.

Hence:

$$\emptyset = 1 + \frac{2\emptyset_{newton}}{c^2} \, {-}{>} \; \Delta\emptyset = \frac{2}{c^2}\Delta\emptyset_{newton} = \frac{2}{c^2}4\pi G\rho = \boldsymbol{\frac{8\pi G\rho}{c^2}}$$

2 Simplification of the application of the Laplace operator

Let us assume a function f(r) on which the Laplace operator will be applied

$$r^2 = x^2 + y^2 + z^2$$

$$\frac{\partial f(r)}{\partial x} = \frac{\partial f(r)}{\partial r}.\frac{\partial r}{\partial x} = \frac{\partial f(r)}{\partial r}.\frac{x}{r}$$

$$\frac{\partial^2 f(r)}{\partial x^2} = \frac{\partial^2 f(r)}{\partial r^2}.\frac{x}{r}.\frac{x}{r} + \frac{\partial f(r)}{\partial r}.\frac{1}{r} - \frac{\partial f(r)}{\partial r}.\frac{x}{r^2}.\frac{x}{r}$$

$$\frac{\partial^2 f(r)}{\partial x^2} = \frac{\partial^2 f(r)}{\partial r^2}.\frac{x^2}{r^2} + \frac{\partial f(r)}{\partial r}.\frac{1}{r}.(1 - \frac{x^2}{r^2})$$

Now for x,y, en z:

$$\frac{\partial^2 f(r)}{\partial x^2} + \frac{\partial^2 f(r)}{\partial y^2} + \frac{\partial^2 f(r)}{\partial z^2} = \frac{\partial^2 f(r)}{\partial r^2}.\frac{x^2+y^2+z^2}{r^2} + \frac{\partial f(r)}{\partial r}.\frac{1}{r}.(3 - \frac{x^2+y^2+z^2}{r^2})$$

$$\boldsymbol{Laplace\ equation:} \qquad \Delta\boldsymbol{f(r)} = \boldsymbol{\frac{\partial^2 f(r)}{\partial r^2} + \frac{2}{r}.\frac{\partial f(r)}{\partial r}} \qquad (\mathbf{1})$$

Let the general form of $\emptyset_{newton}$ be:

$$\emptyset_{newton} = L + Kr^n \qquad (1a)$$

$$\frac{\partial\emptyset_{newton}}{\partial r} = nKr^{n-1}$$

$$\frac{\partial^2\emptyset_{newton}}{\partial r^2} = n(n-1)Kr^{n-2}$$

Hence from equation (1):

$$\Delta\emptyset_{newton} = n(n-1)Kr^{n-2} + \frac{2}{r}.nKr^{n-1} = n(n-1)Kr^{n-2} + 2nKr^{n-2}$$

$$\Delta\emptyset_{newton} = \boldsymbol{n(n+1)Kr^{n-2}} \qquad (\mathbf{2})$$

Let us apply this formula on the gravitational potentials on the outside and inside of a sphere.
Outside a sphere:

$$\emptyset_{newton} = -\frac{GM}{r}$$

So with (1a)

$$\emptyset_{newton} = L + Kr^n$$

Thus n= -1, L=0 and K= -GM. Then with (2):

$$\Delta\emptyset_{newton} = +1(-1+1)GMr^{-1-2} = 0.GMr^{-3} = 0$$

Inside a sphere:

$$\emptyset_{newton} = -\frac{3GM}{2R} + \frac{GM}{2}\frac{r^2}{R^3}$$

So with (1a)

$$\emptyset_{newton} = L + Kr^n$$

Thus n=+2, L=-3GM/2R and K=GM/2R^3

$$\Delta\emptyset_{newton} = +2(2+1)\frac{GM}{2R^3}r^{-2-2} = 6\frac{GM}{2R^3} = \boldsymbol{\frac{3GM}{R^3}} = \frac{3G.\frac{4}{3}\pi R^3\rho}{R^3} = \boldsymbol{4\pi G\rho}$$

This is in accordance with the calculations in the previous chapter.
Furthermore it can be seen that the $\Delta\emptyset_{newton}$ is zero when n=0 or -1, and obviously when r goes to infinity while n<3.

3 Derivation of the gravitational potential inside a static sphere

The gravitational potential inside a static sphere will be derived based on the Poisson equation:

$$\Delta\boldsymbol{\emptyset_{newton}} = \boldsymbol{4\pi G\rho}.$$

And the general form of $\emptyset_{newton}$

$$\emptyset_{newton} = L + Kr^n$$

With formula (2) derived above

$$\Delta\boldsymbol{\emptyset_{newton}} = \boldsymbol{n(n+1)Kr^{n-2}} \quad \mathbf{(2)}$$

$$=> 4\pi G\rho = n(n+1)Kr^{n-2}$$

$$=> n = 2\ so\ 6K = 4\pi G\rho => K = \frac{2}{3}\pi G\rho = \frac{2}{3}\pi G\frac{M}{\frac{4}{3}\pi R^3} = \frac{1}{2}\frac{GM}{R^3}$$

Thus:

$$=> \emptyset_{newton} = L + \frac{2}{3}\pi G\rho r^2$$

On the surface of the sphere where r=R

$$\emptyset_{newton} = -\frac{GM}{r}$$

Thus:

$$\emptyset_{newton} = -\frac{GM}{R} = -\frac{4}{3}\pi\frac{R^3}{R}G\rho = -\frac{4}{3}\pi R^2 G\rho = L + \frac{2}{3}\pi G\rho R^2$$

$$=> L = -\frac{4}{3}\pi R^2 G\rho - \frac{2}{3}\pi G\rho R^2 = -\frac{6}{3}\pi R^2 G\rho = -\frac{6}{3}\pi R^2 G\frac{M}{\frac{4}{3}\pi R^3} = -\frac{3}{2}\frac{MG}{R}$$

$$\emptyset_{newton} = -\frac{3}{2}\frac{MG}{R} + \frac{1}{2}\frac{GM}{R^3}r^2$$

$$acceleration: g_r = \frac{d\emptyset_{newton}}{dr} = \frac{GM}{R^3}r$$

At $r=0$ $acceleration: g_r = 0$ and at $r=R$ then $acceleration: g_r = \frac{GM}{R^2}$.

$\boldsymbol{Gravitational\ \ potential\ inside\ sphere}$:	$\emptyset = 1 + \frac{2\emptyset_{newton}}{c^2} = \boldsymbol{1 - \frac{3MG}{c^2R} + \frac{GM}{c^2R^3}r^2}$	**(3)**

4 Derivation of the Schwarzschild formula to tau (proper time).

Question of R.: Wat ik moeilijk kan accepteren in AR, is de differentiatie naar "*ds*". Het lijnelement is niets anders dan de lichtsnelheid maal het lokaal gemeten tijdsverschil "*dt$_0$*" (*ds* = c. *dt$_0$*).
Ik kan dt/ds nog bevatten (verschil in kloksnelheid) maar wat zegt dx/ds nu?

Answer: The Schwarzschild formule can be split up partiële derivatives like:

Assume the metric tensor components as general components A,B, D and E.

$$c^2 d\tau^2 = Ac^2 dt^2 - Bdx^2 - Ddy^2 - Edz^2$$
$$1 = A\left(\frac{dt}{d\tau}\right)^2 - \frac{B}{c^2}\left(\frac{dx}{dt}\right)^2\left(\frac{dt}{d\tau}\right)^2 - \frac{D}{c^2}\left(\frac{dy}{dt}\right)^2\left(\frac{dt}{d\tau}\right)^2 - \frac{E}{c^2}\left(\frac{dz}{dt}\right)^2\left(\frac{dt}{d\tau}\right)^2$$

Subsequently x,y and z are divided in their own frame and appear to be velocities.

$$1 = A\left(\frac{dt}{d\tau}\right)^2\left\{1 - \frac{B}{Ac^2}\left(\frac{dx}{dt}\right)^2 - \frac{D}{Ac^2}\left(\frac{dy}{dt}\right)^2 - \frac{E}{Ac^2}\left(\frac{dz}{dt}\right)^2\right\}$$
$$1 = A\left(\frac{dt}{d\tau}\right)^2\left\{1 - \frac{v^2}{c^2}\right\} = \frac{A}{\gamma^2}\left(\frac{dt}{d\tau}\right)^2$$
$$\left(\frac{dt}{d\tau}\right)^2 = \frac{\gamma^2}{A}$$

Or:

$$d\tau^2 = \frac{A}{\gamma^2}dt^2 = \frac{\sigma^2}{\gamma^2}dt^2$$
$$d\tau = \frac{\sigma}{\gamma}dt$$

Where:

$$\sigma = \sqrt{1 - \frac{R_s}{r}} \quad and \quad \gamma = 1/\sqrt{(1 - \frac{v^2}{c^2})}$$

5 Elucidation of translation formula of Einstein

Deze formule betekent de translatie tussen twee coordinatenstelsels. Het oude stelsel is wordt weergegeven door x_β, met dus de coordinaat assen x_0, x_1, x_2, x_3. Het nieuwe stelsel x'_α, door $x'_0, x'_{1,} x'_2, x'_3$. Het verband tussen deze twee stelsen wordt weerfgegeven door de volgende formule (covariante componenten):

$$dx'_\alpha = \frac{\partial x'_\alpha}{\partial x_\beta}dx_\beta$$

Deze formule is geschreven volgens de Einstein notatie, dat betekent dat het een sommatie is over β

Hij ziet er dus eigenlijk als volgt uit:

$$dx'_\alpha = \frac{\partial x'_\alpha}{\partial x_0}dx_0 + \frac{\partial x'_\alpha}{\partial x_1}dx_1 + \frac{\partial x'_\alpha}{\partial x_2}dx_2 + \frac{\partial x'_\alpha}{\partial x_3}dx_3$$

Dus iedere nieuwe coordinaat wordt uitgedrukt in alle oude coordinaten..

Totaal:

$$dx'_0 = \frac{\partial x'_0}{\partial x_0}dx_0 + \frac{\partial x'_0}{\partial x_1}dx_1 + \frac{\partial x'_0}{\partial x_2}dx_2 + \frac{\partial x'_0}{\partial x_3}dx_3$$

$$dx'_1 = \frac{\partial x'_1}{\partial x_0} dx_0 + \frac{\partial x'_1}{\partial x_1} dx_1 + \frac{\partial x'_1}{\partial x_2} dx_2 + \frac{\partial x'_1}{\partial x_3} dx_3$$

$$dx'_2 = \frac{\partial x'_2}{\partial x_0} dx_0 + \frac{\partial x'_2}{\partial x_1} dx_1 + \frac{\partial x'_2}{\partial x_2} dx_2 + \frac{\partial x'_2}{\partial x_3} dx_3$$

$$dx'_3 = \frac{\partial x'_3}{\partial x_0} dx_0 + \frac{\partial x'_3}{\partial x_1} dx_1 + \frac{\partial x'_3}{\partial x_2} dx_2 + \frac{\partial x'_3}{\partial x_3} dx_3$$

Kan ook voorgeteld worden als tensor (tensor notation):

$$\begin{pmatrix} dx'_0 \\ dx'_1 \\ dx'_2 \\ dx'_3 \end{pmatrix} = \begin{bmatrix} \frac{\partial x'_0}{\partial x_0} & \frac{\partial x'_0}{\partial x_1} & \frac{\partial x'_0}{\partial x_2} & \frac{\partial x'_0}{\partial x_3} \\ \frac{\partial x'_1}{\partial x_0} & \frac{\partial x'_1}{\partial x_1} & \frac{\partial x'_1}{\partial x_2} & \frac{\partial x'_1}{\partial x_3} \\ \frac{\partial x'_2}{\partial x_0} & \frac{\partial x'_2}{\partial x_1} & \frac{\partial x'_2}{\partial x_2} & \frac{\partial x'_2}{\partial x_3} \\ \frac{\partial x'_3}{\partial x_0} & \frac{\partial x'_3}{\partial x_1} & \frac{\partial x'_3}{\partial x_2} & \frac{\partial x'_3}{\partial x_3} \end{bmatrix} \cdot \begin{pmatrix} dx_0 \\ dx_1 \\ dx_2 \\ dx_3 \end{pmatrix}$$

Dus het is alleen maar een translatie van het ene stelsel naar een andere.

Je kan dit dus gebruiken om bijvoorbeeld bij Schwarzschild van t, r, theta, phi over te gaan naar Schwarzschild t,x,y,z.

6 Deliberations on Schwarzschild metric

Schwarzschild equation with polar co-ordinates:

$$ds^2 = c^2 d\tau^2 = \left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta^2\, d\emptyset^2 \tag{1}$$

Although it looks like the dimensions are not right; formula (1) really means:

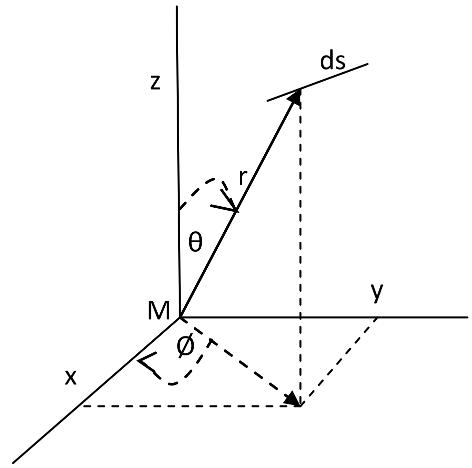
$$ds^2 = \left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 - \frac{r^2}{R_p^2} dR_p^2 . \theta^2 - \frac{r^2}{R_p^2} \sin^2\theta^2\, dR_p^2 . \emptyset^2 \quad where\ R_p = 1\ meter$$

But for practical reasons formula (1) is used.

Here is G the gravitation constant, M the considered mass and *c* is the light velocity.

We would like to know what formula (1) exactly stands for:
In space there is a small, in size, object with mass M, considered as a point mass. This mass has, in standard Newtonian view, a gravitation field and thus force. In Einstein's, and also Schwarzschild's view, this mass deforms space-time and there is no gravitational force. A universal Euclidean coordinate system is chosen with M in the origin. When a particle, considered being massless, is held in space, it experiences a gravitation force due to the mass M. Now we let the particle move at free will. The movement, in standard Newtonian view, will cause acceleration because of the gravitational force. However the particle itself, in its co-moving frame, experiences no force at all; it surrenders itself to space. In

Einstein's view the trajectory follows the curvature of space-time. The trajectory that will be followed by the particle is called a geodesic.



Space coordinates

An Euclidean coordinate system is chosen; either a Cartesian t, x, y, x or a polar t, r, theta, phi system as in (1). The trajectory that is followed by the particle over the geodesic is a function of t, r, theta, and phi. The manner, in which the trajectory is depending on the coordinates, is expressed by coefficients with each coordinate. The coefficients are functions of the coordinate variables, but in this equation limited to *r* and *theta*. They are independent of *t* and *phi*. The equation (1) is spatially symmetric with respect to the origin (M) and thus rotation of the system will lead to the same result.
The coordinate system is a hypothetical system where each coordinate is expressed in units, as if the system is in a space-time completely free of any gravitational influences whatsoever. Schwarzschild now derived a formula that expressed the relationship between the trajectory, *ds*, (in this case space-time path along the time coordinate) and the coordinate system. The geodesic, which is a curved line, is considered as build up out of an infinite number of infinitesimal rectilinear line segments (*ds*). The space-time is curved because of the mass M, but in order to work with an Euclidean coordinate system, the area, build up with *dt, dr, dtheta* and *dphi,* is considered as being infinitesimal small so that the coordinate system is rectilinear and mutually orthogonal in that small area; furthermore the coefficients are considered to be constant in that area. By moving to the next location the same goes, but with slightly different coefficients due to the change of *r* and *theta.* Thus by integrating *ds* the total geodesic trajectory of the particle could be derived.

$$ds^2 = c^2 d\tau^2 = \left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta^2\, d\emptyset^2 \tag{1}$$

$$ds^2 = c^2 d\tau^2 = \sigma^2 c^2 dt^2 - \sigma^{-2} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta^2\, d\emptyset^2 \tag{1a}$$

The actual time is $d\tau$, the *proper time* (is the time measured), elapsed on clocks traveling with the object. The time *dt* is the time in a massless area. This time *dt* is a theoretical time, that cannot be measured but calculated back from the equation. The coordinate time at the location *r* is $\Delta time = \sigma^2 c^2 dt$. The distance covered in time $\Delta time = \sigma^2 c^2 dt$ is:

$$\Delta distance = \sqrt{\sigma^{-2} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta^2\, d\emptyset^2}$$

Thus the velocity of the particle in the frame is:

$$v^2/c^2 = \left(\frac{\Delta distance}{\Delta time}\right)^2 = \frac{(\sigma^{-2}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta^2\, d\emptyset^2)}{\sigma^2 c^2 dt^2}$$

$$ds^2 = c^2 d\tau^2 = \sigma^2 c^2 dt^2 - \frac{\sigma^{-2}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta^2\, d\emptyset^2}{\sigma^2 c^2 dt^2}\sigma^2 c^2 dt^2 \qquad (2)$$

$$c^2 d\tau^2 = \sigma^2 c^2 dt^2 \left(1 - \frac{\sigma^{-4}}{c^2}\left(\frac{dr}{dt}\right)^2 - \frac{\sigma^{-2}r^2}{c^2}\left(\frac{d\theta}{dt}\right)^2 - \frac{\sigma^{-2}r^2 \sin^2\theta^2}{c^2}\left(\frac{d\emptyset}{dt}\right)^2\right) = \sigma^2 c^2 dt^2 \left(1 - \frac{v^2}{c^2}\right) \qquad (3)$$

$$v^2 = \sigma^{-4}\left(\frac{dr}{dt}\right)^2 + \sigma^{-2}r^2\left(\frac{d\theta}{dt}\right)^2 + \sigma^{-2}r^2 \sin^2\theta^2\left(\frac{d\emptyset}{dt}\right)^2 \qquad (3a)$$

$$d\tau = \frac{\sigma}{\gamma}dt \qquad with \qquad \sigma = \left(1 - \frac{2GM}{c^2 r}\right)^{1/2} \quad and\ \gamma = \sqrt{1 - \frac{v^2}{c^2}} \qquad (4)$$

$$\Rightarrow d\tau \le dt \qquad (5)$$

As σ and γ are independent on t then $\boldsymbol{\tau} = \frac{\boldsymbol{\sigma}}{\boldsymbol{\gamma}}\boldsymbol{t}$

In case the particle is a photon $d\tau = 0$:

$$0 = \sigma^2 c^2 dt^2 - \sigma^{-2}dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta^2\, d\emptyset^2 \qquad (6)$$

So path in space is:

$$\Delta distance = \sigma^{-2}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta^2\, d\emptyset^2 \qquad (6b)$$

Thus:

$$c^2 = \frac{\Delta distance}{\Delta time} = \frac{\sigma^{-2}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta^2\, d\emptyset^2}{\sigma^2 dt^2} \qquad (6c)$$

We see here that from the relation between photon and the frame, with M in the origin that the total distance divided by the total time is the light velocity. In the numerator we find the "normal" distance but in the denominator the time is multiplied with sigma, which means a smaller time. Or we consider:

$$c^2 = \frac{\Delta distance}{\Delta time} = \frac{\sigma^{-2}(\sigma^{-2}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta^2\, d\emptyset^2)}{dt^2} \qquad (6d)$$

In this case the total distance is multiplied with σ^{-2} which leads to a greater distance divide by the "normal" time. Now we look at the quotient of the "normal" distance and the "normal" time which leads to a smaller light velocity

$$\sigma^2 c^2 = \frac{\sigma^{-2}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta^2\, d\emptyset^2}{dt^2} = v^2 \qquad (6e)$$

So in the universal frame the light velocity is less than *c*.
The explanation is that due to the curved space the distance between two points is a curve over which the photon passes with the light velocity *c*. So the time over the travelled path is $t = \frac{path}{c}$. Considered from the universal frame the distance between the two points is a straight line thus the velocity of light between the two points $v = \frac{distance}{\frac{path}{c}} =$

$\frac{distance}{path}c$. As the distance is shorter than the path, v is smaller than the light velocity. So the practical light velocity in curved space diminishes.

So from the **Schwarzschild equation** we find:

$$light\ velocity = \sigma^2 c^2 = \frac{\sigma^{-2} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta^2\, d\emptyset^2}{dt^2}$$

Considering now the "**repaired" Schwarzschild** equation:

$$ds^2 = \frac{c^2 dt^2}{\sigma^2} - \frac{dx_1{}^2}{\sigma^2} - \frac{dx_2{}^2}{\sigma^2} - \frac{dx_3{}^2}{\sigma^2}$$

For the photon goes:

$$\frac{c^2 dt^2}{\sigma^2} = \frac{dx_1{}^2}{\sigma^2} + \frac{dx_2{}^2}{\sigma^2} + \frac{dx_3{}^2}{\sigma^2}$$

$$light\ velocity = c^2 = \frac{dx_1{}^2 + dx_2{}^2 + dx_3{}^2}{dt^2}$$

Thus the light velocity from the repaired Schwarzschild is always equal to c independent of the size of mass M.

Or the "**repaired" Schwarzschild** equation 2:

$$c^2 dt_0{}^2 = c^2 \sigma^2 dt_\infty{}^2 - \frac{dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta^2\, d\emptyset^2}{\sigma^2}$$

For the photon goes:

$$c^2 \sigma^2 dt_\infty{}^2 = \frac{dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta^2\, d\emptyset^2}{\sigma^2}$$

$$light\ velocity = \sigma^4 c^2 = \frac{dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta^2\, d\emptyset^2}{dt_\infty{}^2}$$

The velocity of the photon, in the frame of the photon, is zero, and the time $\tau = 0$. From the relation between the photon and the frame, with M in the origin, the velocity of the photon is c. The relation with the coordinates and coefficients are as follows:

$$v^2 = c^2 = \sigma^{-2}\left[\sigma^{-2}\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 + r^2 \sin^2\theta^2 \left(\frac{d\emptyset}{dt}\right)^2\right] \qquad (3b)$$

In case $d\tau = d\theta = d\emptyset = 0$:

$$\sigma^4 c^2 dt^2 = dr^2 \quad so \quad \frac{\sigma^2 dr}{dt} = c \qquad (7)$$

In case of a circle at the equator $d\tau = dr = d\theta = 0$:

$$v = c = \frac{r}{\sigma}\frac{d\emptyset}{dt}$$

Another interesting point is where $r = \infty\ then\ \sigma = 1$ and consequently: $\boldsymbol{d\tau = dt}$. (as mentioned above t is a chosen coordinate as if there is no mass.)

In general at infinity the movement is rectilinear and uniform and the equation becomes:

$$ds^2 = c^2 d\tau^2 = c^2 dt^2 - dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta^2\, d\emptyset^2 \qquad (8)$$

The original approach of Schwarzschild was in Cartesian coordinates. The derivation of the equation resulted in the equation (1) in polar coordinates but this could also be transformed to the original Cartesian coordinates as follows:

$$ds^2 = c^2 d\tau^2 = \sigma^2 c^2 dt^2 - (dx^2 + dy^2 + dz^2) - \frac{1-\sigma^2}{\sigma^2 r^2}(xdx + ydy + zdz)^2 \tag{9}$$

Remark:

The last term on the right hand side is sometimes expressed in a differentiation to τ (differentiation to the local clock) and sometimes to t (differentiation to the universal clock), this could be confusing.

$$ds^2 = c^2 d\tau^2 = \sigma^2 c^2 dt^2 - \sigma^{-2} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta^2\, d\emptyset^2 \tag{1a}$$

Assume $\theta = \pi/2$

$$1 = \sigma^2 \left(\frac{dt}{d\tau}\right)^2 - \sigma^{-2}\left(\frac{dr}{cd\tau}\right)^2 - r^2\left(\frac{d\emptyset}{cd\tau}\right)^2 \tag{10}$$

Or

$$1 = \sigma^2 \left(\frac{dt}{d\tau}\right)^2 \left(1 - \frac{1}{\sigma^4}\left(\frac{dr}{cdt}\right)^2 - \frac{r^2}{\sigma^2}\left(\frac{d\emptyset}{cdt}\right)^2\right) \tag{11}$$

So in the calculation above *v* is the velocity in the universal frame.
If we consider the velocity with respect to the co-located clock $d\tau$ the velocity is:

$${v_{co}}^2 = \frac{\sigma^{-2} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta^2\, d\emptyset^2}{d\tau^2}$$

$$c^2 d\tau^2 = \sigma^2 c^2 dt^2 - \frac{\sigma^{-2} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta^2\, d\emptyset^2}{d\tau^2} d\tau^2 = \sigma^2 c^2 dt^2 - {v_{co}}^2 d\tau^2$$

$$c^2 d\tau^2 + {v_{co}}^2 d\tau^2 = \sigma^2 c^2 dt^2$$

Approximation:

$$d\tau^2 = \frac{\sigma^2}{1 + \left(\frac{v_{co}}{c}\right)^2} dt^2 \approx \sigma^2 \left(1 - \left(\frac{v_{co}}{c}\right)^2\right) dt^2$$

$$d\tau \approx \sigma \sqrt{1 - \left(\frac{v_{co}}{c}\right)^2}\, dt = \; d\tau = \frac{\sigma}{\gamma_{co}} dt$$

Thus for the approximation the result is the same.

In case the trajectory is a circle with $\theta = \pi/2$ then $dr = 0$ and the equation becomes:

$$c^2 d\tau^2 = \sigma^2 c^2 dt^2 - r^2 d\emptyset^2$$

Additional deliberations:

Perhaps we should consider *ds* as an infinitesimal line segment, in space-time, with a size in meters which is measured by the travelling time of a photon over the length of the line segment multiplied with the light velocity. The line segment stays in the origin of its own frame. So the only measurement is time. In this case the line segment ds can be denoted as $ds = cd\tau$. Next we define another frame with an origin, in the Schwarzschild case, in the centre of a mass M. In this frame the distance between the line segment and the origin can be determined by various methods; lasers, rods etc. The only way we can determine the time is by the same clock as the line segment is measured. Thus the first result is: we have the $ds = cd\tau$ (left hand side of the Schwarzschild equation) and we have the distance (in the right hand side of the equation). So considering the Schwarzschild equation, the time part in the new frame is $(c\Delta T)^2 = \left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 =$

$c^2d\tau^2-(\Delta X)^2$ and $c^2dt^2=\frac{(c\Delta T)^2}{\left(1-\frac{2GM}{c^2r}\right)}$. Thus ΔT and dt can only be derived, via the relation in the Schwarzschild equation, but not be measured.

Addition 2

We consider a particle in a co-moving frame, thus the particle is at rest in this frame. The only path, in space-time, is along its t axis. We can express the movement of the particle with respect to another frame, which can be moving with respect to the particle. So the particle can be expressed in t,x,y,z of the new frame. The coordinates t,x,y,z are totally depending on the behavior of the particle so the world-line is naturally a function of τ.

6.1 Summary of Schwarzschild: "On the Gravitational Field of a Mass Point According to Einstein's Theory.

Schwarzschild aim was to find an equation that satisfies Einstein's field equations in vacuum. The equation depicts a point that moves along a geodesic line in a manifold characterized by the line element *ds*.
The conditions that must to be fulfilled as well are:

1. All the components are independent of the time x_4.
2. The equations $g_{\rho 4}=g_{4\rho}=0$ hold exactly for $\rho=1{,}2{,}3$.
3. The solution is spatially symmetric with respect to the origin of the co-ordinate system in the sense that one finds again the same solution when x_1, x_2, x_3 are subjected to an orthogonal transformation (rotation).
4. The $g_{\mu\nu}$ vanish at infinity with the exception of the following limits different from zero:

$$g_{44}=1,\ g_{11}=g_{22}=g_{33}=-1$$

The initial equation was based on rectangular co-ordinates:

$$ds^2=Fdt^2-G(dx^2+dy^2+dz^2)-H(xdx+ydy+zdz)$$

Now he goes over to polar co-ordinates according to $x=r\sin\vartheta\cos\varphi,\ y=r\sin\vartheta\sin\varphi,\ \mathrm{r}\cos\vartheta$, the same element reads:

$$\begin{aligned} ds^2&=Fdt^2-G(dr^2+rd\vartheta^2+r^2\sin\vartheta^2\,d\varphi^2)-Hr^2dr^2\\ &=Fdt^2-(G+Hr^2)dr^2-Gr^2(d\vartheta^2+\sin\vartheta^2\,d\varphi^2)\end{aligned}$$

As the determinant of the metric is unequal to -1 a following transformation is done:
With new variables and polar co-ordinates with determinant 1:

$$ds^2=Fdt^2-\left(\frac{G}{r^4}+\frac{H}{r^2}\right)dx_1{}^2-Gr^2\left[\frac{dx_2{}^2}{1-x_2{}^2}+dx_3{}^2(1-x_2{}^2)\right]$$

Via the Einstein field equations the coefficients are found and results in the following, mostly used, Schwarzschild equation:

$$ds^2=\boldsymbol{c^2d\tau^2}=\left(\mathbf{1}-\frac{\boldsymbol{2GM}}{\boldsymbol{c^2r}}\right)\boldsymbol{c^2dt^2}-\frac{\mathbf{1}}{\left(\mathbf{1}-\frac{\boldsymbol{2GM}}{\boldsymbol{c^2r}}\right)}\boldsymbol{dr^2}-\boldsymbol{r^2d\theta^2}-\boldsymbol{r^2\sin^2\theta^2\,d\emptyset^2} \tag{8}$$

The original approach of Schwarzschild was in Cartesian coordinates. The derivation of the equation resulted in the equation (8) in polar coordinates but this could also be transformed to the original Cartesian coordinates as follows:

$$ds^2=c^2d\tau^2=\left(1-\frac{2GM}{c^2r}\right)c^2dt^2-(dx^2+dy^2+dz^2)-\frac{\frac{2GM}{c^2r}}{\left(1-\frac{2GM}{c^2r}\right)r^2}(xdx+ydy+zdz)^2 \tag{9}$$

However this form is hardly used.

7 Calculation of Hafele & Keating experiment with the Schwarzschild equation.

Derivation based on: A Hafele & Keating like thought experiment, by Paul B. Andersen, date: October 16, 2008 (Anderson, 2008)
https://paulba.no/pdf/H&K_like.pdf

The experiment tries to calculate the time behavior of a cesium clock on various locations and speed with respect Earth. These clocks experience the influence due to the Earth gravity and the speed with respect to Earth.
We will first derive a formula from the Schwarzschild equation based on some approximations. After that we will try an exact solution which obviously will be more complicated but via computer programs like Excel the execution should be easy and the result exact.
The Hafele & Keating experiment exists of two airplanes, both with a cesium clock, and a cesium clock placed on Earth location. The airplanes fly with a constant speed, one to the East and one to the West.

The applicability of the Schwarzschild equation will be scrutinized.

$$c^2 d\tau^2 = \left(1-\frac{2GM}{c^2 r}\right)c^2 dt^2 - \left(1-\frac{2GM}{c^2 r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta^2\, d\emptyset^2 \tag{1}$$

This is a universal frame with the centre of the Earth as the origin. The coordinates are $t, r, \theta, \emptyset$. The Earth is rotating within this frame. The distance to the Earth centre is denoted by *r*. *Theta* is the angle with the North pole and *phi* is the angle with the prime meridian (of the universal frame). $rd\theta$ is an arc length of r meter, thus if *r=1* then $d\theta = 1$ meter. Same goes for $rd\emptyset$. Next *dt* is a small change of *t* when measured in a region free of gravitational influences. Thus *t* is a hypothetical time which is not measured by a clock; it is pure theoretical. The time measured on location *r* is $d\tau$ of the co-located clock.

7.1 First the approximated approach

We assume that the clocks circle around the Earth either at the surface level or at certain heights above the surface of the Earth. Thus for each clock, on a circle, goes that $dr = 0$. Furthermore one assumes the trajectory of the clocks being in the plane of the equator which means $\theta = \pi/2$, so constant and thus $d\theta = 0$.

$$c^2 d\tau^2 = \left(1-\frac{2GM}{c^2 r}\right)c^2 dt^2 - r^2 d\emptyset^2 \tag{2}$$

$$d\tau^2 = \left(\left(1-\frac{2GM}{c^2 r}\right) - \frac{r^2}{c^2}\left(\frac{d\emptyset}{dt}\right)^2\right)dt^2 \tag{3}$$

$$d\tau = \sqrt{\left(1-\frac{2GM}{c^2 r}-\frac{v^2}{c^2}\right)}\,dt \tag{4}$$

Approximation with Taylor

$$d\tau = \left(1-\frac{GM}{c^2 r}-\frac{v^2}{2c^2}\right)dt \tag{5}$$

As *r* and *v* are constant the integration is simple:

$$\tau = \left(1-\frac{GM}{c^2 r}-\frac{v^2}{2c^2}\right)t + \tau(0) \tag{6}$$

The interesting thing here is to compare the proper time of each clock. As a reference we take the proper time of the clock located on surface of the Earth. The other clocks are each located on different airplanes. So each clock has a speed and different location *r* even the clock at the Earth surface has the speed (v_1) of the Earth's rotation.

$$d\tau_1 = \left(1 - \frac{GM}{c^2 r_1} - \frac{{v_1}^2}{2c^2}\right) dt \tag{7}$$

$$d\tau_2 = \left(1 - \frac{GM}{c^2 r_2} - \frac{{v_2}^2}{2c^2}\right) dt \tag{8}$$

$$d\tau_2 = \frac{\left(1 - \frac{GM}{c^2 r_2} - \frac{{v_2}^2}{2c^2}\right)}{\left(1 - \frac{GM}{c^2 r_1} - \frac{{v_1}^2}{2c^2}\right)} d\tau_1 \cong \left(1 + \frac{GM}{c^2}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) + \frac{{v_1}^2 - {v_2}^2}{2c^2}\right) d\tau_1 \tag{9}$$

If we assume that $\tau_2 = 0$ when $\tau_1 = 0$ then the integrating constant is zero:

$$\tau_2 \cong \left(1 + \frac{GM}{c^2}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) + \frac{{v_1}^2 - {v_2}^2}{2c^2}\right) \tau_1 \tag{10}$$

Thus the difference between the proper times of two clocks will be:

$$\tau_2 - \tau_1 = \left(\frac{GM}{c^2}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) + \frac{{v_1}^2 - {v_2}^2}{2c^2}\right) \tau_1 \tag{11}$$

Let us assume that τ_1 is the proper time of the clock which is located at the surface of the Earth, then $r_1 = R$; the radius of the Earth. The distance of the clock τ_2 in an airplane is then $R + h$:

$$\tau_2 - \tau_1 = \left(\frac{GM}{c^2}\left(\frac{1}{R} - \frac{1}{R+h}\right) + \frac{{v_1}^2 - {v_2}^2}{2c^2}\right) \tau_1 \tag{12}$$

Assume $\frac{h}{R} \ll 1$ and the gravitational acceleration is $g = \frac{GM}{R^2}$ then:

$$\boldsymbol{\tau_2 - \tau_1 = \left(\frac{gh}{c^2} + \frac{{v_1}^2 - {v_2}^2}{2c^2}\right) \tau_1} \tag{13}$$

As v_2 is the speed of a plane (to the East) and v_1 the (rotation) velocity of the Earth point (to the East) with respect to our frame, it is probably more practical to measure the ground velocities of the planes with respect to the Earth point velocity. So lets $v_1 = v_{earth}$ and $v_2 = v_{plane} + v_{earth}$ thus:

$${v_1}^2 - {v_2}^2 = {v_{earth}}^2 - \left(v_{plane} + v_{earth}\right)^2 = {v_{earth}}^2 - {v_{plane}}^2 - 2v_{earth}v_{plane} - {v_{earth}}^2$$

$${v_1}^2 - {v_2}^2 = -{v_{plane}}^2 - 2v_{earth}v_{plane} = -v_{plane}\left(v_{plane} + 2v_{earth}\right)$$

Formula used by Hafele & Keating:

$$\boldsymbol{\tau_{plane} - \tau_{earth} = \left(\frac{gh}{c^2} - \frac{v_{plane}\left(v_{plane} + 2v_{earth}\right)}{2c^2}\right) \tau_{earth}} \tag{14}$$

Thus this equation is completely derived from the Schwarzschild equation with some approximations.

Note1:

If the speed of the plane is the ground-speed then, as approximation, at level *h,* $v_2 = \frac{R+h}{R}(v_{plane} + v_{earth})$ the formula (13) should be used with the adapted v_2.

Note2:

According to me it is a better approach to use v_1 and v_2 as elucidated in the chapter below.

7.2 Elaboration on v_1 and v_2 in equation (13)

The speed, v_1 mentioned in equation (13), is the speed of a stationary point at the equator at the surface of the Earth. This speed $v_1 = r_1 \frac{d\emptyset}{dt}$, as mentioned in equation (3), is related to dt , however the measurement is done with respect to the proper time. So a conversion has to be made. Thus the relation between the velocity in the universal frame and the velocity, related to the proper time at level r_1:

$$v_{1t} = r_1 \frac{d\emptyset}{dt} = r_1 \frac{d\emptyset}{d\tau}\frac{d\tau}{dt} = v_{1\tau}\frac{d\tau}{dt}$$

Because *t* is the time in the universal frame, the $\frac{d\emptyset}{dt}$ is the same for each distance *r,* but the velocity at each level is determined by *r* so $v_t = r\frac{d\emptyset}{dt}$

Next we calculate v_{1t} at the surface of the Earth.

$$c^2 d\tau^2 = \left(1 - \frac{2GM}{c^2 r_1}\right) c^2 dt^2 - {r_1}^2 \left(\frac{d\emptyset}{d\tau}\right)^2 d\tau^2$$

$$\left(1 + \frac{{r_1}^2}{c^2}\left(\frac{d\emptyset}{d\tau}\right)^2\right) d\tau^2 = \left(1 - \frac{2GM}{c^2 r_1}\right) dt^2$$

$$\left(1 + \frac{{v_{1\tau}}^2}{c^2}\right) d\tau^2 = \left(1 - \frac{2GM}{c^2 r_1}\right) dt^2 = {\sigma_1}^2 dt^2$$

$$\left(\frac{d\tau}{dt}\right)^2 = \frac{{\sigma_1}^2}{\left(1 + \frac{{v_{1\tau}}^2}{c^2}\right)}$$

So the conversion between the velocity, in the same level, with respect to the time of the universal frame and the proper time of that level is:

$${v_{1t}}^2 = {v_{1\tau}}^2 \left(\frac{d\tau}{dt}\right)^2 = {v_{1\tau}}^2 \frac{{\sigma_1}^2}{\left(1 + \frac{{v_{1\tau}}^2}{c^2}\right)}$$

So $\frac{d\tau}{dt}$ is determined by $v_{1\tau}$ the rotation of the Earth. So in case we consider $v_{1\tau\_plane} = v_{plane\_\tau} + v_{1\tau\_earth}$ it is still in the Earth level so $\frac{d\tau}{dt}$ stays the same:

$$v_{1\tau\_plane} = v_{plane\_\tau} + v_{1\tau\_earth} = r_1 \frac{d\emptyset}{d\tau}$$

$$r_1\frac{d\emptyset}{dt} = r_1\frac{d\emptyset}{d\tau}\frac{d\tau}{dt} = (v_{plane\_\tau} + v_{1\tau\_earth})\frac{d\tau}{dt} = (v_{plane\_\tau} + v_{1\tau\_earth})\frac{\sigma_1}{\sqrt{1+\frac{v_{1\tau\_earth}{}^2}{c^2}}}$$

$$\frac{d\emptyset}{dt} = (v_{plane\_\tau} + v_{1\tau\_earth})\frac{\sigma_1}{r_1\sqrt{1+\frac{v_{1\tau\_earth}{}^2}{c^2}}}$$

Here we have calculated the rotation speed in the universal frame. This is valid for each level, distance from the centre, but the velocity is determined by *r* times this rotation speed.

v_{plane_τ} is the measured speed of the plane on ground level and with respect to the proper time, which is the only time available at that level. v_{earth_τ} is the (rotating) speed of a stationary point on Earth with respect to the universal frame but measured with the proper time on Earth level.
Now we make the conversion to the level of the plane:

$$v_{2t} = r_2\frac{d\emptyset}{dt} = \frac{r_2}{r_1}\frac{\sigma_1(v_{plane\_\tau} + v_{1\tau\_earth})}{\sqrt{1+\frac{v_{1\tau\_earth}{}^2}{c^2}}}$$

Thus the velocity of the plane at level *2* can be considered as build up out of $v_{2t} = v_{2t\_earth} + v_{2t\_plane}$:

$$v_{2t\_earth} = \frac{r_2}{r_1}\frac{\sigma_1 v_{1\tau\_earth}}{\sqrt{1+\frac{v_{1\tau\_earth}{}^2}{c^2}}}$$

And

$$v_{2t\_plane} = v_{2t} - v_{2t\_earth} = \frac{r_2}{r_1}\frac{\sigma_1 v_{plane\_\tau}}{\sqrt{1+\frac{v_{1\tau\_earth}{}^2}{c^2}}}$$

Thus to summarize the result:

Conversion between $\boldsymbol{t}$ and $\boldsymbol{\tau}$ at the same level (in order to use the input data based on τ) :

$$\boldsymbol{v_{1t\_earth} = v_{1\tau\_earth}\frac{\sigma_{earth}}{\sqrt{1+\frac{v_{1\tau\_earth}{}^2}{c^2}}}} \tag{14}$$

Calculation of $\boldsymbol{v_{2t}}$ used in the formula (13), based on the plane velocity on ground level and time τ (which is the input data) and subsequently converted to plane level.

$$\boldsymbol{v_{2t} = \frac{r_2}{r_1}\frac{\sigma_{earth}(v_{plane_\tau} + v_{earth_\tau})}{\sqrt{1+\frac{v_{1\tau_{earth}}{}^2}{c^2}}}} \tag{15}$$

$$\tau_2 - \tau_1 = \left(\frac{gh}{c^2} + \frac{v_1{}^2 - v_2{}^2}{2c^2}\right)\tau_1$$

Thus formula (13) becomes:

$$\boldsymbol{\tau_2 - \tau_1 = \left(\frac{gh}{c^2} + \frac{\sigma_{earth}{}^2}{\left(1 + \frac{v_{1\tau_{earth}}{}^2}{c^2}\right)} \frac{\left[v_{1\tau\_earth}{}^2 - \left(\frac{R+h}{R}\right)^2 \left(v_{plane_\tau} + v_{earth_\tau}\right)^2 \right]}{2c^2} \right) \tau_1} \tag{16}$$

7.3 The exact derivation

Let us start from formula (4)

$$d\tau = \sqrt{\left(1 - \frac{2GM}{c^2 r} - \frac{v^2}{c^2}\right)} dt \tag{4}$$

As *r* and *v* are constant the integration is simple:

$$\tau = \sqrt{\left(1 - \frac{2GM}{c^2 r} - \frac{v^2}{c^2}\right)} t + \tau(0) \tag{6a}$$

The interesting thing here is to compare the proper time of each clock. As a reference we take the proper time of the clock located on surface of the Earth. The other clocks are each located on a different airplane. So each clock has a speed and different location *r* even the clock at the earth surface has the speed of the Earth's rotation.

$$d\tau_1 = \sqrt{\left(1 - \frac{2GM}{c^2 r_1} - \frac{v_1{}^2}{c^2}\right)} dt \tag{7a}$$

$$d\tau_2 = \sqrt{\left(1 - \frac{2GM}{c^2 r_2} - \frac{v_2{}^2}{c^2}\right)} dt \tag{8a}$$

$$d\tau_2 = \sqrt{\frac{\left(1 - \frac{2GM}{c^2 r_2} - \frac{v_2{}^2}{c^2}\right)}{\left(1 - \frac{2GM}{c^2 r_1} - \frac{v_1{}^2}{c^2}\right)}} d\tau_1 \tag{9a}$$

If we assume that $\tau_2 = 0 \; when \; \tau_1 = 0$ then the integrating constant is zero:

$$\tau_2 = \sqrt{\frac{\left(1 - \frac{2GM}{c^2 r_2} - \frac{v_2{}^2}{c^2}\right)}{\left(1 - \frac{2GM}{c^2 r_1} - \frac{v_1{}^2}{c^2}\right)}} \tau_1 \tag{10a}$$

Thus the difference between the proper times of two clocks will be:

$$\tau_2 - \tau_1 = \left(\sqrt{\frac{\left(1 - \frac{2GM}{c^2 r_2} - \frac{v_2{}^2}{c^2}\right)}{\left(1 - \frac{2GM}{c^2 r_1} - \frac{v_1{}^2}{c^2}\right)}} - 1 \right) \tau_1 \tag{11a}$$

Let us assume that τ_1 is the proper time of the clok which is located at the surface of the Earth, then $r_1 = R$; the radius of the Earth. The distance of the clock τ_2 in a plane is then R *+h:*

$$\tau_2 - \tau_1 = \left(\sqrt{\frac{\left(1 - \frac{2GM}{c^2(R+h)} - \frac{{v_2}^2}{c^2}\right)}{\left(1 - \frac{2GM}{c^2R} - \frac{{v_1}^2}{c^2}\right)}} - 1 \right) \tau_1 \tag{12a}$$

As the v_2 is the speed of a plane (to the East with respect to the universal frame) and v_1 the (rotation) velocity of the Earth point (to the East) with respect to the universal frame. v_1 and v_2 are derived in 7.2.14 and 7.2.15.

$$\boldsymbol{\tau_{plane} - \tau_{earth} = \left(\sqrt{\frac{\left(1 - \frac{2GM}{c^2(R+h)} - \frac{{v_2}^2}{c^2}\right)}{\left(1 - \frac{2GM}{c^2R} - \frac{{v_1}^2}{c^2}\right)}} - 1 \right) \tau_{earth}} \tag{14a}$$

Or with the Schwarzschild radius is $R_s = \frac{2GM}{c^2}$:

$$\boldsymbol{\tau_{plane} - \tau_{earth} = \left(\sqrt{\frac{\left(1 - \frac{R_s}{(R+h)} - \frac{{v_2}^2}{c^2}\right)}{\left(1 - \frac{R_s}{R} - \frac{{v_1}^2}{c^2}\right)}} - 1 \right) \tau_{earth}} \tag{15}$$

Thus this equation is completely derived from the Schwarzschild equation and is exact.

Some calculations based on executed experiments:

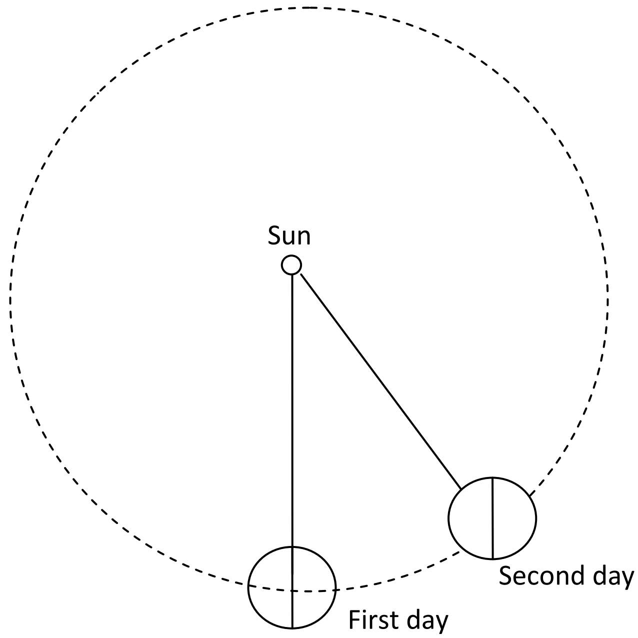
	PaulAnderson	*Re_Spec_92*	*H&K*
Vplane_ground_east_tau	232.55	670	173.98
Vplane_ground_West_tau	-232.55	-670	-124.43
Vplane2_east in dt	232.88	672.00	174.19
Vplane2_west in dt	-232.88	-672.00	-124.62
V_earth_tau	464.58	464.58	464.58
V_earth_t	464.58	464.58	464.58
V_earth_east on plane level dt	465.24	465.97	465.14
V_earth_west on plane level dt	465.24	465.97	465.28
H_east	9000	19000	7664
H_west	9000	19000	9526
t_earth	172328	59746.528	172328
Result (formula 7.1.13):			
Grav_delay (ns)_East	169.46	124.03	144.31
Kin_delay (ns)_East	-260.32	-358.69	-184.94
Total_East	**-9.09E-08**	**-2.35E-07**	**-4.06E-08**
Grav_delay (ns)_West	169.46	124.03	179.37
Kin_delay (ns)_West	155.16	57.63	95.67
Total_West	**3.25E-07**	**1.82E-07**	**2.75E-07**
Exact (Formula: 7.3.15):			
Total_East (ns)	**-9.11E-08**	**-2.35E-07**	**-4.08E-08**
Total_West	**3.24E-07**	**1.81E-07**	**2.75E-07**
diff east	**2.35E-10**	**3.63E-10**	**1.56E-10**
diff west	**2.18E-10**	**3.67E-10**	**2.58E-10**
diff east in %	-0.26%	-0.15%	-0.38%
diff west in %	0.07%	0.20%	0.09%
sidereal day: 23.9344696hr	86164.1	86164.1	86164.1
Lightvelocity	299792458	299792458	299792458
G	6.67E-11	6.67E-11	6.67E-11
M_earth	5.97E+24	5.97E+24	5.97E+24
R_earth	6371000	6371000	6371000
Schwarzschild radius Rs:	8.87E-03	8.87E-03	8.87E-03

Conclusion:
The approximations are correct within less than 0.4%

7.4 Calculation of the velocity of a stationary point at the equator on Earth surface.

Time of Earth rotation (sidereal day):

The time length of a day is the difference in time between two successive highest points of the sun in the sky. This time difference is 24 hours. However because of the orbit around the sun the time of a rotation of the Earth around its own axis, is less than the time of a day. This is shown in the picture below. When the vertical line on the Earth rotates and is back in the same vertical direction then that is the time of an Earth rotation and the time is called sidereal day. A sidereal day is 23.9344696 hours (86164.1 sec). With R_{earth}=6371 km this gives a velocity of the stationary Earth clock of $v_{earth} = \frac{2\pi R_{earth}}{86164.1} = 464.58$ m/s



7.5 Correction on derivation based on Paul Anderson (above)

One of the input data is the speed of the plane with respect to the ground. In the formula 7.1.3 the speed, in the formula of Anderson, is based on *dt*, however the clock in that frame is $d\tau$ so the speed of the plane is also related to the co-moving clock $d\tau$ Thus we should adjust the formula. $d\tau$ is the proper time elapsed on clocks traveling with the object.

Let us start with formula 7.1.2:

$$c^2 d\tau^2 = \left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 - r^2 d\emptyset^2 \tag{2}$$

$$d\tau^2 = \left(1 - \frac{2GM}{c^2 r}\right) dt^2 - \frac{r^2}{c^2}\left(\frac{d\emptyset}{d\tau}\right)^2 d\tau^2 \tag{3b}$$

$$d\tau^2 \left[1 + \frac{r^2}{c^2}\left(\frac{d\emptyset}{d\tau}\right)^2\right] = \left(1 - \frac{2GM}{c^2 r}\right) dt^2 \tag{3b}$$

$$d\tau = \sqrt{\frac{\left(1 - \frac{2GM}{c^2 r}\right)}{1 + \frac{r^2}{c^2}\left(\frac{d\emptyset}{d\tau}\right)^2}}\, dt \tag{4b}$$

$$v_\tau = r\frac{d\emptyset}{d\tau} \tag{4c}$$

$$d\tau_1 = \sqrt{\frac{\left(1 - \frac{2GM}{c^2 r_1}\right)}{1 + \frac{{v_1}^2}{c^2}}}\,dt \tag{7b}$$

$$d\tau_2 = \sqrt{\frac{\left(1 - \frac{2GM}{c^2 r_2}\right)}{1 + \frac{{v_2}^2}{c^2}}}\,dt \tag{8b}$$

$$d\tau_2 = \sqrt{\frac{\left(1 - \frac{2GM}{c^2 r_2}\right)\left(1 + \frac{{v_1}^2}{c^2}\right)}{\left(1 - \frac{2GM}{c^2 r_1}\right)\left(1 + \frac{{v_2}^2}{c^2}\right)}}\,d\tau_1 \tag{9b}$$

$$\tau_2 = \sqrt{\frac{\left(1 - \frac{2GM}{c^2 r_2}\right)\left(1 + \frac{{v_1}^2}{c^2}\right)}{\left(1 - \frac{2GM}{c^2 r_1}\right)\left(1 + \frac{{v_2}^2}{c^2}\right)}}.\tau_1 \tag{10b}$$

$$\boldsymbol{\tau_{plane}} - \boldsymbol{\tau_{earth}} = \left(\sqrt{\frac{\left(1 - \frac{2GM}{c^2 r_2}\right)\left(1 + \frac{{v_1}^2}{c^2}\right)}{\left(1 - \frac{2GM}{c^2 r_1}\right)\left(1 + \frac{{v_2}^2}{c^2}\right)}} - 1\right)\boldsymbol{\tau_{earth}} \tag{11b}$$

Let us assume that τ_1 is the proper time of the clok which is located at the surface of the Earth, then $r_1 = R$; the radius of the Earth. The distance of the clock in a plane is then *R+h*:

$$\boldsymbol{\tau_{plane}} - \boldsymbol{\tau_{earth}} = \left(\sqrt{\frac{\left(1 - \frac{2GM}{c^2 (R+h)}\right)\left(1 + \frac{{v_{earth}}^2}{c^2}\right)}{\left(1 - \frac{2GM}{c^2 R}\right)\left(1 + \frac{{v_2}^2}{c^2}\right)}} - 1\right)\boldsymbol{\tau_{earth}} \tag{14b}$$

Or with the Schwarzschild radius $R_s = \frac{2GM}{c^2}$:

$$\boldsymbol{\tau_{plane}} - \boldsymbol{\tau_{earth}} = \left(\sqrt{\frac{\left(1 - \frac{R_s}{(R+h)}\right)\left(1 + \frac{{v_{earth}}^2}{c^2}\right)}{\left(1 - \frac{R_s}{R}\right)\left(1 + \frac{{v_2}^2}{c^2}\right)}} - 1\right)\boldsymbol{\tau_{earth}} \tag{15b}$$

The given plane velocity is the velocity relative to the ground point so the actual velocity at level *h* is

$$v_2 = \left(v_{earth} + v_{plane\ relative\ to\ earth\ point}\right).\frac{R+h}{R}$$

After the approximations as previous:

$$\boldsymbol{\tau_{plane}} - \boldsymbol{\tau_{earth}} = \left(\left(1 - \frac{GM}{c^2(R+h)}\right)\left(1 + \frac{GM}{c^2R}\right)\left(1 + \frac{{v_{earth}}^2}{2c^2}\right)\left(1 - \frac{{v_2}^2}{2c^2}\right) - 1\right)\boldsymbol{\tau_{earth}} \tag{16}$$

$$\boldsymbol{\tau_{plane}} - \boldsymbol{\tau_{earth}} = \left(\left(1 + \frac{GM}{c^2}\left(\frac{1}{R} - \frac{1}{R+h}\right)\right)\left(1 + \frac{({v_{earth}}^2 - {v_2}^2)}{2c^2}\right) - 1\right)\boldsymbol{\tau_{earth}} \tag{17}$$

$$\boldsymbol{\tau_{plane}} - \boldsymbol{\tau_{earth}} = \left(\left(1 + \frac{GM}{c^2}\frac{h}{R^2}\right)\left(1 + \frac{({v_{earth}}^2 - {v_2}^2)}{2c^2}\right) - 1\right)\boldsymbol{\tau_{earth}} \tag{18}$$

$$\boldsymbol{\tau_{plane}} - \boldsymbol{\tau_{earth}} = \left(\frac{GM}{c^2}\frac{h}{R^2} + \frac{({v_{earth}}^2 - {v_2}^2)}{2c^2}\right)\boldsymbol{\tau_{earth}} \tag{19}$$

$$\boldsymbol{\tau_{plane}} - \boldsymbol{\tau_{earth}} = \left(\frac{gh}{c^2} + \frac{({v_{earth}}^2 - {v_2}^2)}{2c^2}\right)\boldsymbol{\tau_{earth}} \tag{20}$$

Note:

The speed of the airplane is given as the ground speed of the airplane. It is not obvious if this ground speed is measured with respect to the stationary clock on Earth or the clock in the plane. Let us assume the Earth clock is meant. In that case we have to find a conversion to the airplane level subsequently this involves the clock on airplane level. We will do this via the t in the universal frame. If we consider $\frac{d\emptyset_{earth}}{dt}$ then this is the rotation velocity of the Earth in the universal frame. We can find the speed of the Earth at Earth level by multiplying $\frac{d\emptyset_{earth}}{dt}$ with R; the distance from the origin. The speed of the Earth as seen from the airplane level is $(R+h)\frac{d\emptyset_{earth}}{dt}$. For the plane this is similar, at Earth level the relative plane speed is $R\frac{d\emptyset_{plane}}{dt}$ and at airplane level $(R+h)\frac{d\emptyset_{plane}}{dt}$. Now $\frac{d\emptyset_{earth}}{dt}$ and $\frac{d\emptyset_{plane}}{dt}$ have to be found.

We use (4c)

$$v_\tau = r\frac{d\emptyset}{d\tau} = r\frac{d\emptyset}{dt}\frac{dt}{d\tau} => \frac{d\emptyset}{dt} = \frac{v_\tau}{r}\frac{d\tau}{dt}$$

Next we use (4b)

$$\frac{d\tau}{dt} = \sqrt{\frac{\left(1 - \frac{2GM}{c^2r}\right)}{1 + \frac{{v_\tau}^2}{c^2}}}$$

Thus

$$\frac{d\emptyset}{dt} = \frac{v_\tau}{r}\frac{d\tau}{dt} = \frac{v_\tau}{r}\sqrt{\frac{\left(1 - \frac{2GM}{c^2r}\right)}{1 + \frac{{v_\tau}^2}{c^2}}}$$

All the components at the right hand side are known.

At ground level:

$$\frac{d\emptyset_{earth}}{dt} = \frac{v_{earth}}{R}\sqrt{\frac{\left(1-\frac{2GM}{c^2R}\right)}{1+\frac{{v_{earth}}^2}{c^2}}} \quad and \quad \frac{d\emptyset_{plane}}{dt} = \frac{v_{plane}}{R}\sqrt{\frac{\left(1-\frac{2GM}{c^2R}\right)}{1+\frac{{v_{earth}}^2}{c^2}}}$$

Now at the airplane level:

$$v_2 = v_{2\tau\_earth} + v_{2\tau\_plane} = (R+h)\left(\frac{d\emptyset_{earth}}{dt} + \frac{d\emptyset_{plane}}{dt}\right)$$

$$v_2 = v_{2\tau\_earth} + v_{2\tau\_plane} = \frac{(R+h)}{R}\sqrt{\frac{\left(1-\frac{2GM}{c^2R}\right)}{1+\frac{{v_{earth}}^2}{c^2}}}\left(v_{earth} + v_{plane}\right)$$

With first order Taylor approximation:

$$v_2 = \frac{(R+h)}{R}\sqrt{\left(1-\frac{2GM}{c^2R}\right)\left(1-\frac{{v_{earth}}^2}{c^2}\right)}\left(v_{earth} + v_{plane}\right)$$

So the relevant formulas are

$$\boldsymbol{v_2 = \frac{(R+h)}{R}\left(1-\frac{GM}{c^2R}-\frac{{v_{earth}}^2}{2c^2}\right)\left(v_{earth}+v_{plane}\right)}$$

$$\boldsymbol{\tau_{plane} - \tau_{earth} = \left(\frac{gh}{c^2} + \frac{\left({v_{earth}}^2 - {v_2}^2\right)}{2c^2}\right)\tau_{earth}} \qquad (\mathbf{20})$$

Conclusion:

Although this solution above in (20) seems to me the right approach, after some numeric calculations the difference in results are within 0.4%.

Exact (Formula: 7.3.15):	PaulAnderson	Re_Spec_92	H&K
Total_East	**-9.11E-08**	**-2.35E-07**	**-4.08E-08**
Total_West	**3.24E-07**	**1.81E-07**	**2.75E-07**
sidereal day: 23.9344696hr	86164.1	86164.1	86164.1
Lightvelocity	299792458	299792458	299792458
G	6.67E-11	6.67E-11	6.67E-11
M_earth	5.97E+24	5.97E+24	5.97E+24
R_earth	6371000	6371000	6371000
Schwarzschild radius Rs:	8.87E-03	8.87E-03	8.87E-03
Formula: 7.5.20			
Vplane_ground_east_tau	232.55	670	173.98
Vplane_ground_West_tau	-232.55	-670	-124.43
V_earth_tau	464.58	464.58	464.58
H_east	9000	19000	7664
H_west	9000	19000	9526
t_earth	172328	59747	172328
v2_east	698.12	1137.96	639.33
v2_west	232.36	-206.03	340.56
Grav_delay (ns)_East	1.69E-07	1.24E-07	1.44E-07
Kin_delay (ns)_East	-2.60E-07	-3.59E-07	-1.85E-07
Total_East	**-9.09E-08**	**-2.35E-07**	**-4.06E-08**
Grav_delay (ns)_West	1.69E-07	1.24E-07	1.79E-07
Kin_delay (ns)_West	1.55E-07	5.76E-08	9.57E-08
Total_West	**3.25E-07**	**1.82E-07**	**2.75E-07**
diff east	-2.35E-10	-3.63E-10	-1.56E-10
diff west	-2.18E-10	-3.67E-10	-3.23E-10
diff east in %	0.26%	0.15%	0.38%
diff west in %	-0.07%	-0.20%	-0.12%

8 Deliberations on Hafele & Keating experiment and Schwarzschild equation

$$ds^2 = c^2 d\tau^2 = \left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta^2\, d\emptyset^2 \tag{1}$$

$$ds^2 = c^2 d\tau^2 = \sigma^2 c^2 dt^2 - \sigma^{-2} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta^2\, d\emptyset^2 \tag{1a}$$

In the H & K experiment the time of the Naval Observatory Clock (noc) and the speed of an airplane are mentioned. The question is: what is the time in the Schwarzschild equation and what is the airplane speed in the equation?
There is a stationary clock on ground level on the equator and two airplanes in the equator plane; one flying east and one flying west. The flight velocity with respect to ground is equal but opposite for both airplanes.

As the experiment is in the equator plane, $\theta = \frac{\pi}{2}$ and constant, and both airplanes fly in a circular orbit so $r = constant$ the formula (1) simplifies to:

$$c^2 d\tau^2 = \left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 - r^2 d\emptyset^2 \tag{2}$$

The coordinates in the Schwarzschild equation (1) could be considered as a universal frame, without any gravitation, in the direction of the Earth-North pole. The Earth is rotating in this universal frame. The three clocks are in their own proper frame so their time is depicted by τ.

The time in the universal frame cannot be measured but is pure theoretical and is:

$$dt^2 = \frac{d\tau^2 + \frac{r^2}{c^2} d\emptyset^2}{\left(1 - \frac{2GM}{c^2 r}\right)} = \sigma^{-2}\left(d\tau^2 + \frac{r^2}{c^2} d\emptyset^2\right) = \sigma^{-2}\left(1 + \frac{r^2}{c^2}\left(\frac{d\emptyset}{d\tau}\right)^2\right) d\tau^2 \tag{4}$$

If *t=0* when $\tau = 0$ then the integrating constant is zero and:

$$t = \sigma^{-1}\sqrt{1 + \frac{r^2}{c^2}\left(\frac{d\emptyset}{d\tau}\right)^2}\,\tau = \sigma^{-1}\sqrt{1 + \frac{v^2}{c^2}}\,\tau \tag{4a}$$

9 Antwoord op vraag betreffende Schwarzschild.

De eerste vraag is: Waar komt die formulering van AR na 1916 vandaan, die met die Ricci tensor?

In verschillende literatuur wordt vermeld dat $G_{\mu\nu}$ de Einstein tensor wordt genoemd, maar Einstein hield ervan om dingen zo eenvoudig mogelijk op te schrijven en bedoelde met $G_{\mu\nu}$ niets anders dan het onderstaande.

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$$

Hierin is dus altijd de Ricci tensor gebruikt. Hierbij is de Ricci scalar

$$R = g^{\mu\nu} \mathrm{R}_{\mu\nu} = g^{00}\mathrm{R}_{00} + g^{11}\mathrm{R}_{11} + g^{22}\mathrm{R}_{22} + g^{\mu\nu}\mathrm{R}_{33}$$

Wanneer we nu $G_{\mu\nu}$ met $g^{\mu\nu}$ vermenigvuldigen krijgen we:

$$g^{\mu\nu} G_{\mu\nu} = g^{\mu\nu} R_{\mu\nu} - \frac{1}{2} g^{\mu\nu} g_{\mu\nu} R = R - \frac{1}{2} 4R = -R$$

Als we nu $G_{\mu\nu}$ nul willen hebben in het vacuum, dan kan dat alleen als R en $R_{\mu\nu}$ nul zijn.]

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0$$

Einstein probeerde de kromming van de ruimte wiskundig te beschrijven en ging uit van Riemann die dit al eerder had gedaan voor kromme vlakken. De Riemann tensor is bijvoorbeeld: $R_{\mu\beta\rho\nu}$ Deze is 4 dimensionaal en kan je dus nauwelijks voorstellen. (drie dimensionaal is een kubus met elementen maar vier wordt wat lastiger voor te stellen). Met behulp van de metric tensor kan de covariante Riemann veranderd worden naar een gedeeltelijke contravariante vorm:

$$R^{\beta}{}_{\mu\rho\nu} = g^{\beta\beta} R_{\mu\beta\rho\nu}$$

Dit is nodig om zo de gewenste contractie te kunnen uitvoeren. Door nu $\beta = \rho$ te stellen kan contractie plaats vinden waardoor de Ricci tensor $R_{\mu\nu}$ ontstaat. Hier is de Ricci tensor dus de trace (spoor) van de Riemann tensor en zijn er blijkbaar veel elementen van de Riemann tensor overbodig. Deze stap is voor mij ook niet helemaal duidelijk; dat je ongestraft die elementen kan negeren. Het verband met Riemann zie je nog wel terug in de Ricci tensor elementen en Christoffel symbolen:

$$\mathrm{R}_{\mu\nu} = \mathrm{R}^{\rho}_{\mu\rho\nu} = \Gamma^{\rho}_{\mu\nu,\rho} - \Gamma^{\rho}_{\rho\mu,\nu} + \Gamma^{\rho}_{\rho\lambda}\,\Gamma^{\lambda}_{\nu\mu} - \Gamma^{\rho}_{\nu\lambda}\,\Gamma^{\lambda}_{\rho\mu}$$

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\alpha}\left\{\frac{\partial g_{\nu\alpha}}{\partial x^{\mu}} + \frac{\partial g_{\mu\alpha}}{\partial x^{\nu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\alpha}}\right\}$$

Als ik uitreken, met behulp van mijn programma of de Ricci elementen, $\mathrm{R}_{00}, \mathrm{R}_{11}, \mathrm{R}_{22}, \mathrm{R}_{33}$ nul zijn, met de formule zoals ik die gebruik, dan klopt dat, ook theoretisch, maar met de formule zoals door Schwarzschild is gebruikt klopt het niet. Ik heb dat nog niet theoretisch uitgewerkt. Ik ben er van overtuigd dat dat wel zal kloppen omdat Schwarzschild zijn vergelijking met behulp van die formule heeft afgeleid, dus als je zijn resultaat weer in de formule zal invullen dan moet dat kloppen.

Verder uitgezocht:
Schwarschild gebruikt de bekende polar equation, the determinant of the metric tensor (her the product of the coefficients) is not -1. In dat geval wordt voldaan aan De Einstein veldvergelijkingen maar niet aan de beknopte veldvergelijkingen, want daar is vereist dat g=-1. Schwarschild heeft wel een trnsformatie afgeleid, gebaseerd op de polar coordinaten, waarbij hij de transformatie zo gekozen heeft dat er wel aan g=-1 wordt voldaan. In dat geval wordt er wel voldaan aan de beknopte Einstein veldvergelijkingen.

Vraag twee: De consequentie van het verschil in formules is groot. In jouw document tel ik negen Christoffel symbolen, terwijl Karl Schwarzschild er 10 vond. Bij jou lijkt 222 te ontbreken. Dit kan komen omdat deze bij jou nul is. Dit komt weer omdat jouw definitie van de metrische tensor g verschilt met die van Schwarzschild, g22 en g33 zijn bij Schwarzschild -1, terwijl jij er de coördinaat r in zet (bijv. g22 = -r^2). Ook Droste (1917), Eddington (1921), MWT (1975) en OAS (2007) hebben zich aan g = -1 gehouden voor de Schwarzschild oplossing waardoor: g22 = g33 = -1. Dit werpt bij mij de vraag aan jou op: vind je dat g = -1 moet zijn voor de Schwarzschild oplossing?

Schwarzschild leidt zijn vergelijking, in eerste instantie, af, uitgaande van een cartesiaans assenstelsel, x,y,z. In dat geval ontstaat een metric tensor met de volgende elementen:

$$g_{00} = \sigma^2 \quad g_{11} = -\frac{1}{r^4\sigma^2} \quad g_{22} = -\frac{r^2}{\sin^2\theta} \quad g_{33} = -r^2\sin^2\theta$$

Er ontstaan dan 10 (14) relevante Christoffel symbolen.

Ook in mijn formuleoverzicht zie je dat ik de formules heb afgeleid voor de spherical vorm en de x,y,z vorm. In de x,y,z vorm bestaat de 222 wel.

Echter voor de spherical vorm ligt dat anders, daar zijn de metric tensor elementen:

$$g_{00} = \sigma^2 \quad g_{11} = \frac{-1}{\sigma^2} \quad g_{22} = -r^2 \quad g_{33} = -r^2\sin^2\theta$$

Dit geldt ook voor Schwarzschild! De elementen g_{22} *en* g_{33} kunnen niet -1 zijn want dan zouden $\frac{\partial g_{22}}{\partial r}, \frac{\partial g_{22}}{\partial \theta}, \frac{\partial g_{33}}{\partial r}, \frac{\partial g_{33}}{\partial \theta}$ nul zijn en zou het aantal Christoffel symbolen beperkt zijn tot 001 (en 010),100 en 111.

Bij spherical is inderdaad 222 nul omdat g_{22} onafhankelijk is van θ en de afgeleide dus nul is:

$$\Gamma_{22}^{2} = \frac{1}{2} g^{22} \left\{ \frac{\partial g_{22}}{\partial x^2} \right\} = 0$$

Een opmerking moet gemaakt worden dat wanneer gesteld wordt dat $\theta = 90^0$ is dat dit pas aan het eind van de berekeningen moet worden ingevoerd. Bijvoorbeeld:

$$\Gamma_{33}^{2} = \frac{1}{2} g^{22} \left\{ -\frac{\partial g_{33}}{\partial \theta} \right\} = -\boldsymbol{\cos\theta \sin\theta = 0}$$

Maar voor het Ricci element is ook de afgeleide van deze Christoffel sybool nodig en die is dus:

$$\frac{\partial \Gamma_{33}^{2}}{\partial \theta} = \boldsymbol{-\cos^2\theta + \sin^2\theta = 1}$$

En geen nul! En dat geldt voor nog enkele termen.

Waarom Einstein tot een beperking kwam van det(g)=-1 weet ik niet, behalve dat het geheel er wat eenvoudiger/symmetrischer uit komt te zien. Maar blijkbaar hangt dat ook af van welk assenstelsel je kiest. Bijvoorbeel de metric tensor element van t,x,y,z levert inderdaad een det(g) van -1 op

$$\sigma^2 . \left(-\frac{1}{r^4\sigma^2}\right) . \left(-\frac{r^2}{\sin^2\theta}\right) . (-r^2 \sin^2\theta) = -1$$

maar bij de spherial coordinaten is het:

$$\sigma^2 . \frac{-1}{\sigma^2} . (-r^2) . (-r^2 \sin^2\theta) = -\ r^4 \sin^2\theta$$

Dus det $(g) \neq -1$??

Vraag3: De veldvergelijkingen in jouw document op blz. 2 en 3 op basis van de Ricci tensor wijken sterk af van die wij (en Karl Schwarzschild) in appendix E op basis van de G tensor neergezet hebben. Je noemt ook de G tensor in jouw document op blz. 9. Mijn vraag is dus: zou er niet dezelfde uitkomst uit moeten komen?

Ik noem de G tensor formule ook in mijn formule overzicht om hem bij de hand te hebben ter vergelijk, maar ik heb hem niet theoretisch gechecked. Wel laat ik hem ook in mijn programma berekenen maar daar levert hij nooit R00=0 etc. op. Maar ik zou dat verder moeten uit proberen. Zoals ik boven al zei moet het en zal het ook wel kloppen want anders had Schwarzschild nooit tot die configuratie kunnen komen.

10 Angular momentum between Schwarzschild and repaired Schwarzschild frame.

De metriek van de standaard Schwarzschild (ik kan daar gedetailleerd op ingaan als je wilt) blijft onveranderlijk als de tijd verandert of als phi verandert. Dat wilt zeggen dat de coefficienten constant blijven als t of phi veranderen. Dat wilt zeggen dat de ruimte-tijd, wat t en phi betreft, symmetrisch is, i.e. de ruimte blijft in alle richtingen gelijk aan wat die is. Dit betekent ook volgens Noether dat er conservering van natuurgrootheden is. T.g.v de tijd is dit $\sigma^2 c \frac{dt}{d\tau} = constante = E$ en t.g.v phi is dit $r^2 \frac{d\emptyset}{d\tau} = constante = L$.
De berekening van de deflectie met de standaard Schwarzschild is dus gebaseerd op een constante=L bij de zon maar ook in het oneindige dus **constant**! Dit levert een antwoord van deflectie op zoals is vastgesteld/gemeten. Als L niet constant zou zijn dan krijgen we een ander antwoord en dus fout.

Wanneer we nu de berekening doen met de gerepareerde Schwarzschild, dan levert dit wederom voor de tijd dezelfde E op en phi levert dan ook een constante op maar die is dan de oude $\frac{L}{\sigma^2}$. Dus dan is $\frac{L}{\sigma^2}$ constant: overal, zowel bij de zon als in het oneindige. De oude L is dan in dit gerepareerde frame niet constant maar alleen wanneer gedeeld door σ^2.
Wanneer we nu een zelfde berekening doen maar dan gebaseerd op de gerepareerde vergelijking dan levert dat dezelfde deflectie op uitgaande van $\frac{L}{\sigma^2}$.

De verklaring die ik kan vinden is de volgende:
Schwarzschild is een coordinatenstelsel dat zo gekozen is dat de coordinaten zodanig gekromd zijn dat er geen gravitatie kracht wordt ervaren. Dit leidt tot $L = r^2 \frac{d\emptyset}{d\tau}$.
De gerepareerde Schwarschild is een coordinatenstelsel waarvan de kromming enigszins afwijkt van de space-time kromming. Daardoor geeft hij geen echte geodeet weer en daarom voldoet hij niet aan Einstein's veldvergelijkingen. Hierdoor wordt de gravitatiekracht niet volledig gecompenseerd maar blijft men nog wat wat gravitatiekracht ervaren. In de gerepareerde formule wordt dat gecompenseerd door L te delen door σ^2. In het gerepareerde frame/coordinatenstelsel blijft het oude angular moment dus niet meer constant maar verandert met de afstand, dit kan dus gecompenseerd worden met $/\sigma^2$.

11 Check whether the Schwarzschild elements meet the Einstein field equations.

The general form of the Einstein field equations is:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

λ is very small and is only relevant when calculations are done on the total universe. So in general the following form is used:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}$$

The left part of the formula depicts the geometry and the right part the mass and energy. When the calculation is done in a vacuum, so outside a mass, the right side becomes zero. In that case the formula becomes:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0$$

μ and ν depicts the four dimensions of space and time. This means that the Einstein formula consists of 16 equations.

The field equations are totally dependent on metric tensor elements $g_{\mu\nu}$ and its first and second derivatives.

Schwarzschild derived a formula that meets the Einstein field equations in vacuum.

$$ds^2 = \sigma^2 c^2 dt^2 - \frac{dr^2}{\sigma^2} - r^2 d\theta^2 - r^2 \sin^2\theta^2 \, d\emptyset^2$$

$$ds^2 = g_{00} dt^2 + g_{11} dr^2 + g_{22} d\theta^2 + g_{33} d\emptyset^2$$

As can be seen from the formula only four of the sixteen metric tensor elements are relevant; the rest is zero. As a consequence also of the 16 field equations only four are relevant: R_{00}, R_{11}, R_{22} and R_{33}.

$R_{\mu\nu}$ is called the Ricci tensor and consists of sixteen elements. The general form of the Ricci tensor elements is:

$$\mathrm{R}_{\mu\nu} = \mathrm{R}^{\rho}_{\mu\rho\nu} = \Gamma^{\rho}_{\mu\nu,\rho} - \Gamma^{\rho}_{\rho\mu,\nu} + \Gamma^{\rho}_{\rho\lambda}\,\Gamma^{\lambda}_{\nu\mu} - \Gamma^{\rho}_{\nu\lambda}\,\Gamma^{\lambda}_{\rho\mu}$$

In this formula there are the so called Christoffel symbols. The first two items are the derivatives of the Christoffel symbols. The general form of the Christoffel symbol is:

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\alpha}\left\{\frac{\partial g_{\nu\alpha}}{\partial x^{\mu}} + \frac{\partial g_{\mu\alpha}}{\partial x^{\nu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\alpha}}\right\}$$

The Schwarzschild equation is in vacuum and in that case:

$$\mathrm{R}_{\mu\nu} - \frac{1}{2}\mathrm{g}_{\mu\nu}\mathrm{R} = 0$$

Here is R the Ricci scalar and stands for the curvature of the local space-time

$$R = g^{\mu\nu}\mathrm{R}_{\mu\nu}$$

Because of the relation of R to $\mathrm{R}_{\mu\nu}$, it can be worked out that in order to meet

$$\mathrm{R}_{\mu\nu} - \frac{1}{2}\mathrm{g}_{\mu\nu}\mathrm{R} = 0$$

Can be limited to:

$$\mathrm{R}_{\mu\nu} = 0$$

By analyzing the general form of the Ricci elements and the Christoffel symbols the simplification could go even further. First we derived a program so that by computer and numeric application of the equations, we found the relevant form of the Ricci elements. Also by theoretical analysis of the Ricci elements the simplification could be derived. This resulted in the following formulas with the only relevant Christoffel symbols:

$$\mathrm{R}_{00} = \boldsymbol{\Gamma^{1}_{00,1}} + \Gamma^{1}_{00}\,\Gamma^{1}_{11} + \Gamma^{1}_{00}\,\Gamma^{2}_{12} + \Gamma^{1}_{00}\,\Gamma^{3}_{13} - \Gamma^{0}_{01}\,\Gamma^{1}_{00}$$

$$\mathrm{R}_{11} = -\boldsymbol{\Gamma^{0}_{10,1}} - \boldsymbol{\Gamma^{2}_{12,1}} - \boldsymbol{\Gamma^{3}_{13,1}} + \Gamma^{1}_{11}\Gamma^{0}_{10} + \Gamma^{1}_{11}\,\Gamma^{2}_{12} + \Gamma^{1}_{11}\,\Gamma^{3}_{13} - \Gamma^{0}_{10}\Gamma^{0}_{01} - \Gamma^{2}_{12}\,\Gamma^{2}_{21} - \Gamma^{3}_{13}\,\Gamma^{3}_{31}$$

$$\mathrm{R}_{22} = \boldsymbol{\Gamma^{1}_{22,1}} - \boldsymbol{\Gamma^{3}_{23,2}} + \Gamma^{1}_{22}\,\Gamma^{0}_{10} + \Gamma^{1}_{22}\,\Gamma^{1}_{11} + \Gamma^{1}_{22}\,\Gamma^{3}_{13} - \Gamma^{2}_{21}\,\Gamma^{1}_{22} - \Gamma^{3}_{23}\,\Gamma^{3}_{32}$$

$$\mathrm{R}_{33} = +\boldsymbol{\Gamma^{1}_{33,1}} + \boldsymbol{\Gamma^{2}_{33,2}} + \Gamma^{1}_{33}\,\Gamma^{0}_{10} + \Gamma^{1}_{33}\,\Gamma^{1}_{11} + \Gamma^{1}_{33}\,\Gamma^{2}_{12} - \Gamma^{3}_{31}\,\Gamma^{1}_{33} - \Gamma^{3}_{32}\,\Gamma^{2}_{33}$$

First the spherical coordinates are tested. The elements in the 4 formulas above are filled in with Christoffel symbols which are derived and summarized in de table below.

In the literature the Christoffel symbol formula is sometimes shown with the first element -1/2 and sometimes +1/2.

After some calculations it is our experience that the formula with +1/2 can achieve the result of R_{11}=R_{22}=R_{33}=R_{44}=0, which is required by the Einstein field equations in vacuum. Thus the formula in the following format has been applied:

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\alpha}\left\{\frac{\partial g_{\nu\alpha}}{\partial x^{\mu}} + \frac{\partial g_{\mu\alpha}}{\partial x^{\nu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\alpha}}\right\}$$

11.1 Checking of R_{00}, R_{11}, R_{22} and R_{33} with spherical coordinates

$$ds^2 = \sigma^2 c^2 dt^2 - \frac{dr^2}{\sigma^2} - r^2 d\theta^2 - r^2 \sin^2\theta \, d\emptyset^2$$

The Christoffel symbols and its derivatives are used from the table below.

$$\mathbf{R_{00}} = \Gamma^1_{00,1} + \Gamma^1_{00}\,\Gamma^1_{11} + \Gamma^1_{00}\,\Gamma^2_{12} + \Gamma^1_{00}\,\Gamma^3_{13} - \Gamma^0_{01}\,\Gamma^1_{00}$$

$$\mathrm{R}_{00} = \frac{\mathrm{R_s}(3\mathrm{R_s} - 2\mathrm{r})}{2\mathrm{r}^4} + \frac{\sigma^2 R_s}{2r^2}\frac{-R_s}{2r^2\sigma^2} + \frac{\sigma^2 R_s}{2r^2}\frac{1}{r} + \frac{\sigma^2 R_s}{2r^2}\frac{1}{r} - \frac{R_s}{2r^2\sigma^2}\frac{\sigma^2 R_s}{2r^2}$$

$$\mathrm{R}_{00} = \frac{\mathrm{R_s}(3\mathrm{R_s} - 2\mathrm{r})}{2\mathrm{r}^4} - \frac{{R_s}^2}{2r^4} + \frac{2R_s(r - Rs)}{2r^4} = \frac{3{R_s}^2 - 2\mathrm{rRs} - {R_s}^2 + 2R_s r - 2{R_s}^2}{2\mathrm{r}^4} = 0$$

$$\mathbf{R_{00} = 0} \quad \boldsymbol{q.e.d.}$$

$$\mathbf{R_{11}} = -\boldsymbol{\Gamma^0_{10,1}} - \boldsymbol{\Gamma^2_{12,1}} - \boldsymbol{\Gamma^3_{13,1}} + \Gamma^1_{11}\Gamma^0_{10} + \Gamma^1_{11}\,\Gamma^2_{12} + \Gamma^1_{11}\,\Gamma^3_{13} - \Gamma^0_{10}\Gamma^0_{01} - \Gamma^2_{12}\,\Gamma^2_{21} - \Gamma^3_{13}\,\Gamma^3_{31}$$

$$\mathrm{R}_{11} = -\frac{\mathrm{R_s}(\mathrm{R_s} - 2\mathrm{r})}{2\mathrm{r}^4\sigma^4} - \frac{-1}{\mathrm{r}^2} - \frac{-1}{\mathrm{r}^2} + \frac{-R_s}{2r^2\sigma^2}\frac{R_s}{2r^2\sigma^2} + \frac{-R_s}{2r^2\sigma^2}\frac{1}{r} + \frac{-R_s}{2r^2\sigma^2}\frac{1}{r} - \frac{R_s}{2r^2\sigma^2}\frac{R_s}{2r^2\sigma^2} - \frac{1}{r}\frac{1}{r} - \frac{1}{r}\frac{1}{r}$$

$$\mathrm{R}_{11} = -\frac{\mathrm{R_s}(\mathrm{R_s} - 2\mathrm{r})}{2\mathrm{r}^4\sigma^4} + \frac{1}{\mathrm{r}^2} + \frac{1}{\mathrm{r}^2} - \frac{{R_s}^2}{4r^4\sigma^4} - \frac{R_s}{2r^3\sigma^2} - \frac{R_s}{2r^3\sigma^2} - \frac{{R_s}^2}{4r^4\sigma^4} - \frac{1}{\mathrm{r}^2} - \frac{1}{\mathrm{r}^2}$$

$$\mathrm{R}_{11} = -\frac{\mathrm{R_s}(\mathrm{R_s} - 2\mathrm{r})}{2\mathrm{r}^4\sigma^4} - \frac{{R_s}^2}{2r^4\sigma^4} - \frac{2R_s r(1 - \frac{Rs}{r})}{2r^4\sigma^4} = -\frac{\mathrm{R_s}(\mathrm{R_s} - 2\mathrm{r})}{2\mathrm{r}^4\sigma^4} - \frac{{R_s}^2}{2r^4\sigma^4} - \frac{2R_s r - 2{R_s}^2}{2r^4\sigma^4}$$

$$\mathrm{R}_{11} = \frac{-{R_s}^2 + 2\mathrm{rRs}}{2\mathrm{r}^4\sigma^4} + \frac{-{R_s}^2}{2r^4\sigma^4} + \frac{-2R_s r + 2{R_s}^2}{2r^4\sigma^4} = 0$$

$$\mathbf{R_{11} = 0} \quad \boldsymbol{q.e.d.}$$

$$\mathbf{R_{22}} = \boldsymbol{\Gamma^1_{22,1}} - \boldsymbol{\Gamma^3_{23,2}} + \Gamma^1_{22}\,\Gamma^0_{10} + \Gamma^1_{22}\,\Gamma^1_{11} + \Gamma^1_{22}\,\Gamma^3_{13} - \Gamma^2_{21}\Gamma^1_{22} - \Gamma^3_{23}\,\Gamma^3_{32}$$

$$\mathrm{R}_{22} = -1 + 1 - r\sigma^2\,\frac{R_s}{2r^2\sigma^2} + r\sigma^2\,\frac{+R_s}{2r^2\sigma^2} - r\sigma^2\,\frac{1}{r} + \frac{1}{r}\,r\sigma^2 - 0 = 0$$

$$\mathbf{R_{22} = 0} \quad \boldsymbol{q.e.d.}$$

$$\mathbf{R_{33}} = +\boldsymbol{\Gamma^1_{33,1}} + \boldsymbol{\Gamma^2_{33,2}} + \Gamma^1_{33}\,\Gamma^0_{10} + \Gamma^1_{33}\,\Gamma^1_{11} + \Gamma^1_{33}\,\Gamma^2_{12} - \Gamma^3_{31}\,\Gamma^1_{33} - \Gamma^3_{32}\,\Gamma^2_{33}$$

$$\mathrm{R}_{33} = -1 + 1 - r\sigma^2\,\frac{R_s}{2r^2\sigma^2} + r\sigma^2\,\frac{R_s}{2r^2\sigma^2} - r\sigma^2\,\frac{1}{r} + \frac{1}{r}r\sigma^2 - 0 = 0$$

$$\mathbf{R_{33} = 0} \quad \boldsymbol{q.e.d.}$$

11.2 Checking of R_{00}, R_{11}, R_{22} and R_{33} with t, x, y and z coordinates

$$ds^2 = \sigma^2 c^2 {dt_\infty}^2 - \frac{{dx_1}^2}{r^4\sigma^2} - \frac{r^2 {dx_2}^2}{\sin^2\theta} - r^2 \sin^2\theta \, {dx_3}^2$$

The Christoffel symbols and its derivatives are used from the table below.

$$\mathbf{R_{00}} = \Gamma^1_{00,1} + \Gamma^1_{00}\,\Gamma^1_{11} + \Gamma^1_{00}\,\Gamma^2_{12} + \Gamma^1_{00}\,\Gamma^3_{13} - \Gamma^0_{01}\,\Gamma^1_{00}$$

$$\mathrm{R}_{00} = \frac{\mathrm{R_s}^2}{2\mathrm{r}^4} + \frac{R_s\sigma^2}{2}\frac{3R_s - 4r}{2r^4\sigma^2} + \frac{R_s\sigma^2}{2}\,\frac{1}{r^3} + \frac{R_s\sigma^2}{2}\,\frac{1}{r^3} - \frac{R_s}{2r^4\sigma^2}\,\frac{R_s\sigma^2}{2}$$

$$\mathrm{R}_{00} = \frac{2\mathrm{R_s}^2}{4\mathrm{r}^4} + \frac{3\mathrm{R_s}^2 - 4rR_s}{4r^4} + \frac{4R_s r\sigma^2}{4r^4} - \frac{\mathrm{R_s}^2}{4r^4}$$

$$\mathrm{R}_{00} = \frac{2\mathrm{R_s}^2 + 3\mathrm{R_s}^2 - 4rR_s - \mathrm{R_s}^2}{4r^4} + \frac{4R_s(r - R_s)}{4r^4} = \frac{2\mathrm{R_s}^2 + 3\mathrm{R_s}^2 - 4rR_s - \mathrm{R_s}^2}{4r^4} + \frac{4R_s r - 4\mathrm{R_s}^2}{4r^4}$$

$$\mathrm{R}_{00} = \frac{4\mathrm{R_s}^2 - 4rR_s}{4r^4} + \frac{4R_s r - 4\mathrm{R_s}^2}{4r^4} = 0$$

$$\mathbf{R_{00} = 0} \quad \boldsymbol{q.e.d.}$$

$$\mathbf{R_{11}} = -\mathbf{\Gamma^0_{10,1}} - \mathbf{\Gamma^2_{12,1}} - \mathbf{\Gamma^3_{13,1}} + \Gamma^1_{11}\Gamma^0_{10} + \Gamma^1_{11}\,\Gamma^2_{12} + \Gamma^1_{11}\,\Gamma^3_{13} - \Gamma^0_{10}\Gamma^0_{01} - \Gamma^2_{12}\,\Gamma^2_{21} - \Gamma^3_{13}\,\Gamma^3_{31}$$

$$\mathrm{R}_{11} = -\frac{\mathrm{R_s}(3\mathrm{R_s} - 4\mathrm{r})}{2\mathrm{r}^8\sigma^4} - \frac{-3}{\mathrm{r}^6} - \frac{-3}{\mathrm{r}^6} + \frac{3R_s - 4r}{2r^4\sigma^2}\frac{R_s}{2r^4\sigma^2} + \frac{3R_s - 4r}{2r^4\sigma^2}\,\frac{1}{r^3} + \frac{3R_s - 4r}{2r^4\sigma^2}\,\frac{1}{r^3} - \frac{R_s}{2r^4\sigma^2}\frac{R_s}{2r^4\sigma^2} - \frac{1}{r^3}\frac{1}{r^3} - \frac{1}{r^3}\,\frac{1}{r^3}$$

$$\mathrm{R}_{11} = -\frac{2\mathrm{R_s}(3\mathrm{R_s} - 4\mathrm{r})}{4\mathrm{r}^8\sigma^4} + \frac{4}{\mathrm{r}^6} + \frac{R_s(3R_s - 4r)}{4r^8\sigma^4} + \frac{4(3R_s - 4r)r(1 - \frac{Rs}{r})}{4r^8\sigma^4} - \frac{R_s{}^2}{4r^8\sigma^4}$$

$$\mathrm{R}_{11} = \frac{-6R_s{}^2 + 8\mathrm{rR_s} + 3R_s{}^2 - 4r\mathrm{R_s} + 12R_s r - 16r^2 - 12R_s{}^2 + 16rRs - R_s{}^2}{4\mathrm{r}^8\sigma^4} + \frac{4}{\mathrm{r}^6}$$

$$\mathrm{R}_{11} = \frac{-16R_s{}^2 + 32\mathrm{rR_s} - 16r^2}{4\mathrm{r}^8\sigma^4} + \frac{4}{\mathrm{r}^6} = \frac{-16R_s{}^2 + 32\mathrm{rR_s} - 16r^2}{4\mathrm{r}^8\sigma^4} + \frac{16r^2\left(1 - \frac{Rs}{r}\right)^2}{4\mathrm{r}^8\sigma^4}$$

$$\mathrm{R}_{11} = \frac{-16R_s{}^2 + 32\mathrm{rR_s} - 16r^2}{4\mathrm{r}^8\sigma^4} + \frac{4}{\mathrm{r}^6} = \frac{-16R_s{}^2 + 32\mathrm{rR_s} - 16r^2}{4\mathrm{r}^8\sigma^4} + \frac{16r^2(1 - 2\frac{Rs}{r} + \frac{R_s{}^2}{r^2})}{4\mathrm{r}^8\sigma^4} =$$

$$\mathrm{R}_{11} = \frac{-16R_s{}^2 + 32\mathrm{rR_s} - 16r^2}{4\mathrm{r}^8\sigma^4} + \frac{4}{\mathrm{r}^6} = \frac{-16R_s{}^2 + 32\mathrm{rR_s} - 16r^2}{4\mathrm{r}^8\sigma^4} + \frac{16r^2 - 32rRs + 16R_s{}^2}{4\mathrm{r}^8\sigma^4} = 0$$

$$\mathbf{R_{11} = 0} \quad \boldsymbol{q.e.d.}$$

$$\mathbf{R_{22}} = \mathbf{\Gamma^1_{22,1}} - \mathbf{\Gamma^3_{23,2}} + \Gamma^1_{22}\,\Gamma^0_{10} + \Gamma^1_{22}\,\Gamma^1_{11} + \Gamma^1_{22}\,\Gamma^3_{13} - \Gamma^2_{21}\,\Gamma^1_{22} - \Gamma^3_{23}\,\Gamma^3_{32}$$

$$\mathrm{R}_{22} = -3 + \frac{2\mathrm{R_s}}{\mathrm{r}} + 1 - r^3\sigma^2\,\frac{R_s}{2r^4\sigma^2} - r^3\sigma^2\,\frac{3R_s - 4r}{2r^4\sigma^2} - r^3\sigma^2\,\frac{1}{r^3} + \frac{1}{r^3}\,r^3\sigma^2 - 0$$

$$\mathrm{R}_{22} = -3 + \frac{2\mathrm{R_s}}{\mathrm{r}} + 1 - \frac{R_s}{2\mathrm{r}} - \frac{3R_s - 4r}{2\mathrm{r}} - r^3\sigma^2\,\frac{1}{r^3} + \frac{1}{r^3}\,r^3\sigma^2 - 0$$

$$\mathrm{R}_{22} = \frac{-4\mathrm{r}}{2\mathrm{r}} + \frac{4\mathrm{R_s}}{2\mathrm{r}} - \frac{R_s}{2\mathrm{r}} - \frac{3R_s - 4r}{2\mathrm{r}} = 0$$

$$\mathbf{R_{22} = 0} \quad \boldsymbol{q.e.d.}$$

$$R_{33} = +\boldsymbol{\Gamma^{1}_{33,1}} + \boldsymbol{\Gamma^{2}_{33,2}} + \Gamma^{1}_{33}\,\Gamma^{0}_{10} + \Gamma^{1}_{33}\,\Gamma^{1}_{11} + \Gamma^{1}_{33}\,\Gamma^{2}_{12} - \Gamma^{3}_{31}\,\Gamma^{1}_{33} - \Gamma^{3}_{32}\,\Gamma^{2}_{33}$$

$$R_{33} = -3 + \frac{2R_s}{r} + 1 - r^3\sigma^2\,\frac{R_s}{2r^4\sigma^2} - r^3\sigma^2\,\frac{3R_s - 4r}{2r^4\sigma^2} - r^3\sigma^2\,\frac{1}{r^3} + \frac{1}{r^3}\,r^3\sigma^2 - 0$$

$$R_{33} = -3 + \frac{2R_s}{r} + 1 - \frac{R_s}{2r} - \frac{3R_s - 4r}{2r} - r^3\sigma^2\,\frac{1}{r^3} + \frac{1}{r^3}\,r^3\sigma^2 - 0$$

$$R_{33} = \frac{-4r}{2r} + \frac{4R_s}{2r} - \frac{R_s}{2r} - \frac{3R_s - 4r}{2r} = 0$$

$$\mathbf{R_{33} = 0}\quad \boldsymbol{q.e.d.}$$

11.3 Checking of R_{00}, R_{11}, R_{22} and R_{33} with spherical coordinates and "repaired" Schwarzschild

$$\boxed{ds^2 = \sigma^2 c^2 dt_\infty{}^2 - \frac{dr^2}{\sigma^2} - \frac{r^2 d\theta^2}{\sigma^2} - \frac{r^2 \sin^2\theta\, d\varphi^2}{\sigma^2}}$$

$$R_{00} = \boldsymbol{\Gamma^{1}_{00,1}} + \Gamma^{1}_{00}\,\Gamma^{1}_{11} + \Gamma^{1}_{00}\,\Gamma^{2}_{12} + \Gamma^{1}_{00}\,\Gamma^{3}_{13} - \Gamma^{0}_{01}\,\Gamma^{1}_{00}$$

$$R_{00} = \frac{R_s(3Rs - 2r)}{2r^4} + \frac{R_s\sigma^2}{2r^2}\left(\frac{-R_s}{2r^2\sigma^2} + \frac{2r - 3R_s}{2r^2\sigma^2} + \frac{(2r - 3R_s)}{2r^2\sigma^2} - \frac{R_s}{2r^2\sigma^2}\right)$$

$$R_{00} = \frac{R_s(3Rs - 2r)}{2r^4} + \frac{R_s\sigma^2}{2r^2}\left(\frac{4r - 8R_s}{2r^2\sigma^2}\right)$$

$$R_{00} = \frac{R_s(3Rs - 2r)}{2r^4} + \left(\frac{2rR_s - 4R_s{}^2}{2r^4}\right) = \frac{\left(3R_s{}^2 - 2rR_s + 2rR_s - 4R_s{}^2\right)}{2r^4}$$

$$R_{00} = \frac{-R_s{}^2}{2r^4}$$

$$\mathbf{R_{00} \neq 0}$$

Thus this formula does not fulfill the Einstein field equations requirement being zero in vacuum.

$$R_{11} = -\boldsymbol{\Gamma^{0}_{10,1}} - \boldsymbol{\Gamma^{2}_{12,1}} - \boldsymbol{\Gamma^{3}_{13,1}} + \Gamma^{1}_{11}\Gamma^{0}_{10} + \Gamma^{1}_{11}\,\Gamma^{2}_{12} + \Gamma^{1}_{11}\,\Gamma^{3}_{13} - \Gamma^{0}_{10}\Gamma^{0}_{01} - \Gamma^{2}_{12}\,\Gamma^{2}_{21} - \Gamma^{3}_{13}\,\Gamma^{3}_{31}$$

$$\begin{aligned} R_{11} = &-\frac{-R_s(2r - R_s)}{2r^4\sigma^4} - \left(-1 + \frac{3Rs}{r} - \frac{3Rs^2}{2r^2}\right)/r^2\sigma^4 - \left(-1 + \frac{3Rs}{r} - \frac{3Rs^2}{2r^2}\right)/r^2\sigma^4 \\ &+ \frac{-R_s}{2r^2\sigma^2}\left(\frac{R_s}{2r^2\sigma^2} + \frac{2r - 3R_s}{2r^2\sigma^2} + \frac{(2r - 3R_s)}{2r^2\sigma^2}\right) - \frac{R_s}{2r^2\sigma^2}\frac{R_s}{2r^2\sigma^2} - \frac{2r - 3R_s}{2r^2\sigma^2}\frac{2r - 3R_s}{2r^2\sigma^2} \\ &- \frac{(2r - 3R_s)}{2r^2\sigma^2}\frac{(2r - 3R_s)}{2r^2\sigma^2} \end{aligned}$$

$$R_{11} = -\frac{-R_s(2r - R_s)}{2r^4\sigma^4} - 2\left(-1 + \frac{3Rs}{r} - \frac{3Rs^2}{2r^2}\right)/r^2\sigma^4 + \left(\frac{-4rRs + 5Rs^2}{4r^4\sigma^4}\right) - \frac{Rs^2}{4r^4\sigma^4} - 2\frac{(2r - 3R_s)^2}{4r^4\sigma^4}$$

$$R_{11} = -\frac{-R_s(2r - R_s)}{2r^4\sigma^4} - \frac{2\left(-1 + \frac{3Rs}{r} - \frac{3Rs^2}{2r^2}\right)}{r^2\sigma^4} + \frac{-4rRs + 5Rs^2 - Rs^2 - 8r^2 - 18Rs^2 + 24rR_s}{4r^4\sigma^4}$$

$$R_{11} = \frac{(4rR_s - 2Rs^2)}{4r^4\sigma^4} - \frac{2\left(-1 + \frac{3Rs}{r} - \frac{3Rs^2}{2r^2}\right)}{r^2\sigma^4} + \frac{-8r^2 - 14Rs^2 + 20rR_s}{4r^4\sigma^4}$$

$$R_{11} = \frac{(4rR_s - 2Rs^2)}{4r^4\sigma^4} - \frac{(-8r^2 + 24rRs - 12Rs^2)}{4r^4\sigma^4} + \frac{-8r^2 - 14Rs^2 + 20rR_s}{4r^4\sigma^4}$$

$$R_{11} = \frac{(8r^2 - 24Rsr + 12Rs^2)}{4r^4\sigma^4} + \frac{-8r^2 + 24rRs - 12Rs^2}{4r^4\sigma^4}$$

$$\mathbf{R_{11} = 0}$$

Thus this formula fulfills the Einstein field equations requirement being zero in vacuum.

$$R_{22} = \boldsymbol{\Gamma^1_{22,1}} - \boldsymbol{\Gamma^3_{23,2}} + \Gamma^1_{22}\,\Gamma^0_{10} + \Gamma^1_{22}\,\Gamma^1_{11} + \Gamma^1_{22}\,\Gamma^3_{13} - \Gamma^2_{21}\,\Gamma^1_{22} - \Gamma^3_{23}\,\Gamma^3_{32}$$

$$R_{22} = \frac{-1}{\sigma^4} + \frac{2R_s}{r\sigma^4} - \frac{3{R_s}^2}{2r^2\sigma^4} + 1 + \frac{-2r + 3R_s}{2\sigma^2}\left(\frac{R_s}{2r^2\sigma^2} + \frac{-R_s}{2r^2\sigma^2} + \frac{(2r - 3R_s)}{2r^2\sigma^2} - \frac{2r - 3R_s}{2r^2\sigma^2}\right) - 0$$

$$R_{22} = \frac{-1}{\sigma^4} + \frac{2R_s}{r\sigma^4} - \frac{3{R_s}^2}{2r^2\sigma^4} + 1$$

$$R_{22} = \frac{-2r^2 + 4rR_s - 3{R_s}^2 + 2r^2\sigma^4}{2r^2\sigma^4}$$

$$R_{22} = \frac{-2r^2 + 4rR_s - 3{R_s}^2 + 2r^2(1 + \frac{{R_s}^2}{r^2} - 2\frac{R_s}{r})}{2r^2\sigma^4}$$

$$R_{22} = \frac{-2r^2 + 4rR_s - 3{R_s}^2 + 2r^2 + 2{R_s}^2 - 4rR_s}{2r^2\sigma^4} = \frac{-{R_s}^2}{2r^2\sigma^4}$$

$$R_{22} = \frac{-{R_s}^2}{2r^2\sigma^4}$$

$$\mathbf{R_{22} \neq 0}$$

Thus this formula does not fulfill the Einstein field equations requirement being zero in vacuum.

$$R_{33} = +\boldsymbol{\Gamma^1_{33,1}} + \boldsymbol{\Gamma^2_{33,2}} + \Gamma^1_{33}\,\Gamma^0_{10} + \Gamma^1_{33}\,\Gamma^1_{11} + \Gamma^1_{33}\,\Gamma^2_{12} - \Gamma^3_{31}\,\Gamma^1_{33} - \Gamma^3_{32}\,\Gamma^2_{33}$$

$$R_{33} = \frac{-1}{\sigma^4} + \frac{2R_s}{r\sigma^4} - \frac{3{R_s}^2}{2r^2\sigma^4} + 1 + \frac{(-2r + 3R_s)}{2\sigma^2}\left(\frac{R_s}{2r^2\sigma^2} + \frac{-R_s}{2r^2\sigma^2} + \frac{2r - 3R_s}{2r^2\sigma^2} - \frac{(2r - 3R_s)}{2r^2\sigma^2}\right) - 0$$

$$R_{33} = \frac{-1}{\sigma^4} + \frac{2R_s}{r\sigma^4} - \frac{3{R_s}^2}{2r^2\sigma^4} + 1$$

$$R_{33} = \frac{-2r^2 + 4rR_s - 3{R_s}^2 + 2r^2\sigma^4}{2r^2\sigma^4}$$

$$R_{33} = \frac{-2r^2 + 4rR_s - 3{R_s}^2 + 2r^2(1 + \frac{{R_s}^2}{r^2} - 2\frac{R_s}{r})}{2r^2\sigma^4}$$

$$R_{33} = \frac{-2r^2 + 4rR_s - 3{R_s}^2 + 2r^2 + 2{R_s}^2 - 4rR_s}{2r^2\sigma^4} = \frac{-{R_s}^2}{2r^2\sigma^4}$$

$$R_{33} = \frac{-{R_s}^2}{2r^2\sigma^4}$$

$$\mathbf{R_{33} \neq 0}$$

Thus this formula does not fulfill the Einstein field equations requirement being zero in vacuum.

11.4 Checking of R_{00}, R_{11}, R_{22} and R_{33} with t, x, y and z coordinates and "repaired" Schwarzschild version 1

$$\boxed{ds^2 = \frac{c^2 {dt_\infty}^2}{\sigma^2} - \frac{{dx_1}^2}{\sigma^2} - \frac{{dx_2}^2}{\sigma^2} - \frac{{dx_3}^2}{\sigma^2}}$$

$$R_{00} = \boldsymbol{\Gamma^1_{00,1}} + \Gamma^1_{00}\,\Gamma^1_{11} + \Gamma^1_{00}\,\Gamma^2_{12} + \Gamma^1_{00}\,\Gamma^3_{13} - \Gamma^0_{01}\,\Gamma^1_{00}$$

$$R_{00} = \frac{R_s(4r - 3R_s)}{2r^8\sigma^4} + \frac{-R_s}{2r^4\sigma^2}\left(\frac{-R_s}{2r^4\sigma^2} + \frac{-R_s}{2r^4\sigma^2} + \frac{-R_s}{2r^4\sigma^2} + \frac{R_s}{2r^4\sigma^2}\right)$$

$$R_{00} = \frac{R_s(4r - 3R_s)}{2r^8\sigma^4} + \frac{-R_s}{2r^4\sigma^2}\frac{-2R_s}{2r^4\sigma^2}$$

$$R_{00} = \frac{R_s(4r - 3R_s)}{2r^8\sigma^4} + \frac{{R_s}^2}{2r^8\sigma^4}$$

$$R_{00} = \frac{4rR_s - 3{R_s}^2 + {R_s}^2}{2r^8\sigma^4} = \frac{4rR_s - 2{R_s}^2}{2r^8\sigma^4}$$

$$R_{00} = \frac{R_s(2r - R_s)}{r^8\sigma^4}$$

$$\mathbf{R_{00} \neq 0}$$

$$R_{11} = -\boldsymbol{\Gamma^0_{10,1}} - \boldsymbol{\Gamma^2_{12,1}} - \boldsymbol{\Gamma^3_{13,1}} + \Gamma^1_{11}\Gamma^0_{10} + \Gamma^1_{11}\,\Gamma^2_{12} + \Gamma^1_{11}\,\Gamma^3_{13} - \Gamma^0_{10}\Gamma^0_{01} - \Gamma^2_{12}\,\Gamma^2_{21} - \Gamma^3_{13}\,\Gamma^3_{31}$$

$$R_{11} = -\frac{R_s(4r - 3R_s)}{2r^8\sigma^4} - \frac{R_s(4r - 3R_s)}{2r^8\sigma^4} - \frac{R_s(4r - 3R_s)}{2r^8\sigma^4} + \frac{-R_s}{2r^4\sigma^2}\left(\frac{-R_s}{2r^4\sigma^2} + \frac{-R_s}{2r^4\sigma^2} + \frac{-R_s}{2r^4\sigma^2}\right) - 3\left(\frac{-R_s}{2r^4\sigma^2}\right)^2$$

$$R_{11} = -\frac{3R_s(4r - 3R_s)}{2r^8\sigma^4} + 3\left(\frac{-R_s}{2r^4\sigma^2}\right)^2 - 3\left(\frac{-R_s}{2r^4\sigma^2}\right)^2$$

$$R_{11} = -\frac{3R_s(4r - 3R_s)}{2r^8\sigma^4}$$

$$\mathbf{R_{11} \neq 0}$$

$$R_{22} = \mathbf{\Gamma^{1}_{22,1}} - \mathbf{\Gamma^{3}_{23,2}} + \Gamma^{1}_{22}\,\Gamma^{0}_{10} + \Gamma^{1}_{22}\,\Gamma^{1}_{11} + \Gamma^{1}_{22}\,\Gamma^{3}_{13} - \Gamma^{2}_{21}\,\Gamma^{1}_{22} - \Gamma^{3}_{23}\,\Gamma^{3}_{32}$$

$$R_{22} = -\frac{R_s(4r - 3R_s)}{2r^8\sigma^4} - 0 + \frac{R_s}{2r^4\sigma^2}\left(\frac{-R_s}{2r^4\sigma^2} + \frac{-R_s}{2r^4\sigma^2} + \frac{-R_s}{2r^4\sigma^2} - \frac{-R_s}{2r^4\sigma^2}\right) - 0$$

$$R_{22} = -\frac{R_s(4r - 3R_s)}{2r^8\sigma^4} - 0 - 2\left(\frac{R_s}{2r^4\sigma^2}\right)^2 - 0$$

$$R_{22} = -\frac{R_s(4r - 3R_s)}{2r^8\sigma^4} - \frac{{R_s}^2}{2r^8\sigma^4} = \frac{\left(-4rR_s + 3{R_s}^2 - {R_s}^2\right)}{2r^8\sigma^4}$$

$$R_{22} = \frac{\left(-4rR_s + 2{R_s}^2\right)}{2r^8\sigma^4} = \frac{\left(-2rR_s + {R_s}^2\right)}{r^8\sigma^4}$$

$$R_{22} = \frac{R_s(R_s - 2r)}{r^8\sigma^4}$$

$$\mathbf{R_{22} \neq 0}$$

$$R_{33} = +\mathbf{\Gamma^{1}_{33,1}} + \mathbf{\Gamma^{2}_{33,2}} + \Gamma^{1}_{33}\,\Gamma^{0}_{10} + \Gamma^{1}_{33}\,\Gamma^{1}_{11} + \Gamma^{1}_{33}\,\Gamma^{2}_{12} - \Gamma^{3}_{31}\,\Gamma^{1}_{33} - \Gamma^{3}_{32}\,\Gamma^{2}_{33}$$

$$R_{33} = -\frac{R_s(4r - 3R_s)}{2r^8\sigma^4} - +\mathbf{0} + \frac{R_s}{2r^4\sigma^2}\left(\frac{-R_s}{2r^4\sigma^2} + \frac{-R_s}{2r^4\sigma^2} + \frac{-R_s}{2r^4\sigma^2} - \frac{-R_s}{2r^4\sigma^2}\right) - 0$$

$$R_{33} = -\frac{R_s(4r - 3R_s)}{2r^8\sigma^4} - 2\frac{{R_s}^2}{4r^8\sigma^4} = \frac{R_s(-4r + 3R_s - R_s)}{2r^8\sigma^4}$$

$$R_{33} = \frac{R_s(-4r + 3R_s - R_s)}{2r^8\sigma^4} = \frac{-R_s(2r - R_s)}{r^8\sigma^4}$$

$$\mathbf{R_{33} \neq 0}$$

11.5 Checking of R_{00}, R_{11}, R_{22} and R_{33} with t, x, y and z coordinates and "repaired" Schwarzschild version 2

Here $x_1 = r$ only.

$$ds^2 = \frac{c^2{dt_\infty}^2}{\sigma^2} - \frac{{dx_1}^2}{\sigma^2} - \frac{{dx_2}^2}{\sigma^2} - \frac{{dx_3}^2}{\sigma^2}$$

$$R_{00} = \mathbf{\Gamma^{1}_{00,1}} + \Gamma^{1}_{00}\,\Gamma^{1}_{11} + \Gamma^{1}_{00}\,\Gamma^{2}_{12} + \Gamma^{1}_{00}\,\Gamma^{3}_{13} - \Gamma^{0}_{01}\,\Gamma^{1}_{00}$$

$$R_{00} = \frac{R_s(2r - R_s)}{2r^4\sigma^4} + \frac{-R_s}{2r^2\sigma^2}\left(\frac{-R_s}{2r^2\sigma^2} + \frac{-R_s}{2r^2\sigma^2} + \frac{-R_s}{2r^2\sigma^2} - \frac{-R_s}{2r^2\sigma^2}\right)$$

$$R_{00} = \frac{R_s(2r - R_s)}{2r^4\sigma^4} + \frac{-R_s}{2r^2\sigma^2}\frac{-2R_s}{2r^2\sigma^2}$$

$$R_{00} = \frac{R_s(2r - R_s)}{2r^4\sigma^4} + \frac{{R_s}^2}{2r^4\sigma^4}$$

$$R_{00} = \frac{2rR_s - {R_s}^2 + {R_s}^2}{2r^4\sigma^4} = \frac{2rR_s}{2r^4\sigma^4}$$

$$R_{00} = \frac{R_s}{r^3\sigma^4}$$

$$\mathbf{R_{00} \neq 0}$$

$$R_{11} = -\boldsymbol{\Gamma^0_{10,1}} - \boldsymbol{\Gamma^2_{12,1}} - \boldsymbol{\Gamma^3_{13,1}} + \Gamma^1_{11}\Gamma^0_{10} + \Gamma^1_{11}\,\Gamma^2_{12} + \Gamma^1_{11}\,\Gamma^3_{13} - \Gamma^0_{10}\Gamma^0_{01} - \Gamma^2_{12}\,\Gamma^2_{21} - \Gamma^3_{13}\,\Gamma^3_{31}$$

$$R_{11} = -\frac{R_s(2r - R_s)}{2r^4\sigma^4} - \frac{R_s(2r - R_s)}{2r^4\sigma^4} - \frac{R_s(2r - R_s)}{2r^4\sigma^4} + \frac{-R_s}{2r^2\sigma^2}\left(\frac{-R_s}{2r^2\sigma^2} + \frac{-R_s}{2r^2\sigma^2} + \frac{-R_s}{2r^2\sigma^2}\right) - 3\left(\frac{-R_s}{2r^2\sigma^2}\right)^2$$

$$R_{11} = -\frac{3R_s(2r - R_s)}{2r^4\sigma^4} + 3\left(\frac{-R_s}{2r^2\sigma^2}\right)^2 - 3\left(\frac{-R_s}{2r^2\sigma^2}\right)^2$$

$$R_{11} = -\frac{3R_s(2r - R_s)}{2r^4\sigma^4}$$

$$\mathbf{R_{11} \neq 0}$$

$$R_{22} = \boldsymbol{\Gamma^1_{22,1}} - \boldsymbol{\Gamma^3_{23,2}} + \Gamma^1_{22}\,\Gamma^0_{10} + \Gamma^1_{22}\,\Gamma^1_{11} + \Gamma^1_{22}\,\Gamma^3_{13} - \Gamma^2_{21}\,\Gamma^1_{22} - \Gamma^3_{23}\,\Gamma^3_{32}$$

$$R_{22} = -\frac{R_s(2r - R_s)}{2r^4\sigma^4} - 0 + \frac{R_s}{2r^2\sigma^2}\left(\frac{-R_s}{2r^2\sigma^2} + \frac{-R_s}{2r^2\sigma^2} + \frac{-R_s}{2r^2\sigma^2} - \frac{-R_s}{2r^2\sigma^2}\right) - 0$$

$$R_{22} = -\frac{R_s(2r - R_s)}{2r^4\sigma^4} - 0 - 2\left(\frac{R_s}{2r^2\sigma^2}\right)^2 - 0$$

$$R_{22} = -\frac{R_s(2r - R_s)}{2r^4\sigma^4} - \frac{{R_s}^2}{2r^4\sigma^4} = \frac{\left(-2rR_s + {R_s}^2 - {R_s}^2\right)}{2r^4\sigma^4}$$

$$R_{22} = \frac{(-2rR_s)}{2r^4\sigma^4} = \frac{(-R_s)}{r^3\sigma^4}$$

$$R_{22} = \frac{-R_s}{r^3\sigma^4}$$

$$\mathbf{R_{22} \neq 0}$$

$$R_{33} = +\boldsymbol{\Gamma^1_{33,1}} + \boldsymbol{\Gamma^2_{33,2}} + \Gamma^1_{33}\,\Gamma^0_{10} + \Gamma^1_{33}\,\Gamma^1_{11} + \Gamma^1_{33}\,\Gamma^2_{12} - \Gamma^3_{31}\,\Gamma^1_{33} - \Gamma^3_{32}\,\Gamma^2_{33}$$

$$R_{33} = -\frac{R_s(2r - R_s)}{2r^4\sigma^4} - - + \mathbf{0} + \frac{R_s}{2r^2\sigma^2}\left(\frac{-R_s}{2r^2\sigma^2} + \frac{-R_s}{2r^2\sigma^2} + \frac{-R_s}{2r^2\sigma^2} - \frac{-R_s}{2r^2\sigma^2}\right) - 0$$

$$R_{33} = -\frac{R_s(2r - R_s)}{2r^4\sigma^4} - 2\frac{{R_s}^2}{4r^4\sigma^4} = \frac{R_s(-2r + R_s - R_s)}{2r^4\sigma^4}$$

$$R_{33} = \frac{R_s(-2r)}{2r^4\sigma^4} = \frac{-R_s}{r^3\sigma^4}$$

$$\mathbf{R_{33} \neq 0}$$

Conclusion: This configuration does not meet the “extended” field equations of General Relativity.

12 Check whether the Schwarzschild elements meet the Einstein field equations according the limited formula.

In this chapter we will check the Schwarzschild solution with the limited original Einstein formula, which is only valid when $det(g_{\mu\nu}) = -1$:

$$G_{\mu\nu} = \frac{\partial \Gamma^{\alpha}_{\mu\nu}}{\partial x_{\alpha}} + \Gamma^{\alpha}_{\mu\beta}\,\Gamma^{\beta}_{\nu\alpha}$$

Here we use the Christoffel symbol with negative sign as Schwarzschild applied in his derivation.

$$\Gamma^{\rho}_{\mu\nu} = -\frac{1}{2} g^{\rho\alpha}\left\{\frac{\partial g_{\nu\alpha}}{\partial x^{\mu}} + \frac{\partial g_{\mu\alpha}}{\partial x^{\nu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\alpha}}\right\}$$

In that case the formulas of the Christoffel symbols and its derivatives, in the table below, shall change sign as well.

In the derivation of his solution Schwarzschild used the t,x,y,z coordinates, so let us first start with these coordinates.

We first derived the relevant Ricci elements:

$$\mathrm{R}_{00} = \boldsymbol{\Gamma^{1}_{00,1}} + \Gamma^{0}_{01}\,\Gamma^{1}_{00} + \Gamma^{1}_{00}\,\Gamma^{0}_{10}$$

$$\mathrm{R}_{11} = \boldsymbol{\Gamma^{1}_{11,1}} + \Gamma^{0}_{10}\Gamma^{0}_{01} + \Gamma^{1}_{11}\,\Gamma^{1}_{11} + \Gamma^{2}_{12}\,\Gamma^{2}_{21} + \Gamma^{3}_{13}\,\Gamma^{3}_{31}$$

$$\mathrm{R}_{22} = \boldsymbol{\Gamma^{1}_{22,1}} + \boldsymbol{\Gamma^{2}_{22,2}} + \Gamma^{1}_{22}\,\Gamma^{2}_{12} + \Gamma^{2}_{21}\,\Gamma^{1}_{22} + \Gamma^{2}_{22}\,\Gamma^{2}_{22} + \Gamma^{3}_{23}\,\Gamma^{3}_{23}$$

$$\mathrm{R}_{33} = +\boldsymbol{\Gamma^{1}_{33,1}} + \boldsymbol{\Gamma^{2}_{33,2}} + \Gamma^{1}_{33}\,\Gamma^{3}_{13} + \Gamma^{2}_{33}\,\Gamma^{3}_{23} + \Gamma^{3}_{31}\,\Gamma^{1}_{33} + \Gamma^{3}_{32}\,\Gamma^{2}_{33}$$

First:

12.1 t,x,y,z coordinates

$$\mathrm{R}_{00} = \boldsymbol{\Gamma^{1}_{00,1}} + \Gamma^{0}_{01}\,\Gamma^{1}_{00} + \Gamma^{1}_{00}\,\Gamma^{0}_{10}$$

$$\mathrm{R}_{00} = \frac{-\mathrm{R_s}^2}{2\mathrm{r}^4} + \frac{R_s}{2r^4\sigma^2}\,\frac{R_s\sigma^2}{2} + \frac{R_s\sigma^2}{2}\,\frac{R_s}{2r^4\sigma^2} = \frac{-\mathrm{R_s}^2}{2\mathrm{r}^4} + \frac{\mathrm{R_s}^2}{2r^4} = 0$$

$$\mathbf{R_{00} = 0\ q.e.d.}$$

$$\mathrm{R}_{11} = \boldsymbol{\Gamma^{1}_{11,1}} + \Gamma^{0}_{10}\Gamma^{0}_{01} + \Gamma^{1}_{11}\,\Gamma^{1}_{11} + \Gamma^{2}_{12}\,\Gamma^{2}_{21} + \Gamma^{3}_{13}\,\Gamma^{3}_{31}$$

$$\mathrm{R}_{11} = \frac{-6}{\mathrm{r}^6\sigma^4} + \frac{10\mathrm{R_s}}{\mathrm{r}^7\sigma^4} - \frac{4.5\mathrm{R_s}^2}{\mathrm{r}^8\sigma^4} + \frac{R_s}{2r^4\sigma^2}\,\frac{R_s}{2r^4\sigma^2} + \frac{3R_s - 4r}{2r^4\sigma^2}\,\frac{3R_s - 4r}{2r^4\sigma^2} + \frac{1}{r^3}\,\frac{1}{r^3} + \frac{1}{r^3}\,\frac{1}{r^3}$$

$$\mathrm{R}_{11} = \frac{-6}{\mathrm{r}^6\sigma^4} + \frac{10\mathrm{R_s}}{\mathrm{r}^7\sigma^4} - \frac{4.5\mathrm{R_s}^2}{\mathrm{r}^8\sigma^4} + \frac{\mathrm{R_s}^2}{4r^8\sigma^4} + \frac{9\mathrm{R_s}^2 + 16r^2 - 24rR_s}{4r^8\sigma^4} + \frac{2}{r^6}$$

$$\mathrm{R}_{11} = \frac{-24r^2}{4\mathrm{r}^8\sigma^4} + \frac{40\mathrm{rR_s}}{4\mathrm{r}^8\sigma^4} - \frac{18\mathrm{R_s}^2}{4\mathrm{r}^8\sigma^4} + \frac{\mathrm{R_s}^2}{4r^8\sigma^4} + \frac{9\mathrm{R_s}^2 + 16r^2 - 24rR_s}{4r^8\sigma^4} + \frac{2}{r^6}$$

$$\mathrm{R}_{11} = \frac{-8\mathrm{R_s}^2 - 8r^2 + 16rR_s}{4r^8\sigma^4} + \frac{2}{r^6}$$

$$R_{11} = \frac{-8{R_s}^2 - 8r^2 + 16rR_s}{4r^8\sigma^4} + \frac{8r^2\sigma^4}{4r^8\sigma^4}$$

$$R_{11} = \frac{-8{R_s}^2 - 8r^2 + 16rR_s}{4r^8\sigma^4} + \frac{8r^2\left(1-\frac{R_s}{r}\right)^2}{4r^8\sigma^4}$$

$$R_{11} = \frac{-8{R_s}^2 - 8r^2 + 16rR_s}{4r^8\sigma^4} + \frac{8\left(r^2+{R_s}^2 - 2rR_s\right)}{4r^8\sigma^4} = 0$$

$$\mathbf{R_{11} = 0 \quad q.e.d.}$$

$$R_{22} = \boldsymbol{\Gamma^1_{22,1}} + \boldsymbol{\Gamma^2_{22,2}} + \Gamma^1_{22}\,\Gamma^2_{12} + \Gamma^2_{21}\,\Gamma^1_{22} + \Gamma^2_{22}\,\Gamma^2_{22} + \Gamma^3_{23}\,\Gamma^3_{23}$$

$$R_{22} = \frac{-2R_s + 3r}{r\sin^2\theta} + \frac{-1-\cos^2\theta}{\sin^4\theta} + \frac{-r^3\sigma^2}{\sin^2\theta}\,\frac{1}{r^3} + \frac{1}{r^3}\,\frac{-r^3\sigma^2}{\sin^2\theta} + \frac{-\cos(\theta)}{\sin^2(\theta)}\frac{-\cos(\theta)}{\sin^2(\theta)} + \frac{\cos\theta}{\sin^2(\theta)}\frac{\cos\theta}{\sin^2(\theta)}$$

$$R_{22} = \frac{-2R_s + 3r}{r\sin^2\theta} + \frac{-1-\cos^2\theta}{\sin^4\theta} + \frac{-2r^3\sigma^2}{r^3\sin^2\theta} + \frac{2\cos^2\theta}{\sin^4\theta}$$

$$R_{22} = \frac{-2R_s + 3r}{r\sin^2\theta} + \frac{-1-\cos^2\theta}{\sin^4\theta} + \frac{-2(r-R_s)}{r\sin^2\theta} + \frac{2\cos^2\theta}{\sin^4\theta}$$

$$R_{22} = \frac{1}{\sin^2\theta} + \frac{-1-\cos^2\theta}{\sin^4\theta} + \frac{2\cos^2\theta}{\sin^4\theta}$$

$$R_{22} = \frac{\sin^2\theta}{\sin^4\theta} + \frac{-\sin^2\theta - \cos^2\theta - \cos^2\theta}{\sin^4\theta} + \frac{2\cos^2\theta}{\sin^4\theta} = 0$$

$$\mathbf{R_{22} = 0 \quad q.e.d.}$$

$$R_{33} = +\boldsymbol{\Gamma^1_{33,1}} + \boldsymbol{\Gamma^2_{33,2}} + \Gamma^1_{33}\,\Gamma^3_{13} + \Gamma^2_{33}\,\Gamma^3_{23} + \Gamma^3_{31}\,\Gamma^1_{33} + \Gamma^3_{32}\,\Gamma^2_{33}$$

$$R_{33} = \left(3 - \frac{2R_s}{r}\right).\sin^2\theta + 3\cos^2\theta - 1 - r^3\sigma^2\sin^2\theta\,\frac{1}{r^3} + (-\sin^2\theta\cos\theta)\,\frac{\cos\theta}{\sin^2(\theta)}$$
$$-\frac{1}{r^3}\,r^3\sigma^2\sin^2\theta + \frac{\cos\theta}{\sin^2(\theta)}(-\sin^2\theta\cos\theta)$$

$$R_{33} = \left(3 - \frac{2R_s}{r}\right).\sin^2\theta + 3\cos^2\theta - 1 - 2\sigma^2\sin^2\theta - 2\sin^2(\theta)\cos\theta\,\frac{\cos\theta}{\sin^2(\theta)}$$

$$R_{33} = \left(3 - \frac{2R_s}{r}\right).\sin^2\theta + 3\cos^2\theta - 1 - 2\left(1 - \frac{R_s}{r}\right).\sin^2\theta - 2\cos^2\theta$$

$$R_{33} = \left(3 - \frac{2R_s}{r}\right).\sin^2\theta + 3\cos^2\theta - 1 + \left(-2 + \frac{2R_s}{r}\right).\sin^2\theta - 2\cos^2\theta$$

$$R_{33} = \sin^2\theta + 3\cos^2\theta - 1 - 2\cos^2\theta$$

$$R_{33} = \sin^2\theta + 3\cos^2\theta - \sin^2\theta - \cos^2\theta - 2\cos^2\theta = 0$$

$$\mathbf{R_{33} = 0 \quad q.e.d.}$$

12.2 Spherical coordinates

$$R_{00} = \mathbf{\Gamma^{1}_{00,1}} + \Gamma^{0}_{01}\,\Gamma^{1}_{00} + \Gamma^{1}_{00}\,\Gamma^{0}_{10}$$

$$R_{00} = \frac{-R_s(3R_s - 2r)}{2r^4} + \frac{R_s}{2r^2\sigma^2}\frac{\sigma^2 R_s}{2r^2} + \frac{\sigma^2 R_s}{2r^2}\frac{R_s}{2r^2\sigma^2}$$

$$R_{00} = \frac{-R_s(3R_s - 2r)}{2r^4} + \frac{{R_s}^2}{2r^4} =$$

$$R_{00} = \frac{-R_s(2R_s - 2r)}{2r^4} = \frac{-R_s(R_s - r)}{r^4}$$

$$\mathbf{R_{00} \neq 0} \quad ??$$

$$R_{11} = \mathbf{\Gamma^{1}_{11,1}} + \Gamma^{0}_{10}\Gamma^{0}_{01} + \Gamma^{1}_{11}\,\Gamma^{1}_{11} + \Gamma^{2}_{12}\,\Gamma^{2}_{21} + \Gamma^{3}_{13}\,\Gamma^{3}_{31}$$

$$R_{11} = \frac{-R_s(2r - Rs)}{2r^4\sigma^4} + \frac{R_s}{2r^2\sigma^2}\frac{R_s}{2r^2\sigma^2} + \frac{-R_s}{2r^2\sigma^2}\frac{-R_s}{2r^2\sigma^2} + \frac{1}{r}\frac{1}{r} + \frac{1}{r}\frac{1}{r}$$

$$R_{11} = \frac{-R_s(2r - Rs)}{2r^4\sigma^4} + \frac{{R_s}^2}{2r^4\sigma^4} + \frac{2}{r^2}$$

$$R_{11} = \frac{-R_s(2r - Rs)}{2r^4\sigma^4} + \frac{{R_s}^2}{2r^4\sigma^4} + \frac{4\left(r^2 + {R_s}^2 - 2rR_s\right)}{2r^4\sigma^4}$$

$$R_{11} = \frac{-2rR_s + {R_s}^2}{2r^4\sigma^4} + \frac{{R_s}^2}{2r^4\sigma^4} + \frac{4\left(r^2 + {R_s}^2 - 2rR_s\right)}{2r^4\sigma^4}$$

$$R_{11} = \frac{-2rR_s + 2{R_s}^2}{2r^4\sigma^4} + \frac{4\left(r^2 + {R_s}^2 - 2rR_s\right)}{2r^4\sigma^4}$$

$$R_{11} = \frac{-2rR_s + 2{R_s}^2 + 4r^2 + 4{R_s}^2 - 8rR_s}{2r^4\sigma^4}$$

$$R_{11} = \frac{-10rR_s + 6{R_s}^2 + 4r^2}{2r^4\sigma^4} = \frac{3{R_s}^2 + 2r^2 - 5rR_s}{r^4\sigma^4}$$

$$R_{11} = \frac{-10rR_s + 6{R_s}^2 + 4r^2}{2r^4\sigma^4} = \frac{3{R_s}^2 + 2r^2 - 5rR_s}{r^2({R_s}^2 + r^2 - 2rR_s)}$$

$$\mathbf{R_{11} \neq 0} \quad ??$$

$$R_{22} = \mathbf{\Gamma^{1}_{22,1}} + \mathbf{\Gamma^{2}_{22,2}} + \Gamma^{1}_{22}\,\Gamma^{2}_{12} + \Gamma^{2}_{21}\,\Gamma^{1}_{22} + \Gamma^{2}_{22}\,\Gamma^{2}_{22} + \Gamma^{3}_{23}\,\Gamma^{3}_{23}$$

$$R_{22} = 1 + 0 + (-r\sigma^2)\,\frac{1}{r} + \frac{1}{r}\,(-r\sigma^2) + 0 + \frac{\cos\theta}{\sin\theta}\frac{\cos\theta}{\sin\theta}$$

$$R_{22} = 1 - 2\sigma^2 + \frac{\cos^2\theta}{\sin^2\theta} = \frac{\sin^2\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta} - 2\sigma^2$$

$$R_{22} = \frac{1}{\sin^2\theta} - 2\sigma^2$$

$$\mathbf{R_{22} \neq 0} \quad ??$$

$$R_{33} = +\boldsymbol{\Gamma^{1}_{33,1}} + \boldsymbol{\Gamma^{2}_{33,2}} + \Gamma^{1}_{33}\,\Gamma^{3}_{13} + \Gamma^{2}_{33}\,\Gamma^{3}_{23} + \Gamma^{3}_{31}\,\Gamma^{1}_{33} + \Gamma^{3}_{32}\,\Gamma^{2}_{33}$$

$$R_{33} = 1 + \cos^2\theta - \sin^2\theta \quad - r\sigma^2 \sin^2\theta\,\frac{1}{r} - \cos\theta\sin\theta.\,\frac{\cos\theta}{\sin\theta} + \frac{1}{r}\,(-r\sigma^2\sin^2\theta) + \frac{\cos\theta}{\sin\theta}\,(-\cos\theta\sin\theta)$$

$$R_{33} = 1 + \cos^2\theta - \sin^2\theta \quad - 2\sigma^2\sin^2\theta \ - 2\cos\theta\sin\theta.\,\frac{\cos\theta}{\sin\theta}$$

$$R_{33} = 1 + \cos^2\theta - \sin^2\theta \quad - 2\sigma^2\sin^2\theta \ - 2\cos^2\theta$$

$$R_{33} = 1 - \cos^2\theta - \sin^2\theta \quad - 2\sigma^2\sin^2\theta$$

$$R_{33} = -2\sigma^2\sin^2\theta$$

$$\mathbf{R_{33} \neq 0} \quad ??$$

12.3 Check field equations of book “Reparaired Schwarzschild’s Solution”

In this chapter we will check validity of the equations on page 189 till 191.

The coordinate system used is an Cartesian system with t,x,y,z as coordinates.

The four relevant field equations are shown below with the relevant Christoffel symbols and its derivatives.

$$R_{00} = \boldsymbol{\Gamma^{1}_{00,1}} + \Gamma^{0}_{01}\,\Gamma^{1}_{00} + \Gamma^{1}_{00}\,\Gamma^{0}_{10}$$

$$R_{11} = \boldsymbol{\Gamma^{1}_{11,1}} + \Gamma^{0}_{10}\Gamma^{0}_{01} + \Gamma^{1}_{11}\,\Gamma^{1}_{11} + \Gamma^{2}_{12}\,\Gamma^{2}_{21} + \Gamma^{3}_{13}\,\Gamma^{3}_{31}$$

$$R_{22} = \boldsymbol{\Gamma^{1}_{22,1}} + \boldsymbol{\Gamma^{2}_{22,2}} + \Gamma^{1}_{22}\,\Gamma^{2}_{12} + \Gamma^{2}_{21}\,\Gamma^{1}_{22} + \Gamma^{2}_{22}\,\Gamma^{2}_{22} + \Gamma^{3}_{23}\,\Gamma^{3}_{23}$$

$$R_{33} = +\boldsymbol{\Gamma^{1}_{33,1}} + \boldsymbol{\Gamma^{2}_{33,2}} + \Gamma^{1}_{33}\,\Gamma^{3}_{13} + \Gamma^{2}_{33}\,\Gamma^{3}_{23} + \Gamma^{3}_{31}\,\Gamma^{1}_{33} + \Gamma^{3}_{32}\,\Gamma^{2}_{33}$$

The field equations used on page 190 are:

$$G_{00} = R_{00} = \boldsymbol{\Gamma^{1}_{00,1}} + \Gamma^{0}_{01}\,\Gamma^{1}_{00} + \Gamma^{1}_{00}\,\Gamma^{0}_{10} = \boldsymbol{\Gamma^{1}_{00,1}} + 2\Gamma^{0}_{01}\,\Gamma^{1}_{00}$$

$$G_{11} = R_{11} = \boldsymbol{\Gamma^{1}_{11,1}} + \Gamma^{0}_{10}\Gamma^{0}_{01} + \Gamma^{1}_{11}\,\Gamma^{1}_{11} + \Gamma^{2}_{12}\,\Gamma^{2}_{21} + \Gamma^{3}_{13}\,\Gamma^{3}_{31}$$

$$G_{22} = R_{22} = \boldsymbol{\Gamma^{1}_{22,1}} + \boldsymbol{\Gamma^{2}_{22,2}} + \Gamma^{1}_{22}\,\Gamma^{2}_{12} + \Gamma^{2}_{21}\,\Gamma^{1}_{22} + \Gamma^{2}_{22}\,\Gamma^{2}_{22} + \Gamma^{3}_{23}\,\Gamma^{3}_{23}$$

$$G_{33} = R_{33} = +\boldsymbol{\Gamma^{1}_{33,1}} + \boldsymbol{\Gamma^{2}_{33,2}} + \Gamma^{1}_{33}\,\Gamma^{3}_{13} + \Gamma^{2}_{33}\,\Gamma^{3}_{23} + \Gamma^{3}_{31}\,\Gamma^{1}_{33} + \Gamma^{3}_{32}\,\Gamma^{2}_{33}$$

However the red marked elements are missing in the equations on page 190.

The right Christoffel symbols and its derivatives, for a t, x,y,z, coordinate system, can be found in the appropriate table below. As in the overall table a Christoffel formule has been used with a positive sign, here the sign has been changed.

$$\boxed{\Gamma^{2}_{22} = -\frac{1}{2}g^{22}\left\{\frac{\partial g_{22}}{\partial x^2}\right\} = -\frac{1}{2}\left(-\frac{\sin^2\theta}{r^2}\right)\frac{2r^2\cos(\theta)}{sin^4(\theta)} = \frac{\mathbf{-cos(\boldsymbol{\theta})}}{\mathbf{sin^2(\boldsymbol{\theta})}} = \mathbf{0}\ (\boldsymbol{in\ case\ of\ theta\ is\ 90\ degrees})}$$

$$\frac{\partial \Gamma_{22}^{2}}{\partial \mathrm{x}_2} = -\frac{\mathbf{1+\cos^2\theta}}{\mathbf{\sin^4\theta}} = \mathbf{-1}$$

$$\Gamma_{23}^{3} = \Gamma_{32}^{3} = -\frac{1}{2} g^{33}\left\{\frac{\partial g_{33}}{\partial x^2}\right\} = -\frac{1}{2}\left(\frac{-1}{r^2\sin^2\theta}\right)(-2.r^2.\cos(\theta)) = -\frac{\mathbf{\cos\theta}}{\mathbf{\sin^2(\theta)}} = \mathbf{0}$$

$$\frac{\partial \Gamma_{33}^{2}}{\partial \mathrm{x}_2} = 3\,\mathbf{\cos^2\theta - 1 = -1}$$

$$\Gamma_{33}^{2} = -\frac{1}{2} g^{22}\left\{-\frac{\partial g_{33}}{\partial x^2}\right\} = -\frac{1}{2}\left(-\frac{\sin^2\theta}{r^2}\right)(2.r^2.\cos(\theta)) = \mathbf{\sin^2\theta\cos\theta} = 0$$

$$\Gamma_{23}^{3} = \Gamma_{32}^{3} = -\frac{1}{2} g^{33}\left\{\frac{\partial g_{33}}{\partial x^2}\right\} = -\frac{1}{2}\left(\frac{-1}{r^2\sin^2\theta}\right)(-2.r^2.\cos(\theta)) = -\frac{\mathbf{\cos\theta}}{\mathbf{\sin^2(\theta)}} = \mathbf{0}$$

As the G's are zero:

$$\boldsymbol{\Gamma_{00,1}^{1}} = -2\Gamma_{01}^{0}\,\Gamma_{00=}^{1} = \boldsymbol{-2\frac{R_s}{2r^4\sigma^2}\frac{R_s\sigma^2}{2} = \frac{-{R_s}^2}{2r^4}}$$

$$\boldsymbol{\Gamma_{11,1}^{1}} = -\,\Gamma_{10}^{0}\Gamma_{01}^{0} - \Gamma_{11}^{1}\,\Gamma_{11}^{1} - \Gamma_{12}^{2}\,\Gamma_{21}^{2} - \,\Gamma_{13}^{3}\,\Gamma_{31}^{3}$$

$$\boldsymbol{\Gamma_{11,1}^{1} = -\frac{R_s}{2r^4\sigma^2}\frac{R_s}{2r^4\sigma^2} - \frac{3R_s-4r}{2r^4\sigma^2}\frac{3R_s-4r}{2r^4\sigma^2} - \frac{1}{r^3}\frac{1}{r^3} - \frac{1}{r^3}\frac{1}{r^3}}$$

$$\boldsymbol{\Gamma_{11,1}^{1} = -\frac{{R_s}^2}{4r^8\sigma^4} - \frac{9{R_s}^2+16r^2-24rR_s}{4r^8\sigma^4} - \frac{2}{r^6}}$$

$$\boldsymbol{\Gamma_{11,1}^{1} = -\frac{5{R_s}^2+8r^2-12rR_s}{2r^8\sigma^4} - \frac{2}{r^6}}$$

$$\boldsymbol{\Gamma_{11,1}^{1} = -\frac{4}{r^6\sigma^4} - \frac{2}{r^6} + \frac{6R_s}{r^7\sigma^4} - \frac{2.5{R_s}^2}{r^8\sigma^4}\ \textit{in the Book is this E.09.2}}$$

$$\boldsymbol{\Gamma_{11,1}^{1} = -\frac{4}{r^6\sigma^4} - \frac{2\sigma^4}{r^6\sigma^4} + \frac{6R_s}{r^7\sigma^4} - \frac{2.5{R_s}^2}{r^8\sigma^4}}$$

$$\boldsymbol{\Gamma_{11,1}^{1} = -\frac{4r^2}{r^8\sigma^4} - \frac{2({R_s}^2+r^2-2rR_s)}{r^8\sigma^4} + \frac{6R_s}{r^7\sigma^4} - \frac{2.5{R_s}^2}{r^8\sigma^4}}$$

$$\boldsymbol{\Gamma_{11,1}^{1} = -\frac{2({R_s}^2+3r^2-2rR_s)}{r^8\sigma^4} + \frac{6R_s}{r^7\sigma^4} - \frac{2.5{R_s}^2}{r^8\sigma^4}}$$

$$\boldsymbol{\Gamma_{11,1}^{1} = -\frac{2(3r^2-2rR_s)}{r^8\sigma^4} + \frac{6R_sr}{r^8\sigma^4} - \frac{4.5{R_s}^2}{r^8\sigma^4}}$$

$$\boldsymbol{\Gamma_{11,1}^{1} = -\frac{6r^2}{r^8\sigma^4} + \frac{10R_sr}{r^8\sigma^4} - \frac{4.5{R_s}^2}{r^8\sigma^4}}$$

$$\boldsymbol{\Gamma_{11,1}^{1} = -\frac{6}{r^6\sigma^4} + \frac{10R_sr}{r^8\sigma^4} - \frac{4.5{R_s}^2}{r^8\sigma^4}\ \textit{in the book is this E.07.2}}$$

Thus it appears that the equations E.07.2 and E.09.2 are equal to each other.

$$\boldsymbol{\Gamma^{1}_{22,1}} = -\boldsymbol{\Gamma^{2}_{22,2}} - \Gamma^{1}_{22}\,\Gamma^{2}_{12} - \Gamma^{2}_{21}\,\Gamma^{1}_{22} - \Gamma^{2}_{22}\,\Gamma^{2}_{22} - \Gamma^{3}_{23}\,\Gamma^{3}_{23}$$

$$\boldsymbol{\Gamma^{1}_{22,1}} = 1 - (-\boldsymbol{r^3\sigma^2})\frac{\boldsymbol{1}}{\boldsymbol{r^3}} - \frac{\boldsymbol{1}}{\boldsymbol{r^3}}(-\boldsymbol{r^3\sigma^2}) - 0 - 0$$

$$\boldsymbol{\Gamma^{1}_{22,1}} = 1 - (-\boldsymbol{r^3\sigma^2})\frac{\boldsymbol{2}}{\boldsymbol{r^3}} = 1 + 2\boldsymbol{\sigma^2} = \boldsymbol{1} + \boldsymbol{2}\left(\boldsymbol{1} - \frac{\boldsymbol{R_s}}{\boldsymbol{r}}\right) = \boldsymbol{3} - \boldsymbol{2}\frac{\boldsymbol{R_s}}{\boldsymbol{r}}$$

$$\boldsymbol{\Gamma^{1}_{33,1}} = -\boldsymbol{\Gamma^{2}_{33,2}} - \Gamma^{1}_{33}\,\Gamma^{3}_{13} - \Gamma^{2}_{33}\,\Gamma^{3}_{23} - \Gamma^{3}_{31}\,\Gamma^{1}_{33} - \Gamma^{3}_{32}\,\Gamma^{2}_{33}$$

$$\boldsymbol{\Gamma^{1}_{33,1}} = 1 - (-\boldsymbol{r^3\sigma^2})\frac{\boldsymbol{2}}{\boldsymbol{r^3}} - 0 - (-\boldsymbol{r^3\sigma^2})\frac{\boldsymbol{2}}{\boldsymbol{r^3}} - 0$$

$$\boldsymbol{\Gamma^{1}_{33,1}} = 1 - (-\boldsymbol{r^3\sigma^2})\frac{\boldsymbol{2}}{\boldsymbol{r^3}} = 1 + 2\boldsymbol{\sigma^2} = \boldsymbol{1} + \boldsymbol{2}\left(\boldsymbol{1} - \frac{\boldsymbol{R_s}}{\boldsymbol{r}}\right) = \boldsymbol{3} - \boldsymbol{2}\frac{\boldsymbol{R_s}}{\boldsymbol{r}}$$

So the conclusion is that the equations E.07.1,E.07.2, E.07.3 and E.07.4 are right.

The error is because the assumption was made that θ is 90^0 and thus a constant and that consequently de derivatives are zero. But θ is and stays a variable which has a derivative. The best approach is to work out the formulas in general form and fill in numbers in a later stage to avoid these errors.

12.4 Remarks on book "Repairing Schwarzschild's Solution".

As the name suggests the aimed result of the book is to find a formula very similar to the Schwarzschild equation but meeting the Noether requirement which, according to the authors, the original Schwarzschild equation does not.

1. The repaired equation is checked whether it meets the Einstein field equations. Their conclusion is that it does not but they allege that the same goes for the Schwarzschild equation.
2. It is alleged that in the Hafele & Keating experiment a special formula is used where the gravitation effect and the velocity effect are combined. That they could not use the Schwarzschild formula because it was not suitable for explaining the experiment.
3. Shapiro did his experiment where he showed the deflection of an electro-magnetic wave from Earth to a planet and reflected back to Earth passing close to the Sun and so experiencing the gravity effect. In the book it is alleged that the phenomenon could not be explained by the Schwarzschild equation.

Remarks on the three items:

1. In our view it was highly unlikely that Schwarzschild would not meet the Einstein equations because this was on of the starting points. After checking the Schwarzschild equation against the Einstein field equations, we found that the equation perfectly met the field equations as is shown above. Even the equation in polar form, where determinant of g is unequal to -1, does meet the field equations. However because determinant is unequal -1 the general Ricci formula is relevant and not the simplified formula which is only valid for equations where det. g=-1. The authors agreed with our proof and will adapt the chapter in the book accordingly.
2. In My Derivations it is shown that the Hafel & Keating experiment can totally be derived by the Schwarzschild formula. The derivation shows, after an approximation, that the formula is exactly the same as the one mentioned in the "Repairing" book. The authors agreed with this result.
3. The Shapiro experiment can be completely explained and derived from the Schwarzschild equation as is shown in My Derivations.

Furthermore, as is shown in My Derivations, the trajectory of Mercury around the Sun, the deflection of light close to the Sun and the Shapiro experiment can be explained and calculated by means of the Schwarzschild equation.

$$\boxed{\begin{gathered}
\mathrm{R}_{00} = \mathbf{\Gamma^{1}_{00,1}} + \Gamma^{1}_{00}\,\Gamma^{1}_{11} + \Gamma^{1}_{00}\,\Gamma^{2}_{12} + \Gamma^{1}_{00}\,\Gamma^{3}_{13} - \Gamma^{0}_{01}\,\Gamma^{1}_{00} \\
\mathrm{R}_{11} = -\mathbf{\Gamma^{0}_{10,1}} - \mathbf{\Gamma^{2}_{12,1}} - \mathbf{\Gamma^{3}_{13,1}} + \Gamma^{1}_{11}\Gamma^{0}_{10} + \Gamma^{1}_{11}\,\Gamma^{2}_{12} + \Gamma^{1}_{11}\,\Gamma^{3}_{13} - \Gamma^{0}_{10}\Gamma^{0}_{01} - \Gamma^{2}_{12}\,\Gamma^{2}_{21} - \Gamma^{3}_{13}\,\Gamma^{3}_{31} \\
\mathrm{R}_{22} = \mathbf{\Gamma^{1}_{22,1}} - \mathbf{\Gamma^{3}_{23,2}} + \Gamma^{1}_{22}\,\Gamma^{0}_{10} + \Gamma^{1}_{22}\,\Gamma^{1}_{11} + \Gamma^{1}_{22}\,\Gamma^{3}_{13} - \Gamma^{2}_{21}\,\Gamma^{1}_{22} - \Gamma^{3}_{23}\,\Gamma^{3}_{32} \\
\mathrm{R}_{33} = +\mathbf{\Gamma^{1}_{33,1}} + \mathbf{\Gamma^{2}_{33,2}} + \Gamma^{1}_{33}\,\Gamma^{0}_{10} + \Gamma^{1}_{33}\,\Gamma^{1}_{11} + \Gamma^{1}_{33}\,\Gamma^{2}_{12} - \Gamma^{3}_{31}\,\Gamma^{1}_{33} - \Gamma^{3}_{32}\,\Gamma^{2}_{33}
\end{gathered}}$$

13 General Relativity - and Schwarzschild formulas

Einstein notation has been applied.

$$dx^m = \frac{\partial x^m}{\partial y^r} dy^r$$

$$ds^2 = \eta_{mn} d\xi^m d\xi^n$$

$$ds^2 = g_{mn}(x) dx^m dx^n = g_{pq}(y) dy^p dy^q$$

$$g_{pq}(y) = g_{mn}(x) \frac{dx^m}{dy^p} \frac{dx^n}{dy^q}$$

$$V'^n(y) = \frac{\partial y^n}{\partial x^m} V^m(x)$$

$$W'_p(y) = \frac{\partial x^q}{\partial y^p} W'_q$$

$$T_{mn}(x) = \frac{\partial V^m(x)}{\partial x^n}$$

$$T_{mn}(y) = \frac{\partial x^r}{\partial y^m} \frac{\partial x^s}{\partial y^n} T_{rs}(x)$$

$$T^{mn}(y) = \frac{\partial y^m}{\partial x^r} \frac{\partial y^n}{\partial x^s} T^{rs}(x)$$

$$T^{rs}(x) = A_x^r B_x^s$$

$$E_\mu = g_{\mu\vartheta} E^\vartheta$$

$$E^\mu = g^{\mu\vartheta} E_\vartheta = g^{\mu\vartheta} g_{\vartheta\rho} E^\rho = \delta_\rho^\mu E^\rho = E^\mu$$

Lijnsegment in klein gebied geldt: Pythagoras:

$$ds^2 = \delta_{mn} \frac{\partial x^m}{\partial y^n} dy^n . \frac{\partial x^n}{\partial y^s} dy^s$$

Transformeren naar ander frame:

$$ds^2 = \boxed{\delta_{mn} \frac{\partial x^m}{\partial y^r} \cdot \frac{\partial x^n}{\partial y^s}} dy^r dy^s$$

$$\boldsymbol{metric\ tensor}:\ g_{mn} = \delta_{mn} \frac{\partial x^m}{\partial y^r} \cdot \frac{\partial x^n}{\partial y^s}$$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

Geodesic equation:

$$\frac{d^2 x^\lambda}{d\tau^2} + \Gamma_{\mu\nu}^\lambda . \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = 0 \qquad \Gamma_{\mu\nu}^\lambda \equiv \frac{\partial x^\lambda}{\partial \xi^\alpha} \frac{\partial^2 \xi^\alpha}{\partial x^\mu \partial x^\nu}$$

$$T'_{\mu\vartheta}(y) = \frac{\partial x^\alpha}{\partial y^\mu} \frac{\partial x^\beta}{\partial y^\vartheta} T_{\alpha\beta}$$

$$T'^{\mu\vartheta}(y) = \frac{\partial y^\mu}{\partial x^\alpha} \frac{\partial y^\vartheta}{\partial x^\beta} T_{\alpha\beta}$$

$$T'^{\vartheta}_{\mu}(y) = \frac{\partial x^\alpha}{\partial y^\mu} \frac{\partial y^\vartheta}{\partial x^\beta} T_\alpha^\beta$$

$$g_{\mu\alpha} g^{\alpha\vartheta} = \delta_\mu^\vartheta$$

Contraction:

$$A^\mu = g^{\mu\vartheta} A_\vartheta$$

$$A_\mu = g_{\mu\vartheta} A^\vartheta$$

$$so:\ A.B = g_{\mu\vartheta} A^\mu B^\vartheta \equiv A_\vartheta B^\vartheta$$

Ricci Tensor:

$$\mathrm{R}_{\mu\nu} = \mathrm{R}_{\mu\rho\nu}^\rho = \Gamma_{\mu\nu,\rho}^\rho - \Gamma_{\rho\mu,\nu}^\rho + \Gamma_{\rho\lambda}^\rho \Gamma_{\nu\mu}^\lambda - \Gamma_{\nu\lambda}^\rho \Gamma_{\rho\mu}^\lambda$$

$$G_{\mu\nu} = \Gamma_{\mu\nu,\rho}^\rho - \Gamma_{\nu\lambda}^\rho \Gamma_{\rho\mu}^\lambda\ only\ if\ g = \det(g_{\mu\nu}) = -1$$

Christoffel symbol:

$$\Gamma_{\mu\nu}^\rho = \frac{1}{2} g^{\rho\alpha} \left\{ \frac{\partial g_{\nu\alpha}}{\partial x^\mu} + \frac{\partial g_{\mu\alpha}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\alpha} \right\}$$

Ricci scalar:

$$g^{\mu\nu} \mathrm{R}_{\mu\nu} = R_\mu^\mu$$

$$R = g^{ab} (\Gamma_{ab,c}^c - \Gamma_{ac,b}^c + \Gamma_{ab}^d \Gamma_{cd}^c - \Gamma_{ac}^d \Gamma_{bd}^c)$$

$$R = 2 g^{ab} (\Gamma_{a[b,c]}^c + \Gamma_{a[b}^d \Gamma_{c]d}^c)$$

Schwarzschild metric

We add Rp(=1 meter) to get the right dimensions!!

$$ds^2 = \sigma^2 c^2 dt^2 - \frac{dr^2}{\sigma^2} - \frac{r^2}{R_p^2} dR_p^2.\theta^2 - \frac{r^2}{R_p^2}\sin^2\theta^2\, dR_p^2.\emptyset^2$$

$$\sigma^2 = 1 - \frac{R_s}{r} \quad here\ is:\ R_s = \frac{2GM}{c^2} \quad and\ R_p = 1\,m$$

$g_{00} = g_{tt}$ $\quad g_{22} = g_{\theta\theta}$

$g_{11} = g_{rr}$ $\quad g_{33} = g_{\emptyset\emptyset}$

Schwarzschild on polar coordinates

$$g_{00} = \boldsymbol{\sigma^2} \qquad g^{00} = \frac{\mathbf{1}}{\boldsymbol{\sigma^2}}$$

$$g_{11} = \frac{\mathbf{-1}}{\boldsymbol{\sigma^2}} \qquad g^{11} = -\boldsymbol{\sigma^2}$$

$$g_{22} = -\frac{\boldsymbol{r^2}}{\boldsymbol{R_p^2}} \qquad g^{22} = \frac{\boldsymbol{-R_p^2}}{\boldsymbol{r^2}}$$

$$g_{33} = -\frac{\boldsymbol{r^2}}{\boldsymbol{R_p^2}}\sin^2\boldsymbol{\theta} = -\frac{r^2}{R_p^2} \qquad g^{33} = \frac{\boldsymbol{-R_p^2}}{\boldsymbol{r^2\sin^2\theta}} = \frac{-R_p^2}{r^2}$$

$$\frac{d\sigma}{dr} = \frac{\boldsymbol{R_s}}{\boldsymbol{2r^2\sigma}}$$

Metric first derivative on spherical coordinates

$$\frac{\partial g_{00}}{\partial r} = \frac{\boldsymbol{R_s}}{\boldsymbol{r^2}} \qquad \frac{\partial g_{11}}{\partial r} = \frac{\boldsymbol{R_s}}{\boldsymbol{r^2\sigma^4}}$$

$$\frac{\partial g_{22}}{\partial r} = \left(\frac{\boldsymbol{-2r}}{\boldsymbol{R_p^2}}\right) \qquad \frac{\partial g_{33}}{\partial r} = \left(\frac{\boldsymbol{-2r}}{\boldsymbol{R_p^2}}\sin^2\boldsymbol{\theta}\right) = \frac{\boldsymbol{-2r}}{\boldsymbol{R_p^2}}$$

$$\frac{\partial g_{33}}{\partial\theta} = \left(\frac{\boldsymbol{-2r^2}}{\boldsymbol{R_p^2}}.\boldsymbol{sin(\theta)}\mathbf{co\,s}(\boldsymbol{\theta})\right) = \mathbf{0}$$

Metric second derivative on spherical coordinates

$$\frac{\partial^2 g_{00}}{\partial r^2} = \frac{\boldsymbol{-2R_s}}{\boldsymbol{r^3}} \qquad \frac{\partial^2 g_{11}}{\partial r^2} = \frac{\boldsymbol{-2R_s}}{\boldsymbol{r^3\sigma^6}}$$

$$\frac{\partial^2 g_{22}}{\partial r^2} = \frac{\mathbf{-2}}{\boldsymbol{R_p^2}} \qquad \frac{\partial^2 g_{33}}{\partial r^2} = \left(\frac{\mathbf{-2}}{\boldsymbol{R_p^2}}\sin^2\boldsymbol{\theta}\right) = \frac{\mathbf{-2}}{\boldsymbol{R_p^2}}$$

$$\frac{\partial^2 g_{33}}{\partial\theta\partial r} = \left(-\frac{\boldsymbol{4r}}{\boldsymbol{R_p^2}}.\boldsymbol{sin(\theta)}\mathbf{co\,s}(\boldsymbol{\theta})\right) = \mathbf{0}$$

$$\frac{\partial^2 g_{33}}{\partial\theta^2} = \frac{\boldsymbol{2r^2}}{\boldsymbol{R_p^2}}.(\mathbf{sin^2}(\boldsymbol{\theta}) - \mathbf{cos^2}(\boldsymbol{\theta})) = \frac{\boldsymbol{2r^2}}{\boldsymbol{R_p^2}}$$

Schwarzschild polar coordinates:

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2}g^{\rho\alpha}\left\{\frac{\partial g_{\nu\alpha}}{\partial x^\mu} + \frac{\partial g_{\mu\alpha}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\alpha}\right\}$$

$$\Gamma^0_{01} = \Gamma^0_{10} = \frac{1}{2}g^{00}\left\{\frac{\partial g_{00}}{\partial r}\right\} = \frac{\boldsymbol{R_s}}{\boldsymbol{2r^2\sigma^2}}$$

$$\Gamma^1_{00} = \frac{1}{2}g^{11}\left\{-\frac{\partial g_{00}}{\partial r}\right\} = \frac{\boldsymbol{\sigma^2 R_s}}{\boldsymbol{2r^2}}$$

$$\Gamma^1_{11} = \frac{1}{2}g^{11}\left\{\frac{\partial g_{11}}{\partial r}\right\} = \frac{\boldsymbol{-R_s}}{\boldsymbol{2r^2\sigma^2}}$$

$$\Gamma^1_{22} = \frac{1}{2}g^{11}\left\{-\frac{\partial g_{22}}{\partial r}\right\} = -\frac{\boldsymbol{r}}{\boldsymbol{R_p^2}}\boldsymbol{\sigma^2}$$

$$\Gamma^1_{33} = \frac{1}{2}g^{11}\left\{-\frac{\partial g_{33}}{\partial r}\right\} = -\frac{\boldsymbol{r}}{\boldsymbol{R_p^2}}\boldsymbol{\sigma^2}\,\mathbf{sin^2}\,\boldsymbol{\theta} = -\frac{\boldsymbol{r}}{\boldsymbol{R_p^2}}\boldsymbol{\sigma^2}$$

$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{2}g^{22}\left\{\frac{\partial g_{22}}{\partial r}\right\} = \frac{\mathbf{1}}{\boldsymbol{r}}$$

$$\Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{2}g^{33}\left\{\frac{\partial g_{33}}{\partial r}\right\} = \frac{\mathbf{1}}{\boldsymbol{r}}$$

$$\Gamma^2_{33} = \frac{1}{2}g^{22}\left\{-\frac{\partial g_{33}}{\partial\theta}\right\} = -\mathbf{cos}\,\boldsymbol{\theta}\,\mathbf{sin}\,\boldsymbol{\theta} = \mathbf{0}$$

$$\Gamma^3_{23} = \Gamma^3_{32} = \frac{1}{2}g^{33}\left\{\frac{\partial g_{33}}{\partial\theta}\right\} = \frac{\mathbf{cos}\,\boldsymbol{\theta}}{\mathbf{sin}\,\boldsymbol{\theta}} = \mathbf{0}$$

For **r, theta, phi** coordinates:

$$\frac{\partial\Gamma^0_{01}}{\partial r} = \frac{\partial\Gamma^0_{10}}{\partial r} = \frac{\mathbf{R_s(R_s - 2r)}}{\mathbf{2r^4\sigma^4}}$$

$$\frac{\partial\Gamma^1_{00}}{\partial r} = \frac{\mathbf{R_s(3R_s - 2r)}}{\mathbf{2r^4}}$$

$$\frac{\partial \Gamma_{11}^{1}}{\partial r} = \frac{\mathbf{R_s(2r - Rs)}}{\mathbf{2r^4\sigma^4}}$$

$$\frac{\partial \Gamma_{22}^{1}}{\partial r} = -\frac{\mathbf{1}}{\boldsymbol{R_p^2}}$$

$$\frac{\partial \Gamma_{33}^{1}}{\partial r} = -\frac{\mathbf{1}}{\boldsymbol{R_p^2}} \mathbf{\sin^2} \boldsymbol{\theta}$$

$$\frac{\partial \Gamma_{12}^{2}}{\partial r} = \frac{\partial \Gamma_{21}^{2}}{\partial r} = \frac{\partial \Gamma_{13}^{3}}{\partial r} = \frac{\partial \Gamma_{31}^{3}}{\partial r} = \frac{\mathbf{-1}}{\mathbf{r^2}}$$

$$\frac{\partial \Gamma_{33}^{2}}{\partial \theta} = \mathbf{-\cos^2} \boldsymbol{\theta} + \mathbf{\sin^2} \boldsymbol{\theta} = \mathbf{1}$$

$$\frac{\partial \Gamma_{23}^{3}}{\partial \theta} = \frac{\partial \Gamma_{32}^{3}}{\partial \theta} = \frac{-1}{\mathbf{\sin^2} \boldsymbol{\theta}} = \mathbf{-1}$$

Schwarzschild on r, theta, phi coordinates:

$$\Gamma_{\mu\nu}^{\rho} = \frac{1}{2} g^{\rho\alpha} \left\{ \frac{\partial g_{\nu\alpha}}{\partial x^{\mu}} + \frac{\partial g_{\mu\alpha}}{\partial x^{\nu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\alpha}} \right\}$$

First derivative of Christoffel symbol

$$\frac{\partial \Gamma_{\mu\nu}^{\rho}}{\partial x^{\delta}} = \frac{1}{2} \frac{\partial g^{\rho\alpha}}{\partial x^{\delta}} \left\{ \frac{\partial g_{\nu\alpha}}{\partial x^{\mu}} + \frac{\partial g_{\mu\alpha}}{\partial x^{\nu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\alpha}} \right\} + \frac{1}{2} g^{\rho\alpha} \left\{ \frac{\partial^2 g_{\nu\alpha}}{\partial x^{\mu} \partial x^{\delta}} + \frac{\partial^2 g_{\mu\alpha}}{\partial x^{\nu} \partial x^{\delta}} - \frac{\partial^2 g_{\mu\nu}}{\partial x^{\alpha} \partial x^{\delta}} \right\}$$

$$\frac{\partial \Gamma_{\mu\nu}^{\rho}}{\partial x^{\delta}} = \frac{-1}{2} (g^{\rho\alpha})^2 \frac{\partial g_{\rho\alpha}}{\partial x^{\delta}} \left\{ \frac{\partial g_{\nu\alpha}}{\partial x^{\mu}} + \frac{\partial g_{\mu\alpha}}{\partial x^{\nu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\alpha}} \right\} + \frac{1}{2} g^{\rho\alpha} \left\{ \frac{\partial^2 g_{\nu\alpha}}{\partial x^{\mu} \partial x^{\delta}} + \frac{\partial^2 g_{\mu\alpha}}{\partial x^{\nu} \partial x^{\delta}} - \frac{\partial^2 g_{\mu\nu}}{\partial x^{\alpha} \partial x^{\delta}} \right\}$$

Schwarzschild metric on x,y,z

$$x_0 = t_\infty \qquad dx_0 = dt_\infty$$

$$x_1 = \frac{r^3}{3} \qquad dx_1 = r^2.dr \qquad \boldsymbol{\frac{dr}{dx_1} = \frac{1}{r^2}}$$

$$x_2 = -\cos\theta = 0 \quad dx_2 = \sin\theta\,.d\theta = d\theta \quad \boldsymbol{\frac{d\theta}{dx_2} = \frac{1}{\sin\theta}}$$

$$x_3 = \emptyset \qquad dx_3 = d\emptyset$$

$$ds^2 = \sigma^2 c^2 dt_\infty{}^2 - \frac{dx_1{}^2}{r^4\sigma^2} - \frac{r^2 dx_2{}^2}{\sin^2\theta} - r^2\sin^2\theta\, dx_3{}^2$$

Assume at equator level $\theta = 90^0 \Rightarrow \sin\theta = 1$

$$ds^2 = \sigma^2 c^2 dt_\infty{}^2 - \frac{dx_1{}^2}{r^4\sigma^2} - r^2 dx_2{}^2 - r^2 dx_3{}^2$$

Schwarzschild metric on x,y,z

$$g_{00} = \sigma^2 \qquad g^{00} = \frac{1}{\sigma^2}$$

$$g_{11} = -\frac{1}{r^4\sigma^2} \qquad g^{11} = -r^4\sigma^2$$

$$g_{22} = -\frac{r^2}{\sin^2\theta} \qquad g^{22} = -\frac{\sin^2\theta}{r^2}$$

$$g_{33} = -r^2\sin^2\theta = -r^2 \quad g^{33} = \frac{\mathbf{-1}}{r^2\sin^2\theta} = \frac{-1}{r^2}$$

g's are dependent on r (so x_1) and θ *(so x_2):*

$$\frac{dr}{dx_1} = \boldsymbol{\frac{1}{r^2}} \qquad \frac{d\sigma}{dx_1} = \boldsymbol{\frac{R_s}{2r^4\sigma}} \qquad \frac{d\theta}{dx_2} = \boldsymbol{\frac{1}{\sin\theta}}$$

Metric derivative on x,y,z

$$\frac{\partial g_{00}}{\partial x_1} = \frac{\partial g_{00}}{\partial r}\frac{dr}{dx_1} = 2\sigma\frac{R_s}{2r^4\sigma} = \boldsymbol{\frac{R_s}{r^4}}$$

$$\frac{\partial g_{11}}{\partial x_1} = \boldsymbol{\frac{4r - 3R_s}{r^8\sigma^4}}$$

$$\frac{\partial g_{22}}{\partial x_1} = \frac{\partial g_{22}}{\partial r}\frac{dr}{dx_1} = r^{-2}\left(\frac{-2r}{\sin^2\theta}\right) = \boldsymbol{\frac{-2}{r\sin^2\theta} = \frac{-2}{r}}$$

$$\frac{\partial g_{33}}{\partial x_1} = r^{-2}(-2r\sin^2\theta) = \boldsymbol{\frac{-2\sin^2\theta}{r} = \frac{-2}{r}}$$

$$\frac{\partial g_{22}}{\partial x_2} = \frac{2r^2\cos(\theta)}{sin^3(\theta)}.\frac{1}{sin\,\theta} = \boldsymbol{\frac{2r^2\cos(\theta)}{sin^4(\theta)} = 0}$$

$$\frac{\partial g_{33}}{\partial x_2} = \frac{\partial g_{33}}{\partial\theta}\frac{d\theta}{dx_2} = \left(\boldsymbol{-2r^2.sin(\theta)\cos(\theta)}\right)\frac{1}{sin\,\theta}$$
$$= \boldsymbol{-2.r^2.\cos(\theta) = 0}$$

Metric second derivative on x,y,z coordinates

$$\frac{\partial^2 g_{00}}{\partial x_1{}^2} = \boldsymbol{\frac{-4R_s}{r^7}} \qquad \frac{\partial^2 g_{11}}{\partial x_1{}^2} = \boldsymbol{\frac{-2\left(14r^2 + 9R_s{}^2 - 22rR_s\right)}{r^{12}\sigma^6}}$$

$$\frac{\partial^2 g_{22}}{\partial x_1{}^2} = \boldsymbol{\frac{2}{r^4 sin^2(\theta)} = \frac{2}{r^4}}$$

$$\frac{\partial^2 g_{22}}{\partial x_2{}^2} = \boldsymbol{\frac{-2r^2\left(1 + 3\cos^2(\theta)\right)}{sin^6(\theta)} = -2r^2}$$

$$\frac{\partial^2 g_{22}}{\partial x_1\partial x_2} = \frac{\partial^2 g_{22}}{\partial x_2\partial x_1} = \boldsymbol{\frac{4\cos(\theta)}{r sin^4(\theta)} = 0}$$

$$\frac{\partial^2 g_{33}}{\partial x_1{}^2} = \boldsymbol{\frac{2sin^2(\theta)}{r^4} = \frac{2}{r^4}}$$

$$\frac{\partial^2 g_{33}}{\partial x_1\partial x_2} = \frac{\partial^2 g_{33}}{\partial x_2\partial x_1} = \boldsymbol{\frac{-4\cos(\theta)}{r} = 0}$$

$$\frac{\partial^2 g_{33}}{\partial x_2{}^2} = \boldsymbol{2r^2.\sin\theta\frac{1}{\sin\theta} = 2r^2}$$

Schwarzschild on x,y,z

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2}g^{\rho\alpha}\left\{\frac{\partial g_{\nu\alpha}}{\partial x^\mu} + \frac{\partial g_{\mu\alpha}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\alpha}\right\}$$

$$\Gamma^0_{01} = \Gamma^0_{10} = \frac{1}{2}g^{00}\left\{\frac{\partial g_{00}}{\partial x^1}\right\} = \frac{1}{2}\frac{1}{\sigma^2}\frac{R_s}{r^4} = \boldsymbol{\frac{R_s}{2r^4\sigma^2}}$$

$$\Gamma^1_{00} = \frac{1}{2}g^{11}\left\{-\frac{\partial g_{00}}{\partial x^1}\right\} = \frac{1}{2}(-r^4\sigma^2)\frac{-R_s}{r^4} = \boldsymbol{\frac{R_s\sigma^2}{2}}$$

$$\Gamma^1_{11} = \frac{1}{2}g^{11}\left\{\frac{\partial g_{11}}{\partial x^1}\right\} = \frac{1}{2}(-r^4\sigma^2)\frac{4r - 3R_s}{r^8\sigma^4} = \boldsymbol{\frac{3R_s - 4r}{2r^4\sigma^2}}$$

$$\Gamma^1_{22} = \frac{1}{2} g^{11}\left\{-\frac{\partial g_{22}}{\partial x^1}\right\} = \frac{1}{2}(-r^4\sigma^2)\frac{2}{r\sin^2\theta} = \frac{\boldsymbol{-r^3\sigma^2}}{\boldsymbol{\sin^2\theta}} = \boldsymbol{-r^3\sigma^2}$$

$$\Gamma^1_{33} = \frac{1}{2} g^{11}\left\{-\frac{\partial g_{33}}{\partial x^1}\right\} = \frac{1}{2}(-r^4\sigma^2)\frac{2\sin^2\theta}{r} = \boldsymbol{-r^3\sigma^2\sin^2\theta = -r^3\sigma^2}$$

$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{2} g^{22}\left\{\frac{\partial g_{22}}{\partial x^1}\right\} = \frac{1}{2}(-\frac{\sin^2\theta}{r^2})\frac{-2}{r\sin^2\theta} = \frac{\boldsymbol{1}}{\boldsymbol{r^3}}$$

$$\Gamma^2_{33} = \frac{1}{2} g^{22}\left\{-\frac{\partial g_{33}}{\partial x^2}\right\} = \frac{1}{2}\left(-\frac{\sin^2\theta}{r^2}\right)(2.r^2.\cos(\theta)) = \boldsymbol{-\sin^2\theta\cos\theta} = 0$$

$$\Gamma^2_{22} = \frac{1}{2} g^{22}\left\{\frac{\partial g_{22}}{\partial x^2}\right\} = \frac{1}{2}\left(-\frac{\sin^2\theta}{r^2}\right)\frac{2r^2\cos(\theta)}{sin^4(\theta)} = \frac{\boldsymbol{-\cos(\theta)}}{\boldsymbol{\sin^2(\theta)}} = \boldsymbol{0}$$

$$\Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{2} g^{33}\left\{\frac{\partial g_{33}}{\partial x^1}\right\} = \frac{1}{2}(\frac{-1}{r^2\sin^2\theta})\frac{-2\sin^2\theta}{r} = \frac{\boldsymbol{1}}{\boldsymbol{r^3}}$$

$$\Gamma^3_{23} = \Gamma^3_{32} = \frac{1}{2} g^{33}\left\{\frac{\partial g_{33}}{\partial x^2}\right\} = \frac{1}{2}\left(\frac{-1}{r^2\sin^2\theta}\right)(-2.r^2.\cos(\theta)) = \frac{\boldsymbol{\cos\theta}}{\boldsymbol{\sin^2(\theta)}} = \boldsymbol{0}$$

For x,y,z coordinates:

$$\frac{\partial\Gamma^0_{01}}{\partial \mathrm{x}_1} = \frac{\partial\Gamma^0_{10}}{\partial \mathrm{x}_1} = \frac{\mathbf{R_s(3R_s - 4r)}}{\mathbf{2r^8\sigma^4}}$$

$$\frac{\partial\Gamma^1_{00}}{\partial \mathrm{x}_1} = \frac{\mathbf{{R_s}^2}}{\mathbf{2r^4}}$$

$$\frac{\partial\Gamma^1_{11}}{\partial \mathrm{x}_1} = \frac{\mathbf{6}}{\mathbf{r^6\sigma^4}} - \frac{\mathbf{10R_s}}{\mathbf{r^7\sigma^4}} + \frac{\mathbf{4.5{R_s}^2}}{\mathbf{r^8\sigma^4}}$$

$$\frac{\partial\Gamma^1_{22}}{\partial \mathrm{x}_1} = \frac{\mathbf{2R_s - 3r}}{\mathbf{r\sin^2\boldsymbol{\theta}}} = \mathbf{-3} + \frac{\mathbf{2R_s}}{\mathbf{r}}$$

$$\frac{\partial\Gamma^1_{33}}{\partial \mathrm{x}_1} = \left(\mathbf{-3} + \frac{\mathbf{2R_s}}{\mathbf{r}}\right).\boldsymbol{\sin^2\theta} = \mathbf{-3} + \frac{\mathbf{2R_s}}{\mathbf{r}}$$

$$\frac{\partial\Gamma^2_{12}}{\partial \mathrm{x}_1} = \frac{\partial\Gamma^2_{21}}{\partial \mathrm{x}_1} = \frac{\partial\Gamma^3_{13}}{\partial \mathrm{x}_1} = \frac{\partial\Gamma^3_{31}}{\partial \mathrm{x}_1} = \frac{\mathbf{-3}}{\mathbf{r^6}}$$

$$\frac{\partial\Gamma^2_{33}}{\partial \mathrm{x}_1} = \frac{\partial\Gamma^2_{22}}{\partial \mathrm{x}_1} = \frac{\partial\Gamma^3_{23}}{\partial \mathrm{x}_1} = \frac{\partial\Gamma^3_{32}}{\partial \mathrm{x}_1} = \mathbf{0}$$

$$\frac{\partial\Gamma^1_{22}}{\partial \mathrm{x}_2} = \frac{\mathbf{2r^3\sigma^2\cos\boldsymbol{\theta}}}{\mathbf{\sin^4\boldsymbol{\theta}}} = \mathbf{0}$$

$$\frac{\partial\Gamma^1_{33}}{\partial \mathrm{x}_2} = \mathbf{-2r^3\sigma^2\cos\boldsymbol{\theta} = 0}$$

$$\frac{\partial\Gamma^2_{33}}{\partial \mathrm{x}_2} = -3\,\mathbf{\cos^2\boldsymbol{\theta} + 1 = 1}$$

$$\frac{\partial\Gamma^2_{22}}{\partial \mathrm{x}_2} = \frac{\mathbf{1 + \cos^2\boldsymbol{\theta}}}{\mathbf{\sin^4\boldsymbol{\theta}}} = \mathbf{1}$$

$$\frac{\partial\Gamma^3_{23}}{\partial \mathrm{x}_2} = \frac{\partial\Gamma^3_{32}}{\partial \mathrm{x}_2} = \frac{\mathbf{-1 - \cos^2\boldsymbol{\theta}}}{\mathbf{\sin^4\boldsymbol{\theta}}} = \mathbf{-1}$$

Repaired Schwarzschild metric on spherical coordinates:

$$ds^2 = \sigma^2 c^2 dt_\infty{}^2 - \frac{dr^2}{\sigma^2} - \frac{r^2 d\theta^2}{\sigma^2} - \frac{r^2 \sin^2\theta\, d\varphi^2}{\sigma^2}$$

Repaired Schwarzschild metric on spherical coordinates:

$$g_{00} = \sigma^2 \qquad g^{00} = \frac{1}{\sigma^2}$$

$$g_{11} = -\frac{1}{\sigma^2} \qquad g^{11} = -\sigma^2$$

$$g_{22} = -\frac{r^2}{\sigma^2} \qquad g^{22} = -\frac{\sigma^2}{r^2}$$

$$g_{33} = -\frac{r^2\sin^2\theta}{\sigma^2} \qquad g^{33} = -\frac{\sigma^2}{r^2\sin^2\theta}$$

g's are dependent on r (so x_1) and *θ (so x_2):*

$$\frac{d\sigma}{dr} = \boldsymbol{\frac{R_s}{2r^2\sigma}}$$

Repaired Schwarzschild metric on spherical coordinates:

$$\frac{\partial g_{00}}{\partial r} = \frac{\partial g_{00}}{\partial \sigma}\frac{\partial \sigma}{\partial r} = 2\sigma\frac{R_s}{2r^2\sigma} = \boldsymbol{\frac{R_s}{r^2}}$$

$$\frac{\partial g_{11}}{\partial r} = \frac{\partial g_{11}}{\partial \sigma}\frac{\partial \sigma}{\partial r} = \frac{2}{\sigma^3}\frac{R_s}{2r^2\sigma} = \boldsymbol{\frac{R_s}{r^2\sigma^4}}$$

$$\frac{\partial g_{22}}{\partial r} = -\frac{2r}{\sigma^2} + \frac{2r^2}{\sigma^3}\frac{R_s}{2r^2\sigma} = -\frac{2r(1-\frac{R_s}{r})}{\sigma^4} + \frac{R_s}{\sigma^4} = \boldsymbol{\frac{-2r+3R_s}{\sigma^4}}$$

$$\frac{\partial g_{33}}{\partial r} = \boldsymbol{\frac{(-2r+3R_s)\sin^2\theta}{\sigma^4}} = \boldsymbol{\frac{-2r+3R_s}{\sigma^4}}$$

$$\frac{\partial g_{33}}{\partial \theta} = -\boldsymbol{\frac{2r^2}{\sigma^2}\sin(\theta)\cos(\theta) = 0}$$

Repaired Schwarzschild metric on spherical coordinates:

$$\frac{\partial^2 g_{00}}{\partial r^2} = \boldsymbol{\frac{-2R_s}{r^3}} \qquad \frac{\partial^2 g_{11}}{\partial r^2} = \boldsymbol{\frac{-2R_s}{r^3\sigma^6}}$$

$$\frac{\partial^2 g_{22}}{\partial r^2} = \boldsymbol{\frac{-2}{\sigma^6} + \frac{6R_s}{r\sigma^6} - \frac{6{R_s}^2}{r^2\sigma^6}}$$

$$\frac{\partial^2 g_{33}}{\partial r^2} = \boldsymbol{\left(\frac{-2}{\sigma^6} + \frac{6R_s}{r\sigma^6} - \frac{6{R_s}^2}{r^2\sigma^6}\right)\sin^2(\theta)}$$

$$\frac{\partial^2 g_{33}}{\partial r \partial \theta} = \boldsymbol{\frac{2(-2r+3R_s)}{\sigma^4}\sin(\theta)\cos(\theta) = 0}$$

$$\frac{\partial^2 g_{33}}{\partial \theta^2} = \boldsymbol{\frac{2r^2}{\sigma^2}\left(\sin^2(\theta) - \cos^2(\theta)\right) = \frac{2r^2}{\sigma^2}}$$

Repaired Schwarzschild metric on spherical coordinates:

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2}g^{\rho\alpha}\left\{\frac{\partial g_{\nu\alpha}}{\partial x^\mu} + \frac{\partial g_{\mu\alpha}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\alpha}\right\}$$

$$\Gamma^0_{01} = \Gamma^0_{10} = \frac{1}{2}g^{00}\left\{\frac{\partial g_{00}}{dr}\right\} = \frac{1}{2}\frac{1}{\sigma^2}\frac{R_s}{r^2} = \boldsymbol{\frac{R_s}{2r^2\sigma^2}}$$

$$\Gamma^1_{00} = \frac{1}{2}g^{11}\left\{-\frac{\partial g_{00}}{dr}\right\} = \frac{1}{2}(-\sigma^2)\frac{-R_s}{r^2} = \boldsymbol{\frac{R_s\sigma^2}{2r^2}}$$

$$\Gamma^1_{11} = \frac{1}{2}g^{11}\left\{\frac{\partial g_{11}}{dr}\right\} = \frac{1}{2}(-\sigma^2)\frac{R_s}{r^2\sigma^4} = \boldsymbol{\frac{-R_s}{2r^2\sigma^2}}$$

$$\Gamma^1_{22} = \frac{1}{2}g^{11}\left\{-\frac{\partial g_{22}}{dr}\right\} = \frac{1}{2}(-\sigma^2)\frac{2r-3R_s}{\sigma^4} = \boldsymbol{\frac{-2r+3R_s}{2\sigma^2}}$$

$$\Gamma^1_{33} = \frac{1}{2}g^{11}\left\{-\frac{\partial g_{33}}{dr}\right\} = \frac{1}{2}(-\sigma^2)\frac{(2r-3R_s)\sin^2\theta}{\sigma^4} = \boldsymbol{\frac{(-2r+3R_s)\sin^2\theta}{2\sigma^2}} = \boldsymbol{\frac{(-2r+3R_s)}{2\sigma^2}}$$

$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{2}g^{22}\left\{\frac{\partial g_{22}}{dr}\right\} = \frac{1}{2}\left(-\frac{\sigma^2}{r^2}\right)\frac{-2r+3R_s}{\sigma^4} = \boldsymbol{\frac{2r-3R_s}{2r^2\sigma^2}}$$

$$\Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{2}g^{33}\left\{\frac{\partial g_{33}}{\partial r}\right\} = \frac{1}{2}\left(-\frac{\sigma^2}{r^2\sin^2\theta}\right)\frac{(-2r+3R_s)\sin^2\theta}{\sigma^4} = \boldsymbol{\frac{(2r-3R_s)}{2r^2\sigma^2}}$$

$$\Gamma^2_{33} = \frac{1}{2}g^{22}\left\{-\frac{\partial g_{33}}{\partial\theta}\right\} = \frac{1}{2}\left(-\frac{\sigma^2}{r^2}\right)\left(\frac{2r^2}{\sigma^2}\sin(\theta)\cos(\theta)\right) = \boldsymbol{-\sin(\theta)\cos(\theta) = 0}$$

$$\Gamma_{22}^{2} = \frac{1}{2} g^{22} \left\{ \frac{\partial g_{22}}{\partial \theta} \right\} = \frac{1}{2} \left(-\frac{\sigma^2}{r^2} \right) 0 = \mathbf{0}$$

$$\Gamma_{23}^{3} = \Gamma_{32}^{3} = \frac{1}{2} g^{33} \left\{ \frac{\partial g_{33}}{\partial \theta} \right\}$$
$$= \frac{1}{2} \left(-\frac{\sigma^2}{r^2 \sin^2 \theta} \right) \left(\frac{-2r^2}{\sigma^2} sin(\theta) \cos(\theta) \right) = \frac{\mathbf{cos(\boldsymbol{\theta})}}{\boldsymbol{sin(\theta)}}$$
$$= \mathbf{0}$$

Repaired Schwarzschild metric on spherical coordinates:

$$\frac{\partial \Gamma_{01}^{0}}{\partial \mathrm{r}} = \frac{\partial \Gamma_{10}^{0}}{\partial \mathrm{r}} = \frac{\mathbf{-R_s(2r - R_s)}}{\mathbf{2r^4 \boldsymbol{\sigma}^4}}$$

$$\frac{\partial \Gamma_{00}^{1}}{\partial \mathrm{r}} = \frac{\mathbf{R_s(3Rs - 2r)}}{\mathbf{2r^4}}$$

$$\frac{\partial \Gamma_{11}^{1}}{\partial \mathrm{r}} = \frac{\mathbf{R_s(2r - R_s)}}{\mathbf{2r^4 \boldsymbol{\sigma}^4}}$$

$$\frac{\partial \Gamma_{22}^{1}}{\partial \mathrm{r}} = \frac{\mathbf{-1}}{\boldsymbol{\sigma}^{\mathbf{4}}} + \frac{\mathbf{2R_s}}{\mathbf{r}\boldsymbol{\sigma}^{\mathbf{4}}} - \frac{\mathbf{3R_s}^{\mathbf{2}}}{\mathbf{2r^2}\boldsymbol{\sigma}^{\mathbf{4}}}$$

$$\frac{\partial \Gamma_{33}^{1}}{\partial \mathrm{r}} = \left(\frac{\mathbf{-1}}{\boldsymbol{\sigma}^{\mathbf{4}}} + \frac{\mathbf{2R_s}}{\mathbf{r}\boldsymbol{\sigma}^{\mathbf{4}}} - \frac{\mathbf{3R_s}^{\mathbf{2}}}{\mathbf{2r^2}\boldsymbol{\sigma}^{\mathbf{4}}} \right) \mathbf{sin^2}\, \theta$$

$$\frac{\partial \Gamma_{12}^{2}}{\partial \mathrm{r}} = \frac{\partial \Gamma_{21}^{2}}{\partial \mathrm{r}} = \frac{\partial \Gamma_{13}^{3}}{\partial \mathrm{r}} = \frac{\partial \Gamma_{31}^{3}}{\partial \mathrm{r}}$$
$$= \left(-1 + \frac{3\mathrm{Rs}}{\mathrm{r}} - \frac{3\mathrm{Rs}^2}{\mathbf{2r^2}} \right) / \mathbf{r^2 \boldsymbol{\sigma}^4}$$

$$\frac{\partial \Gamma_{33}^{2}}{\partial \mathrm{r}} = \frac{\partial \Gamma_{22}^{2}}{\partial \mathrm{r}} = \frac{\partial \Gamma_{23}^{3}}{\partial \mathrm{r}} = \frac{\partial \Gamma_{32}^{3}}{\partial \mathrm{r}} = \mathbf{0}$$

$$\frac{\partial \Gamma_{33}^{1}}{\partial \theta} = \frac{(-2\mathrm{r} + 3\mathrm{Rs}) 2 \boldsymbol{sin(\theta)} \mathbf{cos(\boldsymbol{\theta})}}{\mathbf{2}\boldsymbol{\sigma}^{\mathbf{2}}}$$

$$\frac{\partial \Gamma_{33}^{2}}{\partial \theta} = \mathbf{sin^2}\, \theta - \mathbf{cos^2}\, \theta = 1$$

$$\frac{\partial \Gamma_{23}^{3}}{\partial \theta} = \frac{\partial \Gamma_{32}^{3}}{\partial \theta} = -\left(\mathbf{1} + \frac{\mathbf{cos^2}\, \theta}{\mathbf{sin^2}\, \theta} \right) = -1$$

Repaired Schwarzschild metric on x,y,z version 1

$x_0 = t_\infty \qquad dx_0 = dt_\infty$

$x_1 = \frac{r^3}{3} \qquad dx_1 = r^2.dr \qquad \boldsymbol{\frac{dr}{dx_1} = \frac{1}{r^2}}$

$x_2 = -\cos\theta = 0 \quad dx_2 = \sin\theta \,.\, d\theta = d\theta \qquad \boldsymbol{\frac{d\theta}{dx_2} = \frac{1}{\sin\theta}}$

$x_3 = \emptyset \qquad dx_3 = d\emptyset$

$$ds^2 = \frac{c^2 dt_\infty{}^2}{\sigma^2} - \frac{dx_1{}^2}{\sigma^2} - \frac{dx_2{}^2}{\sigma^2} - \frac{dx_3{}^2}{\sigma^2}$$

Repaired Schwarzschild metric on x,y,z version 1

$g_{00} = \frac{1}{\sigma^2} \qquad g^{00} = \sigma^2$

$g_{11} = -\frac{1}{\sigma^2} \qquad g^{11} = -\sigma^2$

$g_{22} = -\frac{1}{\sigma^2} \qquad g^{22} = -\sigma^2$

$g_{33} = -\frac{1}{\sigma^2} \quad g^{33} = -\sigma^2$

g's are dependent on r (so x_1) and θ (so x_2):

$$\frac{dr}{dx_1} = \boldsymbol{\frac{1}{r^2}} \qquad \frac{d\sigma}{dx_1} = \boldsymbol{\frac{R_s}{2r^4\sigma}} \qquad \frac{d\theta}{dx_2} = \boldsymbol{\frac{1}{\sin\theta}}$$

Repaired Metric derivative on x,y,z version 1

$$\frac{\partial g_{00}}{\partial x_1} = \frac{\partial g_{00}}{\partial\sigma}\frac{d\sigma}{dx_1} = \frac{-2}{\sigma^3}\frac{R_s}{2r^4\sigma} = \boldsymbol{\frac{-R_s}{r^4\sigma^4}}$$

$$\frac{\partial g_{11}}{\partial x_1} = \frac{\partial g_{11}}{\partial\sigma}\frac{\partial\sigma}{dx_1} = \boldsymbol{\frac{R_s}{r^4\sigma^4}}$$

$$\frac{\partial g_{22}}{\partial x_1} = \frac{\partial g_{22}}{\partial\sigma}\frac{\partial\sigma}{dx_1} = \boldsymbol{\frac{R_s}{r^4\sigma^4}}$$

$$\frac{\partial g_{33}}{\partial x_1} = \frac{\partial g_{33}}{\partial\sigma}\frac{\partial\sigma}{dx_1} = \boldsymbol{\frac{R_s}{r^4\sigma^4}}$$

$$\frac{\partial g_{33}}{\partial x_2} = \mathbf{0}$$

$$\frac{\partial g_{22}}{\partial x_2} = \mathbf{0}$$

Repaired Metric second derivative on x,y,z version 1

$$\frac{\partial^2 g_{00}}{\partial x_1{}^2} = \boldsymbol{\frac{R_s(4r - 2R_s)}{r^8\sigma^6}} \qquad \frac{\partial^2 g_{11}}{\partial x_1{}^2} = \boldsymbol{\frac{-R_s(4r - 2R_s)}{r^8\sigma^6}}$$

$$\frac{\partial^2 g_{22}}{\partial x_1{}^2} = \boldsymbol{\frac{-R_s(4r - 2R_s)}{r^8\sigma^6}}$$

$$\frac{\partial^2 g_{22}}{\partial x_2{}^2} = \mathbf{0}$$

$$\frac{\partial^2 g_{33}}{\partial x_1{}^2} = \boldsymbol{\frac{-R_s(4r - 2R_s)}{r^8\sigma^6}}$$

Repaired Schwarzschild metric on x,y,z version 1

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2}g^{\rho\alpha}\left\{\frac{\partial g_{\nu\alpha}}{\partial x^\mu} + \frac{\partial g_{\mu\alpha}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\alpha}\right\}$$

$$\Gamma^0_{01} = \Gamma^0_{10} = \frac{1}{2}g^{00}\left\{\frac{\partial g_{00}}{\partial x^1}\right\} = \frac{1}{2}\sigma^2\frac{-R_s}{r^4\sigma^4} = \boldsymbol{\frac{-R_s}{2r^4\sigma^2}}$$

$$\Gamma^1_{00} = \frac{1}{2}g^{11}\left\{-\frac{\partial g_{00}}{\partial x^1}\right\} = \frac{1}{2}(-\sigma^2)\frac{R_s}{r^4\sigma^4} = \boldsymbol{\frac{-R_s}{2r^4\sigma^2}}$$

$$\Gamma^1_{11} = \frac{1}{2}g^{11}\left\{\frac{\partial g_{11}}{\partial x^1}\right\} = \frac{1}{2}(-\sigma^2)\frac{R_s}{r^4\sigma^4} = \boldsymbol{\frac{-R_s}{2r^4\sigma^2}}$$

$$\Gamma^1_{22} = \frac{1}{2}g^{11}\left\{-\frac{\partial g_{22}}{\partial x^1}\right\} = \frac{1}{2}(-\sigma^2)\frac{-R_s}{r^4\sigma^4} = \boldsymbol{\frac{R_s}{2r^4\sigma^2}}$$

$$\Gamma^1_{33} = \frac{1}{2}g^{11}\left\{-\frac{\partial g_{33}}{\partial x^1}\right\} = \frac{1}{2}(-\sigma^2)\frac{-R_s}{r^4\sigma^4} = \boldsymbol{\frac{R_s}{2r^4\sigma^2}}$$

$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{2}g^{22}\left\{\frac{\partial g_{22}}{\partial x^1}\right\} = \frac{1}{2}(-\sigma^2)\frac{R_s}{r^4\sigma^4} = \boldsymbol{\frac{-R_s}{2r^4\sigma^2}}$$

$$\Gamma^2_{33} = \frac{1}{2}g^{22}\left\{-\frac{\partial g_{33}}{\partial x^2}\right\} = \frac{1}{2}(-\sigma^2)(0) = \mathbf{0}$$

$$\Gamma^2_{22} = \frac{1}{2}g^{22}\left\{\frac{\partial g_{22}}{\partial x^2}\right\} = \frac{1}{2}(-\sigma^2)0 = \mathbf{0}$$

$$\Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{2}g^{33}\left\{\frac{\partial g_{33}}{\partial x^1}\right\} = \frac{1}{2}(-\sigma^2)\frac{R_s}{r^4\sigma^4} = \boldsymbol{\frac{-R_s}{2r^4\sigma^2}}$$

$$\Gamma^3_{23} = \Gamma^3_{32} = \frac{1}{2} g^{33} \left\{ \frac{\partial g_{33}}{\partial x^2} \right\} = \frac{1}{2} (-\sigma^2)(0) = \mathbf{0}$$

Repaired Schwarzschild metric on x,y,z version 1

$$\frac{\partial \Gamma^0_{01}}{\partial \mathrm{x}_1} = \frac{\partial \Gamma^0_{10}}{\partial \mathrm{x}_1} = \frac{\mathbf{R_s(4r - 3R_s)}}{\mathbf{2r^8\sigma^4}}$$

$$\frac{\partial \Gamma^1_{00}}{\partial \mathrm{x}_1} = \frac{\mathbf{R_s(4r - 3R_s)}}{\mathbf{2r^8\sigma^4}}$$

$$\frac{\partial \Gamma^1_{11}}{\partial \mathrm{x}_1} = \frac{\mathbf{R_s(4r - 3R_s)}}{\mathbf{2r^8\sigma^4}}$$

$$\frac{\partial \Gamma^1_{22}}{\partial \mathrm{x}_1} = \frac{\mathbf{-R_s(4r - 3R_s)}}{\mathbf{2r^8\sigma^4}}$$

$$\frac{\partial \Gamma^1_{33}}{\partial \mathrm{x}_1} = \frac{\mathbf{-R_s(4r - 3R_s)}}{\mathbf{2r^8\sigma^4}}$$

$$\frac{\partial \Gamma^2_{12}}{\partial \mathrm{x}_1} = \frac{\partial \Gamma^2_{21}}{\partial \mathrm{x}_1} = \frac{\partial \Gamma^3_{13}}{\partial \mathrm{x}_1} = \frac{\partial \Gamma^3_{31}}{\partial \mathrm{x}_1} = \frac{\mathbf{R_s(4r - 3R_s)}}{\mathbf{2r^8\sigma^4}}$$

$$\frac{\partial \Gamma^2_{33}}{\partial \mathrm{x}_1} = \frac{\partial \Gamma^2_{22}}{\partial \mathrm{x}_1} = \frac{\partial \Gamma^3_{23}}{\partial \mathrm{x}_1} = \frac{\partial \Gamma^3_{32}}{\partial \mathrm{x}_1} = \mathbf{0}$$

$$\frac{\partial \Gamma^1_{22}}{\partial \mathrm{x}_2} = \mathbf{0}$$

$$\frac{\partial \Gamma^1_{33}}{\partial \mathrm{x}_2} = \mathbf{0}$$

$$\frac{\partial \Gamma^2_{33}}{\partial \mathrm{x}_2} = \mathbf{0}$$

$$\frac{\partial \Gamma^2_{22}}{\partial \mathrm{x}_2} = \mathbf{0}$$

$$\frac{\partial \Gamma^3_{23}}{\partial \mathrm{x}_2} = \frac{\partial \Gamma^3_{32}}{\partial \mathrm{x}_2} = \mathbf{0}$$

Repaired Schwarzschild metric on x,y,z version 2

Here a coordinate system has been chosen where

$$x_1 = r$$

$$ds^2 = \frac{c^2 dt_\infty{}^2}{\sigma^2} - \frac{dx_1{}^2}{\sigma^2} - \frac{dx_2{}^2}{\sigma^2} - \frac{dx_3{}^2}{\sigma^2}$$

Repaired Schwarzschild metric on x,y,z version 2

$$g_{00} = \frac{1}{\sigma^2} \qquad g^{00} = \sigma^2$$

$$g_{11} = -\frac{1}{\sigma^2} \qquad g^{11} = -\sigma^2$$

$$g_{22} = -\frac{1}{\sigma^2} \qquad g^{22} = -\sigma^2$$

$$g_{33} = -\frac{1}{\sigma^2} \qquad g^{33} = -\sigma^2$$

g's are dependent on r (so x_1):

$$\frac{d\sigma}{dx_1} = \boldsymbol{\frac{R_s}{2r^2\sigma}}$$

Repaired Metric derivative on x,y,z version 2

$$\frac{\partial g_{00}}{\partial x_1} = \frac{\partial g_{00}}{\partial \sigma}\frac{d\sigma}{dx_1} = \frac{-2}{\sigma^3}\frac{R_s}{2r^2\sigma} = \boldsymbol{\frac{-R_s}{r^2\sigma^4}}$$

$$\frac{\partial g_{11}}{\partial x_1} = \frac{\partial g_{11}}{\partial \sigma}\frac{\partial \sigma}{dx_1} = \boldsymbol{\frac{R_s}{r^2\sigma^4}}$$

$$\frac{\partial g_{22}}{\partial x_1} = \frac{\partial g_{22}}{\partial \sigma}\frac{\partial \sigma}{dx_1} = \boldsymbol{\frac{R_s}{r^2\sigma^4}}$$

$$\frac{\partial g_{33}}{\partial x_1} = \frac{\partial g_{33}}{\partial \sigma}\frac{\partial \sigma}{dx_1} = \boldsymbol{\frac{R_s}{r^2\sigma^4}}$$

$$\frac{\partial g_{33}}{\partial x_2} = \mathbf{0}$$

$$\frac{\partial g_{22}}{\partial x_2} = \mathbf{0}$$

Repaired Metric second derivative on x,y,z version 2

$$\frac{\partial^2 g_{00}}{\partial x_1{}^2} = \boldsymbol{\frac{2R_s}{r^3\sigma^6}} \qquad \frac{\partial^2 g_{11}}{\partial x_1{}^2} = \boldsymbol{\frac{-2R_s}{r^3\sigma^6}}$$

$$\frac{\partial^2 g_{22}}{\partial x_1{}^2} = \boldsymbol{\frac{-2R_s}{r^3\sigma^6}}$$

$$\frac{\partial^2 g_{22}}{\partial x_2{}^2} = \mathbf{0}$$

$$\frac{\partial^2 g_{33}}{\partial x_1{}^2} = \boldsymbol{\frac{-2R_s}{r^3\sigma^6}}$$

Repaired Schwarzschild metric on x,y,z version 2

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2}g^{\rho\alpha}\left\{\frac{\partial g_{\nu\alpha}}{\partial x^\mu} + \frac{\partial g_{\mu\alpha}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\alpha}\right\}$$

$$\Gamma^0_{01} = \Gamma^0_{10} = \frac{1}{2}g^{00}\left\{\frac{\partial g_{00}}{\partial x^1}\right\} = \frac{1}{2}\sigma^2\frac{-R_s}{r^2\sigma^4} = \boldsymbol{\frac{-R_s}{2r^2\sigma^2}}$$

$$\Gamma^1_{00} = \frac{1}{2}g^{11}\left\{-\frac{\partial g_{00}}{\partial x^1}\right\} = \frac{1}{2}(-\sigma^2)\frac{R_s}{r^2\sigma^4} = \boldsymbol{\frac{-R_s}{2r^2\sigma^2}}$$

$$\Gamma^1_{11} = \frac{1}{2}g^{11}\left\{\frac{\partial g_{11}}{\partial x^1}\right\} = \frac{1}{2}(-\sigma^2)\frac{R_s}{r^2\sigma^4} = \boldsymbol{\frac{-R_s}{2r^2\sigma^2}}$$

$$\Gamma^1_{22} = \frac{1}{2}g^{11}\left\{-\frac{\partial g_{22}}{\partial x^1}\right\} = \frac{1}{2}(-\sigma^2)\frac{-R_s}{r^2\sigma^4} = \boldsymbol{\frac{R_s}{2r^2\sigma^2}}$$

$$\Gamma^1_{33} = \frac{1}{2}g^{11}\left\{-\frac{\partial g_{33}}{\partial x^1}\right\} = \frac{1}{2}(-\sigma^2)\frac{-R_s}{r^2\sigma^4} = \boldsymbol{\frac{R_s}{2r^2\sigma^2}}$$

$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{2}g^{22}\left\{\frac{\partial g_{22}}{\partial x^1}\right\} = \frac{1}{2}(-\sigma^2)\frac{R_s}{r^2\sigma^4} = \boldsymbol{\frac{-R_s}{2r^2\sigma^2}}$$

$$\Gamma^2_{33} = \frac{1}{2}g^{22}\left\{-\frac{\partial g_{33}}{\partial x^2}\right\} = \frac{1}{2}(-\sigma^2)(0) = \mathbf{0}$$

$$\Gamma^2_{22} = \frac{1}{2}g^{22}\left\{\frac{\partial g_{22}}{\partial x^2}\right\} = \frac{1}{2}(-\sigma^2)0 = \mathbf{0}$$

$$\Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{2}g^{33}\left\{\frac{\partial g_{33}}{\partial x^1}\right\} = \frac{1}{2}(-\sigma^2)\frac{R_s}{r^2\sigma^4} = \boldsymbol{\frac{-R_s}{2r^2\sigma^2}}$$

$$\Gamma^3_{23} = \Gamma^3_{32} = \frac{1}{2}g^{33}\left\{\frac{\partial g_{33}}{\partial x^2}\right\} = \frac{1}{2}(-\sigma^2)(0) = \mathbf{0}$$

Repaired Schwarzschild metric on x,y,z version 2

$$\frac{\partial\Gamma_{01}^{0}}{\partial x_1}=\frac{\partial\Gamma_{10}^{0}}{\partial x_1}=\frac{\mathbf{R_s(2r-R_s)}}{\mathbf{2r^4\sigma^4}}$$

$$\frac{\partial\Gamma_{00}^{1}}{\partial x_1}=\frac{\mathbf{R_s(2r-R_s)}}{\mathbf{2r^4\sigma^4}}$$

$$\frac{\partial\Gamma_{11}^{1}}{\partial x_1}=\frac{\mathbf{R_s(2r-R_s)}}{\mathbf{2r^4\sigma^4}}$$

$$\frac{\partial\Gamma_{22}^{1}}{\partial x_1}=\frac{\mathbf{-R_s(2r-R_s)}}{\mathbf{2r^4\sigma^4}}$$

$$\frac{\partial\Gamma_{33}^{1}}{\partial x_1}=\frac{\mathbf{-R_s(2r-R_s)}}{\mathbf{2r^4\sigma^4}}$$

$$\frac{\partial\Gamma_{12}^{2}}{\partial x_1}=\frac{\partial\Gamma_{21}^{2}}{\partial x_1}=\frac{\partial\Gamma_{13}^{3}}{\partial x_1}=\frac{\partial\Gamma_{31}^{3}}{\partial x_1}=\frac{\mathbf{R_s(2r-R_s)}}{\mathbf{2r^4\sigma^4}}$$

$$\frac{\partial\Gamma_{33}^{2}}{\partial x_1}=\frac{\partial\Gamma_{22}^{2}}{\partial x_1}=\frac{\partial\Gamma_{23}^{3}}{\partial x_1}=\frac{\partial\Gamma_{32}^{3}}{\partial x_1}=\mathbf{0}$$

$$\frac{\partial\Gamma_{22}^{1}}{\partial x_2}=\mathbf{0}$$

$$\frac{\partial\Gamma_{33}^{1}}{\partial x_2}=\mathbf{0}$$

$$\frac{\partial\Gamma_{33}^{2}}{\partial x_2}=\mathbf{0}$$

$$\frac{\partial\Gamma_{22}^{2}}{\partial x_2}=\mathbf{0}$$

$$\frac{\partial\Gamma_{23}^{3}}{\partial x_2}=\frac{\partial\Gamma_{32}^{3}}{\partial x_2}=\mathbf{0}$$

$$\mathrm{R}^i_{jkl} = \Gamma^i_{jl,k} - \Gamma^i_{jk,l} + \Gamma^u_{jl}\,\Gamma^{iuk}_{\nu\mu} - \Gamma^u_{jk}\,\Gamma^i_{ul}$$

$$\mathrm{R}_{\mu\nu} = \mathrm{R}^\rho_{\mu\rho\nu} = \Gamma^\rho_{\mu\nu,\rho} - \Gamma^\rho_{\rho\mu,\nu} + \Gamma^\rho_{\rho\lambda}\,\Gamma^\lambda_{\nu\mu} - \Gamma^\rho_{\nu\lambda}\,\Gamma^\lambda_{\rho\mu}$$

$$\mathrm{R}_{\mu\nu} = \mathrm{R}^\rho_{\mu\nu\rho} = -\Gamma^\rho_{\nu\mu,\rho} + \Gamma^\rho_{\mu\rho,\nu} - \Gamma^\rho_{\rho\lambda}\,\Gamma^\lambda_{\nu\mu} + \Gamma^\rho_{\nu\lambda}\,\Gamma^\lambda_{\mu\rho}$$

After some calculations the conclusion was that in order to achieve all Ricci tensor elements being zero in vacuum the Christoffel symbol formula should start with a positive +1/2:

$$\Gamma^\rho_{\mu\nu} = +\frac{1}{2} g^{\rho\alpha}\left\{\frac{\partial g_{\nu\alpha}}{\partial x^\mu} + \frac{\partial g_{\mu\alpha}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\alpha}\right\}$$

The start sign of the Christoffel symbols has no influence on the product of the Christoffel symbols in the Ricci tensor element but only on the sign of the first two terms: the derivatives of the Christoffel symbols.

Schwarschild symmetry

$$\mathrm{R}_{\mu\nu} = \Gamma^0_{\mu\nu,0} - \Gamma^0_{0\mu,\nu} + \Gamma^0_{0\lambda}\,\Gamma^\lambda_{\nu\mu} - \Gamma^0_{\nu\lambda}\,\Gamma^\lambda_{0\mu}$$

$$+\Gamma^1_{\mu\nu,1} - \Gamma^1_{1\mu,\nu} + \Gamma^1_{1\lambda}\,\Gamma^\lambda_{\nu\mu} - \Gamma^1_{\nu\lambda}\,\Gamma^\lambda_{1\mu}$$

$$+\Gamma^2_{\mu\nu,2} - \Gamma^2_{2\mu,\nu} + \Gamma^2_{2\lambda}\,\Gamma^\lambda_{\nu\mu} - \Gamma^2_{\nu\lambda}\,\Gamma^\lambda_{2\mu}$$

$$+\Gamma^3_{\mu\nu,3} - \Gamma^3_{3\mu,\nu} + \Gamma^3_{3\lambda}\,\Gamma^\lambda_{\nu\mu} - \Gamma^3_{\nu\lambda}\,\Gamma^\lambda_{3\mu}$$

$$\mathrm{R}_{\mu\nu} = \Gamma^\rho_{\mu\nu,\rho} - \Gamma^\rho_{\rho\mu,\nu} + \Gamma^\rho_{\rho\lambda}\,\Gamma^\lambda_{\nu\mu} - \Gamma^\rho_{\nu\lambda}\,\Gamma^\lambda_{\rho\mu}$$

$$\mathrm{R}_{00} = \Gamma^1_{00,1} + \Gamma^1_{11}\,\Gamma^1_{00} + \Gamma^2_{21}\,\Gamma^1_{00} + \Gamma^3_{31}\,\Gamma^1_{00} - \Gamma^1_{00}\,\Gamma^0_{10} = \frac{\mathrm{R_s}^2}{2\mathrm{r}^4} - \frac{1}{2}\frac{4r-3R_s}{r^4\sigma^2}\frac{1}{2}R_s\sigma^2 - \frac{1}{2}R_s\sigma^2\frac{1}{2}\frac{R_s}{r^4\sigma^2} - \frac{1}{2}\frac{R_s}{r^4\sigma^2}\frac{1}{2}R_s\sigma^2$$

$$\mathrm{R}_{11} = -\Gamma^0_{01,1} - \Gamma^2_{21,1} - \Gamma^3_{31,1} + \Gamma^0_{01}\,\Gamma^1_{11} + \Gamma^2_{21}\,\Gamma^1_{11} + \Gamma^3_{31}\,\Gamma^1_{11} - \Gamma^0_{10}\,\Gamma^0_{01} - \Gamma^2_{12}\,\Gamma^2_{21} - \Gamma^3_{13}\,\Gamma^3_{31}$$

$$\mathrm{R}_{22} = \Gamma^1_{22,1} - \Gamma^3_{32,2} + \Gamma^0_{01}\,\Gamma^1_{22} + \Gamma^1_{11}\,\Gamma^1_{22} + \Gamma^2_{21}\,\Gamma^1_{22} + \Gamma^3_{31}\,\Gamma^1_{22} - \Gamma^1_{22}\,\Gamma^2_{12} - \Gamma^2_{21}\,\Gamma^1_{22}$$

$$\mathrm{R}_{33} = \Gamma^1_{33,1} + \Gamma^0_{01}\,\Gamma^1_{33} + \Gamma^1_{11}\,\Gamma^1_{33} + \Gamma^2_{21}\,\Gamma^1_{33} - \Gamma^1_{33}\,\Gamma^3_{13}$$

For spherical coordinates and Schwarzschild configuration with theta is 90^0, the following Ricci tensor elements are relevant:

$$\mathrm{R}_{00} = \boldsymbol{\Gamma^1_{00,1}} + \Gamma^1_{00}\,\Gamma^1_{11} + \Gamma^1_{00}\,\Gamma^2_{12} + \Gamma^1_{00}\,\Gamma^3_{13} - \Gamma^0_{01}\,\Gamma^1_{00}$$

$$\mathrm{R}_{11} = -\boldsymbol{\Gamma^0_{10,1}} - \boldsymbol{\Gamma^2_{12,1}} - \boldsymbol{\Gamma^3_{13,1}} + \Gamma^1_{11}\Gamma^0_{10} + \Gamma^1_{11}\,\Gamma^2_{12} + \Gamma^1_{11}\,\Gamma^3_{13} - \Gamma^0_{10}\Gamma^0_{01} - \Gamma^2_{12}\,\Gamma^2_{21} - \Gamma^3_{13}\,\Gamma^3_{31}$$

$$\mathrm{R}_{22} = \boldsymbol{\Gamma^1_{22,1}} - \boldsymbol{\Gamma^3_{23,2}} + \Gamma^1_{22}\,\Gamma^0_{10} + \Gamma^1_{22}\,\Gamma^1_{11} + \Gamma^1_{22}\,\Gamma^3_{13} + \Gamma^2_{22}\,\Gamma^3_{32} - \Gamma^2_{21}\,\Gamma^1_{22} - \Gamma^3_{23}\,\Gamma^3_{32}$$

$$\mathrm{R}_{33} = +\boldsymbol{\Gamma^1_{33,1}} + \boldsymbol{\Gamma^2_{33,2}} + \Gamma^1_{33}\,\Gamma^0_{10} + \Gamma^1_{33}\,\Gamma^1_{11} + \Gamma^1_{33}\,\Gamma^2_{12} + \Gamma^2_{33}\,\Gamma^2_{22} - \Gamma^3_{31}\,\Gamma^1_{33} - \Gamma^3_{32}\,\Gamma^2_{33}$$

$$\mathrm{R}_{33} = \sin^2\theta\,.\,\mathrm{R}_{22}$$

Wanneer theta niet 90^0 is dan komt er bij R_{22} en R_{33} de term $\Gamma^2_{22}\,\Gamma^3_{32}\; en\; resp. +\Gamma^2_{33}\,\Gamma^2_{22}$ bij.

14 Derivation of derivative of Christoffel symbol in general form

It is shown how the Christoffel symbol only depends on the metric tensor elements and its derivatives. This is handy when used in a spreadsheet or program.

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\alpha}\left\{\frac{\partial g_{\nu\alpha}}{\partial x^{\mu}} + \frac{\partial g_{\mu\alpha}}{\partial x^{\nu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\alpha}}\right\}$$

$$\frac{\partial \Gamma^{\rho}_{\mu\nu}}{\partial x^{\gamma}} = \frac{1}{2}\frac{\partial g^{\rho\alpha}}{\partial x^{\gamma}}\left\{\frac{\partial g_{\nu\alpha}}{\partial x^{\mu}} + \frac{\partial g_{\mu\alpha}}{\partial x^{\nu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\alpha}}\right\} + \frac{1}{2} g^{\rho\alpha}\left\{\frac{\partial^2 g_{\nu\alpha}}{\partial x^{\mu}\partial x^{\gamma}} + \frac{\partial^2 g_{\mu\alpha}}{\partial x^{\nu}\partial x^{\gamma}} - \frac{\partial^2 g_{\mu\nu}}{\partial x^{\alpha}\partial x^{\gamma}}\right\}$$

$$\frac{\partial g^{\rho\alpha}}{\partial x^{\gamma}} = \frac{\partial \dfrac{1}{g_{\rho\alpha}}}{\partial x^{\gamma}} = \frac{-1}{{g_{\rho\alpha}}^2}.\frac{\partial g_{\rho\alpha}}{\partial x^{\gamma}} = -(g^{\rho\alpha})^2.\frac{\partial g_{\rho\alpha}}{\partial x^{\gamma}}$$

$$\frac{\partial \Gamma^{\rho}_{\mu\nu}}{\partial x^{\gamma}} = \frac{-1}{2}(g^{\rho\alpha})^2.\frac{\partial g_{\rho\alpha}}{\partial x^{\gamma}}\left\{\frac{\partial g_{\nu\alpha}}{\partial x^{\mu}} + \frac{\partial g_{\mu\alpha}}{\partial x^{\nu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\alpha}}\right\} + \frac{1}{2} g^{\rho\alpha}\left\{\frac{\partial^2 g_{\nu\alpha}}{\partial x^{\mu}\partial x^{\gamma}} + \frac{\partial^2 g_{\mu\alpha}}{\partial x^{\nu}\partial x^{\gamma}} - \frac{\partial^2 g_{\mu\nu}}{\partial x^{\alpha}\partial x^{\gamma}}\right\}$$

$$\boldsymbol{\frac{\partial \Gamma^{\rho}_{\mu\nu}}{\partial x^{\gamma}} = \frac{1}{2} g^{\rho\alpha}\left[-g^{\rho\alpha}.\frac{\partial g_{\rho\alpha}}{\partial x^{\gamma}}\left\{\frac{\partial g_{\nu\alpha}}{\partial x^{\mu}} + \frac{\partial g_{\mu\alpha}}{\partial x^{\nu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\alpha}}\right\} + \left\{\frac{\partial^2 g_{\nu\alpha}}{\partial x^{\mu}\partial x^{\gamma}} + \frac{\partial^2 g_{\mu\alpha}}{\partial x^{\nu}\partial x^{\gamma}} - \frac{\partial^2 g_{\mu\nu}}{\partial x^{\alpha}\partial x^{\gamma}}\right\}\right]}$$

15 Detailed elaboration of Schwarzschild

Schwarzschild on r, theta, phi coordinates:

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\alpha}\left\{\frac{\partial g_{\nu\alpha}}{\partial x^{\mu}} + \frac{\partial g_{\mu\alpha}}{\partial x^{\nu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\alpha}}\right\}$$

$\Gamma^0_{01} = \Gamma^0_{10} = \frac{1}{2} g^{00}\left\{\frac{\partial g_{00}}{\partial r}\right\}$ $\Gamma^1_{00} = \frac{1}{2} g^{11}\left\{-\frac{\partial g_{00}}{\partial r}\right\}$ $\Gamma^1_{11} = \frac{1}{2} g^{11}\left\{\frac{\partial g_{11}}{\partial r}\right\}$ $\Gamma^1_{22} = \frac{1}{2} g^{11}\left\{-\frac{\partial g_{22}}{\partial r}\right\}$

$\Gamma^1_{33} = \frac{1}{2} g^{11}\left\{-\frac{\partial g_{33}}{\partial r}\right\}$ $\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{2} g^{22}\left\{\frac{\partial g_{22}}{\partial r}\right\}$ $\Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{2} g^{33}\left\{\frac{\partial g_{33}}{\partial r}\right\}$ $\Gamma^2_{33} = \frac{1}{2} g^{22}\left\{-\frac{\partial g_{33}}{\partial \theta}\right\}$

$\Gamma^3_{23} = \Gamma^3_{32} = \frac{1}{2} g^{33}\left\{\frac{\partial g_{33}}{\partial \theta}\right\}$

All elements in the metric tensor are zero apart from the elements in the trace. This means that the contravariant elements the directly inverse are of the covariant components. Thus e.g. $g^{00} = \frac{1}{g_{00}}$ *etcetera.*

For **r, theta, phi** coordinates:

Derivatives of gamma to x_1=r:

$$0011 = 0101 = \frac{\partial\Gamma^0_{01}}{\partial r} = \frac{\partial\Gamma^0_{10}}{\partial r} = \frac{1}{2}\left\{\frac{-1}{{g_{00}}^2}\left(\frac{\partial g_{00}}{\partial r}\right)^2 + \frac{1}{g_{00}}\frac{\partial^2 g_{00}}{\partial r^2}\right\} = \frac{1}{2g_{00}}\left\{\frac{-1}{g_{00}}(g_{00}')^2 + g_{00}''\right\}$$

$$1001 = \frac{\partial\Gamma^1_{00}}{\partial r} = \frac{-1}{2}\left\{\frac{-1}{{g_{11}}^2}\frac{\partial g_{11}}{\partial r}\frac{\partial g_{00}}{\partial r} + \frac{1}{g_{11}}\frac{\partial^2 g_{00}}{\partial r^2}\right\} = \frac{-1}{2g_{11}}\left\{\frac{-1}{g_{11}}g_{11}'g_{00}' + g_{00}''\right\}$$

$$1111 = \frac{\partial\Gamma^1_{11}}{\partial r} = \frac{1}{2}\left\{\frac{-1}{{g_{11}}^2}\left(\frac{\partial g_{11}}{\partial r}\right)^2 + \frac{1}{g_{11}}\frac{\partial^2 g_{11}}{\partial r^2}\right\} = \frac{1}{2g_{11}}\left\{\frac{-1}{g_{11}}(g_{11}')^2 + g_{11}''\right\}$$

$$1221 = \frac{\partial\Gamma^1_{22}}{\partial r} = \frac{-1}{2}\left\{\frac{-1}{{g_{11}}^2}\frac{\partial g_{11}}{\partial r}\frac{\partial g_{22}}{\partial r} + \frac{1}{g_{11}}\frac{\partial^2 g_{22}}{\partial r^2}\right\} = \frac{-1}{2g_{11}}\left\{\frac{-1}{g_{11}}g_{11}'g_{22}' + g_{22}''\right\}$$

$$1331 = \frac{\partial\Gamma^1_{33}}{\partial r} = \frac{-1}{2}\left\{\frac{-1}{{g_{11}}^2}\frac{\partial g_{11}}{\partial r}\frac{\partial g_{33}}{\partial r} + \frac{1}{g_{11}}\frac{\partial^2 g_{33}}{\partial r^2}\right\} = \frac{-1}{2g_{11}}\left\{\frac{-1}{g_{11}}g_{11}'g_{33}' + g_{33}''\right\}$$

$$2121 = 2211 = \frac{\partial\Gamma^2_{12}}{\partial r} = \frac{\partial\Gamma^2_{21}}{\partial r} = \frac{1}{2}\left\{\frac{-1}{{g_{22}}^2}\left(\frac{\partial g_{22}}{\partial r}\right)^2 + \frac{1}{g_{22}}\frac{\partial^2 g_{22}}{\partial r^2}\right\} = \frac{1}{2g_{22}}\left\{\frac{-1}{g_{22}}(g_{22}')^2 + g_{22}''\right\}$$

$$3131 = 3311 = \frac{\partial\Gamma^3_{13}}{\partial r} = \frac{\partial\Gamma^3_{31}}{\partial r} = \frac{1}{2}\left\{\frac{-1}{{g_{33}}^2}\left(\frac{\partial g_{33}}{\partial r}\right)^2 + \frac{1}{g_{33}}\frac{\partial^2 g_{33}}{\partial r^2}\right\} = \frac{1}{2g_{33}}\left\{\frac{-1}{g_{33}}(g_{33}')^2 + g_{33}''\right\}$$

$$2331 = \frac{\partial\Gamma^2_{33}}{\partial r} = \frac{-1}{2}\left\{\frac{-1}{{g_{22}}^2}\frac{\partial g_{22}}{\partial r}\frac{\partial g_{33}}{\partial \theta} + \frac{1}{g_{22}}\frac{\partial^2 g_{33}}{\partial r\partial\theta}\right\} = \frac{-1}{2g_{22}}\left\{\frac{-1}{g_{22}}g_{22}'\frac{\partial g_{33}}{\partial\theta} + \frac{\partial^2 g_{33}}{\partial r\partial\theta}\right\}$$

$$3231 = 3321 = \frac{\partial\Gamma^3_{23}}{\partial r} = \frac{\partial\Gamma^3_{32}}{\partial r} = \frac{1}{2}\left\{\frac{-1}{{g_{33}}^2}\frac{\partial g_{33}}{\partial r}\frac{\partial g_{33}}{\partial \theta} + \frac{1}{g_{33}}\frac{\partial^2 g_{33}}{\partial r\partial\theta}\right\} = \frac{1}{2g_{33}}\left\{\frac{-1}{g_{33}}g_{33}'\frac{\partial g_{33}}{\partial\theta} + \frac{\partial^2 g_{33}}{\partial r\partial\theta}\right\}$$

Derivatives of gamma to $x_2=\theta$:

$$1222 = \frac{\partial\Gamma^1_{22}}{\partial\theta} = \frac{-1}{2g_{11}}\frac{\partial^2 g_{22}}{\partial r\partial\theta}$$

$$1332 = \frac{\partial\Gamma^1_{33}}{\partial\theta} = \frac{-1}{2g_{11}}\frac{\partial^2 g_{33}}{\partial r\partial\theta}$$

$$2332 = \frac{\partial\Gamma^2_{33}}{\partial\theta} = \frac{-1}{2}\left\{\frac{-1}{{g_{22}}^2}\frac{\partial g_{22}}{\partial \theta}\frac{\partial g_{33}}{\partial \theta} + \frac{1}{g_{22}}\frac{\partial^2 g_{33}}{\partial\theta^2}\right\} = \frac{-1}{2g_{22}}\left\{\frac{-1}{g_{22}}\frac{\partial g_{22}}{\partial\theta}\frac{\partial g_{33}}{\partial\theta} + \frac{\partial^2 g_{33}}{\partial\theta^2}\right\}$$

$$2222 = \frac{\partial\Gamma^2_{22}}{\partial\theta} = \frac{1}{2}\left\{\frac{-1}{{g_{22}}^2}\left(\frac{\partial g_{22}}{\partial\theta}\right)^2 + \frac{1}{g_{22}}\frac{\partial^2 g_{22}}{\partial\theta^2}\right\} = \frac{1}{2g_{22}}\left\{\frac{-1}{g_{22}}\left(\frac{\partial g_{22}}{\partial\theta}\right)^2 + \frac{\partial^2 g_{22}}{\partial\theta^2}\right\}$$

$$3312 = 3132 = \frac{\partial\Gamma^3_{31}}{\partial\theta} = \frac{\partial\Gamma^3_{13}}{\partial\theta} = \frac{1}{2}\left\{\frac{-1}{{g_{33}}^2}\frac{\partial g_{33}}{\partial r}\frac{\partial g_{33}}{\partial \theta} + \frac{1}{g_{33}}\frac{\partial^2 g_{33}}{\partial r\partial\theta}\right\} = \frac{1}{2g_{33}}\left\{\frac{-1}{g_{33}}g_{33}'\frac{\partial g_{33}}{\partial\theta} + \frac{\partial^2 g_{33}}{\partial r\partial\theta}\right\}$$

$$3232 = 3322 = \frac{\partial\Gamma^3_{23}}{\partial\theta} = \frac{\partial\Gamma^3_{32}}{\partial\theta} = \frac{1}{2}\left\{\frac{-1}{{g_{33}}^2}\left(\frac{\partial g_{33}}{\partial\theta}\right)^2 + \frac{1}{g_{33}}\frac{\partial^2 g_{33}}{\partial\theta^2}\right\} = \frac{1}{2g_{33}}\left\{\frac{-1}{g_{33}}\left(\frac{\partial g_{33}}{\partial\theta}\right)^2 + \frac{\partial^2 g_{33}}{\partial\theta^2}\right\}$$

16 Tweelichamenprobleem.

16.1 Planeetbeweging. Wetten van Kepler.

De zon *S* (massa *M*) en de Planeet S (massa *m*) bevinden zich op een afstand *SP=r* van elkaar (fig.1). Ze trekken elkaar aan volgens de wet van Newton met een attractiekracht:

$$K = \gamma \frac{Mm}{r^2}$$

Hierdoor ondervindt *P* van *S* een versnelling a_1 in de richting *PS* van een bedrag $a_1=(\gamma M/r^2)$.
S ondervindt van P een versnelling a_2 in de richting *SP*, gelijk aan $a_2=(\gamma m/r^2)$.
De relatieve versnelling van P ten opzichte van S bedraagt dan

$$a_r = a_1 + a_2 = \gamma \frac{(M+m)}{r^2}\,.$$

Deze is gericht van *P* naar *S.* Kiest men een rechthoekig coördinatenstelsel (in het vlak van de beweging) met *S* als oorsprong, en de richting van de *X*-as willekeurig, dan zijn de componenten van de versnelling van *P* in *X*- en *Y*-richting achtereenvolgens

$$a_x = -a_r \cos\phi = -\gamma \frac{(M+m)}{r^2}.\frac{x}{r},$$
$$a_y = -a_r \sin\phi = -\gamma \frac{(M+m)}{r^2}.\frac{y}{r},$$

Waarin *Φ* de hoek tussen de voerstraal *r=SP* en de positieve *X*-as voorstelt.

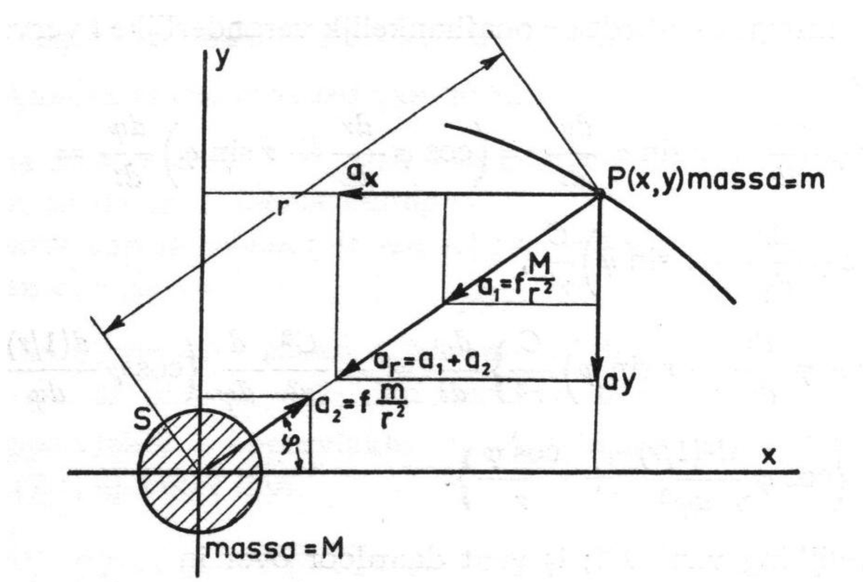


Fig.1

De differentiaalvergelijkingen van de beweging van *P* ten opzichte van *S* zijn dan

$$\begin{cases} \dfrac{d^2x}{dt^2} + \gamma \dfrac{(M+m)x}{r^3} = 0, \\ \dfrac{d^2y}{dt^2} + \gamma \dfrac{(M+m)y}{r^3} = 0. \end{cases} \tag{1}$$

Uit het stelsel (1) volgt

$$x\frac{d^2x}{dt^2} - y\frac{d^2x}{dt^2} = \frac{d}{dt}(x\frac{dy}{dt} - \gamma\frac{dx}{dt}) = 0,$$

dus na integratie

$$x\frac{dy}{dt} - y\frac{dx}{dt} = C. \quad (2)$$

De meetkundige fysische betekenis van deze integratieconstante C wordt reeds voor een deel duidelijk door invoering van poolcoördinaten

$$x\frac{dy}{dt} - y\frac{dx}{dt} = r^2\frac{d\phi}{dt} = C, \quad \text{of} \quad r^2 d\phi = Cdt, \tag{3}$$

$r^2 d\phi =$ tweemaal het oppervlakte-element doorlopen door de voerstraal in *dt* seconden. Uit (3) volgt $\int r^2 d\phi = Ct$ = tweemaal de sector (of het Perk) door de voerstraal in *t* seconden doorlopen. (4)

Door (4) wordt de beroemde *perkenwet= tweede wet van Kepler* in de traditionele telling uitgedrukt:

De oppervlakte van het perk dat door de voerstraal wordt beschreven, is recht evenredig met de daarvoor benodigde tijd.

De constante *C* uit (3) en (4) draagt de naam van *perkenconstante*. Voor de verdere integratie wordt de onafhankelijk veranderlijke *t* vervangen door ϕ , volgens (3)

$$\frac{dx}{dt} = \cos\phi\frac{dr}{dt} - r\sin\phi\frac{d\phi}{dt} = \left(\cos\phi\frac{dr}{d\phi} - r\sin\phi\right)\frac{d\phi}{dt} = \left(\cos\phi\frac{dr}{d\phi} - r\sin\phi\right)\frac{C}{r^2},$$

$$\frac{d^2x}{dt^2} = \frac{d}{d\phi}\left\{\left(\cos\phi\frac{dr}{d\phi} - r\sin\phi\right)\frac{C}{r^2}\right\}\frac{d\phi}{dt} = -\frac{C^2}{r^2}\frac{d}{d\phi}\left\{\left(\cos\phi\frac{d(1/r)}{d\phi} + \frac{\sin\phi}{r}\right)\right\} = -\frac{C^2}{r^2}\left(\cos\phi\frac{d^2(1/r)}{d\phi^2} + \frac{\cos\phi}{r}\right),$$ De eerste vergelijking van (1) gat daardoor over in

$$-\frac{C^2}{r^2}\cos\phi\left(\frac{d^2(1/r)}{d\phi^2} + \frac{1}{r}\right) + \frac{\gamma(M+m)\cos\phi}{r^2} = 0,$$

of

$$\left(\frac{d^2(1/r)}{d\phi^2} + \frac{1}{r}\right) = \frac{\gamma(M+m)}{C^2} = \text{constant}, \tag{5}$$

Het resultaat is een gewone volledige differentiaalvergelijking van de tweede orde in1/*r*, met constante coëfficiënten. (De tweede vergelijking van (1) voert tot hetzelfde resultaat.) De integratie van (5) levert

$$\frac{1}{r} = \frac{\gamma(M+m)}{C^2} + A\cos\phi + \mathrm{B}\sin\phi = \frac{\gamma(M+m)}{C^2} + E\cos(\phi - \alpha), \tag{6}$$

waarin *A, B, E* en *α* nieuwe integratieconstanten voortellen. In plaats van *C* en *E* worden twee nieuwe constanten *α* en *e* ingevoerd, bepaald door

$$E = \frac{\gamma(M+m)}{C^2}.e \tag{7a}$$

$$\frac{\gamma(M+m)}{C^2}=\frac{1}{a\left(1-e^2\right)}. \tag{7b}$$

De uitkomst (6) wordt dan

$$\boxed{r=\frac{a\left(1-e^2\right)}{1+e\cos(\phi-\alpha)}.} \tag{8}$$

Dit is de *poolvergelijking* van een kegelsnede, waarvan het éne brandpunt samenvalt met de pool *S*. In het normale geval is *e < 1*. Dan wordt de kegelsnede een ellips. De betekenis van de parameters *a* en *e* is duidelijk.

$$r_{\min}=\textit{perihelium}\text{ - afstand}=\frac{a\left(1-e^2\right)}{1+e}=a(1-e)\text{ voor }\phi=\alpha.\ \text{ (eigenlijk ABS(e))} \tag{8a}$$

$$r_{\max}=\textit{aphelium}\text{ - afstand}=\frac{a\left(1-e^2\right)}{1-e}=a(1+e)\text{ voor }\phi=\alpha+\pi.\ \text{ (eigenlijk ABS(e))} \tag{8b}$$

Dus

a is de lengte van de *halve grote* as van de ellips,
e is de *numerieke excentriciteit* van de ellips.

De halve kleine as $b=a\sqrt{1-e^2}.$ Het resultaat wordt uitgedrukt door de *eerste wet van Kepler,* in de traditionele telling:

De baankromme van de planeet is een ellips, waarvan één van de brandpunten wordt ingenomen door de zon.

De betekenis van de *perkenconstante C* kan nu worden verduidelijkt. Noemt men de *omlooptijd* van de planeet =*T*, dan wordt in de tijd *T* een perk beschreven waarvan de oppervlakte = oppervlakte van de ellips = $\pi ab=\pi a^2\sqrt{1-e^2}.$ Uit de perkenwet (4) volgt dan

$$CT=2\pi a^2\sqrt{1-e^2},\ \text{dus } C=\frac{2\pi a^2\sqrt{1-e^2}}{T}. \tag{9}$$

Tenslotte volgt uit (9) en (7b)

$$\frac{a^3}{T^2}=\frac{\gamma(M+m)}{4\pi^2}. \tag{10}$$

Deze vergelijking geeft in *verbeterde* vorm de *derde wet van Kepler* aan. In de regel wordt deze wet in elementaire leerboeken der kosmografie aldus geformuleerd:

Bij de verschillende planeten verhouden zich de kwadraten der omloopstijden als de derdemachten der halve grote assen.

Wanneer dus de halve grote assen van twee planeten a_1 en a_2 zijn en de overeenkomstige omloopstijden T_1 en T_2, dan zou volgens de derde wet van Kepler in deze primitieve vorm

$$\frac{a_1^3}{T_1^2}=\frac{a_2^3}{T_2^2}=\text{constant}$$

moeten zijn. Het blijkt echter uit (10) dat deze uitkomst niet van planeet tot planeet constant is, maar afhankelijk is van de individuele massa der planeet. Als de massa's der betrokken planeten m_1 en m_2 zijn, dan is

$$\frac{a_1^3}{T_1^2}=\frac{\gamma M\left\{1+\left(m_1/M\right)\right\}}{4\pi^2};\quad \frac{a_2^3}{T_2^2}=\frac{\gamma M\left\{1+\left(m_2/M\right)\right\}}{4\pi^2}.$$

Beide uitkomsten verschillen echter *zeer* weinig van de *constante* waarde $\gamma M / 4\pi^2$, omdat in het ergste geval (Jupiter) *(m/M)* < 0.001 is, zodat de factor 1+*(m/M)* ∞ 1 is. Uit de aard der zaak is hierbij afgezien van de zeer geringe storingsinvloed die beide planeten op elkaar uitoefenen.

16.2 Het bepalen van de planeetcurve als de positie op de X-as en Y-as en de snelheden Vx en Vy bekend zijn.

De curve is dus gebaseerd op vergelijk (8)

$$\boxed{r = \frac{a\left(1-e^2\right)}{1+e\cos(\varphi-\alpha)}.}$$

De posities en de snelheden op een bepaald tijdstip noemen we: *Xo, Yo, Vxo* en *Vyo.* De onbekenden *a, e, φ* en *α* moeten dus uitgedrukt worden in deze beginwaarden.

$$r_0 = \sqrt{x_0^2 + y_0^2}$$

$$\cos\varphi_0 = \frac{x_0}{r_0}, \quad \sin\varphi_0 = \frac{y_0}{r_0}, \quad tg\varphi_0 = \frac{y_0}{x_0}. \qquad \text{Dus } \varphi_0 = arctg\,\frac{y_0}{x_0}$$

(In het onderstaande worden voor het gemak de indices (o) weggelaten. De formules blijven n.l. universeel geldig en de beginvoorwaarden kunnen later ingevuld worden.)

Uit $x = r\cos\varphi$ en $y = r\sin\varphi$ kan afgeleid worden:

$$\frac{dx}{dt} = V_x = \frac{dr}{dt}\cos\varphi - r\sin\varphi\,\frac{d\varphi}{dt} \tag{11}$$

$$\frac{dy}{dt} = V_y = \frac{dr}{dt}\sin\varphi + r\cos\varphi\,\frac{d\varphi}{dt} \tag{12}$$

Hieruit volgen:

$$\frac{dr}{dt} = V_x x\cos\varphi + V_y\sin\varphi \tag{13}$$

$$r\,\frac{d\varphi}{dt} = V_y\cos\varphi - V_x\sin\varphi \tag{14}$$

Differentiëren van (8) geeft:

$$\frac{dr}{dt} = \frac{re\sin(\varphi-\alpha)}{1+e\cos(\phi-\alpha)}.\frac{d\varphi}{dt} \tag{15}$$

Verder is uit (2):

$$\boxed{C = xV_y - yV_x} \tag{16}$$

Uit (7b)

$$\boxed{a = \frac{C^2}{\gamma(M+m)(1-e^2)}} \tag{17}$$

Uit (13), (15) en (15):

$$V_x x + V_y y = \frac{e\sin(\varphi-\alpha)}{1+e\cos(\varphi-\alpha)}.(V_y x - V_x y) = \frac{e\sin(\varphi-\alpha)}{1+e\cos(\varphi-\alpha)}C$$

$$\Rightarrow (xV_x + yV_y)(1+e\cos(\varphi-\alpha)) = eC\sin(\varphi-\alpha)$$

$$\Rightarrow (xV_x + yV_y) = eC\sin(\varphi-\alpha) - e\cos(\varphi-\alpha)(xV_x + yV_y)$$

$$\Rightarrow \boxed{e = \frac{xV_x + yV_y}{C\sin(\varphi-\alpha) - (xV_x + yV_y)\cos(\varphi-\alpha)}} \tag{18}$$

Vanuit (8) en (17):

$$r = \sqrt{x^2+y^2} = \frac{C^2}{\gamma(M+m)(1-e^2)}.\frac{(1-e^2)}{1+e\cos(\varphi-\alpha)} = \frac{C^2}{\gamma(M+m)(1+e\cos(\varphi-\alpha))}$$

$$\Rightarrow 1+e\cos(\varphi-\alpha) = \frac{C^2}{\gamma(M+m)\sqrt{x^2+y^2}}$$

$$\Rightarrow e = \frac{\dfrac{C^2}{\gamma(M+m)\sqrt{x^2+y^2}} - 1}{\cos(\varphi-\alpha)} \tag{19}$$

Uit (18) en (19):

$$\frac{xV_x + yV_y}{C\sin(\varphi-\alpha) - (xV_x + yV_y)\cos(\varphi-\alpha)} = \frac{\dfrac{C^2}{\gamma(M+m)\sqrt{x^2+y^2}} - 1}{\cos(\varphi-\alpha)}$$

$$\Rightarrow \frac{1}{\dfrac{C^2}{\gamma(M+m)\sqrt{x^2+y^2}} - 1} = \frac{C\sin(\varphi-\alpha) - (xV_x + yV_y).\cos(\varphi-\alpha)}{(xV_x + yV_y).\cos(\varphi-\alpha)}$$

$$\Rightarrow \frac{1}{\dfrac{C^2}{\gamma(M+m)\sqrt{x^2+y^2}} - 1} = \frac{C}{xV_x + yV_y}.tg(\varphi-\alpha) - 1$$

$$\Rightarrow tg(\varphi-\alpha=)\frac{\dfrac{\dfrac{1}{C^2}}{\gamma(M+m)\sqrt{x^2+y^2}}-1+1}{\dfrac{C}{xV_x+yV_y}}$$

$$\Rightarrow tg(\varphi-\alpha=)\frac{\dfrac{\gamma(M+m)\sqrt{x^2+y^2}}{C^2-\gamma(M+m)\sqrt{x^2+y^2}}+1}{\dfrac{C}{xV_x+yV_y}}==\Rightarrow=\frac{\gamma(M+m)\sqrt{x^2+y^2}+C^2-\lambda(M+m)\sqrt{x^2+y^2}}{C^2-\gamma(M+m)\sqrt{x^2+y^2}}.\frac{xV_x+yV_y}{C}$$

$$\Rightarrow tg(\varphi-\alpha)=\frac{C(xV_x+yV_y)}{C^2-\gamma(M+m)\sqrt{x^2+y^2}}=P=\frac{tg\varphi-tg\alpha}{1+tg\varphi.tg\alpha}$$

$$\Rightarrow tg\alpha=\frac{\dfrac{y}{x}-P}{\dfrac{y}{x}.P+1}=\frac{y-Px}{yp+x} \qquad (20)$$

Dus hierbij is gesteld: $\boxed{P=\dfrac{C(xV_x+yV_y)}{C^2-\gamma(M+m)\sqrt{x^2+y^2}}}$

Uit (20):

$$\boxed{\alpha=arctg\left(\frac{y-Px}{yp+x}\right)} \qquad (21)$$

Afleiding om snelheid van de planeet te bepalen.
Afgeleide van (8):

$$\frac{dr}{dt}=\frac{-r}{1+e\cos(\varphi-\alpha)}.(-e\sin(\varphi-\alpha)).\frac{d\varphi}{dt} \qquad (22)$$

Uit (11) en (22):

$$\Rightarrow V_x=\frac{dr}{dt}.\cos\varphi-r.\sin\varphi.\frac{d\varphi}{dt}=\frac{re\sin(\varphi-\alpha)}{1+e\cos(\varphi-\alpha)}.\frac{d\varphi}{dt}.\cos\varphi-r\sin\varphi.\frac{d\varphi}{dt}$$

$$\Rightarrow V_x=\frac{re\sin(\varphi-\alpha)}{1+e\cos(\varphi-\alpha)}.\frac{d\varphi}{dt}.\cos\varphi-r\sin\varphi.\frac{d\varphi}{dt}=r\frac{d\varphi}{dt}.\left[\frac{e\sin(\varphi-\alpha)}{1+e\cos(\varphi-\alpha)}.\cos\varphi-\sin\varphi\right]$$

Uit (12) en (22):

$$\Rightarrow V_y=\frac{re\sin(\varphi-\alpha)}{1+e\cos(\varphi-\alpha)}.\frac{d\varphi}{dt}.\sin\varphi+r\cos\varphi.\frac{d\varphi}{dt}=r\frac{d\varphi}{dt}.\left[\frac{e\sin(\varphi-\alpha)}{1+e\cos(\varphi-\alpha)}.\sin\varphi+\cos\varphi\right]$$

$$\Rightarrow V_x = V_y.\frac{\left[\dfrac{e\sin(\varphi-\alpha)\cos\varphi}{1+e\cos(\varphi-\alpha)} - \sin\varphi\right]}{\left[\dfrac{e\sin(\varphi-\alpha).\sin\varphi}{1+e\cos(\varphi-\alpha)} + \cos\varphi\right]}$$

$$\Rightarrow V_x = V_y.\frac{e\sin(\varphi-\alpha).\cos\varphi - \sin\varphi - e\cos(\varphi-\alpha).\sin\varphi}{e\sin(\varphi-\alpha).\sin\varphi + \cos\varphi + e\cos(\varphi-\alpha).\cos\varphi} = K..V_y \quad (23)$$

uit (16) en (23):

$$C = xV_y - yV_x \Rightarrow V_y = \frac{C+yV_x}{x} \Rightarrow V_x = K.\frac{C+yV_x}{x}$$
$$\Rightarrow xV_x = K.C + K.yV_x \Rightarrow V_x.(x-Ky) = K.C$$

$$\boxed{\Rightarrow V_x = \frac{K.C}{x-y.K} \quad \text{en} \quad V_y = \frac{C}{x-y.K}}$$

Ellips relateren aan de inputgegevens: langste afstand; Rmax, en de kortste afstand; Rmin (of eventueel aan de omlooptijd T)

Uit (8a) en (8b):

$$\frac{1+e}{1-e} = \frac{R\max}{R\min} \Rightarrow R\min + eR\min = R\max - eR\max \Rightarrow$$

$$\boxed{\Rightarrow e = \frac{R\max - R\min}{R\max + R\min}} \quad \text{en} \quad \boxed{a = \frac{1}{2}(R\max + R\min)} \qquad (23)$$

Uit (7b):

$$\boxed{C = \sqrt{\gamma(M+m)a(1-e^2)}} \quad \text{en} \quad \boxed{T = \frac{2\pi a^2\sqrt{1-e^2}}{C}}$$

Hiermee kan de curve beschreven worden.

Nu als de afstand (Rmax) bekend is en de omlooptijd.

Uit (10) volgt:

$$\frac{a^3}{T^2} = \frac{\gamma(M+m)}{4\pi^2}. \qquad \Rightarrow a^3 = \frac{T^2\gamma(M+m)}{4\pi^2}. \qquad \boxed{\Rightarrow a = \sqrt[3]{\frac{T^2\chi(M+m)}{4\pi^2}}.}$$

Uit (23) volgt:

$$a=\frac{1}{2}(R\max+R\min)\Rightarrow R\min=2a-R\max$$

$$\boxed{\Rightarrow R\min=2\sqrt[3]{\frac{T^2\chi(M+m)}{4\pi^2}}-R\max.}$$

$$\boxed{\Rightarrow e=\frac{2R\max}{2\sqrt[3]{\frac{T^2\chi(M+m)}{4\pi^2}}}-1}$$

17 Co-ordinate systems

17.1 Rectangular co-ordinate system

In order to distinguish between points in space a coordinate system is created. The main characteristics of a coordinate system are an origin and the co-ordinate axis. The origin may be chosen what is most practical and for the axis mostly a Cartesian system is chosen because of its simplicity. In a Cartesian co-ordinate system:

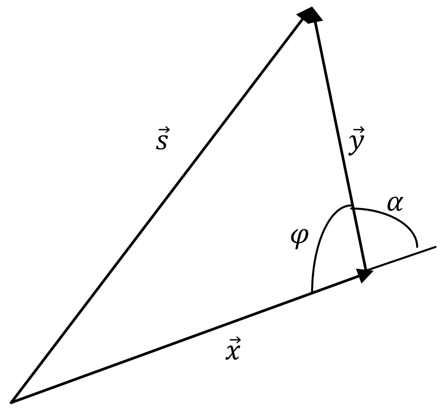
- the axis are perpendicular on each other.
- The axis are independent from each other. i.e. by changing the size of one co-ordinate does not influence the others.
- The axis have a direction and size and therefore they could be considered as vectors.

A point in space is depicted by its coordinates f.i. A(x_a,y_a) The x_a can be found by drawing a line, parallel to the y-ax; where that line intersects with the x-ax that point is x_a The same for the y_a.
The distance of point A to the origin can be found by Pythagoras. $(A\text{-origin})^2=x_a^2+y_a^2$.
In case one works with a line segment between A en B then the size are: $(A\text{-}B)^2=(x_a\text{-}x_b)^2+(y_a\text{-}y_b)^2$. The advantage here is that the length of the line segment is independent of the arbitrary chosen origin.

17.2 Non-rectangular co-ordinate system

Because of practical reasons also a co-ordinate system can be chosen of which the axis are not orthogonal. Now again we have to realize that the segment s is build up out of vectors:

$$\vec{s}=\vec{x}+\vec{y}$$



The size s of $\vec{s}$ can be found by the in-product of $\vec{s}$ with itself:

$$\vec{s}.\vec{s} = (\vec{x}+\vec{y}).(\vec{x}+\vec{y}) = \vec{x}.\vec{x}+\vec{x}.\vec{y}+\vec{y}.\vec{x}+\vec{y}.\vec{y}$$
$$s^2 = x^2 + 2\cos\alpha\, xy + y^2$$
$$\cos\alpha = \cos(180^0 - \varphi) = -\cos\varphi$$
$$s^2 = x^2 + y^2 - 2\cos\varphi \cdot xy$$

This is the well known cosine rule.

17.3 Curved co-ordinates

Instead of co-ordinate axes that are not orthogonal it could also be practical to have curved co-ordinates. To work with these is obviously more complicated but Einstein had the following approach:
A curved line could be considered as a line build up out of infinitesimal straight lines. Looking at an infinitesimal small area these curved co-ordinates could be considered as a coordinate system with straight co-ordinates; but not necessarily orthogonal. Because the co-ordinate system here concerns infinitesimal co-ordinates the co-ordinates are depicted as dx, dy etcetera. Furthermore these co-ordinates have coefficients and these coefficients contain information about the curvature of the coordinate lines. So the coefficients, in case of curvature, are not constants anymore but variables depending on their location along the co-ordinate lines. It is said that the gravity bend the co-ordinate lines but, as I gather it, the gravity deforms the space and creates gravitational force and thus acceleration. However by choosing a curved co-ordinate system in such a way that it moves and curves according to the direction of the gravity field, no force or gravity is experienced. In the same way as a moving co-ordinate system was chosen, in case of special relativity, to nullify the speed of the moving object.

17.4 General form for a co-ordinate system.

Let us derive an equation for the relation between a line segment and its curved co-ordinate system. As mentioned before an infinitesimal line segment d$\vec{s}$ is a vector and the size can be calculated as shown above:

$$\overrightarrow{ds}.\overrightarrow{ds} = \left(\overrightarrow{dx}+\overrightarrow{dy}\right).\left(d\vec{x}+\overrightarrow{dy}\right) = \overrightarrow{dx}.\overrightarrow{dx}+\overrightarrow{dx}.\overrightarrow{dy}+\overrightarrow{dy}.\overrightarrow{dx}+\overrightarrow{dy}.\overrightarrow{dy}$$

In order to have a more general form it is assume that each term has a coefficient $g_{\mu v}$:

$$ds^2 = g_{xx}\overrightarrow{dx}.\overrightarrow{dx} + g_{xy}\overrightarrow{dx}.\overrightarrow{dy} + g_{yx}\overrightarrow{dy}.\overrightarrow{dx} + g_{yy}\overrightarrow{dy}.\overrightarrow{dy}$$
$$Here\ g_{xx} = g_{yy=1}\ \ and\ \ g_{xy} = g_{yx} = -\cos\varphi$$

The $g_{\mu\nu}$ is called the metric tensor and could be regarded in this two dimensional co-ordinate system as a matrix of 2x2 elements. For a general form:

$$ds^2 = \sum_{\mu\nu} g_{\mu\nu}\, dx^\mu dx^\nu$$

In Einstein notation:

$$\boldsymbol{ds^2 = g_{\mu\nu} dx^\mu dx^\nu}$$

For a space-time four dimensional co-ordinate system $\mu\ and\ \nu$ can be 0,1,2,3 or t,x,y,z. So this formula shows the product of each co-ordinate and the cross products between each co-ordinate pair. In case the co-ordinate system is orthogonal then $\mu = \nu$. As is said before this local co-ordinate system consists of straight lines but the information about the curvature may not be lost and will be part of the $g_{\mu\nu}$ elements.

In case a different co-ordinate system is used then it still describes the same line segment, in that case the relation between the two co-ordinate systems is shown in:

$$\boldsymbol{ds^2 = g_{\mu\nu}(x) dx^\mu dx^\nu = g_{mn}(y) dy^m dy^n}$$

17.5 Transformation between two co-ordinate systems.

As is mentioned above that in case of a curved co-ordinate system "locally" in an infinitesimal area a co-ordinate system with straight lines can be used. For an four dimensional co-ordinate system then each new co-ordinate has a relation with all old co-ordinates.

$$dx^0 = \frac{\partial x^0}{\partial y^0} dy^0 + \frac{\partial x^0}{\partial y^1} dy^1 + \frac{\partial x^0}{\partial y^2} dy^2 + \frac{\partial x^0}{\partial y^3} dy^3$$

The same goes for the three other co-ordinates and leads to the general formula:

$$\boldsymbol{dx^m = \frac{\partial x^m}{\partial y^r} dy^r}$$

18 Infinitesimal transformation between Cartesian and polar coordinates.

It is assumed that the reader knows the following relation between polar and Cartesian coordinates:

$$x = r \sin\theta \cos\varphi \qquad y = r \sin\theta \sin\varphi \qquad z = r\cos\theta$$

Derivation of the dx, dy en dz:

$$dx = \sin\theta \cos\varphi\, dr + r\cos\theta \cos\varphi\, d\theta - r\sin\theta \sin\varphi\, d\varphi$$

$$dy = \sin\theta \sin\varphi\, dr + r\cos\theta \sin\varphi\, d\theta + r\sin\theta \cos\varphi\, d\varphi$$

$$dz = \cos\theta\, dr - r\sin\theta\, d\theta$$

Because the coordinates $r, \theta\ and\ \varphi$ are perpendicular the cross terms are zero, thus remaining:

$$dx^2 = \sin^2\theta \cos^2\varphi\, dr^2 + r^2\cos^2\theta \cos^2\varphi\, d\theta^2 + r^2 \sin^2\theta \sin^2\varphi\, d\varphi^2$$

$$dy^2 = \sin^2\theta \sin^2\varphi \, dr^2 + r^2 \cos^2\theta \sin^2\varphi \, d\theta^2 + r^2 \sin^2\theta \cos^2\varphi \, d\varphi^2$$

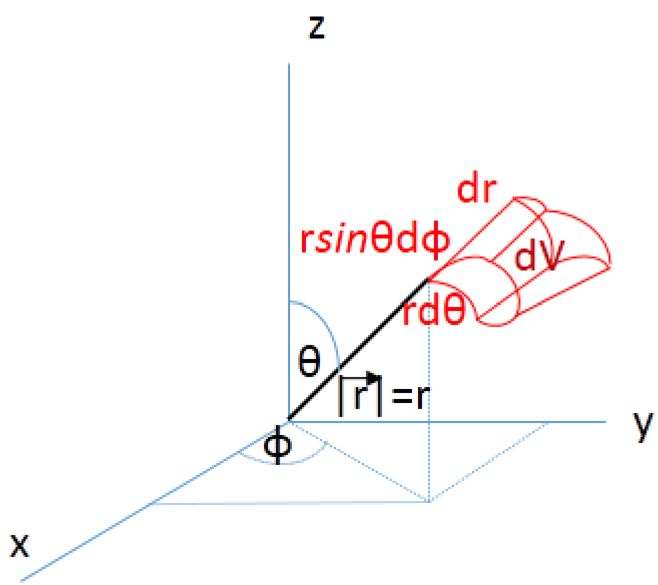
$$dz^2 = \cos^2\theta \, dr^2 + r^2 \sin^2\theta \, d\theta^2$$

Now summation of dx^2+dy^2+dz^2:

$$dx^2+dy^2+dz^2 = \sin^2\theta \cos^2\varphi \, dr + \sin^2\theta \sin^2\varphi \, dr + \cos^2\theta \, dr^2 + r^2 \cos^2\theta \cos^2\varphi \, d\theta^2 + r^2 \cos^2\theta \sin^2\varphi \, d\theta^2 + r^2 \sin^2\theta \, d\theta^2 + r^2 \sin^2\theta \sin^2\varphi \, d\varphi^2 + r^2 \sin^2\theta \cos^2\varphi \, d\varphi^2$$

$$\boldsymbol{dx^2+dy^2+dz^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\varphi^2}$$

Volume element dxdydz:



$$\boldsymbol{dV = dxdydz} = dr.r\,d\theta.\mathrm{r}\sin\varphi\,d\varphi = \boldsymbol{r^2 \sin\varphi \, dr d\theta d\varphi}$$

19 Deliberations on the Minkowsky and Schwarzschild formula.

19.1 Minkowsky

Assume a point K in space-time with his own coordinate system t,x,y,z. The point K stays in the origin of its coordinate system, so x=y=z=0. The only thing that moves, i.e., progresses, is the time and because it is in space-time, the distance, or interval, is **s=ct.** An observer is at another location with his/her own coordinate system but there is a relative movement between the two coordinate systems. The relation between the two system is:

$$v^2 = \frac{(x^2 + y^2 + z^2)}{t^2}$$

This means the observer sees K moving with a speed v.

$$s^2 = c^2t^2 - x^2 - y^2 - z^2$$
$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$$

It has be realized that t,x,y and z have a size and direction, they are vectors. Thus finding the size of s is adding the four vectors. If this co-ordinate system is an orthogonal system than Pythagoras theorem can be applied.

To find a general form for the relation between line segment s and its co-ordinates:

$$\vec{s} = a_1\vec{x}_1 + a_2\vec{x}_2$$

To find the size of s we find the in-product of s by multiplying s with itself:

$$\vec{s}.\vec{s} = (a_1\vec{x}_1 + a_2\vec{x}_2).(a_1\vec{x}_1 + a_2\vec{x}_2)$$

$$s^2 = a_1{}^2x_{1+}^2a_1a_2\vec{x}_1.\vec{x}_2 + a_1a_2\vec{x}_2.\vec{x}_{1+}a_2{}^2x_2^2$$

This was for two dimensions but to generalize this to four dimensions:

$$s^2 = \sum_{\mu}\sum_{\nu} g_{\mu\nu}\, x^{\mu}x^{\nu}$$

Or in Einstein notation:

$$s^2 = g_{\mu\nu}x^{\mu}x^{\nu}$$

When an orthogonal co-ordinate system is used then all products where $\mu \neq \nu$ vanish. When only a infinitesimal small local “area” is considered dx is used instead of x etcetera.
Finally, when an orthogonal co-ordinate system is used, the equation results in a Minkowsky or Schwarzschild form:

$$ds^2 = (cdx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2$$
$$ds^2 = g_{00}(cdx^0)^2 - g_{11}(dx^1)^2 - g_{22}(dx^2)^2 - g_{33}(dx^3)^2$$

What does the Minkowsky formula actually mean?

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2 = c^2dt'^2 - dx'^2 - dy'^2 - dz'^2$$

The ds term signifies an object which is in its own co-ordinate system where x=y=z=0 only t progresses. An observer is in system, t,x,y,z he perceives that ds moves with an velocity of

$$dv^2 = \frac{(dx^2 + dy^2 + dz^2)}{dt^2}$$

with respect to the origin of the observer’s co-ordinate system. Another observer in the t', x', y', z' perceives ds moving with a velocity of

$$v'^2 = \frac{(dx'^2 + dy'^2 + dz'^2)}{dt'^2}$$

Thus if the observer is in t,x,y,z the when ds changes the effect for the observer is dt,dx,dy,dz if we jump back to t,x,y,z axis then x,y,z are the distances to s and t is the time in the t,x,y,z system while the time of ds can change differently:

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2 = c^2$$
$$c^2d\tau^2 = c^2dt^2\left(1 - \frac{v^2}{c^2}\right) = \frac{c^2dt^2}{\gamma}$$

The relation between the time in the ds system with respect to the observer:

$$d\tau^2 = \frac{dt^2}{\gamma}$$
$$dt^2 = \gamma d\tau^2$$

As γ is 1 or greater then $d\tau$ is always equal or smaller than dt. Thus the clock of ds goes slower than the clock of the observer.

20 Exercise to formally apply the metric transformation formula.

$$\textit{The relation between the "new" and "old" coordinates}: dx^m = \frac{\partial x^m}{\partial y^r} dy^r$$
$$\textit{The relation between a line segment and its coordinates}: ds^2 = \eta_{mn} d\xi^m d\xi^n$$
$$\textit{The relation between two different coordinate systems}: ds^2 = g_{mn}(x) dx^m dx^n = g_{pq}(y) dy^p dy^q$$
$$\textit{The relation between the "new" metric tensor and the "old"}: \boldsymbol{g_{pq}(y) = g_{mn}(x) \frac{dx^m}{dy^p} \frac{dx^n}{dy^q}}$$

For the exercise we consider the transformation between a Cartesian- and a polar system.

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$$
$$ds^2 = c^2 dt^2 - dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta \, d\varphi^2$$

The old metric tensor has $g_{00} = 1$, $g_{11} = -1$, $g_{22} = -1$, $g_{33} = -1$ as, Cartesian, elements and the rest is zero.
Now we have to find, via the formula, the new polar metric tensor elements,

$$g_{00} = 1\,, \quad g_{11} = -1\,, \; g_{22} = -r^2 \quad , g_{33} = -r^2 \sin^2\theta$$

As mentioned in a previous chapter, the relationship between polar and Cartesian coordinates is:

$$x = r\sin\theta\cos\varphi \qquad y = r\sin\theta\sin\varphi \qquad z = r\cos\theta$$

In general:

$$dx^m = \frac{\partial x^m}{\partial y^r} dy^r$$

Worked out for this example:

$$dt = \frac{\partial t}{\partial t} dt + \frac{\partial t}{\partial r} dr + \frac{\partial t}{\partial \theta} d\theta + \frac{\partial t}{\partial \varphi} d\varphi$$
$$dx = \frac{\partial x}{\partial t} dt + \frac{\partial x}{\partial r} dr + \frac{\partial x}{\partial \theta} d\theta + \frac{\partial x}{\partial \varphi} d\varphi$$
$$dy = \frac{\partial y}{\partial t} dt + \frac{\partial y}{\partial r} dr + \frac{\partial y}{\partial \theta} d\theta + \frac{\partial y}{\partial \varphi} d\varphi$$
$$dz = \frac{\partial z}{\partial t} dt + \frac{\partial z}{\partial r} dr + \frac{\partial z}{\partial \theta} d\theta + \frac{\partial z}{\partial \varphi} d\varphi$$

Derivation of the dx, dy en dz:

$$dt = dt$$
$$dx = \sin\theta\cos\varphi\, dr + r\cos\theta\cos\varphi\, d\theta - r\sin\theta\sin\varphi\, d\varphi$$
$$dy = \sin\theta\sin\varphi\, dr + r\cos\theta\sin\varphi\, d\theta + r\sin\theta\cos\varphi\, d\varphi$$
$$dz = \cos\theta\, dr - r\sin\theta\, d\theta$$

Thus the metric tensor elements are:

$$\frac{\partial t}{\partial t}=1 \quad \frac{\partial t}{\partial r}=0 \quad \frac{\partial t}{\partial \theta}=0 \quad \frac{\partial t}{\partial \varphi}=0$$
$$\frac{\partial x}{\partial t}=0 \quad \frac{\partial x}{\partial r}=+\sin\theta\cos\varphi \quad \frac{\partial x}{\partial \theta}=+r\cos\theta\cos\varphi \quad \frac{\partial x}{\partial \varphi}=-r\sin\theta\sin\varphi$$
$$\frac{\partial y}{\partial t}=0 \quad \frac{\partial y}{\partial r}=+\sin\theta\sin\varphi \quad \frac{\partial y}{\partial \theta}=+r\cos\theta\sin\varphi \quad \frac{\partial y}{\partial \varphi}=+r\sin\theta\cos\varphi$$
$$\frac{\partial z}{\partial t}=0 \quad \frac{\partial z}{\partial r}=+\cos\theta \quad \frac{\partial z}{\partial \theta}=-r\sin\theta \quad \frac{\partial z}{\partial \varphi}=0$$

Now we apply:

$$g_{pq}(y)=g_{mn}(x)\frac{dx^m}{dy^p}\frac{dx^n}{dy^q}$$

Worked out for the metric tensor element:

$$\begin{gathered}
g_{00}(y)=g_{00}(x)\frac{dx^0}{dy^0}\frac{dx^0}{dy^0}+g_{01}(x)\frac{dx^0}{dy^0}\frac{dx^1}{dy^0}+g_{02}(x)\frac{dx^0}{dy^0}\frac{dx^2}{dy^0}+g_{03}(x)\frac{dx^0}{dy^0}\frac{dx^3}{dy^0}+\\
g_{10}(x)\frac{dx^1}{dy^0}\frac{dx^0}{dy^0}+g_{11}(x)\frac{dx^1}{dy^0}\frac{dx^1}{dy^0}+g_{12}(x)\frac{dx^1}{dy^0}\frac{dx^2}{dy^0}+g_{13}(x)\frac{dx^1}{dy^0}\frac{dx^3}{dy^0}+\\
g_{20}(x)\frac{dx^2}{dy^0}\frac{dx^0}{dy^0}+g_{21}(x)\frac{dx^2}{dy^0}\frac{dx^1}{dy^0}+g_{22}(x)\frac{dx^2}{dy^0}\frac{dx^2}{dy^0}+g_{23}(x)\frac{dx^2}{dy^0}\frac{dx^3}{dy^0}+\\
g_{30}(x)\frac{dx^3}{dy^0}\frac{dx^0}{dy^0}+g_{31}(x)\frac{dx^3}{dy^0}\frac{dx^1}{dy^0}+g_{32}(x)\frac{dx^3}{dy^0}\frac{dx^2}{dy^0}+g_{33}(x)\frac{dx^3}{dy^0}\frac{dx^3}{dy^0}
\end{gathered}$$

Now we fill in, as an example, the appropriate, polar and Cartesian, coordinates in the element g_{11}

$$\begin{gathered}
g_{rr}=g_{tt}\frac{dt}{dr}\frac{dt}{dr}+g_{tx}\frac{dt}{dr}\frac{dx}{dr}+g_{ty}\frac{dt}{dr}\frac{dy}{dr}+g_{tz}\frac{dt}{dr}\frac{dz}{dr}+\\
g_{xt}\frac{dx}{dr}\frac{dt}{dr}+g_{xx}\frac{dx}{dr}\frac{dx}{dr}+g_{xy}\frac{dx}{dr}\frac{dy}{dr}+g_{xz}\frac{dx}{dr}\frac{dz}{dr}+\\
g_{yt}\frac{dy}{dr}\frac{dt}{dr}+g_{yx}\frac{dy}{dr}\frac{dx}{dr}+g_{yy}\frac{dy}{dr}\frac{dy}{dr}+g_{yz}\frac{dy}{dr}\frac{dz}{dr}+\\
g_{zt}\frac{dz}{dr}\frac{dt}{dr}+g_{zx}\frac{dz}{dr}\frac{dx}{dr}+g_{zy}\frac{dz}{dr}\frac{dy}{dr}+g_{zz}\frac{dz}{dr}\frac{dz}{dr}
\end{gathered}$$

Because the coordinate system is an orthogonal system, only the elements with equal indices are non zero. Thus the matrix above boils down to:

$$g_{tt}=g_{tt}\frac{dt}{dt}\frac{dt}{dt}+g_{xx}\frac{dx}{dt}\frac{dx}{dt}+g_{yy}\frac{dy}{dt}\frac{dy}{dt}+g_{zz}\frac{dz}{dt}\frac{dz}{dt}$$
$$g_{tt}=1+0+0+0=1$$
$$g_{rr}=g_{tt}\frac{dt}{dr}\frac{dt}{dr}+g_{xx}\frac{dx}{dr}\frac{dx}{dr}+g_{yy}\frac{dy}{dr}\frac{dy}{dr}+g_{zz}\frac{dz}{dr}\frac{dz}{dr}$$
$$g_{rr}=0-1(+\sin\theta\cos\varphi)^2-1(+\sin\theta\sin\varphi)^2-1(+\cos\theta)^2=-\sin^2\varphi-\cos^2\varphi=-1$$
$$g_{\theta\theta}=g_{tt}\frac{dt}{d\theta}\frac{dt}{d\theta}+g_{xx}\frac{dx}{d\theta}\frac{dx}{d\theta}+g_{yy}\frac{dy}{d\theta}\frac{dy}{d\theta}+g_{zz}\frac{dz}{d\theta}\frac{dz}{d\theta}$$
$$g_{\theta\theta}=0-1(+r\cos\theta\cos\varphi)^2-1(+r\cos\theta\sin\varphi)^2-1(-r\sin\theta)^2=-r^2\cos^2\theta-r^2\sin^2\theta=-r^2$$
$$g_{\varphi\varphi}=g_{tt}\frac{dt}{d\varphi}\frac{dt}{d\varphi}+g_{xx}\frac{dx}{d\varphi}\frac{dx}{d\varphi}+g_{yy}\frac{dy}{d\varphi}\frac{dy}{d\varphi}+g_{zz}\frac{dz}{d\varphi}\frac{dz}{d\varphi}$$
$$g_{rr}=0-1(-r\sin\theta\sin\varphi)^2-1(+r\sin\theta\cos\varphi)^2-0=r^2\sin^2\varphi$$

Thus the transformation from Cartesian to polar metric tenor elements is:

$$\boldsymbol{g_{00}=1 \quad g_{11}=-1 \quad g_{22}=-1 \quad g_{33}=-1}$$
$$\boldsymbol{g_{tt}=1 \quad g_{rr}=-1 \quad g_{\theta\theta}=-r^2 \quad g_{rr}=r^2\sin^2\varphi}$$

20.1 Transformations performed by Schwarzschild

Here we consider Schwarzschild equation and the transformation to new x,y and z coordinates:
He starts from Cartesian coordinates and transforms to polar coordinates according to the method followed above an d resulting in:

$$ds^2 = \sigma^2 c^2 dt^2 - \frac{dr^2}{\sigma^2} - r^2 d\theta^2 - r^2 \sin^2\theta^2\, d\emptyset^2$$

Here he realizes that the product of the metric tensor elements, the g determinant, is not -1 as wished by Einstein.
Because $g = \sigma^2.\left(\frac{-1}{\sigma^2}\right).(-r^2).(-r^2\sin^2\theta) = -r^4\sin^2\theta$

To meet the desire to have g=-1, he wants to perform the transformation with $\frac{dr}{dx_1} = \frac{1}{r^2}$ *and* $\frac{d\theta}{dx_2} = \frac{1}{\sin\theta}$ *and* $\frac{d\emptyset}{dx_3} = 1$

And as Schwarzschild mentioned: "The new variables are the *polar co-ordinates with the determinant 1"*.

In order to get these derivatives he finds the relations $x_1 = \frac{r^3}{3}$, $x_2 = -\cos\theta$, $x_3 = \emptyset$ and transforms accordingly.

$$ds^2 = \sigma^2 c^2 dt^2 - \frac{{dx_1}^2}{r^4\sigma^2} - r^2\frac{1}{\sin^2\theta}{dx_2}^2 - r^2\sin^2\theta\,{dx_3}^2$$

As $x_2 = -\cos\theta$ *then* ${x_2}^2 = \cos^2\theta = 1 - \sin^2\theta \Rightarrow \sin^2\theta = 1 - {x_2}^2$

$$ds^2 = \sigma^2 c^2 dt^2 - \frac{{dx_1}^2}{r^4\sigma^2} - r^2\frac{1}{1-{x_2}^2}{dx_2}^2 - r^2(1-{x_2}^2){dx_3}^2$$

Thus the metric tensor elements are:

$$\boldsymbol{g_{00}} = \sigma^2 \quad \boldsymbol{g_{11}} = -\frac{1}{r^4\sigma^2} \quad \boldsymbol{g_{22}} = \frac{-1}{1-{x_2}^2} \quad \boldsymbol{g_{33}} = -r^2(1-{x_2}^2)$$

So indeed now g=-1 and the performed transformations are legitimate. In the special case $\theta = 90^0$ then $x_2 = 0$

20.2 Transformation for the repaired Schwarzschild solution

Next we elaborate on the transformations executed in the appendix F of the book "Repairing Schwarzschild's Solution" page 194.
Here is started from:

$$ds^2 = \sigma^2 c^2 dt^2 - \frac{dr^2}{\sigma^2} - r^2 d\theta^2 - r^2\sin^2\theta^2\, d\emptyset^2 \qquad 15.2.1$$

The performed correction leads to:

$$ds^2 = \frac{c^2dt^2}{\sigma^2} - \frac{dr^2}{\sigma^2} - \frac{r^2d\theta^2}{\sigma^2} - \frac{r^2\sin^2\theta^2\,d\emptyset^2}{\sigma^2} \qquad 15.2.2$$

To go to new coordinates the approach is:

$$dy = d\alpha = rd\theta \textit{ and } \; dz = d\beta = r\sin\theta\, d\emptyset$$

In the book it is assumed that the result is:

$$ds^2 = \frac{c^2dt^2}{\sigma^2} - \frac{dx^2}{\sigma^2} - \frac{dy^2}{\sigma^2} - \frac{dz^2}{\sigma^2} \qquad 15.2.3$$

The relation between the new and the old co-ordinates is that t=t and x=r but to find the y and the z then integration has to be done:

$$\int dy = \int r d\theta \ \ and \ \int dz = \int r \sin\theta \, d\emptyset$$

As r, θ *and* $\emptyset$ are independent the result is:

$$y = \ r\theta + const1$$
$$z = \ \emptyset . r . \sin\theta + const2$$

As the new coordinates y and z are dependent on the old co-ordinates r, θ *and* $\emptyset$, the derivatives of t, x, y and z are:

$$dt = \ dt$$
$$dx = \ dr$$
$$dy = \ \theta dr + r d\theta$$
$$dz = \ \emptyset \sin\theta dr + r\emptyset \cos\theta d\theta + r \sin\theta d\emptyset$$

Next because of the mutually independency of r, θ *and* $\emptyset$ this results in:

$$dy^2 = \ \theta^2 dr^2 + r^2 d\theta^2$$
$$dz^2 = \ \emptyset^2 \sin^2\theta dr^2 + r^2\emptyset^2 \cos^2\theta d\theta^2 + r^2 \sin^2\theta d\emptyset^2$$

Filling in:

$$ds^2 = \frac{c^2dt^2}{\sigma^2} - \frac{dr^2}{\sigma^2} - \frac{\theta^2 dr^2 + r^2 d\theta^2}{\sigma^2} - \frac{\emptyset^2 \sin^2\theta dr^2 + r^2\emptyset^2 \cos^2\theta d\theta^2 + r^2 \sin^2\theta d\emptyset^2}{\sigma^2}$$

$$ds^2 = \frac{c^2dt^2}{\sigma^2} - \frac{dr^2 + \theta^2 dr^2 + \emptyset^2 \sin^2\theta dr^2}{\sigma^2} - \frac{+r^2 d\theta^2 + r^2\emptyset^2 \cos^2\theta d\theta^2 +}{\sigma^2} - \frac{r^2 \sin^2\theta d\emptyset^2}{\sigma^2}$$

$$\boldsymbol{ds^2 = \frac{c^2dt^2}{\sigma^2} - \frac{(1 + \theta^2 + \emptyset^2 \sin^2\theta)dr^2}{\sigma^2} - \frac{r^2(1 + \emptyset^2 \cos^2\theta)d\theta^2}{\sigma^2} - \frac{r^2 \sin^2\theta d\emptyset^2}{\sigma^2}} \qquad \mathbf{15.2.4}$$

The aim was:

$$ds^2 = \frac{c^2dt^2}{\sigma^2} - \frac{dr^2}{\sigma^2} - \frac{r^2 d\theta^2}{\sigma^2} - \frac{r^2 \sin^2\theta d\emptyset^2}{\sigma^2} \qquad 15.2.2$$

The equations 15.2.2 and 15.2.4 shall be equivalent. Equivalency occurs if the red items vanish.
This is true under the condition that $\theta = 0$ *and* $\emptyset = 0$ but in that case the third and the fourth term would vanish as well, with the result:

$$ds^2 = \frac{c^2dt^2}{\sigma^2} - \frac{dr^2}{\sigma^2}$$

The equation 15.2.2 becomes then:

$$ds^2 = \frac{c^2dt^2}{\sigma^2} - \frac{dx^2}{\sigma^2} - \frac{dy^2}{\sigma^2} - \frac{dz^2}{\sigma^2}$$

But because θ *and* $\emptyset$ are always zero, also $d\theta$, $d\emptyset$ *and consequently* dy *and* dz are zero as well so:

Hence 15.2.2 becomes:

$$\boldsymbol{ds^2 = \frac{c^2dt^2}{\sigma^2} - \frac{dx^2}{\sigma^2}}$$

Thus 15.2.3 is not a proper transformation from 15.2.2 and is quite different from the aimed 15.2.3 equation. Also for the transition from 15.2.1 to 15.2.2 no proper transformation can be found.

Now a check will be done whether the metric tensor elements from the contemplated formula 15.2.3 would meet Einstein's field equations. The metric tensor elements would have been:

$$g_{00} = \frac{1}{\sigma^2} \quad g_{11} = -\frac{1}{\sigma^2} \quad g_{22} = -\frac{1}{\sigma^2} \quad g_{33} = -\frac{1}{\sigma^2}$$

The g becomes:

$$g = -\frac{1}{\sigma^8}$$

This means that not the simplified field equations but the extended field equations are applicable.

Next we can check whether the field equations requirements are met. The formulas from the previous table "Repaired Schwarzschild metric for the x, y, z version 2" are used:

$$\frac{d\sigma}{dr} = \frac{\boldsymbol{R_s}}{\boldsymbol{2r^2\sigma}}$$

$$\frac{\partial g_{00}}{\partial r} = \frac{-R_s}{r^2\sigma^4} \qquad\qquad \frac{\partial g_{11}}{\partial r} = \frac{\partial g_{22}}{\partial r} = \frac{\partial g_{33}}{\partial r} = \frac{+\boldsymbol{R_s}}{\boldsymbol{r^2\sigma^4}}$$

$$\mathrm{R}_{00} = \boldsymbol{\Gamma^1_{00,1}} + \Gamma^1_{00}\,\Gamma^1_{11} + \Gamma^1_{00}\,\Gamma^2_{12} + \Gamma^1_{00}\,\Gamma^3_{13} - \Gamma^0_{01}\,\Gamma^1_{00}$$

$$\mathrm{R}_{00} = \frac{\mathrm{R_s(2r - R_s)}}{2\mathrm{r}^4\sigma^4} + \frac{-R_s}{2r^2\sigma^2}\left(\frac{-R_s}{2r^2\sigma^2} + \frac{-R_s}{2r^2\sigma^2} + \frac{-R_s}{2r^2\sigma^2} - \frac{-R_s}{2r^2\sigma^2}\right)$$

$$\mathrm{R}_{00} = \frac{\mathrm{R_s(2r - R_s)}}{2\mathrm{r}^4\sigma^4} + \frac{-2R_s}{2r^2\sigma^2}\left(\frac{-R_s}{2r^2\sigma^2}\right)$$

$$\mathrm{R}_{00} = \frac{\mathrm{R_s(2r - R_s)}}{2\mathrm{r}^4\sigma^4} + \frac{R_sR_s}{2r^4\sigma^4}$$

$$\mathrm{R}_{00} = \frac{\mathrm{R_s(2r)}}{2\mathrm{r}^4\sigma^4} - \frac{\mathrm{R_sR_s}}{2\mathrm{r}^4\sigma^4} + \frac{R_sR_s}{2r^4\sigma^4}$$

$$\mathrm{R}_{00} = \frac{\mathrm{R_s}}{\mathrm{r}^3\sigma^4}$$

This is unequal zero so does not meet the field equations requirements.

$$\mathrm{R}_{11} = -\boldsymbol{\Gamma^0_{10,1}} - \boldsymbol{\Gamma^2_{12,1}} - \boldsymbol{\Gamma^3_{13,1}} + \Gamma^1_{11}\Gamma^0_{10} + \Gamma^1_{11}\,\Gamma^2_{12} + \Gamma^1_{11}\,\Gamma^3_{13} - \Gamma^0_{10}\Gamma^0_{01} - \Gamma^2_{12}\,\Gamma^2_{21} - \Gamma^3_{13}\,\Gamma^3_{31}$$

$$\mathrm{R}_{11} = -\frac{\mathrm{R_s(2r - R_s)}}{2\mathrm{r}^4\sigma^4} - \frac{\mathrm{R_s(2r - R_s)}}{2\mathrm{r}^4\sigma^4} - \frac{\mathrm{R_s(2r - R_s)}}{2\mathrm{r}^4\sigma^4} + \frac{-R_s}{2r^2\sigma^2}\left(\frac{-R_s}{2r^2\sigma^2} + \frac{-R_s}{2r^2\sigma^2} + \frac{-R_s}{2r^2\sigma^2}\right) - \frac{-R_s}{2r^2\sigma^2}\frac{-R_s}{2r^2\sigma^2} - \frac{-R_s}{2r^2\sigma^2}\frac{-R_s}{2r^2\sigma^2} - \frac{-R_s}{2r^2\sigma^2}\frac{-R_s}{2r^2\sigma^2}$$

$$\mathrm{R}_{11} = -\frac{3\mathrm{R_s(2r - R_s)}}{2\mathrm{r}^4\sigma^4} + \frac{-3R_s}{2r^2\sigma^2}\left(\frac{-R_s}{2r^2\sigma^2}\right) - \frac{-3R_s}{2r^2\sigma^2}\frac{-R_s}{2r^2\sigma^2}$$

$$\mathrm{R}_{11} = -\frac{3\mathrm{R_s(2r - R_s)}}{2\mathrm{r}^4\sigma^4}$$

This is unequal zero so does not meet the field equations requirements.

$$\mathrm{R}_{22} = \boldsymbol{\Gamma^1_{22,1}} - \boldsymbol{\Gamma^3_{23,2}} + \Gamma^1_{22}\,\Gamma^0_{10} + \Gamma^1_{22}\,\Gamma^1_{11} + \Gamma^1_{22}\,\Gamma^3_{13} - \Gamma^2_{21}\,\Gamma^1_{22} - \Gamma^3_{23}\,\Gamma^3_{32}$$

$$\mathrm{R}_{22} = \frac{-\mathrm{R_s(2r - R_s)}}{2\mathrm{r}^4\sigma^4} - 0 + \frac{R_s}{2r^2\sigma^2}\left(\frac{-R_s}{2r^2\sigma^2} + \frac{-R_s}{2r^2\sigma^2} + \frac{R_s}{2r^2\sigma^2} - \frac{-R_s}{2r^2\sigma^2}\right) - \Gamma^3_{23}\,\Gamma^3_{32}$$

$$\mathrm{R}_{22} = \frac{-\mathrm{R_s(2r - R_s)}}{2\mathrm{r}^4\sigma^4}$$

This is unequal zero so does not meet the field equations requirements.

$$R_{33} = +\boldsymbol{\Gamma^{1}_{33,1}} + \boldsymbol{\Gamma^{2}_{33,2}} + \Gamma^{1}_{33}\,\Gamma^{0}_{10} + \Gamma^{1}_{33}\,\Gamma^{1}_{11} + \Gamma^{1}_{33}\,\Gamma^{2}_{12} - \Gamma^{3}_{31}\,\Gamma^{1}_{33} - \Gamma^{3}_{32}\,\Gamma^{2}_{33}$$

$$R_{33} = \frac{-R_s(2r - R_s)}{2r^4\sigma^4} + \frac{R_s}{2r^2\sigma^2}\left(\frac{-R_s}{2r^2\sigma^2} + \frac{-R_s}{2r^2\sigma^2} + \frac{-R_s}{2r^2\sigma^2} - \frac{-R_s}{2r^2\sigma^2}\right)$$

$$R_{33} = \frac{-R_s(2r - R_s)}{2r^4\sigma^4} - \frac{R_s R_s}{2r^4\sigma^4}$$

$$R_{33} = \frac{-R_s}{r^3\sigma^4}$$

This is unequal zero so does not meet the field equations requirements.

Conclusion: No proper transformation for the “Repaired Schwarzschild Solution” equation has been found. In case an equation as 15.2.3 would have been found it does not meet the Einstein field equations requirements as is shown above. The Schwarzschild equations always meet the Einstein field equations independent whether theta is 90^0. The transformation between the r, theta and phi and the x, y and z co-ordinates of the “Repaired Schwarzschild Solution” equation is not valid because it leads to a different and a two dimensional equation; only dependent on t and x.

20.3 Time relation between Earth observer and universal frame with centre of Sun

When, in other chapters, the deflection of light or orbits of planets around the Sun are considered, a frame is used with the centre in the middle of the Sun. While we observe the phenomenon from the Earth and have a **rotation velocity with respect to the Sun.** Here we consider this effect and calculate the correction factor.

Starting point is the Schwarzschild metric:

$$ds^2 = \sigma^2 c^2 dt^2 - \frac{dr^2}{\sigma^2} - r^2 d\theta^2 - r^2 \sin^2\theta^2\, d\emptyset^2$$

Centre of the frame is the Sun centre. The orbit of the Earth around the Sun is assumed to be a circle. The observed physical movement is in the equatorial plane of the frame. Thus the radius is constant and theta is $\pi/2$.
The equation simplifies then to:

$$ds^2 = c^2 d\tau^2 = \sigma^2 c^2 dt^2 - r^2 d\emptyset^2$$

τ Is the proper time of the observer on Earth (on North or South Pole), while t is the time of the universal Sun frame. So everything, the Earth observer inclusive, is related to the universal Sun frame.

$$d\tau^2 = \sigma^2 dt^2 - \frac{r^2}{c^2}\left(\frac{d\emptyset}{dt}\right)^2 dt^2 = \left(1 - \frac{R_s}{r} - \frac{r^2}{c^2}\left(\frac{d\emptyset}{dt}\right)^2\right) dt^2$$

$$d\tau^2 = \left(1 - \frac{R_s}{r} - \frac{v^2}{c^2}\right) dt^2$$

$$\tau = \sqrt{\left(1 - \frac{R_s}{r} - \frac{v^2}{c^2}\right)}\, t$$

First order Taylor expansion:

$$\tau = \left(1 - \frac{R_s}{2r} - \frac{v^2}{2c^2}\right) t$$

$R_s = 2950m$, this is the Schwarzschild radius of the Sun. The rotation velocity of the Earth around the sun is $v = 30000m/s$. The distance from the observer to the sun is $r = 150 * 10^9$m.
The second term at the right hand side is due to the Sun gravity and the third term is due to the Earth velocity.

$$\tau = (1 - 99.10^{-10} - 50.10^{-10})t$$

$$\boldsymbol{\tau} \approx (\mathbf{1} - \mathbf{15.10^{-9}})\boldsymbol{t}$$

This is the relation between the time of the Earth observer and the universal Sun frame time t.
Here is:

$$\sigma = \sqrt{1 - \frac{2GM}{c^2 r}} \qquad R_s = \frac{2GM}{c^2}$$

As the Earth observer is also influenced by the **gravity of the Earth**, while standing on one of the poles, $dr = d\theta = d\emptyset = 0$ then

$$d\tau = \sqrt{1 - \frac{2GM}{c^2 r}}dt = \sqrt{1 - 1.3908 * 10^{-9}}dt = (1 - 0.6954 * 10^{-9})dt$$

At the equator is the radius r_e=6,378,137 m. In addition, the rotation of the Earth needs to be taken into account. This imparts on an observer an angular velocity of $\frac{d\emptyset}{dt}$ of 2π divided by the sidereal period of the Earth's rotation, 86162.4 seconds. So $d\emptyset = 7.2923 * 10^{-5}dt$. The proper time equation then produces

$$d\tau = \sqrt{(1 - 1.3908 * 10^{-9}) - 2.4059 * 10^{-12}}dt = (1 - 0.6966 * 10^{-9})dt.$$

M_e=5.9742x10^{24}kg, r_e=6,356,752 m, G=6.674x10^{-11}Nkg^{-2}m^2, c=299,792,458 m/s.

---

https://sites.math.washington.edu/~morrow/papers/Genrel.pdf
The Precession of Mercury's Perihelion
Owen Biesel
January 25, 2008 (Biesel, 2008)
https://arxiv.org/pdf/0712.3709.pdf
Christian Magnan: Complete calculations of the perihelion precession of Mercury and the deflection of light by the Sun in General Relativity (Magnan)
http://www.physics.ucc.ie/apeer/PY4112/Sch.pdf
Schwarzschild Solution and Black Holes
Asaf Pe'er1
February 19, 2014 (Pe'er1, 2014)

21 Motion of Particles in Schwarzschild Geometry

Here follows the derivation of an equation for the motion of particles and in particular the perihelion precession of Mercury and the deflection of light by the Sun. As a starting point the Schwarzschild equation is used because it meets the Einstein field equations and of its proven applicability. As the metric in the Schwarzschild geometry is symmetric in time *t* and in the polar coordinate Ø i.e. none of the coefficients in the equation is depending on either *t* or Ø and therefore it meets the Noëther theorem. The Noëther theorem says that symmetry leads to conservation and in this case the independency of *t* leads to conservation of *E* (momentum) and the independency of Ø leads to conservation of the angular momentum.

Schwarzschild metric:,

$$ds^2 = \sigma^2 c^2 dt^2 - \frac{dr^2}{\sigma^2} - \frac{r^2}{R_p^2} dR_p^2.\theta^2 - \frac{r^2}{R_p^2}\sin^2\theta^2\, dR_p^2.\emptyset^2$$

Rp=1m is added to get the dimensions right but more practical:

$$\boldsymbol{ds^2 = \sigma^2 c^2 dt^2 - \frac{dr^2}{\sigma^2} - r^2 d\theta^2 - r^2 \sin^2\theta^2\, d\emptyset^2}$$

$$\sigma = \sqrt{1 - \frac{2GM}{c^2 r}} = \sqrt{1 - \frac{R_s}{r}} \quad Schwarzschild\ radius: R_s = \frac{2GM}{c^2}$$

The Schwarzschild metric for polar co-ordinates

$$g_{00} = \boldsymbol{\sigma^2},\ g_{11} = \frac{\boldsymbol{-1}}{\boldsymbol{\sigma^2}},\ g_{22} = \boldsymbol{-r^2},\ g_{33} = \boldsymbol{-r^2\sin^2\theta} = -r^2;\ g^{00} = \frac{\boldsymbol{1}}{\boldsymbol{\sigma^2}},\ g^{11} = \boldsymbol{-\sigma^2},\ g^{22} = \frac{\boldsymbol{-1}}{\boldsymbol{r^2}},\ g^{33} = \frac{\boldsymbol{-1}}{\boldsymbol{r^2\sin^2\theta}} = \frac{-1}{r^2}$$

$$\frac{d\sigma}{dr} = \frac{\boldsymbol{R_s}}{\boldsymbol{2r^2\sigma}}$$

Metric first derivative on polar co-ordinates

$$\frac{\partial g_{00}}{\partial r} = \frac{\boldsymbol{R_s}}{\boldsymbol{r^2}},\ \frac{\partial g_{11}}{\partial r} = \frac{\boldsymbol{R_s}}{\boldsymbol{r^2\sigma^4}},\quad \frac{\partial g_{22}}{\partial r} = (\boldsymbol{-2r}),\ \frac{\partial g_{33}}{\partial r} = \left(\boldsymbol{-2r\sin^2\theta}\right) = \boldsymbol{-2r},\quad \frac{\partial g_{33}}{\partial\theta} = (\boldsymbol{-2r.sin(\theta)\cos(\theta)}) = \boldsymbol{0}$$

The relevant (non-zero) Christoffel symbols for Schwarzschild polar co-ordinates:

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2}g^{\rho\alpha}\left\{\frac{\partial g_{\nu\alpha}}{\partial x^{\mu}} + \frac{\partial g_{\mu\alpha}}{\partial x^{\nu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\alpha}}\right\}$$

$$\Gamma^0_{01} = \Gamma^0_{10} = \frac{1}{2}g^{00}\left\{\frac{\partial g_{00}}{\partial r}\right\} = \frac{\boldsymbol{R_s}}{\boldsymbol{2r^2\sigma^2}},\quad \Gamma^1_{00} = \frac{1}{2}g^{11}\left\{-\frac{\partial g_{00}}{\partial r}\right\} = \frac{\boldsymbol{\sigma^2 R_s}}{\boldsymbol{2r^2}},\quad \Gamma^1_{11} = \frac{1}{2}g^{11}\left\{\frac{\partial g_{11}}{\partial r}\right\} = \frac{\boldsymbol{-R_s}}{\boldsymbol{2r^2\sigma^2}}$$

$$\Gamma^1_{22} = \frac{1}{2}g^{11}\left\{-\frac{\partial g_{22}}{\partial r}\right\} = \boldsymbol{-r\sigma^2},\quad \Gamma^1_{33} = \frac{1}{2}g^{11}\left\{-\frac{\partial g_{33}}{\partial r}\right\} = \boldsymbol{-r\sigma^2\sin^2\theta}$$

$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{2}g^{22}\left\{\frac{\partial g_{22}}{\partial r}\right\} = \frac{\boldsymbol{1}}{\boldsymbol{r}},\quad \Gamma^2_{33} = \frac{1}{2}g^{22}\left\{-\frac{\partial g_{33}}{\partial\theta}\right\} = \boldsymbol{-\cos\theta\sin\theta}$$

$$\Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{2}g^{33}\left\{\frac{\partial g_{33}}{\partial r}\right\} = \frac{\boldsymbol{1}}{\boldsymbol{r}},\quad \Gamma^3_{23} = \Gamma^3_{32} = \frac{1}{2}g^{33}\left\{\frac{\partial g_{33}}{\partial\theta}\right\} = \frac{\boldsymbol{\cos\theta}}{\boldsymbol{\sin\theta}}$$

All the other Christoffel symbols are zero.

The geodesic equations:

$$\frac{d^2x^{\alpha}}{d\lambda^2} + \Gamma^{\alpha}_{\mu\nu}.\frac{dx^{\mu}}{d\lambda}\frac{dx^{\nu}}{d\lambda} = 0$$

Work-out for the four coordinates, where λ is the affine parameter:

$$\frac{d^2t}{d\lambda^2} + \Gamma^{t}_{\mu\nu}.\frac{dx^{\mu}}{d\lambda}\frac{dx^{\nu}}{d\lambda} = \frac{d^2t}{d\lambda^2} + 2\Gamma^0_{01}.\frac{dt}{d\lambda}\frac{dr}{d\lambda} = \boldsymbol{\frac{d^2t}{d\lambda^2} + 2\frac{R_s}{2r^2\sigma^2}.\frac{dt}{d\lambda}\frac{dr}{d\lambda} = 0}$$

$$\frac{d^2r}{d\lambda^2} + \Gamma^{r}_{\mu\nu}.\frac{dx^{\mu}}{d\lambda}\frac{dx^{\nu}}{d\lambda} = \frac{d^2r}{d\lambda^2} + \Gamma^1_{00}.\left(\frac{dt}{d\lambda}\right)^2 + \Gamma^1_{11}.\left(\frac{dr}{d\lambda}\right)^2 + \Gamma^1_{22}.\left(\frac{d\theta}{d\lambda}\right)^2 + \Gamma^1_{33}.\left(\frac{d\varphi}{d\lambda}\right)^2 = 0$$

$$\frac{d^2r}{d\lambda^2}+\Gamma^r_{\mu\nu}.\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}=\boldsymbol{\frac{d^2r}{d\lambda^2}+\frac{\sigma^2R_s}{2r^2}.\left(\frac{dt}{d\lambda}\right)^2-\frac{R_s}{2r^2\sigma^2}.\left(\frac{dr}{d\lambda}\right)^2-r\sigma^2.\left(\frac{d\theta}{d\lambda}\right)^2-r\sigma^2\sin^2\theta.\left(\frac{d\varphi}{d\lambda}\right)^2=0}$$

$$\frac{d^2\theta}{d\lambda^2}+\Gamma^\theta_{\mu\nu}.\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}=\frac{d^2\theta}{d\lambda^2}+2\Gamma^2_{12}.\frac{dr}{d\lambda}\frac{d\theta}{d\lambda}+\Gamma^2_{33}.\left(\frac{d\varphi}{d\lambda}\right)^2=0$$

$$\boldsymbol{\frac{d^2\theta}{d\lambda^2}+2\frac{1}{r}.\frac{dr}{d\lambda}\frac{d\theta}{d\lambda}-\cos\theta\sin\theta\ .\left(\frac{d\varphi}{d\lambda}\right)^2=0}$$

$$\frac{d^2\varphi}{d\lambda^2}+\Gamma^\varphi_{\mu\nu}.\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}=\frac{d^2\varphi}{d\lambda^2}+2\Gamma^3_{13}.\frac{dr}{d\lambda}\frac{d\varphi}{d\lambda}+2\Gamma^3_{23}.\frac{d\theta}{d\lambda}\frac{d\varphi}{d\lambda}=0$$

$$\boldsymbol{\frac{d^2\varphi}{d\lambda^2}+2\frac{1}{r}.\frac{dr}{d\lambda}\frac{d\varphi}{d\lambda}+2\frac{\cos\theta}{\sin\theta}.\frac{d\theta}{d\lambda}\frac{d\varphi}{d\lambda}=0}$$

To summarize the resulting four equations:

$$\frac{d^2t}{d\lambda^2}+2\frac{R_s}{2r^2\sigma^2}\frac{dt}{d\lambda}\frac{dr}{d\lambda}=0 \tag{1}$$

$$\frac{d^2r}{d\lambda^2}+\frac{\sigma^2R_s}{2r^2}\left(\frac{dt}{d\lambda}\right)^2-\frac{R_s}{2r^2\sigma^2}\left(\frac{dr}{d\lambda}\right)^2-r\sigma^2\left(\frac{d\theta}{d\lambda}\right)^2-r\sigma^2\sin^2\theta\left(\frac{d\varphi}{d\lambda}\right)^2=0 \tag{2}$$

$$\frac{d^2\theta}{d\lambda^2}+2\frac{1}{r}\frac{dr}{d\lambda}\frac{d\theta}{d\lambda}-\cos\theta\sin\theta\ \left(\frac{d\varphi}{d\lambda}\right)^2=0 \tag{3}$$

$$\frac{d^2\varphi}{d\lambda^2}+2\frac{1}{r}\frac{dr}{d\lambda}\frac{d\varphi}{d\lambda}+2\frac{\cos\theta}{\sin\theta}\frac{d\theta}{d\lambda}\frac{d\varphi}{d\lambda}=0 \tag{4}$$

Now according to Asaf Pe'er in his article "Schwarzschild Solution and Black Holes" (Pe'er1, 2014):
At first sight, there does not seem to be much hope for simply solving this set of 4 coupled equations by inspection. Fortunately our task is greatly simplified by the high degree of symmetry of the Schwarzschild metric. We know that there are four Killing vectors: three for the spherical symmetry, and one for time translations. Each of these will lead to a constant of the motion for a free particle. Recall that if K_μ is a Killing vector, we know that

$$K_\mu\frac{dx^\mu}{d\lambda}=constant. \tag{34}$$

In addition, there is another constant of the motion that we always have for geodesics (there is no acceleration); metric compatibility implies that along the path the quantity

$$ds^2=-g_{\mu\nu}dx^\mu dx^\nu$$

$$\left(\frac{ds}{d\lambda}\right)^2=\varepsilon=-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} \tag{35}$$

is constant. (This is simply normalization of the 4-velocity: take $\lambda=\tau$ and get $g_{\mu\nu}U^\mu U^\nu=-\varepsilon$, with $\varepsilon=1$ for massive particles and $\varepsilon=0$ for mass-less particles. We may also consider space-like geodesics, for which $\varepsilon=-1$).

We will work out (1):

$$\frac{d^2t}{d\lambda^2}+2\frac{R_s}{2r^2\sigma^2}\frac{dt}{d\lambda}\frac{dr}{d\lambda}=0$$

$$\frac{d^2t}{d\lambda^2}(1-\frac{R_s}{r})+\frac{R_s}{r^2}\frac{dt}{d\lambda}\frac{dr}{d\lambda}=0$$

$$\frac{d^2t}{d\lambda^2}+\frac{R_s}{r^2}\frac{dt}{d\lambda}\frac{dr}{d\lambda}-\frac{R_s}{r}\frac{d^2t}{d\lambda^2}=0$$

$$\frac{d}{d\lambda}\left(\frac{dt}{d\lambda}-\frac{R_s}{r}\frac{dt}{d\lambda}\right)=0$$

$$\boldsymbol{\frac{dt}{d\lambda}\left(1-\frac{R_s}{r}\right)=constant=\frac{E}{c^2}\ \ (total\ energy)}$$

We will work out 4, but to make live a bit easier we assume $\theta=\frac{\pi}{2}$:

$$\frac{d^2\varphi}{d\lambda^2}+2\frac{1}{r}\frac{dr}{d\lambda}\frac{d\varphi}{d\lambda}+2\frac{\cos\theta}{\sin\theta}\frac{d\theta}{d\lambda}\frac{d\varphi}{d\lambda}=0$$

$$\frac{d^2\varphi}{d\lambda^2}+2\frac{1}{r}\frac{dr}{d\lambda}\frac{d\varphi}{d\lambda}=0$$

$$\frac{1}{r^2}\frac{d}{d\lambda}\left(r^2\frac{d\varphi}{d\lambda}\right)=0$$

$$\boldsymbol{r^2\frac{d\varphi}{d\lambda}=constant=L\ (angular\ momentum)}$$

21.1 The Gravitational Potential

Instead of trying to solve directly the geodesic equations using the four conserved quantities associated with Killing vectors, let us first analyze the constraints.

In flat space-time, the symmetries represented by the Killing vectors, and according to Noether's theorem, lead to very familiar conserved quantities: Invariance under **time translations** leads to ***conservation of energy***, while invariance under **spatial rotations** leads to conservation of the three components of ***angular momentum***.

Essentially the same applies to the Schwarzschild metric. We can think of the angular momentum as a three-vector with a magnitude (one component) and direction (two components). Conservation of the *direction* of angular momentum means that the particle will move in a plane. We can choose this to be the equatorial plane of our coordinate system; if the particle is not in this plane, we can rotate coordinates until it

$$\theta=\frac{\pi}{2} \tag{36}$$

The other two Killing vectors correspond to **energy** and the magnitude of **angular momentum**. The time-like Killing vector is $K^\mu=(1,0,0,0)^T$, and thus

$$K_\mu=K^\nu g_{\mu\nu}=\left(-\left(1-\frac{2GM}{r}\right),0,0,0\right) \tag{37}$$

This gives rise to conservation of energy, since using Equation 34,

$$K_\mu\frac{dx^\mu}{d\lambda}=\left(1-\frac{2GM}{c^2r}\right)\frac{dt}{d\lambda}=\frac{E}{mc^2}\,, \tag{38}$$

Where *E* is constant of motion.

Similarly, the Killing vector whose conserved quantity is the magnitude of the angular momentum is $L=\partial_\phi(L^\mu=(0,0,0,1)^T)$, and thus

$$L_\mu=(0,0,0,r^2\sin^2\theta)\,. \tag{39}$$

Using $sin\,\theta\ =\ 1$ derived from Equation 36, one finds

$$r^2\frac{d\phi}{d\lambda}=\frac{L}{m}\ . \tag{40}$$

where *L*, the total angular momentum, is the second conserved quantity. (For mass-less particles these can be thought of as the energy and angular momentum; for massive particles they are the energy and angular momentum per unit mass of the particle.)

Further note that the constancy of the angular momentum in Equation 40 is the GR equivalent of Kepler's second law (equal areas are swept out in equal times).

Armed with this information, we can now analyze the orbits of particles in Schwarzschild metric. We begin by writing explicitly Equation 35, using Equation 36,

$$-\left(1-\frac{2GM}{c^2r}\right)c^2\left(\frac{dt}{d\lambda}\right)^2+\left(1-\frac{2GM}{c^2r}\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2+r^2\left(\frac{d\phi}{d\lambda}\right)^2=-c^2\varepsilon. \tag{41}$$

Multiply this Equation by (1−2GM/r) and use the expressions for E and L (Equations 38 and 40) to write

$$\frac{-E^2}{c^2}+\left(\frac{dr}{d\lambda}\right)^2+\left(1-\frac{2GM}{c^2r}\right)\left(\frac{L^2}{r^2}+c^2\varepsilon\right)=0\,. \tag{42}$$

Clearly, we have made a great progress: instead of the 4 geodesic Equations, we obtain one differential equation for r(λ).

We can re-write Equation 42 as

$$\frac{1}{2}\left(\frac{dr}{d\lambda}\right)^2+V(r)=\frac{1}{2}\frac{E^2}{c^2}\,. \tag{43}$$

Where

$$V(r)=\frac{1}{2}c^2\varepsilon-\varepsilon\frac{GM}{r}+\frac{L^2}{2r^2}-\frac{GML^2}{c^2r^3}\,. \tag{44}$$

Equation 43 is identical to the classical equation describing the motion of a (unit mass) particle moving in a 1-dimensional potential *V (r),* provided its "energy" is $\frac{1}{2}E^2$. (Of course, the true energy is E, but we use this form due to the potential).

Looking at the potential (Equation 44) we see that it only differs from the Newtonian potential by the last term (note that this potential is *exact*, not a power series in *1/r*!). The first term is just a constant ($\varepsilon=1{,}0$) the 2nd term corresponds exactly to the Newtonian gravitational potential, and the third term is a contribution from angular momentum which takes the same form in Newtonian gravity and general relativity. It is the last term, though, which contains the GR contribution, which turns out to make a great deal of difference, especially at small *r.*

It is important not to get confused, though: the physical situation is quite different from a classical particle moving in one dimension. The trajectories under consideration are orbits around a star or other object (see Figure 1). The quantities of interest to us are not only *r(λ),* but also *t(λ)* and $\phi(\lambda)$*.* Nevertheless, obviously it is great help that the radial behavior reduces this to a problem which we know how to solve.

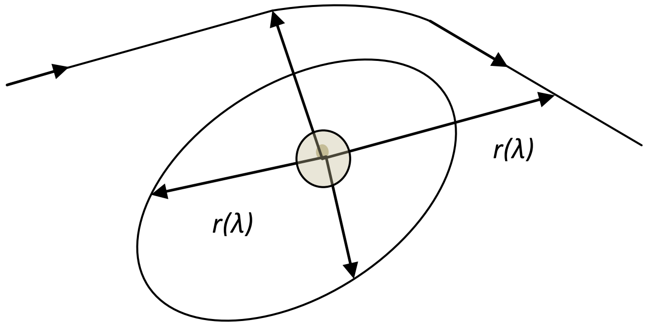


Fig. 1.— Trajectories of particles in a gravitational potential.

21.2 Intermezzo on Energy

Here we will consider the energy as mentioned in 2.1.1 (38)

$$K_\mu \frac{dx^\mu}{d\lambda} = \left(1 - \frac{2GM}{c^2 r}\right)\frac{dt}{d\lambda} = \frac{E}{mc^2} = \sigma^2\,\frac{dt}{d\lambda}, \tag{38}$$

$$E = \sigma^2 mc^2\,\frac{dt}{d\lambda}$$

$$ds^2 = c^2 d\tau^2 = \sigma^2 c^2 dt^2 - \frac{dr^2}{\sigma^2} - r^2 d\theta^2 - r^2 \sin^2\theta^2\, d\emptyset^2$$

$$-\sigma^2 c^2\left(\frac{dt}{d\lambda}\right)^2 + \sigma^{-2}\left(\frac{dr}{d\lambda}\right)^2 + r^2\left(\frac{d\phi}{d\lambda}\right)^2 = -c^2\varepsilon. \tag{41}$$

$$-\sigma^2 c^2\left(\frac{dt}{d\lambda}\right)^2 + \sigma^{-2}\left(\frac{dr}{dt}\right)^2\left(\frac{dt}{d\lambda}\right)^2 + r^2\left(\frac{d\phi}{dt}\right)^2\left(\frac{dt}{d\lambda}\right)^2 = -c^2\varepsilon$$

$$\sigma^2\left(\frac{dt}{d\lambda}\right)^2\left(1 - \frac{\sigma^{-2}\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\phi}{dt}\right)^2}{\sigma^2 c^2}\right) = \varepsilon = 1$$

$$\left(\frac{dt}{d\lambda}\right)^2\left(1 - \frac{v^2}{\sigma^2 c^2}\right) = \frac{1}{\sigma^2} \Rightarrow \frac{dt}{d\lambda} = \frac{1}{\sigma\sqrt{\left(1 - \frac{v^2}{\sigma^2 c^2}\right)}}$$

$$\boldsymbol{E = \frac{\sigma mc^2}{\sqrt{\left(1 - \frac{v^2}{\sigma^2 c^2}\right)}}}\ \textit{is conserved energy} \tag{41a}$$

$$\boldsymbol{E = \gamma_\sigma \sigma mc^2} \tag{41b}$$

$$\boldsymbol{E = \sigma mc^2}\ \textit{is rest energy} \tag{41c}$$

$$\boldsymbol{E = \sigma mc^2\left[\frac{1}{\sqrt{\left(1 - \frac{v^2}{\sigma^2 c^2}\right)}} - 1\right]}\ \textit{is kinetic energy} \tag{41d}$$

For $v \ll c$:

$$\boldsymbol{E} = \sigma m c^2 \left[1 + \frac{v^2}{2\sigma^2 c^2} - 1 \right] = \frac{\boldsymbol{m v^2}}{\boldsymbol{2\sigma}} \; is\ kinetic\ energy \qquad (41e)$$

Another approach:

$$ds^2 = c^2 d\tau^2 = \sigma^2 c^2 dt^2 - \frac{dr^2}{\sigma^2} - r^2 d\theta^2 - r^2 \sin^2\theta^2 \, d\emptyset^2$$

$$-\sigma^2 c^2 \left(\frac{dt}{d\lambda}\right)^2 + \sigma^{-2}\left(\frac{dr}{d\lambda}\right)^2 + r^2\left(\frac{d\phi}{d\lambda}\right)^2 = -c^2\varepsilon \qquad (41)$$

$$\sigma^2 c^2 \left(\frac{dt}{d\lambda}\right)^2 - \sigma^{-2}\left(\frac{dr}{d\lambda}\right)^2 - r^2\left(\frac{d\phi}{d\lambda}\right)^2 = c^2\varepsilon$$

$$\sigma^4 c^2 \left(\frac{dt}{d\lambda}\right)^2 - \left(\frac{dr}{d\lambda}\right)^2 - \sigma^2 r^2\left(\frac{d\phi}{d\lambda}\right)^2 = \sigma^2 c^2\varepsilon$$

$$E = \sigma^2 m c^2 \frac{dt}{d\lambda}$$

$$\left(\frac{E}{mc}\right)^2 = \left(\frac{dr}{d\lambda}\right)^2 + \sigma^2 r^2\left(\frac{d\phi}{d\lambda}\right)^2 + \sigma^2 c^2\varepsilon$$

$$r^2 \frac{d\phi}{d\lambda} = \frac{L}{m} \; => \; r^2\left(\frac{d\phi}{d\lambda}\right)^2 = \frac{L^2}{r^2 m^2} = \frac{(m v_t r)^2}{r^2 m^2} = {v_t}^2$$

Now we make $\lambda = \tau$ and $\varepsilon = 1$:

$$\left(\frac{E}{mc}\right)^2 = \left(\frac{dr}{d\tau}\right)^2 + \sigma^2 {v_t}^2 + \sigma^2 c^2 = {v_r}^2 + \sigma^2 {v_t}^2 + \sigma^2 c^2$$

$$\boldsymbol{m v_r} \; is\ the\ radial\ momentum$$

$$\boldsymbol{m\sigma v_t} is\ the\ tangential\ momentum$$

$$\boldsymbol{\sigma m c^2} \; is\ the\ rest\ energy$$

So kinetic energy is:

$$\boldsymbol{E_{kin}} = \boldsymbol{m}\sqrt{{v_r}^2 + \sigma^2 {v_t}^2}$$

Note: Klopt nog niet dus verder op kauwen.

Another:

$$-\sigma^2 c^2 \left(\frac{dt}{d\lambda}\right)^2 + \sigma^{-2}\left(\frac{dr}{d\lambda}\right)^2 + r^2\left(\frac{d\phi}{d\lambda}\right)^2 = -c^2\varepsilon \qquad (41)$$

$$\left(\frac{E}{c}\right)^2 - \sigma^{-2}\left(\frac{dr}{d\lambda}\right)^2 - r^2\left(\frac{d\phi}{d\lambda}\right)^2 = c^2\varepsilon$$

$$\left(\frac{E}{c}\right)^2 - p^2 = c^2\varepsilon => \left(\frac{E}{c}\right)^2 = c^2\varepsilon + p^2$$

$$E^2 = c^4\varepsilon + p^2c^2$$

$$E = c^2 \text{ when in rest}$$

$$E = pc \text{ in case of photon}$$

Or

$$E^2 = c^4\varepsilon + U^2c^2$$

U is the relativistic speed $\frac{dx}{d\tau}$.

21.3 Deflection of Light

Historically, this was the first independent test of GR. While in Newtonian gravity photons move in straight lines, in GR their paths are deflected. This can be observed when we look at the light coming from a distant star which is "nearly behind" the sun, and ½ a year later when the earth is on the other side of the sun. From practical reasons, the first measurement can be done only during solar eclipse. The location of the star in the sky (=relatively to other stars) will change.

Consider a light ray that approaches from infinity. Using Equations 43 and 44, we find that (with $\epsilon = 0$ for a photon)

$$\frac{1}{2}\left(\frac{dr}{d\lambda}\right)^2 + V(r) = \frac{1}{2}\frac{E^2}{c^2}. \tag{43}$$

With

$$V(r) = \frac{1}{2}c^2\varepsilon - \varepsilon\frac{GM}{r} + \frac{L^2}{2r^2} - \frac{GML^2}{c^2r^3}. \tag{44}$$

$$\frac{1}{2}\left(\frac{dr}{d\lambda}\right)^2 + \frac{L^2}{2r^2} - \frac{GML^2}{c^2r^3} = \frac{1}{2}\frac{E^2}{c^2}. \tag{44a}$$

$$\frac{1}{L^2}\left(\frac{dr}{d\lambda}\right)^2 + \frac{1}{r^2} - \frac{2GM}{c^2r^3} = \frac{E^2}{c^2L^2}. \tag{44b}$$

$$\frac{1}{L^2}\left(\frac{dr}{d\lambda}\right)^2 + \frac{1}{r^2}\left(1 - \frac{2GM}{c^2r}\right) = \frac{E^2}{c^2L^2} \tag{57}$$

$$\left(\frac{dr}{d\lambda}\right)^2 = L^2\left[\frac{E^2}{c^2L^2} - \frac{1}{r^2}\left(1 - \frac{2GM}{c^2r}\right)\right] \tag{57a}$$

It is necessary to specify the parameters found in the formulae. First the angular momentum of the moving particle at infinity is equal by definition to the product of its linear momentum ***p*** by what is called the *impact parameter* ***b***, which represents the distance between the center of attraction (the sun in the present case) and the initial direction of the velocity of the particle (see the figure 2).

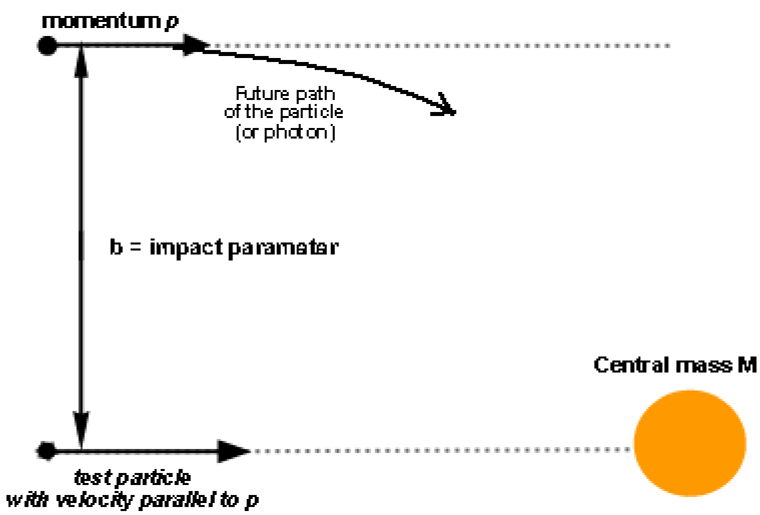


Figure 2. Definition of the impact parameter *b*. The moving particle approaches the mass M from a great distance with vector momentum p. A test particle with a parallel velocity plunges radially onto the mass M. The distance b between their initially parallel paths at 'infinity' is the impact parameter b.

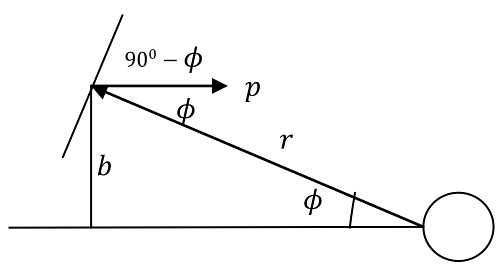
In other words

$$L = p\, b \tag{23}$$

In addition it is known that the momentum ***p*** of a photon is equal to its energy ***E*** (with the units that were chosen). It results at once from this formula that

$$b = \frac{L}{E/c}, \tag{58}$$

Additional elucidation of relationship:



The angular momentum is $L = p \sin\phi . r = p . r \sin\phi = p . b$

Energy $E^2 = p^2c^2 + m^2c^4$, for a photon is *m=0* thus *E=pc*.

So

$$\frac{L}{E/c} = \frac{pb}{pc/c} = b$$

and using Equation 40, $\left(r^2 \frac{d\phi}{d\lambda} = L\ \right)$ we find:

$$\frac{d\phi}{d\lambda} = \frac{d\phi}{dr}\frac{dr}{d\lambda} = \frac{L}{r^2}$$

With (57a):

$$\frac{d\phi}{dr} = \frac{L}{r^2}\left(\frac{dr}{d\lambda}\right)^{-1} = \pm\frac{L}{r^2}\frac{1}{L}\left[\frac{E^2}{L^2} - \frac{1}{r^2}\left(1 - \frac{2GM}{r}\right)\right]^{-1/2}$$

$$\frac{d\phi}{dr} = \pm\frac{1}{r^2}\left[\frac{1}{b^2} - \frac{1}{r^2}\left(1 - \frac{2GM}{r}\right)\right]^{-1/2} \tag{59}$$

Or

$$\left(\frac{1}{r^2}\frac{dr}{d\phi}\right)^2 = \frac{1}{b^2} - \frac{1}{r^2}\left(1 - \frac{2GM}{r}\right) \tag{59a}$$

(see Figure 9).

Getting the maximum deflection angle is now a matter of simple integration,(from infinity to r_1, closest to the Sun, and this distance 2 times). From (59):

$$\Delta\phi = 2\int_{r_1}^{\infty}\frac{dr}{r^2}\left[\frac{1}{b^2} - \frac{1}{r^2}\left(1 - \frac{2GM}{r}\right)\right]^{\frac{-1}{2}} \tag{60}$$

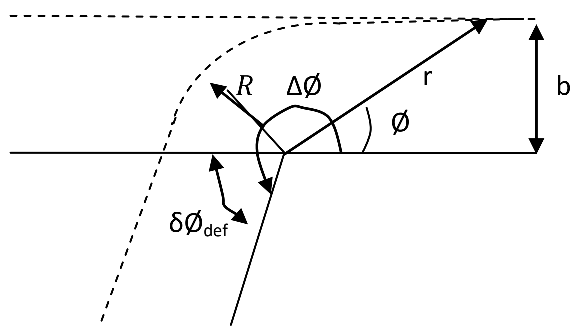


Fig. 9.— Deflection of light by angle $\delta\varphi_{def}$.

Where $r = R$ is the turning point, which is the radius where $\frac{dr}{d\phi} = 0$ (see formula (59a)) and thus $\frac{1}{b^2} = \frac{1}{R^2}\left(1 - \frac{2GM}{c^2 R}\right)$.

For deflection of light by the sun, the impact parameter b cannot be smaller than the stellar radius, $b \geq R_{sun} \approx 7 * 10^8\ m$, and thus $\frac{2GM_{sun}}{c^2 b} \leq 10^{-6}$

Formula (59a) will allow us to determine the change in the direction of a light pulse caused by the gravitational field of the sun. To achieve this aim we have to sum up the successive infinitesimal increments $d\phi$ of the azimuthal angle ϕ along the path. This means that we have to carry out the integration of $\frac{1}{dr}\left(\frac{d\phi}{dr}\right)$ when r varies from the minimum distance denoted R (R is the radius of the sun if the light ray grazes its surface). We should still multiply that quantity par 2 to account for both symmetrical "legs" of the trajectory (the photon first approaches the Sun then recedes from it).

It is necessary to stipulate a further point, namely the relation existing between the two quantities ***b*** and R that we have introduced and that are not independent. The point ***r=R*** corresponds to the place where the light photon is closest to the sun. There the photon moves tangentially. Since at that point there is no radial component, we can write that the derivative $\frac{dr}{dt}$ vanishes. It suffices to take the element dr from Equation (59a) to find immediately

$$\frac{1}{b^2} = \frac{1}{R^2}\left(1 - \frac{2GM}{c^2 R}\right) \tag{61}$$

So that this same equation (59a) becomes

$$\left(\frac{1}{r^2}\frac{dr}{d\phi}\right)^2 = \frac{1}{R^2}\left(1 - \frac{2GM}{R}\right) - \frac{1}{r^2}\left(1 - \frac{2GM}{r}\right) \tag{62}$$

The form of the expression dictates to us to pose

$$u = R/r$$

Where u varies between 1 and 0. The last equation (62) then becomes

$$\left(\frac{du}{d\phi}\right)^2 = \left(1 - \frac{2GM}{R}\right) - u^2\left(1 - \frac{2GMu}{R}\right)$$

Or

$$\left(\frac{du}{d\phi}\right)^2 = 1 - u^2 - \frac{2GM}{R}(1 - u^3) \tag{63}$$

Consequently the infinitesimal variation $d\phi$ of the azimuth is given in terms of the variation du of $\frac{R}{r}$ by

$$\begin{aligned} d\phi &= \left[1 - u^2 - \frac{2GM}{R}(1 - u^3)\right]^{-\frac{1}{2}} du \\ &= \frac{(1-u^2)^{-1/2} du}{\left[1 - \frac{2GM}{R}(1-u^3)(1-u^2)^{-1}\right]^{\frac{1}{2}}} \end{aligned} \tag{64}$$

The presence of the term $(1 - u^2)$ in Expression (64) encourages us to make the change of variable

$$\mathrm{u} = \cos\alpha, 0 < u < 1, 0 < \alpha < \pi/2$$

This leads to

$$d\phi = -\left[1 - \frac{2GM}{R}(1 - \cos^3\alpha)\sin^{-2}\alpha\right]^{-\frac{1}{2}} d\alpha \tag{65}$$

By observing that

$$\frac{1-\cos^3\alpha}{\sin^2\alpha} = \frac{(1-\cos\alpha)(1+\cos\alpha+\cos^2\alpha)}{(1-\cos\alpha)(1+\cos\alpha)} = \frac{1+\cos\alpha\,(1+\cos\alpha)}{(1+\cos\alpha)} = \cos\alpha + \frac{1}{(1+\cos\alpha)}$$

We end up with the final equation of the trajectory under the form

$$d\phi = -\left[1 - \frac{2GM}{R}\left(\cos\alpha + \frac{1}{(1+\cos\alpha)}\right)\right]^{-\frac{1}{2}} d\alpha \tag{66}$$

with

$$\cos\alpha = R/r$$

It is interesting to emphasize that so far there have been no approximation. This is quite rewarding.

21.4 Approximations and integration

The small value of the term $2GM/c^2R = 4.27 \cdot 10^{-6}$ will allow us to make an approximation and in this way will make us able to complete the integration.

In Equation (66) we can thus use the classical (Taylor) approximation $(1+\epsilon)^p \simeq 1 + p\epsilon$ to arrive at

$$d\phi = -\left[1 + \frac{GM}{R}\left(\cos\alpha + \frac{1}{(1+\cos\alpha)}\right)\right] d\alpha \tag{67}$$

Therefore the total variation of the azimuth ϕ along the path of the photon is

$$\Delta\phi = 2\int_0^{\frac{\pi}{2}} \left[1 + \frac{GM}{R}\left(\cos\alpha + \frac{1}{(1+\cos\alpha)}\right)\right] d\alpha \tag{68}$$

$$= 2\left[\alpha + \frac{GM}{R}\left(\sin\alpha + \tan\frac{\alpha}{2}\right)\right]_0^{\frac{\pi}{2}} \tag{69}$$

$$= \pi + \frac{4GM}{R} \tag{70}$$

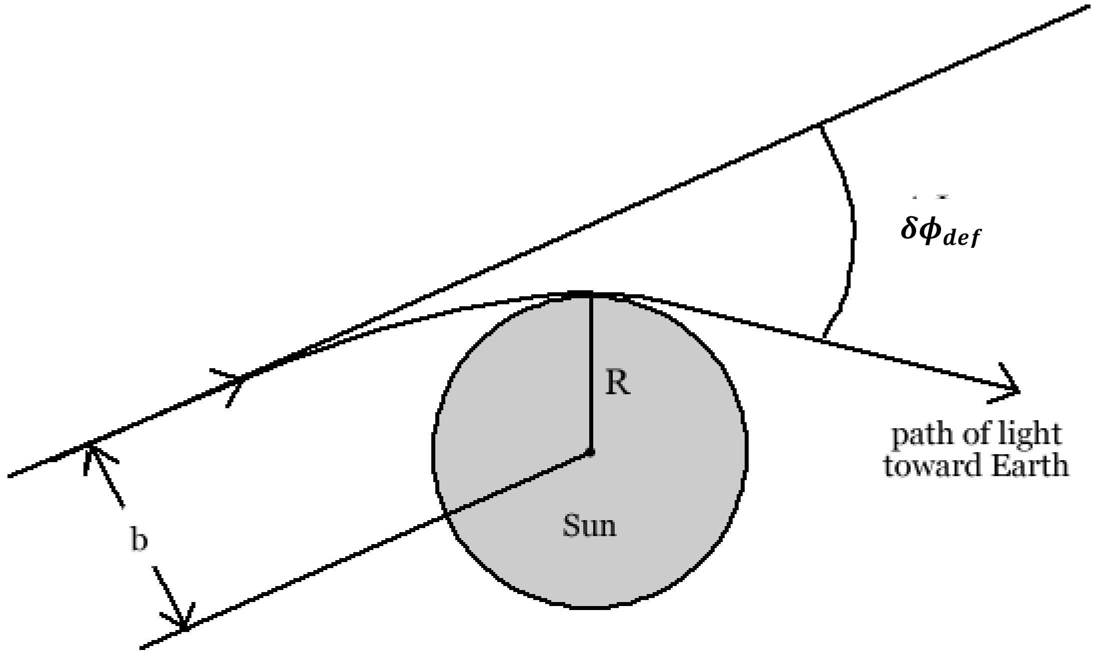
Remark: the integral should be from infinity to R and thus α from $\frac{\pi}{2}$ to 0, by changing the integral to $0\ to\ \frac{\pi}{2}$ the sign changes and the minus sign disappears.

Check whether $\frac{\mathrm{d}\left(\tan\frac{\alpha}{2}\right)}{\mathrm{d}\alpha} = \frac{1}{(1+\cos\alpha)}$, in the formula above, is correct:

$$\frac{\mathrm{d}\left(\tan\frac{\alpha}{2}\right)}{\mathrm{d}\alpha} = \left(\frac{\cos\frac{\alpha}{2}}{\cos\frac{\alpha}{2}} + \frac{\sin^2\frac{\alpha}{2}}{\cos^2\frac{\alpha}{2}}\right)\frac{\mathrm{d}\left(\frac{\alpha}{2}\right)}{\mathrm{d}\alpha} = \frac{1}{2}\left(1 + \frac{\sin^2\frac{\alpha}{2}}{\cos^2\frac{\alpha}{2}}\right) = \frac{1}{2\cos^2\frac{\alpha}{2}}$$

$$\frac{1}{(1+\cos\alpha)} = \frac{1}{1+\cos\left(\frac{\alpha}{2}+\frac{\alpha}{2}\right)} = \frac{1}{\cos^2\frac{\alpha}{2} + \sin^2\frac{\alpha}{2} + \cos^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2}} = \frac{1}{2\cos^2\frac{\alpha}{2}}$$

Thus the integration is correct!



gravitational deflection angle $\delta\phi_{def}$ of starlight by Sun

The first term π of (formula 70) gives the total change in the azimuthal angle of the photon where there is no Sun present, since in that case the photon follows a straight path. But the second term gives the additional angle of deflection $\delta\phi_{def}$ with respect to this straight line (see the figure)

Thus the actual deflection is

$$\delta\phi_{def} = \Delta\phi - \pi \approx \frac{4GM}{c^2R} \qquad (62)$$

Numerically at the surface of the sun (with the values of the mass and the radius given above) one finds $\delta\phi_{def} = 8.5 \cdot 10^{-6}$ radian, or (knowing that π radians equal 180 degrees and that there are 60 minutes of arc in one degree and 60 seconds of arc in one minute of arc)

$$\delta\phi_{def}\phi \lesssim 1.75'' \quad (arc.sec = \frac{\pi}{648000})$$

This effect is also seen outside our solar system, as part of what is known as "gravitational lensing".

21.5 Precession of the Perihelia

From Owen Biesel "The Precession of Mercury's Perihelion" January 25, 2008 (Biesel, 2008)

In the general relativistic case, we assume that the particle is a test particle traveling along a geodesic through space-time. It can be described with the Schwarzschild metric:

$$\boldsymbol{ds^2 = c^2d\tau^2 = -\sigma^2c^2dt^2 + \frac{dr^2}{\sigma^2} + r^2d\theta^2 + r^2\sin^2\theta^2\, d\emptyset^2} \qquad (5)$$

where

$$\sigma = \sqrt{1 - \frac{2GM}{c^2 r}} = \sqrt{1 - \frac{R_s}{r}} \quad with\ the\ Schwarzschild\ radius: R_s = \frac{2GM}{c^2}$$

Now if we parameterize a curve $x(\tau) = \big(t(\tau), r(\tau), \theta(\tau), \emptyset(\tau)\big)$ by proper time, then we find that letting $\mathcal{L} = \langle \frac{dx}{d\tau} . \frac{dx}{d\tau} \rangle$ (differentiation with respect to proper time), $\mathcal{L}$ is both a constant of the motion (−1, in fact) and also satisfies the Euler-Lagrange equations so that $I = \int \mathcal{L}\, d\tau$ is stationary. By exactly the same reasoning as in the classical case, we may restrict our attention to motion in the equatorial plane and assume that $\theta(\tau) \equiv \pi/2$, so that the "Lagrangian" becomes (set c=1):

$$\mathcal{L} = -\left[1 - \frac{R_s}{r}\right]\left(\frac{dt}{d\tau}\right)^2 + \left[1 - \frac{R_s}{r}\right]^{-1}\left(\frac{dr}{d\tau}\right)^2 + r^2\left(\frac{d\emptyset}{d\tau}\right)^2 \qquad (6)$$

Euler-Lagrange operation:

$$Here\ for\ \emptyset{:}\ \frac{d}{d\tau}\left(\frac{\partial \mathcal{L}}{\partial \dot{\emptyset}}\right) = \frac{\partial \mathcal{L}}{\partial \emptyset} = 0 \qquad for\ t{:}\ \frac{d}{d\tau}\left(\frac{\partial \mathcal{L}}{\partial \dot{t}}\right) = \frac{\partial \mathcal{L}}{\partial t} = 0$$

Then the Euler-Lagrange equations for ∅ and *t* read:

$$0 = \frac{d}{d\tau}\left(2r^2 \frac{d\emptyset}{d\tau}\right)$$

$$0 = \frac{d}{d\tau}\left(-2\left(1 - \frac{R_s}{r}\right)\frac{dt}{d\tau}\right)$$

This implies that the *angular momentum (per mass unit)* $L = r^2 \frac{d\emptyset}{d\tau}$ and the *momentum (per mass unit)* $E = \frac{dt}{d\tau}\left(\frac{R_s}{r} - 1\right)$ are two constants of the motion. Then the relation $\mathcal{L} = -1$ gives us:

$$1 = \left[1 - \frac{R_s}{r}\right]\left(\frac{dt}{d\tau}\right)^2 - \left[1 - \frac{R_s}{r}\right]^{-1}\left(\frac{dr}{d\tau}\right)^2 - r^2\left(\frac{d\emptyset}{d\tau}\right)^2$$

$$1 = \frac{E^2}{1 - \frac{R_s}{r}} - \frac{\left(\frac{dr}{d\tau}\right)^2}{1 - \frac{R_s}{r}} - \frac{L^2}{r^2} \quad i.e.$$

$$1 - \frac{R_s}{r} = E^2 - \left(\frac{dr}{d\tau}\right)^2 - \frac{L^2}{r^2} + \frac{L^2}{r^2}\frac{R_s}{r} \quad i.e.$$

$$\left(\frac{dr}{d\tau}\right)^2 = (E^2 - 1) + \frac{R_s}{r} - \frac{L^2}{r^2} + \frac{R_s L^2}{r^3}$$

Once again, assuming $L \neq 0$ allows us to invert $\emptyset = \emptyset(\tau)$, so we may obtain *r* as a function of ∅ with

$$\frac{dr}{d\tau} = \frac{dr}{d\emptyset}\frac{d\emptyset}{d\tau} = r' \frac{L}{r^2} => (r')^2 = \frac{r^4}{L^2}\left(\frac{dr}{d\tau}\right)^2 \ and\ d\emptyset = \frac{1}{r'}dr$$

and hence we have

$$(r')^2 = \frac{E^2 - 1}{L^2} r^4 + \frac{R_s}{L^2} r^3 - r^2 + R_s r$$

Now the requirement that of a closed orbit with $(r')^2 \geq 0$ imposes some constraints on L, E, and R_S; we need a connected component of $\{r: r' \geq 0\}$ to be a compact subset of $\mathbb{R}^+$. This means there exist at least two values R_+ and R_- where r' = 0, i.e. aphelion and perihelion. Then the angle shift from R_+ and R_- is given, as in the classical case, by

$$d\emptyset = \frac{1}{r'}dr$$

$$\emptyset_+ - \emptyset_- = \int_{R_-}^{R_+} \frac{dr}{\sqrt{\frac{E^2-1}{L^2}r^4 + \frac{R_s}{L^2}r^3 - r^2 + R_s r}} \tag{7}$$

Given that ($r - R_+$) and ($r - R_-$) are factors of $\frac{E^2-1}{L^2}r^4 + \frac{R_s}{L^2}r^3 - r^2 + R_s r$, we can solve for $E^2 - 1$ and L^2 in terms of $R_\pm$ and R_S:

$$(E^2-1)R_+^4 + (L^2)(-R_+^2 + R_s R_+) = -R_s R_+^3$$
$$(E^2-1)R_-^4 + (L^2)(-R_-^2 + R_s R_-) = -R_s R_-^3$$

Which give

$$E^2 - 1 = \frac{-R_+R_-R_s + (R_+ + R_-){R_s}^2}{R_+R_-(R_+ + R_- + R_s) - (R_+ + R_-)^2 R_s}$$

$$L^2 = \frac{R^2{}_+R^2{}_-R_s}{R_+R_-(R_+ + R_- + R_s) - (R_+ + R_-)^2 R_s}$$

It is convenient to introduce the combination

$$D = \frac{R_+R_-}{R_+ + R_-},$$

Which has units of distance. Then the above expressions for $E^2 - 1$ and L^2 become:

$$E^2 - 1 = \frac{(-R_s/R_+R_-) + ({R_s}^2/DR_+R_-)}{\frac{1}{D} + \left(\frac{R_s}{R_+R_-}\right) - \left(\frac{R_s}{D^2}\right)}$$

$$L^2 = \frac{R_s}{\frac{1}{D} + \left(\frac{R_s}{R_+R_-}\right) - \left(\frac{R_s}{D^2}\right)}$$

$$\frac{L^2}{E^2-1} = \frac{R_s}{(-R_s/R_+R_-) + ({R_s}^2/DR_+R_-)} = \frac{R_+R_-}{-1 + R_s/D}$$

$$\frac{L^2/R_+R_-}{1-E^2} = \frac{1}{1 - R_s/D}$$

We would like an expression for $\mathcal{E}$, the third nonzero root of $\frac{E^2-1}{L^2}r^4 + \frac{R_s}{L^2}r^3 - r^2 + R_s r = 0$

We know that the sum of the three nonzero roots is $\frac{R_s}{E^2-1}$ (the coefficient of r^3 with the polynomial in standard form); using the above expressions we can swiftly obtain:

$$\varepsilon = \frac{R_s}{1 - \frac{R_s}{D}}$$

Now we can approximate (7), by writing

$$\frac{E^2-1}{L^2}r^4 + \frac{R_s}{L^2}r^3 - r^2 + R_s r = \frac{1-E^2}{L^2}(R_+ - r)(r - R_-)(r - \varepsilon)r.$$

We obtain:

$$\emptyset_+ - \emptyset_- = \sqrt{\frac{L^2}{1-E^2}} \int_{R_-}^{R_+} \frac{1}{\sqrt{r(R_+ - r)(r - R_-)(r - \varepsilon)}} dr$$

$$= \sqrt{\frac{L^2}{1-E^2}} \int_{R_-}^{R_+} \frac{1}{r\sqrt{(R_+ - r)(r - R_-)}} \left(1 - \frac{\varepsilon}{r}\right)^{-1/2} dr$$

Now use the Taylor series expansion $\left(1 - \frac{\varepsilon}{r}\right)^{-1/2} \approx 1 + \frac{\varepsilon}{2r}$, with an error ε bounded by $|\varepsilon| \leq \frac{3}{8}\left(1 - \frac{\varepsilon}{r}\right)^{-5/2}\left(\frac{\varepsilon}{r}\right)^2 \leq \frac{3}{8}\left(1 - \frac{\varepsilon}{R_+}\right)^{-5/2}\left(\frac{\varepsilon}{R_-}\right)^2$ which produces:

$$= \sqrt{\frac{L^2}{1-E^2}} \int_{R_-}^{R_+} \frac{1+\varepsilon}{r\sqrt{(R_+ - r)(r - R_-)}} + \frac{\frac{\varepsilon}{2}}{r^2\sqrt{(R_+ - r)(r - R_-)}} dr$$

Note: According to me the ε, as in the Owen Biesel derivation, should be zero.

The first integral in closed form:

$$\emptyset_+ - \emptyset_- = \int_{R_-}^{R_+} \frac{1+\varepsilon}{r\sqrt{(R_+ - r)(r - R_-)}} dr$$

$$= \frac{1+\varepsilon}{\sqrt{R_+R_-}} \arctan\left[\frac{(R_+ - r)(r - R_-) + r^2 - R_+R_-}{2\sqrt{(R_+ - r)(r - R_-)R_+R_-}}\right]_{R_-}^{R_+}$$

$$\rightarrow \frac{1+\varepsilon}{\sqrt{R_+R_-}}[\arctan[+\infty] - \arctan[-\infty]] = \frac{1+\varepsilon}{\sqrt{R_+R_-}}\left[\frac{\pi}{2} + \frac{\pi}{2}\right] = \frac{1+\varepsilon}{\sqrt{R_+R_-}}\pi$$

Check integrand

$$\frac{d}{dr}\left\{\frac{1}{\sqrt{R_+R_-}} \arctan\left[\frac{(R_+ - r)(r - R_-) + r^2 - R_+R_-}{2\sqrt{(R_+ - r)(r - R_-)R_+R_-}}\right]\right\} ?=? \frac{1}{r\sqrt{(R_+ - r)(r - R_-)}}$$

Pose $a = R_+$ and $b = R_-$

$$\frac{\mathrm{darctanx}}{dx} = \frac{1}{1+x^2}$$

$$\frac{1}{\sqrt{ab}} \frac{d}{dr}\left\{\arctan\left[\frac{(a-r)(r-b) + r^2 - ab}{2\sqrt{(a-r)(r-b)ab}}\right]\right\} = \frac{1}{\sqrt{ab}} \frac{1}{1 + \left[\frac{(a-r)(r-b) + r^2 - ab}{2\sqrt{(a-r)(r-b)ab}}\right]^2} \frac{d}{dr}\left[\frac{(a-r)(r-b) + r^2 - ab}{2\sqrt{(a-r)(r-b)ab}}\right]$$

$$= \frac{1}{\sqrt{ab}} \frac{4(a-r)(r-b)ab}{4(a-r)(r-b)ab + [(a-r)(r-b) + r^2 - ab]^2} \frac{d}{dr}\left[\frac{(a-r)(r-b) + r^2 - ab}{2\sqrt{(a-r)(r-b)ab}}\right]$$

$$= \frac{1}{\sqrt{ab}} \frac{4(a-r)(r-b)ab}{4(a-r)(r-b)ab + [(a-r)(r-b) + r^2 - ab]^2}\left[\frac{-(r-b) + (a-r) + 2r}{2\sqrt{(a-r)(r-b)ab}} - \frac{ab\{(a-r)(r-b) + r^2 - ab\}\{-(r-b) + (a-r)\}}{4\{(a-r)(r-b)ab\}^{3/2}}\right]$$

$$= \frac{1}{\sqrt{ab}} \frac{4(a-r)(r-b)ab}{4(a-r)(r-b)ab + [(a-r)(r-b) + r^2 - ab]^2}\left[\frac{a+b}{2\sqrt{(a-r)(r-b)ab}} - \frac{ab\{ar - ab - r^2 + rb + r^2 - ab\}\{-r + b + a - r\}}{4\{(a-r)(r-b)ab\}^{3/2}}\right]$$

$$= \frac{1}{\sqrt{ab}} \frac{4(a-r)(r-b)ab}{4(a-r)(r-b)ab + [(a-r)(r-b) + r^2 - ab]^2}\left[\frac{a+b}{2\sqrt{(a-r)(r-b)ab}} - \frac{ab\{ar - 2ab + rb\}\{b + a - 2r\}}{4\{(a-r)(r-b)ab\}^{3/2}}\right]$$

$$=\frac{1}{\sqrt{ab}}\frac{4(a-r)(r-b)ab}{4(a-r)(r-b)ab+[(a-r)(r-b)+r^2-ab]^2}\frac{1}{\sqrt{(a-r)(r-b)ab}}\left[\frac{2(a+b)}{4}-\frac{ab\{ar-2ab+rb\}\{b+a-2r\}}{4(a-r)(r-b)ab}\right]$$

$$=\frac{1}{\sqrt{ab}}\frac{4(a-r)(r-b)ab}{4(a-r)(r-b)ab+[(a-r)(r-b)+r^2-ab]^2}\frac{1}{\sqrt{(a-r)(r-b)ab}}*$$

$$\left[\frac{2(a+b)(a-r)(r-b)ab-ab\{ar-2ab+rb\}\{b+a-2r\}}{4(a-r)(r-b)ab}\right]$$

$$=\frac{1}{\sqrt{ab}}\frac{2(a+b)(a-r)(r-b)ab-ab\{ar-2ab+rb\}\{b+a-2r\}}{4(a-r)(r-b)ab+[(a-r)(r-b)+r^2-ab]^2}\frac{1}{\sqrt{(a-r)(r-b)ab}}$$

$$=\frac{1}{ab\sqrt{(a-r)(r-b)}}\frac{2(a+b)(a-r)(r-b)ab-ab\{ar-2ab+rb\}\{b+a-2r\}}{4(a-r)(r-b)ab+[(a-r)(r-b)+r^2-ab]^2}$$

$$=\frac{1}{\sqrt{(a-r)(r-b)}}\frac{2(a+b)(a-r)(r-b)-\{ar-2ab+rb\}\{b+a-2r\}}{4(a-r)(r-b)ab+[(a-r)(r-b)+r^2-ab]^2}$$

$$=\frac{1}{\sqrt{(a-r)(r-b)}}\frac{(2a^2-2ar+2ab-2br)(r-b)-\{abr-2ab^2+rb^2+a^2r-2a^2b+abr-2ar^2+4abr-2br^2\}}{4a^2br-4abr^2-4a^2b^2+4ab^2r+[ar-r^2-ab+br+r^2-ab]^2}$$

$$=\frac{1}{\sqrt{(a-r)(r-b)}}\frac{(2a^2-2ar+2ab-2br)(r-b)-\{6abr-2ab^2+b^2r+a^2r-2a^2b-2ar^2-2br^2\}}{4a^2br-4abr^2-4a^2b^2+4ab^2r+[ar-2ab+br]^2}$$

$$=\frac{1}{\sqrt{(a-r)(r-b)}}\frac{2a^2r-2ar^2+4abr-2br^2-2a^2b-2ab^2+2b^2r-6abr+2ab^2-b^2r-a^2r+2a^2b+2ar^2+2br^2}{4a^2br-4abr^2-4a^2b^2+4ab^2r+[ar-2ab+br]^2}$$

$$=\frac{1}{\sqrt{(a-r)(r-b)}}\frac{a^2r-2abr++b^2r}{4a^2br-4abr^2-4a^2b^2+4ab^2r+[ar-2ab+br]^2}$$

$$=\frac{1}{\sqrt{(a-r)(r-b)}}\frac{r(a^2-2ab++b^2)}{4a^2br-4abr^2-4a^2b^2+4ab^2r+[ar-2ab+br]^2}$$

$$=\frac{1}{\sqrt{(a-r)(r-b)}}\frac{r(a-b)^2}{4a^2br-4abr^2-4a^2b^2+4ab^2r+a^2r^2+4a^2b^2+b^2r^2-4a^2br+2abr^2-4ab^2r}$$

$$=\frac{1}{\sqrt{(a-r)(r-b)}}\frac{r(a-b)^2}{-2abr^2+a^2r^2+b^2r^2}=\frac{1}{\sqrt{(a-r)(r-b)}}\frac{r(a-b)^2}{r^2(-2ab+a^2+b^2)}$$

$$=\frac{1}{\sqrt{(a-r)(r-b)}}\frac{r(a-b)^2}{r^2(a-b)^2}$$

$$=\frac{1}{r\sqrt{(a-r)(r-b)}}$$

Thus

$$\frac{1}{r\sqrt{(R_+-r)(r-R_-)}}$$

Thus the integrand operation is correct!

===.

The second integral is trickier, but can be evaluated in closed form:

$$\int_{R_-}^{R_+} \frac{\varepsilon/2}{r^2\sqrt{(R_+ - r)(r - R_-)}} dr = \frac{\pi\varepsilon/2}{2\sqrt{R_+R_-}} \frac{R_+ + R_-}{R_+R_-} = \frac{1}{\sqrt{R_+R_-}} \frac{\pi\varepsilon}{4D}$$

Then if we recognize that $\frac{L^2/R_+R_-}{1-E^2} = \frac{1}{1-R_s/D}$, we find that

$$\emptyset_+ - \emptyset_- = \pi(1+\varepsilon)\sqrt{\frac{L^2/R_+R_-}{1-E^2}} + \sqrt{\frac{L^2/R_+R_-}{1-E^2}} \frac{\pi\varepsilon}{4D}$$

$$= \frac{\pi}{\sqrt{1-R_s/D}}\left(1 + \frac{1}{4}\frac{R_s/D}{1-R_s/D}\right) + \frac{\pi}{\sqrt{1-R_s/D}}\varepsilon$$

Using the observed values $R_+ = 69.8.10^6\text{km}$, $R_- = 46.0.10^6\text{km}$ (from which we obtain $D = 27.7.10^6\text{km}$, $and\ R_s = \frac{2GM}{c^2} = 2.95km$, we find that the second term is bounded above by $\pi\frac{3}{8}(1-\varepsilon/R_+)^{-5/2}(\varepsilon/R_-)^2/\sqrt{1-R_s/D} \approx 4.88 \cdot 10^{-15}$, making the first term $\frac{\pi}{\sqrt{1-R_s/D}}\left(1+\frac{1}{4}\frac{R_s/D}{1-R_s/D}\right) \approx \pi + 2.515 \cdot 10^{-7}$ a trustworthy estimate of $\emptyset_+ - \emptyset_-$ (half a revolution, in radians). Since Mercury completes 415.2 revolutions each century, and there are $360 \cdot 60 \cdot 60/2\pi$ arcseconds per radian, we find that Mercury's perihelion advances by $(2.515 \cdot 10^{-7})\left(\frac{360\cdot60\cdot60}{\pi}\right) \cdot 415.2 = 43.084$ arcseconds per century.

==

From Asaf Pe'er: For small deflection angle, the result is

$$\delta\phi_{prec} = \frac{6\pi GM}{c^2 a(1-\varepsilon^2)} \tag{63}$$

where a is the semi-major axis and ε is the eccentricity. Obviously, this effect is largest for small a. For mercury, it predicts 43 arc-sec per century, which is consistent with observations.

==

21.6 Shapiro Time Delay

The derivation of the Shapiro time delay is based on the Schwarzschild equation and use will be made on derivations made above.

Instead of solving for $\frac{d\phi}{dr}$, one can solve for $\frac{dt}{dr}$ (using equation 38 and 57a). The result is

$$\left(1 - \frac{2GM}{c^2 r}\right)\frac{dt}{d\lambda} = \frac{E}{mc^2}, \tag{38}$$

$$\left(\frac{dr}{d\lambda}\right)^2 = L^2\left[\frac{E^2}{c^2L^2} - \frac{1}{r^2}\left(1 - \frac{2GM}{c^2 r}\right)\right] \tag{57a}$$

$$\frac{dt}{d\lambda} = \frac{dt}{dr}\frac{dr}{d\lambda} \qquad \frac{dt}{dr} = \frac{dt}{d\lambda}\left(\frac{dr}{d\lambda}\right)^{-1}$$

$$\frac{dt}{dr} = \frac{dt}{d\lambda}\left(\frac{dr}{d\lambda}\right)^{-1} = \frac{E}{c^2}\left(1 - \frac{2GM}{c^2 r}\right)^{-1} \cdot \frac{1}{L}\left(\frac{E^2}{c^2L^2} - \frac{1}{r^2}\left(1 - \frac{2GM}{c^2 r}\right)\right)^{-1/2}$$

$$\frac{dt}{dr} = \pm\frac{1}{cb}\left(1 - \frac{2GM}{c^2 r}\right)^{-1}\left[\frac{1}{b^2} - \frac{1}{r^2}\left(1 - \frac{2GM}{c^2 r}\right)\right]^{-\frac{1}{2}} \tag{64}$$

Intermezzo to try to work out the integral:

$$\frac{dt}{dr} = \pm\frac{1}{cb}\left(1-\frac{2GM}{c^2r}\right)^{-1}\left[\frac{1}{b^2}-\frac{1}{r^2}\left(1-\frac{2GM}{c^2r}\right)\right]^{-\frac{1}{2}} = \pm\frac{1}{cb}\frac{1}{\left(1-\frac{2GM}{c^2r}\right)}\frac{1}{\sqrt{\frac{1}{b^2}-\frac{1}{r^2}\left(1-\frac{2GM}{c^2r}\right)}}$$

Apply Taylor:

$$\frac{dt}{dr} = \pm\frac{1}{cb}\frac{1}{\left(1-\frac{2GM}{c^2r}\right)}\frac{1}{\frac{1}{b}\sqrt{1-\frac{b^2}{r^2}\left(1-\frac{2GM}{c^2r}\right)}} = \pm\frac{\left(1+\frac{2GM}{c^2r}\right)}{c\sqrt{1-\frac{b^2}{r^2}\left(1-\frac{2GM}{c^2r}\right)}} \qquad (65)$$

First approximation:

$$\frac{dt}{dr} = \frac{1}{c}\left(1+\frac{2GM}{c^2r}\right) => \ dt = \frac{1}{c}\int_{r_s}^{r_p}\left(1+\frac{2GM}{c^2r}\right)dr$$

Here is r_p the distance between the centre of the Sun and the reflecting planet. r_s is the shortest distance to the centre of the Sun.

$$dt = \frac{1}{c}\int_{r_s}^{r_p}\left(1+\frac{2GM}{c^2r}\right)dr = \left[\frac{r}{c}+\frac{2GM}{c^3}\ln{(r)}\right]_{r_s}^{r_p} = \frac{r_p-r_s}{c}+\frac{2GM}{c^3}\ln{(\frac{r_p}{r_s})}$$

For the distance from Earth to the sun:

$$dt = \frac{1}{c}\int_{r_s}^{r_e}\left(1+\frac{2GM}{c^2r}\right)dr = \left[\frac{r}{c}+\frac{2GM}{c^3}\ln(r)\right]_{r_s}^{r_e} = \frac{r_e-r_s}{c}+\frac{2GM}{c^3}\ln\left(\frac{r_e}{r_s}\right)$$

To find the total time, the times have to be added and multiplied by two because two times the path is crossed.

$$dt = 2\frac{r_p-r_s}{c}+2\frac{r_e-r_s}{c}+\frac{4GM}{c^3}\ln\left(\frac{r_pr_e}{{r_s}^2}\right)$$

The first two terms on the right hand side are because of the distances between the Sun, planet and Earth. But the third term is due to the curvature caused by the mass of the Sun.

Thus the extra time delay:

$$\boldsymbol{dt = \frac{4GM}{c^3}\ln\left(\frac{r_pr_e}{{r_s}^2}\right)}$$

Second approximation:

$$\frac{dt}{dr} = \frac{\left(1+\frac{2GM}{c^2r}\right)}{c\sqrt{1-\frac{b^2}{r^2}\left(1-\frac{2GM}{c^2r}\right)}} \qquad (65a)$$

b can be big so no Taylor? Let's try anyway:

$$\frac{dt}{dr} = \frac{1}{c}\left(1+\frac{2GM}{c^2r}\right)\left[1+\frac{b^2}{2r^2}\left(1-\frac{2GM}{c^2r}\right)\right]$$

$$\frac{dt}{dr} = \frac{1}{c}\left(1+\frac{2GM}{c^2r}\right)\left[1+\frac{b^2}{2r^2}-\frac{b^2}{r^3}\frac{GM}{c^2}\right]$$

$$\frac{dt}{dr} = \frac{1}{c}\left(1+\frac{b^2}{2r^2}-\frac{b^2}{r^3}\frac{GM}{c^2}+\frac{2GM}{c^2r}+\frac{GM}{c^2}\frac{b^2}{r^3}-2\frac{b^2}{r^4}\left(\frac{GM}{c^2}\right)^2\right)$$

$$\frac{dt}{dr} = \frac{1}{c}\left(1 + \frac{2GM}{c^2 r} + \frac{b^2}{2r^2} - 2\frac{b^2}{r^4}\left(\frac{GM}{c^2}\right)^2\right)$$

Sun to planet:

$$dt = \frac{1}{c}\int_{r_s}^{r_p}\left(1 + \frac{2GM}{c^2 r} + \frac{b^2}{2r^2} - 2\frac{b^2}{r^4}\left(\frac{GM}{c^2}\right)^2\right)dr$$

$$\Delta t = \left[\frac{r}{c} + \frac{2GM}{c^3}\ln(r) - \frac{b^2}{2cr} + \frac{2}{3}\frac{b^2}{cr^3}\left(\frac{GM}{c^2}\right)^2\right]_{r_s}^{r_p} = \frac{r_p - r_s}{c} + \frac{2GM}{c^3}\ln\left(\frac{r_p}{r_s}\right) - \frac{b^2}{2c}\left(\frac{1}{r_p} - \frac{1}{r_s}\right) + \frac{2}{3}\frac{b^2}{c}\left(\frac{GM}{c^2}\right)^2\left(\frac{1}{{r_p}^3} - \frac{1}{{r_s}^3}\right)$$

Sun to Earth:

$$\Delta t = \frac{r_e - r_s}{c} + \frac{2GM}{c^3}\ln\left(\frac{r_e}{r_s}\right) - \frac{b^2}{2c}\left(\frac{1}{r_e} - \frac{1}{r_s}\right) + \frac{2}{3}\frac{b^2}{c}\left(\frac{GM}{c^2}\right)^2\left(\frac{1}{{r_e}^3} - \frac{1}{{r_s}^3}\right)$$

Add the two *dt's* and multiply with two:

$$\Delta t = 2\frac{r_p - r_s}{c} + 2\frac{r_e - r_s}{c} + \frac{4GM}{c^3}\ln\left(\frac{r_p r_e}{{r_s}^2}\right) - \frac{b^2}{c}\left(\frac{1}{r_e} + \frac{1}{r_p} - \frac{2}{r_s}\right) + \frac{4}{3}\frac{b^2}{c}\left(\frac{GM}{c^2}\right)^2\left(\frac{1}{{r_p}^3} + \frac{1}{{r_e}^3} - \frac{2}{{r_s}^3}\right)$$

Again the first two terms are the straight lines but the rest of the right hand side is the extra delay:

$$\boldsymbol{\Delta t = \frac{4GM}{c^3}\ln\left(\frac{r_p r_e}{{r_s}^2}\right) - \frac{b^2}{c}\left(\frac{1}{r_e} + \frac{1}{r_p} - \frac{2}{r_s}\right) + \frac{4}{3}\frac{b^2 G^2 M^2}{c^5}\left(\frac{1}{{r_p}^3} + \frac{1}{{r_e}^3} - \frac{2}{{r_s}^3}\right)}$$

Where r_p is distance from Sun to planet, r_e from Sun to Earth, r_s is shortest distance to centre of the Sun.

Thus, if one sends an electromagnetic signal in a gravitational field, which can be reflected back, and one knows the location of the emitter and reflector, it is possible to calculate the GR correction to the time of the returned signal. As emitter we take a radar located at earth, and as reflector we take one of the inner planets (Mercury or Venus). The GR excess time delay is

$$(\Delta t)_{\text{excess}} \approx \frac{4GM}{c^3}\left[log\left(\frac{4r_R r_{earth}}{r_1^2}\right) + 1\right] \tag{65}$$

Where $r_1 = b$ is the radius of closest approach to the center.

This experiment was proposed and carried by Irwin Shapiro, and is known as Shapiro time delay. In the solar system, the effect is a few hundred microseconds, over a period of ~ 1 hour it takes the signal to reach the planets and return; thus a minimum accuracy of $\sim 10^{-7}$ is needed to see this effect!

21.7 Shapiro Delay – Hobson et al.

Schwarzschild:

$$ds^2 = c^2 d\tau^2 = -\sigma^2 c^2 dt^2 + \frac{dr^2}{\sigma^2} + r^2 d\theta^2 + r^2 \sin^2\theta^2\, d\emptyset^2$$

$$\sigma = \sqrt{1 - \frac{2GM}{c^2 r}} = \sqrt{1 - \frac{2\mu}{r}}$$

In the equatorial plane $\theta = \pi/2$:

$$c^2 d\tau^2 = -\sigma^2 c^2 dt^2 + \frac{dr^2}{\sigma^2} + r^2 d\emptyset^2$$

For photon or radar echoes goes that $d\tau = 0$:

$$\sigma^2 c^2 dt^2 = \frac{dr^2}{\sigma^2} + r^2 d\emptyset^2$$

$$\sigma^2 c^2 \left(\frac{dt}{d\lambda}\right)^2 = \frac{1}{\sigma^2}\left(\frac{dr}{d\lambda}\right)^2 + r^2 \left(\frac{d\emptyset}{d\lambda}\right)^2$$

$$\sigma^2 c^2 \left(\frac{dt}{d\lambda}\right)^2 = \frac{1}{\sigma^2}\left(\frac{dr}{d\lambda}\right)^2 + \frac{L^2}{r^2}$$

$$\sigma^4 c^2 \left(\frac{dt}{d\lambda}\right)^2 = \left(\frac{dr}{d\lambda}\right)^2 + \frac{L^2}{r^2}\sigma^2$$

Pose

$$k^2 = \sigma^4 \left(\frac{dt}{d\lambda}\right)^2$$

$$\left(\frac{dr}{d\lambda}\right)^2 + \frac{L^2}{r^2}\sigma^2 = k^2 c^2$$

The “energy” equation for a photon orbit in the Schwarzschild geometry is:

$$\left(\frac{dr}{d\lambda}\right)^2 + \frac{L^2}{r^2}\left(1 - \frac{2\mu}{r}\right) = k^2 c^2$$

Using

$$\left(\frac{dr}{d\lambda}\right)^2 = \left(\frac{dr}{dt}\frac{dt}{d\lambda}\right)^2 = \frac{k^2}{\left(1 - \frac{2\mu}{r}\right)^2}\left(\frac{dr}{dt}\right)^2$$

We can rewrite the energy equation as

$$\frac{1}{\left(1 - \frac{2\mu}{r}\right)^3}\left(\frac{dr}{dt}\right)^2 + \frac{L^2}{k^2 r^2} - \frac{c^2}{1 - \frac{2\mu}{r}} = 0 \qquad (1)$$

Now consider a photon path from Earth to another planet (say Venus), as shown in Figure 6. Evidently the photon path will be deflected by the gravitational field of the Sun (assuming that the planets are in a configuration like that shown in the figure, where the photon has to pass close to the Sun in order to reach Venus). Let r_0 be the coordinate distance of the closest approach of the photon to the Sun; then

$$\left(\frac{dr}{dt}\right)_{r_0} = 0$$

And so from (1) we get

$$\frac{L^2}{k^2{r_0}^2} = \frac{c^2}{1-\frac{2\mu}{r_0}}$$

Thus, after rearrangement, we can write (1) as

$$\left(\frac{dr}{dt}\right)^2 = \left(1-\frac{2\mu}{r}\right)^3\left(-\frac{L^2}{k^2r^2}+\frac{c^2}{1-\frac{2\mu}{r}}\right) = \left(1-\frac{2\mu}{r}\right)^3\left(-\frac{k^2{r_0}^2c^2}{k^2r^2\left(1-\frac{2\mu}{r_0}\right)}+\frac{c^2}{1-\frac{2\mu}{r}}\right)$$

$$\left(\frac{dr}{dt}\right)^2 = \left(1-\frac{2\mu}{r}\right)^2\left(c^2-\frac{{r_0}^2c^2\left(1-\frac{2\mu}{r}\right)}{r^2\left(1-\frac{2\mu}{r_0}\right)}\right) = c^2\left(1-\frac{2\mu}{r}\right)^2\left(1-\frac{{r_0}^2\left(1-\frac{2\mu}{r}\right)}{r^2\left(1-\frac{2\mu}{r_0}\right)}\right)$$

$$\frac{dr}{dt} = c\left(1-\frac{2\mu}{r}\right)\left[1-\frac{{r_0}^2\left(1-\frac{2\mu}{r}\right)}{r^2\left(1-\frac{2\mu}{r_0}\right)}\right]^{1/2}$$

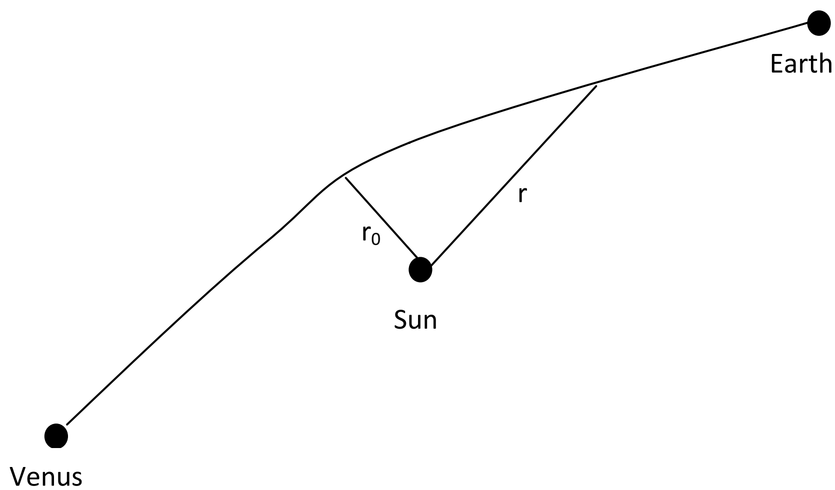


Figure 6 Photon path from Earth to Venus deflected by the Sun.

Which can be integrated to give for the time taken to travel between points r_0 and r

$$t(r,r_0) = \int_{r_0}^{r}\frac{1}{c\left(1-\frac{2\mu}{r}\right)}\left[1-\frac{{r_0}^2\left(1-\frac{2\mu}{r}\right)}{r^2\left(1-\frac{2\mu}{r_0}\right)}\right]^{-\frac{1}{2}}dr$$

First order expansion of

$$\frac{\left(1-\frac{2\mu}{r}\right)}{\left(1-\frac{2\mu}{r_0}\right)} \approx \left(1-\frac{2\mu}{r}\right)\left(1+\frac{2\mu}{r_0}\right) = 1-\frac{2\mu}{r}+\frac{2\mu}{r_0}-\frac{4\mu^2}{rr_0}$$

The integrand can be expanded to the first order in μ/r to obtain

$$t(r,r_0) = \int_{r_0}^{r} \frac{1}{c\left(1-\frac{2\mu}{r}\right)\left[1-\frac{r_0{}^2}{r^2}\left(1-\frac{2\mu}{r}+\frac{2\mu}{r_0}-\frac{4\mu^2}{rr_0}\right)\right]^{\frac{1}{2}}}dr$$

$$t(r,r_0) = \int_{r_0}^{r} \frac{r}{c\left(1-\frac{2\mu}{r}\right)\left[r^2-r_0{}^2\left(1-\frac{2\mu}{r}+\frac{2\mu}{r_0}-\frac{4\mu^2}{rr_0}\right)\right]^{\frac{1}{2}}}dr$$

$$t(r,r_0) = \int_{r_0}^{r} \frac{r}{c\left(1-\frac{2\mu}{r}\right)\left[r^2-r_0{}^2-2\mu r_0+\frac{2\mu r_0{}^2}{r}+\frac{4\mu^2 r_0}{r}\right]^{\frac{1}{2}}}dr$$

$$t(r,r_0) = \int_{r_0}^{r} \frac{r}{c\sqrt{r^2-r_0{}^2}\left(1-\frac{2\mu}{r}\right)\left[1-\frac{2\mu r_0(1-\frac{r_0}{r}-\frac{2\mu}{r}}{r^2-r_0{}^2}\right]^{\frac{1}{2}}}dr$$

$$t(r,r_0) = \int_{r_0}^{r} \frac{r}{c\sqrt{r^2-r_0{}^2}\left[\left(1-\frac{4\mu}{r}+\frac{4\mu^2}{r^2}\right)\left(1-\frac{2\mu r_0(1-\frac{r_0}{r}-\frac{2\mu}{r}}{r^2-r_0{}^2}\right)\right]^{\frac{1}{2}}}dr$$

First work out the right hand-side of the denumerator

$$\left(1-\frac{4\mu}{r}+\frac{4\mu^2}{r^2}\right)\left(1-\frac{2\mu r_0(1-\frac{r_0}{r}-\frac{2\mu}{r}}{r^2-r_0{}^2}\right)$$
$$= \mathbf{1}-\frac{\boldsymbol{4\mu}}{\boldsymbol{r}}+\frac{4\mu^2}{r^2}-\frac{\boldsymbol{2\mu r_0\left(1-\frac{r_0}{r}-\frac{2\mu}{r}\right)}}{\boldsymbol{r^2-r_0{}^2}}+\frac{8\mu^2 r_0(1-\frac{r_0}{r}-\frac{2\mu}{r}}{r(r^2-r_0{}^2)}-\frac{8\mu^3 r_0\left(1-\frac{r_0}{r}-\frac{2\mu}{r}\right)}{r^2(r^2-r_0{}^2)}$$

After ignoring smallest

$$\left(1-\frac{4\mu}{r}+\frac{4\mu^2}{r^2}\right)\left(1-\frac{2\mu r_0(1-\frac{r_0}{r}-\frac{2\mu}{r}}{r^2-r_0{}^2}\right) = \mathbf{1}-\frac{\boldsymbol{4\mu}}{\boldsymbol{r}}-\frac{\boldsymbol{2\mu r_0\left(1-\frac{r_0}{r}\right)}}{\boldsymbol{r^2-r_0{}^2}}$$

$$\left(1-\frac{4\mu}{r}+\frac{4\mu^2}{r^2}\right)\left(1-\frac{2\mu r_0(1-\frac{r_0}{r}-\frac{2\mu}{r}}{r^2-r_0{}^2}\right) = \mathbf{1}-\frac{\boldsymbol{4\mu}}{\boldsymbol{r}}-\frac{\boldsymbol{2\mu r_0(r-r_0)}}{\boldsymbol{r(r+r_0)(1-r_0)}}$$

$$\left(1-\frac{4\mu}{r}+\frac{4\mu^2}{r^2}\right)\left(1-\frac{2\mu r_0(1-\frac{r_0}{r}-\frac{2\mu}{r}}{r^2-{r_0}^2}\right)=\mathbf{1-\frac{4\mu}{r}-\frac{2\mu r_0}{r(r+r_0)}}$$

Fill in the denumerator

$$t(r,r_0)=\int_{r_0}^{r}\frac{r}{c\sqrt{r^2-{r_0}^2}\left[\mathbf{1-\frac{4\mu}{r}-\frac{2\mu r_0}{r(r+r_0)}}\right]^{\frac{1}{2}}}dr$$

Approximation

$$t(r,r_0)=\int_{r_0}^{r}\frac{r}{c\sqrt{r^2-{r_0}^2}}\left[1+\frac{2\mu}{r}+\frac{\mu r_0}{r(r+r_0)}\right]dr$$

This can be evaluated to give

$$t(r,r_0)=\frac{(r^2-{r_0}^2)^{\frac{1}{2}}}{c}+\frac{2\mu}{c}ln\left[\frac{r+(r^2-{r_0}^2)^{\frac{1}{2}}}{r_0}\right]+\frac{\mu}{c}\left(\frac{r-r_0}{r+r_0}\right)^{\frac{1}{2}}$$

==

Check of this formula:

$$\frac{dt(r,r_0)}{dr}=\frac{r}{c(r^2-{r_0}^2)^{\frac{1}{2}}}+\frac{2\mu}{c}\frac{\left(\frac{1}{r_0}+\frac{r}{r_0(r^2-{r_0}^2)^{\frac{1}{2}}}\right)}{\frac{r+(r^2-{r_0}^2)^{\frac{1}{2}}}{r_0}}+\frac{\mu}{c}\frac{\frac{1}{2}\left(\frac{1}{r+r_0}-\frac{(r-r_0)}{(r+r_0)^2}\right)}{\left(\frac{r-r_0}{r+r_0}\right)^{\frac{1}{2}}}$$

$$\frac{dt(r,r_0)}{dr}=\frac{r}{c(r^2-{r_0}^2)^{\frac{1}{2}}}+\frac{2\mu}{c}\frac{\left(1+\frac{r}{(r^2-{r_0}^2)^{\frac{1}{2}}}\right)}{r+(r^2-{r_0}^2)^{\frac{1}{2}}}+\frac{\mu}{c}\frac{\frac{1}{2}\left(\frac{r+r_0-r+r_0}{(r+r_0)^2}\right)}{\left(\frac{r-r_0}{r+r_0}\right)^{\frac{1}{2}}}$$

$$\frac{dt(r,r_0)}{dr}=\frac{r}{c(r^2-{r_0}^2)^{\frac{1}{2}}}+\frac{2\mu}{c}\frac{\left(r+(r^2-{r_0}^2)^{\frac{1}{2}}\right)}{\left(r+(r^2-{r_0}^2)^{\frac{1}{2}}\right)(r^2-{r_0}^2)^{\frac{1}{2}}}+\frac{\mu}{c}\frac{\frac{1}{2}(r+r_0-r+r_0)}{\left(\frac{r-r_0}{r+r_0}\right)^{\frac{1}{2}}(r+r_0)^2}$$

$$\frac{dt(r,r_0)}{dr}=\frac{r}{c(r^2-{r_0}^2)^{\frac{1}{2}}}+\frac{2\mu}{c}\frac{1}{(r^2-{r_0}^2)^{\frac{1}{2}}}+\frac{\mu}{c}\frac{r_0}{\frac{(r^2-{r_0}^2)^{\frac{1}{2}}}{r+r_0}(r+r_0)^2}$$

$$\frac{dt(r,r_0)}{dr}=\frac{r}{c(r^2-{r_0}^2)^{\frac{1}{2}}}+\frac{2\mu}{c}\frac{1}{(r^2-{r_0}^2)^{\frac{1}{2}}}+\frac{\mu}{c}\frac{r_0}{(r^2-{r_0}^2)^{\frac{1}{2}}(r+r_0)}$$

$$\frac{dt(r,r_0)}{dr} = \frac{r}{c(r^2 - {r_0}^2)^{\frac{1}{2}}}\left[1 + \frac{2\mu}{r} + \frac{\mu r_0}{r(r+r_0)}\right]$$

Thus the formula is correct!

==

The first term on the right-hand side is just what we would have expected if the light had been travelling in a straight line. The second and third terms give us the extra coordinate time taken for the photon to travel along the *curved* path to the point *r*. So, you can see from Figure 6 that if we bounce a radar beam to Venus and back then the excess coordinate-time delay over a straight-line path is

$$\Delta t = 2\left[t(r_E, r_0) + t(r_V, r_0) - \frac{({r_E}^2 - {r_0}^2)^{\frac{1}{2}}}{c} - \frac{({r_V}^2 - {r_0}^2)^{\frac{1}{2}}}{c}\right]$$

Where the factor 2 is included because the photon has to go to Venus and back.

Since $r_{E} \gg r_0$ and $r_{v} \gg r_0$ we have

$$t(r_E, r_0) - \frac{({r_E}^2 - {r_0}^2)^{\frac{1}{2}}}{c} \approx \frac{2\mu}{c} ln\left(\frac{r_E + r_E}{r_0}\right) + \frac{\mu}{c} = \frac{2\mu}{c} ln\left(\frac{2r_E}{r_0}\right) + \frac{\mu}{c}$$

And likewise for t_v and $t_{v.}$ Thus

$$\Delta t \approx \frac{2\mu}{c} ln\left(\frac{2r_E}{r_0}\right) + \frac{2\mu}{c} ln\left(\frac{2r_V}{r_0}\right) + \frac{2\mu}{c} = \frac{4GM}{c^3}\left[ln\left(\frac{4r_E r_V}{{r_0}^2}\right) + 1\right]$$

Thus, the excess coordinate-time delay is

$$\boxed{\boldsymbol{\Delta t \approx \frac{4GM}{c^3}\left[ln\left(\frac{4r_E r_V}{{r_0}^2}\right) + 1\right]}}$$

Of course, clocks on Earth do not measure coordinate time but the corresponding proper time of the signal, this is given by

$$\Delta\tau = \left(1 - \frac{2GM}{c^2 r_E}\right)^{\frac{1}{2}} \Delta t$$

However, since $r_{E} \gg \frac{GM}{c^2}$, we can ignore this effect to the accuracy of our calculation. For Venus, when it is opposite to the Earth on the far side of the Sun,

$$\Delta\tau \approx 220\mu s.$$

The effect of the rotation of the Earth around the Sun gives a delay of 15*nsec/sec* , as is shown in a previous chapter (20.3).

The time (Earth, Sun, Venus) without delay is 1720sec. Thus the delay due to the orbit of the Earth is $1720 * 15 * 10^{-9} = 25.8 * 10^{-6}\ sec$.

_(Asmodelle, 2017)

2.1 Shapiro delay

The light travel time delay, or Shapiro delay, is sometimes called the fourth classical test of GR and was first introduced by Shapiro in 1964. This delay in the arrival time of light passing nearby a massive object is the casual result of general relativistic time dilation, due to a significant gravitational potential. Shapiro realized, the speed of light is dependent on the gravitational potential along its travel path, according to GR, and could be tested with a close conjunction of the Sun (Shapiro, 1964). These tests were measured in the weak-field regime. The physical definition of r_p, r_e and r_0, during a test scenario are represented by Figure 8.

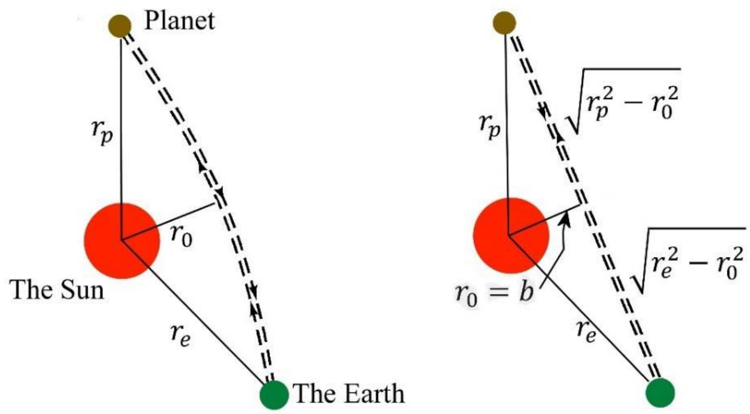


Figure 8 The radar reflection of photons from the Earth to a planet and back. The left image is the actual path, exaggerated. The right image is the Euclidean form. Illustration E. Asmodelle.

To define Shapiro delay, assume the Earth and the planet are stationary, while the total time for the round trip of the radar signal is Δt, in coordinate time. The value of t must be represented in terms of r along the entire pathway, while r_0 is the closest approach to the Sun or its radius. The travel time from Earth to the reflecting planet and back to Earth is given in Equation 19, where k is the Schwarzschild metric parameter: $k = GM_{\odot}/c^2$. Also r_e is the orbital radius of the Earth and r_p is the orbital radius of the reflecting planet[17] (Shapiro, 1964; Lambourne, 2010).

$$\Delta t \approx \frac{2}{c}\left[\left(r_e^2 + r_0^2\right)^{1/2} + \left(r_p^2 + r_0^2\right)^{1/2}\right] + \frac{4k}{c}\left\{\ln\left(4\frac{r_e r_p}{r_0^2}\right) + 1\right\} \quad \text{eq.19}$$

This leads to a viable solution for the excess time delay, or Shapiro delay Δt_{excess}, provided in Equation 20. An alternative notational derivation is presented in Appendix 2.1.1.

$$\Delta t_{excess} \approx \frac{4GM}{c^3}\left[\ln\left(\frac{r_p r_e}{r_0}\right) + 1\right]$$

Eq. 20

In 1965, Shapiro et al., using the Lincoln Laboratory Haystack radar system, began a program to measure the delay in reflected radar signals off solar system planets. The radar system had ~200 μsec accuracy. In 1966, they measured the Shapiro delay of signals bounced off Venus, and also from Mercury in 1967, both during a super-conjunction when the Shapiro delay signal is the *strongest* as the gravitational potential is *highest*. See Figure 9, for the Mercury data. The time delay corresponded with GR predicted value to within, ~20% (Shapiro et al., 1968).

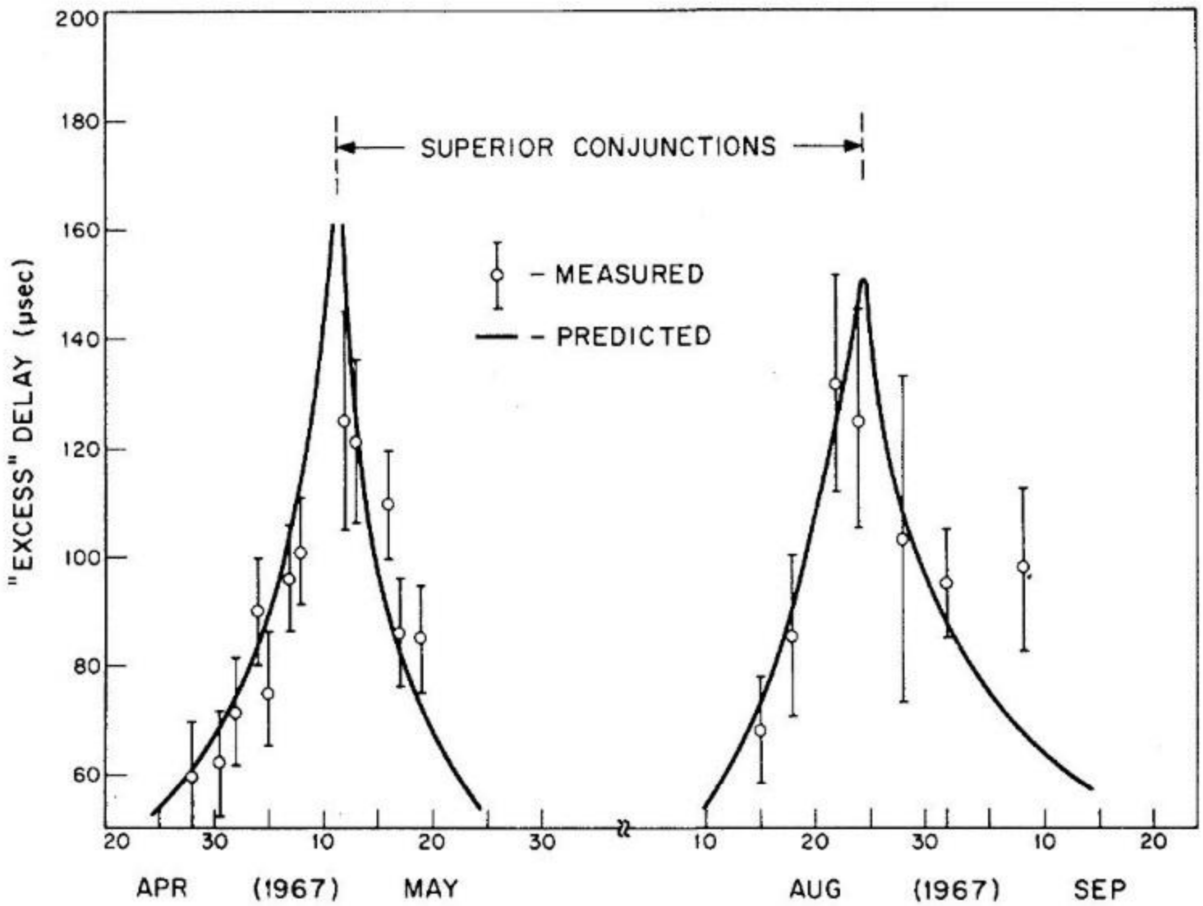


Figure 9: shows measured and predicted Shapiro delays, in msec, based on GR for Earth-Mercury. (Shapiro et al., 1968).

Subsequent measurements in 1971 further reduced the deviation from GR to ~ 5%. Then in 1979, transponders were placed on the Viking spacecraft, whereby radar signals were bounced off Mars from the Viking lander. This enabled Shapiro and Reasenberg to improve the accuracy of the Shapiro delay results, to better than ~0.1% with GR (Rindler, 2006).
In 2003, the Cassini spacecraft flew by Saturn and measured the Shapiro time delay of bounced signals. The deviation of the PPN parameter γ, being the amount of curvature produced per unit mass, was 1.000021 ± 0.000023, against the GR value of $\gamma = 1$ (Ni, 2005). This result confirmed the GR effect of Shapiro delay to approximately 20 parts per million (Cooperstock, 2009).
Fundamentally, Shapiro time delay was not originally thought of by Einstein and his contemporaries, and so its later introduction and experimental verification elucidated GR.

21.8 Trajectories of massive particles-Second Derivation

From "General Relativity an introduction for Physicits" by M.P. Hobson, G. Efstathlou A.N. Lasenby Pag. 230 (M.P Hobson, 2006)

As derived above there are the following equations available:

$$\left(1 - \frac{2GM}{c^2 r}\right)\frac{dt}{d\lambda} = \frac{E}{c^2}$$

$$c^2\left(1-\frac{2GM}{c^2r}\right)\left(\frac{dt}{d\lambda}\right)^2-\left(1-\frac{2GM}{c^2r}\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2-r^2\left(\frac{d\emptyset}{d\lambda}\right)^2=c^2$$

$$r^2\frac{d\emptyset}{d\lambda}=L$$

By substituting the first and the third equation into the second equation:

$$c^2\left(1-\frac{2GM}{c^2r}\right)^2\left(\frac{dt}{d\lambda}\right)^2-\left(\frac{dr}{d\lambda}\right)^2-r^2\left(\frac{d\emptyset}{d\lambda}\right)^2\left(1-\frac{2GM}{c^2r}\right)=c^2\left(1-\frac{2GM}{c^2r}\right)$$

$$\left(\frac{dr}{d\lambda}\right)^2+r^2\left(\frac{d\emptyset}{d\lambda}\right)^2\left(1-\frac{2GM}{c^2r}\right)-c^2\left(1-\frac{2GM}{c^2r}\right)^2\left(\frac{dt}{d\lambda}\right)^2=c^2\left(\frac{2GM}{c^2r}-1\right)$$

$$\left(\frac{dr}{d\lambda}\right)^2+\frac{L^2}{r^2}\left(1-\frac{2GM}{c^2r}\right)-E^2=c^2\left(\frac{2GM}{c^2r}-1\right)=\frac{2GM}{r}-c^2$$

$$\left(\frac{dr}{d\lambda}\right)^2+\frac{L^2}{r^2}\left(1-\frac{2GM}{c^2r}\right)-\frac{2GM}{r}=c^2\left(\frac{E^2}{c^2}-1\right)$$

$$\frac{dr}{d\lambda}=\frac{dr}{d\emptyset}\frac{d\emptyset}{d\lambda}=\frac{L}{r^2}\frac{dr}{d\emptyset}$$

$$\left(\frac{dr}{d\lambda}\right)^2+\frac{L^2}{r^2}=c^2\left(\frac{E^2}{c^2}-1\right)+\frac{2GM}{r}+\frac{2GML^2}{c^2r^3}$$

$$\left(\frac{L}{r^2}\frac{dr}{d\emptyset}\right)^2+\frac{L^2}{r^2}=c^2\left(\frac{E^2}{c^2}-1\right)+\frac{2GM}{r}+\frac{2GML^2}{c^2r^3}$$

$$\left(\frac{1}{r^2}\frac{dr}{d\emptyset}\right)^2+\frac{1}{r^2}=\frac{c^2}{L^2}\left(\frac{E^2}{c^2}-1\right)+\frac{2GM}{rL^2}+\frac{2GM}{c^2r^3}$$

Substitute by $u=1/r$

$$\frac{dr}{d\emptyset}=\frac{dr}{du}\frac{du}{d\emptyset}=\frac{-1}{u^2}\frac{du}{d\emptyset}=-r^2\frac{du}{d\emptyset}=>\frac{1}{r^2}\frac{dr}{d\emptyset}=-\frac{du}{d\emptyset}$$

$$\left(\frac{du}{d\emptyset}\right)^2+u^2=\frac{c^2}{L^2}\left(\frac{E^2}{c^2}-1\right)+\frac{2GMu}{L^2}+\frac{2GMu^3}{c^2}$$

Now we differentiate this equation with respect to Ø to obtain:

$$2\frac{du}{d\emptyset}\frac{d^2u}{d\emptyset^2}+2u\frac{du}{d\emptyset}=\frac{2GM}{L^2}\frac{du}{d\emptyset}+\frac{6GMu^2}{c^2}\frac{du}{d\emptyset}$$

Divide by $2\frac{du}{d\emptyset}$:

$$\frac{d^2u}{d\emptyset^2}+u=\frac{GM}{L^2}+\frac{3GMu^2}{c^2} \qquad (1)$$

If we ignore the last term, we get the equation according to the Newtonian theory, the solution is:

$$u = \frac{GM}{L^2}(1 + e\cos\emptyset)$$

Which describes an ellipse, the parameter *e* measures the *ellipticity* of the orbit. Thus, for example, we can draw the orbit of a planet around the Sun as in figure below. We can write the distance of closest approach (*perihelion*) as $r_1 = a(1-e)$ and the distance of furthest approach (*aphelion*) as $r_2 = a(1+e)$. The equation of motion then requires that the semi-major axis is given by

$$a = \frac{L^2}{GM(1-e^2)} \tag{2}$$

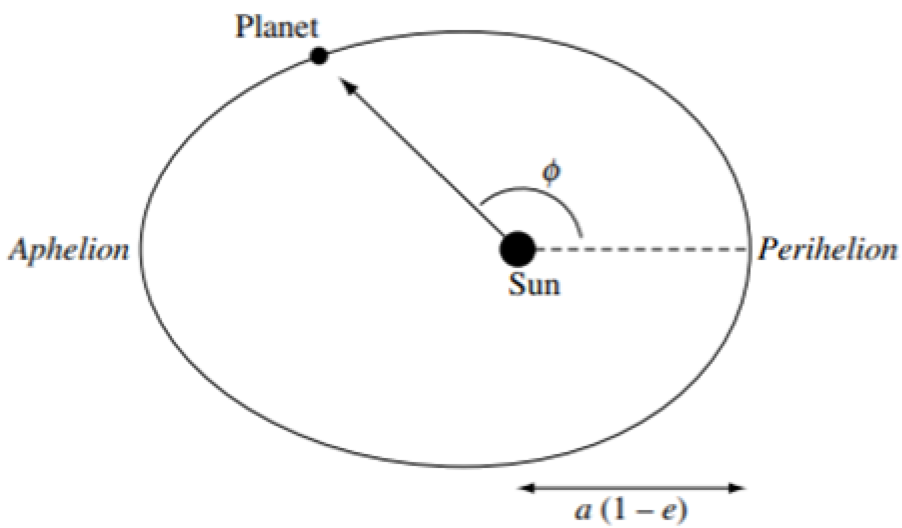
Derivation:

$$r = \frac{L^2}{GM(1 + e\cos\emptyset)} \quad => \quad r_{max} = \frac{L^2}{GM(1+e)} \quad and \ r_{min} = \frac{L^2}{GM(1-e)}$$

$$a = \frac{r_{max} + r_{min}}{2} = \frac{L^2}{2GM}\left(\frac{1}{(1-e)} + \frac{1}{(1+e)}\right) = \frac{L^2}{2GM}\left(\frac{1+e+1-e}{(1-e)(1+e)}\right) = \frac{L^2}{GM(1-e^2)}$$

Hence

$$r_{max} = \frac{L^2}{GM(1-e)} = a(1+e) \quad and \ r_{min} = \frac{L^2}{GM(1+e)} = a(1-e)$$



The elliptical orbit of a planet around the Sun; *e* is the ellipticity of the orbit

Now to include the third term as well the solution looks like:

$$u = \frac{GM}{L^2}(1 + e\cos\emptyset) + \Delta u \tag{3}$$

$$\frac{du}{d\emptyset} = -\frac{GM}{L^2}e\sin\emptyset + \frac{d\Delta u}{d\emptyset}$$

$$\frac{d^2u}{d\emptyset^2} = -\frac{GM}{L^2} e\cos\emptyset + \frac{d^2\Delta u}{d\emptyset^2}$$

Substitute this in formula (1):

$$\frac{d^2u}{d\emptyset^2} + u = \frac{GM}{L^2} + \frac{3GMu^2}{c^2} \tag{1}$$

$$\frac{d^2u}{d\emptyset^2} + u = \frac{GM}{L^2}(1 + e\cos\emptyset - e\cos\emptyset) + \frac{d^2\Delta u}{d\emptyset^2} + \Delta\mathrm{u} = \frac{GM}{L^2} + \frac{d^2\Delta u}{d\emptyset^2} + \Delta\mathrm{u}$$

$$\frac{d^2\Delta u}{d\emptyset^2} + \Delta\mathrm{u} = -\frac{GM}{L^2} + \frac{d^2u}{d\emptyset^2} + u = -\frac{GM}{L^2} + \frac{GM}{L^2} + \frac{3GMu^2}{c^2} = \frac{3GMu^2}{c^2}$$

$$\frac{d^2\Delta u}{d\emptyset^2} + \Delta\mathrm{u} = \frac{3GM}{c^2}\left(\left(\boldsymbol{\frac{GM}{L^2}}\right)^{\mathbf{2}} + \left(\boldsymbol{\frac{GM}{L^2} e\cos}\,\emptyset\right)^{\mathbf{2}} + (\Delta\mathrm{u})^2 + \mathbf{2}\left(\boldsymbol{\frac{GM}{L^2}}\right)^{\mathbf{2}} \boldsymbol{e\cos}\,\emptyset + 2\frac{GM}{L^2}\Delta\mathrm{u} + 2\frac{GM}{L^2} e\cos\emptyset.\Delta\mathrm{u}\right)$$

We find that, to first-order in Δu,

$$\frac{d^2\Delta u}{d\emptyset^2} + \Delta\mathrm{u} = \frac{3(GM)^3}{c^2L^4}\left(\mathbf{1} + (\boldsymbol{e\cos}\,\emptyset)^{\mathbf{2}} + \mathbf{2}\boldsymbol{e\cos}\,\emptyset\right)$$

A particular integral of the equation is found to be:

$$\boldsymbol{\Delta\mathrm{u} = \frac{3(GM)^3}{c^2L^4}\left[1 + e^2\left(\frac{1}{2} - \frac{1}{6}\cos 2\,\emptyset\right) + \mathrm{e}\,\emptyset\sin\emptyset\right]} \tag{4}$$

This can be checked by direct differentiation:

$$\frac{d\Delta u}{d\emptyset} = \frac{3(GM)^3}{c^2L^4}\left[\frac{1}{3}e^2\sin 2\,\emptyset + \mathrm{e}\sin\emptyset + \mathrm{e}\,\emptyset\cos\emptyset\right]$$

$$\frac{d^2\Delta u}{d\emptyset^2} = \frac{3(GM)^3}{c^2L^4}\left[\frac{2}{3}e^2\cos 2\,\emptyset + \mathrm{e}\cos\emptyset + \mathrm{e}\cos\emptyset - \mathrm{e}\,\emptyset\sin\emptyset\right]$$

$$\frac{d^2\Delta u}{d\emptyset^2} = \frac{3(GM)^3}{c^2L^4}\left[\frac{2}{3}e^2\cos 2\,\emptyset + 2\mathrm{e}\cos\emptyset - \mathrm{e}\,\emptyset\sin\emptyset\right]$$

$$\frac{d^2\Delta u}{d\emptyset^2} + \Delta\mathrm{u} = \frac{3(GM)^3}{c^2L^4}\left[\frac{2}{3}e^2\cos 2\,\emptyset + 2\mathrm{e}\cos\emptyset - \mathrm{e}\,\emptyset\sin\emptyset + 1 + e^2\left(\frac{1}{2} - \frac{1}{6}\cos 2\,\emptyset\right) + \mathrm{e}\,\emptyset\sin\emptyset\right]$$

$$\frac{d^2\Delta u}{d\emptyset^2} + \Delta\mathrm{u} = \frac{3(GM)^3}{c^2L^4}\left[1 + \frac{1}{2}e^2 + \frac{1}{2}e^2\cos 2\,\emptyset + 2\mathrm{e}\cos\emptyset\right]$$

$$\frac{d^2\Delta u}{d\emptyset^2} + \Delta\mathrm{u} = \frac{3(GM)^3}{c^2L^4}\left[1 + \frac{1}{2}e^2(1 + \cos 2\,\emptyset) + 2\mathrm{e}\cos\emptyset\right]$$

$$\frac{d^2\Delta u}{d\emptyset^2} + \Delta\mathrm{u} = \frac{3(GM)^3}{c^2L^4}\left[1 + \frac{1}{2}e^2(\sin^2\emptyset + \cos^2\emptyset + \cos^2\emptyset - \sin^2\emptyset) + 2\mathrm{e}\cos\emptyset\right]$$

$$\frac{d^2\Delta u}{d\emptyset^2} + \Delta\mathrm{u} = \frac{3(GM)^3}{c^2L^4}[1 + e^2\cos^2\emptyset + 2\mathrm{e}\cos\emptyset]$$

So equation (4) is correct.

$$u = \frac{GM}{L^2}(1 + e\cos\emptyset) + \Delta\mathrm{u} = \frac{GM}{L^2}(1 + e\cos\emptyset) + \frac{3(GM)^3}{c^2L^4}\left[1 + e^2\left(\frac{1}{2} - \frac{1}{6}\cos 2\,\emptyset\right) + \mathrm{e}\,\emptyset\sin\emptyset\right]$$

Since the constant $\frac{3(GM)^3}{c^2L^4}$ is very small, the last three terms on the right-hand side are tiny, and of no use in testing the theory. However, the last term $e\frac{3(GM)^3}{c^2L^4}\emptyset\sin\emptyset$ might be tiny at first but will gradually grow with time, since the factor $\emptyset$ means that it is cumulative. We must therefore retain it.

$$u=\frac{GM}{L^2}\left[1+e\left(\cos\emptyset+\frac{3(GM)^2}{c^2L^2}\emptyset\sin\emptyset\right)\right]+\frac{3(GM)^3}{c^2L^4}\left[1+e^2\left(\frac{1}{2}-\frac{1}{6}\cos2\emptyset\right)\right]$$

So our approximate solution reads:

$$u=\frac{GM}{L^2}\left[1+e\left(\cos\emptyset+\frac{3(GM)^2}{c^2L^2}\emptyset\sin\emptyset\right)\right]$$

Using the relation

$$cos\left[\emptyset\left(1-\frac{3(GM)^2}{c^2L^2}\right)\right]=cos\left(\emptyset-\frac{3(GM)^2}{c^2L^2}\emptyset\right)=\cos\emptyset\cos\frac{3(GM)^2}{c^2L^2}\emptyset+\sin\emptyset\sin\frac{3(GM)^2}{c^2L^2}\emptyset$$

$$\approx\cos\emptyset+\frac{3(GM)^2}{c^2L^2}\emptyset\sin\emptyset\qquad\text{for }\frac{3(GM)^2}{c^2L^2}\ll1\,,$$

We can therefore write

$$u\approx\frac{GM}{L^2}\left\{1+ecos\left[\emptyset\left(1-\frac{3(GM)^2}{c^2L^2}\right)\right]\right\}=\frac{GM}{L^2}\{1+ecos[\emptyset(1-a)]\}$$

$$r=\frac{L^2}{GM\{1+ecos[\emptyset(1-a)]\}}\tag{5}$$

Here is

$$\alpha=\frac{3(GM)^2}{c^2L^2}$$

From this expression, we see the orbit is periodic, but with a period $2\pi/(1-\alpha)$, i.e. the *r*-values repeat on a cycle that is larger than 2π. The result is that the orbit cannot 'close', and so ellipse *precesses* (see figure below). In one revolution, the ellipse will rotate about the focus by an amount

$$\Delta\emptyset=\frac{2\pi}{1-\alpha}-2\pi=\frac{2\pi a}{1-\alpha}\approx2\pi a=\frac{6\pi(GM)^2}{c^2L^2}$$

Substituting for *L* from (2)

$$a=\frac{L^2}{GM(1-e^2)}\tag{2}$$

Substituting in (5)

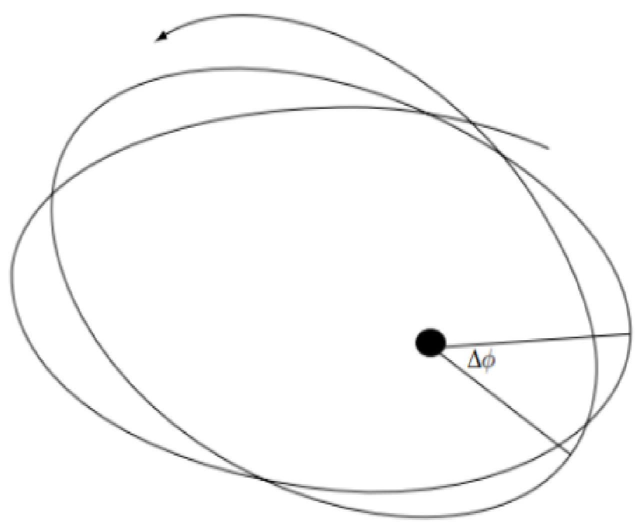
$$r=\frac{L^2}{GM\{1+ecos[\emptyset(1-a)]\}}=\frac{\boldsymbol{a(1-e^2)}}{\boldsymbol{1+ecos[\emptyset(1-a)]}}$$

$$L^2 = aGM(1-e^2)$$

$$\Delta\emptyset = \frac{6\pi(GM)^2}{c^2 aGM(1-e^2)}$$

We finally obtain

$$\boldsymbol{\Delta\emptyset = \frac{6\pi GM}{a(1-e^2)c^2}} \tag{3}$$



Precession of an elliptical orbit (greatly exaggerated)

Let us apply equation (3) to the orbit of Mercury, which has the following parameters: period=88 days, a=5.8x10^{10}m, e=0.2. Using M_s=2x10^{30}kg, we find

$$\Delta\emptyset = 43'' \text{per century.}$$

In fact, the measured precession is:

$$5599''.7 \pm 0''.4 \text{ per century,}$$

But almost all of this is caused by perturbations from other planets. The residual, after taking perturbations into account, is in remarkable agreement with general relativity. The residuals for a number of planets (and Icarus, which is a large asteroid with a perihelion that lies within the orbit of Mercury) may also be calculated (in arcseconds per century):

	Observed residual	Predicted residual
Mercury	43.1+/-0.5	43.03
Venus	8+/-5	8.6
Earth	5+/-1	3.8

Icarus	10+/-1	10.3

In each case, the results are in excellent agreement with the predictions of general relativity. Einstein included this calculation regarding Mercury in his 1915 paper on general relativity. He had solved one of the major problems of celestial mechanics in the very first application of his complicated theory to an empirically testable problem. As you can imagine, this gave him tremendous confidence in his new theory.

21.9 Motion of Particles in Repaired Schwarzschild Geometry

Allereerst kiezen we een coordinatenstelsel:

$$ds^2 = \sigma^2 c^2 dt^2 - \frac{dx^2}{\sigma^2} - \frac{1}{\sigma^2} dy^2 - \frac{1}{\sigma^2} dz^2 \tag{1}$$

Als oorsprong van het coordinatenstelsel wordt het centrum van de zon gekozen.
Verder geldt:

$$\sigma = \sqrt{1 - \frac{2GM}{c^2 r}} = \sqrt{1 - \frac{R_s}{r}} \quad Schwarzschild\ radius: R_s = \frac{2GM}{c^2}$$

$$r = \sqrt{x^2 + y^2 + z^2}$$

$\frac{GM}{r}$ is hier de gravitatie potentiaal. Hierbij is *r* de afstand tot het zonnecentrum. Uiteraard kan het coordinaten-oorsprong ook elders liggen maar dan moet de lengte van r bepaald worden door

$$r = \sqrt{({x_2}^2 - {x_1}^2) + ({y_2}^2 - {y_1}^2) + ({z_2}^2 - {z_1}^2)}$$

De locatie van het zonnecentrum is (x_1,y_1,z_1) en eindpunt van *r* (x_2,y_2,z_2).
Kortom het leven wordt eenvoudiger door de oorsprong in het zonnecentrum te kiezen.

Om gebruik te maken van reeds bestaande afleidingen gaan we over naar een polair coordinatenstelsel.
We kunnen hierbij later ongestraft $\theta = \pi/2$ kiezen om het eenvoudiger te houden. Maar voorlopig houden we hem nog even variabel om ongewisse verschijnselen te voorkomen.
Het verband tussen het coordinatenstelsel in vergelijking (1) en het polaire stelsel is als volgt:

$$x = r \sin\Theta \cos\phi, \;\; y = \; r \sin\Theta \sin\phi\,, \;\; z = \cos\Theta$$

The Repaired Schwarzschild metric for polar co-ordinates

$$ds^2 = \sigma^2 c^2 dt^2 - \frac{dr^2}{\sigma^2} - \frac{r^2}{\sigma^2} d\theta^2 - \frac{r^2 \sin^2\theta}{\sigma^2} d\phi^2$$

$$g_{00} = \boldsymbol{\sigma^2},\; g_{11} = \frac{\mathbf{-1}}{\boldsymbol{\sigma^2}},\; g_{22} = -\frac{\boldsymbol{r^2}}{\sigma^2},\; g_{33} = -\frac{\boldsymbol{r^2 \sin^2\theta}}{\sigma^2} = -\frac{\boldsymbol{r^2}}{\sigma^2};\;\; g^{00} = \frac{\mathbf{1}}{\boldsymbol{\sigma^2}},\; g^{11} = -\boldsymbol{\sigma^2},\; g^{22} = \frac{-\sigma^2}{\boldsymbol{r^2}},\; g^{33} = \frac{-\sigma^2}{\boldsymbol{r^2 \sin^2\theta}} = \frac{-\sigma^2}{r^2}$$

$$\frac{d\sigma}{dr} = \frac{\boldsymbol{R_s}}{\boldsymbol{2r^2\sigma}}$$

In eerste instantie kijken we of de geodeet een eenvoudige oplossing kan bieden. Hiervoor moeten we de Christoffel symbolen berekenen.

Metric first derivative on polar co-ordinates

$$\frac{\partial g_{00}}{\partial r}=\boldsymbol{\frac{R_s}{r^2}},\quad \frac{\partial g_{11}}{\partial r}=\boldsymbol{\frac{R_s}{r^2\sigma^4}},\qquad \frac{\partial g_{22}}{\partial r}=\boldsymbol{\left(\frac{-2r+3R_s}{\sigma^4}\right)},\quad \frac{\partial g_{33}}{\partial r}=\boldsymbol{\frac{(-2r+3R_s)\sin^2\theta}{\sigma^4}}=\boldsymbol{\frac{(-2r+3R_s)}{\sigma^4}},$$
$$\frac{\partial g_{33}}{\partial \theta}=\boldsymbol{\left(-2\frac{r^2}{\sigma^2}.sin(\theta)\cos(\theta)\right)=0}$$

The relevant (non-zero) Christoffel symbols for Schwarzschild polar co-ordinates:

$$\Gamma^{\rho}_{\mu\nu}=\frac{1}{2}g^{\rho\alpha}\left\{\frac{\partial g_{\nu\alpha}}{\partial x^{\mu}}+\frac{\partial g_{\mu\alpha}}{\partial x^{\nu}}-\frac{\partial g_{\mu\nu}}{\partial x^{\alpha}}\right\}$$

$$\Gamma^0_{01}=\Gamma^0_{10}=\frac{1}{2}g^{00}\left\{\frac{\partial g_{00}}{\partial r}\right\}=\boldsymbol{\frac{R_s}{2r^2\sigma^2}},\qquad \Gamma^1_{00}=\frac{1}{2}g^{11}\left\{-\frac{\partial g_{00}}{\partial r}\right\}=\boldsymbol{\frac{\sigma^2R_s}{2r^2}},\qquad \Gamma^1_{11}=\frac{1}{2}g^{11}\left\{\frac{\partial g_{11}}{\partial r}\right\}=\boldsymbol{\frac{-R_s}{2r^2\sigma^2}}$$

$$\Gamma^1_{22}=\frac{1}{2}g^{11}\left\{-\frac{\partial g_{22}}{\partial r}\right\}=\frac{-2r+3\boldsymbol{R_s}}{\boldsymbol{2\sigma^2}},\qquad \Gamma^1_{33}=\frac{1}{2}g^{11}\left\{-\frac{\partial g_{33}}{\partial r}\right\}=\frac{(-2r+3\boldsymbol{R_s})\boldsymbol{\sin^2\theta}}{2\boldsymbol{\sigma^2}}$$

$$\Gamma^2_{12}=\Gamma^2_{21}=\frac{1}{2}g^{22}\left\{\frac{\partial g_{22}}{\partial r}\right\}=\boldsymbol{\frac{2r-3R_s}{2r^2\sigma^2}},\qquad \Gamma^2_{33}=\frac{1}{2}g^{22}\left\{-\frac{\partial g_{33}}{\partial \theta}\right\}=\boldsymbol{\frac{(-2r+3R_s)\,r^2\cos\theta\sin\theta}{\sigma^6}}$$

$$\Gamma^3_{13}=\Gamma^3_{31}=\frac{1}{2}g^{33}\left\{\frac{\partial g_{33}}{\partial r}\right\}=\boldsymbol{\frac{2r-3R_s}{2r^2\sigma^2}},\qquad ,\qquad \Gamma^3_{23}=\Gamma^3_{32}=\frac{1}{2}g^{33}\left\{\frac{\partial g_{33}}{\partial \theta}\right\}=\boldsymbol{\frac{\cos\theta}{\sin\theta}}$$

All the other Christoffel symbols are zero.

The geodesic equations:

$$\frac{d^2x^{\alpha}}{d\lambda^2}+\Gamma^{\alpha}_{\mu\nu}.\frac{dx^{\mu}}{d\lambda}\frac{dx^{\nu}}{d\lambda}=0$$

Work-out for the four coordinates, where λ is the affine parameter:

$$\frac{d^2t}{d\lambda^2}+\Gamma^{t}_{\mu\nu}.\frac{dx^{\mu}}{d\lambda}\frac{dx^{\nu}}{d\lambda}=\frac{d^2t}{d\lambda^2}+2\Gamma^0_{01}.\frac{dt}{d\lambda}\frac{dr}{d\lambda}=\boldsymbol{\frac{d^2t}{d\lambda^2}+2\frac{R_s}{2r^2\sigma^2}.\frac{dt}{d\lambda}\frac{dr}{d\lambda}=0}$$

$$\frac{d^2r}{d\lambda^2}+\Gamma^{r}_{\mu\nu}.\frac{dx^{\mu}}{d\lambda}\frac{dx^{\nu}}{d\lambda}=\frac{d^2r}{d\lambda^2}+\Gamma^1_{00}.\left(\frac{dt}{d\lambda}\right)^2+\Gamma^1_{11}.\left(\frac{dr}{d\lambda}\right)^2+\Gamma^1_{22}.\left(\frac{d\theta}{d\lambda}\right)^2+\Gamma^1_{33}.\left(\frac{d\varphi}{d\lambda}\right)^2=0$$

$$\frac{d^2r}{d\lambda^2}+\Gamma^{r}_{\mu\nu}.\frac{dx^{\mu}}{d\lambda}\frac{dx^{\nu}}{d\lambda}=\boldsymbol{\frac{d^2r}{d\lambda^2}+\frac{\sigma^2R_s}{2r^2}.\left(\frac{dt}{d\lambda}\right)^2-\frac{R_s}{2r^2\sigma^2}.\left(\frac{dr}{d\lambda}\right)^2}+\frac{-2r+3\boldsymbol{R_s}}{\boldsymbol{2\sigma^2}}.\boldsymbol{\left(\frac{d\theta}{d\lambda}\right)^2}+\frac{(-2r+3\boldsymbol{R_s})\boldsymbol{\sin^2\theta}}{2\boldsymbol{\sigma^2}}.\boldsymbol{\left(\frac{d\varphi}{d\lambda}\right)^2=0}$$

$$\frac{d^2\theta}{d\lambda^2}+\Gamma^{\theta}_{\mu\nu}.\frac{dx^{\mu}}{d\lambda}\frac{dx^{\nu}}{d\lambda}=\frac{d^2\theta}{d\lambda^2}+2\Gamma^2_{12}.\frac{dr}{d\lambda}\frac{d\theta}{d\lambda}+\Gamma^2_{33}.\left(\frac{d\varphi}{d\lambda}\right)^2=0$$

$$\boldsymbol{\frac{d^2\theta}{d\lambda^2}+\frac{2r-3R_s}{r^2\sigma^2}.\frac{dr}{d\lambda}\frac{d\theta}{d\lambda}+\frac{(-2r+3R_s)\,r^2\cos\theta\sin\theta}{\sigma^6}.\left(\frac{d\varphi}{d\lambda}\right)^2=0}$$

$$\frac{d^2\varphi}{d\lambda^2}+\Gamma^{\varphi}_{\mu\nu}.\frac{dx^{\mu}}{d\lambda}\frac{dx^{\nu}}{d\lambda}=\frac{d^2\varphi}{d\lambda^2}+2\Gamma^3_{13}.\frac{dr}{d\lambda}\frac{d\varphi}{d\lambda}+2\Gamma^3_{23}.\frac{d\theta}{d\lambda}\frac{d\varphi}{d\lambda}=0$$

$$\boldsymbol{\frac{d^2\varphi}{d\lambda^2}+\frac{2r-3R_s}{r^2\sigma^2}.\frac{dr}{d\lambda}\frac{d\varphi}{d\lambda}+2\frac{\cos\theta}{\sin\theta}.\frac{d\theta}{d\lambda}\frac{d\varphi}{d\lambda}=0}$$

To summarize the resulting four equations:

$$\frac{d^2t}{d\lambda^2}+2\frac{R_s}{2r^2\sigma^2}\frac{dt}{d\lambda}\frac{dr}{d\lambda}=0$$

$$\frac{d^2r}{d\lambda^2}+\frac{\sigma^2R_s}{2r^2}.\left(\frac{dt}{d\lambda}\right)^2-\frac{R_s}{2r^2\sigma^2}.\left(\frac{dr}{d\lambda}\right)^2+\frac{-2r+3R_s}{2\sigma^2}.\left(\frac{d\theta}{d\lambda}\right)^2+\frac{(-2r+3R_s)\sin^2\theta}{2\sigma^2}.\left(\frac{d\varphi}{d\lambda}\right)^2=0$$

$$\frac{d^2\theta}{d\lambda^2}+\frac{2r-3R_s}{r^2\sigma^2}.\frac{dr}{d\lambda}\frac{d\theta}{d\lambda}+\frac{(-2r+3R_s)\,r^2\cos\theta\sin\theta}{\sigma^6}.\left(\frac{d\varphi}{d\lambda}\right)^2=0$$

$$\frac{d^2\varphi}{d\lambda^2}+\frac{2r-3R_s}{r^2\sigma^2}.\frac{dr}{d\lambda}\frac{d\varphi}{d\lambda}+2\frac{\cos\theta}{\sin\theta}.\frac{d\theta}{d\lambda}\frac{d\varphi}{d\lambda}=0$$

Now according to Asaf Pe'er in his article "Schwarzschild Solution and Black Holes":
At first sight, there does not seem to be much hope for simply solving this set of 4 coupled equations by inspection. Fortunately our task is greatly simplified by the high degree of symmetry of the Schwarzschild metric. We know that there are four Killing vectors: three for the spherical symmetry, and one for time translations. Each of these will lead to a constant of the motion for a free particle. Recall that if K_μ is a Killing vector, we know that

$$K_\mu\frac{dx^\mu}{d\lambda}=constant. \tag{34}$$

In addition, there is another constant of the motion that we always have for geodesics (there is no acceleration); metric compatibility implies that along the path the quantity

$$ds^2=-g_{\mu\nu}dx^\mu dx^\nu$$

$$\left(\frac{ds}{d\lambda}\right)^2=\varepsilon=-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} \tag{35}$$

is constant. (This is simply normalization of the 4-velocity: take $\lambda = \tau$ and get $g_{\mu\nu}U^\mu U^\nu=-\varepsilon$, with $\varepsilon=1$ for massive particles and $\varepsilon=0$ for mass-less particles. We may also consider space-like geodesics, for which $\varepsilon=-1$).

21.9.1 Intermezzo: analyse van het verschil tussen formule (1) en de Schwarzschild vergelijking.

Vergelijking (1):
De vier coëfficiënten in vergelijking bevatten alleen, door $\sigma=\sqrt{1-\frac{2GM}{c^2r}}$, de variabele *r*, de afstand tussen zonnecentrum en de locatie van het infinitesimaal kleine lijnsegment ds. Er is dus, wat de coëfficiënten betreft, geen afhankelijkheid van de tijd, Dus de ruimte is symmetrisch in tijd en ruimte maar wel afhankelijk van de afstand.

De Schwarzschild vergelijking in polaire coördinaten:
In deze vergelijking zijn de coëfficiënten afhankelijk van de *r* coördinaat en θ coördinaat. Dus de vergelijking is symmetrsich in tijd en voor de coödinaat φ. Door nu $\theta=\pi/2$ te kiezen zijn de coëfficiënten alleen afhankelijk van *r*

Conclusie:
Voor de formule (1) geldt dat er bolsymmetrie is, dus symmetrie in alle richtingen en alleen de metriek (coëfficiënten) is afhankelijk van de afstand.
Voor Schwarzschild geldt dat de symmetrie alleen in het vlak van een meridiaan bestaat.
Dus zolang we verschijnselen beschouwen die in een plat vlak, door de mediaan, liggen, zijn vergelijking (1) en de Schwarzschild vergelijking identiek in symmetrie. Zoals we weten geldt volgens Emmy Noether dat symmetrie leidt tot behoud van natuurkundige grootheden zoals bijvoorbeeld momentum en angular momentum.

Dus vergelijking (1) is algemener omdat bolsymmetrie geldt en terwijl dat voor Schwarzschild zich beperkt in een vlak. Wel weten we, door een eerdere analyse, dat Schwarzschild voldoet aan de Veldvergelijkingen van de Algemene Relativiteitsleer en dit geldt niet voor formule (1). Echter door het overgaan van cartesian naar spherical coordinaten geldt volgens mij ook dat de repaired schwarzschild vergelijking alleen symmetrisch is in één vlak door de meridiaan.

21.9.2 The Gravitational Potential in Repaired Schwarzschild Geometry

Instead of trying to solve directly the geodesic equations using the four conserved quantities associated with Killing vectors, let us first analyze the constraints.

In flat space-time, the symmetries represented by the Killing vectors, and according to Noether's theorem, lead to very familiar conserved quantities: Invariance under **time translations** leads to ***conservation of energy***, while invariance under **spatial rotations** leads to conservation of the three components of ***angular momentum***.

Essentially the same applies to the Schwarzschild metric. We can think of the angular momentum as a three-vector with a magnitude (one component) and direction (two components). Conservation of the *direction* of angular momentum means that the particle will move in a plane. We can choose this to be the equatorial plane of our coordinate system; if the particle is not in this plane, we can rotate coordinates until it

$$\theta = \frac{\pi}{2} \tag{36}$$

The other two Killing vectors correspond to **energy** and the magnitude of **angular momentum**. The time-like Killing vector is $K^\mu = (1,0,0,0)^T$, and thus

$$K_\mu = K^\nu g_{\mu\nu} = \left(-\left(1 - \frac{R_s}{r}\right), 0,0,0\right) \tag{37}$$

which gives rise to conservation of energy, since using Equation 34,

$$K_\mu \frac{dx^\mu}{d\lambda} = \left(1 - \frac{R_s}{r}\right)\frac{dt}{d\lambda} = E\ , \tag{38}$$

where E is constant of motion.

Similarly, The Killing vector whose conserved quantity is the magnitude of the angular momentum is $L = \partial_\phi (L^\mu = (0,0,0,1)^T)$, and thus

$$L_\mu = \left(0,0,0, \frac{r^2 \sin^2\theta}{\sigma^2}\right). \tag{39}$$

Using $sin\ \theta\ =\ 1$ derived from Equation 36, one finds

$$\frac{r^2}{\sigma^2}\frac{d\phi}{d\lambda} = L\ \ . \tag{40}$$

Wijkt hier af van de bekende angular momentum $L = r^2 \frac{d\phi}{d\lambda}$. Uiteraard kunnen we hier ook nemen

$\frac{L}{\sigma^2} = \frac{r^2}{\sigma^2}\frac{d\phi}{d\lambda}$ maar de rechtse, vierde term dient constant te zijn. We weten dat normaliter *L* ook constant moet zijn maar de *r* in de sigma-term is variabel en verstoort dus het behoud van angular momentum.
Dus het angular momentum hier is σ^2 maal zo groot dan de standaard *L*.

Thus L, (in 40) the total angular momentum, is the second conserved quantity. (For mass-less particles these can be thought of as the energy and angular momentum; for massive particles they are the energy and angular momentum per unit mass of the particle.)

Further note that the constancy of the angular momentum in Equation 40 is the GR equivalent of Kepler's second law (equal areas are swept out in equal times). (Nu door de sigma niet meer!)

Armed with this information, we can now analyze the orbits of particles in Schwarzschild metric.
We begin by writing explicitly Equation 35, using Equation 36,

$$-\left(1-\frac{R_s}{r}\right)\left(\frac{dt}{d\lambda}\right)^2+\left(1-\frac{R_s}{r}\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2+\frac{r^2}{\sigma^2}\left(\frac{d\phi}{d\lambda}\right)^2=-\varepsilon\ . \tag{41}$$

Multiply this Equation by (1−2GM/r) and use the expressions for E and L (Equations 38 and 40) to write

$$-E^2+\left(\frac{dr}{d\lambda}\right)^2+\left(1-\frac{R_s}{r}\right)\left(\frac{\sigma^2L^2}{r^2}+\varepsilon\ \right)=0\ . \tag{42}$$

L is hierbij de *L* zoals gedefinieërd in (40) en wijkt af van de standaard angular momentum.

Clearly, we have made a great progress: instead of the 4 geodesic Equations, we obtain one differential equation for r(λ).

We can re-write Equation 42 as

$$\frac{1}{2}\left(\frac{dr}{d\lambda}\right)^2+V(r\ )=\frac{1}{2}E^2\ . \tag{43}$$

Where

$$V(r\ )=\frac{1}{2}\varepsilon-\varepsilon\frac{R_s}{2r}+\frac{\sigma^2L^2}{2r^2}-\frac{\sigma^2R_sL^2}{2r^3}\ . \tag{44}$$

Equation 43 is identical to the classical equation describing the motion of a (unit mass) particle moving in a 1-dimensional potential *V (r),* provided its “energy” is $\frac{1}{2}E^2$. (Of course, the true energy is E, but we use this form due to the potential).

Looking at the potential (Equation 44) we see that it only differs from the Newtonian potential by the last term (note that this potential is *exact*, not a power series in *1/r*!). The first term is just a constant ($\varepsilon=1{,}0$) the 2nd term corresponds exactly to the Newtonian gravitational potential, and the third term is a contribution from angular momentum which takes the same form in Newtonian gravity and general relativity, apart from the factorσ^2. It is the last term, though, which contains the GR contribution, which turns out to make a great deal of difference, especially at small *r.*

It is important not to get confused, though: the physical situation is quite different from a classical particle moving in one dimension. The trajectories under consideration are orbits around a star or other object (see Figure 1). The quantities of interest to us are not only *r(λ),* but also *t(λ)* and $\phi(\lambda)$*.* Nevertheless, obviously it is great help that the radial behavior reduces this to a problem which we know how to solve.

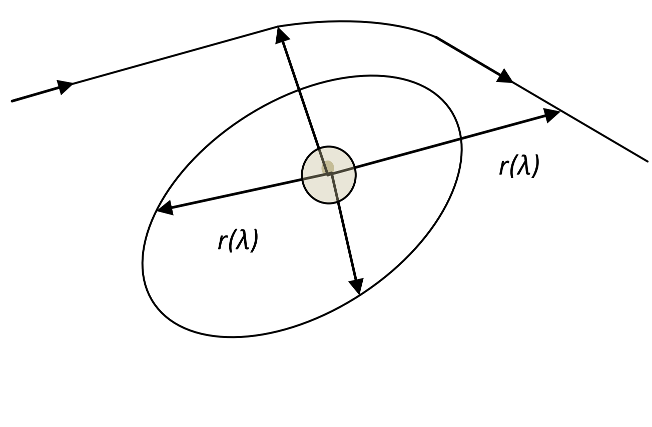


Fig. 1.— Trajectories of particles in a gravitational potential.

21.9.3 Deflection of Light in Repaired Schwarzschild Geometry

Historically, this was the first independent test of GR. While in Newtonian gravity photons move in straight lines, in GR their paths are deflected. This can be observed when we look at the light coming from a distant star which is "nearly behind" the sun, and ½ a year later when the earth is in the other side of the sun. From practical reasons, the first measurement can be done only during solar eclipse. The location of the star in the sky (=relatively to other stars) will change.

Consider a light ray that approaches from infinity. Using Equations 43 and 44, we find that (with $\epsilon = 0, because\ for\ fotons\ \tau = 0$)

$$\frac{1}{2}\left(\frac{dr}{d\lambda}\right)^2 + V(r) = \frac{1}{2}E^2 . \tag{43}$$

With

$$V(r) = \frac{1}{2}\varepsilon - \varepsilon\frac{R_s}{2r} + \frac{\sigma^2 L^2}{2r^2} - \frac{\sigma^2 R_s L^2}{2r^3} . \tag{44}$$

$$\frac{1}{2}\left(\frac{dr}{d\lambda}\right)^2 + \frac{\sigma^2 L^2}{2r^2} - \frac{\sigma^2 R_s L^2}{2r^3} = \frac{1}{2}E^2 . \tag{44a}$$

$$\frac{1}{L^2}\left(\frac{dr}{d\lambda}\right)^2 + \frac{\sigma^2}{r^2} - \frac{\sigma^2 R_s}{r^3} = \frac{E^2}{L^2} . \tag{44b}$$

$$\frac{1}{L^2}\left(\frac{dr}{d\lambda}\right)^2 + \frac{\sigma^2}{r^2}\left(1 - \frac{R_s}{r}\right) = \frac{E^2}{L^2} \tag{57}$$

$$\left(\frac{dr}{d\lambda}\right)^2 = L^2\left[\frac{E^2}{L^2} - \frac{\sigma^2}{r^2}\left(1 - \frac{R_s}{r}\right)\right] \tag{57a}$$

It is necessary to specify the parameters found in the formulae. First the angular momentum of the moving particle at infinity is equal by definition to the product of its linear momentum ***p*** by what is called the *impact parameter* ***b***, which represents the distance between the center of attraction (the sun in the present case) and the initial direction of the velocity of the particle (see the figure).

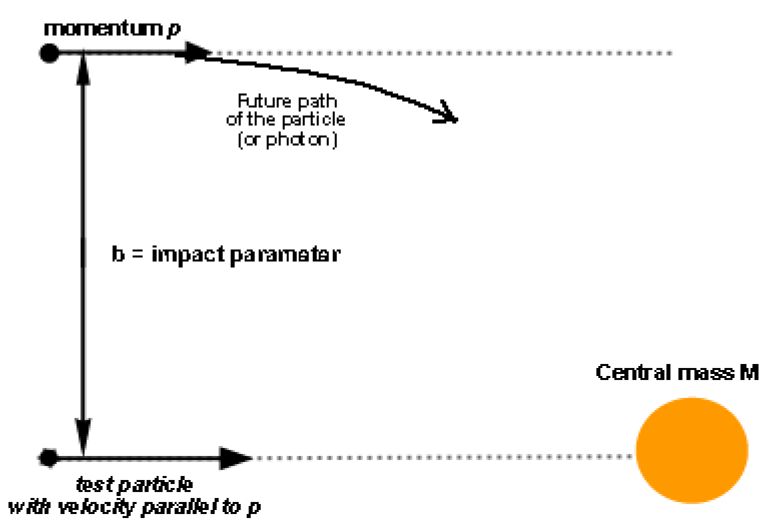


Figure 2. Definition of the impact parameter b. The moving particle approaches the mass M from a great distance with vector momentum p. A test particle with a parallel velocity plunges radially onto the mass M. The distance b between their initially parallel paths at "infinity" is the impact parameter b.

In other words

$$L = p\, b \tag{23}$$

In addition it is known that the momentum ***p*** of a photon is equal to its energy ***E*** (with the units that were chosen). It results at once from this formula that

$$b = \frac{L}{E}, \tag{58}$$

and using Equation 40, $\frac{r^2}{\sigma^2}\frac{d\phi}{d\lambda} = L$ we find:

$$\frac{d\phi}{d\lambda} = \frac{d\phi}{dr}\frac{dr}{d\lambda} = \frac{\sigma^2 L}{r^2}$$

$$\frac{d\phi}{dr} = \frac{\sigma^2 L}{r^2}\left(\frac{dr}{d\lambda}\right)^{-1} = \pm\frac{\sigma^2 L}{r^2}\frac{1}{L}\left[\frac{E^2}{L^2} - \frac{\sigma^2}{r^2}\left(1 - \frac{R_s}{r}\right)\right]^{-1/2}$$

$$\frac{d\phi}{dr} = \pm\frac{\sigma^2}{r^2}\left[\frac{1}{b^2} - \frac{\sigma^2}{r^2}\left(1 - \frac{R_s}{r}\right)\right]^{-1/2} \tag{59}$$

Or

$$\left(\frac{\sigma^2}{r^2}\frac{dr}{d\phi}\right)^2 = \frac{1}{b^2} - \frac{\sigma^2}{r^2}\left(1 - \frac{R_s}{r}\right) \tag{59a}$$

(see Figure 9).
Getting the maximum deflection angle is now a matter of simple integration,

$$\Delta\phi = 2\int_{r_1}^{\infty}\frac{\sigma^2 dr}{r^2}\left[\frac{1}{b^2} - \frac{\sigma^2}{r^2}\left(1 - \frac{R_s}{r}\right)\right]^{\frac{-1}{2}} \tag{60}$$

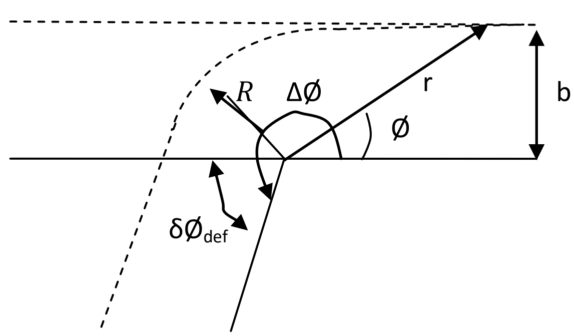


Fig. 9.— Deflection of light by angle $\delta\varphi_{def}$.

where *r* = *R* is the turning point, which is the radius where $\frac{dr}{d\phi} = 0$ and thus $\frac{1}{b^2} = \frac{\sigma^2}{R^2}\left(1 - \frac{R_s}{R}\right)$.

For deflection of light by the sun, the impact parameter b cannot be smaller than the stellar radius, $b \geq R_{sun} \approx 7 * 10^8\ m, and\ thus\ \frac{2GM_{sun}}{c^2 b} \leq 10^{-6}$

Formula (59a) will allow us to determine the change in the direction of a light pulse caused by the gravitational field of the sun. To achieve this aim we have to sum up the successive infinitesimal increments $d\phi$ of the azimuthal angle ϕ along the path. This means that we have to carry out the integration of $\frac{1}{dr}\left(\frac{d\phi}{dr}\right)$ when *r* varies from the minimum distance denoted *R* (*R* is the radius of the sun if the light ray grazes its surface). We should still multiply that quantity par 2 to account for both symmetrical "legs" of the trajectory (the photon first approaches the Sun then recedes from it).
It is necessary to stipulate a further point, namely the relation existing between the two quantities ***b*** and *R* that we have introduced and that are not independent. The point ***r=R*** corresponds to the place where the light photon is closest to the sun. There the photon moves tangentially. Since at that point there is no radial component, we can write that the derivative $\boldsymbol{\frac{dr}{dt}}$ vanishes. It suffices to take the element $\boldsymbol{dr}$ from Equation (59a) to find immediately

$$\frac{1}{b^2} = \frac{\sigma^2}{R^2}\left(1 - \frac{2GM}{c^2R}\right) \tag{61}$$

so that this same equation (59a) becomes

$$\left(\frac{\sigma^2}{r^2}\frac{dr}{d\phi}\right)^2 = \frac{\sigma^2}{R^2}\left(1 - \frac{R_s}{R}\right) - \frac{\sigma^2}{r^2}\left(1 - \frac{R_s}{r}\right) \tag{62}$$

$$\left(\frac{1}{r^2}\frac{dr}{d\phi}\right)^2 = \frac{1}{R^2}\left(1 - \frac{R_s}{R}\right) - \frac{1}{r^2}\left(1 - \frac{R_s}{r}\right) \tag{62a}$$

The form of the expression dictates to us to pose

$$u = R/r$$

where ***u*** varies between 1 and 0. The last equation (62) then becomes

$$\left(\frac{du}{d\phi}\right)^2 = \left(1 - \frac{R_s}{R}\right) - u^2\left(1 - \frac{R_s u}{R}\right)$$

Or

$$\left(\frac{du}{d\phi}\right)^2 = \left[1 - u^2 - \frac{R_s}{R}(1 - u^3)\right] \tag{63}$$

Consequently the infinitesimal variation $d\phi$ of the azimuth is given in terms of the variation du of $\frac{R}{r}$ by

$$d\phi = \left[1 - u^2 - \frac{R_s}{R}(1 - u^3)\right]^{-\frac{1}{2}} du$$

$$= \frac{(1 - u^2)^{-1/2} du}{\left[1 - \frac{R_s}{R}(1 - u^3)(1 - u^2)^{-1}\right]^{\frac{1}{2}}} \tag{64}$$

The presence of the term $(1-u^2)$ in Expression (64) encourages us to make the change of variable

$$\mathrm{u} = \cos\alpha, 0 < u < 1, 0 < \alpha < \pi/2$$

This leads to

$$d\phi = -\left[1 - \frac{R_s}{R}(1-\cos^3\alpha)\sin^{-2}\alpha\right]^{-\frac{1}{2}} d\alpha \tag{65}$$

By observing that

$$\frac{1-\cos^3\alpha}{\sin^2\alpha} = \frac{(1-\cos\alpha)(1+\cos\alpha+\cos^2\alpha)}{(1-\cos\alpha)(1+\cos\alpha)} = \cos\alpha + \frac{1}{(1+\cos\alpha)}$$

we end up with the final equation of the trajectory under the form

$$d\phi = -\left[1 - \frac{R_s}{R}\left(\cos\alpha + \frac{1}{(1+\cos\alpha)}\right)\right]^{-\frac{1}{2}} d\alpha \tag{66}$$

with

$$\cos\alpha = R/r$$

It is interesting to emphasize that so far there have been no approximation. This is quite rewarding.

21.9.4 Approximations and integration in Repaired Schwarzschild Geometry

The small value of the term $R_s/R = 2GM/c^2R = 4.27 \cdot 10^{-6}$ will allow us to make an approximation and in this way will make us able to complete the integration.

In Equation (66) we can thus use the classical (Taylor) approximation $(1+\epsilon)^p \simeq 1 + p\epsilon$ to arrive at

$$d\phi = -\left[1 + \frac{R_s}{2R}\left(\cos\alpha + \frac{1}{(1+\cos\alpha)}\right)\right] d\alpha \tag{67}$$

Therefore the total variation of the azimuth ϕ along the path of the photon is

$$\Delta\phi = 2\int_0^{\frac{\pi}{2}} \left[1 + \frac{R_s}{2R}\left(\cos\alpha + \frac{1}{(1+\cos\alpha)}\right)\right] d\alpha \tag{68}$$

$$= 2\left[\alpha + \frac{R_s}{2R}\left(\sin\alpha + \tan\frac{\alpha}{2}\right)\right]_0^{\frac{\pi}{2}} \tag{69}$$

$$= \pi + \frac{2R_s}{R} \tag{70}$$

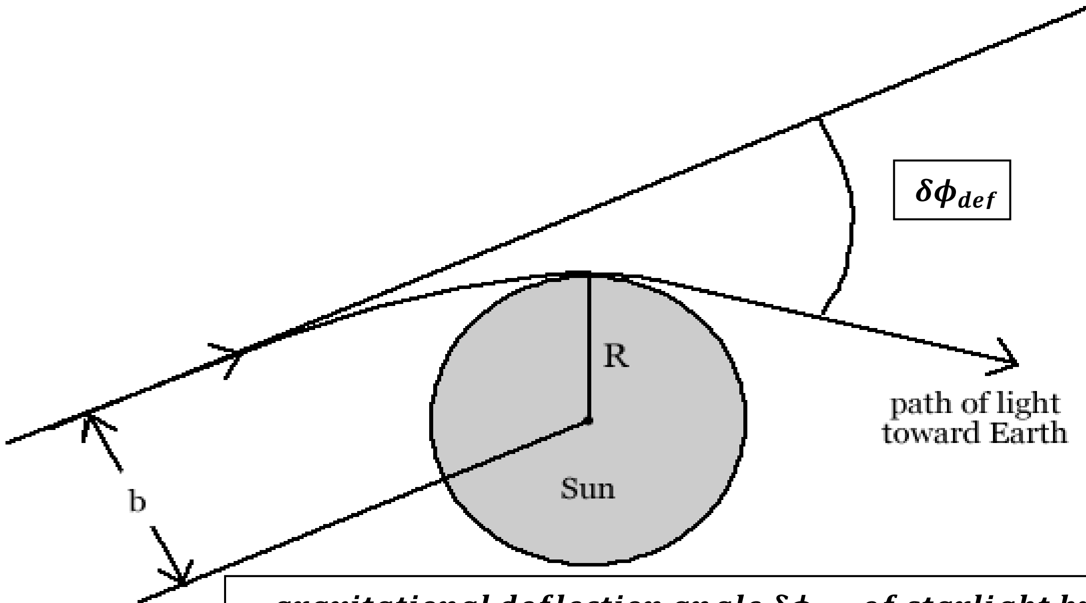
Remark: the integral should be from infinity to R and thus $\alpha\ from\ \frac{\pi}{2}\ to\ 0\ \left(\int_{\frac{\pi}{2}}^{0}\right)$, by changing the integral to $\int_0^{\frac{\pi}{2}}$ the sign changes and the minus sign disappears.

Check whether $\frac{\mathrm{d}\left(\tan\frac{\alpha}{2}\right)}{\mathrm{d}\alpha} = \frac{1}{(1+\cos\alpha)}$ in the formula above is correct:

$$\frac{\mathrm{d}\left(\tan\frac{\alpha}{2}\right)}{\mathrm{d}\alpha} = \left(\frac{\cos\frac{\alpha}{2}}{\cos\frac{\alpha}{2}} + \frac{\sin^2\frac{\alpha}{2}}{\cos^2\frac{\alpha}{2}}\right)\frac{\mathrm{d}\left(\frac{\alpha}{2}\right)}{\mathrm{d}\alpha} = \frac{1}{2}\left(1 + \frac{\sin^2\frac{\alpha}{2}}{\cos^2\frac{\alpha}{2}}\right) = \frac{1}{2\cos^2\frac{\alpha}{2}}$$

$$\frac{1}{(1+\cos\alpha)} = \frac{1}{1+\cos\left(\frac{\alpha}{2}+\frac{\alpha}{2}\right)} = \frac{1}{\cos^2\frac{\alpha}{2}+\sin^2\frac{\alpha}{2}+\cos^2\frac{\alpha}{2}-\sin^2\frac{\alpha}{2}} = \frac{1}{\cos^2\frac{\alpha}{2}+\sin^2\frac{\alpha}{2}+\cos^2\frac{\alpha}{2}-\sin^2\frac{\alpha}{2}} = \frac{1}{2\cos^2\frac{\alpha}{2}}$$

Thus the integration is correct!



$$\boldsymbol{gravitational\ deflection\ angle\ \delta\phi_{def}\ of\ starlight\ by\ Sun}$$
$$(\boldsymbol{greatly\ exaggerated})$$

The first term π gives the total change in the azimuthal angle of the photon where there is no Sun present, since in that case the photon follows a straight path. But the second term gives the additional angle of deflection $\delta\phi_{def}$ with respect to this straight line (see the figure)

Thus the actual deflection is

$$\delta\phi_{def} = \Delta\phi - \pi \approx \frac{4GM}{c^2 b} = \frac{4GM}{c^2 R} = \frac{2R_s}{R} \tag{62}$$

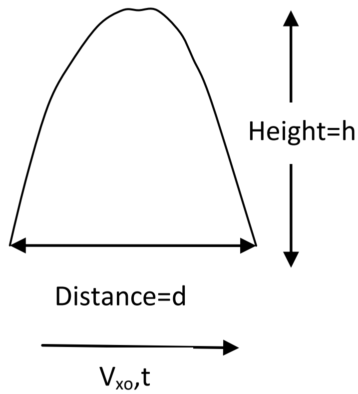
Numerically at the surface of the sun (with the values of the mass and the radius given above) one finds $\delta\phi_{def} = 8.5 \cdot 10^{-6}$ radian, or (knowing that π radians equal 180 degrees and that there are 60 minutes of arc in one degree and 60 seconds of arc in one minute of arc)

$$\delta\phi_{def}\phi \lesssim 1.75'' \quad (arc.sec = \frac{\pi}{648000})$$

This effect is also seen outside our solar system, as part of what is known as “gravitational lensing”.

22 Calculation of trajectory of a bullet

22.1 Via Newton approach



The time for the bullet to cover distance d:

$$v_{xo} = \frac{d}{t} \;=>\; t = \frac{d}{v_{xo}}$$

To fall from the highest point to Earth takes 1/2t seconds:

$$h = \frac{g}{2}\left(\frac{t}{2}\right)^2 = \frac{g}{2}\left(\frac{d}{2v_{xo}}\right)^2 = \frac{g}{8}\left(\frac{d}{v_{xo}}\right)^2$$

To reach the highest point:

$$v_{yo} = g\frac{t}{2} = g\sqrt{\frac{2h}{g}} = \sqrt{2hg}$$

$$v^2 = v_{xo}{}^2 + v_{yo}^2 = \frac{gd^2}{8h} + 2hg = g\left(\frac{d^2 + 16h^2}{8h}\right)$$

$$v = \sqrt{g\left(\frac{d^2 + 16h^2}{8h}\right)}$$

22.2 Via Schwarzschild approach

$$ds^2 = c^2 d\tau^2 = -\sigma^2 c^2 dt^2 + \frac{dr^2}{\sigma^2} + r^2 d\theta^2 + r^2 \sin^2\theta^2\, d\emptyset^2$$

Line up of the frame with the bullet trajectory. So $\theta = \pi/2$.

$$c^2 d\tau^2 = \sigma^2 c^2 dt^2 - \frac{dr^2}{\sigma^2} - r^2 d\emptyset^2$$

Assume we are at the equator and let us determine our time with respect to the universal frame. Because we are in an circular orbit dr=0. And our time is here τ_{earth}

$$c^2 d\tau_{earth}{}^2 = \sigma^2 c^2 dt^2 - r^2 d\emptyset^2$$

$$r d\emptyset = \frac{2\pi R}{sidereal\ day} = \frac{2\pi R}{t_{sd}}$$

Sidereal day=86164.1 sec.

$$\sigma^2 c^2 dt^2 = c^2 d\tau_{earth}{}^2 + r^2 d\emptyset^2$$

$$dt^2 = \sigma^{-2}\left(d\tau_{earth}{}^2 + \frac{r^2}{c^2}d\emptyset^2\right) = \sigma^{-2}d\tau_{earth}{}^2\left[1 + \frac{r^2}{c^2}\left(\frac{d\emptyset}{d\tau_{earth}}\right)^2\right]$$

Now we look at the Bullet situation:

$$c^2 d\tau^2 = \sigma^2 c^2 dt^2 - \frac{dr^2}{\sigma^2} - r^2 d\emptyset^2$$

$$\left(1 - \frac{2GM}{c^2 r}\right)\frac{dt}{d\lambda} = \frac{E}{mc^2}$$

$$r^2\frac{d\phi}{d\lambda} = \frac{L}{m}$$

$$1 = \frac{\sigma^2 c^2 dt^2}{c^2 d\tau^2} - \frac{dr^2}{\sigma^2 c^2 d\tau^2} - \frac{r^2 d\emptyset^2}{c^2 d\tau^2}$$

$$\sigma^2 = \frac{\sigma^4 c^2 dt^2}{c^2 d\tau^2} - \frac{dr^2}{c^2 d\tau^2} - \sigma^2\frac{r^2 d\emptyset^2}{c^2 d\tau^2}$$

$$\sigma^2 = \frac{E^2}{c^2} - \frac{dr^2}{c^2 d\tau^2} - \sigma^2\frac{L^2}{c^2 r^2}$$

$$\frac{dr^2}{c^2 d\tau^2} = \frac{E^2}{c^4} - \sigma^2 - \sigma^2\frac{L^2}{c^2 r^2}$$

23 Bibliography

Anderson, P. B. (2008, October 16). *A Hafele & Keating like thought experiment*. Retrieved from https://paulba.no/pdf/H&K like.pdf

Asmodelle, E. (2017). *Tests of General Relativity: A Review.* University of Central Lancashire (UCLAN).

Biesel, O. (2008, January 25). *The Precession of Mercury's Perihelion*. Retrieved from https://sites.math.washington.edu/~morrow/papers/Genrel.pdf

M.P Hobson, G. E. (2006). *General Relativity an introduction for Physics.* Cambridge University Press.

Magnan, C. (n.d.). *Complete calculations of the perihelion precession of Mercury an the degelection of light by the Sun in General Relativity*. Retrieved from https://arxiv.org/pdf/0712.3709.pdf

Pe'er1, A. (2014, February 19). *Schwarzschild Solution and Black holes*. Retrieved from http://www.physics.ucc.ie/apeer/PY4112/Sch.pdf

Interessante sites:

http://voorbijeinstein.nl/html/artikel_art_paragraaf_05.htm
http://star-www.st-and.ac.uk/~hz4/gr/HeavensGR.pdf
https://einsteinrelativelyeasy.com/index.php/general-relativity/22-geodesics-and-christoffel-symbols
http://jamesowenweatherall.com/SCPPRG/LehmkuhlDennis2008Man_MassEnergyMomentumGR.pdf
http://www.damtp.cam.ac.uk/user/reh10/lectures/nst-mmii-chapter2.pdf Chapter 2 Poisson's Equation - University of Cambridge
http://www.mathpages.com/home/kmath711/kmath711.htm Poisson's Equation and the Universe
https://web.stanford.edu/~oas/SI/SRGR/notes/SchwarzschildSolution.pdf Gary Oas
https://ned.ipac.caltech.edu/level5/March01/Carroll3/Carroll7.html

To do:

1. –geodesic
2. Afleiding stelsel gelijk aan vallend object. Versnelling nul. Afleiding Christoffel symbol
3.