A Special Relativity Paradox: The Barn and the Pole.

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Introduction of the paradox

A pole moves in the direction of a barn. The pole is longer than the length of the barn but due to the Lorentz-contraction the pole fits, at a certain, great, velocity, exactly in the barn. This is in case the observer is on the ground so in rest with respect to the barn.
In case the observer sits on the pole, the barn moves, relatively, with great speed towards the pole. In this case the length of the barn will be even shorter due to the Lorentz-contraction. In this case the pole certainly will not fit; so this is the paradox.
However the times of the events that coincide in the first instance will not coincide in the second case depending on their speed and location; this appears from the formulae of the Lorentz-contraction. Below it is shown that the both cases are not conflicting.

Applied formulae

The calculation is based upon the formulae of the Lorentz-transformation. The derivation of the formulae can be found at the Lorentz Transformations.

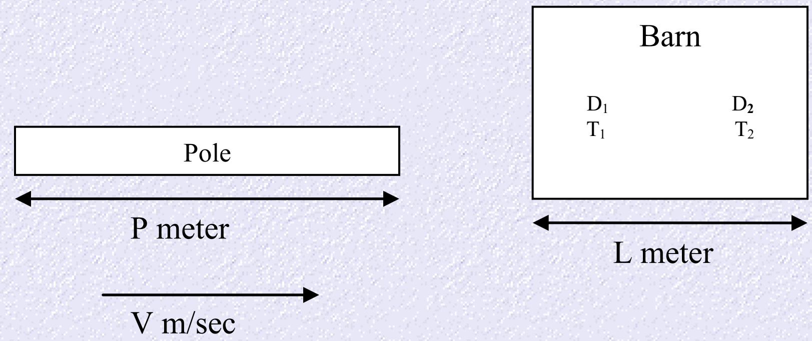
$$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}}$$; *y'=y; z'=z (movement in x-direction)*

$$t' = \frac{t - \frac{v}{c^2}x}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$Length_{pole} = Length'_{pole} \cdot \sqrt{1 - \frac{v^2}{c^2}}$$ (see: Lorentz-Fitzgerald contraction)

x: location on the x-ax in meters
y: location on the y-ax in meters
z: location on the z-ax in meters
t: time in seconds
v: velocity of object in m/sec
c: speed of light in m/sec

Calculation



First consideration: the barn forms the frame of reference.

A pole moves with a velocity *v* towards the barn, where the door D_1 is open and door D_2 closed. Due to the high velocity there is a noticeable length contraction. According to the Lorentz transformation is $P' = P\sqrt{(1-v^2/c^2)} = P/\gamma$ where (to simplify) $\gamma = 1/\sqrt{(1-v^2/c^2)}$. Here is P the length of the pole in a coordinate system where the pole is moving and P' the length of the pole as experienced in the coordinate system of the barn.
C is velocity of light. Assume the pole has a length P = 2L. The pole will only fit in the barn when the velocity is such that the length contraction is half or differently $\gamma = 2$.

$$P' = P/\gamma = 2L/\gamma = L$$

So the calculation of the required size of v:

$$\sqrt{(1-v^2/c^2)} = 1/2$$
$$1-v^2/c^2 = 1/4$$
$$4c^2 - 4v^2 = c^2$$
$$4v^2 = 3c^2$$
$$v = c\sqrt{3}/2 \qquad \text{(a)}$$

On the moment that the pole is exactly in the barn, the door D_2 at the front of the pole opens (time event T_2) and the door D_1, at the back of the pole, closes (at time event T_1). Obviously here is $T_1 = T_2$. This distinction is made because at transformation to an other frame of reference, these time events will be different due to the difference of location. This is, according to the Lorentz transformation:

$$t' = (t-vx/c^2)/\sqrt{(1-v^2/c^2)} = \gamma\,(t-vx/c^2) \qquad \text{(b)}$$

Second consideration: the pole forms the frame of reference.

Now the pole is at rest and the barn moves to the left towards the pole. In this case the length of the barn will be:

$$L' = L/\gamma$$
$$\text{so } L' = L/2$$

$$P = 2L$$
$$\text{so } L' = P/4$$

The length of the pole is now four times the length of the barn. This is only then not in conflict with the above, if the front of the pole is at the door D_2 at the time T_2' and the back of the pole at time T_1' at the door D_1. This will be calculated below.

According to (b):

The velocity of the barn is now towards the pole, so v = -v

$$T_2' = \gamma\,(T_2 + vx_2/\,c^2) \text{ where is } x_2 = L$$
$$T_1' = \gamma\,(T_1 + vx_1/\,c^2) \text{ where is } x_1 = 0$$
Hence
$$T_2' = \gamma\,(T_2 + vL/\,c^2)$$
$$T_1' = \gamma\,T_1$$
$$T_2' - T_1' = \gamma\,T_2 + \gamma\,vL/\,c^2 - \gamma\,T_1$$
$$T_1 = T_2$$
Hence
$$T_2' - T_1' = \gamma\,vL/\,c^2$$

Because the length of the barn is P/4, the barn has to shift **3P/4** in the time difference $T_2' - T_1'$; thus the time difference between opening of door D_2 and closing of the door D_1. The length of the total shift of the barn is:

$$x = vt$$
$$v = c\sqrt{3}/2 \text{ according to (a)}$$
$$t = T_2' - T_1' = \gamma\,vL/\,c^2$$
$$x = vt = \gamma\,v^2L/\,c^2$$
$$x = 2(3c^2/4)\,L/\,c^2$$
$$x = (3/2)L$$
$$L = P/2$$
Hence
$$\mathbf{x = 3P/4}$$
q.e.d.

Above the assumption was made that length of the pole is two times the length of the barn, however now the same will be done but more general where the length of the pole is n times the length of the barn.

First consideration: the barn forms the frame of reference.

A pole moves with a velocity *v* towards the barn, where the door D_1 is open and door D_2 closed. Due to the high velocity there is a noticeable length contraction. According to the Lorentz transformation is $P = P'\sqrt{(1-v^2/c^2)} = P'/\gamma$ where (to simplify) $\gamma = 1/\sqrt{(1-v^2/c^2)}$.

Here is P the length of the pole in a coordinate system where the pole is moving and P' the length of the pole as experienced in the coordinate system of the barn.
C is velocity of light. Assume the pole has a length P = nL. The pole will only fit in the barn when the velocity is such that the length contraction is 1/n or differently $\gamma = n$.

$$P' = P/\gamma = nL/\gamma = L$$

So here is:

$$\sqrt{(1-v^2/c^2)} = 1/n$$
$$1-v^2/c^2 = 1/n^2$$
$$n^2 c^2 - n^2 v^2 = c^2$$
$$n^2 v^2 = (n^2 - 1) c^2$$
$$v = c\frac{\sqrt{(n^2-1)}}{n} \qquad \text{(a)}$$

On the moment that the pole is exactly in the barn, the door D_2 at the front of the pole opens (time event T_2) and the door D_1, at the back of the pole, closes (at time event T_1). Obviously here is $T_1 = T_2$. This distinction is made because at transformation to an other frame of reference, these time events will be different due to the difference of location. This is, according to the Lorentz transformation:

$$t' = (t - vx/c^2)/\sqrt{(1-v^2/c^2)} = \gamma\,(t - vx/c^2) \qquad \text{(b)}$$

Second consideration: the pole forms the frame of reference.

Now the pole is at rest and the barn moves to the left towards the pole. In this case the length of the barn will be:

$$L' = L/\gamma$$
$$\text{so } L' = L/n$$
$$P = nL$$
$$\text{so } L' = P/n^2$$

The length of the pole is now n^2 times the length of the barn. This is only then not in conflict with the above, if the front of the pole is at the door D_2 at the time T_2' and the back of the pole at time T_1' at the door D_1. This will be calculated below.

According to (b):

$$T_2' = \gamma\,(T_2 + vx/c^2) \text{ where is } x = L$$
$$T_1' = \gamma\,(T_1 + vx/c^2) \text{ where is } x = 0$$

Hence

$$T_2' = \gamma\,(T_2 + vL/c^2)$$
$$T_1' = \gamma\,T_1$$
$$T_2' - T_1' = \gamma\,T_2 + \gamma\,vL/c^2 - \gamma\,T_1$$
$$T_1 = T_2$$

Hence

$$T_2' - T_1' = \gamma\,vL/c^2$$

Because the length of the barn is P/n^2, the barn has to shift **(n^2-1)P/ n^2** in the time difference T_2' - T_1'; thus the time difference between opening of door D_2 and closing of the door D_1.
The length of the total shift of the barn is:

$$x = vt$$

$$v = c\frac{\sqrt{(n^2-1)}}{n}$$

According to (a)

$$t = T_2' - T_1' = \gamma\, vL/c^2$$

Hence

$$x = \gamma\, v^2L/c^2$$

$$x = n\{(n^2-1)c^2/n^2\}L/c^2$$

$$x = \{(n^2-1)/n\}L$$

$$L = P/n$$

Hence

$$\mathbf{x = (n^2-1)P/n^2}$$

q.e.d.

Alternative second consideration: the pole forms the frame of reference.

Now the pole is at rest and the barn moves to the left towards the pole. The length of the barn in its own frame is L while the length of the pole in its own frame is P. In the first consideration it has been shown that the pole only fits in the barn if $P = \gamma . L$

The barn is the "moving" system. In the system the barn, the doors of the barn act at the same moments. Thus $T_1^{'} = T_2^{'}$ this results in the following equation:

$$T_2 - T_1 = \frac{v}{c^2}.(x_2 - x_1)$$

$$x_1^{'} = 0$$

$$x_2^{'} = L$$

$$x_2^{'} - x_1^{'} = L = \frac{(x_2 - x_1) - v.(T_2 - T_1)}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{(x_2 - x_1).(1 - \frac{v^2}{c^2})}{\sqrt{1 - \frac{v^2}{c^2}}} = (x_2 - x_1).\sqrt{1 - \frac{v^2}{c^2}}$$

$$x_2 - x_1 = \frac{L}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma . L = P$$

q.e.d.

Although the doors of the barn act on the same moments $T_1^{'} = T_2^{'}$ in its own system it shows that the doors of the barn do not act on the same moments in the system of the pole. Thus when one door of the barn hits one side of the pole, the barn continues to move such that when the second door of the barn hits the other side of the pole, this door acts. So the time difference between the actions of the two doors is exactly the time that the second side of the barn moves to the second side of the pole.