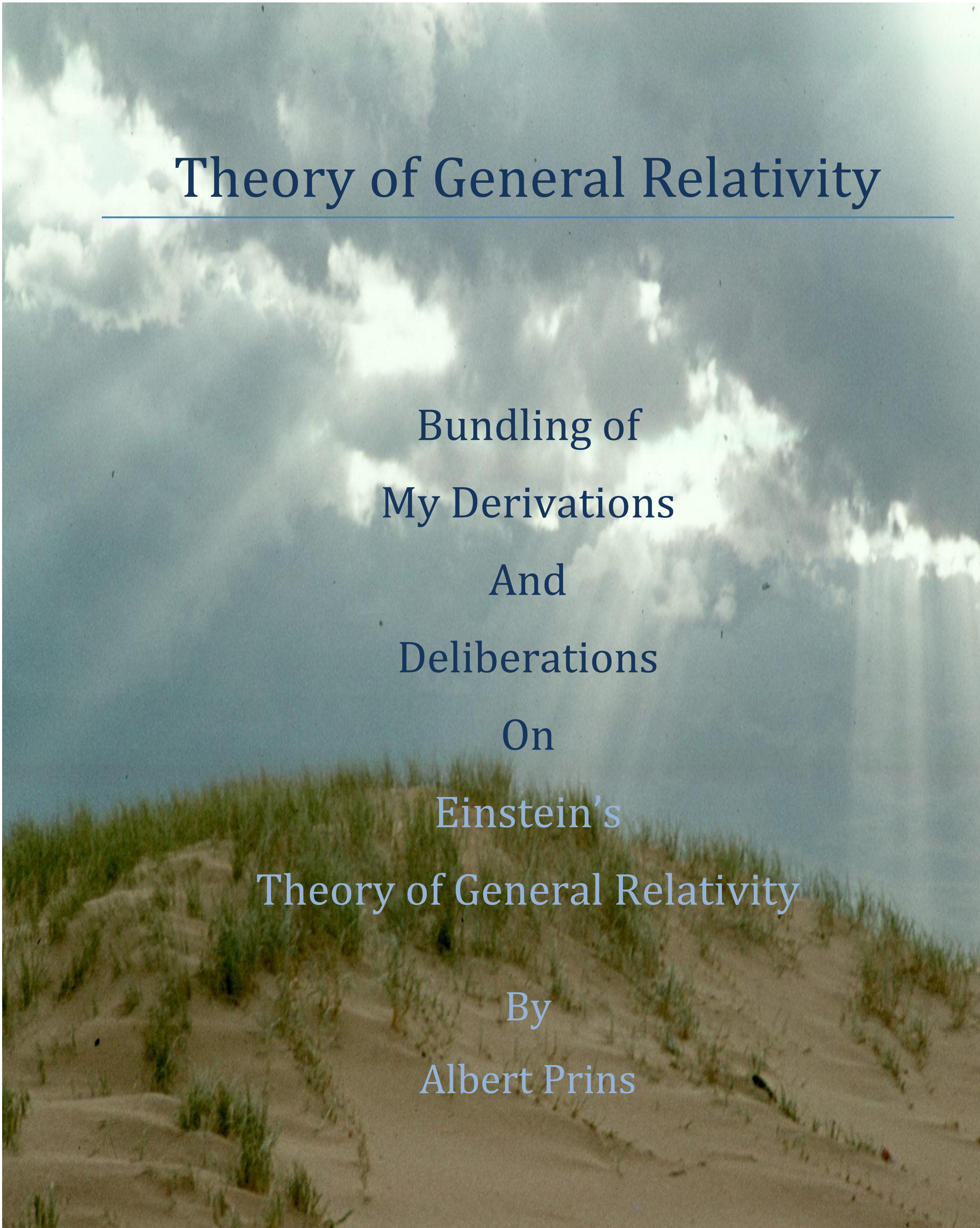


Theory of General Relativity

Bundling of

My Derivations

And

Deliberations

On

Einstein's

Theory of General Relativity

By

Albert Prins

Preface

As we received many questions on Albert Einstein's Theory of General Relativity, we produced as many answers. In order to avoid a repetition of the same responses we decided to construct an overview of the theory of General Relativity.

So this document gives a concise description of the derivation of the Einstein's Field Equations. It describes also a number of derivations and experiments corroborating the theory of General Relativity like the trajectory of Mercury around the Sun, deflection of light brushing the Sun, Shapiro's experiment and an ordinary bullet trajectory calculated with the Schwarzschild equation.

This document can be found at:

https://issuu.com/home/published/generalrelativity_albertprins_part

and

http://www.prinikx.synology.me/familyprins/Astronomy/GR/GeneralRelativity_AlbertPrins.pdf

Albert Prins

Your feedback is welcome at aprins@hotmail.com

Table of Contents

Bundling of

My Derivations

And

Deliberations

On

Einstein's

Theory of General Relativity

By

Albert Prins

1. Introduction

The aim of this paper is to give an overview of the General Relativity (GR) Theory of Einstein; with the main purpose to derive the mathematics for a number of experiments that supports this theory.

Making calculations based on this theory of General Relativity is quite cumbersome because Einstein strived to make his theory as general as possible i.e. all possible types of coordinate systems could be used. This independency of the applied coordinate system is called the principle of covariance.
Fortunately, in the year 1915 the same year as Einstein invented his GR theory, Karl Schwarzschild derived a solution based on this theory but more concise and mainly based on polar coordinates (elucidated below). With the Schwarzschild equation most of the experiments can be explained and calculated. Also, in this document, it is shown that the Schwarzschild equation meets the Einstein GR rules.
A number of chapters is added, which contains answers on frequently asked GR questions. Bundling these answers can help by understanding the GR theory.

2. Concise description of the General Relativity Theory

Before Einstein came in 1915 to his famous theory, he developed in 1905 his theory of Special Relativity (see Appendix 7). In this theory of Special Relativity, Einstein only considered coordinate frames that moved uniformly, thus with constant speed with respect to each other; the effect of masses, and thus gravitation, was not taken into account. The premises where the Special Relativity is based on are:

- The maximum possible speed, in each coordinate frame, is the velocity of light c=299792458 m/s.
- The laws of physics are valid in each uniformly (not accelerating) moving coordinate frame.

In Newton's approach the increments of time were equal in the "rest" frame and the moving frame. However via the Special Relativity Theory it was revealed that the **increments of time** in a moving frame were different and smaller than in a frame at "rest", i.e. the rate of the ticks of a clock, and everything else, in a moving frame slows down with respect to those in a frame at rest. Furthermore the **length of an object** is influenced by its velocity and decreases, with respect to the "rest" frame, in the moving direction.
These were both consequences of the observation that the velocity of light in vacuum was always the same in each frame independent of its speed.

One of the results of the theory is the well known $E = mc^2$, the relation between energy and mass (see Appendix 7.6).

In his next project Einstein philosophized about accelerated frames and the effect of masses, leading to the Theory of General Relativity in 1915.

To get a sneak preview of the final formula of the field equations, derived by Einstein, we refer to chapter 2.15, where a summary is given. The chapters below will explain the details necessary to achieve and understand the final results.

2.1. The principle of Equivalence

By studying the effect of masses, Newton came to the formulation of gravitational forces and how masses accelerate due to these forces.
When we compare the effect of the gravitational force with, for instance, the electric and magnetic force, we see great similarities but also distinct differences.
We will look at how the force is formed and what kind of acceleration it causes:

For an electric force between two electrically charged particles goes (the Coulomb's law):

$$F = k_e \frac{q_1 q_2}{r^2}$$

Here q_1 and q_2 are the charges of two particles that attract or repel each other, depending on the difference of polarity of the charges. The distance between the two particles is *r* and k_e is a constant. Due to this force the particles will undergo an acceleration which could be repelling or attracting; again depending on the difference of polarity between the charges of the particles. The magnitude of this acceleration does not only depend on the charges but also on the masses of the particles.

So there is an attraction force due to the charges but the acceleration is determined by both the size of the masses and the attraction force.

For instance the acceleration of a particle with charge q_1 with mass $m_1$is depicted by:

$$F = m_1 a_1 = k_e \frac{q_1 q_2}{r^2} => \; a_1 = k_e \frac{q_1 q_2}{m_1 r^2}$$

Similarly for magnetic forces it applies that they cause acceleration, which is depending on the positive or negative sign of particles and the polarization of the magnetic field ***and*** the masses of the particles.

When we consider now the gravitational force between two objects with mass $m_1$and mass m_2 respectively, then we get the Newton law:

$$F = G \frac{m_1 m_2}{r^2}$$

Where *G* is the gravitational constant and *r* is the distance between the two masses.

However if we compare the gravitational forces with the electric and magnetic forces, then we would expect a gravitational part of the mass i.e. m_{grav} that leads to the attraction force and a part of the mass m_{inert} that undergoes the acceleration force, with result:

$$F = m_{inert\,1} a_1 = G \frac{m_{grav\,1} m_{grav\,2}}{r^2} => a_1 = G \frac{m_{grav\,1} m_{grav\,2}}{m_{inert\,1} r^2}$$

There seems to be no reason why $m_{inert\,1} \equiv m_{grav\,1,}$ however after many experiments by various researchers, around 1885 by Eötvös and later by others, it was found that they are always equal.

Another difference between the gravitational force and the electric and magnetic force is that there is not a positive and a negative gravitational force; the force between two masses is always attracting.

So this equality $m_{inert\,1} \equiv m_{grav\,1}$ leads to:

$$F = ma = G\frac{mM}{r^2}$$

Resulting in acceleration on the particle with mass m:

$$a = G\frac{M}{r^2}$$

As the mass *m* now disappeared from the equation, the acceleration on mass *m* is independent of the size of m and is completely determined by the other mass *M.* So in case of the Earth (M), ignoring the effect of the air, everything falls to the Earth with exactly the same acceleration(a), only determined by mass M of the Earth. (See also Note and the end of this chapter)

Inspired by the phenomena mentioned above Einstein followed a somewhat other approach. He compared a person standing still on Earth, experiencing a gravitational acceleration of $g\frac{m}{sec^2}$, and a person in a rocket, far from any force of gravity, which happens to accelerate with exactly the same acceleration $g\frac{m}{sec^2}$. In such a case the person cannot distinguish whether he, or she, is attracted by Earth, i.e. gravitation, or by the acceleration caused by the rocket motor, i.e. inertia (apart from tidal forces that is. See Appendix 6).
Einstein decided therefore that such a distinction should not be made and concluded that there is no gravitational force, but that the geography of space-time is locally curved due to the presence of mass. This is the *Einstein Equivalence Principle,* thus the, local, equivalence of gravitation and inertia.
If there are no (gravitational or whatever) forces, a moving particle follows a straight line, which was already known by Galileo and Newton. But also when space-time is curved, due to masses, the particle follows a "straight line" in space-time, called a geodesic, although this latter path is curved in accordance with the size of the neighboring mass. So if one falls freely from a height to the Earth, following the curved line, one experiences no force but feels him- or herself floating (until one hits the Earth ☹).

Note:
Of course there is also an opposite acceleration on M

$$F = Ma = G\frac{mM}{r^2} \text{ resulting in } a = G\frac{m}{r^2}$$

But assuming that in this case M>>m the acceleration on M is very small, although the forces on both particles are equal but opposite.

Based on the approach of Newton only particles with a mass exert a force on each other; which implies that a mass has no influence on a mass-less particle like a photon.
However based on the General Relativity Theory the space-time is curved due to masses, which implies that everything follows the curvature of space-time even when it is mass-less in case of a photon. This phenomenon was demonstrated in 1919 by Arthur Eddington during a solar eclipse. He showed that the positions of the stars, which were visible during the solar eclipse, seemed to be shifted exactly according to the prediction of Einstein.

We will mathematically show the bending of light according to Einstein's theory in **Experiment 3 - Deflection of Light**.

2.2. Curvature of Space-Time

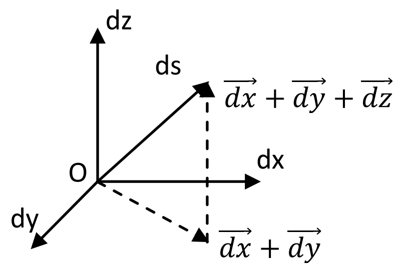
To grasp the importance of the paradigm shift from the Newtonian gravitational force approach to the geographical approach we start by explaining it in a slightly different way.

When a particle is in free space, far from any mass and any force, the particle continues moving with its initial velocity and direction. This effect was already known by Galileo Galilei around the year 1600.
If we envisage space-time as build up out of rectangular grid lines the trajectory of this particle moves along a straight line. Einstein's premise was that in case there is a big mass the geography of space-time becomes curved; in this way ignoring any gravitational force. So instead of rectangular grid lines the grid lines become bent. This curvature is depending on the size of the big mass. In case the particle comes in the vicinity of the mass it keeps on following the line of the curvature. So now the particle follows its curved "straight" line. Later on we will find that this line is called a geodesic.
So instead of a gravitational force the effect on the particle, in the theory of General Relativity, is described by the curvature of space-time.

So Einstein's challenge was to describe the geography of space-time as a function of mass and more general, mass and energy. He also would hope to find this function being independent of the chosen coordinate frame. In the following chapters we will follow the line of thoughts taken by Einstein and derive the Einstein Field Equations describing the curvature of space-time.

2.2.1. Independency of chosen frame

To consider a point in space we are interested in its location and its movement, but to establish its location we need to determine a point of reference and the distance of our point to that point of reference. A common approach is to choose a Cartesian coordinate system or frame with three axes, normally called the x-ax, y-ax and z-ax, which are orthogonal to each other.
We can describe the location of that point in space with for instance *(x, y, z),* in which *x, y,* and *z* are distances along their respective coordinates to the origin (*O*) of the frame. Here the total distance to the origin is $s = \sqrt{x^2 + y^2 + z^2}$ according to the well known Pythagoras equation. When another frame is chosen and thus another origin, the indications of the location will change accordingly and so will the distance *s*. But if our presumption is not a point in space but a (small) distance between two points in space, then this distance is always the same, independent of whatever frame we choose. We denote this distance as $ds = \sqrt{dx^2 + dy^2 + dz^2}$. So the distance can be described by the Pythagoras theorem for a frame with rectangular axes. To be more specific *ds*, *dx*, *dy* and *dz* could be considered being vectors, because they have all a direction and size, and not necessarily orthogonal. So:



$$\overrightarrow{ds}=\overrightarrow{dx}+\overrightarrow{dy}+\overrightarrow{dz}$$

To find the size of *ds* the usual approach is to calculate the dot product $\overrightarrow{ds}.\overrightarrow{ds}$ thus

$$ds^2=\overrightarrow{ds}.\overrightarrow{ds}\ =(\overrightarrow{dx}+\overrightarrow{dy}+\overrightarrow{dz}).(\overrightarrow{dx}+\overrightarrow{dy}+\overrightarrow{dz})$$

For you recollection: the dot product of two vectors is:

$$\overrightarrow{dA}.\overrightarrow{dB}=AB\cos(\varphi)\,where\ \varphi\ is\ the\ angle\ between\ the\ two\ vectors.$$

Giving the general form:

$$ds^2=\overrightarrow{\boldsymbol{dx}}.\overrightarrow{\boldsymbol{dx}}+\overrightarrow{dx}.\overrightarrow{dy}+\overrightarrow{dx}.\overrightarrow{dz}+\overrightarrow{dy}.\overrightarrow{dx}+\overrightarrow{\boldsymbol{dy}}.\overrightarrow{\boldsymbol{dy}}+\overrightarrow{dy}.\overrightarrow{dz}+\overrightarrow{dz}.\overrightarrow{dx}+\overrightarrow{dz}.\overrightarrow{dy}+\overrightarrow{\boldsymbol{dz}}.\overrightarrow{\boldsymbol{dz}}$$

When *dx*, *dy* and *dz* are orthogonal then the dot products of different axes are zero, so the result is

$$ds^2=\overrightarrow{dx}.\overrightarrow{dx}+\overrightarrow{dy}.\overrightarrow{dy}++\overrightarrow{dz}.\overrightarrow{dz}=\cos(\alpha)\,dx^2+\cos(\beta)\,dy^2+\cos(\gamma)\,dz^2$$

As in this orthogonal case $\alpha=\beta=\gamma=0$

$$ds^2=dx^2+dy^2+dz^2$$

In case we allow for non orthogonal axes then also the dot products of different axes are non zero, resulting in the general form mentioned above. Assuming that each dot product produces a coefficient, we get to a more general form:

$$ds^2=Adx^2+Bdxdy+Cdxdz+Ddydx+Edy^2+Fdydz+Gdzdx+Hdzdy+Idz^2 \tag{1}$$

As Einstein aimed for an even more general formula for a coordinate system with one axis for time and three axes for space, where the axes were not necessarily orthogonal, he came to the following formula:

$$ds^2=\sum_{\mu}\sum_{\nu}g_{\mu\nu}\,dx^{\mu}\,dx^{\nu}$$

Or in **Einstein notation**

$$ds^2=g_{\mu\nu}\,dx^{\mu}\,dx^{\nu} \tag{2}$$

We need to give a little explanation. Here μ and ν are just indices and not exponents; μ and ν are, for practical reasons, each denoted as *0, 1, 2 or 3* instead of *t, x, y, z.* So a more common notation is x^{μ} where μ could be 0, 1, 2, 3 and thus $ct=x^0, x=x^1, y=x^2, z=x^3$ and the same holds for x^{ν}. Einstein also used his, so called, Einstein notation where a summation is done over the same (repeated) indices, also known as dummy indices, if they appear as a lower and higher index at the same side of the equation.

So formula (2) works out to be

$$\begin{aligned} ds^2 = {} & g_{00}dx^0dx^0 + g_{01}dx^0dx^1 + g_{02}dx^0dx^2 + dg_{03}dx^0dx^3 + \\ & g_{10}dx^1dx^0 + g_{11}dx^1dx^1 + g_{12}dx^1dx^2 + dg_{13}dx^1dx^3 + \\ & g_{20}dx^2dx^0 + g_{21}dx^2dx^1 + g_{22}dx^2dx^2 + dg_{23}dx^2dx^3 + \\ & g_{30}dx^3dx^0 + g_{31}dx^3dx^1 + g_{32}dx^3dx^2 + dg_{33}dx^3dx^3 \end{aligned} \tag{3}$$

This is comparable with equation (1) but now for a four, instead of three, dimensional coordinate system. (For more detailed information see chapter 4)

Note:
So ds^2contains 16 elements, but in case of symmetry the elements $g_{\mu\nu}\,dx^\mu\,dx^\nu$and $g_{\nu\mu}\,dx^\nu\,dx^\mu$ are equal, thus instead of 16 unknown elements the uncertainty is now limited to 10 elements.

2.3. Contra-variant and covariant vectors and dual-vectors

In the theory of General Relativity the terms contra-variant and covariant are often used, so here below we give an elucidation of their significance and usage.
As we have mentioned before, the main properties of vectors and fields should stay the same in the theory of General Relativity regardless of what coordinate frame is used. If it is for some reason more convenient to use a certain coordinate frame instead of the current one is using, we will study the impact on the properties of the vectors and fields due to this transformation.
The main items we consider are the *scalars*, *vectors* and the *fields*.

Scalars, like temperature, can be different in magnitude at each location but they do not have a direction. But the values per location can have a tendency of increasing or decreasing in a certain direction, forming a *field*. By taking the derivative of a scalar-field, then this derivative results in a kind of vector, a so-called dual-vector. This differentiation is related to the used coordinate system. In case a transformation to another frame is done, the dual-vector stays the same but the components (coefficients of each coordinate) of the dual-vector change accordingly; the dual-vectors are so-called **covariant**.

As for *vectors* is concerned, if for instance the current coordinate frame rotates, or translates, to another frame, then the components of the vector change in opposite direction, because the frame changes but the vector stays the same, so they are called **contra-variant**.

A covariant vector is denoted, by convention, with a lower index (A_μ) while the contra-variant has an upper index (A^μ). By definition $A_\mu A^\mu = I$.

Now we consider the rules of transformation from one coordinate system to another coordinate system.

Assume the coordinates of the current frame are x^0, x^1, x^2, x^3, and the transformation is to a frame with coordinates: y^0, y^1, y^2, y^3. Then there is the following relation between both coordinate frames y^n *and* x^m :

$$y^n = \frac{\partial y^n}{\partial x^0} x^0 + \frac{\partial y^n}{\partial x^1} x^1 + \frac{\partial y^n}{\partial x^2} x^2 + \frac{\partial y^n}{\partial x^3} x^3$$

In Einstein notation:

$$y^n = \frac{\partial y^n}{\partial x^m} x^m$$

(Here *n* and *m* can be 0, 1, 2 or 3)

Let φ be a scalar field then $d\varphi = \frac{\partial\varphi}{\partial x^m} dx^m$ or, according to the Einstein notation, completely:

$$d\varphi = \frac{\partial\varphi}{\partial x^0} dx^0 + \frac{\partial\varphi}{\partial x^1} dx^1 + \frac{\partial\varphi}{\partial x^2} dx^2 + \frac{\partial\varphi}{\partial x^3} dx^3$$

The dual-vector in the new frame is, with partial derivatives (chain rule):

$$\frac{d\varphi}{dy^n} = \frac{\partial\varphi}{\partial x^m} \frac{dx^m}{dy^n}$$

$$=> A_n^{(y)} = \frac{dx^m}{dy^n} B_m^{(x)} \tag{1}$$

(Here is $A_n^{(y)}$ the (covariant) vector $A_n = \frac{d\varphi}{dy^n}$ in the y frame and the (covariant) vector $B_m = \frac{\partial\varphi}{\partial x^m}$ in the x frame).

To write equation (1) out completely:

$$\begin{pmatrix} A_0 \\ A_1 \\ A_2 \\ A_3 \end{pmatrix}_y = \begin{pmatrix} \frac{dx^0}{dy^0} & \frac{dx^1}{dy^0} & \frac{dx^2}{dy^0} & \frac{dx^3}{dy^0} \\ \frac{dx^0}{dy^1} & \frac{dx^1}{dy^1} & \frac{dx^2}{dy^1} & \frac{dx^3}{dy^1} \\ \frac{dx^0}{dy^2} & \frac{dx^1}{dy^2} & \frac{dx^2}{dy^2} & \frac{dx^3}{dy^2} \\ \frac{dx^0}{dy^3} & \frac{dx^1}{dy^3} & \frac{dx^2}{dy^3} & \frac{dx^3}{dy^3} \end{pmatrix} \begin{pmatrix} B_0 \\ B_1 \\ B_2 \\ B_3 \end{pmatrix}_x$$

Here is $\frac{dx^m}{dy^n}$ the transformation matrix and $A_n^{(y)}$ is the resulting dual-vector after transformation and $B_m^{(x)}$ is the original dual-vector. Thus $A_n^{(y)} = \frac{dx^m}{dy^n} B_m^{(x)}$ is called the covariant transformation.

The process for a contra-variant vector is defined as a ***contra-variant transformation*** and is follows:

$$W_{(y)}^n = \frac{dy^n}{dx^m} B_{(x)}^m$$

Written out:

$$\begin{pmatrix} W^0 \\ W^1 \\ W^2 \\ W^3 \end{pmatrix}_y = \begin{pmatrix} \frac{dy^0}{dx^0} & \frac{dy^0}{dx^1} & \frac{dy^0}{dx^2} & \frac{dy^0}{dx^3} \\ \frac{dy^1}{dx^0} & \frac{dy^1}{dx^1} & \frac{dy^1}{dx^2} & \frac{dy^1}{dx^3} \\ \frac{dy^2}{dx^0} & \frac{dy^2}{dx^1} & \frac{dy^2}{dx^2} & \frac{dy^2}{dx^3} \\ \frac{dy^3}{dx^0} & \frac{dy^3}{dx^1} & \frac{dy^3}{dx^2} & \frac{dy^3}{dx^3} \end{pmatrix} \begin{pmatrix} B^0 \\ B^1 \\ B^2 \\ B^3 \end{pmatrix}_x$$

Thus the difference, between covariant and contra-variant vectors, is how the transformations are performed.

Now similar for a contra-variant **tensor** (product of two vectors):

$$T^{mn}_{(y)} = A^m_{(y)} B^n_{(y)} = \frac{dy^m}{dx^r} A^r_{(x)} \frac{dy^n}{dx^s} B^s_{(x)} = \frac{dy^m}{dx^r} \frac{dy^n}{dx^s} A^r_{(x)} B^s_{(x)} = \frac{dy^m}{dx^r} \frac{dy^n}{dx^s} T^{rs}_{(x)}$$

Thus **contra-variant transformation** of a contra-variant tensor is:

$$T^{mn}_{(y)} = \frac{dy^m}{dx^r} \frac{dy^n}{dx^s} T^{rs}_{(x)}$$

For a covariant tensor is this:

$$T^{(y)}_{mn} = A^{(y)}_m B^{(y)}_n = \frac{dx^r}{dy^m} A^{(x)}_r \frac{dx^s}{dy^n} B^{(x)}_s = \frac{dx^r}{dy^m} \frac{dx^s}{dy^n} A^{(x)}_r B^{(x)}_s = \frac{dx^r}{dy^m} \frac{dx^s}{dy^n} T^{(x)}_{rs}$$

Thus **covariant transformation** of a covariant tensor:

$$T^{(y)}_{mn} = \frac{dx^r}{dy^m} \frac{dx^s}{dy^n} T^{(x)}_{rs}$$

There is also a mixed form; a tensor as product of a contra-variant and a covariant vector:

$$T^m_n(y) = A^m_{(y)} B^{(y)}_n = \frac{dy^m}{dx^r} A^r_{(x)} \frac{dx^s}{dy^n} B^{(x)}_s = \frac{dy^m}{dx^r} \frac{dx^s}{dy^n} A^r_{(x)} B^{(x)}_s = \frac{dy^m}{dx^r} \frac{dx^s}{dy^n} T^r_s(x)$$

Thus the **mixed transformation** for a tensor is:

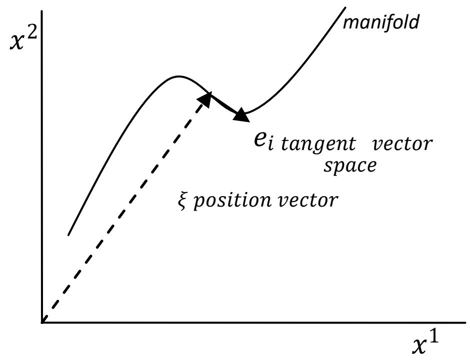
$$T^m_n(y) = \frac{dy^m}{dx^r} \frac{dx^s}{dy^n} T^r_s(x)$$

Note:
The tensor T^{mn} is called a contra-variant tensor with rank-2 while V^m is a contra-variant vector but could be considered as a contra-variant tensor with rank-1. A scalar is a tensor with rank-0.
And vice-versa for the covariant situation.

2.4. Derivation of Christoffel symbol and Covariant Derivative

As Einstein wanted to describe the effect of gravitation in terms of geography of space-time, or in other words curvature of space-time, he needed a function that would describe the extent and manner of the curvature at each location in space-time. This function is the Christoffel Symbol which also helps to define the *covariant derivative*, which will be discussed later. Here we will derive the Christoffel symbol and derive the various forms.

2.4.1. Christoffel symbol



We start from a coordinate system x^i with ξ (ksi) as the position vector and introduce the Christoffel symbol Γ_{ij}^k . The tangent vectors are

$$\vec{e}_i = \frac{\partial \vec{\xi}}{\partial x^i} => \frac{\partial \vec{e}_i}{\partial x^j} = \frac{\partial^2 \vec{\xi}}{\partial x^i \partial x^j} = \Gamma_{ij}^k \vec{e}_k \qquad (1)$$

Written out:

$$\frac{\partial \vec{e}_i}{\partial x^j} = \Gamma_{ij}^k \vec{e}_k = \Gamma_{ij}^0 \vec{e}_0 + \Gamma_{ij}^1 \vec{e}_1 + \Gamma_{ij}^2 \vec{e}_2 + \Gamma_{ij}^3 \vec{e}_3$$

(For reasons of convenience we ignore the vector sign (arrow) from here on.)

From the definition of covariant and contra-variant vectors:

$$e^k e_k = 1 \qquad (2)$$

So multiplying both sides of (1) with e^k we get:

$$\boldsymbol{\Gamma_{ij}^k = e^k \frac{\partial e_i}{\partial x^j}} \qquad \boldsymbol{is\ the\ definition\ of\ the\ Christoffel\ symbol} \qquad (3)$$

Because of symmetry in equation (1) the lower indices of the Christoffel symbol can be swapped:

$$\frac{\partial e_i}{\partial x^j} = \frac{\partial e_j}{\partial x^i} => e^k \frac{\partial e_i}{\partial x^j} = e^k \frac{\partial e_j}{\partial x^i} => \Gamma_{ij}^k = \Gamma_{ji}^k \qquad (4)$$

$$e_k = \frac{\partial \xi}{\partial x^k} => e^k = \frac{1}{e_k} = \frac{\partial x^k}{\partial \xi} \qquad (5)$$

From (1) and (5) we find:

$$e^k \frac{\partial^2 \xi}{\partial x^i \partial x^j} = \Gamma_{ij}^k => \boldsymbol{\Gamma_{ij}^k = \frac{\partial x^k}{\partial \xi} \frac{\partial^2 \xi}{\partial x^i \partial x^j}} \qquad (6)$$

Now we introduce a new term the metric tensor.
The **metric tensor** can be defined as the dot product of the base vectors e_i and e_k:

$$g_{ik} = e_i.e_k = \left(g^{ik}\right)^{-1} \tag{7}$$

$$e_i.\frac{1}{e^k} = \left(g^{ik}\right)^{-1}$$

$$e^k = g^{ik}e_i \tag{8}$$

Note:

The vectors e_i do not reside in the manifold but in the tangent space touching the manifold at one point.
The vector e_i is the tangent vector, so the derivative of the position vector or the derivative of the trajectory. In case the trajectory is a straight line then the derivative of the position vector is a constant; and consequently the derivative of e_i is zero.

2.4.2. Covariant derivative

The covariant derivative is similar to a normal derivative in a Euclidian frame, but now extended to specify a derivative along tangent vectors in curved space-time.

As Einstein required covariance, that is that the results are independent of the chosen coordinate frame, then this leads to the consequence that if the derivative of a tensor is zero in one frame then the derivative of that tensor shall be zero in any frame. To meet this requirement the covariant derivative is defined, where the normal derivative is corrected in such a way that the **covariant derivative ∇** meets this requirement.

For each metric (coordinate frame) there is a unique torsion-free *covariant derivative*, depicted by ∇, called the *Levi-Civita connection* such that the covariant derivative of the metric is zero. If the $\nabla g_{mn} = 0$ in flat space then it shall be zero in any space (see note at the end of this chapter).

Now let us derive the covariant derivative.
The general form of the metric tensor (7) is:

$$g_{mn} = e_m.e_n \tag{9}$$

$$\frac{\partial g_{mn}}{\partial \mathrm{x}^{\mathrm{s}}} = \frac{\partial(e_m.e_n)}{\partial \mathrm{x}^{\mathrm{s}}} = e_m\frac{\partial e_n}{\partial \mathrm{x}^{\mathrm{s}}} + e_n\frac{\partial e_m}{\partial \mathrm{x}^{\mathrm{s}}} \tag{10}$$

Because of symmetry as mentioned above (see equation 4):

$$\frac{\partial g_{mn}}{\partial \mathrm{x}^{\mathrm{s}}} = e_m\frac{\partial e_n}{\partial \mathrm{x}^{\mathrm{s}}} + e_n\frac{\partial e_m}{\partial \mathrm{x}^{\mathrm{s}}} = e_m\frac{\partial e_s}{\partial \mathrm{x}^{\mathrm{n}}} + e_n\frac{\partial e_s}{\partial \mathrm{x}^{\mathrm{m}}} \tag{11}$$

Thus

$$\frac{\partial g_{mn}}{\partial \mathrm{x}^{\mathrm{s}}} - e_m\frac{\partial e_s}{\partial \mathrm{x}^{\mathrm{n}}} - e_n\frac{\partial e_s}{\partial \mathrm{x}^{\mathrm{m}}} = 0 \tag{12}$$

We define the covariant derivative as follows:

$$\nabla_{\mathrm{s}} g_{mn} = \frac{\partial g_{mn}}{\partial \mathrm{x^s}} - e_m \frac{\partial e_s}{\partial \mathrm{x^n}} - e_n \frac{\partial e_s}{\partial \mathrm{x^m}} = 0 \tag{13}$$

Now we will express the covariant derivative in Christoffel symbols:

$$\nabla_{\mathrm{s}} g_{mn} = \frac{\partial g_{mn}}{\partial \mathrm{x^s}} - e_m \frac{\boldsymbol{\partial e_s}}{\boldsymbol{\partial \mathrm{x^n}}} \boldsymbol{e^t} e_t - e_n \frac{\boldsymbol{\partial e_s}}{\boldsymbol{\partial \mathrm{x^m}}} \boldsymbol{e^t} e_t = 0 \tag{14}$$

As seen in the previous chapter:

$$\Gamma^t_{sn} = e^t \frac{\partial e_s}{\partial x^n} \quad and \quad g_{mt} = e_m \cdot e_t$$

So here we get the **covariant derivative** of the metric tensor, expressed in the normal derivative corrected with two terms which are products of the metric tensor and the appropriate Christoffel symbol:

$$\boldsymbol{\nabla_s g_{mn} = \frac{\partial g_{mn}}{\partial \mathrm{x^s}} - g_{mt}\Gamma^t_{sn} - g_{nt}\Gamma^t_{sm} = 0} \tag{15}$$

Thus in the same way by cyclic permutation:

$$\nabla_{\mathrm{m}} g_{ns} = \frac{\partial g_{ns}}{\partial \mathrm{x^m}} - g_{nt}\Gamma^t_{ms} - g_{st}\Gamma^t_{mn} = 0 \tag{16}$$

$$\nabla_{\mathrm{n}} g_{sm} = \frac{\partial g_{sm}}{\partial \mathrm{x^n}} - g_{st}\Gamma^t_{nm} - g_{mt}\Gamma^t_{ns} = 0 \tag{17}$$

Now we do the following operation (17)+(16)-(15) keeping in mind, from (4), that $\Gamma^k_{ij} = \Gamma^k_{ji}$, resulting in:

$$\frac{\partial g_{sm}}{\partial \mathrm{x^n}} + \frac{\partial g_{ns}}{\partial \mathrm{x^m}} - \frac{\partial g_{mn}}{\partial \mathrm{x^s}} - 2 g_{st}\Gamma^t_{nm} = 0 \tag{18}$$

$$g_{st}\Gamma^t_{nm} = \frac{1}{2}\left(\frac{\partial g_{sm}}{\partial \mathrm{x^n}} + \frac{\partial g_{ns}}{\partial \mathrm{x^m}} - \frac{\partial g_{mn}}{\partial \mathrm{x^s}}\right) \tag{19}$$

This results in the **Christoffel symbol**, completely determined by the metric tensor and its normal derivatives:

$$\boldsymbol{\Gamma^t_{nm} = \frac{1}{2} g^{st}\left(\frac{\partial g_{sm}}{\partial \mathrm{x^n}} + \frac{\partial g_{ns}}{\partial \mathrm{x^m}} - \frac{\partial g_{mn}}{\partial \mathrm{x^s}}\right)} \tag{20}$$

Note:

To support the above statement that the covariant derivative of the metric tensor is zero, we give an extra elucidation.

According to the definition of a covariant derivative then $\nabla \mathrm{A}_\mu$ is a vector. So the following transformation operation is valid (see 8):

$$\nabla \mathrm{A}_\mu = \mathrm{g}_{\mu\nu} \nabla \mathrm{A}^\nu \tag{20a}$$

But also the covariant derivative of the vector can be taken, with the following result:
As we know:

$$\mathrm{A}_\mu = \mathrm{g}_{\mu\nu} \mathrm{A}^\nu$$

Thus:

$$\nabla \mathrm{A}_\mu = \nabla(\mathrm{g}_{\mu\nu} \mathrm{A}^\nu) = \mathrm{g}_{\mu\nu} \nabla \mathrm{A}^\nu + \mathrm{A}^\nu \nabla \mathrm{g}_{\mu\nu} \tag{20b}$$

So from the equations (20a) and (20b):

$$g_{\mu\nu}\nabla A^{\nu} = g_{\mu\nu}\nabla A^{\nu} + A^{\nu}\nabla g_{\mu\nu} \;\; => \;\; \nabla g_{\mu\nu} = 0$$

Thus it can be deduced that the covariant derivative of the metric tensor leads to $\boldsymbol{\nabla g_{\mu\nu} = 0}$. This is a consequence of the definition of covariant derivative and metric tensor.

2.4.2.1. *Covariant differentiation for a contra-variant vector*

Now the covariant derivative for a contra-variant vector field V^m:

$$\vec{V} = V^m \vec{e}_m \tag{21}$$

$$\frac{\partial \vec{V}}{\partial x^l} = \frac{\partial V^m}{\partial x^l}\vec{e}_m + V^m \frac{\partial \vec{e}_m}{\partial x^l} \tag{22}$$

As seen from above (1)

$$\Gamma^k_{ml}\,\vec{e}_k = \frac{\partial \vec{e}_m}{\partial x^l} \tag{23}$$

Thus

$$\frac{\partial \vec{V}}{\partial x^l} = \frac{\partial V^m}{\partial x^l}\vec{e}_m + V^m \Gamma^k_{ml}\,\vec{e}_k \tag{24}$$

The right hand side has two dummy indices, *k* and *m*. (*When in the formula on the right hand side there is a product of an element with an upper index together with an element with the same lower index we call this a dummy index; then according to the Einstein notation a summation over this index shall be done*. *In that case the name of the index is not important.*)
So the formula can be changed by *k* to m, and *m* to γ (see note below).

$$\frac{\partial \vec{V}}{\partial x^l} = \frac{\partial V^m}{\partial x^l}\vec{e}_m + V^{\gamma}\Gamma^m_{\gamma l}\,\vec{e}_m = \left(\frac{\partial V^m}{\partial x^l} + V^{\gamma}\Gamma^m_{\gamma l}\right)\vec{e}_m \tag{25}$$

The covariant derivative of a vector field V^m (**contra-variant vector**) is

$$\boldsymbol{\nabla_l V^m = \frac{\partial V^m}{\partial x^l} + V^{\gamma}\Gamma^m_{\gamma l}} \tag{26}$$

So the covariant derivative is the usual derivative along the coordinates with a correction term that contains the information about the change in the coordinates. The covariant derivative $\nabla_l V^m$ transforms like a tensor and is independent of its coordinate frame.

Note:
If an equation has an element with dummy indices, then these indices can be renamed to any name which is preferable.
For instance $V^{\mu}A_{\mu}$ is in Einstein notation and actually stands for:

$$V^{\mu}A_{\mu} = V^0 A_0 + V^1 A_1 + V^2 A_2 + V^3 A_3$$

Thus whatever dummy index name was chosen it would always lead to the same result!

2.4.2.2. *Covariant differentiation for a covariant vector*

Now let's take the scalar product $A^\mu B_\mu$ of two arbitrary vectors, one contra-variant *A* and the other covariant *B.* Applying the derivation rules we get:

$$\nabla_\alpha\left(A^\mu B_\mu\right) = (\nabla_\alpha A^\mu)B_\mu + A^\mu\left(\nabla_\alpha B_\mu\right) \tag{27}$$

$$\nabla_\alpha\left(A^\mu B_\mu\right) = \left(\frac{\partial A^\mu}{\partial x^\alpha} + \Gamma^\mu_{\alpha\nu} A^\nu\right) B_\mu + A^\mu\left(\nabla_\alpha B_\mu\right) \tag{28}$$

As the value of a scalar in a point in space-time does not depend on the basis vectors, the covariant derivative of a scalar equals to its ordinary derivative:

$$\nabla_\alpha\left(A^\mu B_\mu\right) = \frac{\partial\left(A^\mu B_\mu\right)}{\partial x^\alpha} = \frac{\partial A^\mu}{\partial x^\alpha} B_\mu + A^\mu \frac{\partial B_\mu}{\partial x^\alpha} \tag{29}$$

By renaming some of the dummy indices these last two equations, (29) and (28), become:

$$\cancel{\frac{\partial A^\mu}{\partial x^\alpha}} B_\mu + A^\mu \frac{\partial B_\mu}{\partial x^\alpha} = \left(\cancel{\frac{\partial A^\mu}{\partial x^\alpha}} + \Gamma^\mu_{\alpha\nu} A^\nu\right) B_\mu + A^\mu\left(\nabla_\alpha B_\mu\right) \tag{30}$$

Interchanging the dummy indices in the second right-hand term by μ to σ, and ν to μ:

$$=> \; A^\mu \left[-\frac{\partial B_\mu}{\partial x^\alpha} + \Gamma^\sigma_{\alpha\mu} B_\sigma + \left(\nabla_\alpha B_\mu\right)\right] = 0 \tag{31}$$

$$\boldsymbol{\nabla_\alpha B_\mu = \frac{\partial B_\mu}{\partial x^\alpha} - B_\sigma \Gamma^\sigma_{\alpha\mu}} \tag{32}$$

This is the covariant derivative of a **covariant vector**.

2.4.3. **Relation with tensor**

Consider the transformation of tensor $\mathrm{T_{mn}}$ from the *x* frame to the *y* frame.

$$\mathrm{T_{mn}}(\mathrm{x}) = \frac{\partial \mathrm{V_m}(\mathrm{x})}{\partial \mathrm{x^n}} \tag{33}$$

$$\mathrm{T_{mn}}(\mathrm{y}) = \frac{\partial \mathrm{V_m}(\mathrm{y})}{\partial \mathrm{y^n}} \tag{34}$$

Are these two equations the same?

The normal covariant transformation rule for a tensor leads to:

$$\mathrm{T_{mn}}(\mathrm{y}) = \frac{\partial \mathrm{x^r}}{\partial \mathrm{y^m}}\frac{\partial \mathrm{x^s}}{\partial \mathrm{y^n}}\, \mathrm{T_{rs}}(\mathrm{x}) \tag{35}$$

$$\mathrm{T_{mn}}(\mathrm{y}) = \frac{\partial \mathrm{x^r}}{\partial \mathrm{y^m}}\frac{\partial \mathrm{x^s}}{\partial \mathrm{y^n}}\,\frac{\partial \mathrm{V_r}(\mathrm{x})}{\partial \mathrm{x^s}} = \frac{\partial \mathrm{x^r}}{\partial \mathrm{y^m}}\,\frac{\partial \mathrm{V_r}(\mathrm{x})}{\partial \mathrm{y^n}} \tag{36}$$

$$T_{mn}(y) = \boxed{\frac{\partial x^r}{\partial y^m}\frac{\partial V_r(x)}{\partial y^n}} \overset{?}{=} \frac{\partial V_m(y)}{\partial y^n} \tag{37}$$

Now:

$$\frac{\partial V_m(y)}{\partial y^n} = \frac{\partial}{\partial y^n}\left(\frac{\partial x^r}{\partial y^m} V_r(x)\right) = \boxed{\frac{\partial x^r}{\partial y^m}\frac{\partial V_r(x)}{\partial y^n}} + \left(\frac{\partial}{\partial y^n}\frac{\partial x^r}{\partial y^m}\right) V_r(x) = \boxed{\frac{\partial x^r}{\partial y^m}\frac{\partial V_r(x)}{\partial y^n}} + \frac{\partial^2 x^r}{\partial y^n \partial y^m} V_r(x) \tag{38}$$

As we know:

$$V_r(x) = \frac{\partial y^a}{\partial x^r} V_a(y) \tag{39}$$

$$\frac{\partial V_m(y)}{\partial y^n} = \boxed{\frac{\partial x^r}{\partial y^m}\frac{\partial V_r(x)}{\partial y^n}} + \frac{\partial y^a}{\partial x^r}\frac{\partial^2 x^r}{\partial y^n \partial y^m} V_a(y) \tag{40}$$

From (6) we know:

$$\Gamma^a_{nm} = \frac{\partial y^a}{\partial x^r}\frac{\partial^2 x^r}{\partial y^n \partial y^m}$$

$$\boxed{\frac{\partial x^r}{\partial y^m}\frac{\partial V_r(x)}{\partial y^n}} = \frac{\partial V_m(y)}{\partial y^n} - \frac{\partial y^a}{\partial x^r}\frac{\partial^2 x^r}{\partial y^n \partial y^m} V_a(y) = \frac{\partial V_m(y)}{\partial y^n} - \Gamma^a_{nm} V_a(y) \tag{41}$$

Thus $T_{mn}(y) \neq \frac{\partial V_m(y)}{\partial y^n}$.

According to (32), (36) and (41):

$$T_{mn}(y) = \frac{\partial V_m(y)}{\partial y^n} - \Gamma^a_{nm} V_a(y) = \nabla_n V_m(y)$$

$$\mathbf{T_{mn}(y) = \nabla_n \boldsymbol{V_m}(y)} \tag{42}$$

2.4.3.1. ***Covariant differentiation for a covariant tensor***

$$T_{\mu\nu} = A_\mu B_\nu$$

$$\nabla_\alpha T_{\mu\nu} = B_\nu \nabla_\alpha A_\mu + A_\mu \nabla_\alpha B_\nu$$

$$\nabla_\alpha T_{\mu\nu} = B_\nu \left\{\frac{\partial A_\mu}{\partial x^\alpha} - A_\beta \Gamma^\beta_{\alpha\mu}\right\} + A_\mu \left\{\frac{\partial B_\nu}{\partial x^\alpha} - B_\gamma \Gamma^\gamma_{\alpha\nu}\right\}$$

$$\nabla_\alpha T_{\mu\nu} = B_\nu \frac{\partial A_\mu}{\partial x^\alpha} - A_\beta B_\nu \Gamma^\beta_{\alpha\mu} + A_\mu \frac{\partial B_\nu}{\partial x^\alpha} - A_\mu B_\gamma \Gamma^\gamma_{\alpha\nu}$$

$$\nabla_\alpha T_{\mu\nu} = B_\nu \frac{\partial A_\mu}{\partial x^\alpha} + A_\mu \frac{\partial B_\nu}{\partial x^\alpha} - A_\beta B_\nu \Gamma^\beta_{\alpha\mu} - A_\mu B_\gamma \Gamma^\gamma_{\alpha\nu}$$

$$\nabla_\alpha T_{\mu\nu} = \frac{\partial\left(A_\mu B_\nu\right)}{\partial \mathrm{x}^\alpha} - A_\beta B_\nu \Gamma^\beta_{\alpha\mu} - A_\mu B_\gamma \Gamma^\gamma_{\alpha\nu}$$

$$\boldsymbol{\nabla_\alpha T_{\mu\nu} = \frac{\partial T_{\mu\nu}}{\partial x^\alpha} - T_{\beta\nu}\Gamma^\beta_{\alpha\mu} - T_{\mu\gamma}\Gamma^\gamma_{\alpha\nu}} \tag{43}$$

2.4.3.2. ***Covariant differentiation for a contra-variant tensor***

$$T^{\mu\nu} = A^\mu B^\nu$$

$$\nabla_\alpha T^{\mu\nu} = B^\nu \nabla_\alpha A^\mu + A^\mu \nabla_\alpha B^\nu$$

$$\nabla_\alpha T^{\mu\nu} = B^\nu \left\{\frac{\partial A^\mu}{\partial \mathrm{x}^\alpha} + A^\beta \Gamma^\mu_{\beta\alpha}\right\} + A^\mu \left\{\frac{\partial B^\nu}{\partial \mathrm{x}^\alpha} + B^\gamma \Gamma^\nu_{\gamma\alpha}\right\}$$

$$\nabla_\alpha T^{\mu\nu} = B^\nu \frac{\partial A^\mu}{\partial \mathrm{x}^\alpha} + A^\beta B^\nu \Gamma^\mu_{\beta\alpha} + A^\mu \frac{\partial B^\nu}{\partial \mathrm{x}^\alpha} + A^\mu B^\gamma \Gamma^\nu_{\gamma\alpha}$$

$$\nabla_\alpha T^{\mu\nu} = B^\nu \frac{\partial A^\mu}{\partial \mathrm{x}^\alpha} + A^\mu \frac{\partial B^\nu}{\partial \mathrm{x}^\alpha} + A^\beta B^\nu \Gamma^\mu_{\beta\alpha} + A^\mu B^\gamma \Gamma^\nu_{\gamma\alpha}$$

$$\nabla_\alpha T^{\mu\nu} = \frac{\partial(A^\mu B^\nu)}{\partial \mathrm{x}^\alpha} + A^\beta B^\nu \Gamma^\mu_{\beta\alpha} + A^\mu B^\gamma \Gamma^\nu_{\gamma\alpha}$$

$$\boldsymbol{\nabla_\alpha T^{\mu\nu} = \frac{\partial T^{\mu\nu}}{\partial x^\alpha} + T^{\beta\nu}\Gamma^\mu_{\beta\alpha} + T^{\mu\gamma}\Gamma^\nu_{\gamma\alpha}} \tag{44}$$

2.4.3.3. ***Covariant differentiation for a mixed tensor***

$$T^\mu_\nu = A^\mu B_\nu$$

$$\nabla_\alpha T^\mu_\nu = B_\nu \nabla_\alpha A^\mu + A^\mu \nabla_\alpha B_\nu$$

$$\nabla_\alpha T^\mu_\nu = B_\nu \left\{\frac{\partial A^\mu}{\partial \mathrm{x}^\alpha} + A^\beta \Gamma^\mu_{\beta\alpha}\right\} + A^\mu \left\{\frac{\partial B_\nu}{\partial \mathrm{x}^\alpha} - B_\gamma \Gamma^\gamma_{\alpha\nu}\right\}$$

$$\nabla_\alpha T^\mu_\nu = B_\nu \frac{\partial A^\mu}{\partial \mathrm{x}^\alpha} + A^\beta B_\nu \Gamma^\mu_{\beta\alpha} + A^\mu \frac{\partial B_\nu}{\partial \mathrm{x}^\alpha} - A^\mu B_\gamma \Gamma^\gamma_{\alpha\nu}$$

$$\nabla_\alpha T^\mu_\nu = B_\nu \frac{\partial A^\mu}{\partial \mathrm{x}^\alpha} + A^\mu \frac{\partial B_\nu}{\partial \mathrm{x}^\alpha} + A^\beta B_\nu \Gamma^\mu_{\beta\alpha} - A^\mu B_\gamma \Gamma^\gamma_{\alpha\nu}$$

$$\nabla_\alpha T^\mu_\nu = \frac{\partial(A^\mu B_\nu)}{\partial \mathrm{x}^\alpha} + A^\beta B_\nu \Gamma^\mu_{\beta\alpha} - A^\mu B_\gamma \Gamma^\gamma_{\alpha\nu}$$

$$\boldsymbol{\nabla_\alpha T^\mu_\nu = \frac{\partial T^\mu_\nu}{\partial x^\alpha} + T^\beta_\nu \Gamma^\mu_{\beta\alpha} - T^\mu_\gamma \Gamma^\gamma_{\nu\alpha}} \tag{45}$$

2.5. Geodesic equation and Christoffel symbols

As said before Einstein tried to describe space-time geography such that when one makes a free fall in space-time one follows a “straight” space-time line. Along this line the person does not experience any gravitational effects. Such a line, where no gravitational influences are felt, is called a geodesic. This free movement of this person or particle is given by the acceleration four-vector’s magnitude and is equal to zero.

$$\frac{d^2\xi^\alpha}{d\tau^2} = 0 \text{ and } ds^2 = c^2 d\tau^2$$

Where τ refers to the time, as measured by an observer at rest in his own (free falling) coordinate frame, also called the *proper time*. This observer is the free falling object and his frame moves in accordance with the local acceleration due to the gravitational “force”. You could say that the origin of the frame surrenders to the gravity forces at that location.

In general

$$\xi^\alpha = \frac{\partial\xi^\alpha}{\partial x^\mu}x^\mu \text{ thus } \xi^\alpha = \sum_{\mu=0}^{3}\frac{\partial\xi^\alpha}{\partial x^\mu}x^\mu \Rightarrow \frac{d\xi^\alpha}{d\tau} = \sum_{\mu=0}^{3}\frac{\partial\xi^\alpha}{\partial x^\mu}\frac{dx^\mu}{d\tau}$$

In Einstein notation:

$$\frac{d\xi^\alpha}{d\tau} = \frac{\partial\xi^\alpha}{\partial x^\mu}\frac{dx^\mu}{d\tau}$$

As an example written out:

$$\frac{d\xi^\alpha}{d\tau} = \frac{\partial\xi^\alpha}{\partial x^0}\frac{dx^0}{d\tau} + \frac{\partial\xi^\alpha}{\partial x^1}\frac{dx^1}{d\tau} + \frac{\partial\xi^\alpha}{\partial x^2}\frac{dx^2}{d\tau} + \frac{\partial\xi^\alpha}{\partial x^3}\frac{dx^3}{d\tau}$$

Now we apply the chain rule to the free falling equation with the knowledge that the acceleration is zero:

$$0 = \frac{d}{d\tau}\left(\frac{d\xi^\alpha}{d\tau}\right) = \frac{d}{d\tau}\left(\frac{\partial\xi^\alpha}{\partial x^\mu}\frac{dx^\mu}{d\tau}\right) = \frac{d}{d\tau}\left(\frac{d\xi^\alpha}{dx^\mu}\right)\frac{dx^\mu}{d\tau} + \frac{\partial\xi^\alpha}{\partial x^\mu}\frac{d^2x^\mu}{d\tau^2} =$$

$$0 = \left(\frac{d^2\xi^\alpha}{dx^\mu dx^\nu}\right)\frac{dx^\nu}{d\tau}\frac{dx^\mu}{d\tau} + \frac{\partial\xi^\alpha}{\partial x^\mu}\frac{d^2x^\mu}{d\tau^2}$$

Multiply with $\frac{dx^\beta}{d\xi^\alpha}$:

$$0 = \left(\frac{dx^\beta}{d\xi^\alpha}\right)\left(\frac{d^2\xi^\alpha}{dx^\mu dx^\nu}\right)\frac{dx^\nu}{d\tau}\frac{dx^\mu}{d\tau} + \frac{\partial\xi^\alpha}{\partial x^\mu}\frac{d^2x^\mu}{d\tau^2}\left(\frac{dx^\beta}{d\xi^\alpha}\right)$$

Here is

$$\frac{\partial\xi^\alpha}{\partial x^\mu}\frac{\partial x^\beta}{\partial\xi^\alpha} = \frac{\partial x^\beta}{\partial x^\mu} = \delta^\beta_\mu \ (the\ Kronecker\ delta)$$

So

$$0 = \frac{\partial x^\beta}{\partial\xi^\alpha}\frac{\partial^2\xi^\alpha}{\partial x^\mu\partial x^\nu}\frac{dx^\mu}{d\tau}\frac{\partial x^\nu}{\partial\tau} + \delta^\beta_\mu\frac{d^2x^\mu}{d\tau^2}$$

The ***Kronecker delta*** is defined as 1 only if $\beta = \mu$, and 0 if $\beta \neq \mu$. $\frac{\partial x^\beta}{\partial x^\mu} = \delta^\beta_\mu = 0$, because x^β and x^μ are perpendicular in case $\beta \neq \mu$. This means that we can replace the μ index by β in the last term.

So

$$0=\frac{\partial x^{\beta}}{\partial \xi^{\alpha}}\frac{\partial^2 \xi^{\alpha}}{\partial x^{\mu}\partial x^{\nu}}\frac{dx^{\mu}}{d\tau}\frac{\partial x^{\nu}}{\partial \tau}+\frac{d^2 x^{\beta}}{d\tau^2}$$

$$\boldsymbol{0=\frac{d^2 x^{\beta}}{d\tau^2}+\Gamma^{\beta}_{\mu\nu}\frac{\partial x^{\mu}}{\partial \tau}\frac{\partial x^{\nu}}{\partial \tau}\ Geodesic\ equation} \tag{1}$$

So the acceleration $\frac{d^2 x^{\beta}}{d\tau^2}$ is compensated by the gamma term. In case there is no gravity then there is no curvature, so the term $\Gamma^{\beta}_{\mu\nu}$ is zero and consequently $\frac{d^2 x^{\beta}}{d\tau^2}=0$.

$$where\ \Gamma^{\beta}_{\mu\nu}=\frac{\partial x^{\beta}}{\partial \xi^{\alpha}}\frac{\partial^2 \xi^{\alpha}}{\partial x^{\mu}\partial x^{\nu}}\ \ is\ the\ Christoffel\ symbol$$

Summary:

So in general the relation between the acceleration in a moving frame along the trajectory and the acceleration with respect to the rest frame is:

$$\frac{d^2 \xi^{\beta}}{d\tau^2}=\frac{d^2 x^{\beta}}{d\tau^2}+\Gamma^{\beta}_{\mu\nu}\frac{\partial x^{\mu}}{\partial \tau}\frac{\partial x^{\nu}}{\partial \tau}$$

For a trajectory of a geodesic the acceleration along the trajectory shall be zero:

$$0=\frac{d^2 x^{\beta}}{d\tau^2}+\Gamma^{\beta}_{\mu\nu}\frac{\partial x^{\mu}}{\partial \tau}\frac{\partial x^{\nu}}{\partial \tau}$$

Resulting in acceleration in the rest frame:

$$\frac{d^2 x^{\beta}}{d\tau^2}=-\Gamma^{\beta}_{\mu\nu}\frac{\partial x^{\mu}}{\partial \tau}\frac{\partial x^{\nu}}{\partial \tau}$$

Where the Christoffel symbol contains the relation between the moving frame ξ^{α} and the “rest” frame x^{β}.

$$\Gamma^{\beta}_{\mu\nu}=\frac{\partial x^{\beta}}{\partial \xi^{\alpha}}\frac{\partial^2 \xi^{\alpha}}{\partial x^{\mu}\partial x^{\nu}}$$

Remark:

In the geodesic equation, the derivation is done to the proper time τ however this parameter could be awkward in case of propagation of photons where $\tau=0$. Therefore mostly a so-called affine parameter λ is used as:

$$0=\frac{d^2 x^{\beta}}{d\lambda^2}+\Gamma^{\beta}_{\mu\nu}\frac{\partial x^{\mu}}{\partial \lambda}\frac{\partial x^{\nu}}{\partial \lambda}$$

In general in the final result of calculations the lambda disappears so the awkwardness of being zero is circumvented.

Note:

In the literature it is often custom to choose the light-velocity *c*=1 for reasons of convenience. However we try to include the *c* in the equations because it is a handy tool to check whether the dimensions are correct and so to avoid errors.

2.6. Christoffel symbols in terms of the metric tensor

As mentioned before all the information about the space-time geography is contained in the metric. Now we will express the Christoffel symbols in terms of this metric.

So far, we have defined both the metric and the Christoffel symbols as respectively:

$$g_{\mu\nu} = \eta_{\alpha\beta} \frac{\partial \xi^\alpha}{\partial x^\mu} \frac{\partial \xi^\beta}{\partial x^\nu} \quad and \quad \Gamma^\beta_{\mu\nu} = \frac{\partial x^\beta}{\partial \xi^\lambda} \frac{\partial^2 \xi^\lambda}{\partial x^\mu \partial x^\nu}$$

From chapter 4.5.1 we know:

$$\eta_{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

We begin by rewriting the metric tensor in a slightly different form $g_{\alpha\mu}$:

$$g_{\mu\nu} = \eta_{\alpha\beta} \frac{\partial \xi^\alpha}{\partial x^\mu} \frac{\partial \xi^\beta}{\partial x^\nu} \quad by\ symmetry => \quad g_{\nu\mu} = \eta_{\alpha\beta} \frac{\partial \xi^\alpha}{\partial x^\mu} \frac{\partial \xi^\beta}{\partial x^\nu}$$

Replacing the dummy index α by σ:

$$\sigma => \quad g_{\nu\mu} = \eta_{\sigma\beta} \frac{\partial \xi^\sigma}{\partial x^\mu} \frac{\partial \xi^\beta}{\partial x^\nu}$$

Replacing the index ν by α:

$$\alpha => \quad g_{\alpha\mu} = \eta_{\sigma\beta} \frac{\partial \xi^\sigma}{\partial x^\mu} \frac{\partial \xi^\beta}{\partial x^\alpha} \qquad (2)$$

Now we rewrite the Christoffel symbol by multiplying each part of the equation by the partial derivative of ξ^σ relative to x^β :

$$\left(\frac{\partial \xi^\sigma}{\partial x^\beta}\right) \Gamma^\beta_{\mu\nu} = \frac{\partial x^\beta}{\partial \xi^\lambda} \frac{\partial^2 \xi^\lambda}{\partial x^\mu \partial x^\nu} \left(\frac{\partial \xi^\sigma}{\partial x^\beta}\right) = \left(\frac{\partial x^\beta}{\partial \xi^\lambda} \frac{\partial \xi^\sigma}{\partial x^\beta}\right) \frac{\partial^2 \xi^\lambda}{\partial x^\mu \partial x^\nu} \qquad (3a)$$

Or

$$\left(\frac{\partial x^\beta}{\partial \xi^\lambda} \frac{\partial \xi^\sigma}{\partial x^\beta}\right) = \frac{\partial \xi^\sigma}{\partial \xi^\lambda} = \delta^\sigma_\lambda \ or\ \delta^\sigma_\lambda\ \{= 1\ if\ \sigma = \lambda\ and = 0\ if\ \ \sigma \neq \lambda\}$$

So together with (3a)

$$\left(\frac{\partial \xi^\sigma}{\partial x^\beta}\right) \Gamma^\beta_{\mu\nu} = \delta^\sigma_\lambda \frac{\partial^2 \xi^\lambda}{\partial x^\mu \partial x^\nu}$$

If $\sigma = \lambda$ we replace σ by λ:

$$\left(\frac{\partial \xi^\lambda}{\partial x^\beta}\right) \Gamma^\beta_{\mu\nu} = \frac{\partial^2 \xi^\lambda}{\partial x^\mu \partial x^\nu} \qquad (3b)$$

Thus from (2):

With (3b) we can derive:

$$\frac{\partial\, g_{\alpha\mu}}{\partial x^{\nu}} = \eta_{\sigma\beta}\,\frac{\partial^2 \xi^{\sigma}}{\partial x^{\nu}\partial x^{\mu}}\frac{\partial \xi^{\beta}}{\partial x^{\alpha}} + \eta_{\sigma\beta}\,\frac{\partial \xi^{\sigma}}{\partial x^{\mu}}\frac{\partial^2 \xi^{\beta}}{\partial x^{\nu}\partial x^{\alpha}}$$

$$\frac{\partial^2 \xi^{\sigma}}{\partial x^{\nu}\partial x^{\mu}} = \frac{\partial \xi^{\sigma}}{\partial x^{\rho}}\Gamma^{\rho}_{\mu\nu} \;\; and \;\; \frac{\partial^2 \xi^{\beta}}{\partial x^{\nu}\partial x^{\alpha}} = \frac{\partial \xi^{\beta}}{\partial x^{\rho}}\Gamma^{\rho}_{\nu\alpha}$$

Now rewrite the partial derivative of $g_{\alpha\mu}$ by x^{ν} as follows:

$$\frac{\partial\, g_{\alpha\mu}}{\partial x^{\nu}} = \eta_{\sigma\beta}\,\frac{\partial \xi^{\beta}}{\partial x^{\alpha}}\frac{\partial \xi^{\sigma}}{\partial x^{\rho}}\Gamma^{\rho}_{\mu\nu} + \eta_{\sigma\beta}\,\frac{\partial \xi^{\sigma}}{\partial x^{\mu}}\frac{\partial \xi^{\beta}}{\partial x^{\rho}}\Gamma^{\rho}_{\nu\alpha}$$

We know from above

$$metric\ tensor\colon\ g_{\mu\nu} = \eta_{\alpha\beta}\,\frac{\partial \xi^{\alpha}}{\partial x^{\mu}}\frac{\partial \xi^{\beta}}{\partial x^{\nu}}$$

Thus

$$\eta_{\sigma\beta}\,\frac{\partial \xi^{\beta}}{\partial x^{\alpha}}\frac{\partial \xi^{\sigma}}{\partial x^{\rho}} = g_{\rho\alpha} \;\; and \;\; \eta_{\sigma\beta}\,\frac{\partial \xi^{\sigma}}{\partial x^{\mu}}\frac{\partial \xi^{\beta}}{\partial x^{\rho}} = g_{\mu\rho}$$

$$\frac{\partial\, g_{\alpha\mu}}{\partial x^{\nu}} = g_{\rho\alpha}\Gamma^{\rho}_{\mu\nu} + g_{\mu\rho}\Gamma^{\rho}_{\nu\alpha} \qquad (3c)$$

$$\frac{\partial\, g_{\alpha\nu}}{\partial x^{\mu}} = g_{\rho\alpha}\Gamma^{\rho}_{\nu\mu} + g_{\nu\rho}\Gamma^{\rho}_{\mu\alpha} \qquad \mu\ and\ \nu\ \text{ are swapped} \qquad (3d)$$

$$\frac{\partial\, g_{\mu\nu}}{\partial x^{\alpha}} = g_{\rho\mu}\Gamma^{\rho}_{\nu\alpha} + g_{\nu\rho}\Gamma^{\rho}_{\alpha\mu} \qquad \alpha\ and\ \mu\ \text{ are swapped} \qquad (3e)$$

Next we perform (3c)+(3d)-(3e):

$$\frac{\partial\, g_{\alpha\mu}}{\partial x^{\nu}} + \frac{\partial\, g_{\alpha\nu}}{\partial x^{\mu}} - \frac{\partial\, g_{\mu\nu}}{\partial x^{\alpha}} = g_{\rho\alpha}\Gamma^{\rho}_{\mu\nu} + g_{\mu\rho}\Gamma^{\rho}_{\nu\alpha} + g_{\rho\alpha}\Gamma^{\rho}_{\nu\mu} + g_{\nu\rho}\Gamma^{\rho}_{\mu\alpha} - g_{\rho\mu}\Gamma^{\rho}_{\nu\alpha} - g_{\nu\rho}\Gamma^{\rho}_{\alpha\mu}$$

Resulting in:

$$\frac{\partial\, g_{\alpha\mu}}{\partial x^{\nu}} + \frac{\partial\, g_{\alpha\nu}}{\partial x^{\mu}} - \frac{\partial\, g_{\mu\nu}}{\partial x^{\alpha}} = 2g_{\rho\alpha}\Gamma^{\rho}_{\mu\nu}$$

$$g_{\rho\alpha}\Gamma^{\rho}_{\mu\nu} = \frac{1}{2}\left(\frac{\partial\, g_{\alpha\mu}}{\partial x^{\nu}} + \frac{\partial\, g_{\alpha\nu}}{\partial x^{\mu}} - \frac{\partial\, g_{\mu\nu}}{\partial x^{\alpha}}\right)$$

The last step consists in multiplying both sides of the equations by the inverse metric $g^{\rho\alpha}$ to find the Christoffel symbol:

$$g^{\rho\alpha} g_{\rho\alpha}\Gamma^{\rho}_{\mu\nu} = \Gamma^{\rho}_{\mu\nu} = \frac{1}{2}\, g^{\rho\alpha}\left(\frac{\partial\, g_{\alpha\mu}}{\partial x^{\nu}} + \frac{\partial\, g_{\alpha\nu}}{\partial x^{\mu}} - \frac{\partial\, g_{\mu\nu}}{\partial x^{\alpha}}\right)$$

Swapping ρ to β:

$$\Gamma^{\beta}_{\mu\nu} = \frac{1}{2}\, g^{\beta\alpha}\left(\frac{\partial\, g_{\alpha\mu}}{\partial x^{\nu}} + \frac{\partial\, g_{\alpha\nu}}{\partial x^{\mu}} - \frac{\partial\, g_{\mu\nu}}{\partial x^{\alpha}}\right)$$

Usually the following convention for writing partial derivatives is adopted:

$$\frac{\partial\, g_{\alpha\mu}}{\partial x^{\nu}} \equiv g_{\alpha\mu,\nu}$$

Thus the **Christoffel symbol** is:

$$\boldsymbol{\Gamma^{\beta}_{\mu\nu} = \frac{1}{2}\, g^{\beta\alpha}(\, g_{\alpha\mu,\nu} + g_{\alpha\nu,\mu} - g_{\mu\nu,\alpha})}$$

Note:
The equation shows that the Christoffel symbol is totally determined by the metric tensor and its derivatives!

2.7. Geodesic Equation in the Newtonian Limit

Newtonian gravity tells us how matter produces gravity and how gravity affects matter.
From Newton's second law it can be derived that:

$$\vec{a} = -\nabla\Phi$$

Here is Φ the gravitational potential, caused by matter. ∇ is the Euclidean gradient operator $\left(\frac{\partial}{\partial x}\hat{e}_x + \frac{\partial}{\partial y}\hat{e}_y + \frac{\partial}{\partial z}\hat{e}_z\right)$ and $\vec{a}$ is the resulting acceleration vector. $Here\ \hat{e}_x\ is\ the\ unity\ vector\ along\ the\ x\ axis.$
Now we will derive an approximation of the Newtonian gravitational equation with the mathematics of the theory of General Relativity.
There are three assumptions for this Newtonian limit:

- The particle is moving relatively slow with respect to the light velocity.
- The gravitational field is weak
- The field is static, so it does not change with time.

The geodesic equation describes the world-line of a particle that is affected only by gravity. We will now show that in the context of the Newtonian limit, the geodesic equation reduces to the Newton's gravity equation.
From the previous chapter we know that the geodesic equations, using proper time as the parameter of the world-line are:

$$\frac{d^2x^\beta}{d\tau^2} + \Gamma^\beta_{\mu\nu}\frac{\partial x^\mu}{\partial\tau}\frac{\partial x^\nu}{\partial\tau} = 0$$

The second term comprises a sum in μ and ν over all indices, which are 16 terms. Because the particle is moving very slowly, with regard to the light velocity, the time-component, i.e. the 0th component of the particle's vector, dominates the other spatial components. We come thus to the following approximation:

$$with\ \frac{dx^i}{d\tau} \ll \frac{dt}{d\tau} \quad (as\ we\ know\ c\partial t = \partial x^0)$$

$$\frac{d^2x^\beta}{d\tau^2} + \Gamma^\beta_{\mu\nu}\frac{\partial x^\mu}{\partial\tau}\frac{\partial x^\nu}{\partial\tau} = 0$$

$$=> \frac{d^2x^\beta}{d\tau^2} + \Gamma^\beta_{00}\left(\frac{cdt}{d\tau}\right)^2 = 0$$

For the description of the four dimensional space-time normally Greek letters are used for the indices, but in case we consider only the three dimensional space it is custom to use Latin letters.
So by restricting to the Newtonian 3-D space, meaning that we assign β to spatial dimensions only, we can replace β by the Latin letter *i* (i=x, y, z) giving:

$$\frac{d^2x^i}{d\tau^2} + \Gamma^i_{00}\left(\frac{cdt}{d\tau}\right)^2 = 0 \qquad (1)$$

From the chapter *Christoffel symbols in terms of the metric tensor*, we know how to calculate the Christoffel symbol with respect to the components of a given metric $(x^0 \equiv \tau)$:

$$\Gamma^i_{00} = \frac{1}{2}\, g^{ij} \left(\frac{\partial\, g_{j\,0}}{\partial x^0} + \frac{\partial\, g_{j\,0}}{\partial x^0} - \frac{\partial\, g_{00}}{\partial x^j} \right)$$

Because the field is static, second assumption of the Newtonian limit, the time derivative $\frac{\partial\, g_{j\,0}}{\partial x^0}$ is zero, so that the Christoffel symbol can be simplified to:

$$\Gamma^i_{00} = -\frac{1}{2}\, g^{ij}\, \frac{\partial\, g_{00}}{\partial x^j} \tag{2}$$

If the gravitational field is weak enough, space-time will only be slightly deformed from the gravity-free Minkowski space-time of Special Relativity. So we can consider the space-time metric as a small perturbation from the Minkowski metric $\eta_{\mu\nu}$

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad with \;\; \lceil h_{\mu\nu} \rceil \ll 1$$

$$\frac{d\, g_{00}}{dx^j} = \frac{d(\eta_{00} + h_{00})}{dx^j}$$

$$\frac{d\, g_{00}}{dx^j} = \frac{d\eta_{00}}{dx^j} + \frac{d\, h_{00}}{dx^j} = 0 + \frac{d\, h_{00}}{dx^j} \;\; as\, \eta_{00} = 1$$

$$=> \frac{d\, g_{00}}{dx^j} = \frac{dh_{00}}{dx^j} \tag{3}$$

So from (2) and (3) equation (1) becomes:

$$\frac{d^2 x^i}{d\tau^2} = -\Gamma^i_{00} \left(\frac{cdt}{d\tau}\right)^2$$

$$\frac{d^2 x^i}{d\tau^2} = \frac{1}{2}\, g^{ij}\, \frac{\partial\, h_{00}}{\partial x^j} \left(\frac{cdt}{d\tau}\right)^2$$

Defining $g^{ij} = \eta^{ij} - h^{ij}$, we find that $g^{\mu\sigma}\, g_{\sigma\nu} = \delta^\mu_\nu$ to within the first order of h_{ij}, defining an inverse metric. We then obtain

$$\frac{d^2 x^i}{d\tau^2} = \frac{1}{2}\, \eta^{ij}\, \frac{\partial\, h_{00}}{\partial x^j} \left(\frac{cdt}{d\tau}\right)^2$$

But as η^{ij} is not zero for j=i, then $\eta^{ii} = -1$ (I refers to the spatial components)

$$\frac{d^2 x^i}{d\tau^2} = -\frac{1}{2} \frac{\partial\, h_{00}}{\partial x^i} \left(\frac{cdt}{d\tau}\right)^2$$

We will now change the derivative on the left hand-side from τ to t , we do that as follows:
First we replace *i* by 0 in the above equation:

$$c^2 \frac{d^2 \mathrm{t}}{d\tau^2} = -\frac{1}{2} \frac{\partial\, h_{00}}{\partial x^0} \left(\frac{cdt}{d\tau}\right)^2$$

As the gravitational field is constant then $\frac{\partial\, h_{00}}{\partial x^0} = 0$:

$$c^2 \frac{d^2 \mathrm{t}}{d\tau^2} = 0 \;\; => \;\; \frac{d^2 \mathrm{t}}{d\tau^2} = 0 \tag{4}$$

Next we need to work on the partial derivatives with respect to tau (τ):

$$\frac{d^2x^i}{d\tau^2} = \frac{d}{d\tau}\frac{dx^i}{d\tau} = \frac{d}{d\tau}\left(\frac{dt}{d\tau}\frac{dx^i}{dt}\right)$$

$$= \frac{dt}{d\tau}\left(\frac{d}{d\tau}\frac{dx^i}{dt}\right) + \frac{dx^i}{dt}\left(\frac{d}{d\tau}\frac{d\mathrm{t}}{d\tau}\right)$$

$$= \frac{dt}{d\tau}\left(\frac{dt}{d\tau}\frac{d}{dt}\frac{dx^i}{dt}\right) + \frac{dx^i}{dt}\left(\frac{d}{d\tau}\frac{d\mathrm{t}}{d\tau}\right)$$

$$= \left(\frac{dt}{d\tau}\right)^2\left(\frac{d^2x^i}{dt^2}\right) + \frac{dx^i}{dt}\left(\frac{d^2\mathrm{t}}{d\tau^2}\right)$$

As we have seen from (4) above $\frac{d^2\mathrm{t}}{d\tau^2} = 0$:

$$\frac{d^2x^i}{d\tau^2} = \left(\frac{dt}{d\tau}\right)^2\left(\frac{d^2x^i}{dt^2}\right) = -\frac{1}{2}\frac{\partial\, h_{00}}{\partial x^i}\left(\frac{cdt}{d\tau}\right)^2 = -\frac{c^2}{2}\frac{\partial\, h_{00}}{\partial x^i}\left(\frac{dt}{d\tau}\right)^2$$

$$=> \left(\frac{d^2x^i}{dt^2}\right)\left(\cancel{\frac{dt}{d\tau}}\right)^2 = -\frac{c^2}{2}\frac{\partial\, h_{00}}{\partial x^i}\left(\cancel{\frac{dt}{d\tau}}\right)^2$$

$$=> \frac{d^2x^i}{dt^2} = -\frac{c^2}{2}\frac{\partial\, h_{00}}{\partial x^i}$$

More general:

$$\frac{d^2\mathrm{x}}{dt^2}i + \frac{d^2\mathrm{y}}{dt^2}j + \frac{d^2\mathrm{z}}{dt^2}k = -\frac{\partial}{\partial\mathrm{x}}\left(\frac{c^2\, h_{00}}{2}\right)i - \frac{\partial}{\partial\mathrm{y}}\left(\frac{c^2\, h_{00}}{2}\right)j - \frac{\partial}{\partial\mathrm{z}}\left(\frac{c^2\, h_{00}}{2}\right)k$$

$$\frac{d^2\mathrm{x}}{dt^2}i + \frac{d^2\mathrm{y}}{dt^2}j + \frac{d^2\mathrm{z}}{dt^2}k = -\left[\frac{\partial}{\partial\mathrm{x}}i + \frac{\partial}{\partial\mathrm{y}}j + \frac{\partial}{\partial\mathrm{z}}k\right]\left(\frac{c^2\, h_{00}}{2}\right) = -\nabla\left(\frac{c^2\, h_{00}}{2}\right)$$

Expressing this in vector form:

$$\boxed{\boldsymbol{\frac{d^2\vec{\mathrm{r}}}{dt^2} = -\nabla\phi \quad or \quad \frac{d^2\vec{\mathrm{r}}}{dt^2} = -\overrightarrow{\mathbf{grad}}\,\phi}}$$

$$\textbf{with } \boldsymbol{\phi = \frac{c^2 h_{00}}{2}} \textbf{ and thus } \boldsymbol{h_{00} = \frac{2\phi}{c^2}}.$$

This is another way of writing the Newtonian gravitational equation $\vec{a} = -\nabla\Phi$.

Note:

By writing the metric component g_{00} as:

$$g_{00} = \eta_{00} + h_{00} = 1 + \frac{2\phi}{c^2} \tag{5}$$

We see the direct link between the metric tensor (component $_{00}$) on the left hand-side and the gravitational potential ϕ on the right hand-side.

Example:

We can calculate the h_{00} value on the Earth and check whether its value is infinitesimal, meaning that the deviation relative to the Minkowski metric due to the gravitational field is negligible.

$$h_{00} = \frac{2\phi}{c^2} \; with \; \phi = \frac{GM_{earth}}{R_{earth}}$$

$$G = 6.67 \times 10^{-11} \, m^3.kg^{-1}.s^{-2}$$

$$M_{earth} \simeq 6 \times 10^{24} kg \quad R_{earth} \simeq 6400 \, km$$

$$c \simeq 3 \times 10^8 \, m.s^{-1}$$

$$h_{00} \simeq 10^{-9}$$

Doing the same calculation for the surface of the Sun and of a white dwarf, the correction to the Minkowski metric is -10^{-6} *and* -10^{-4} respectively. So we may conclude that the weak-field limit is an excellent approximation.

2.8. Generalization of the definition of the metric tensor

Above we have generalized the formulation of a geodesic equation from an inertial frame to an arbitrary frame. In the same way we will generalize the definition of the metric tensor from a Minkowski space-time to the one of a so called pseudo Riemann manifold, which is the mathematical structure by which the theory of General Relativity can be modeled.

Again we will name the space-time coordinates ξ^α in the local inertial frame, i.e.: $\xi^0 = ct,\ \xi^1 = x,\ \xi^2 = y,\ \xi^3 = z,$ we can then write the Minkowski line element as follows (see also 2.2.1 equation 2)

$$ds^2 = \eta_{\alpha\beta} \, d\xi^\alpha d\xi^\beta \; where \; \eta_{\alpha\beta} \equiv \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix}$$

We name x^μ the coordinates in the new, non inertial, frame, where $\xi^\alpha = \xi^\alpha(x^0, x^1, x^2, x^3)$, and so the infinitesimal variation $d\xi^\alpha$ is:

$$d\xi^\alpha = \frac{\partial \xi^\alpha}{\partial x^0} dx^0 + \frac{\partial \xi^\alpha}{\partial x^1} dx^1 + \frac{\partial \xi^\alpha}{\partial x^2} dx^2 + \frac{\partial \xi^\alpha}{\partial x^3} dx^3$$

By using Einstein summation convention:

$$d\xi^\alpha = \frac{\partial \xi^\alpha}{\partial x^\mu} dx^\mu \text{and } d\xi^\beta = \frac{\partial \xi^\beta}{\partial x^\nu} dx^\nu$$

$$ds^2 = \eta_{\alpha\beta} \frac{\partial \xi^\alpha}{\partial x^\mu} \frac{\partial \xi^\beta}{\partial x^\nu} dx^\mu dx^\nu$$

Let us name

$$\boldsymbol{metric\ tensor}:\ \boldsymbol{g_{\mu\nu} = \eta_{\alpha\beta} \frac{\partial \xi^\alpha}{\partial x^\mu} \frac{\partial \xi^\beta}{\partial x^\nu}}$$

So

$$\boldsymbol{ds^2 = g_{\mu\nu} dx^\mu dx^\nu}$$

The properties of the metric tensor are:

- It is symmetric in the sense that $g_{\mu\nu} = g_{\nu\mu}$. The covariant metric element.

- The inverse matrix is noted $g^{\mu\nu}$, the contra-variant element, and is defined as follows $g^{\mu\nu} g_{\mu\nu} = \delta^{\mu}_{\nu}$ (Kronecker delta)

The metric tensor $g_{\mu\nu}$ is of fundamental importance: it contains all the information of the space-time. Because space-time curvature is equivalent to gravitation, **the metric contains all the information about the gravitational field.**

The goal of the theory of General Relativity could therefore be defined as to be able to calculate this metric. For symmetry reason, it is easy to see that the 16 metric components can be reduced to only **10 independent values.**

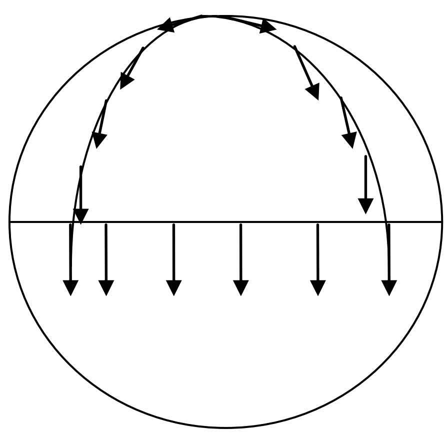
2.9. Riemann Curvature Tensor

The most important tensor in the theory of General Relativity is the *Riemann tensor.* This tensor contains all the information regarding the curvature of space-time. Thus in case of a Euclidean, flat, space the Riemann tensor vanishes. In this chapter we derive the Riemann tensor via two methods: via the "covariant derivative commutator" and via an alternative method the "geodesic deviation".

2.9.1. Derivation of the Riemann tensor from the Covariant Derivative Commutator

By means of the concept of parallel transport of vectors, or tensors, we will derive the expression for the Riemann tensor.

North Pole



Parallel transport of a vector around a closed loop

As an example of curved space we can use the Earth. Assume that we start from the North Pole, holding a stick horizontally and pointing in one direction and moving towards the equator via a meridian. We constantly keep the stick pointing at the same direction and holding it horizontally with respect to the Earth. At the equator we move over a certain distance and then change direction with 90 degrees and go upwards via a meridian again towards the North Pole. Finally back at the North Pole it turns out that the stick is pointing in a different direction. This caused by the fact that the Earth is not flat.

Now we can do the same in an infinitesimal loop on a manifold. By parallel transport of a vector around this infinitesimal loop, the vector will be equal to the vector we started with in case of flat space and in case the space is curved the start- and end vector direction will differ.

Parallel transport has a very precise definition in curved space: it is defined as transport for which the *covariant derivative* is zero. So holding the covariant derivative zero while transporting a vector around a small loop is one way to derive the Riemann tensor.
However there is also another more indirect way using what is called the commutator of the covariant derivative of a vector. So the different approaches have been performed below.

2.9.1.1. *Covariant derivative commutator*

Commutator here refers to the difference between two operations, one in one direction and the other in the opposite direction. The commutator is defined as

$$[AB] = AB - BA$$

So the commutator is only zero when the sequence of the two operations is irrelevant.
To get the Riemann tensor, the operation of choice is the covariant derivative. The commutator of two covariant derivatives measures the difference between parallel transporting the tensor first one way and then in the opposite way. So as a measure for the difference of the tensor along the path, the covariant derivative of the tensor is used.
In flat space, the order of covariant differentiation makes no difference, as covariant differentiation reduces to partial differentiation, and so the commutator must yield zero. Inversely any non-zero result of applying the commutator to covariant differentiation can be attributed to the curvature of the space, and therefore is indicated as the Riemann tensor.

2.9.1.2. *Derivation of the Riemann tensor*

So it is our aim to derive the Riemann tensor by finding the commutator.

$$\nabla_{\mathrm{c}}\nabla_{\mathrm{b}}V_a - \nabla_{\mathrm{b}}\nabla_{\mathrm{c}}V_a$$

We know that the covariant derivative of V_a is given by (see 32):

$$\nabla_{\mathrm{b}}V_a = \frac{\partial V_a}{\partial x^b} - \Gamma^d_{ab}V_d$$

As we have seen in the previous chapter:
(see 42)

$$\mathrm{T_{mn}}(\mathrm{y}) = \nabla_{\mathrm{n}}V_m = \frac{\partial \mathrm{V_m}}{\partial \mathrm{y^n}} - \Gamma^r_{nm}\mathrm{V_r}(\mathrm{x})$$

$$=> \ T_{\mathrm{ab}}(\mathrm{y}) = \nabla_{\mathrm{b}}V_a = \frac{\partial \mathrm{V_a}}{\partial \mathrm{y^b}} - \Gamma^r_{ba}\mathrm{V_r}(\mathrm{x})$$

Thus the covariant derivative of a vector ($\nabla_{\mathrm{b}}V_a$) is a tensor (see 42).

Now the covariant derivative of a tensor is (see 43) :

$$\nabla_\alpha T_{\mu\nu} = \frac{\partial T_{\mu\nu}}{\partial x^\alpha} - T_{\beta\nu}\Gamma^\beta_{\alpha\mu} - T_{\mu\gamma}\Gamma^\gamma_{\alpha\nu}$$

$$=> \nabla_c T_{ab} = \frac{\partial T_{ab}}{\partial x^c} - T_{eb}\Gamma^e_{ca} - T_{ae}\Gamma^e_{cb}$$

Resulting in:

$$\nabla_c\nabla_b V_a = \frac{\partial}{\partial x^c}(\nabla_b V_a) - \Gamma^e_{ac}\nabla_b V_e - \Gamma^e_{bc}\nabla_e V_a \tag{1}$$

The first right-hand term:

$$\frac{\partial}{\partial x^c}(\nabla_b V_a) = \frac{\partial^2 V_a}{\partial x^c \partial x^b} - \frac{\partial}{\partial x^c}(\Gamma^d_{ab} V_d) \tag{1a}$$

$$\frac{\partial}{\partial x^c}(\nabla_b V_a) = \frac{\partial^2 V_a}{\partial x^c \partial x^b} - \Gamma^d_{ab}\frac{\partial V_d}{\partial x^c} - V_d\frac{\partial \Gamma^d_{ab}}{\partial x^c} \tag{1b}$$

The second and third right-hand terms:

$$\Gamma^e_{ac}\nabla_b V_e = \Gamma^e_{ac}\left(\frac{\partial V_e}{\partial x^b} - \Gamma^d_{be}V_d\right) \tag{1c}$$

$$\Gamma^e_{bc}\nabla_e V_a = \Gamma^e_{bc}\left(\frac{\partial V_a}{\partial x^e} - \Gamma^d_{ae}V_d\right) \tag{1d}$$

Putting the three terms (1b, 1c, 1d) together in (1) gives:

$$\nabla_c\nabla_b V_a = \frac{\partial^2 V_a}{\partial x^c \partial x^b} - \Gamma^d_{ab}\frac{\partial V_d}{\partial x^c} - V_d\frac{\partial \Gamma^d_{ab}}{\partial x^c} - \Gamma^e_{ac}\left(\frac{\partial V_e}{\partial x^b} - \Gamma^d_{be}V_d\right) - \Gamma^e_{bc}\left(\frac{\partial V_a}{\partial x^e} - \Gamma^d_{ae}V_d\right) \tag{1e}$$

By interchanging b and c we find:

$$\nabla_b\nabla_c V_a = \frac{\partial^2 V_a}{\partial x^b \partial x^c} - \Gamma^d_{ac}\frac{\partial V_d}{\partial x^b} - V_d\frac{\partial \Gamma^d_{ac}}{\partial x^b} - \Gamma^e_{ab}\left(\frac{\partial V_e}{\partial x^c} - \Gamma^d_{ce}V_d\right) - \Gamma^e_{cb}\left(\frac{\partial V_a}{\partial x^e} - \Gamma^d_{ae}V_d\right) \tag{2}$$

Subtract (1e)-(2), the first term and the last term compensate each other; as we know the Christoffel symbol is symmetric relative to the lower indices, therefore the result is:

$$\nabla_c\nabla_b V_a - \nabla_b\nabla_c V_a = -\Gamma^d_{ab}\frac{\partial V_d}{\partial x^c} - V_d\frac{\partial \Gamma^d_{ab}}{\partial x^c} - \Gamma^e_{ac}\left(\frac{\partial V_e}{\partial x^b} - \Gamma^d_{be}V_d\right) + \Gamma^d_{ac}\frac{\partial V_d}{\partial x^b} + V_d\frac{\partial \Gamma^d_{ac}}{\partial x^b} + \Gamma^e_{ab}\left(\frac{\partial V_e}{\partial x^c} - \Gamma^d_{ce}V_d\right)$$

Multiplying out the brackets in the last terms and factorizing out the terms with V_d

$$\nabla_c\nabla_b V_a - \nabla_b\nabla_c V_a = -\Gamma^d_{ab}\frac{\partial V_d}{\partial x^c} - V_d\frac{\partial \Gamma^d_{ab}}{\partial x^c} - \Gamma^e_{ac}\frac{\partial V_e}{\partial x^b} + \Gamma^e_{ac}\Gamma^d_{be}V_d + \Gamma^d_{ac}\frac{\partial V_d}{\partial x^b} + V_d\frac{\partial \Gamma^d_{ac}}{\partial x^b} + \Gamma^e_{ab}\frac{\partial V_e}{\partial x^c} - \Gamma^e_{ab}\Gamma^d_{ce}V_d$$

$$= \Gamma^d_{ac}\frac{\partial V_d}{\partial x^b} - \Gamma^d_{ab}\frac{\partial V_d}{\partial x^c} + \Gamma^e_{ab}\frac{\partial V_e}{\partial x^c} - \Gamma^e_{ac}\frac{\partial V_e}{\partial x^b} + \left(\frac{\partial \Gamma^d_{ac}}{\partial x^b} - \frac{\partial \Gamma^d_{ab}}{\partial x^c} + \Gamma^e_{ac}\Gamma^d_{be} - \Gamma^e_{ab}\Gamma^d_{ce}\right)V_d$$

From equation ([2.4.1 1](#)) in the previous chapter we know:

$$\frac{\partial e_i}{\partial x^j} = \Gamma^k_{ij}e_k \tag{3}$$

Therefore

$$\frac{\partial V_e}{\partial x^c} = \Gamma^d_{ec}V_d => \Gamma^e_{ab}\frac{\partial V_e}{\partial x^c} = \Gamma^e_{ab}\Gamma^d_{ec}V_d \ and\ \frac{\partial V_e}{\partial x^b} = \Gamma^d_{eb}V_d => \Gamma^e_{ac}\frac{\partial V_e}{\partial x^b} = \Gamma^e_{ac}\Gamma^d_{eb}V_d$$

$$\nabla_c\nabla_b V_a - \nabla_b\nabla_c V_a = \Gamma^d_{ac}\frac{\partial V_d}{\partial x^b} - \Gamma^d_{ab}\frac{\partial V_d}{\partial x^c} + \boxed{\cancel{\Gamma^e_{ab}\frac{\partial V_e}{\partial x^c}}} - \boxed{\cancel{\Gamma^e_{ac}\frac{\partial V_e}{\partial x^b}}} + \left(\frac{\partial \Gamma^d_{ac}}{\partial x^b} - \frac{\partial \Gamma^d_{ab}}{\partial x^c}\boxed{\cancel{+\Gamma^e_{ac}\Gamma^d_{be}}} - \boxed{\cancel{\Gamma^e_{ab}\Gamma^d_{ce}}}\right)V_d$$

$$\nabla_c\nabla_b V_a - \nabla_b\nabla_c V_a = \Gamma^d_{ac}\frac{\partial V_d}{\partial x^b} + V_d\frac{\partial \Gamma^d_{ac}}{\partial x^b} - \Gamma^d_{ab}\frac{\partial V_d}{\partial x^c} - V_d\frac{\partial \Gamma^d_{ab}}{\partial x^c}$$

After swapping d by e in the first and the third term on the right hand-side:

$$\nabla_c \nabla_b V_a - \nabla_b \nabla_c V_a = \Gamma^e_{ac} \frac{\partial V_e}{\partial x^b} + V_d \frac{\partial \Gamma^d_{\alpha c}}{\partial x^b} - \Gamma^e_{\alpha b} \frac{\partial V_e}{\partial x^c} - V_d \frac{\partial \Gamma^d_{\alpha b}}{\partial x^c} =$$

$$\nabla_c \nabla_b V_a - \nabla_b \nabla_c V_a = \Gamma^e_{ac} \Gamma^d_{eb} V_d + V_d \frac{\partial \Gamma^d_{\alpha c}}{\partial x^b} - \Gamma^e_{\alpha b} \Gamma^d_{ec} V_d - V_d \frac{\partial \Gamma^d_{\alpha b}}{\partial x^c} =$$

$$\nabla_c \nabla_b V_a - \nabla_b \nabla_c V_a = \left(\frac{\partial \Gamma^d_{\alpha c}}{\partial x^b} - \frac{\partial \Gamma^d_{\alpha b}}{\partial x^c} + \Gamma^e_{\alpha c} \Gamma^d_{be} - \Gamma^e_{\alpha b} \Gamma^d_{ce} \right) V_d$$

We define the expression inside the brackets on the right-hand side to be **the Riemann tensor**, meaning

$$\boldsymbol{\nabla_c \nabla_b V_a - \nabla_b \nabla_c V_a = R^d_{abc} V_d}$$

$$\boldsymbol{R^d_{abc} = \frac{\partial \Gamma^d_{\alpha c}}{\partial x^b} - \frac{\partial \Gamma^d_{\alpha b}}{\partial x^c} + \Gamma^e_{\alpha c} \Gamma^d_{be} - \Gamma^e_{\alpha b} \Gamma^d_{ce}}$$

$$\boldsymbol{R^d_{abc} = \Gamma^d_{\alpha c,b} - \Gamma^d_{\alpha b,c} + \Gamma^e_{\alpha c} \Gamma^d_{be} - \Gamma^e_{\alpha b} \Gamma^d_{ce}}$$

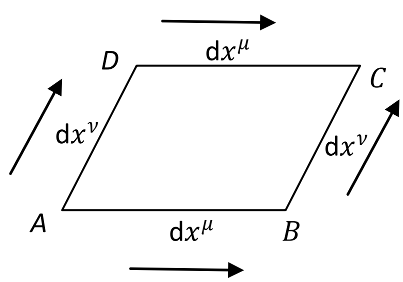
Note:
So here the commutator could be considered as the subtraction of two vectors. The magnitude of the resulting vector is the Riemann tensor.

2.9.1.3. *Alternative Derivation of the Riemann tensor via commutator*

We consider an infinitesimal area over which a vector is moved (parallel transported) via two different paths. When the manifold is flat the difference between the two end vectors would be zero. However in case the manifold is intrinsically curved this would lead to a difference between the end vectors.

First we move a vector $\overrightarrow{V}$ from point A via B to C. To find the direction of the movement of the vector we take the derivative of the vector with respect to dx^μ and next we look at the change of this result with respect to dx^ν .

Next we do the same from A via D to C, now first with respect to dx^ν and next to dx^μ. Then we subtract both results from each other which should lead to the Riemann tensor.



$$\overrightarrow{V} = V^m \cdot \vec{e}_m$$

The vector e_m is the tangent vector, so the derivative of the position vector or the derivative of the trajectory. In case the trajectory is a straight line then the derivative of e_m is a constant; and consequently the derivative of e_m , and thus the Christoffel symbol, is zero.

First from A to B to find the direction we take the derivative (see also 3):

$$\frac{\partial \overrightarrow{V}}{\partial x^\mu} = \frac{\partial V^m}{\partial x^\mu} \cdot \vec{e}_m + V^m \frac{\partial \vec{e}_m}{\partial x^\mu} = \frac{\partial V^m}{\partial x^\mu} \cdot \vec{e}_m + V^m \Gamma^k_{m\mu} \vec{e}_k$$

Change the two dummy indices, *k* and *m*. So the formula can be changed by *k* to m, and *m* to γ.

$$\frac{\partial \vec{V}}{\partial x^\mu} = \frac{\partial V^m}{\partial x^\mu} \vec{e}_m + V^\gamma \Gamma^m_{\gamma\mu} \vec{e}_m = \left(\frac{\partial V^m}{\partial x^\mu} + V^\gamma \Gamma^m_{\gamma\mu} \right) \vec{e}_m$$

Which is the covariant derivative of the contra-variant vector $\vec{V}$. And from the definition of the Christoffel symbol in the previous chapters we know that $\frac{\partial \vec{e}_m}{\partial x^\mu} = \Gamma^k_{m\mu} \vec{e}_k$

Next the change of the direction from B to C with respect to dx^ν:

$$\frac{\partial^2 \vec{V}}{\partial x^\nu \, \partial x^\mu} = \frac{\partial^2 V^m}{\partial x^\nu \, \partial x^\mu} \vec{e}_m + \frac{\partial V^m}{\partial x^\mu} \frac{\vec{e}_m}{\partial x^\nu} + \frac{\partial V^\gamma}{\partial x^\nu} \Gamma^m_{\gamma\mu} \vec{e}_m + V^\gamma \frac{\partial \Gamma^m_{\gamma\mu}}{\partial x^\nu} \vec{e}_m + V^\gamma \Gamma^m_{\gamma\mu} \frac{\partial \vec{e}_m}{\partial x^\nu}$$

$$\frac{\partial^2 \vec{V}}{\partial x^\nu \, \partial x^\mu} = \frac{\partial^2 V^m}{\partial x^\nu \, \partial x^\mu} \vec{e}_m + \frac{\partial V^m}{\partial x^\mu} \Gamma^k_{m\nu} \vec{e}_k + \frac{\partial V^\gamma}{\partial x^\nu} \Gamma^m_{\gamma\mu} \vec{e}_m + V^\gamma \frac{\partial \Gamma^m_{\gamma\mu}}{\partial x^\nu} \vec{e}_m + V^\gamma \Gamma^m_{\gamma\mu} \Gamma^k_{m\nu} \vec{e}_k$$

Replace in the right hand side, of the equation, the second term the indices *k* with *m* and *m* with γ and in the fifth term swap the *k* and *m*:

$$\frac{\partial^2 \vec{V}}{\partial x^\nu \, \partial x^\mu} = \frac{\partial^2 V^m}{\partial x^\nu \, \partial x^\mu} \vec{e}_m + \frac{\partial V^\gamma}{\partial x^\mu} \Gamma^m_{\gamma\nu} \vec{e}_m + \frac{\partial V^\gamma}{\partial x^\nu} \Gamma^m_{\gamma\mu} \vec{e}_m + V^\gamma \frac{\partial \Gamma^m_{\gamma\mu}}{\partial x^\nu} \vec{e}_m + V^\gamma \Gamma^k_{\gamma\mu} \Gamma^m_{k\nu} \vec{e}_m$$

$$\frac{\partial^2 \vec{V}}{\partial x^\nu \, \partial x^\mu} = \frac{\partial \Gamma^m_{\gamma\mu}}{\partial x^\nu} V^\gamma \vec{e}_m + \Gamma^k_{\gamma\mu} \Gamma^m_{k\nu} V^\gamma \vec{e}_m + \frac{\partial^2 V^m}{\partial x^\nu \, \partial x^\mu} \vec{e}_m + \frac{\partial V^\gamma}{\partial x^\mu} \Gamma^m_{\gamma\nu} \vec{e}_m + \frac{\partial V^\gamma}{\partial x^\nu} \Gamma^m_{\gamma\mu} \vec{e}_m$$

Now for the other direction just swap μ and ν

$$\frac{\partial^2 \vec{V}}{\partial x^\mu \, \partial x^\nu} = \frac{\partial \Gamma^m_{\gamma\nu}}{\partial x^\mu} V^\gamma \vec{e}_m + \Gamma^k_{\gamma\nu} \Gamma^m_{k\mu} V^\gamma \vec{e}_m + \frac{\partial^2 V^m}{\partial x^\mu \, \partial x^\nu} \vec{e}_m + \frac{\partial V^\gamma}{\partial x^\nu} \Gamma^m_{\gamma\mu} \vec{e}_m + \frac{\partial V^\gamma}{\partial x^\mu} \Gamma^m_{\gamma\nu} \vec{e}_m$$

Now subtract the last two equations:

$$\frac{\partial^2 \vec{V}}{\partial x^\mu \, \partial x^\nu} - \frac{\partial^2 \vec{V}}{\partial x^\nu \, \partial x^\mu} =$$

$$= \frac{\partial \Gamma^m_{\gamma\nu}}{\partial x^\mu} V^\gamma \vec{e}_m - \frac{\partial \Gamma^m_{\gamma\mu}}{\partial x^\nu} V^\gamma \vec{e}_m + \Gamma^k_{\gamma\nu} \Gamma^m_{k\mu} V^\gamma \vec{e}_m - \Gamma^k_{\gamma\mu} \Gamma^m_{k\nu} V^\gamma \vec{e}_m + \frac{\partial^2 V^m}{\partial x^\mu \, \partial x^\nu} \vec{e}_m - \frac{\partial^2 V^m}{\partial x^\nu \, \partial x^\mu} \vec{e}_m + \frac{\partial V^\gamma}{\partial x^\nu} \Gamma^m_{\gamma\mu} \vec{e}_m$$
$$- \frac{\partial V^\gamma}{\partial x^\nu} \Gamma^m_{\gamma\mu} \vec{e}_m + \frac{\partial V^\gamma}{\partial x^\mu} \Gamma^m_{\gamma\nu} \vec{e}_m - \frac{\partial V^\gamma}{\partial x^\mu} \Gamma^m_{\gamma\nu} \vec{e}_m$$
$$= \frac{\partial \Gamma^m_{\gamma\nu}}{\partial x^\mu} V^\gamma \vec{e}_m - \frac{\partial \Gamma^m_{\gamma\mu}}{\partial x^\nu} V^\gamma \vec{e}_m + \Gamma^k_{\gamma\nu} \Gamma^m_{k\mu} V^\gamma \vec{e}_m - \Gamma^k_{\gamma\mu} \Gamma^m_{k\nu} V^\gamma \vec{e}_m$$

$$=> \frac{\partial^2 \vec{V}}{\partial x^\mu\, \partial x^\nu} - \frac{\partial^2 \vec{V}}{\partial x^\nu\, \partial x^\mu} = \left(\frac{\boldsymbol{\partial \Gamma^m_{\gamma\nu}}}{\boldsymbol{\partial x^\mu}} - \frac{\boldsymbol{\partial \Gamma^m_{\gamma\mu}}}{\boldsymbol{\partial x^\nu}} + \boldsymbol{\Gamma^k_{\gamma\nu}\Gamma^m_{k\mu}} - \boldsymbol{\Gamma^k_{\gamma\mu}\Gamma^m_{k\nu}} \right) V^\gamma \vec{e}_m$$

$$\frac{\partial^2 \vec{V}}{\partial x^\mu\, \partial x^\nu} - \frac{\partial^2 \vec{V}}{\partial x^\nu\, \partial x^\mu} = \boldsymbol{R^m_{\gamma\mu\nu}} V^\gamma \vec{e}_m$$

So the Riemann tensor is:

$$\boldsymbol{R^m_{\gamma\mu\nu}} = \frac{\boldsymbol{\partial \Gamma^m_{\gamma\nu}}}{\boldsymbol{\partial x^\mu}} - \frac{\boldsymbol{\partial \Gamma^m_{\gamma\mu}}}{\boldsymbol{\partial x^\nu}} + \boldsymbol{\Gamma^k_{\gamma\nu}\Gamma^m_{k\mu}} - \boldsymbol{\Gamma^k_{\gamma\mu}\Gamma^m_{k\nu}}$$

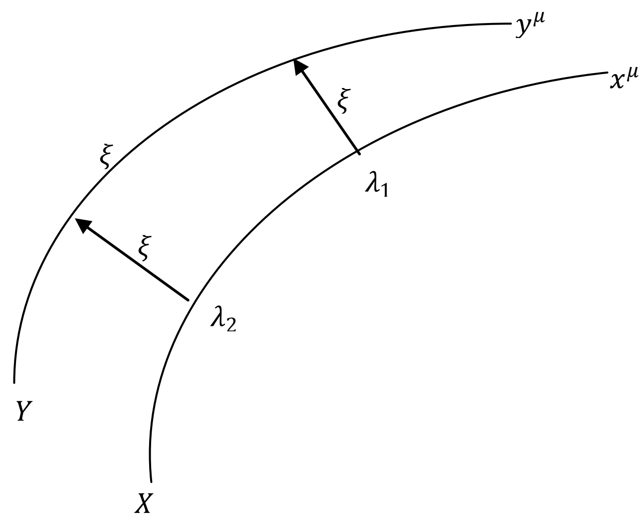
2.9.2. Derivation of the Riemann tensor from the Geodesic Deviation

In the previous chapter we have shown a way to derive the Riemann tensor from the covariant derivative commutator, which physically corresponds to the difference of parallel transporting a vector first in one way and then the other. Another interpretation is in terms of relative acceleration of nearby particles in free-fall. Imagine a cloud of particles in free fall. Let us suppose that an observer is travelling with one of the particles. He looks at a nearby particle and measures its position in local inertial coordinates. In special relativity, it will move in a straight line at constant speed with no acceleration. But what happens in a gravitational field?

As we recall from the previous chapter, a geodesic generalizes the notion of a “straight line” to curved space-time.

Here we will show how the evolution of the separation measured between two adjacent geodesics, also known as geodesic deviation, can indeed be related to a non-zero curvature of the space-time, or to use a Newtonian term, to the presence of *tidal force*. So let us pick out two particles following two very close geodesics.

Their respective path could be described by the functions $x^\mu(\tau)$ (for the reference particle) and $y^\mu(\tau) \equiv x^\mu(\tau) + \xi^\mu(\tau)$ (for the second particle) where τ (tau) is the proper time along the reference particle’s world-line and where ξ refers to the deviation four-vector joining one particle to the other at each given time τ.



The relative acceleration A^α of the two objects is defined roughly as the second derivative of the separation vector ξ^α as the objects advance along their respective geodesics.

Our goal in this chapter is to show that this relative acceleration is related to the Riemann tensor by the following equation

$$\left(\frac{d^2\xi}{d\tau^2}\right)^\alpha = -R^\alpha_{\mu\sigma\nu}\, u^\nu u^\mu \xi^\sigma$$

In case space-time is flat, the Riemann tensor is zero which results in a null relative acceleration.

As each particle follows a geodesic, the equation of their respective coordinate is (see 2.5 **1**):

$$0 = \frac{d^2x^\alpha}{d\tau^2} + \Gamma^\alpha_{\mu\nu}\left(x^\alpha(\tau)\right)\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}$$

$$0 = \frac{d^2y^\alpha}{d\tau^2} + \Gamma^\alpha_{\mu\nu}\left(y^\alpha(\tau)\right)\frac{dy^\mu}{d\tau}\frac{dy^\nu}{d\tau}$$

In each of these equations, the Christoffel symbol is equivalent at each particle's x and y respective position. As the separation among particles is infinitesimal, we therefore evaluate Christoffel symbol at $y^\alpha(\tau)$ position by a Taylor series development

$$f(x) = f(a) + \frac{f^{'}(a)}{1!}(x-a) + \frac{f^{''}(a)}{2!}(x-a)^2 \ldots .\frac{f^n(a)}{n!}(x-a)^n$$

Approximation till the first derivative because ξ is infinitesimal.

$$\Gamma^\alpha_{\mu\nu}\left(y^\alpha(\tau)\right) \approx \Gamma^\alpha_{\mu\nu}\left(x^\alpha(\tau)\right) + \xi^\sigma\left[\partial_\sigma \Gamma^\alpha_{\mu\nu}\right]\left(x^\alpha(\tau)\right)$$

==

This could also be approximated as follows for infinitesimal Δx:

$$\frac{d\Gamma^\alpha_{\mu\nu}(x)}{dx} = \frac{\Gamma^\alpha_{\mu\nu}(x+\Delta x) - \Gamma^\alpha_{\mu\nu}(x)}{\Delta x}$$

$$\Gamma^{\alpha}_{\mu\nu}(x+\Delta x) = \Gamma^{\alpha}_{\mu\nu}(x) + \Delta x \frac{d\Gamma^{\alpha}_{\mu\nu}(x)}{dx}$$
$$\Delta x = \xi$$
$$\Gamma^{\alpha}_{\mu\nu}(x+\xi) = \Gamma^{\alpha}_{\mu\nu}(x) + \xi \frac{d\Gamma^{\alpha}_{\mu\nu}(x)}{dx}$$

==
=

With the assumption that $y^{\alpha}(\tau) = x^{\alpha}(\tau) + \xi^{\alpha}(\tau)$ and by replacing this last expression in the *y* particle's geodesic equation we get

$$0 = \frac{d^2 y^{\alpha}}{d\tau^2} + \Gamma^{\alpha}_{\mu\nu}\left(y^{\alpha}(\tau)\right)\frac{dy^{\mu}}{d\tau}\frac{dy^{\nu}}{d\tau}$$

$$0 = \frac{d^2(x^{\alpha}+\xi^{\alpha})}{d\tau^2} + \left[\Gamma^{\alpha}_{\mu\nu} + \xi^{\sigma}\left(\partial_{\sigma}\Gamma^{\alpha}_{\mu\nu}\right)\right]\frac{d(x^{\mu}+\xi^{\mu})}{d\tau}\frac{d(x^{\nu}+\xi^{\nu})}{d\tau}$$

$$0 = \frac{d^2 x^{\alpha}}{d\tau^2} + \frac{d^2 \xi^{\alpha}}{d\tau^2} + \left[\Gamma^{\alpha}_{\mu\nu} + \xi^{\sigma}\left(\partial_{\sigma}\Gamma^{\alpha}_{\mu\nu}\right)\right]\left(\frac{dx^{\mu}}{d\tau} + \frac{d\xi^{\mu}}{d\tau}\right)\left(\frac{dx^{\nu}}{d\tau} + \frac{d\xi^{\nu}}{d\tau}\right)$$

Where the Christoffel Symbol and its first order derivatives are now evaluated in $x^{\alpha}(\tau)$

By developing all the terms in parenthesis, and cancelling out those terms in second order with respect to ξ we get

$$0 = \frac{d^2 x^{\alpha}}{d\tau^2} + \frac{d^2 \xi^{\alpha}}{d\tau^2} + \Gamma^{\alpha}_{\mu\nu}\left(\frac{dx^{\mu}}{d\tau}\frac{dx^{\nu}}{d\tau} + \frac{dx^{\mu}}{d\tau}\frac{d\xi^{\nu}}{d\tau} + \frac{d\xi^{\mu}}{d\tau}\frac{dx^{\nu}}{d\tau} + \cancel{\frac{d\xi^{\mu}}{d\tau}\frac{d\xi^{\nu}}{d\tau}}\right) +$$

$$+\xi^{\sigma}\left(\partial_{\sigma}\Gamma^{\alpha}_{\mu\nu}\right)\left(\frac{dx^{\mu}}{d\tau}\frac{dx^{\nu}}{d\tau} + \cancel{\frac{dx^{\mu}}{d\tau}\frac{d\xi^{\nu}}{d\tau}} + \cancel{\frac{d\xi^{\mu}}{d\tau}\frac{dx^{\nu}}{d\tau}} + \cancel{\frac{d\xi^{\mu}}{d\tau}\frac{d\xi^{\nu}}{d\tau}}\right)$$

As mentioned before, we know the Christoffel symbol is symmetric relative to the lower indices, thus these lower indices may be swapped.

$$0 = \frac{d^2 x^{\alpha}}{d\tau^2} + \frac{d^2 \xi^{\alpha}}{d\tau^2} + \Gamma^{\alpha}_{\mu\nu}\left(\frac{dx^{\mu}}{d\tau}\frac{dx^{\nu}}{d\tau} + 2\frac{dx^{\mu}}{d\tau}\frac{d\xi^{\nu}}{d\tau}\right) + \xi^{\sigma}\left(\partial_{\sigma}\Gamma^{\alpha}_{\mu\nu}\right)\frac{dx^{\mu}}{d\tau}\frac{dx^{\nu}}{d\tau}$$

By using the geodesic equation of particle *x* (see 2.51):

$$\frac{d^2 x^{\alpha}}{d\tau^2} = -\Gamma^{\alpha}_{\mu\nu}\frac{dx^{\mu}}{d\tau}\frac{dx^{\nu}}{d\tau}$$

Then the first and third term cancel. We finally can write:

$$0 = \frac{d^2 \xi^{\alpha}}{d\tau^2} + 2\Gamma^{\alpha}_{\mu\nu}u^{\mu}\frac{d\xi^{\nu}}{d\tau} + \xi^{\sigma}\left(\partial_{\sigma}\Gamma^{\alpha}_{\mu\nu}\right)u^{\mu}u^{\nu}$$

$$\frac{d^2 \xi^{\alpha}}{d\tau^2} = -2\Gamma^{\alpha}_{\mu\nu}u^{\mu}\frac{d\xi^{\nu}}{d\tau} - \xi^{\sigma}\left(\partial_{\sigma}\Gamma^{\alpha}_{\mu\nu}\right)u^{\mu}u^{\nu}$$

Here $u^{\mu} = \frac{dx^{\mu}}{d\tau}$ is the four-velocity vector of the reference particle.

We now have an expression for $\frac{d\xi^\alpha}{d\tau}$, but this isn't the total derivative of the four-vector ξ, since its derivative could also get a contribution from the change of the basis vectors as the object moves along its geodesic. To get the total derivative, we have

$$\frac{d\xi}{d\tau} = \frac{d}{d\tau}(\xi^\alpha \boldsymbol{e}_\alpha) = \frac{d\xi^\alpha}{d\tau}\boldsymbol{e}_\alpha + \xi^\alpha \frac{d\boldsymbol{e}_\alpha}{d\tau} = \frac{d\xi^\alpha}{d\tau}\boldsymbol{e}_\alpha + \xi^\alpha \frac{dx^\mu}{d\tau}\frac{d\boldsymbol{e}_\alpha}{dx^\mu}$$

By replacing the dummy index α by σ in the second term and from the definition of the Christoffel Symbol we get

$$\xi^\sigma \frac{dx^\mu}{d\tau}\frac{d\boldsymbol{e}_\sigma}{dx^\mu} = \xi^\sigma \frac{dx^\mu}{d\tau}\Gamma^\alpha_{\mu\sigma}\boldsymbol{e}_\alpha = \xi^\sigma u^\mu \Gamma^\alpha_{\mu\sigma}\boldsymbol{e}_\alpha$$

$$\frac{d\xi}{d\tau} = \frac{d\xi^\alpha}{d\tau}\boldsymbol{e}_\alpha + \xi^\sigma u^\mu \Gamma^\alpha_{\mu\sigma}\boldsymbol{e}_\alpha = \left(\frac{d\xi^\alpha}{d\tau} + \Gamma^\alpha_{\mu\sigma}\xi^\sigma u^\mu\right)\boldsymbol{e}_\alpha$$

So that

$$\left(\frac{d\xi}{d\tau}\right)^\alpha = \frac{d\xi^\alpha}{d\tau} + \Gamma^\alpha_{\mu\sigma}\xi^\sigma u^\mu$$

Since we're still dealing with the condition that ξ is a four-vector its derivative with respect to proper time is also a four-vector, so we can find the second absolute derivative by using the same development as for the first order derivative.

$$\left(\frac{d}{d\tau}\left[\frac{d\xi}{d\tau}\right]\right)^\alpha = \frac{d}{d\tau}\left(\left[\frac{d\xi}{d\tau}\right]^\alpha\right) + \Gamma^\alpha_{\mu\sigma}\left(\frac{d\xi}{d\tau}\right)^\sigma u^\mu$$

$$\left(\frac{d^2\xi}{d\tau^2}\right)^\alpha = \left(\frac{d}{d\tau}\left[\frac{d\xi}{d\tau}\right]\right)^\alpha = \frac{d}{d\tau}\left(\frac{d\xi^\alpha}{d\tau} + \Gamma^\alpha_{\mu\sigma}u^\mu\xi^\sigma\right) + \Gamma^\alpha_{\mu\sigma}u^\mu\left(\frac{d\xi^\sigma}{d\tau} + \Gamma^\sigma_{\beta\gamma}u^\beta\xi^\gamma\right)$$

$$= \frac{d^2\xi^\alpha}{d\tau^2} + \frac{d\Gamma^\alpha_{\mu\sigma}}{d\tau}u^\mu\xi^\sigma + \Gamma^\alpha_{\mu\sigma}\frac{du^\mu}{d\tau}\xi^\sigma + \Gamma^\alpha_{\mu\sigma}u^\mu\frac{d\xi^\sigma}{d\tau} + \Gamma^\alpha_{\mu\sigma}u^\mu\frac{d\xi^\sigma}{d\tau} + \Gamma^\alpha_{\mu\sigma}\Gamma^\sigma_{\beta\gamma}u^\mu u^\beta\xi^\gamma$$

By using the equation of Christoffel symbol Taylor's series above and replacing ν *by* σ in the **first term**, we get

$$\frac{d^2\xi^\alpha}{d\tau^2} = -2\Gamma^\alpha_{\mu\nu}u^\mu\frac{d\xi^\nu}{d\tau} - \left(\frac{d\Gamma^\alpha_{\mu\nu}}{dx^\sigma}\right)u^\mu u^\nu\xi^\sigma$$

Interchange in the first term, on the right hand side, ν and σ

$$\frac{d^2\xi^\alpha}{d\tau^2} = -2\Gamma^\alpha_{\mu\sigma}u^\mu\frac{d\xi^\sigma}{d\tau} - \left(\frac{d\Gamma^\alpha_{\mu\nu}}{dx^\sigma}\right)u^\mu u^\nu\xi^\sigma$$

We can rewrite the **second term** as the Christoffel symbols depends on τ by depending on the position of the reference particle

$$=> \frac{d\Gamma^\alpha_{\mu\sigma}}{d\tau}u^\mu\xi^\sigma = \frac{d\Gamma^\alpha_{\mu\sigma}}{dx^\nu}\frac{dx^\nu}{d\tau}u^\mu\xi^\sigma = \frac{d\Gamma^\alpha_{\mu\sigma}}{dx^\nu}u^\nu u^\mu\xi^\sigma$$

By using the geodesic equation, we can rewrite the **third term**, i.e. working out $\frac{du^\mu}{d\tau}$

$$u^\mu = \frac{dx^\mu}{d\tau}$$
$$\frac{du^\mu}{d\tau} = \frac{d^2x^\mu}{d\tau^2}$$
$$geodesic: \quad \frac{d^2x^\mu}{d\tau^2} = -\Gamma^\mu_{\nu\gamma}\frac{dx^\nu}{d\tau}\frac{dx^\gamma}{d\tau} = -\Gamma^\mu_{\nu\gamma}u^\nu u^\gamma = \frac{du^\mu}{d\tau}$$
$$=> \Gamma^\alpha_{\mu\sigma}\frac{du^\mu}{d\tau}\xi^\sigma = -\Gamma^\alpha_{\mu\sigma}\Gamma^\mu_{\nu\gamma}u^\nu u^\gamma \xi^\sigma$$

Interchange, in the right hand term, μ and γ

$$\Gamma^\alpha_{\mu\sigma}\frac{du^\mu}{d\tau}\xi^\sigma = -\Gamma^\alpha_{\gamma\sigma}\Gamma^\gamma_{\nu\mu}u^\nu u^\mu \xi^\sigma$$

Also in order to obtain an expression $u^\nu u^\mu \xi^\sigma$, so with μ, ν and σ only, we can rewrite the **last term** by renaming the dummy indices σ and β

$$\Gamma^\alpha_{\mu\sigma}\Gamma^\sigma_{\beta\gamma}u^\mu u^\beta \xi^\gamma =$$
$$(\sigma \leftrightarrow \gamma) = \Gamma^\alpha_{\mu\gamma}\Gamma^\gamma_{\beta\sigma}u^\mu u^\beta \xi^\sigma$$
$$(\beta \leftrightarrow \nu) = \Gamma^\alpha_{\mu\gamma}\Gamma^\gamma_{\nu\sigma}u^\mu u^\nu \xi^\sigma$$
$$(\mu \leftrightarrow \nu) = \Gamma^\alpha_{\nu\gamma}\Gamma^\gamma_{\mu\sigma}u^\nu u^\mu \xi^\sigma$$

So finally we can write, replacing all the terms

$$\left(\frac{d^2\xi}{d\tau^2}\right)^\alpha = \frac{d^2\xi^\alpha}{d\tau^2} + \frac{d\Gamma^\alpha_{\mu\sigma}}{d\tau}u^\mu\xi^\sigma + \Gamma^\alpha_{\mu\sigma}\frac{du^\mu}{d\tau}\xi^\sigma + \left(\Gamma^\alpha_{\mu\sigma}u^\mu\frac{d\xi^\sigma}{d\tau} + \Gamma^\alpha_{\mu\sigma}u^\mu\frac{d\xi^\sigma}{d\tau}\right) + \Gamma^\alpha_{\mu\sigma}\Gamma^\sigma_{\beta\gamma}u^\mu u^\beta\xi^\gamma$$

$$= \cancel{-2\Gamma^\alpha_{\mu\sigma}u^\mu\frac{d\xi^\sigma}{d\tau}} - \left(\frac{d\Gamma^\alpha_{\mu\nu}}{dx^\sigma}\right)u^\mu u^\nu\xi^\sigma + \frac{d\Gamma^\alpha_{\mu\sigma}}{dx^\nu}u^\nu u^\mu\xi^\sigma - \Gamma^\alpha_{\gamma\sigma}\Gamma^\gamma_{\nu\mu}u^\nu u^\mu\xi^\sigma + \cancel{2\Gamma^\alpha_{\mu\sigma}u^\mu\frac{d\xi^\sigma}{d\tau}} + \Gamma^\alpha_{\nu\gamma}\Gamma^\gamma_{\mu\sigma}u^\nu u^\mu\xi^\sigma$$

By cancelling out the first and the fifth terms and taking out the common factor $u^\nu u^\mu \xi^\sigma$

$$\left(\frac{d^2\xi}{d\tau^2}\right)^\alpha = -\left(\frac{d\Gamma^\alpha_{\mu\nu}}{dx^\sigma} - \frac{d\Gamma^\alpha_{\mu\sigma}}{dx^\nu} + \Gamma^\alpha_{\gamma\sigma}\Gamma^\gamma_{\nu\mu} - \Gamma^\alpha_{\nu\gamma}\Gamma^\gamma_{\mu\sigma}\right)u^\nu u^\mu\xi^\sigma$$

Since this is still a tensor equation, the quantity in the brackets is a tensor and we can define the ***Riemann tensor*** as

$$\boxed{\boldsymbol{R^\alpha_{\mu\sigma\nu} = \left(\frac{d\Gamma^\alpha_{\mu\nu}}{dx^\sigma} - \frac{d\Gamma^\alpha_{\mu\sigma}}{dx^\nu} + \Gamma^\alpha_{\sigma\gamma}\Gamma^\gamma_{\mu\nu} - \Gamma^\alpha_{\nu\gamma}\Gamma^\gamma_{\mu\sigma}\right)}}$$

Then we can rewrite the above equation in a shorter expression, known as the ***geodesic deviation equation***

$$\boldsymbol{\left(\frac{d^2\xi}{d\tau^2}\right)^\alpha = -R^\alpha_{\mu\sigma\nu}u^\nu u^\mu\xi^\sigma}$$

Since the only quantity in this equation that depends intrinsically on the metric is the Riemann tensor, we see that if it is identically zero, space-time is flat, but if only one component of this tensor is non-zero, space-time is curved.

Summary:

So for a geodesic line goes the following property:

$$0 = \frac{d^2x^\beta}{d\tau^2} + \Gamma^\beta_{\mu\nu}\frac{\partial x^\mu}{\partial\tau}\frac{\partial x^\nu}{\partial\tau} \quad \textit{Geodesic equation}$$

Or

$$\frac{d^2x^\beta}{d\tau^2} = -\boldsymbol{\Gamma^\beta_{\mu\nu}} u^\nu u^\mu$$

While for the deviation from one geodesic to an infinitesimal near geodesic line goes:

$$\left(\frac{d^2\xi}{d\tau^2}\right)^\alpha = -\boldsymbol{R^\alpha_{\mu\sigma\nu}} u^\nu u^\mu \xi^\sigma \quad \textit{Geodesic deviation equation}$$

2.10. Symmetries and Independent Components

In the previous chapters, the rather complicated expression of the Riemann curvature tensor has been derived; a mixture of derivatives and products of Christoffel symbols, with 256 (=4^4) components in four dimensional space-time. In this chapter we will demonstrate that the Riemann tensor has only 20 independent components and that the components are precisely a combination of these non zero second derivatives.
The methodology here is to study the Riemann tensor symmetries in a Local Inertial Frame, where all Christoffel symbols are zero. To generalize these symmetries to any reference frame, as by definition a tensor equation valid in a given referential is valid in any referential frame.

Riemann tensor symmetries lead to Riemann tensor independent components:
Using the definition of the Riemann tensor as seen in the previous chapters:

$$R^\alpha_{\beta\mu\nu} \equiv \frac{d\Gamma^\alpha_{\beta\nu}}{dx^\mu} - \frac{d\Gamma^\alpha_{\beta\mu}}{dx^\nu} + \Gamma^\alpha_{\mu\gamma}\Gamma^\gamma_{\beta\nu} - \Gamma^\alpha_{\nu\gamma}\Gamma^\gamma_{\beta\mu}$$

And knowing that all the Christoffel symbols are zero at the origin of the Local Inertial Frame, this expression get simplified to:

$$R^\alpha_{\beta\mu\nu} \equiv \frac{d\Gamma^\alpha_{\beta\nu}}{dx^\mu} - \frac{d\Gamma^\alpha_{\beta\mu}}{dx^\nu}$$

By applying the contraction mechanism we can rewrite the Riemann tensor with all indices lowered:

$$R_{\alpha\beta\mu\nu} \equiv g_{\alpha\sigma}R^\sigma_{\beta\mu\nu} \equiv g_{\alpha\sigma}\left[\frac{d\Gamma^\sigma_{\beta\nu}}{dx^\mu} - \frac{d\Gamma^\sigma_{\beta\mu}}{dx^\nu}\right]$$

As we know we can write the Christoffel symbol with respect to the metric derivatives:

$$\Gamma^\sigma_{\beta\nu} = \frac{1}{2}g^{\sigma\gamma}\left(\frac{\partial g_{\nu\gamma}}{\partial \mathrm{x}^\beta} + \frac{\partial g_{\gamma\beta}}{\partial \mathrm{x}^\nu} - \frac{\partial g_{\beta\nu}}{\partial \mathrm{x}^\gamma}\right)$$

So that we can write

$$g_{\alpha\sigma}\frac{d\Gamma^\sigma_{\beta\nu}}{dx^\mu} = \frac{1}{2}g_{\alpha\sigma}g^{\sigma\gamma}\left(\frac{\partial}{\partial x^\mu}\frac{\partial g_{\nu\gamma}}{\partial \mathrm{x}^\beta} + \frac{\partial}{\partial x^\mu}\frac{\partial g_{\gamma\beta}}{\partial \mathrm{x}^\nu} - \frac{\partial}{\partial x^\mu}\frac{\partial g_{\beta\nu}}{\partial \mathrm{x}^\gamma}\right) + \frac{1}{2}g_{\alpha\sigma}\frac{\partial g^{\sigma\gamma}}{\partial x^\mu}\left(\frac{\partial g_{\nu\gamma}}{\partial \mathrm{x}^\beta} + \frac{\partial g_{\gamma\beta}}{\partial \mathrm{x}^\nu} - \frac{\partial g_{\beta\nu}}{\partial \mathrm{x}^\gamma}\right)$$

The **second term** is zero because the Christoffel symbols are zero at the origin of the local inertial frame, as is mentioned above:

$$\frac{1}{2}g_{\alpha\sigma}\frac{\partial g^{\sigma\gamma}}{\partial x^{\mu}}\left(\frac{\partial g_{\nu\gamma}}{\partial \mathrm{x}^{\beta}}+\frac{\partial g_{\gamma\beta}}{\partial \mathrm{x}^{\nu}}-\frac{\partial g_{\beta\nu}}{\partial \mathrm{x}^{\gamma}}\right)=g_{\alpha\sigma}\frac{\partial g^{\sigma\gamma}}{\partial x^{\mu}}g_{\sigma\gamma}\frac{1}{2}g^{\sigma\gamma}\left(\frac{\partial g_{\nu\gamma}}{\partial \mathrm{x}^{\beta}}+\frac{\partial g_{\gamma\beta}}{\partial \mathrm{x}^{\nu}}-\frac{\partial g_{\beta\nu}}{\partial \mathrm{x}^{\gamma}}\right)=g_{\alpha\sigma}\frac{\partial g^{\sigma\gamma}}{\partial x^{\mu}}g_{\sigma\gamma}\Gamma^{\sigma}_{\beta\nu}=0$$

Thus continuing:

$$g_{\alpha\sigma}\frac{d\Gamma^{\sigma}_{\beta\nu}}{dx^{\mu}}=\frac{1}{2}\delta^{\gamma}_{\alpha}\left(\frac{\partial}{\partial x^{\mu}}\frac{\partial g_{\nu\gamma}}{\partial \mathrm{x}^{\beta}}+\frac{\partial}{\partial x^{\mu}}\frac{\partial g_{\gamma\beta}}{\partial \mathrm{x}^{\nu}}-\frac{\partial}{\partial x^{\mu}}\frac{\partial g_{\beta\nu}}{\partial \mathrm{x}^{\gamma}}\right)$$

$$=\frac{1}{2}\left(\frac{\partial}{\partial x^{\mu}}\frac{\partial g_{\nu\alpha}}{\partial \mathrm{x}^{\beta}}+\frac{\partial}{\partial x^{\mu}}\frac{\partial g_{\alpha\beta}}{\partial \mathrm{x}^{\nu}}-\frac{\partial}{\partial x^{\mu}}\frac{\partial g_{\beta\nu}}{\partial \mathrm{x}^{\alpha}}\right)$$

Interchanging indices μ and ν, leads to the second term of the Riemann tensor expression:

$$g_{\alpha\sigma}\frac{d\Gamma^{\sigma}_{\beta\mu}}{dx^{\nu}}=\frac{1}{2}\left(\frac{\partial}{\partial x^{\nu}}\frac{\partial g_{\mu\alpha}}{\partial \mathrm{x}^{\beta}}+\frac{\partial}{\partial x^{\nu}}\frac{\partial g_{\alpha\beta}}{\partial \mathrm{x}^{\mu}}-\frac{\partial}{\partial x^{\nu}}\frac{\partial g_{\beta\mu}}{\partial \mathrm{x}^{\alpha}}\right)$$

The middle terms vanish after subtracting the last two expressions, resulting in:

$$R_{\alpha\beta\mu\nu}=g_{\alpha\sigma}\left[\frac{d\Gamma^{\sigma}_{\beta\nu}}{dx^{\mu}}-\frac{d\Gamma^{\sigma}_{\beta\mu}}{dx^{\nu}}\right]$$

$$R_{\alpha\beta\mu\nu}=\frac{1}{2}\left[\frac{\partial}{\partial x^{\mu}}\frac{\partial g_{\nu\alpha}}{\partial \mathrm{x}^{\beta}}+\frac{\partial}{\partial x^{\nu}}\frac{\partial g_{\beta\mu}}{\partial \mathrm{x}^{\alpha}}-\frac{\partial}{\partial x^{\nu}}\frac{\partial g_{\mu\alpha}}{\partial \mathrm{x}^{\beta}}-\frac{\partial}{\partial x^{\mu}}\frac{\partial g_{\beta\nu}}{\partial \mathrm{x}^{\alpha}}\right] \tag{1}$$

Multiplied with -1:

$$R_{\alpha\beta\mu\nu}=-\frac{1}{2}\left[\frac{\partial}{\partial x^{\nu}}\frac{\partial g_{\mu\alpha}}{\partial \mathrm{x}^{\beta}}+\frac{\partial}{\partial x^{\mu}}\frac{\partial g_{\beta\nu}}{\partial \mathrm{x}^{\alpha}}-\frac{\partial}{\partial x^{\mu}}\frac{\partial g_{\nu\alpha}}{\partial \mathrm{x}^{\beta}}-\frac{\partial}{\partial x^{\nu}}\frac{\partial g_{\beta\mu}}{\partial \mathrm{x}^{\alpha}}\right] \tag{2}$$

Swapping μ and ν in (1):

$$R_{\alpha\beta\nu\mu}=\frac{1}{2}\left[\frac{\partial}{\partial x^{\nu}}\frac{\partial g_{\mu\alpha}}{\partial \mathrm{x}^{\beta}}+\frac{\partial}{\partial x^{\mu}}\frac{\partial g_{\beta\nu}}{\partial \mathrm{x}^{\alpha}}-\frac{\partial}{\partial x^{\mu}}\frac{\partial g_{\nu\alpha}}{\partial \mathrm{x}^{\beta}}-\frac{\partial}{\partial x^{\nu}}\frac{\partial g_{\beta\mu}}{\partial \mathrm{x}^{\alpha}}\right] \tag{3}$$

Thus from (2) and (3) we get:

$$\boxed{R_{\alpha\beta\mu\nu}=-R_{\alpha\beta\nu\mu}}$$

Be aware that this equation is only valid at the origin of the Local Inertial Frame. But these are tensor equations and as we know, that if these tensor equations are valid in one frame they shall be valid in any frame.

Now we will show, in a similar way that the Riemann tensor is symmetric by swapping the first two indices:

$$R_{\alpha\beta\mu\nu}=\frac{1}{2}\left[\frac{\partial}{\partial x^{\mu}}\frac{\partial g_{\nu\alpha}}{\partial \mathrm{x}^{\beta}}+\frac{\partial}{\partial x^{\nu}}\frac{\partial g_{\beta\mu}}{\partial \mathrm{x}^{\alpha}}-\frac{\partial}{\partial x^{\nu}}\frac{\partial g_{\mu\alpha}}{\partial \mathrm{x}^{\beta}}-\frac{\partial}{\partial x^{\mu}}\frac{\partial g_{\beta\nu}}{\partial \mathrm{x}^{\alpha}}\right]$$

$$R_{\alpha\beta\mu\nu}=-\frac{1}{2}\left[\frac{\partial}{\partial x^{\nu}}\frac{\partial g_{\mu\alpha}}{\partial \mathrm{x}^{\beta}}+\frac{\partial}{\partial x^{\mu}}\frac{\partial g_{\beta\nu}}{\partial \mathrm{x}^{\alpha}}-\frac{\partial}{\partial x^{\mu}}\frac{\partial g_{\nu\alpha}}{\partial \mathrm{x}^{\beta}}-\frac{\partial}{\partial x^{\nu}}\frac{\partial g_{\beta\mu}}{\partial \mathrm{x}^{\alpha}}\right]$$

$$R_{\beta\alpha\mu\nu} = \frac{1}{2}\left[\frac{\partial}{\partial x^{\mu}}\frac{\partial g_{\nu\beta}}{\partial x^{\alpha}} + \frac{\partial}{\partial x^{\nu}}\frac{\partial g_{\alpha\mu}}{\partial x^{\beta}} - \frac{\partial}{\partial x^{\nu}}\frac{\partial g_{\mu\beta}}{\partial x^{\alpha}} - \frac{\partial}{\partial x^{\mu}}\frac{\partial g_{\alpha\nu}}{\partial x^{\beta}}\right]$$

$$\boxed{R_{\alpha\beta\mu\nu} = -R_{\beta\alpha\mu\nu}}$$

Swapping the first and third indices ($\alpha \leftrightarrow \mu$), and also the second and fourth ($\beta \leftrightarrow \nu$), we get:

$$R_{\mu\nu\alpha\beta} = \frac{1}{2}\left[\frac{\partial}{\partial x^{\alpha}}\frac{\partial g_{\beta\mu}}{\partial x^{\nu}} + \frac{\partial}{\partial x^{\beta}}\frac{\partial g_{\nu\alpha}}{\partial x^{\alpha}} - \frac{\partial}{\partial x^{\beta}}\frac{\partial g_{\alpha\mu}}{\partial x^{\nu}} - \frac{\partial}{\partial x^{\alpha}}\frac{\partial g_{\nu\beta}}{\partial x^{\mu}}\right]$$

$$\boxed{R_{\mu\nu\alpha\beta} = R_{\alpha\beta\mu\nu}}$$

If we cyclically permute the last 3 indices β, μ and ν and add up the 3 terms, we get:

$$R_{\alpha\beta\mu\nu} + R_{\alpha\nu\beta\mu} + R_{\alpha\mu\nu\beta} = \frac{1}{2}\left[\cancel{\frac{\partial}{\partial x^{\beta}}\frac{\partial g_{\alpha\nu}}{\partial x^{\mu}}} + \cancel{\frac{\partial}{\partial x^{\alpha}}\frac{\partial g_{\beta\mu}}{\partial x^{\nu}}} - \frac{\partial}{\partial x^{\beta}}\frac{\partial g_{\alpha\mu}}{\partial x^{\nu}} - \cancel{\frac{\partial}{\partial x^{\alpha}}\frac{\partial g_{\beta\nu}}{\partial x^{\mu}}}\right]$$

$$+\frac{1}{2}\left[\frac{\partial}{\partial x^{\nu}}\frac{\partial g_{\alpha\mu}}{\partial x^{\beta}} + \cancel{\frac{\partial}{\partial x^{\alpha}}\frac{\partial g_{\nu\beta}}{\partial x^{\mu}}} - \cancel{\frac{\partial}{\partial x^{\nu}}\frac{\partial g_{\alpha\beta}}{\partial x^{\mu}}} - \cancel{\frac{\partial}{\partial x^{\alpha}}\frac{\partial g_{\mu\nu}}{\partial x^{\beta}}}\right]$$

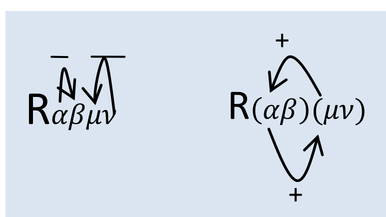
$$+\frac{1}{2}\left[\cancel{\frac{\partial}{\partial x^{\mu}}\frac{\partial g_{\alpha\beta}}{\partial x^{\nu}}} + \cancel{\frac{\partial}{\partial x^{\alpha}}\frac{\partial g_{\mu\nu}}{\partial x^{\beta}}} - \cancel{\frac{\partial}{\partial x^{\mu}}\frac{\partial g_{\alpha\nu}}{\partial x^{\beta}}} - \cancel{\frac{\partial}{\partial x^{\alpha}}\frac{\partial g_{\mu\beta}}{\partial x^{\nu}}}\right]$$

$$\boxed{\boldsymbol{R_{\alpha\beta\mu\nu} + R_{\alpha\nu\beta\mu} + R_{\alpha\mu\nu\beta} = 0}}$$

Summarized:

$$R_{\alpha\beta\mu\nu} = -R_{(\beta)(\alpha)\mu\nu} = -R_{\alpha\beta(\nu)(\mu)} \quad \textit{is anti symmetric}$$
$$R_{(\alpha\beta)(\mu\nu)} = R_{(\mu\nu)(\alpha\beta)} \quad \textit{is symmetric}$$
$$R_{\alpha\beta\mu\nu} + R_{\alpha\nu\beta\mu} + R_{\alpha\mu\nu\beta} = 0$$

Or depicted as:



2.11. Bianchi Identity and Ricci Tensor

The **Bianchi Identity** equation will be derived in order to find the Einstein field equations.
The Riemann curvature tensor does not appear in the Einstein field equations. But by contraction of the Riemann tensor, two other important quantities of the curvature, known as the **Ricci tensor** and the **Ricci scalar,** will be derived.
In this chapter we will define these three important Riemann tensor derivatives.
First we will derive the **Bianchi identity**

$$\nabla_{\sigma} R_{\alpha\beta\mu\nu} + \nabla_{\nu} R_{\alpha\beta\sigma\mu} + \nabla_{\mu} R_{\alpha\beta\nu\sigma} = 0$$

From the previous chapter *Symmetries and independent components* we know that at the origin of a Local Inertial Frame, we have

$$R_{\alpha\beta\mu\nu} = \frac{1}{2}\left[\frac{\partial}{\partial x^\beta}\frac{\partial g_{\nu\alpha}}{\partial x^\mu} + \frac{\partial}{\partial x^\alpha}\frac{\partial g_{\beta\mu}}{\partial x^\nu} - \frac{\partial}{\partial x^\beta}\frac{\partial g_{\mu\alpha}}{\partial x^\nu} - \frac{\partial}{\partial x^\alpha}\frac{\partial g_{\beta\nu}}{\partial x^\mu}\right]$$

We also know that at the origin of Local Inertial Frame, the Christoffel symbols do all vanish, and then the covariant derivative becomes the ordinary derivative:

$$\nabla_\beta V^\alpha = \frac{\partial V^\alpha}{\partial x^\beta}$$

Therefore, we get, at the origin of a Local Inertial Frame:

$$\nabla_\sigma R_{\alpha\beta\mu\nu} = \frac{\partial}{\partial x^\sigma} R_{\alpha\beta\mu\nu} = \frac{1}{2}\left[\frac{\partial}{\partial x^\sigma}\frac{\partial}{\partial x^\beta}\frac{\partial g_{\nu\alpha}}{\partial x^\mu} + \frac{\partial}{\partial x^\sigma}\frac{\partial}{\partial x^\alpha}\frac{\partial g_{\beta\mu}}{\partial x^\nu} - \frac{\partial}{\partial x^\sigma}\frac{\partial}{\partial x^\beta}\frac{\partial g_{\mu\alpha}}{\partial x^\nu} - \frac{\partial}{\partial x^\sigma}\frac{\partial}{\partial x^\alpha}\frac{\partial g_{\beta\nu}}{\partial x^\mu}\right]$$

By cyclically permuting the index of the derivative with the last two indices of the tensor, we get:

$$\nabla_\nu R_{\alpha\beta\sigma\mu} = \frac{\partial}{\partial x^\nu} R_{\alpha\beta\sigma\mu} = \frac{1}{2}\left[\frac{\partial}{\partial x^\nu}\frac{\partial}{\partial x^\beta}\frac{\partial g_{\mu\alpha}}{\partial x^\sigma} + \frac{\partial}{\partial x^\nu}\frac{\partial}{\partial x^\alpha}\frac{\partial g_{\beta\sigma}}{\partial x^\mu} - \frac{\partial}{\partial x^\nu}\frac{\partial}{\partial x^\beta}\frac{\partial g_{\alpha\sigma}}{\partial x^\mu} - \frac{\partial}{\partial x^\nu}\frac{\partial}{\partial x^\nu}\frac{\partial g_{\beta\mu}}{\partial x^\sigma}\right]$$

$$\nabla_\mu R_{\alpha\beta\nu\sigma} = \frac{\partial}{\partial x^\mu} R_{\alpha\beta\nu\sigma} = \frac{1}{2}\left[\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\beta}\frac{\partial\,\partial g_{\sigma\alpha}}{\partial x^\nu} + \frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\alpha}\frac{\partial g_{\beta\nu}}{\partial x^\sigma} - \frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\beta}\frac{\partial g_{\alpha\nu}}{\partial x^\sigma} - \frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\alpha}\frac{\partial g_{\beta\sigma}}{\partial x^\nu}\right]$$

Adding these three equations and using the commutativity of partial derivatives, we see that the terms cancel in pairs, so we get ***the Bianchi identity***:

$$\boxed{\boldsymbol{\nabla_\sigma R_{\alpha\beta\mu\nu} + \nabla_\nu R_{\alpha\beta\sigma\mu} + \nabla_\mu R_{\alpha\beta\nu\sigma} = 0}}$$

2.11.1. The Ricci tensor

In the next chapter we will deal with the Energy-momentum tensor. This tensor is a rank two tensor, for this reason we have to modify the rank-4 Riemann tensor to a rank-2 tensor which is called the Ricci tensor. This can be done by multiplying the, in this case, covariant Riemann tensor with a rank-2 contra-variant metric tensor, having two commonly shared indices. This process is called contraction.

$$Ricci\ tensor\ \ R_{\mu\nu} = g^{\alpha\beta} R_{\alpha\mu\beta\nu} = R^{\beta}_{\mu\beta\nu} = R_{\mu\nu}$$

2.11.2. The Ricci scalar

When the Ricci tensor again is multiplied with the metric tensor with the same indices, contraction of the Ricci tensor is done, resulting in the Ricci scalar:

$$Ricci\ scalar\ \ R = g^{\mu\nu} R_{\mu\nu}$$

This scalar curvature *R* is the trace of the Ricci tensor.

2.12. Energy-Momentum Tensor

The final goal is to formulate a relation between the space-time geometry and its content; first the right mathematical tool to describe this space-time content has to be found.

In Special Relativity it has been shown that mass, energy and momentum are related, as expressed in the energy momentum relation:

$$|P|^2 = (m_0 c)^2$$

$$|P|^2 = \eta_{\mu\nu} P^\mu P^\nu = \frac{E^2}{c^2} - p_x^2 - p_y^2 - p_z^2 = \frac{E^2}{c^2} - p^2$$

$$=> \ (m_0 c)^2 = \frac{E^2}{c^2} - p^2$$

$$\boxed{=> \ E^2 = p^2 c^2 + {m_0}^2 c^4}$$

It therefore seems reasonable to make the hypothesis that the source of the gravitational field in the theory of General Relativity should include momentum and energy as well as mass.
On the other hand, the equivalent Newtonian equation describing the gravitational potential Φ caused by a mass density ρ, as expressed by Poisson's equation is (see: **Appendix 5**):

$$-\vec{\nabla} \cdot \vec{g} = -\vec{\nabla} \cdot \left(-\vec{\nabla}\Phi\right) = 4\pi G \rho$$

So the following question arises: is the equivalent relativistic energy density to be found also a scalar quantity or a tensor component?
To answer this question, consider a volume *dx* x *dy* x *dz* of non-interacting particles at rest with respect to each other, commonly referred to as "dust cloud".
In its own referential S, this cloud has an energy density $\boldsymbol{\rho_0 = m_0 n_0}$ where m_0 refers to the mass of a dust particle and n_0 kinetic term.
In a different Lorentz referential S$'$ moving with constant velocity *v* in the *x*-direction, Lorentz transformation will lead to:

$$m_0 \to m_0 \gamma,$$

$$n_0 \to n_0 \gamma,$$

$$=> \ \rho_0 \to \rho = \rho_0 \gamma^2.$$

The first factor γ relates to the Lorentz energy transformation. The second γ factor is caused by the length contraction in the *x* direction, which makes the new dust volume observed from S' equals to $\frac{dx}{\gamma} \times dy \times dz$ and therefore multiply the new density by the same factor.
Obviously, ρ is not a scalar, because then it would be invariant. Neither it can be a four-vector component; in that case it would transform linearly with γ factor. Actually ρ behaves like a component of a rank-2 tensor (rank-2 tensors transform like a product of two Lorentz transformations). More precisely, in this case, ρ behaves like a *tt*-component of a rank-2 tensor.

Writing the four-velocity vector of the dust cloud in the S' referential, we get:

$$u^\mu = \frac{\partial x^\mu}{\partial \tau} = \frac{\partial x^\mu}{\partial t}\frac{dt}{\partial \tau} = v^\mu \frac{dt}{\partial \tau} = v^\mu u^t$$

$$u^\mu = \gamma(1, v) = \begin{pmatrix} \gamma \\ v_x\gamma \\ v_y\gamma \\ v_z\gamma \end{pmatrix} = \begin{pmatrix} u^t \\ v_x u^t \\ v_y u^t \\ v_z u^t \end{pmatrix}$$

So we can set $\gamma = u^t$

Using this equality and the fact that the energy of each particle equals $p^t = mu^t$, the total energy density in S' can be rewritten as:

$$\rho \equiv np^t = (n_0 u^t)(mu^t) = (n_0 m)u^t u^t = \rho_0 u^t u^t$$

We therefore are able to confirm that this quantity could be interpreted as the *tt*-component of a symmetric rank-2 tensor:

$$T^{\mu\nu} = T^{\nu\mu} = \rho_0 u^\mu u^\nu$$

This is the **energy-momentum tensor**, also known as the **stress-energy tensor** for the dust.

2.12.1. Physical meaning of the energy-momentum tensor

Because the stress-energy tensor is of the order two, its components can be displayed in 4 x 4 matrix form:

$$T^{\mu\nu} = \begin{pmatrix} T^{tt} & T^{tx} & T^{ty} & T^{tz} \\ T^{xt} & T^{xx} & T^{xy} & T^{xz} \\ T^{yt} & T^{yx} & T^{yy} & T^{yz} \\ T^{zt} & T^{zx} & T^{zy} & T^{zz} \end{pmatrix}$$

As seen previously, T^{tt} represents the density of relativistic mass, i.e. the energy density. But what can represent all the other 15 components of the energy momentum tensor?

Let us consider the T^{tx} components first; we can write:

$$T^{tx} = \rho_0 u^t u^x = (n_0 m)u^t u^x = (n_0 u^t)(mu^t)v_x = np^t v_x = \frac{(nAv_x dt)p^t}{Adt}$$

The quantity $Av_x dt$ represents the dust volume which passes through the surface *A* perpendicular to the x-direction during the *dt* time interval, so that $Av_x dt$ represents the total number of particles which goes through this surface. We can then interpret $T^{tx} = T^{xt}$ as the total energy per unit of surface and of time, i.e. the flux of energy per unit area per unit time in the *x*-direction. A similar argument applies to T^{ty} and T^{tz}, respectively the flux of energy across per unit area per unit time of *y* and *z*=constant.

For the other components, let us consider T^{kl} where *k* and *l* are spatial indices. In this case, we have

$$T^{kl} = \rho_0 u^k u^l = (n_0 m)u^k u^l = (n_0 m)u^t v_k u^l$$

$$= (n_0 u^t)v_k\left(mu^l\right) = nv_k\left(mu^l\right) = nv_k p^l = \frac{(nAv_k dt)p^l}{Adt}$$

The first factor is the flux per unit area per unit time of particles in the *k* direction, so $\boldsymbol{T^{kl}}$ **is the flux in the *k* direction of *l* momentum**. For example, T^{xz} is the flux of *z* momentum in the *x* direction(or, since *T* is symmetric, the flux of *x* momentum in the *z*-direction).

2.12.2. Covariant differentiation of the energy-momentum tensor

In flat space-time of special relativity, the fundamental laws of conservation of energy and momentum (i.e. there are flows but no sources or sinks of energy-momentum) could be expressed by saying:

$$0 = \frac{\partial\, T^{\mu\nu}}{\partial x^{\nu}} = \partial^{\nu}\, T^{\mu\nu} = T^{\mu\nu}_{,\nu}$$

That is a consequence of Noether's theorem concerning the invariance of physical systems with respect to spatial translation (in other words, that the laws of physics do not vary with locations in space), which gives the law of conservation of linear momentum.

Using the 'comma goes to semi-colon' rule, we get the following tensor equation, which by the Principle of General Covariance is valid for any coordinate system, therefore will stay valid in curved space-time of the theory of General Relativity.

$$\boxed{0 = \nabla_{\nu}\, T^{\mu\nu} = T^{\mu\nu}_{;\nu}}$$

2.13. Einstein Tensor

The Poisson's equation for the gravitational field in the Newton view is:

$$-\vec{\nabla}\cdot\vec{\mathrm{g}} = -\vec{\nabla}\cdot\left(-\vec{\nabla}\Phi\right) = 4\pi G\rho$$

Where Φ refers to the gravitational potential and ρ to the mass density.

Now we have to find the relativistic equivalent.

In the previous chapter "The **Energy-Momentum Tensor**", we have seen that the generalization of the mass density (right hand term of the Einstein field equation) corresponds to the energy-momentum $T^{\mu\nu}$.

It seems reasonable then to assume that our equation should take the form of:

$$G^{\mu\nu} = k\, T^{\mu\nu}$$

Where *k* stands for a scalar and $G^{\mu\nu}$, called the Einstein tensor represents a rank-2 tensor describing the space-time curvature.
As far as we know, $G^{\mu\nu}$ should obey the following constraints:

- It should be zero in flat space-time
- It should describe the space-time curvature and be linear with respect to the Riemann tensor
- It should be symmetric and of rank 2 (as $T^{\mu\nu}$)
- It should have a zero divergence (as $T^{\mu\nu}$)
- And finally, in Newtonian limits, it should reduce to $4\pi G\rho$

2.13.1. First attempt with Ricci tensor as solution

We know (see chapter 2.7) that the gravitational potential Φ is linked to the 00-component of the metric

$$\boxed{\frac{d^2\vec{r}}{dt^2} = -\overrightarrow{grad}\Phi \text{ with } \Phi = \frac{c^2 h_{00}}{2}}$$

It seems then natural to look for a tensor which involves the second derivatives of the metric, which is the case for the Riemann tensor. Furthermore, the Riemann tensor is the only candidate that we know so far being able to describe the space-time curvature (conform second constraint above).

As we have to find a rank-2 tensor (the third constraint), and if we assume that we have to find a solution solely in terms of the Riemann tensor, it seems natural to first consider the contracted form of the Riemann tensor, known as the Ricci tensor. To see this, we recall the expression of the Riemann tensor:

$$\boxed{R^{\alpha}_{\mu\sigma\nu} = \left(\frac{d\Gamma^{\alpha}_{\mu\nu}}{dx^{\sigma}} - \frac{d\Gamma^{\alpha}_{\mu\sigma}}{dx^{\nu}} + \Gamma^{\alpha}_{\sigma\gamma}\Gamma^{\gamma}_{\mu\nu} - \Gamma^{\alpha}_{\nu\gamma}\Gamma^{\gamma}_{\mu\sigma}\right)}$$

$$\boxed{R_{\mu\nu} = R^{\alpha}_{\mu\alpha\,\nu} = \left(\frac{d\Gamma^{\alpha}_{\mu\nu}}{dx^{\alpha}} - \frac{d\Gamma^{\alpha}_{\mu\alpha}}{dx^{\nu}} + \Gamma^{\alpha}_{\alpha\gamma}\Gamma^{\gamma}_{\mu\nu} - \Gamma^{\alpha}_{\nu\gamma}\Gamma^{\gamma}_{\mu\alpha}\right)}$$

In the limit of a static and weak gravitational field, only one term contributes to R_{00}:

$$R_{00} = R^{\alpha}_{00\alpha} = \Gamma^{\alpha}_{00,\alpha} - \Gamma^{\alpha}_{0\alpha,0} + O(h^2) = \Gamma^{i}_{00,i}$$

After calculation it is found that this Christoffel symbol is:

$$\Gamma^{i}_{00} = -\frac{1}{2}\mathrm{g}^{\mathrm{ij}}g_{00,j}$$

With the approximation of $\mathrm{g}^{\mathrm{ij}} = \eta^{\mathrm{ij}}$ and $g_{00,j} = h_{00,j}$ we get:

$$\Gamma^{i}_{00} = -\frac{1}{2}\eta^{ij}h_{00,j} = \frac{1}{2}\delta^{i}_{j}h_{00,j}$$

$$\Gamma^{i}_{00,i} = \frac{1}{2}\delta^{i}_{j}h_{00,ij} = \frac{1}{2}h_{00,ii}$$

$$R_{00} = \Gamma^{i}_{00,i} = \frac{1}{2}(\partial_1^2 h_{00} + \partial_2^2 h_{00} + \partial_3^2 h_{00})$$

$$R_{00} = \frac{1}{2}\nabla^2 h_{00} = \frac{1}{\mathrm{c}^2}\nabla^2\Phi = \frac{4\pi G\rho}{\mathrm{c}^2}$$

The identification of R_{00} with $\nabla\Phi$ (Laplacian operator) suggests that the field equation in the theory of General Relativity should equate R_{ab} to a constant multiple of T_{ab}.

In 1915, using this equation, Einstein was even able to resolve the long standing problem of Mercury perihelion precession, causing to write in November of that year that, “For a few days, I was beside myself with joyous excitement.”

Eventually, Einstein had to reject this first attempt, due to the fact that in general **divergence of R_{ab} does not become zero.**

2.13.2. Second attempt

There is a tensor closely related to the Ricci scalar which can be put on the left-hand side. This is the Einstein tensor defined as follows:

$$G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu}$$

Where $R = R^a_a$ is the *Ricci scalar or scalar curvature*.

This form of G_{ab} is symmetrical and rank-2 and obviously describes the space-time curvature. **So it just remains to show that the total derivative is zero.** (See also chapter 2.4.2 equation 15)

To do this, we start with the Bianchi identity

$$\nabla_\sigma R_{\alpha\beta\mu\nu} + \nabla_\nu R_{\alpha\beta\,\sigma\mu} + \nabla_\mu R_{\alpha\beta\nu\,\sigma} = 0$$

Multiplying through by $g^{\gamma\sigma} g^{\alpha\mu} g^{\beta\nu}$ (the metrics derivatives are zero, so they act as constants and can be taken inside the derivatives), yielding:

$$\nabla_\sigma g^{\gamma\sigma} g^{\alpha\mu} g^{\beta\nu} R_{\alpha\beta\mu\nu} + \nabla_\nu g^{\gamma\sigma} g^{\alpha\mu} g^{\beta\nu} R_{\alpha\beta\,\sigma\mu} + \nabla_\mu g^{\gamma\sigma} g^{\alpha\mu} g^{\beta\nu} R_{\alpha\beta\nu\,\sigma} = 0$$

$$\nabla_\sigma g^{\gamma\sigma} R + \nabla_\nu g^{\gamma\sigma} g^{\alpha\mu} g^{\beta\nu} R_{\alpha\beta\,\sigma\mu} + \nabla_\mu g^{\gamma\sigma} g^{\alpha\mu} g^{\beta\nu} R_{\alpha\beta\nu\,\sigma} = 0$$

$$\nabla_\sigma g^{\gamma\sigma} R + \nabla_\nu g^{\gamma\sigma} g^{\alpha\mu} g^{\beta\nu} R_{\sigma\mu\alpha\beta} + \nabla_\mu g^{\gamma\sigma} g^{\alpha\mu} g^{\beta\nu} R_{\nu\sigma\alpha\beta} = 0$$

$$\nabla_\sigma g^{\gamma\sigma} R - \nabla_\nu g^{\gamma\sigma} g^{\alpha\mu} g^{\beta\nu} R_{\mu\sigma\alpha\beta} - \nabla_\mu g^{\gamma\sigma} g^{\alpha\mu} g^{\beta\nu} R_{\nu\sigma\beta\alpha} = 0$$

By using the Ricci tensor definition $R^{\mu\nu} = g^{\mu\beta} g^{\nu\sigma} R_{\beta\sigma}$ (step 3) and by renaming the indices (step 4), we get

$$\nabla_\sigma g^{\gamma\sigma} R - \nabla_\nu g^{\gamma\sigma} g^{\alpha\mu} g^{\beta\nu} R_{\mu\sigma\alpha\beta} - \nabla_\mu g^{\gamma\sigma} g^{\alpha\mu} g^{\beta\nu} R_{\nu\sigma\beta\alpha} = 0$$

$$\nabla_\sigma g^{\gamma\sigma} R - \nabla_\nu g^{\gamma\sigma} g^{\beta\nu} R_{\sigma\beta} - \nabla_\mu g^{\gamma\sigma} g^{\alpha\mu} R_{\sigma\alpha} = 0$$

$$\nabla_\sigma g^{\gamma\sigma} R - \nabla_\nu R^{\gamma\nu} - \nabla_\mu R^{\gamma\mu} = 0$$

$$\nabla_\sigma g^{\gamma\sigma} R - \nabla_\sigma R^{\gamma\sigma} - \nabla_\sigma R^{\gamma\sigma} = 0$$

$$\nabla_\sigma g^{\gamma\sigma} R - 2\nabla_\sigma R^{\gamma\sigma} = 0$$

$$\nabla_\sigma (2R^{\gamma\sigma} - g^{\gamma\sigma} R) = 0$$

$$\nabla_\sigma \left(R^{\gamma\sigma} - \frac{1}{2} g^{\gamma\sigma} R \right) = 0$$

This is what we wanted to demonstrate: the divergence of Einstein tensor is zero, and we have found the right candidate for the left hand side of our curvature/mass-energy equation.

2.14. Einstein Field Equations

In the last two chapters, we have derived the $G^{\mu\nu}$ (Einstein tensor) and $T^{\mu\nu}$ (energy-momentum tensor) components of the Einstein equation:

$$G^{\mu\nu} = kT^{\mu\nu}$$

We have yet to determine the constant *k*.
To achieve this, we need to show that the Einstein equation reduces to Newton's law of gravity for weak and static gravitational fields (Newtonian limit).

The first step consists of writing the previous Einstein equation in a slightly different form that is sometimes more practical to use in equations.

That is actually under this second form that Einstein published in his article "The Field Equations of Gravitation" submitted on November 25, 1915 in *Königlish Preussiche Akademie der Wissenschaften*

Ist in dem betrachteten Raume "Materie" vorhanden, so tritt deren Energietensor auf der rechten Seite von (2) bis zum weiter (3) auf. Wir setzen

$$\boldsymbol{G_{im} = -\chi\left(T_{im} - \frac{1}{2}g_{im}T\right),} \qquad (2a)$$

Wobei

$$\boldsymbol{\sum_{tv} g^{tv} T_{tv} = \sum_{v} T_v^v = T} \qquad (5)$$

Gesetzt ist;*T* ist der Skalar des Energietensors der "Materie", die rechte Seite von (2a) ein Tensor.

2.14.1. Einstein's equation alternative form

Replacing Einstein tensor by its full expression:

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = kT^{\mu\nu}$$

Multiplying both sides by $g_{\mu\nu}$ yields:

$$g_{\mu\nu}R^{\mu\nu} - \frac{1}{2}g_{\mu\nu}g^{\mu\nu}R = kg_{\mu\nu}T^{\mu\nu}$$

By definition of the metric contraction, $g_{\mu\nu}R^{\mu\nu} = R$ and $g_{\mu\nu}T^{\mu\nu} = T$ so

$$R - \frac{1}{2}Rg_{\mu\nu}g^{\mu\nu} = kT$$

Because the tensor $g^{\mu\nu}$ is the inverse of $g_{\mu\nu}$, their product gives the identity matrix of rank-4 $\delta_\nu^\mu = 1$ (this can be seen by doing the calculation in a local inertial frame where $g_{\mu\nu} = \eta_{\mu\nu}$ and noting that it's a tensor equation, it's valid in all coordinate systems). By contracting the δ^μ_ν tensor we just sum up its diagonal elements and since these are all one, we get

$$g_{\mu\nu}\, g^{\mu\nu} = \delta^\nu_\nu = 1 + 1 + 1 + 1 = 4$$

Therefore

$$R - \frac{1}{2} R \times 4 = kT$$

$$R - 2R = kT$$

$$R = -kT$$

Replacing *R* by –*kT* in Einstein original equations gives:

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} \times (-kT) = kT^{\mu\nu}$$

$$R^{\mu\nu} + \frac{1}{2} k g^{\mu\nu} T = kT^{\mu\nu}$$

$$R^{\mu\nu} = k\left(T^{\mu\nu} - \frac{1}{2} g^{\mu\nu} T\right)$$

Or

$$g_{\alpha\mu}\, g_{\beta\nu}\, R^{\mu\nu} = g_{\alpha\mu}\, g_{\beta\nu}\; k\left(T^{\mu\nu} - \frac{1}{2} g^{\mu\nu} T\right)$$

$$R_{\alpha\beta} = k\left(T_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} T\right)$$

Replace the dummy indices $\alpha\beta$ with $\mu\nu$:

$$R_{\mu\nu} = k\left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\right)$$

$$R_{\mu\nu} = kT_{\mu\nu} - \frac{1}{2} g_{\mu\nu}\, kT$$

Together with

$$R = -kT$$

Gives

$$R_{\mu\nu} = kT_{\mu\nu} + \frac{1}{2} g_{\mu\nu} R$$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = kT_{\mu\nu}$$

2.14.2. Newtonian limit

In the context of Newtonian limit, we have also already demonstrated from the previous chapter that the component R_{00} of the Riemann tensor approximates to

$$R_{00} \approx \frac{1}{c^2} \nabla^2 \Phi$$

But we can remark that when the metric reduces to the metric η from flat space-time, we get the definition of the Ricci tensor:

$$R^{\mu\nu} \equiv g^{0\mu} g^{0\nu} R_{\mu\nu} \approx \eta^{0\mu} \eta^{0\nu} R_{\mu\nu} = (-1)(-1) R_{00} = R_{00}$$

$$R_{00} \approx \frac{1}{c^2}\nabla^2\Phi = \frac{4\pi G\rho}{c^2}$$

The Newtonian limit implies also that the only non negligible component of the stress-energy tensor $T^{\mu\nu}$ is $T^{00} = \rho c^2$ ($T^{\mu\nu} = \rho u^\mu u^\nu$ with $u^i \ll u^0 = c$)

Then we can write

$$T = g_{\mu\nu}T^{\mu\nu} \approx g_{00}T^{00} \approx \eta_{00}T^{00} = T^{00} = \rho c^2$$

This yields by developing the 00-th component of the Einstein equation to

$$R_{00} = k\left(T_{00} - \frac{1}{2}\eta_{00}T\right)$$

$$\frac{4\pi G\rho}{c^2} = k\left(\rho c^2 - \frac{1}{2} \times 1 \times \rho c^2\right)$$

$$\frac{4\pi G\rho}{c^2} = \frac{1}{2}k\rho c^2$$

$$k = \frac{8\pi G}{c^4}$$

We can finally formulate the Einstein equation both in its standard and alternative form

$$\boxed{R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = \frac{8\pi G}{c^4}T^{\mu\nu}}$$

$$\boxed{R^{\mu\nu} = \frac{8\pi G}{c^4}\left(T^{\mu\nu} - \frac{1}{2}g^{\mu\nu}T\right)}$$

Or

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4}T_{\mu\nu}$$

$$R_{\mu\nu} = \frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)$$

Remark 1: When calculated, the value of *G* is very small, meaning that the space-time is very rigid or stiff: a huge value of mass/energy is required to set a ‘noticeable’ curvature.

$$k = \frac{8\pi G}{c^4} \approx 2.071 \times 10^{-43}s^2m^{-1}kg^{-1}$$

Remark 2: Despite the simple appearance of the equations they are actually quite complicated. Given a specified distribution of matter and energy in the form of a stress-energy tensor, the Einstein Equation or Einstein Field

Equations (EFE) are understood to be a set of equations for the metric tensor $g^{\mu\nu}$, as both the Ricci tensor and scalar curvature depend on the metric on a complicated nonlinear manner. In fact, when fully written out, the EFE represent a system of 10 coupled, non linear second-order partial differential equations for the metric tensor, corresponding to the 10 independent component of the symmetric tensor $g^{\mu\nu}$.

Remark 3: The non-linearity of the EFE has a profound physical meaning. It relates to the auto-referential role of space-time in this theory, because it constitutes both the dynamical object and the context within which the dynamics are defined. In other terms, gravitation itself gravitates. As stated by Kevin Brown in his Reflection on Relativity, "the self-referential quantity of the metric field equations also manifests itself in their non-linearity. This is really unavoidable for a theory in which the metrical relations between entities determine the "positions" of those entities, and those positions in turn influence the metric.

This non-linearity means also, as we will see later, that two gravitations are able to exchange a graviton, which would not be possible in the case of a set of linear equations; for example, electromagnetism's linearity does not allow two photons to exchange another (virtual) photon to interact.

Remark 4: Finally and to be accurate, the EFE do not determine completely and uniquely all ten components of the metric. The Einstein equation must place only six independent constraints on the ten $g^{\mu\nu}(P)$, leaving four arbitrary functions to be adjusted by man's specialization of the four coordinate functions $x_\alpha(P)$, the fact that ten distinct differential equations lead to the setting of only six constraints is precisely due to the zero divergence of the Einstein tensor *G*.

2.15. Summary of the Final Formula for the theory of General Relativity

In the previous chapters the derivation of the Einstein Field Equations (EFE) has been performed; including all the tools necessary for executing this derivation. This chapter is meant to give you a brief overview over the achieved result.

The main idea of Einstein was that there is no force of gravity but that space-time is curved due to the presence of mass and energy. The degree of curvature depends on the size of mass and energy. The goal of Einstein was to develop a mathematical description of this curvature and to find the relationship between the curvature, i.e. the geography, of space-time and the amount of mass and energy

Without going through the total derivation of Einstein's formula we show here the final result

$$\boldsymbol{R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}} \tag{4}$$

Here the term $\lambda g_{\mu\nu}$ is very small and is only relevant when we consider total space-time (cosmos), so in general this can be ignored. (The cosmological constant $\lambda = 1.1056 \times 10^{-52} m^{-2}$.)
So in general:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu} \tag{5}$$

Here the left hand-side denotes the geometry of space-time, while the right-hand side stands for the mass and energy.
In case we consider a location outside a mass the right hand side becomes zero.

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0 \tag{6}$$

As we have seen in **2.14.1 is:**

$$R = -\frac{8\pi G}{c^4} T$$

So in case the right hand side is naught

$$\mathrm{R} = 0$$

And consequently

$$R_{\mu\nu} = 0 \tag{7}$$

So outside a mass in vacuum both terms *R* and $R_{\mu\nu}$ are zero.

Now back to the general formula. As we have mentioned above μ and ν are indices denoting the four dimensions *t, x, y* and *z* by means of 0, 1, 2 and 3.
So equation (5) actually consists of 16 equations:

$$R_{00} - \frac{1}{2} g_{00} R = \frac{8\pi G}{c^4} T_{00}\,, \quad R_{01} - \frac{1}{2} g_{01} R = \frac{8\pi G}{c^4} T_{01}, \quad R_{02} - \frac{1}{2} g_{02} R = \frac{8\pi G}{c^4} T_{02}\,, \quad R_{03} - \frac{1}{2} g_{03} R = \frac{8\pi G}{c^4} T_{03}$$
$$R_{10} - \frac{1}{2} g_{10} R = \frac{8\pi G}{c^4} T_{10}\,, \quad R_{11} - \frac{1}{2} g_{11} R = \frac{8\pi G}{c^4} T_{11}, \quad R_{12} - \frac{1}{2} g_{12} R = \frac{8\pi G}{c^4} T_{12}\,, \quad R_{13} - \frac{1}{2} g_{13} R = \frac{8\pi G}{c^4} T_{13}$$
$$R_{20} - \frac{1}{2} g_{20} R = \frac{8\pi G}{c^4} T_{20}\,, \quad R_{21} - \frac{1}{2} g_{21} R = \frac{8\pi G}{c^4} T_{21}, \quad R_{22} - \frac{1}{2} g_{22} R = \frac{8\pi G}{c^4} T_{22}\,, \quad R_{23} - \frac{1}{2} g_{23} R = \frac{8\pi G}{c^4} T_{23}$$
$$R_{30} - \frac{1}{2} g_{30} R = \frac{8\pi G}{c^4} T_{30}\,, \quad R_{31} - \frac{1}{2} g_{31} R = \frac{8\pi G}{c^4} T_{31}, \quad R_{32} - \frac{1}{2} g_{32} R = \frac{8\pi G}{c^4} T_{32}\,, \quad R_{33} - \frac{1}{2} g_{33} R = \frac{8\pi G}{c^4} T_{33}$$

Because for most systems goes, that there is symmetry for $\mu\nu = \nu\mu$, so that the number of equations diminishes to 10.
$\boldsymbol{R_{\mu\nu}}$ is called the Ricci tensor and could be denoted in tensor form (very similar to a matrix) as

$$\boldsymbol{R_{\mu\nu}} = \begin{vmatrix} R_{00} & R_{01} & R_{02} & R_{03} \\ R_{10} & R_{11} & R_{12} & R_{13} \\ R_{20} & R_{21} & R_{22} & R_{23} \\ R_{30} & R_{31} & R_{32} & R_{33} \end{vmatrix}$$

$\boldsymbol{g_{\mu\nu}}$ is called the metric tensor and could be denoted in tensor form as

$$\boldsymbol{g_{\mu\nu}} = \begin{vmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} \end{vmatrix}$$

This metric tensor is very important because it contains all the information about the curvature of the space-time considered. $R_{\mu\nu}$ is based on this metric tensor $g_{\mu\nu}$ as we will show below.
R is the Ricci scalar and could be derived via

$$\boldsymbol{R} = \mathrm{g}^{\mu\nu} R_{\mu\nu}$$

All the elements of the left hand side of the equation (5) describe the geography of the space-time considered. At the right hand side we find the energy-momentum tensor $\boldsymbol{T_{\mu\nu}}$ which contains the elements describing the energy, mass density and momentum.

$$\boldsymbol{T_{\mu\nu}} = \begin{vmatrix} T_{00} & T_{01} & T_{02} & T_{03} \\ T_{10} & T_{11} & T_{12} & T_{13} \\ T_{20} & T_{21} & T_{22} & T_{23} \\ T_{30} & T_{31} & T_{32} & T_{33} \end{vmatrix}$$

In formula (5) ***c*** stands for the light velocity ($2.99792458 * 10^8$ m/s) and ***G*** is the well known gravity constant $6.674 * 10^{-11}$ $\mathrm{m^3kg^{-1}s^{-2}}$.
As is sometimes said: the mass/energy determines how the geography of space-time looks like and the geography of space-time determines how the mass will move.
To show the relation between $R_{\mu\nu}$ and $g_{\mu\nu}$, every element in the Ricci tensor is:

$$\mathrm{R}_{\mu\nu} = \mathrm{R}^{\rho}_{\mu\rho\nu} = \frac{\partial\Gamma^{\rho}_{\mu\nu}}{\partial x^{\rho}} - \frac{\partial\Gamma^{\rho}_{\rho\mu}}{\partial x^{\nu}} + \Gamma^{\rho}_{\rho\lambda}\,\Gamma^{\lambda}_{\nu\mu} - \Gamma^{\rho}_{\nu\lambda}\,\Gamma^{\lambda}_{\rho\mu} \quad \text{(note 1)}$$

Here is $\boldsymbol{\Gamma}$ the so called Christoffel symbol that is only zero if space-time is not curved, so in case of the absence of gravity.

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\alpha} \left\{ \frac{\partial g_{\nu\alpha}}{\partial x^{\mu}} + \frac{\partial g_{\mu\alpha}}{\partial x^{\nu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\alpha}} \right\} \quad \text{(note 1)}$$

And so the geography is determined by the metric tensor and its derivatives.

We realize that in such an abbreviated form it is perhaps difficult to grasp it all but it was the intension to give a little insight in the formulae for the field equations forming the theory of General Relativity. The main goal is to get acquainted with the Schwarzschild metric because with the Schwarzschild equation, most experiments could be explained. In 1915 Schwarzschild derived a solution for the Einstein field equations in vacuum. This resulted in a very manageable equation that can be used for many practical applications.

To give an example of the field equations formula, outside a mass, we could use the Schwarzschild equation mentioned below (see chapter 2.16):

$$ds^2 = \left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta^2\, d\emptyset^2$$

Schwarzschild has chosen for a coordinate system much less general than the field equations of Einstein but still meets all the requirements made by the theory of General Relativity. The chosen frame consists of four perpendicular coordinates so that all the cross products disappear. The metric tensor consists then of the elements:

$$g_{00} = \left(1 - \frac{2GM}{c^2 r}\right), \qquad g_{11} = -\left(1 - \frac{2GM}{c^2 r}\right)^{-1}, \qquad g_{22} = -r^2, \qquad g_{33} = -r^2 \sin^2\theta^2$$

This is the so-called *trace* of the tensor. Or in tensor form

$$g_{\mu\nu} = \begin{vmatrix} \left(1-\frac{2GM}{c^2r}\right) & 0 & 0 & 0 \\ 0 & -\left(1-\frac{2GM}{c^2r}\right)^{-1} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2\theta^2 \end{vmatrix}$$

As the Schwarzschild equation is used outside a mass the right hand-side of the Einstein field equations becomes zero ($T_{\mu\nu} = 0$). Thus the field equations become equation (6) and because R is derived from $R_{\mu\nu}$, equation (6) only can be zero when $R_{\mu\nu} = 0$. Thus the only relevant equation is $R_{\mu\nu} = 0$. As we mentioned before the $R_{\mu\nu}$ tensor is build up out of Christoffel symbols and its derivatives. All the relevant Christoffel symbols for this metric we have derived and summarized in Appendix 1.2: *Schwarzschild metric – polar coordinates.*

The Schwarzschild equation uses the polar or spherical coordinate frame describing total space-time; however a physical motion happens, because of angular momentum conservation, in one surface. So by choosing the right polar coordinate frame, this frame could be rotated in such a way that the equatorial plane coincides with the studied surface. In that case the angle $\theta = \pi/2$, in which case the metric tensor further simplifies to

$$g_{\mu\nu} = \begin{vmatrix} \left(1-\frac{2GM}{c^2r}\right) & 0 & 0 & 0 \\ 0 & -\left(1-\frac{2GM}{c^2r}\right)^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -r^2 \end{vmatrix}$$

(See also chapter 7.3 "Answer on question concerning Schwarzschild")

Note 1: Einstein uses in his document for the Christoffel symbol $\Gamma^{\rho}_{\mu\nu}$ with an opposite sign and also the Ricci tensor $\mathrm{R}_{\mu\nu}$ has a opposite sign for the third and the fourth term on the right hand side of the equation. We have used for the metric the so called (+ - - -) notation a.k.a. the West Coast convention.

2.16. Schwarzschild metric

To work with the Einstein formula is in general rather complicated because of its general set up. Luckily Karl Schwarzschild came up in 1915 with the first exact solution to the Einstein field equations. (See chapter 5 : Check whether the Schwarzschild elements meet the Einstein field equations)

Einstein considered all possible configurations of masses, but Schwarzschild limits himself to a location in vacuum, so mass is zero, but regarding the effect on a "particle" by one big massive object in the vicinity; for instance the effect of the Sun on its planets or influence on the passing photons.
(See for more detailed information the chapter below and chapter 4.9.

$$ds^2 = c^2 d\tau^2 = \left(1-\frac{2GM}{c^2r}\right)c^2dt^2 - \left(1-\frac{2GM}{c^2r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2\sin^2\theta^2\, d\emptyset^2 \tag{8}$$

This formula consists of four coordinates which are curved in space-time but in an infinitesimal area a frame can be formed where the coordinates $cdt, dr, d\theta, d\phi$ are linear and orthogonal to one another in that local area. The coefficients are constant in the local area but depending on r and θ thus they differ per location.
For more musings over the Schwarzschild metric see the following chapter.

For the full derivation of the Schwarzschild equation: (Schwarzschild, On the Gravitational Field of a Point-Mass, According to Einstein's Theory, 13 January 1916) and (Oas):
(Schwarzschild, On the Gravitational Field of a Point-Mass, According to Einstein's Theory, 13 January 1916)
(Oas)

2.16.1. Deliberations on Schwarzschild metric

Schwarzschild equation with polar coordinates:

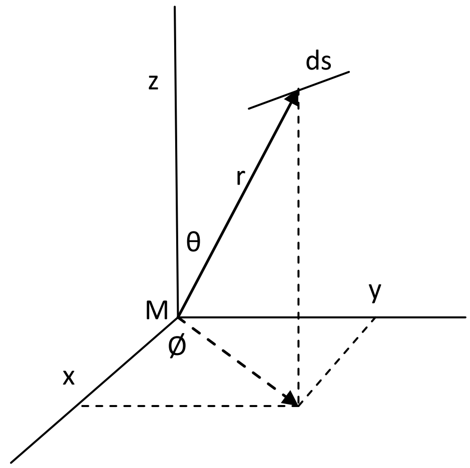
$$ds^2 = c^2 d\tau^2 = \left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta^2\, d\emptyset^2 \qquad (1)$$

We would like to have the coefficients dimensionless and the coordinates with the same dimension (here m^2).
Although it looks like the dimensions are not right; formula (1) really means:

$$ds^2 = \left(1 - \frac{2GM}{c^2 r}\right) dc^2 t^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 - \frac{r^2}{R_p^2} dR_p^2 .\theta^2 - \frac{r^2}{R_p^2} \sin^2\theta^2\, dR_p^2 .\emptyset^2 \quad where\ R_p = 1\ meter$$

So here the dimensions of the coordinates are all in meters while the coefficients are all dimensionless.
But for practical reasons formula (1) is used but keep in mind that actually the theta and phi are here in meters.
Here is G the gravitational constant, M the considered mass and the constant *c* is the light velocity.

We would like to know what formula (1) exactly stands for:
In space there is an object with mass M, considered as a point mass. This mass has, in standard Newtonian view, a gravitational field and thus force. In Einstein's, and also Schwarzschild's view, this mass deforms space-time and there is no gravitational force. A universal Euclidean coordinate system is chosen with M in the origin. When a particle, having a negligible mass, is held in space, it experiences a gravitational force due to the mass M. Now we let the particle move at free will. The movement, in standard Newtonian view, will cause acceleration because of the gravitational force. However the particle itself, in its co-moving frame, experiences no force at all; it surrenders itself to space. In Einstein's view the trajectory follows the curvature of space-time. The trajectory that will be followed by the particle is called a geodesic.



Space coordinates

A Euclidean coordinate system is chosen; either a Cartesian (t, x, y, x) or a polar (t, r, theta, phi) system as in (1). In case a polar coordinate system is chosen, the trajectory that is followed by the particle over the geodesic is a function of t, r, theta, and phi. The manner, in which the trajectory is depending on the coordinates, is expressed by coefficients with each coordinate. The coefficients are functions of the coordinate variables, but in this equation limited to *r* and *theta*. They are independent of *t* and *phi*. The equation (1) is spatially symmetric with respect to the origin (M) and thus rotation of the system will lead to the same result.
The coordinate system is a hypothetical system where each coordinate is expressed in units, as if the system is in a space-time completely free of any gravitational influences whatsoever. Schwarzschild now derived a formula that expressed the relationship between the trajectory, *ds*, (space-time path along the time coordinate) and the coordinate system.
The geodesic, which is a curved line, is considered as build up out of an infinite number of infinitesimal rectilinear line segments (*ds*). The space-time is curved because of the mass M, but in order to work with an Euclidean coordinate system, the area, build up with c*dt, dr, dtheta* and *dphi,* is considered as being infinitesimal small so that the coordinate system is rectilinear and mutually orthogonal in that small area; furthermore the coefficients are considered to be constant in that area. By moving to the next location the same goes, but with slightly different coefficients due to the change of *r* and *theta.* Thus by integrating *ds,* the total geodesic trajectory of the particle could be derived.

As we have seen

$$ds^2 = c^2 d\tau^2 = \left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta^2\, d\emptyset^2 \qquad (1)$$

Or in a shorter form

$$ds^2 = c^2 d\tau^2 = \sigma^2 c^2 dt^2 - \sigma^{-2} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta^2\, d\emptyset^2 \qquad (1a)$$

Here is: $\sigma = \sqrt{1 - \frac{2GM}{c^2 r}}$

The actual time is $d\tau$, the *proper time* (is the time measured), elapsed on clocks traveling with the object. The time *dt* is the time in a massless area or at infinity $r = \infty$. This time *dt* is a theoretical time, that cannot be measured but calculated back from the equation. The coordinate time at the location *r* is $\Delta time = \sigma^2 dt$. The distance covered in time $\Delta time = \sigma^2 dt$ is:

$$\Delta distance = \sqrt{\sigma^{-2} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta^2\, d\emptyset^2}$$

Thus the velocity of the particle in the frame is:

$$v^2/c^2 = \frac{1}{c^2}\left(\frac{\Delta distance}{\Delta time}\right)^2 = \frac{(\sigma^{-2} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta^2\, d\emptyset^2)}{\sigma^2 c^2 dt^2}$$

Filled in formula (1a):

$$ds^2 = c^2 d\tau^2 = \sigma^2 c^2 dt^2 - \frac{\sigma^{-2} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta^2\, d\emptyset^2}{\sigma^2 c^2 dt^2} \sigma^2 c^2 dt^2 \qquad (2)$$

$$c^2 d\tau^2 = \sigma^2 c^2 dt^2 \left(1 - \frac{\sigma^{-4}}{c^2}\left(\frac{dr}{dt}\right)^2 - \frac{\sigma^{-2} r^2}{c^2}\left(\frac{d\theta}{dt}\right)^2 - \frac{\sigma^{-2} r^2 \sin^2 \theta^2}{c^2}\left(\frac{d\emptyset}{dt}\right)^2\right) = \sigma^2 c^2 dt^2 \left(1 - \frac{v^2}{c^2}\right) \qquad (3)$$

$$v^2 = \sigma^{-4}\left(\frac{dr}{dt}\right)^2 + \sigma^{-2} r^2 \left(\frac{d\theta}{dt}\right)^2 + \sigma^{-2} r^2 \sin^2 \theta^2 \left(\frac{d\emptyset}{dt}\right)^2 \qquad (3a)$$

Thus from (3) we get:

$$d\tau = \frac{\sigma}{\gamma} dt \qquad with \qquad \sigma = \left(1 - \frac{2GM}{c^2 r}\right)^{1/2} \qquad and\ \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \qquad (4)$$

$$\Longrightarrow d\tau \leq dt \qquad (5)$$

As σ and γ are independent of *t* then $\boldsymbol{\tau = \frac{\sigma}{\gamma} t}$

(5a)

In case the particle is a photon $d\tau = 0$:

$$0 = \sigma^2 c^2 dt^2 - \sigma^{-2} dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta^2\, d\emptyset^2 \qquad (6)$$

So path in space is:

$$(\Delta distance)^2 = \sigma^{-2} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta^2\, d\emptyset^2 \qquad (6b)$$

Thus:

$$c^2 = \left(\frac{\Delta distance}{\Delta time}\right)^2 = \frac{\sigma^{-2} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta^2\, d\emptyset^2}{\sigma^2 dt^2} = v^2 \qquad (6c)$$

We see here that from the relation between photon and the frame, with M in the origin, that the total distance divided by the total time, is the light velocity. In the numerator we find the “normal” distance but in the denominator the time is multiplied with sigma, which means a ***smaller time***. Or we consider:

$$c^2 = \left(\frac{\Delta distance}{\Delta time}\right)^2 = \frac{\sigma^{-2}(\sigma^{-2} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta^2\, d\emptyset^2)}{dt^2} \qquad (6d)$$

In this case the total distance is multiplied with σ^{-2} $(\sigma \leq 1)$ which leads to a ***greater distance*** divided by the “normal” time. Now we look at the quotient of the “normal” distance and the “normal” time which leads to a smaller light velocity

$$\sigma^2 c^2 = \frac{\sigma^{-2} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta^2\, d\emptyset^2}{dt^2} \tag{6e}$$

So in the universal frame the light velocity is less than *c*.

The explanation is that due to the curved space-time the distance between two points is a curve over which the photon passes with the light velocity *c*. So the time over the travelled path is $t = \frac{path}{c}$. Considered from the universal frame the *distance* between the two points is a straight line thus the velocity of light between the two points

$$v = \frac{distance}{t} = \frac{distance}{\frac{path}{c}} = \frac{distance}{path} c$$

As the distance is shorter than the path, *v* is smaller than the light velocity. So the practical light velocity in the universal frame diminishes due to curved space-time.

So from the Schwarzschild equation we find:

$$(light\ velocity)^2 = \sigma^2 c^2 = \frac{\sigma^{-2} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta^2\, d\emptyset^2}{dt^2}$$

Above the coordinate time *dt* was mentioned as being a hypothetical time. As we do the measurement from the Earth a relation between the time of the Earth observer and the theoretical frame time can be derived as is done in chapter 3.4. The relation is also shown in (5a) $\boldsymbol{d\tau_{earth}} = \frac{\boldsymbol{\sigma_{earth}}}{\boldsymbol{\gamma_{earth}}} \boldsymbol{dt}$ or $\boldsymbol{dt} = \frac{\boldsymbol{\gamma_{earth}}}{\boldsymbol{\sigma_{earth}}} \boldsymbol{d\tau_{earth}}$.

Thus the time in a moving object slows down due to its velocity and the influence of the mass in the origin of the universal frame, all with respect to the universal frame. For somebody moving along with the moving frame the time is always the same i.e. the proper time with each second is equal to a second but its seconds are stretched out with respect to the seconds in the universal frame.

The frame of the photon moves with the photon so the distance must be zero but the velocity of the photon is *c*. As the distance is $cd\tau$, equal to 0, and *c* is not zero then it must be that $d\tau = 0$.

From the relation between the photon and the frame, with M in the origin, the velocity of the photon is *c*. The relation with the coordinates and coefficients are as follows:

$$v^2 = c^2 = \sigma^{-2}\left[\sigma^{-2}\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 + r^2 \sin^2\theta^2 \left(\frac{d\emptyset}{dt}\right)^2\right] \tag{3b}$$

In case $d\tau = d\theta = d\emptyset = 0$:

$$\sigma^4 c^2 dt^2 = dr^2 \quad so \quad \frac{\sigma^{-2} dr}{dt} = c \tag{7}$$

In case of a circle at the equator $(\theta = \frac{\pi}{2})$ $d\tau = dr = d\theta = 0$:

$$v = c = \frac{r}{\sigma}\frac{d\emptyset}{dt}$$

Another interesting point is where $r = \infty$ then $\sigma = 1$ and consequently: $\boldsymbol{d\tau = dt}$. (as mentioned above *t* is a chosen coordinate as if there is no mass.)

In general at infinity the movement is rectilinear and uniform and the equation becomes:

$$ds^2 = c^2d\tau^2 = c^2dt^2 - dr^2 - r^2d\theta^2 - r^2\sin^2\theta^2\, d\emptyset^2 \tag{8}$$

The original approach of Schwarzschild was in Cartesian coordinates. The derivation of the equation resulted in the equation (1) in polar coordinates but this could also be transformed to the original Cartesian coordinates as follows:

$$ds^2 = c^2d\tau^2 = \sigma^2c^2dt^2 - (dx^2 + dy^2 + dz^2) - \frac{1-\sigma^2}{\sigma^2r^2}(xdx + ydy + zdz)^2 \tag{9}$$

Remark:

The last term on the right hand side of (8) is sometimes expressed in a differentiation to τ (differentiation to the local clock) and sometimes to t (differentiation to the universal clock), this could be confusing.

$$ds^2 = c^2d\tau^2 = \sigma^2c^2dt^2 - \sigma^{-2}dr^2 - r^2d\theta^2 - r^2\sin^2\theta^2\, d\emptyset^2 \tag{1a}$$

Assume $\theta = \pi/2$

$$1 = \sigma^2\left(\frac{dt}{d\tau}\right)^2 - \sigma^{-2}\left(\frac{dr}{cd\tau}\right)^2 - r^2\left(\frac{d\emptyset}{cd\tau}\right)^2 \tag{10}$$

Or with partial derivatives:

$$1 = \sigma^2\left(\frac{dt}{d\tau}\right)^2 - \sigma^{-2}\left(\frac{dr}{cdt}\frac{dt}{d\tau}\right)^2 - r^2\left(\frac{d\emptyset}{cdt}\frac{dt}{d\tau}\right)^2$$

Then:

$$1 = \sigma^2\left(\frac{dt}{d\tau}\right)^2\left(1 - \frac{1}{\sigma^4}\left(\frac{dr}{cdt}\right)^2 - \frac{r^2}{\sigma^2}\left(\frac{d\emptyset}{cdt}\right)^2\right) \tag{11}$$

So in the calculation above is v the velocity in the universal frame.

If we consider the velocity with respect to the co-located clock $d\tau$ the velocity is:

$${v_{co}}^2 = \frac{\sigma^{-2}dr^2 + r^2d\theta^2 + r^2\sin^2\theta^2\, d\emptyset^2}{d\tau^2}$$

$$c^2d\tau^2 = \sigma^2c^2dt^2 - \frac{\sigma^{-2}dr^2 + r^2d\theta^2 + r^2\sin^2\theta^2\, d\emptyset^2}{d\tau^2}d\tau^2 = \sigma^2c^2dt^2 - {v_{co}}^2d\tau^2$$

$$c^2d\tau^2 + {v_{co}}^2d\tau^2 = \sigma^2c^2dt^2$$

Approximation (Taylor series with $v_{co} \ll$ C):

$$d\tau^2 = \frac{\sigma^2}{1 + \left(\frac{v_{co}}{c}\right)^2}dt^2 \approx \sigma^2\left(1 - \left(\frac{v_{co}}{c}\right)^2\right)dt^2$$

$$d\tau \approx \sigma\sqrt{1 - \left(\frac{v_{co}}{c}\right)^2}\,dt = \ d\tau = \frac{\sigma}{\gamma_{co}}dt$$

Thus for the approximation the result is the same.

In general a trajectory occurs in one flat plane, in that case the polar system can always be chosen such that the equator plane coincides with the trajectory plane; in that case $\theta = \pi/2$.

In case the trajectory is a circle with $\theta = \pi/2$ then $dr = 0$ and the equation becomes:

$$c^2d\tau^2 = \sigma^2c^2dt^2 - r^2d\emptyset^2$$

Additional deliberations:

Addition 1

Perhaps we should consider *ds* as an infinitesimal line segment, in space-time, with a size in meters which is measured by the travelling time of a photon over the length of the line segment multiplied with the light velocity. The line segment stays in the origin of its own frame. So the only measurement is time. In this case the line segment ds can be denoted as $ds = cd\tau$. Next we define another frame with an origin, in the Schwarzschild case, in the centre of a mass M. In this frame the distance between the line segment and the origin can be determined by various methods; lasers, rods etc. The only way we can determine the time is by the same clock as the line segment is measured. Thus the first result is: we have the $ds = cd\tau$ (left hand side of the Schwarzschild equation) and we have the distance (in the right hand side of the equation). So considering the Schwarzschild equation, the time part in the new frame is $(\mathrm{c}\Delta T)^2 = \left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 = c^2 d\tau^2 - (\Delta X)^2$ and $c^2 dt^2 = \frac{(\mathrm{c}\Delta T)^2}{\left(1 - \frac{2GM}{c^2 r}\right)}$. Thus ΔT and dt can only be derived, via the relation in the Schwarzschild equation, but not be measured.

Addition 2

We consider a particle in a co-moving frame, thus the particle is at rest in this frame. The only path, in space-time, is along its τ axis. We can express the movement of the particle with respect to another frame, which can be moving with respect to the particle. So the particle can be expressed in *t, x, y, z* of the new frame. The coordinates *t, x, y, z* are totally depending on the behavior of the particle so the world-line is naturally a function of τ.

Example

Next we calculate the time difference, at the Earth surface, between time at the Poles and at a location on the Equator, due to relativistic effects.

We start with the general Schwarzschild equation:

$$c^2 d\tau^2 = \sigma^2 c^2 dt^2 - \sigma^{-2} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta^2\, d\emptyset^2$$

At the poles $dr = 0, \theta = 0, d\theta = 0, \sin\theta = 0$ thus:

$$c^2 d\tau_{poles}{}^2 = \sigma^2 c^2 dt^2$$

$$d\tau_{poles} = \sigma dt$$

At the equator $dr = 0, \theta = \pi/2, d\theta = 0, \sin\theta = 1$

$$c^2 d\tau_{equator}{}^2 = \sigma^2 c^2 dt^2 - r^2 d\emptyset^2$$

$$c^2 d\tau_{equator}{}^2 = c^2 d\tau_{poles}{}^2 - r^2 d\emptyset^2$$

$$c^2 d\tau_{equator}{}^2 = c^2 d\tau_{poles}{}^2 \left\{1 - \frac{r^2}{c^2}\left(\frac{d\emptyset}{d\tau_{poles}}\right)^2\right\} = c^2 d\tau_{poles}{}^2 \left\{1 - \frac{v_{equator}{}^2}{c^2}\right\}$$

$$d\tau_{equator} = d\tau_{poles}\sqrt{1 - \frac{v_{equator}{}^2}{c^2}} \approx d\tau_{poles}\,(1 - \frac{1}{2}\frac{v_{equator}{}^2}{c^2})$$

The velocity on the equator is approximately 1672km/hr or 465m/s. Thus the time rate at the equator with respect to the poles is slightly lower:

$$d\tau_{equator} = d\tau_{poles}\,(1 - 1.2 * 10^{-12})$$

Consequently a person who would live 100 years at the North Pole would have lived 3.75 milliseconds longer at the Equator, ceteris paribus.

Addition 3

We can also give some special attention to the form

$$1 - \frac{2GM}{c^2 r}$$

This form reminds us of the relationship of the minimum velocity a mass needs to have in order to bring a mass from the Earth into space.

The calculation of this minimum speed is as follows:

The mass needs to have initially a kinetic energy that is equal to the energy or work done to bring the mass into infinity, i.e. so far away that the gravitational influence of *M* goes to zero.
We take as an example the Earth with mass *M* and the ejected mass as *m.* For calculation purposes we assume here that all the mass *M* is concentrated in the centre, as a point-mass.
Kinetic energy:

$$\frac{1}{2}mv^2$$

The gravitational force:

$$F = G\frac{Mm}{r^2}$$

Work done to bring the projectile from a distance *r* of the Earth centre to infinity:

$$\int_r^\infty F ds = \int_r^\infty G\frac{Mm}{s^2}ds = -G\frac{Mm}{s}\int_r^\infty = G\frac{Mm}{r}$$

Thus the minimum velocity shall be determined by the expression:

$$\frac{1}{2}mv^2 = G\frac{Mm}{r}$$

$$v^2 = \frac{2GM}{r}$$

$$v = \sqrt{\frac{2GM}{r}}$$

Or the distance from the centre of the mass is:

$$r = \frac{2GM}{v^2}$$

The maximum that the velocity *v* can have is the light velocity *c*, resulting in:

$$r = \frac{2GM}{c^2}$$

So if the distance *r* is shorter that the expression here above it is impossible to bring something outside the influence of the mass *M*. Even light photons cannot escape. This is what is called a “black hole”. The radius

$$r = R_s = \frac{2GM}{c^2}$$

is called the Schwarzschild radius or the radius of the event horizon surrounding a non rotating black hole.
It is said that even Einstein and Schwarzschild were not aware of the effect this coefficient in their formula had.

So now back to the formula:

$$r = \frac{2GM}{c^2}$$

$$1 = \frac{2GM}{c^2 r}$$

$$1 - \frac{2GM}{c^2 r} = 0$$

Which is the coefficient used in the Schwarzschild formula.
So normally the coefficient is between one and zero but in the special case that this coefficient is zero it is a black hole.
We will here not speculate what it means when the coefficient becomes negative.

2.17. Experiments

Various experiments have taken place to proof the validity of Einstein’s field equations. For the calculations use was made of this Schwarzschild equation.
Experiments that we have studied are:

- Hafele & Keating experiment (see chapter 3.1)
- Motion of Particles (see chapter 3.2)
- Deflection of Light (see chapter 3.2.2)
- Precession of the Perihelia (Mercury) (see chapter 3.2.3)
- Shapiro time delay (see chapter 3.3)
- Calculation of trajectory of bullet (see chapter3.6)

All calculations based on the Schwarzschild equation agreed with the found results of the experiments.

3. Experiments corroborating Einstein's theory

In this chapter we will study a number of experiments that corroborate the GR theory of Einstein. A rather important tool for this exercise is the Schwarzschild equation.
The relevant experiments are:

Hafele & Keating experiment (see chapter 3.1)
Motion of Particles (see chapter 3.2)
Deflection of Light (see chapter 3.2.2)
Precession of the Perihelia (Mercury) (see chapter 3.2.3)
Shapiro time delay (see chapter 3.3)
Calculation of trajectory of bullet (see chapter 3.6)

3.1. Experiment 1 - Calculation of Hafele & Keating experiment with the Schwarzschild equation

Derivation based on: A Hafele & Keating like thought experiment, by Paul B. Andersen, date: October 16, 2008 (Anderson, 2008)
Hafele and Keating were testing specific quantitative predictions of relativity, in particular the time distortion due to motion and gravity.
Two travelling clocks, in planes, experienced effects when flying in opposite directions, and this suggests that the rate at which time progresses depends on the motion of the observer. The east-going clock was moving in the same direction as the earth's rotation, so its velocity relative to the earth's non-rotating center was greater than that of the clock that remained in Washington, while the west-going clock's velocity was correspondingly reduced. The fact that the east-going clock fell behind (rotation speed of the earth plus the speed of the plane, relative to earth), and the west-going one (rotation speed of the earth minus the speed of the plane, relative to earth), got ahead, shows that the effect of motion causes time to progress more slowly. This effect of motion on time was predicted by Einstein in his original 1905 paper on relativity, written when he was 26.

All three clocks are moving to the east. Even though the west –going plane is moving to the west relative to the air, the air is moving to the east due to the earth's rotation.
From: (Crowell, Mar 11, 2018)

The experiment tries to calculate the time behavior of a cesium clock on various locations and speed with respect to Earth. These clocks experience the influence due to the Earth gravity and the speed with respect to Earth.
We will first derive a formula from the Schwarzschild equation based on some approximations. After that we will try to find an exact solution. Calculations with the exact solution will, obviously, be more complicated but via computer programs like Excel the execution should be easy and the result exact.
The Hafele & Keating experiment exists of two airplanes, both with a cesium clock, and a cesium clock placed on Earth location. The airplanes fly with a constant speed, one to the East and one to the West.

The applicability of the Schwarzschild equation will be scrutinized.

$$c^2 d\tau^2 = \left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta^2\, d\emptyset^2 \tag{1}$$

This is a universal frame with the centre of the Earth as the origin. The coordinates are $t, r, \theta, \emptyset$. The Earth is rotating within this frame. The distance to the Earth centre is denoted by *r*. *Theta* is the angle with the North pole and *phi* is the angle with the prime meridian (of the universal frame). $rd\theta$ is an arc length of r meter, thus if *r=1* then $d\theta = 1$ meter. Same holds for $rd\emptyset$. Next *dt* is a small change of *t* when measured in a region free of gravitational influences. Thus *t* is a hypothetical time which is not measured by a clock; it is pure theoretical. The time measured on location *r* is $d\tau$ of the co-located clock.

3.1.1. First the approximated approach

We assume that the clocks circle around the Earth either at the surface level or at certain heights above the surface of the Earth. Thus for each clock, on a circle, holds that $dr = 0$. Furthermore one assumes the trajectory of the clocks being in the plane of the equator which means $\theta = \pi/2$, so constant and thus $d\theta = 0$.

$$c^2 d\tau^2 = \left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 - r^2 d\emptyset^2 \tag{2}$$

$$d\tau^2 = \left(\left(1 - \frac{2GM}{c^2 r}\right) - \frac{r^2}{c^2}\left(\frac{d\emptyset}{dt}\right)^2\right) dt^2 \tag{3}$$

$$d\tau = \sqrt{\left(1 - \frac{2GM}{c^2 r} - \frac{v^2}{c^2}\right)}\, dt \tag{4}$$

Approximation with the first-order Taylor polynomials:

$$d\tau = \left(1 - \frac{GM}{c^2 r} - \frac{v^2}{2c^2}\right) dt \tag{5}$$

As *r* and *v* are constant the integration is simple:

$$\tau = \left(1 - \frac{GM}{c^2 r} - \frac{v^2}{2c^2}\right) t + \tau(0) \tag{6}$$

The interesting thing here is to compare the proper time of each clock. As a reference we take the proper time of the clock located on surface of the Earth. The other clocks are each located on different airplanes. So each clock has a speed and different location *r* even the clock at the Earth surface has the speed (v_1) of the Earth's rotation.

$$d\tau_1 = \left(1 - \frac{GM}{c^2 r_1} - \frac{{v_1}^2}{2c^2}\right) dt \tag{7}$$

$$d\tau_2 = \left(1 - \frac{GM}{c^2 r_2} - \frac{{v_2}^2}{2c^2}\right) dt \tag{8}$$

Now with Taylor and (7) and (8):

$$d\tau_2 = \frac{\left(1 - \frac{GM}{c^2 r_2} - \frac{{v_2}^2}{2c^2}\right)}{\left(1 - \frac{GM}{c^2 r_1} - \frac{{v_1}^2}{2c^2}\right)} d\tau_1 \cong \left(1 - \frac{GM}{c^2 r_2} - \frac{{v_2}^2}{2c^2}\right)\left(1 + \frac{GM}{c^2 r_1} + \frac{{v_1}^2}{2c^2}\right) d\tau_1$$

$$d\tau_2 \cong \left(1 + \frac{GM}{c^2 r_1} + \frac{{v_1}^2}{2c^2} - \frac{GM}{c^2 r_2}\left(1 + \frac{GM}{c^2 r_1} + \frac{{v_1}^2}{2c^2}\right) - \frac{{v_2}^2}{2c^2}\left(1 + \frac{GM}{c^2 r_1} + \frac{{v_1}^2}{2c^2}\right)\right) d\tau_1$$

As the terms $\frac{GM}{c^2 r_1}, \frac{{v_1}^2}{2c^2}, \frac{GM}{c^2 r_1}$ and $\frac{{v_1}^2}{2c^2}$ are very small their products could be ignored

$$d\tau_2 \cong \left(1 + \frac{GM}{c^2 r_1} + \frac{{v_1}^2}{2c^2} - \frac{GM}{c^2 r_2} - \frac{{v_2}^2}{2c^2}\right) d\tau_1$$

$$d\tau_2 \cong \left(1 + \frac{GM}{c^2}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) + \frac{{v_1}^2 - {v_2}^2}{2c^2}\right) d\tau_1 \tag{9}$$

If we assume that the clocks start at the same moment then $\tau_2 = 0$ when $\tau_1 = 0$ then the integrating constant is zero (see equation (6)):

$$\tau_2 \cong \left(1 + \frac{GM}{c^2}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) + \frac{{v_1}^2 - {v_2}^2}{2c^2}\right) \tau_1 \tag{10}$$

Thus the difference between the proper times of two clocks will be:

$$\tau_2 - \tau_1 = \left(\frac{GM}{c^2}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) + \frac{{v_1}^2 - {v_2}^2}{2c^2}\right) \tau_1 \tag{11}$$

Let us assume that τ_1 is the proper time of the clock which is located at the surface of the Earth, then $r_1 = R$; the radius of the Earth. The distance of the clock τ_2 in an airplane is then $R + h$:

$$\tau_2 - \tau_1 = \left(\frac{GM}{c^2}\left(\frac{1}{R} - \frac{1}{R+h}\right) + \frac{{v_1}^2 - {v_2}^2}{2c^2}\right) \tau_1 = \left(\frac{GM}{c^2}\left(\frac{\mathrm{h}}{\mathrm{R}^2}\right) + \frac{{v_1}^2 - {v_2}^2}{2c^2}\right) \tau_1 \tag{12}$$

Assume $\frac{h}{R} \ll 1$ and the gravitational acceleration is $g = \frac{GM}{R^2}$ then:

$$\boldsymbol{\tau_2 - \tau_1 = \left(\frac{gh}{c^2} - \frac{{v_2}^2 - {v_1}^2}{2c^2}\right) \tau_1} \tag{13}$$

As v_2 is the speed of a plane (to the East) and v_1 the (rotation) velocity of the Earth point (to the East) with respect to our frame, it is probably more practical to measure the ground velocities of the planes with respect to the Earth point velocity. So lets $v_1 = v_{earth}$ and $v_2 = v_{plane} + v_{earth}$ thus:

$${v_1}^2 - {v_2}^2 = {v_{earth}}^2 - \left(v_{plane} + v_{earth}\right)^2 = {v_{earth}}^2 - {v_{plane}}^2 - 2v_{earth}v_{plane} - {v_{earth}}^2$$

$$v_1{}^2 - v_2{}^2 = -v_{plane}{}^2 - 2v_{earth}v_{plane} = -v_{plane}\left(v_{plane} + 2v_{earth}\right)$$

Which corresponds with the formula used by **Hafele & Keating**:

$$\boldsymbol{\tau_{plane} - \tau_{earth} = \left(\frac{gh}{c^2} - \frac{v_{plane}\left(v_{plane} + 2v_{earth}\right)}{2c^2}\right)\tau_{earth}} \tag{14}$$

Thus this equation is completely derived from the Schwarzschild equation with some approximations.

Note1:

If the speed of the plane (v_2) is the ground-speed then, as approximation, at level *h*, $v_2 = \frac{R+h}{R}\left(v_{plane} + v_{earth}\right)$ the formula (13) should be used with the adapted v_2.

Note2:

According to me it is a better approach to use v_1 and v_2 as elucidated in the chapter below.

3.1.2. Elaboration on v_1 and v_2 in equation (13)

The speed, v_1 mentioned in equation (13), is the speed of a stationary point at the equator at the surface of the Earth. This speed $v_1 = r_1 \frac{d\emptyset}{dt}$, as mentioned in equation (3), is related to dt , however the measurement is done with respect to the proper time, so here at the Earth level. So a conversion has to be made. Thus the relation between the velocity in the universal frame and the velocity, related to the proper time at level r_1:

$$v_{1t} = r_1\frac{d\emptyset}{dt} = r_1\frac{d\emptyset}{d\tau}\frac{d\tau}{dt} = v_{1\tau}\frac{d\tau}{dt} \tag{14a}$$

Because *t* is the time in the universal frame, the $\frac{d\emptyset}{dt}$ is the same for each distance *r,* but the velocity at each level is determined by *r* so $v_t = r\frac{d\emptyset}{dt}$

Next we calculate v_{1t} at the surface of the Earth.

$$c^2d\tau^2 = \left(1 - \frac{2GM}{c^2r_1}\right)c^2dt^2 - r_1{}^2\left(\frac{d\emptyset}{d\tau}\right)^2 d\tau^2$$

$$\left(1 + \frac{r_1{}^2}{c^2}\left(\frac{d\emptyset}{d\tau}\right)^2\right)d\tau^2 = \left(1 - \frac{2GM}{c^2r_1}\right)dt^2$$

We define:

$$\sigma^2 = 1 - \frac{2GM}{c^2r}$$

$$\left(1 + \frac{v_{1\tau}{}^2}{c^2}\right)d\tau^2 = \left(1 - \frac{2GM}{c^2r_1}\right)dt^2 = \sigma_1{}^2dt^2$$

$$\left(\frac{d\tau}{dt}\right)^2 = \frac{\sigma_1{}^2}{\left(1 + \frac{v_{1\tau}{}^2}{c^2}\right)} \tag{14b}$$

So the conversion between the velocity, at the same level, with respect to the time of the universal frame and the proper time of that level is (from equation 14a and 14b):

$$v_{1t}{}^2 = v_{1\tau}{}^2 \left(\frac{d\tau}{dt}\right)^2 = v_{1\tau}{}^2 \frac{\sigma_1{}^2}{\left(1 + \frac{v_{1\tau}{}^2}{c^2}\right)}$$

So $\frac{d\tau}{dt}$ is determined by $v_{1\tau}$ the rotation of the Earth. So in case we consider $v_{1\tau\_plane} = v_{plane\ \_\tau} + v_{1\tau\_eart\,h}$ it is still at the Earth level so $\frac{d\tau}{dt}$ stays the same:

$$v_{1\tau\_plane} = v_{plane\ \_\tau} + v_{1\tau\_eart\,h} = r_1 \frac{d\emptyset}{d\tau}$$

$$r_1 \frac{d\emptyset}{dt} = r_1 \frac{d\emptyset}{d\tau} \frac{d\tau}{dt} = \left(v_{plane\ \_\tau} + v_{1\tau\_eart\,h}\right) \frac{d\tau}{dt} = \left(v_{plane\ \_\tau} + v_{1\tau\_eart\,h}\right) \frac{\sigma_1}{\sqrt{1 + \frac{v_{1\tau\_eart\,h}{}^2}{c^2}}}$$

$$\frac{d\emptyset}{dt} = \left(v_{plane\ \_\tau} + v_{1\tau\_eart\,h}\right) \frac{\sigma_1}{r_1 \sqrt{1 + \frac{v_{1\tau\_eart\,h}{}^2}{c^2}}}$$

Here we have calculated the rotation speed (angular velocity) in the universal frame. This is valid for each level, distance from the centre, but the velocity itself is determined by *r* times this rotation speed.
$v_{plane\ \_\tau}$ is the measured speed of the plane on ground level and with respect to the proper time, which is the only time available at that level. $v_{eart\,h\_\tau}$ is the (rotating) speed of a stationary point on Earth with respect to the universal frame but measured with the proper time on Earth level.
Now we make the conversion to the level of the plane:

$$v_{2t} = r_2 \frac{d\emptyset}{dt} = \frac{r_2}{r_1} \frac{\sigma_1\left(v_{plane\ \_\tau} + v_{1\tau\_eart\,h}\right)}{\sqrt{1 + \frac{v_{1\tau\_eart\,h}{}^2}{c^2}}}$$

Thus the velocity of the plane at level 2 can be considered as build up out of $v_{2t} = v_{2t\_eart\,h} + v_{2t\_plane}$:

$$v_{2t\_eart\,h} = \frac{r_2}{r_1} \frac{\sigma_1 v_{1\tau\_eart\,h}}{\sqrt{1 + \frac{v_{1\tau\_eart\,h}{}^2}{c^2}}}$$

And

$$v_{2t\_plane} = v_{2t} - v_{2t\_eart\,h} = \frac{r_2}{r_1} \frac{\sigma_1 v_{plane\ \_\tau}}{\sqrt{1 + \frac{v_{1\tau\_eart\,h}{}^2}{c^2}}}$$

Thus to summarize the result:
Conversion between $\boldsymbol{t}$ and $\boldsymbol{\tau}$ at the same level (in order to use the input data based on local measurements):

$$\boldsymbol{v_{1t\_earth} = v_{1\tau\_earth} \frac{\sigma_{earth}}{\sqrt{1 + \frac{v_{1\tau\_earth}{}^2}{c^2}}}} \tag{15}$$

Calculation of $\boldsymbol{v_{2t}}$ used in the formula (13), based on the plane velocity on ground level and time τ (which is the input data) and subsequently converted to plane level.

$$\boldsymbol{v_{2t} = \frac{r_2}{r_1} \frac{\sigma_{earth}\left(v_{plane_\tau} + v_{earth_\tau}\right)}{\sqrt{1 + \frac{v_{1\tau_{earth}}{}^2}{c^2}}}} \tag{16}$$

$$\tau_2 - \tau_1 = \left(\frac{gh}{c^2} - \frac{v_2{}^2 - v_1{}^2}{2c^2}\right) \tau_1 \qquad (13)$$

Thus formula (13) becomes:

$$\boldsymbol{\tau_2 - \tau_1 = \left(\frac{gh}{c^2} - \frac{\sigma_{earth}{}^2}{\left(1 + \frac{v_{1\tau_{earth}}{}^2}{c^2}\right)} \frac{\left[\left(\frac{R+h}{R}\right)^2 \left(v_{plane_\tau} + v_{earth_\tau}\right)^2 - v_{1\tau\_earth} \right]^2}{2c^2} \right) \tau_1} \tag{17}$$

3.1.3. The exact derivation

Let us start from formula (4)

$$d\tau = \sqrt{\left(1 - \frac{2GM}{c^2 r} - \frac{v^2}{c^2}\right)} dt \tag{4}$$

As *r* and *v* are constant the integration is simple:

$$\tau = \sqrt{\left(1 - \frac{2GM}{c^2 r} - \frac{v^2}{c^2}\right)} t + \tau(0) \tag{6a}$$

The interesting thing here is to compare the proper time of each clock. As a reference we take the proper time of the clock located on surface of the Earth. The other clocks are each located on a different airplane. So each clock has a speed and different location *r* even the clock at the earth surface has the speed of the Earth's rotation.

$$d\tau_1 = \sqrt{\left(1 - \frac{2GM}{c^2 r_1} - \frac{v_1{}^2}{c^2}\right)} dt \tag{7a}$$

$$d\tau_2 = \sqrt{\left(1 - \frac{2GM}{c^2 r_2} - \frac{v_2{}^2}{c^2}\right)} dt \tag{8a}$$

$$d\tau_2 = \sqrt{\frac{\left(1 - \frac{2GM}{c^2 r_2} - \frac{v_2{}^2}{c^2}\right)}{\left(1 - \frac{2GM}{c^2 r_1} - \frac{v_1{}^2}{c^2}\right)}} d\tau_1 \tag{9a}$$

If we assume that $\tau_2 = 0 \; when \; \tau_1 = 0$ then the integrating constant is zero:

$$\tau_2 = \sqrt{\frac{\left(1 - \frac{2GM}{c^2 r_2} - \frac{v_2{}^2}{c^2}\right)}{\left(1 - \frac{2GM}{c^2 r_1} - \frac{v_1{}^2}{c^2}\right)}} \tau_1 \tag{10a}$$

Thus the difference between the proper times of two clocks will be:

$$\tau_2 - \tau_1 = \left(\sqrt{\frac{\left(1 - \frac{2GM}{c^2 r_2} - \frac{v_2{}^2}{c^2}\right)}{\left(1 - \frac{2GM}{c^2 r_1} - \frac{v_1{}^2}{c^2}\right)}} - 1 \right) \tau_1 \tag{11a}$$

Let us assume that τ_1 is the proper time of the clock which is located at the surface of the Earth, then $r_1 = R$; the radius of the Earth. The distance of the clock τ_2 in a plane is then R *+h:*

$$\tau_2 - \tau_1 = \left(\sqrt{\frac{\left(1 - \frac{2GM}{c^2(R+h)} - \frac{{v_2}^2}{c^2}\right)}{\left(1 - \frac{2GM}{c^2R} - \frac{{v_1}^2}{c^2}\right)}} - 1 \right) \tau_1 \tag{12a}$$

As the v_2 is the speed of a plane (to the East with respect to the universal frame) and v_1 the (rotation) velocity of the Earth point (to the East) with respect to the universal frame. v_1 and v_2 are derived in chapter 3.1.5 equations $14b$ and $15b$.

$$\boldsymbol{\tau_{plane} - \tau_{earth} = \left(\sqrt{\frac{\left(1 - \frac{2GM}{c^2(R+h)} - \frac{{v_2}^2}{c^2}\right)}{\left(1 - \frac{2GM}{c^2R} - \frac{{v_1}^2}{c^2}\right)}} - 1 \right) \tau_{earth}} \tag{14b}$$

Or with the Schwarzschild radius is $R_s = \frac{2GM}{c^2}$:

$$\boldsymbol{\tau_{plane} - \tau_{earth} = \left(\sqrt{\frac{\left(1 - \frac{R_s}{(R+h)} - \frac{{v_2}^2}{c^2}\right)}{\left(1 - \frac{R_s}{R} - \frac{{v_1}^2}{c^2}\right)}} - 1 \right) \tau_{earth}} \tag{15b}$$

Thus this equation is completely derived from the Schwarzschild equation and is exact.

Some calculations based on executed experiments:

	PaulAnderson	Re_Spec_92	H&K
Vplane_ground_east_tau	232.55	670	173.98
Vplane_ground_West_tau	-232.55	-670	-124.43
Vplane2_east in dt	232.88	672.00	174.19
Vplane2_west in dt	-232.88	-672.00	-124.62
V_earth_tau	464.58	464.58	464.58
V_earth_t	464.58	464.58	464.58
V_earth_east on plane level dt	465.24	465.97	465.14
V_earth_west on plane level dt	465.24	465.97	465.28
H_east	9000	19000	7664
H_west	9000	19000	9526
t_earth	172328	59746.528	172328
Result (formula 7.1.13):			
Grav_delay (ns)_East	169.46	124.03	144.31
Kin_delay (ns)_East	-260.32	-358.69	-184.94
Total_East	-9.09E-08	-2.35E-07	-4.06E-08
Grav_delay (ns)_West	169.46	124.03	179.37
Kin_delay (ns)_West	155.16	57.63	95.67
Total_West	3.25E-07	1.82E-07	2.75E-07
Exact (Formula: 7.3.15):			
Total_East (ns)	-9.11E-08	-2.35E-07	-4.08E-08
Total_West	3.24E-07	1.81E-07	2.75E-07
diff east	2.35E-10	3.63E-10	1.56E-10
diff west	2.18E-10	3.67E-10	2.58E-10
diff east in %	-0.26%	-0.15%	-0.38%
diff west in %	0.07%	0.20%	0.09%
sidereal day: 23.9344696hr	86164.1	86164.1	86164.1
Lightvelocity	299792458	299792458	299792458
G	6.67E-11	6.67E-11	6.67E-11
M_earth	5.97E+24	5.97E+24	5.97E+24
R_earth	6371000	6371000	6371000
Schwarzschild radius Rs:	8.87E-03	8.87E-03	8.87E-03

Conclusion:

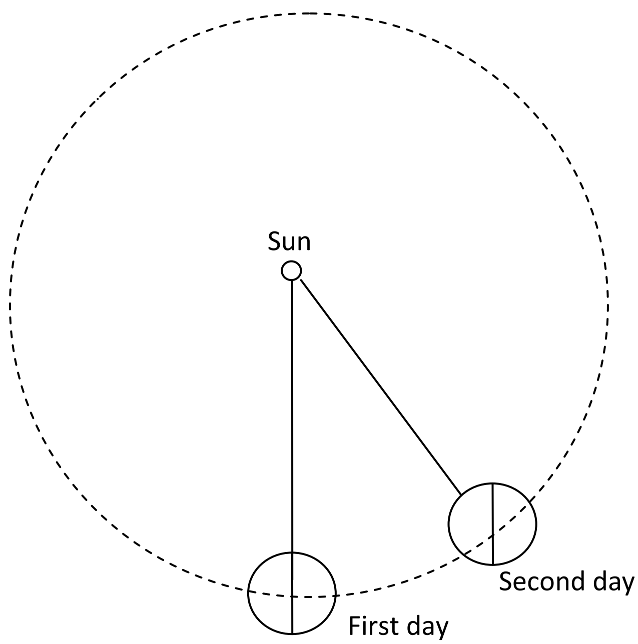
The approximations are correct within less than 0.4%

3.1.4. Calculation of the velocity of a stationary point at the equator on Earth surface

First we calculate the rotation time of the Earth; the so-called *sidereal day*:
The time length of a day is the difference in time between two successive highest points of the sun in the sky. This time difference is 24 hours. However because of the orbit around the sun the time of a rotation of the Earth around its own axis, is less than the time of a day. This is shown in the picture below. When the vertical line on the Earth rotates and is back in the same vertical direction then that is the time of an Earth rotation and the time is called sidereal day. So in a year there are on average 365.25 days but because of this offset there is an extra rotation resulting in 366.25 rotations in one year.
Thus

$$\frac{365.25}{366.25} * 24 * 3600 = 86164.1\ seconds. => \frac{86164.1}{3600} = 23.93447\ hours$$

A sidereal day is 23.9344696 hours (86164.1 sec). With R_{earth}=6371 km this gives a velocity of the stationary Earth clock of $v_{eart\,h} = \frac{2\pi R_{eart\,h}}{86164.1} = 464.58$ m/s (against 463.3 m/s for 24 hours).



3.1.5. Correction on derivation based on Paul Anderson (above)

One of the input data is the speed of the plane with respect to the ground. In the formula 3 in chapter 673.1.1 the speed, in the formula of Anderson, is based on *dt*, however the clock in that frame is $d\tau$ so the speed of the plane is also related to the co-moving clock $d\tau$ Thus we should adjust the formula. $d\tau$ is the proper time elapsed on clocks traveling with the object.
Let us start with the non-approximated formula 2 in chapter 3.1.1:

$$c^2 d\tau^2 = \left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 - r^2 d\emptyset^2 \quad (2)$$

$$d\tau^2 = \left(1 - \frac{2GM}{c^2 r}\right) dt^2 - \frac{r^2}{c^2}\left(\frac{d\emptyset}{d\tau}\right)^2 d\tau^2 \tag{3b}$$

$$d\tau^2\left[1 + \frac{r^2}{c^2}\left(\frac{d\emptyset}{d\tau}\right)^2\right] = \left(1 - \frac{2GM}{c^2 r}\right) dt^2 \tag{3c}$$

$$d\tau = \sqrt{\frac{\left(1 - \frac{2GM}{c^2 r}\right)}{1 + \frac{r^2}{c^2}\left(\frac{d\emptyset}{d\tau}\right)^2}}\, dt \tag{4b}$$

$$v_\tau = r\frac{d\emptyset}{d\tau} \tag{4c}$$

$$d\tau_1 = \sqrt{\frac{\left(1 - \frac{2GM}{c^2 r_1}\right)}{1 + \frac{{v_1}^2}{c^2}}}\, dt \tag{7b}$$

$$d\tau_2 = \sqrt{\frac{\left(1 - \frac{2GM}{c^2 r_2}\right)}{1 + \frac{{v_2}^2}{c^2}}}\, dt \tag{8b}$$

$$d\tau_2 = \sqrt{\frac{\left(1 - \frac{2GM}{c^2 r_2}\right)\left(1 + \frac{{v_1}^2}{c^2}\right)}{\left(1 - \frac{2GM}{c^2 r_1}\right)\left(1 + \frac{{v_2}^2}{c^2}\right)}}\, d\tau_1 \tag{9b}$$

$$\tau_2 = \sqrt{\frac{\left(1 - \frac{2GM}{c^2 r_2}\right)\left(1 + \frac{{v_1}^2}{c^2}\right)}{\left(1 - \frac{2GM}{c^2 r_1}\right)\left(1 + \frac{{v_2}^2}{c^2}\right)}}\,.\,\tau_1 \tag{10b}$$

$$\boldsymbol{\tau_{plane}} - \boldsymbol{\tau_{earth}} = \left(\sqrt{\frac{\left(1 - \frac{2GM}{c^2 r_2}\right)\left(1 + \frac{{v_1}^2}{c^2}\right)}{\left(1 - \frac{2GM}{c^2 r_1}\right)\left(1 + \frac{{v_2}^2}{c^2}\right)}} - 1\right)\boldsymbol{\tau_{earth}} \tag{11b}$$

Let us assume that τ_1 is the proper time of the clock which is located at the surface of the Earth, then $r_1 = R$; the radius of the Earth. The distance of the clock in a plane is then $R + h$:

$$\boldsymbol{\tau_{plane}} - \boldsymbol{\tau_{earth}} = \left(\sqrt{\frac{\left(1 - \frac{2GM}{c^2 (R+h)}\right)\left(1 + \frac{{v_{earth}}^2}{c^2}\right)}{\left(1 - \frac{2GM}{c^2 R}\right)\left(1 + \frac{{v_2}^2}{c^2}\right)}} - 1\right)\boldsymbol{\tau_{earth}} \tag{14b}$$

Or with the Schwarzschild radius $R_s = \frac{2GM}{c^2}$:

$$\boldsymbol{\tau_{plane}} - \boldsymbol{\tau_{earth}} = \left(\sqrt{\frac{\left(1 - \frac{R_s}{(R+h)}\right)\left(1 + \frac{{v_{eart\,h}}^2}{c^2}\right)}{\left(1 - \frac{R_s}{R}\right)\left(1 + \frac{{v_2}^2}{c^2}\right)}} - 1 \right) \boldsymbol{\tau_{earth}} \qquad (15b)$$

The given plane velocity is the velocity relative to the ground point so the actual velocity at level h is $v_2 = (v_{eart\,h} + v_{plane\ \ relative\ \ to\ eart\,h\ point}).\frac{R+h}{R}$

Up till now the formula is without any approximation.

After first order Taylor approximations of 14b, as was done previously, the result becomes:

$$\boldsymbol{\tau_{plane}} - \boldsymbol{\tau_{earth}} = \left(\left(1 - \frac{GM}{c^2(R+h)}\right)\left(1 + \frac{GM}{c^2 R}\right)\left(1 + \frac{{v_{eart\,h}}^2}{2c^2}\right)\left(1 - \frac{{v_2}^2}{2c^2}\right) - 1 \right) \boldsymbol{\tau_{earth}} \qquad (16)$$

$$\boldsymbol{\tau_{plane}} - \boldsymbol{\tau_{earth}} = \left(\left(1 + \frac{GM}{c^2}\left(\frac{1}{R} - \frac{1}{R+h}\right)\right)\left(1 + \frac{({v_{eart\,h}}^2 - {v_2}^2)}{2c^2}\right) - 1 \right) \boldsymbol{\tau_{earth}} \qquad (17)$$

$$\boldsymbol{\tau_{plane}} - \boldsymbol{\tau_{earth}} = \left(\left(1 + \frac{GM}{c^2}\frac{h}{R^2}\right)\left(1 + \frac{({v_{eart\,h}}^2 - {v_2}^2)}{2c^2}\right) - 1 \right) \boldsymbol{\tau_{earth}} \qquad (18)$$

$$\boldsymbol{\tau_{plane}} - \boldsymbol{\tau_{earth}} = \left(\frac{GM}{c^2}\frac{h}{R^2} + \frac{({v_{eart\,h}}^2 - {v_2}^2)}{2c^2} \right) \boldsymbol{\tau_{earth}} \qquad (19)$$

$$\boldsymbol{\tau_{plane}} - \boldsymbol{\tau_{earth}} = \left(\frac{gh}{c^2} - \frac{\left({v_2}^2 - {v_{eart\,h}}^2\right)}{2c^2} \right) \boldsymbol{\tau_{earth}} \qquad (20)$$

Note:

The speed of the airplane is given as the ground speed of the airplane. It is not obvious if this ground speed is measured with respect to the stationary clock on Earth or the clock in the plane. Let us assume the Earth clock is meant. In that case we have to find a conversion to the airplane level subsequently this involves the clock on airplane level. We will do this via the t in the universal frame. If we consider $\frac{d\emptyset_{eart\,h}}{dt}$ then this is the rotation velocity of the Earth in the universal frame. We can find the speed of the Earth at Earth level by multiplying $\frac{d\emptyset_{eart\,h}}{dt}$ with R; the distance from the origin. The speed of the Earth as seen from the airplane level is $(R+h)\frac{d\emptyset_{eart\,h}}{dt}$. For the plane this is similar, at Earth level the relative plane speed is $R\frac{d\emptyset_{plane}}{dt}$ and at airplane level $(R+h)\frac{d\emptyset_{plane}}{dt}$. Now $\frac{d\emptyset_{eart\,h}}{dt}$ and $\frac{d\emptyset_{plane}}{dt}$ have to be found.

We use chapter 3.1.5 equation $(4c)$

$$v_\tau = r\frac{d\emptyset}{d\tau} = r\frac{d\emptyset}{dt}\frac{dt}{d\tau} => \frac{d\emptyset}{dt} = \frac{v_\tau}{r}\frac{d\tau}{dt}$$

Next we use chapter 3.1.5 equation $(4b)$

$$\frac{d\tau}{dt} = \sqrt{\frac{\left(1 - \frac{2GM}{c^2 r}\right)}{1 + \frac{{v_\tau}^2}{c^2}}}$$

Thus

$$\frac{d\emptyset}{dt} = \frac{v_\tau}{r}\frac{d\tau}{dt} = \frac{v_\tau}{r}\sqrt{\frac{\left(1-\frac{2GM}{c^2 r}\right)}{1+\frac{{v_\tau}^2}{c^2}}}$$

All the components at the right hand side are known.

At ground level:

$$\frac{d\emptyset_{eart\,h}}{dt} = \frac{v_{eart\,h}}{R}\sqrt{\frac{\left(1-\frac{2GM}{c^2R}\right)}{1+\frac{{v_{eart\,h}}^2}{c^2}}} \quad and \quad \frac{d\emptyset_{plane}}{dt} = \frac{v_{plane}}{R}\sqrt{\frac{\left(1-\frac{2GM}{c^2R}\right)}{1+\frac{{v_{eart\,h}}^2}{c^2}}}$$

Now at the airplane level:

$$v_2 = v_{2\tau\_eart\,h} + v_{2\tau\_plane} = (R+h)\left(\frac{d\emptyset_{eart\,h}}{dt}+\frac{d\emptyset_{plane}}{dt}\right)$$

$$v_2 = v_{2\tau\_eart\,h} + v_{2\tau\_plane} = \frac{(R+h)}{R}\sqrt{\frac{\left(1-\frac{2GM}{c^2R}\right)}{1+\frac{{v_{eart\,h}}^2}{c^2}}}\left(v_{eart\,h}+v_{plane}\right)$$

With first order Taylor approximation:

$$v_2 = \frac{(R+h)}{R}\sqrt{\left(1-\frac{2GM}{c^2R}\right)\left(1-\frac{{v_{eart\,h}}^2}{c^2}\right)}\left(v_{eart\,h}+v_{plane}\right)$$

So the relevant formulae are

$$\boldsymbol{v_2 = \frac{(R+h)}{R}\left(1-\frac{GM}{c^2R}-\frac{{v_{earth}}^2}{2c^2}\right)\left(v_{earth}+v_{plane}\right)}$$

$$\boldsymbol{\tau_{plane} - \tau_{earth} = \left(\frac{gh}{c^2}-\frac{\left({v_2}^2-{v_{earth}}^2\right)}{2c^2}\right)\tau_{earth}} \qquad (\mathbf{20})$$

Conclusion:
Although this solution above in (20) seems to me the right approach; after some numeric calculations the difference in results are within 0.4%.

Exact (Formula: 7.3.15):	PaulAnderson	Re_Spec_92	H&K
Total_East	-9.11E-08	-2.35E-07	-4.08E-08
Total_West	3.24E-07	1.81E-07	2.75E-07
sidereal day: 23.9344696hr	86164.1	86164.1	86164.1
Lightvelocity	299792458	299792458	299792458
G	6.67E-11	6.67E-11	6.67E-11
M_earth	5.97E+24	5.97E+24	5.97E+24
R_earth	6371000	6371000	6371000
Schwarzschild radius Rs:	8.87E-03	8.87E-03	8.87E-03
Formula: 7.5.20			
Vplane_ground_east_tau	232.55	670	173.98
Vplane_ground_West_tau	-232.55	-670	-124.43
V_earth_tau	464.58	464.58	464.58
H_east	9000	19000	7664
H_west	9000	19000	9526
t_earth	172328	59747	172328
v2_east	698.12	1137.96	639.33
v2_west	232.36	-206.03	340.56
Grav_delay (ns)_East	1.69E-07	1.24E-07	1.44E-07
Kin_delay (ns)_East	-2.60E-07	-3.59E-07	-1.85E-07
Total_East	-9.09E-08	-2.35E-07	-4.06E-08
Grav_delay (ns)_West	1.69E-07	1.24E-07	1.79E-07
Kin_delay (ns)_West	1.55E-07	5.76E-08	9.57E-08
Total_West	3.25E-07	1.82E-07	2.75E-07
diff east	-2.35E-10	-3.63E-10	-1.56E-10
diff west	-2.18E-10	-3.67E-10	-3.23E-10
diff east in %	0.26%	0.15%	0.38%
diff west in %	-0.07%	-0.20%	-0.12%

3.1.6. Deliberations on the Hafele & Keating experiment and the Schwarzschild equation

$$ds^2 = c^2 d\tau^2 = \left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta^2 \, d\emptyset^2 \qquad (1)$$

$$ds^2 = c^2 d\tau^2 = \sigma^2 c^2 dt^2 - \sigma^{-2} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta^2 \, d\emptyset^2 \qquad (1a)$$

In the H & K experiment the time of the Naval Observatory Clock (NOC) and the speed of an airplane are mentioned. The question is: what is the time in the Schwarzschild equation and what is the airplane speed in the equation?
There is a stationary clock on ground level on the equator and in the two airplanes in the equator plane; one flying east and the other flying west. The flight velocity with respect to the ground is equal but opposite for both airplanes.

As the experiment is in the equator plane, $\theta = \frac{\pi}{2}$ and constant, and both airplanes fly in a circular orbit so $r = constant$ the formula (1) simplifies to:

$$c^2 d\tau^2 = \left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 - r^2 d\emptyset^2 \qquad (2)$$

The coordinates in the Schwarzschild equation (1) could be considered as a universal frame, without any gravitation, in the direction of the Earth-North pole. The Earth is rotating in this universal frame. The three clocks are in their own proper frame so their time is denoted by τ.
The time in the universal frame cannot be measured but is pure theoretical and is:

$$dt^2 = \frac{d\tau^2 + \frac{r^2}{c^2} d\emptyset^2}{\left(1 - \frac{2GM}{c^2 r}\right)} = \sigma^{-2}\left(d\tau^2 + \frac{r^2}{c^2} d\emptyset^2\right) = \sigma^{-2}\left(1 + \frac{r^2}{c^2}\left(\frac{d\emptyset}{d\tau}\right)^2\right) d\tau^2 \qquad (4)$$

If *t=0* when $\tau = 0$ then the integrating constant is zero and:

$$t = \sigma^{-1}\sqrt{1 + \frac{r^2}{c^2}\left(\frac{d\emptyset}{d\tau}\right)^2}\,\tau = \sigma^{-1}\sqrt{1 + \frac{v^2}{c^2}}\,\tau \qquad (4a)$$

Or first order Taylor approximated for $v^2 \ll c^2$

$$\boldsymbol{t} = \sigma^{-1}\sqrt{1 + \frac{v^2}{c^2}}\,\tau = \frac{1}{\sigma\sqrt{1 - \frac{v^2}{c^2}}}\,\tau = \frac{\boldsymbol{\gamma}}{\boldsymbol{\sigma}}\,\boldsymbol{\tau} \qquad (\boldsymbol{4b})$$

3.2. Experiment 2 - Motion of Particles in Schwarzschild Geometry

The derivations in this chapter are for a great deal based on information out of the following articles:
(Biesel, 2008) The Precession of Mercury's Perihelion
Owen Biesel
January 25, 2008 (Biesel, 2008)
(Magnan) Christian Magnan: Complete calculations of the perihelion precession of Mercury and the deflection of light by the Sun in General Relativity (Magnan)
(Pe'er1, 2014) Schwarzschild Solution and Black Holes
Asaf Pe'er1
February 19, 2014 (Pe'er1, 2014)

Here follows the derivation of equations for the motion of particles and in particular the **perihelion precession of Mercury**, the **deflection of light by the Sun**, the **Shapiro experiment** and the calculation of a **trajectory of a bullet**.

As a starting point the Schwarzschild equation is used because it meets the Einstein field equations and of its proven applicability. As the metric in the Schwarzschild geometry is symmetric in time *t* and in the polar coordinate Ø i.e. none of the coefficients in the equation is depending on either *t* or Ø and therefore it meets the Noether theorem. The Noether theorem says that symmetry leads to conservation and in this case the independency of *t* leads to conservation of *E* (momentum) and the independency of Ø leads to conservation of the angular momentum.

Schwarzschild metric:

$$ds^2 = \sigma^2 c^2 dt^2 - \frac{dr^2}{\sigma^2} - \frac{r^2}{R_p^2} dR_p^2.\theta^2 - \frac{r^2}{R_p^2}\sin^2\theta^2\, dR_p^2.\emptyset^2$$

By adding *Rp=1*m we get the dimensions right; the coefficients are dimensionless and the coordinates are in meters (also *cdt=d(ct)* is in meters).

However the formula is usually applied in a more practical form:

$$\boldsymbol{ds^2 = \sigma^2 c^2 dt^2 - \frac{dr^2}{\sigma^2} - r^2 d\theta^2 - r^2 \sin^2\theta^2\, d\emptyset^2}$$

$$\sigma = \sqrt{1 - \frac{2GM}{c^2 r}} = \sqrt{1 - \frac{R_s}{r}} \quad Schwarzschild\ radius: R_s = \frac{2GM}{c^2} \tag{1a}$$

First we derive a number of useful formulas.

The Schwarzschild metric for polar coordinates

$$g_{00} = \boldsymbol{\sigma^2};\ \ g_{11} = \frac{\boldsymbol{-1}}{\boldsymbol{\sigma^2}};\ \ g_{22} = \boldsymbol{-r^2};\ \ g_{33} = \boldsymbol{-r^2\sin^2\theta} = -r^2$$

$$g^{00} = \frac{\boldsymbol{1}}{\boldsymbol{\sigma^2}};\ \ g^{11} = \boldsymbol{-\sigma^2};\ \ g^{22} = \frac{\boldsymbol{-1}}{\boldsymbol{r^2}};\ \ g^{33} = \frac{\boldsymbol{-1}}{\boldsymbol{r^2\sin^2\theta}}$$

$$\frac{d\sigma}{dr} = \frac{\boldsymbol{R_s}}{\boldsymbol{2r^2\sigma}}$$

Metric first derivative on polar coordinates

$$\frac{\partial g_{00}}{\partial r} = \frac{\boldsymbol{R_s}}{\boldsymbol{r^2}};\ \ \frac{\partial g_{11}}{\partial r} = \frac{\boldsymbol{R_s}}{\boldsymbol{r^2\sigma^4}};\ \ \frac{\partial g_{22}}{\partial r} = (\boldsymbol{-2r});\ \ \frac{\partial g_{33}}{\partial r} = \left(\boldsymbol{-2r\sin^2\theta}\right) = \boldsymbol{-2r};\ \frac{\partial g_{33}}{\partial\theta} = (\boldsymbol{-2r.sin(\theta)\cos(\theta)})$$

The relevant (non-zero) Christoffel symbols for Schwarzschild polar coordinates:

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\alpha}\left\{\frac{\partial g_{\nu\alpha}}{\partial x^{\mu}} + \frac{\partial g_{\mu\alpha}}{\partial x^{\nu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\alpha}}\right\}$$

$$\Gamma^0_{01} = \Gamma^0_{10} = \frac{1}{2} g^{00}\left\{\frac{\partial g_{00}}{\partial r}\right\} = \frac{\boldsymbol{R_s}}{\boldsymbol{2r^2\sigma^2}};\ \ \Gamma^1_{00} = \frac{1}{2} g^{11}\left\{-\frac{\partial g_{00}}{\partial r}\right\} = \frac{\boldsymbol{\sigma^2 R_s}}{\boldsymbol{2r^2}};\ \ \Gamma^1_{11} = \frac{1}{2} g^{11}\left\{\frac{\partial g_{11}}{\partial r}\right\} = \frac{\boldsymbol{-R_s}}{\boldsymbol{2r^2\sigma^2}}$$

$$\Gamma^1_{22} = \frac{1}{2}g^{11}\left\{-\frac{\partial g_{22}}{\partial r}\right\} = -\boldsymbol{r\sigma^2};\ \ \Gamma^1_{33} = \frac{1}{2}g^{11}\left\{-\frac{\partial g_{33}}{\partial r}\right\} = -\boldsymbol{r\sigma^2\sin^2\theta}$$

$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{2}g^{22}\left\{\frac{\partial g_{22}}{\partial r}\right\} = \boldsymbol{\frac{1}{r}};\ \ \Gamma^2_{33} = \frac{1}{2}g^{22}\left\{-\frac{\partial g_{33}}{\partial \theta}\right\} = -\boldsymbol{\cos\theta\sin\theta}$$

$$\Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{2}g^{33}\left\{\frac{\partial g_{33}}{\partial r}\right\} = \boldsymbol{\frac{1}{r}};\ \ \Gamma^3_{23} = \Gamma^3_{32} = \frac{1}{2}g^{33}\left\{\frac{\partial g_{33}}{\partial \theta}\right\} = \boldsymbol{\frac{\cos\theta}{\sin\theta}}$$

All other Christoffel symbols are zero

The Schwarzschild equation meets Einstein's field equations (see 5.1) and therefore there is no gravity effect others than curvature of space-time. So the geodesic equations are zero.
The geodesic equations are:

$$\frac{d^2x^\alpha}{d\lambda^2} + \Gamma^\alpha_{\mu\nu}.\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = 0$$

Work-out for the four coordinates, where λ is the affine parameter:

$$\boldsymbol{For\ t}:\ \frac{d^2t}{d\lambda^2} + \Gamma^t_{\mu\nu}.\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = \frac{d^2t}{d\lambda^2} + 2\Gamma^0_{01}.\frac{dt}{d\lambda}\frac{dr}{d\lambda} = \boldsymbol{\frac{d^2t}{d\lambda^2} + 2\frac{R_s}{2r^2\sigma^2}.\frac{dt}{d\lambda}\frac{dr}{d\lambda} = 0}$$

$$\boldsymbol{For\ r}:\ \frac{d^2r}{d\lambda^2} + \Gamma^r_{\mu\nu}.\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = \frac{d^2r}{d\lambda^2} + \Gamma^1_{00}.\left(\frac{dt}{d\lambda}\right)^2 + \Gamma^1_{11}.\left(\frac{dr}{d\lambda}\right)^2 + \Gamma^1_{22}.\left(\frac{d\theta}{d\lambda}\right)^2 + \Gamma^1_{33}.\left(\frac{d\varphi}{d\lambda}\right)^2 = 0$$

$$\frac{d^2r}{d\lambda^2} + \Gamma^r_{\mu\nu}.\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = \boldsymbol{\frac{d^2r}{d\lambda^2} + \frac{\sigma^2R_s}{2r^2}.\left(\frac{dt}{d\lambda}\right)^2 - \frac{R_s}{2r^2\sigma^2}.\left(\frac{dr}{d\lambda}\right)^2 - r\sigma^2.\left(\frac{d\theta}{d\lambda}\right)^2 - r\sigma^2\sin^2\theta.\left(\frac{d\varphi}{d\lambda}\right)^2 = 0}$$

$$\boldsymbol{For\ \theta}:\ \frac{d^2\theta}{d\lambda^2} + \Gamma^\theta_{\mu\nu}.\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = \frac{d^2\theta}{d\lambda^2} + 2\Gamma^2_{12}.\frac{dr}{d\lambda}\frac{d\theta}{d\lambda} + \Gamma^2_{33}.\left(\frac{d\varphi}{d\lambda}\right)^2 = 0$$

$$\boldsymbol{\frac{d^2\theta}{d\lambda^2} + 2\frac{1}{r}.\frac{dr}{d\lambda}\frac{d\theta}{d\lambda} - \cos\theta\sin\theta\ \ .\left(\frac{d\varphi}{d\lambda}\right)^2 = 0}$$

$$\boldsymbol{For\ \varphi}:\ \frac{d^2\varphi}{d\lambda^2} + \Gamma^\varphi_{\mu\nu}.\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = \frac{d^2\varphi}{d\lambda^2} + 2\Gamma^3_{13}.\frac{dr}{d\lambda}\frac{d\varphi}{d\lambda} + 2\Gamma^3_{23}.\frac{d\theta}{d\lambda}\frac{d\varphi}{d\lambda} = 0$$

$$\boldsymbol{\frac{d^2\varphi}{d\lambda^2} + 2\frac{1}{r}.\frac{dr}{d\lambda}\frac{d\varphi}{d\lambda} + 2\frac{\cos\theta}{\sin\theta}.\frac{d\theta}{d\lambda}\frac{d\varphi}{d\lambda} = 0}$$

To summarize the resulting four equations:

$$\frac{d^2t}{d\lambda^2} + 2\frac{R_s}{2r^2\sigma^2}\frac{dt}{d\lambda}\frac{dr}{d\lambda} = 0 \tag{1}$$

$$\frac{d^2r}{d\lambda^2} + \frac{\sigma^2R_s}{2r^2}\left(\frac{dt}{d\lambda}\right)^2 - \frac{R_s}{2r^2\sigma^2}\left(\frac{dr}{d\lambda}\right)^2 - r\sigma^2\left(\frac{d\theta}{d\lambda}\right)^2 - r\sigma^2\sin^2\theta\left(\frac{d\varphi}{d\lambda}\right)^2 = 0 \tag{2}$$

$$\frac{d^2\theta}{d\lambda^2} + 2\frac{1}{r}\frac{dr}{d\lambda}\frac{d\theta}{d\lambda} - \cos\theta\sin\theta\ \left(\frac{d\varphi}{d\lambda}\right)^2 = 0 \tag{3}$$

$$\frac{d^2\varphi}{d\lambda^2} + 2\frac{1}{r}\frac{dr}{d\lambda}\frac{d\varphi}{d\lambda} + 2\frac{\cos\theta}{\sin\theta}\frac{d\theta}{d\lambda}\frac{d\varphi}{d\lambda} = 0 \tag{4}$$

First we will not withhold you from the derivation of Asaf Pe'er in his article "Schwarzschild Solution and Black Holes" (Pe'er1, 2014) but after that we will show a more simple approach.

Now according to Asaf Pe'er:

At first sight, there does not seem to be much hope for simply solving this set of 4 coupled equations by inspection. Fortunately our task is greatly simplified by the high degree of symmetry of the Schwarzschild metric. We know that there are four Killing vectors: three for the spherical symmetry, and one for time translations. Each of these will lead to a constant of the motion for a free particle. Recall that if K_μ is a Killing vector, we know that

$$K_\mu \frac{dx^\mu}{d\lambda} = constant. \tag{5}$$

In addition, there is another constant of the motion that we always have for geodesics (there is no acceleration); metric compatibility implies that along the path the quantity

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu$$

$$\left(\frac{ds}{d\lambda}\right)^2 = \left(\frac{cd\tau}{d\lambda}\right)^2 = c^2\varepsilon = g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} \tag{6}$$

is constant. (This is simply normalization of the 4-velocity: take $\lambda = \tau$ and get $g_{\mu\nu}\, U^\mu U^\nu = c^2\varepsilon$, with $\varepsilon = 1$ for massive particles and $\varepsilon = 0$ for mass-less particles. We may also consider space-like geodesics, for which $\varepsilon = -1$).

Instead of trying to solve directly the geodesic equations using the four conserved quantities associated with Killing vectors, let us first analyze the constraints.
In flat space-time, the symmetries represented by the Killing vectors, and according to Noether's theorem, lead to very familiar conserved quantities: Invariance under **time translations** leads to ***conservation of energy***, while invariance under **spatial rotations** leads to conservation of the three components of ***angular momentum***.

Essentially the same applies to the Schwarzschild metric. We can think of the angular momentum as a three-vector with a magnitude (one component) and direction (two components). Conservation of the *direction* of angular momentum means that the particle will move in a plane. We can choose this to be the equatorial plane of our coordinate system; if the particle is not in this plane, we can rotate coordinates until it is

$$\theta = \frac{\pi}{2} \tag{7}$$

The other two Killing vectors correspond to **energy** and the magnitude of **angular momentum**. The time-like Killing vector is $K^\mu = (1,0,0,0)^T$, and thus

$$K_\mu = K^\nu g_{\mu\nu} = \left(\left(1 - \frac{2GM}{r}\right), 0,0,0\right) \tag{8}$$

This gives rise to conservation of energy (per unit mass of the particle), since using Equation 5 in chapter 3.2,

$$K_\mu \frac{dx^\mu}{d\lambda} = \left(1 - \frac{2GM}{c^2 r}\right)\frac{dt}{d\lambda} = \frac{E}{c^2}\,, \tag{9}$$

Where *E* is constant of motion.

Similarly, the Killing vector whose conserved quantity is the magnitude of the angular momentum is $L = \partial_\phi (L^\mu = (0,0,0,-1)^T)$, and thus

$$L_\mu = (0,0,0,-r^2 \sin^2\theta)\,. \tag{10}$$

Using $sin\,\theta = 1$ because $\theta = \frac{\pi}{2}$, one finds

$$r^2 \frac{d\phi}{d\lambda} = L\ . \tag{11}$$

Where L, the total angular momentum, is the second conserved quantity. (For mass-less particles these can be thought of as the energy and angular momentum; for massive particles they are the energy and angular momentum per unit mass of the particle.) (For more information on the angular momentum see **Appendix 8 page: **194)

Further note that the constancy of the angular momentum in Equation 11 is the GR equivalent of Kepler's second law (equal areas are swept out in equal times).

Alternative derivation:

Despite the remark from Asaf Pe'er above, it is not so complicated to solve a part of the geodesic.
Now let's solve the geodesic equations from the equations (1) and (4).
We can work out equation (1):

$$\frac{d^2t}{d\lambda^2} + 2\frac{R_s}{2r^2\sigma^2}\frac{dt}{d\lambda}\frac{dr}{d\lambda} = 0$$

Multiply with (1a)

$$\sigma^2 = 1 - \frac{2GM}{c^2 r} = 1 - \frac{R_s}{r}$$

$$\frac{d^2t}{d\lambda^2}\sigma^2 + \frac{R_s}{r^2}\frac{dt}{d\lambda}\frac{dr}{d\lambda} = 0$$

$$\frac{d^2t}{d\lambda^2}(1 - \frac{R_s}{r}) + \frac{R_s}{r^2}\frac{dt}{d\lambda}\frac{dr}{d\lambda} = 0$$

$$\frac{d^2t}{d\lambda^2} + \frac{R_s}{r^2}\frac{dt}{d\lambda}\frac{dr}{d\lambda} - \frac{R_s}{r}\frac{d^2t}{d\lambda^2} = 0$$

$$\Rightarrow \frac{d}{d\lambda}\left(\frac{dt}{d\lambda} - \frac{R_s}{r}\frac{dt}{d\lambda}\right) = 0 \ \Rightarrow \frac{d}{d\lambda}\left[\frac{dt}{d\lambda}\left(1 - \frac{R_s}{r}\right)\right] = 0$$

This means that $\boldsymbol{\frac{dt}{d\lambda}\left(1 - \frac{R_s}{r}\right)}, or\ \boldsymbol{\left(1 - \frac{R_s}{r}\right)\frac{Cdt}{d\lambda}}$ to be precise, is constant with respect to time, so it is a conserved quantity. We recognize here the conserved quantity $\boldsymbol{\frac{E}{c}}$, which is the energy or momentum per unit mass.
Thus:

$$\boldsymbol{\frac{cdt}{d\lambda}\left(1 - \frac{R_s}{r}\right) = constant = \frac{E}{c}\ \ (total\ energy)} \tag{9}$$

Next we will work out equation (4), but to make life a bit easier we assume to be in the equatorial plane and so $\theta = \frac{\pi}{2}$:

$$\frac{d^2\varphi}{d\lambda^2} + 2\frac{1}{r}\frac{dr}{d\lambda}\frac{d\varphi}{d\lambda} + 2\frac{\cos\theta}{\sin\theta}\frac{d\theta}{d\lambda}\frac{d\varphi}{d\lambda} = 0$$

$$\frac{d^2\varphi}{d\lambda^2} + 2\frac{1}{r}\frac{dr}{d\lambda}\frac{d\varphi}{d\lambda} = 0$$

$$\Rightarrow \frac{1}{r^2}\frac{d}{d\lambda}\left(r^2\frac{d\varphi}{d\lambda}\right) = 0$$

So again $r^2\frac{d\varphi}{d\lambda}$ is constant with respect to time and thus a conserved quantity. We see that $r^2\frac{d\varphi}{d\lambda} = r^2\omega = vr$ and recognize that this is the angular momentum per unit mass.

Thus:

$$\boldsymbol{r^2\frac{d\varphi}{d\lambda} = constant = L\ (angular\ momentum)} \qquad \mathbf{(11)}$$

3.2.1. The Gravitational Potential

Armed with this information, we can now analyze the orbits of particles in Schwarzschild metric.

We begin by writing explicitly Equation 6, using Equation 7,

$$\left(1-\frac{2GM}{c^2r}\right)c^2\left(\frac{dt}{d\lambda}\right)^2 - \left(1-\frac{2GM}{c^2r}\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2 - r^2\left(\frac{d\phi}{d\lambda}\right)^2 = c^2\varepsilon. \qquad (12)$$

$$\left(1-\frac{2GM}{c^2r}\right)c^2\left(\frac{dt}{d\lambda}\right)^2 - \left(1-\frac{2GM}{c^2r}\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2 - \frac{L^2}{r^2} = c^2\varepsilon$$

Multiply this Equation by (1−2GM/r) and use the expressions for *E* and *L* (Equations 10 and 11) to write

$$\left(1-\frac{2GM}{c^2r}\right)^2 c^2\left(\frac{dt}{d\lambda}\right)^2 - \left(\frac{dr}{d\lambda}\right)^2 - \left(1-\frac{2GM}{c^2r}\right)\left(\frac{L^2}{r^2}+c^2\varepsilon\right) = 0$$

$$\frac{E^2}{c^2} - \left(\frac{dr}{d\lambda}\right)^2 - \left(1-\frac{2GM}{c^2r}\right)\left(\frac{L^2}{r^2}+c^2\varepsilon\right) = 0\,. \qquad (13)$$

Clearly, we have made a great progress: instead of the 4 geodesic Equations, we obtain one differential equation for r(λ).

We can re-write Equation (13) as

$$\frac{1}{2}\left(\frac{dr}{d\lambda}\right)^2 + V(r) = \frac{1}{2}\frac{E^2}{c^2}\,. \qquad (14)$$

Where

$$V(r) = \frac{1}{2}c^2\varepsilon - \varepsilon\frac{GM}{r} + \frac{L^2}{2r^2} - \frac{GML^2}{c^2r^3}\,. \qquad (15)$$

Equation 14 is identical to the classical equation describing the motion of a (unit mass) particle moving in a 1-dimensional potential *V (r),* provided its “energy” is $\frac{1}{2}E^2$. (Of course, the true energy is E, but we use this form due to the potential). The first term at the left hand side of the form looks like the kinetic energy, the second term is the potential energy, while the sum of both is constant.

Looking at the potential (Equation 15) we see that it only differs from the Newtonian potential by the last term (note that this potential is *exact*, not a power series in *1/r*!). The first term is just a constant ($\varepsilon = 1\ or\ 0$) the 2nd term corresponds exactly to the Newtonian gravitational potential, and the third term is a contribution from angular momentum which takes the same form in Newtonian gravity and the theory of General Relativity. It is the last term, though, which contains the GR contribution, which turns out to make a great deal of difference, especially at small *r.*

It is important not to get confused though: the physical situation is quite different from a classical particle moving in one dimension. The trajectories under consideration are orbits around a star or other object (see Figure 1). The quantities of interest to us are not only *r(λ),* but also *t(λ)* and $\phi(\lambda)$. Nevertheless, it is great help that the radial behavior reduces this to a problem which we know how to solve.

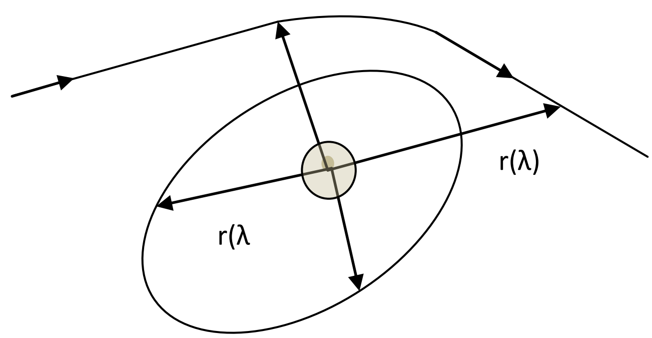


Figure 1 — Trajectories of particles in a gravitational potential.

3.2.1.1. *Interlude on Energy*

Here we will consider the energy as mentioned in equation 9 in chapter 3.2.

$$\left(1 - \frac{2GM}{c^2 r}\right)\frac{dt}{d\lambda} = \frac{E}{mc^2} = \sigma^2\ \frac{dt}{d\lambda}, \tag{9}$$

$$E = \sigma^2 mc^2\ \frac{dt}{d\lambda}$$

Schwarzschild equation:

$$ds^2 = c^2 d\tau^2 = \sigma^2 c^2 dt^2 - \frac{dr^2}{\sigma^2} - r^2 d\theta^2 - r^2 \sin^2\theta^2\, d\emptyset^2$$

Use the affine parameter λ to circumvent the situation $\tau = 0$:

$$d\tau = d\lambda$$

In the equatorial plane, set:

$$\theta = \pi/2$$

$$\sigma^2 c^2 \left(\frac{dt}{d\lambda}\right)^2 - \sigma^{-2}\left(\frac{dr}{d\lambda}\right)^2 - r^2\left(\frac{d\phi}{d\lambda}\right)^2 = c^2\varepsilon.$$

$$\sigma^2 c^2 \left(\frac{dt}{d\lambda}\right)^2 - \sigma^{-2}\left(\frac{dr}{dt}\right)^2\left(\frac{dt}{d\lambda}\right)^2 - r^2\left(\frac{d\phi}{dt}\right)^2\left(\frac{dt}{d\lambda}\right)^2 = c^2\varepsilon$$

For mass particles is $\varepsilon = 1$:

$$\sigma^2\left(\frac{dt}{d\lambda}\right)^2\left(1 - \frac{\sigma^{-2}\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\phi}{dt}\right)^2}{\sigma^2 c^2}\right) = \varepsilon = 1 \tag{9a}$$

$$\left(\frac{dt}{d\lambda}\right)^2\left(1 - \frac{v^2}{\sigma^2 c^2}\right) = \frac{1}{\sigma^2} => \frac{dt}{d\lambda} = \frac{1}{\sigma\sqrt{\left(1 - \frac{v^2}{\sigma^2 c^2}\right)}}$$

We have seen in equation 9 above that

$$E = \sigma^2 mc^2 \frac{dt}{d\lambda}$$

So:

$$\boldsymbol{E = \frac{\sigma mc^2}{\sqrt{\left(1 - \frac{v^2}{\sigma^2 c^2}\right)}}} \quad \textit{is total conserved energy}$$

$$\boldsymbol{E = \gamma_\sigma \sigma mc^2}$$

$$\boldsymbol{E = \sigma mc^2} \textit{ is } \text{rest } \textit{energy}$$

$$\boldsymbol{E = \sigma mc^2\left[\frac{1}{\sqrt{\left(1 - \frac{v^2}{\sigma^2 c^2}\right)}} - 1\right]} \textit{ is } \text{kinetic } \textit{energy}$$

For $v \ll c$ the Taylor expansion:

$$\boldsymbol{E} = \sigma mc^2\left[1 + \frac{v^2}{2\sigma^2 c^2} - 1\right] = \boldsymbol{\frac{mv^2}{2\sigma}} \textit{ is kinetic energy}$$

Another approach:

$$ds^2 = c^2 d\tau^2 = \sigma^2 c^2 dt^2 - \frac{dr^2}{\sigma^2} - r^2 d\theta^2 - r^2 \sin^2\theta^2\, d\emptyset^2$$

$$\sigma^2 c^2\left(\frac{dt}{d\lambda}\right)^2 - \sigma^{-2}\left(\frac{dr}{d\lambda}\right)^2 - r^2\left(\frac{d\phi}{d\lambda}\right)^2 = c^2\varepsilon$$

$$\sigma^4 c^2\left(\frac{dt}{d\lambda}\right)^2 - \left(\frac{dr}{d\lambda}\right)^2 - \sigma^2 r^2\left(\frac{d\phi}{d\lambda}\right)^2 = \sigma^2 c^2\varepsilon$$

$$E = \sigma^2 mc^2 \frac{dt}{d\lambda} => \frac{E}{mc} = \sigma^2 c \frac{dt}{d\lambda}$$

$$\left(\frac{E}{mc}\right)^2 = \left(\frac{dr}{d\lambda}\right)^2 + \sigma^2 r^2\left(\frac{d\phi}{d\lambda}\right)^2 + \sigma^2 c^2\varepsilon$$

$$r^2\frac{d\phi}{d\lambda} = \frac{L}{m} => r^2\left(\frac{d\phi}{d\lambda}\right)^2 = \frac{L^2}{r^2 m^2} = \frac{(mv_t r)^2}{r^2 m^2} = {v_t}^2$$

Now we make $\lambda = \tau$ and $\varepsilon = 1$:

$$\left(\frac{E}{mc}\right)^2 = \left(\frac{dr}{d\tau}\right)^2 + \sigma^2 {v_t}^2 + \sigma^2 c^2 = {v_r}^2 + \sigma^2 {v_t}^2 + \sigma^2 c^2$$

$$\left(\frac{E}{c}\right)^2 = \mathrm{m}^2 {v_r}^2 + \mathrm{m}^2\sigma^2 {v_t}^2 + \mathrm{m}^2\sigma^2 c^2$$

$$\boldsymbol{mv_r}\ is\ the\ radial\ momentum$$

$$\boldsymbol{m\sigma v_t} is\ the\ tangential\ momentum$$

$$\boldsymbol{\sigma mc^2}\ is\ the\ rest\ energy$$

So kinetic energy is:

$$\boldsymbol{E_{kin}} = \boldsymbol{m}c\sqrt{{v_r}^2 + \sigma^2 {v_t}^2}$$

Another:

$$\sigma^2 c^2 \left(\frac{dt}{d\lambda}\right)^2 - \sigma^{-2}\left(\frac{dr}{d\lambda}\right)^2 - r^2\left(\frac{d\phi}{d\lambda}\right)^2 = c^2\varepsilon$$

$$\left(\frac{E}{\sigma c}\right)^2 - \sigma^{-2}\left(\frac{dr}{d\lambda}\right)^2 - r^2\left(\frac{d\phi}{d\lambda}\right)^2 = c^2\varepsilon$$

$$\left(\frac{E}{\sigma c}\right)^2 - p^2 = c^2\varepsilon => \left(\frac{E}{\sigma c}\right)^2 = c^2\varepsilon + p^2$$

$$E^2 = \sigma^2 c^4\varepsilon + \sigma^2 p^2 c^2$$

$$E = \sigma c^2\ when\ in\ rest$$

$$E = \sigma pc\ in\ case\ of\ photon$$

Or

$$E^2 = \sigma^2 c^4\varepsilon + \sigma^2 U^2 c^2$$

U is the relativistic speed $\frac{dx}{d\tau}$. $Here\ \ U^2 = \left(\frac{dx}{d\tau}\right)^2 = \sigma^{-2}\left(\frac{dr}{d\tau}\right)^2 + r^2\left(\frac{d\phi}{d\tau}\right)^2$.

This energy above is the energy per mass unity (kg). So in general

$$E^2 = \sigma^2 m^2 c^4\varepsilon + \sigma^2 U^2 m^2 c^2$$

3.2.2. Experiment 3 - Deflection of Light

Historically, this was the first independent test of GR. While in Newtonian gravity photons move in straight lines, in GR their paths are deflected. This can be observed when we look at the light coming from a distant star which is “nearly behind” the sun, and ½ a year later when the earth is on the other side of the sun. From practical reasons, the first measurement can be done only during solar eclipse. The location of the star in the sky (relatively to other stars) will change. This phenomenon was first shown by Arthur Eddington in 1919.

Consider a light ray that approaches from infinity. Using Equations 14 and 15 in chapter 3.2.1, we find that (with $\varepsilon\ =\ 0$ for a photon)

$$\frac{1}{2}\left(\frac{dr}{d\lambda}\right)^2 + V(r) = \frac{1}{2}\frac{E^2}{c^2}. \tag{14}$$

Together with

$$V(r) = \frac{1}{2}c^2\varepsilon - \varepsilon\frac{GM}{r} + \frac{L^2}{2r^2} - \frac{GML^2}{c^2 r^3} \tag{15}$$

$$\frac{1}{2}\left(\frac{dr}{d\lambda}\right)^2 + \frac{L^2}{2r^2} - \frac{GML^2}{c^2 r^3} = \frac{1}{2}\frac{E^2}{c^2}.$$

Divide by $L^2$and multiply with 2:

$$\frac{1}{L^2}\left(\frac{dr}{d\lambda}\right)^2 + \frac{1}{r^2} - \frac{2GM}{c^2r^3} = \frac{E^2}{c^2L^2}.$$

$$\frac{1}{L^2}\left(\frac{dr}{d\lambda}\right)^2 + \frac{1}{r^2}\left(1 - \frac{2GM}{c^2r}\right) = \frac{E^2}{c^2L^2}$$

$$\left(\frac{dr}{d\lambda}\right)^2 = L^2\left[\frac{E^2}{c^2L^2} - \frac{1}{r^2}\left(1 - \frac{2GM}{c^2r}\right)\right] \tag{16}$$

It is necessary to specify the parameters found in the formulae. First the angular momentum of the moving particle at infinity is equal by definition to the product of its linear momentum ***p*** by what is called the *impact parameter* ***b***, which represents the distance between the center of attraction (the sun in the present case) and the initial direction of the velocity of the particle (see the figure 2).

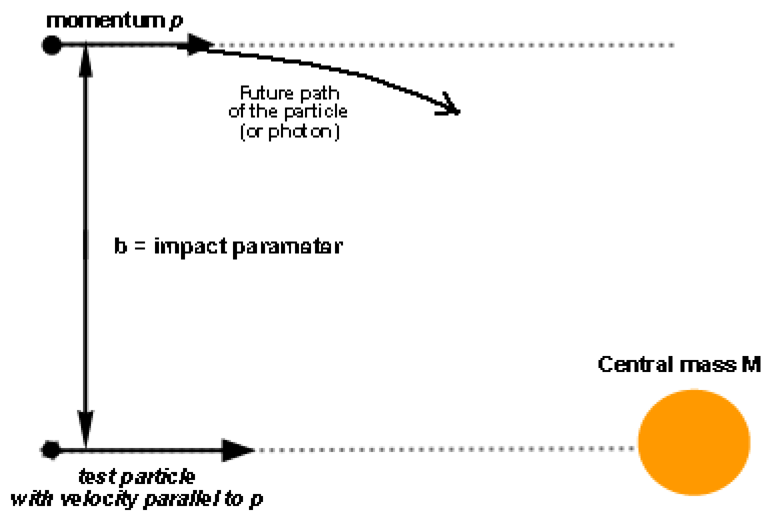


Figure 2. Definition of the impact parameter *b*. The moving particle approaches the mass M from a great distance with vector momentum p. A test particle with a parallel velocity plunges radially onto the mass M. The distance b between their initially parallel paths at ‘infinity’ is the impact parameter b.

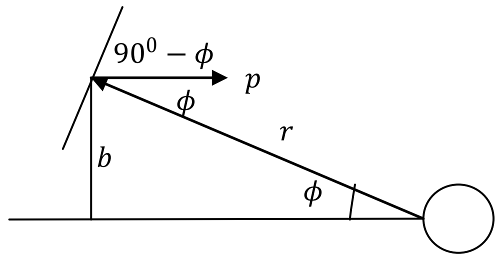
In other words

$$L = pb \tag{17}$$

In addition it is known that the momentum ***p*** of a photon is equal to its energy ***E/c***. It results at once from this formula that

$$b = \frac{L}{E/c}, \tag{18}$$

Additional elucidation of relationship (17) and (18):



The angular momentum is $L = p\sin\phi \cdot r = p \cdot r\sin\phi = p \cdot b$

The energy in general is $E^2 = p^2c^2 + m^2c^4$; and for a photon is *m=0* hence *E=pc*.
So

$$\frac{L}{E/c} = \frac{pb}{pc/c} = b$$

Using Equation 9.11, $\left(r^2\frac{d\phi}{d\lambda} = L\ \right)$ we find:

$$\frac{d\phi}{d\lambda} = \frac{d\phi}{dr}\frac{dr}{d\lambda} = \frac{L}{r^2} \ => \frac{d\phi}{dr} = \frac{L}{r^2}\left(\frac{dr}{d\lambda}\right)^{-1}$$

Together with (16):

$$\frac{d\phi}{dr} = \frac{L}{r^2}\left(\frac{dr}{d\lambda}\right)^{-1} = \pm\frac{L}{r^2}\frac{1}{L}\left[\frac{E^2}{c^2L^2} - \frac{1}{r^2}\left(1 - \frac{2GM}{c^2r}\right)\right]^{-1/2}$$

$$\frac{d\phi}{dr} = \pm\frac{1}{r^2}\left[\frac{1}{b^2} - \frac{1}{r^2}\left(1 - \frac{2GM}{c^2r}\right)\right]^{-1/2} \tag{19}$$

Or

$$\left(\frac{1}{r^2}\frac{dr}{d\phi}\right)^2 = \frac{1}{b^2} - \frac{1}{r^2}\left(1 - \frac{2GM}{c^2r}\right) \tag{20}$$

(see Figure 9).
Getting the maximum deflection angle is now a matter of simple integration, (from infinity to $r_{1,}$ closest to the Sun, and this distance 2 times). From (19):

$$\Delta\phi = 2\int_{r_1}^{\infty}\frac{dr}{r^2}\left[\frac{1}{b^2} - \frac{1}{r^2}\left(1 - \frac{2GM}{c^2r}\right)\right]^{-\frac{1}{2}} \tag{21}$$

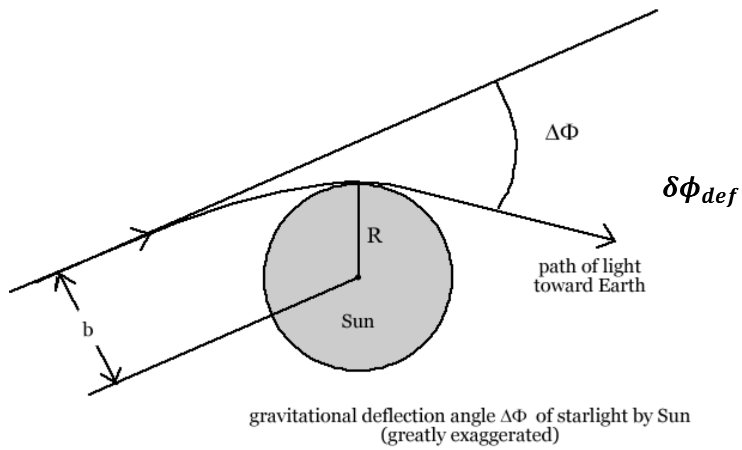


Fig. 9— Deflection of light by angle $\delta\varphi_{def}$

Where *r = R* is the turning point, which is the radius where $\frac{dr}{d\phi} = 0$ (see formula (20)) and thus

$$\frac{1}{b^2} = \frac{1}{R^2}\left(1 - \frac{2GM}{c^2R}\right)$$

For deflection of light by the sun, the impact parameter *b* cannot be smaller than the stellar radius, $b \geq R_{sun} \approx 7 * 10^8\ m$, and thus $\frac{2GM_{sun}}{c^2b} \leq 10^{-6}$

Formula (20) will allow us to determine the change in the direction of a light pulse caused by the gravitational field of the sun. To achieve this aim we have to sum up the successive infinitesimal increments $d\phi$ of the azimuthal angle ϕ along the path. This means that we have to carry out the integration of $\frac{1}{dr}\left(\frac{d\phi}{dr}\right)$ when *r* varies from the minimum distance denoted *R* (*R* is the radius of the sun if the light ray grazes its surface). We should still multiply that quantity with 2 to account for both symmetrical "legs" of the trajectory (the photon first approaches the Sun then recedes from it).
It is necessary to stipulate a further point, namely the relation existing between the two quantities ***b*** and *R* that we have introduced and that are not independent. The point ***r=R*** corresponds to the place where the light photon is closest to the sun. There the photon moves tangentially. Since at that point there is no radial component, we can write that the derivative $\frac{\boldsymbol{dr}}{\boldsymbol{d\phi}}$ vanishes. It suffices to take the element $\boldsymbol{dr}$ from Equation (20) to find immediately

$$\frac{1}{b^2} = \frac{1}{R^2}\left(1 - \frac{2GM}{c^2R}\right) \tag{22}$$

So that this same equation (20) becomes

$$\left(\frac{1}{r^2}\frac{dr}{d\phi}\right)^2 = \frac{1}{R^2}\left(1 - \frac{2GM}{c^2R}\right) - \frac{1}{r^2}\left(1 - \frac{2GM}{c^2r}\right) \tag{23}$$

The form of the expression dictates to us to pose

$$u = \frac{R}{r}$$

$$\frac{du}{d\phi}=\frac{du}{dr}\frac{dr}{d\phi}=\frac{-R}{r^2}\frac{dr}{d\phi}\Rightarrow\left(\frac{du}{d\phi}\right)^2=\left(\frac{R}{r^2}\frac{dr}{d\phi}\right)^2$$

Where ***u*** varies between 1 (r=R) and 0 (r=∞). The last equation (23) then becomes

$$\left(\frac{du}{d\phi}\right)^2=\left(1-\frac{2GM}{c^2R}\right)-u^2\left(1-\frac{2GMu}{c^2R}\right)$$

Or

$$\left(\frac{du}{d\phi}\right)^2=1-u^2-\frac{2GM}{c^2R}(1-u^3) \tag{24}$$

Consequently the infinitesimal variation $d\phi$ of the azimuth is given in terms of the variation du of $\frac{R}{r}$ by

$$\begin{aligned} d\phi &= \left[1-u^2-\frac{2GM}{c^2R}(1-u^3)\right]^{-\frac{1}{2}}du \\ &= \frac{(1-u^2)^{-1/2}du}{\left[1-\frac{2GM}{c^2R}(1-u^3)(1-u^2)^{-1}\right]^{\frac{1}{2}}} \end{aligned} \tag{25}$$

The presence of the term $(1-u^2)$ in Expression (25) encourages us to make the change of variable

$$\mathrm{u}=\cos\alpha, 0<u<1, 0<\alpha<\pi/2$$

This leads to

$$d\phi=-\left[1-\frac{2GM}{c^2R}(1-\cos^3\alpha)\sin^{-2}\alpha\right]^{-\frac{1}{2}}d\alpha \tag{26}$$

By observing that

$$\frac{1-\cos^3\alpha}{\sin^2\alpha}=\frac{(1-\cos\alpha)(1+\cos\alpha+\cos^2\alpha)}{(1-\cos\alpha)(1+\cos\alpha)}=\frac{1+\cos\alpha\,(1+\cos\alpha)}{(1+\cos\alpha)}=\cos\alpha+\frac{1}{(1+\cos\alpha)}$$

We end up with the final equation of the trajectory under the form

$$d\phi=-\left[1-\frac{2GM}{c^2R}\left(\cos\alpha+\frac{1}{(1+\cos\alpha)}\right)\right]^{-\frac{1}{2}}d\alpha \tag{27}$$

With

$$\cos\alpha=R/r$$

It is interesting to emphasize that so far there have been no approximations.

3.2.2.1. *Approximations and integration*

The small value of the term $2GM/c^2R=4.24\cdot10^{-6}$ will allow us to make an approximation and in this way will enable us to complete the integration.

In Equation (27) we can thus use the classical (Taylor) approximation $(1+\epsilon)^p\simeq1+p\epsilon$ (or here $\frac{1}{\sqrt{1-k}}\simeq1+\frac{1}{2}k$) to arrive at

$$d\phi=-\left[1+\frac{GM}{c^2R}\left(\cos\alpha+\frac{1}{(1+\cos\alpha)}\right)\right]d\alpha \tag{28}$$

Therefore the total variation of the azimuth ϕ along the path of the photon is

$$\Delta\phi = 2\int_0^{\frac{\pi}{2}}\left[1+\frac{GM}{c^2R}\left(\cos\alpha+\frac{1}{(1+\cos\alpha)}\right)\right]d\alpha \tag{29}$$

To find the integral $\int\frac{1}{(1+\cos\alpha)}d\alpha$:

$$\frac{1}{(1+\cos\alpha)} = \frac{1}{1+\cos\left(\frac{\alpha}{2}+\frac{\alpha}{2}\right)} = \frac{1}{\cos^2\frac{\alpha}{2}+\sin^2\frac{\alpha}{2}+\cos^2\frac{\alpha}{2}-\sin^2\frac{\alpha}{2}} = \frac{1}{2\cos^2\frac{\alpha}{2}}$$

$$\frac{1}{2\cos^2\frac{\alpha}{2}} = \frac{1}{2}\left(1+\frac{\sin^2\frac{\alpha}{2}}{\cos^2\frac{\alpha}{2}}\right) = \frac{1}{2}\left(\frac{\cos\frac{\alpha}{2}}{\cos\frac{\alpha}{2}}+\frac{\sin^2\frac{\alpha}{2}}{\cos^2\frac{\alpha}{2}}\right) = \frac{1}{2}\frac{\mathrm{d}\left(\tan\frac{\alpha}{2}\right)}{\mathrm{d}\left(\frac{\alpha}{2}\right)} = \frac{\mathrm{d}\left(\tan\frac{\alpha}{2}\right)}{\mathrm{d}\alpha}$$

Thus

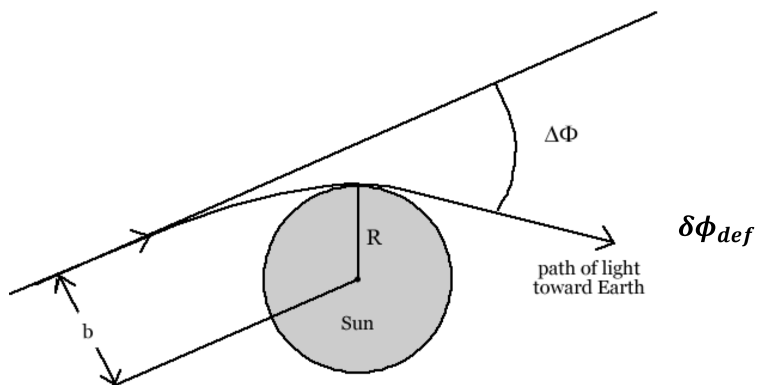
$$\int\frac{1}{(1+\cos\alpha)}d\alpha = \tan\frac{\alpha}{2}$$

Now fill in equation (29)

$$\Delta\phi = 2\left[\alpha+\frac{GM}{c^2R}\left(\sin\alpha+\tan\frac{\alpha}{2}\right)\right]_0^{\frac{\pi}{2}} \tag{30}$$

$$\Delta\phi = \pi+\frac{4GM}{c^2R} \tag{31}$$

Remark: the integral should be from *r* is infinity to R, or *u* goes from 0 to 1, and thus α from $\frac{\pi}{2}$ to 0. By changing the integral to $0\ to\ \frac{\pi}{2}$ the sign changes and the minus sign disappears.



gravitational deflection angle $\delta\phi_{def}$ of starlight by Sun

The first term π of (formula 31) gives the total change in the azimuth of the photon where there is no Sun present, since in that case the photon follows a straight path. But the second term gives the additional angle of deflection $\delta\phi_{def}$ with respect to this straight line (see the figure)
Thus the actual deflection is

$$\delta\phi_{def} = \Delta\phi-\pi \approx \frac{4GM}{c^2R} \tag{32}$$

Numerically at the surface of the sun (with the values of the mass and the radius given above) one finds $\delta\phi_{def} = 8.5\cdot10^{-6}$ radian, or (knowing that π radians equal 180 degrees and that there are 60 minutes of arc in one degree and 60 seconds of arc in one minute of arc)

$$\delta\phi_{def}\,\phi \lesssim 1.75'' \quad (arc.sec = \frac{\pi}{648000})$$

This effect is also seen outside our solar system, as part of what is known as “gravitational lensing”.

3.2.3. Experiment 4 - Precession of the Perihelia (Mercurius)

Based on Owen Biesel (Biesel, 2008)

In the general relativistic case, we assume that the particle is a test particle traveling along a geodesic through space-time. It can be described with the Schwarzschild metric:

$$\boldsymbol{ds^2 = c^2 d\tau^2 = \sigma^2 c^2 dt^2 - \frac{dr^2}{\sigma^2} - r^2 d\theta^2 - r^2 \sin^2\theta^2\, d\emptyset^2} \qquad \mathbf{(33)}$$

where

$$\sigma = \sqrt{1 - \frac{2GM}{c^2 r}} = \sqrt{1 - \frac{R_s}{r}} \quad with\ the\ Schwarzschild\ radius\colon R_s = \frac{2GM}{c^2}$$

The derivation of (33) with respect to τ and with $\theta = \frac{\pi}{2}$, becomes:

$$1 = \left(1 - \frac{R_s}{r}\right)\left(\frac{dt}{d\tau}\right)^2 - \frac{1}{c^2}\left(1 - \frac{R_s}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 - \frac{1}{c^2} r^2 \left(\frac{d\emptyset}{d\tau}\right)^2$$

Lagrange approach. **(see Appendix 10)**

Although we already derived above the equations for *E* (equation 9) and *L* (equation 11) it is also interesting to find these two constants via the Lagrange approach:

Now if we parameterize a curve $x(\tau) = \big(t(\tau), r(\tau), \theta(\tau), \emptyset(\tau)\big)$ by proper time, then we find that letting $\mathcal{L} = \langle \frac{dx}{d\tau} . \frac{dx}{d\tau} \rangle$ (differentiation with respect to proper time), $\mathcal{L}$ is both a constant of motion and also satisfies the Euler-Lagrange equations so that $I = \int \mathcal{L}\, d\tau$ is stationary. By exactly the same reasoning as in the classical case, we may restrict our attention to motion in the equatorial plane and assume that $\theta(\tau) \equiv \pi/2$, so that the “Lagrangian” becomes:

$$\mathcal{L} = \left(1 - \frac{R_s}{r}\right)\left(\frac{dt}{d\tau}\right)^2 - \frac{1}{c^2}\left(1 - \frac{R_s}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 - \frac{1}{c^2} r^2 \left(\frac{d\emptyset}{d\tau}\right)^2 \qquad (34)$$

$$\frac{d\emptyset}{d\tau} = \dot{\emptyset}\ \ and\ \ \frac{dt}{d\tau} = \dot{t}$$

$$\mathcal{L} = \left(1 - \frac{R_s}{r}\right)\dot{t}^{\,2} - \frac{1}{c^2}\left(1 - \frac{R_s}{r}\right)^{-1}\dot{r}^2 - \frac{1}{c^2} r^2 \dot{\emptyset}^2$$

Euler-Lagrange operation:

$$Here\ for\ \emptyset\colon\ \frac{d}{d\tau}\left(\frac{\partial \mathcal{L}}{\partial \dot{\emptyset}}\right) = \frac{\partial \mathcal{L}}{\partial \emptyset} = 0$$

$$for\ \boldsymbol{t}\colon\ \frac{d}{d\tau}\left(\frac{\partial \mathcal{L}}{\partial \dot{t}}\right) = \frac{\partial \mathcal{L}}{\partial t} = 0$$

Then the Euler-Lagrange equations for ∅ and *t* read:

$$0=\frac{d}{d\tau}\left(2\frac{1}{c^2}r^2\frac{d\emptyset}{d\tau}\right)=>\ r^2\frac{d\emptyset}{d\tau}=constant$$
$$0=\frac{d}{d\tau}\left(2\left(1-\frac{R_s}{r}\right)\frac{dt}{d\tau}\right)=>\ \left(1-\frac{R_s}{r}\right)\frac{dt}{d\tau}=constant$$
This implies that the angular momentum (per mass unit) $L=r^2\frac{d\emptyset}{d\tau}$ and the momentum (per mass unit) $\frac{E}{c^2}=\frac{dt}{d\tau}\left(1-\frac{R_s}{r}\right)$ are two constants of motion. Then the relation $\mathcal{L}=1$ gives us:
$$1=\left(1-\frac{R_s}{r}\right)\left(\frac{dt}{d\tau}\right)^2-\frac{1}{c^2}\left(1-\frac{R_s}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2-\frac{1}{c^2}r^2\left(\frac{d\emptyset}{d\tau}\right)^2$$
$$1=\frac{\frac{E^2}{c^4}}{1-\frac{R_s}{r}}-\frac{1}{c^2}\frac{\left(\frac{dr}{d\tau}\right)^2}{1-\frac{R_s}{r}}-\frac{1}{c^2}\frac{L^2}{r^2}$$
$$1-\frac{R_s}{r}=\frac{E^2}{c^4}-\frac{1}{c^2}\left(\frac{dr}{d\tau}\right)^2-\frac{1}{c^2}\frac{L^2}{r^2}+\frac{1}{c^2}\frac{L^2}{r^2}\frac{R_s}{r}$$
$$\left(\frac{dr}{d\tau}\right)^2=c^2\left(\frac{E^2}{c^4}-1\right)+c^2\frac{R_s}{r}-\frac{L^2}{r^2}+\frac{R_sL^2}{r^3}$$
Once again, assuming $L\neq 0$ allows us to invert $\emptyset=\emptyset(\tau)$, so we may obtain *r* as a function of $\emptyset$ with
$$\frac{dr}{d\tau}=\frac{dr}{d\emptyset}\frac{d\emptyset}{d\tau}=\frac{dr}{d\emptyset}\frac{L}{r^2}=>\ \left(\frac{dr}{d\emptyset}\right)^2=\frac{r^4}{L^2}\left(\frac{dr}{d\tau}\right)^2$$
And hence we have
$$\left(\frac{dr}{d\emptyset}\right)^2=c^2\frac{\frac{E^2}{c^4}-1}{L^2}r^4+c^2\frac{R_s}{L^2}r^3-r^2+R_sr$$
Now the requirement that of a closed orbit with $\left(\frac{dr}{d\emptyset}\right)^2\geq 0$ imposes some constraints on *L, E*, and R_S; we need a connected component of $\left\{r:\frac{dr}{d\emptyset}\geq 0\right\}$ to be a compact subset of $\mathbb{R}^+$. This means there exist at least two values *A* and *P* where $\frac{dr}{d\emptyset}$= 0, i.e. aphelion (*A*) and perihelion (*P*). Then the angle shift from *A* and *P* is given, as in the classical case, by
$$d\emptyset=\frac{1}{\sqrt{c^2\frac{\frac{E^2}{c^4}-1}{L^2}r^4+c^2\frac{R_s}{L^2}r^3-r^2+R_sr}}dr$$
$$\emptyset_{\boldsymbol{A}}-\emptyset_{\boldsymbol{P}}=\int_P^A\frac{dr}{\sqrt{c^2\frac{\frac{E^2}{c^4}-1}{L^2}r^4+c^2\frac{R_s}{L^2}r^3-r^2+R_sr}}\tag{35}$$
Given that (r −A) and (r −P) are factors of $c^2\frac{\frac{E^2}{c^4}-1}{L^2}r^4+c^2\frac{R_s}{L^2}r^3-r^2+R_sr$, we can solve for $\frac{E^2}{c^4}-1$ and $\frac{L^2}{c^2}$ in terms of A, P and R_S:
$$c^2\left(\frac{E^2}{c^4}-1\right)A^4+(L^2)(-A^2+R_sA)=-c^2R_sA^3\tag{36}$$
$$c^2\left(\frac{E^2}{c^4}-1\right)P^4+(L^2)(-P^2+R_sP)=-c^2R_sP^3\tag{37}$$

Multiply (36) with $(-P^2 + R_s P)$

$$c^2\left(\frac{E^2}{c^4} - 1\right)A^4(-P^2 + R_s P) + (L^2)(-A^2 + R_s A)(-P^2 + R_s P) = -c^2 R_s A^3(-P^2 + R_s P)$$

Multiply (37) with $(-A^2 + R_s A)$

$$c^2\left(\frac{E^2}{c^4} - 1\right)P^4(-A^2 + R_s A) + (L^2)(-P^2 + R_s P)(-A^2 + R_s A) = -c^2 R_s P^3(-A^2 + R_s A)$$

Subtract

$$c^2\left(\frac{E^2}{c^4} - 1\right)[A^4(-P^2 + R_s P) - P^4(-A^2 + R_s A)] = -c^2 R_s A^3(-P^2 + R_s P) + c^2 R_s P^3(-A^2 + R_s A)$$

$$c^2\left(\frac{E^2}{c^4} - 1\right) = \frac{-c^2 R_s A^3(-P^2 + R_s P) + c^2 R_s P^3(-A^2 + R_s A)}{[A^4(-P^2 + R_s P) - P^4(-A^2 + R_s A)]}$$

$$\left(\frac{E^2}{c^4} - 1\right) = \frac{-R_s[A^3(-P^2 + R_s P) - P^3(-A^2 + R_s A)]}{[A^4(-P^2 + R_s P) - P^4(-A^2 + R_s A)]}$$

$$\left(\frac{E^2}{c^4} - 1\right) = \frac{-R_s[A^3 P(-P + R_s) - P^3 A(-A + R_s)]}{[A^4 P(-P + R_s) - P^4 A(-A + R_s)]}$$

$$\left(\frac{E^2}{c^4} - 1\right) = \frac{-R_s AP[A^2(-P + R_s) - P^2(-A + R_s)]}{AP[A^3(-P + R_s) - P^3(-A + R_s)]}$$

$$\left(\frac{E^2}{c^4} - 1\right) = \frac{-R_s[A^2(-P + R_s) - P^2(-A + R_s)]}{[A^3(-P + R_s) - P^3(-A + R_s)]}$$

$$\left(\frac{E^2}{c^4} - 1\right) = \frac{-R_s[-PA^2 + R_s A^2 + AP^2 - R_s P^2]}{[-PA^3 + R_s A^3 + AP^3 - R_s P^3]}$$

$$\left(\frac{E^2}{c^4} - 1\right) = \frac{-R_s[-AP(A - P) + R_s(A^2 - P^2)]}{[-AP(A^2 - P^2) + R_s(A^3 - P^3)]}$$

$$\left(\frac{E^2}{c^4} - 1\right) = \frac{-R_s(A - P)[-AP + R_s(A + P)]}{(A - P)\left[-AP(A + P) + R_s\frac{(A^3 - P^3)}{A - P}\right]}$$

Interlude to work out $\frac{(A^3-P^3)}{A-P}$

$$(A^2 - P^2)(A + P) = A^3 - AP^2 + A^2 P - P^3$$
$$A^3 - P^3 = (A^2 - P^2)(A + P) - AP(A - P)$$
$$A^3 - P^3 = (A - P)(A + P)(A + P) - AP(A - P)$$
$$\frac{A^3 - P^3}{A - P} = (A + P)^2 - AP$$

Fill in result

$$\left(\frac{E^2}{c^4} - 1\right) = \frac{-R_s(A - P)[-AP + R_s(A + P)]}{(A - P)[-AP(A + P) + R_s(A + P)^2 - R_s AP]}$$

$$\left(\frac{E^2}{c^4} - 1\right) = \frac{-R_s[-AP + R_s(A + P)]}{[-AP(A + P + R_s) + R_s(A + P)^2]}$$

$$\boldsymbol{\left(\frac{E^2}{c^4} - 1\right) = \frac{R_s[-AP + R_s(A + P)]}{AP(A + P + R_s) - R_s(A + P)^2}}$$

Now find *L* from chapter 3.2.3 equations (36 and 37)

$$c^2\left(\frac{E^2}{c^4}-1\right)A^4+(L^2)(-A^2+R_sA)=-c^2R_sA^3 \tag{36}$$

$$c^2\left(\frac{E^2}{c^4}-1\right)P^4+(L^2)(-P^2+R_sP)=-c^2R_sP^3 \tag{37}$$

Multiply 36 with A

$$c^2\left(\frac{E^2}{c^4}-1\right)A^4P^4+(L^2)(-A^2+R_sA)P^4=-c^2R_sA^3P^4$$

Multiply 37 with P

$$c^2\left(\frac{E^2}{c^4}-1\right)A^4P^4+(L^2)(-P^2+R_sP)A^4=-c^2R_sA^4P^3$$

Subtract

$$(L^2)[(-A^2+R_sA)P^4-(-P^2+R_sP)A^4]=-c^2R_sA^3P^4+c^2R_sA^4P^3$$

$$L^2=\frac{c^2R_sA^3P^3[-P+A]}{(-A^2+R_sA)P^4-(-P^2+R_sP)A^4}$$

$$L^2=\frac{c^2R_sA^3P^3[-P+A]}{(-A+R_s)AP^4-(-P+R_s)PA^4}$$

$$L^2=\frac{c^2R_sA^3P^3[-P+A]}{AP[(-A+R_s)P^3-(-P+R_s)A^3]}$$

$$L^2=\frac{c^2R_sA^2P^2[-P+A]}{[(-A+R_s)P^3-(-P+R_s)A^3]}$$

$$L^2=\frac{c^2R_sA^2P^2[-P+A]}{A^3P-AP^3-(A^3-P^3)R_s}$$

$$L^2=\frac{c^2R_sA^2P^2[-P+A]}{AP(A^2-P^2)-(A^3-P^3)R_s}$$

$$L^2=\frac{c^2R_sA^2P^2}{AP(A+P)-R_s(A+P)^2+APR_s}$$

$$L^2=\frac{c^2R_sA^2P^2}{AP(A+P+R_s)-R_s(A+P)^2}$$

$$\boldsymbol{\frac{L^2}{c^2}=\frac{R_sA^2P^2}{AP(A+P+R_s)-R_s(A+P)^2}}$$

Thus the result is

$$\frac{E^2}{c^4}-1=\frac{-APR_s+(A+P){R_s}^2}{AP(A+P+R_s)-R_s(A+P)^2}$$

$$\frac{L^2}{c^2}=\frac{A^2P^2R_s}{AP(A+P+R_s)-R_s(A+P)^2}$$

It is convenient to introduce the combination

$$D=\frac{AP}{A+P}$$

This has units of distance. Then the above expressions for E^2-1 and L^2 become:

$$\frac{E^2}{c^4}-1=\frac{(-R_s/AP)+\left({R_s}^2/DAP\right)}{\frac{1}{D}+\left(\frac{R_s}{AP}\right)-\left(\frac{R_s}{D^2}\right)}$$

$$\frac{L^2}{c^2} = \frac{R_s}{\frac{1}{D} + \left(\frac{R_s}{AP}\right) - \left(\frac{R_s}{D^2}\right)}$$

$$\frac{\frac{L^2}{c^2}}{\frac{E^2}{c^4} - 1} = \frac{R_s}{(-R_s/AP) + \left({R_s}^2/DAP\right)} = \frac{AP}{-1 + R_s/D}$$

$$\frac{\frac{L^2}{c^2 AP}}{1 - \frac{E^2}{c^4}} = \frac{1}{1 - R_s/D} \tag{38}$$

We would like an expression for Ɛ, the third nonzero root of $\frac{E^2/c^4-1}{L^2/c^2} r^4 + \frac{R_s}{L^2/c^2} r^3 - r^2 + R_s r = 0$

$$\frac{E^2/c^4 - 1}{L^2/c^2}\left[r^4 + \frac{R_s}{\frac{E^2}{c^4} - 1} r^3 - \frac{\frac{L^2}{c^2}}{\frac{E^2}{c^4} - 1} r^2 + \frac{\frac{L^2}{c^2}}{\frac{E^2}{c^4} - 1} R_s r\right] = 0$$

So this gives three non-zero roots: A, P and Ɛ.

The total expression is:

$$\frac{E^2/c^4 - 1}{L^2/c^2}(r - A)(r - P)(r - \varepsilon)r$$

Let us work out the four factors:

$$\frac{E^2/c^4 - 1}{L^2/c^2}[r^4 - (\mathrm{A} + \mathrm{P} + \varepsilon)\mathrm{r}^3 + \{\mathrm{AP} + \varepsilon(\mathrm{A} + \mathrm{P})\}\mathrm{r}^2 - \varepsilon APr]$$

So we know that the sum of the three nonzero roots is $\frac{R_s}{E^2/c^4-1}$ (the coefficient of r^3 with the polynomial in standard form); hence we obtain:

$$-(\mathrm{A} + \mathrm{P} + \varepsilon) = R_s \frac{1}{E^2/c^4 - 1}$$

From above we know:

$$\left(\frac{E^2}{c^4} - 1\right) = \frac{R_s[-AP + (A + P)R_s]}{AP(A + P + R_s) - R_s(A + P)^2}$$

So we will fill this in the equation above:

$$\mathrm{A} + \mathrm{P} + \varepsilon = R_s \frac{-1}{E^2/c^4 - 1}$$

$$\mathrm{A} + \mathrm{P} + \varepsilon = R_s \frac{-AP(A + P + R_s) + R_s(A + P)^2}{R_s[-AP + (A + P)R_s]} = \frac{-AP(A + P + R_s) + R_s(A + P)^2}{-AP + (A + P)R_s}$$

$$\varepsilon = \frac{-AP(A + P + R_s) + R_s(A + P)^2}{-AP + (A + P)R_s} - (\mathrm{A} + \mathrm{P})$$

$$= \frac{-AP(A + P + R_s) + R_s(A + P)^2 + \mathrm{AP}(A + P) - (A + P)^2 R_s}{-AP + (A + P)R_s}$$

$$\varepsilon = \frac{-AP(A + P + R_s) + \mathrm{AP}(A + P)}{-AP + (A + P)R_s} = \frac{-APR_s}{-AP + (A + P)R_s} = \frac{R_s}{1 - \frac{(A + P)R_s}{AP}} = \frac{R_s}{1 - \frac{R_s}{D}}$$

Giving:

$$\varepsilon = \frac{R_s}{1 - \frac{R_s}{D}} \tag{39}$$

Now we can approximate (35), by writing

$$\frac{E^2/c^4 - 1}{L^2/c^2} r^4 + \frac{R_s}{L^2/c^2} r^3 - r^2 + R_s r = \frac{E^2/c^4 - 1}{L^2/c^2}(r - A)(r - P)(r - \varepsilon)r$$

$$= \frac{1 - E^2/c^4}{L^2/c^2}(A - r)(r - P)(r - \varepsilon)r.$$

We obtain:

$$\emptyset_{\boldsymbol{A}} - \emptyset_{\boldsymbol{P}} = \sqrt{\frac{L^2/c^2}{1 - E^2/c^4}} \int_P^A \frac{1}{\sqrt{r(A - r)(r - P)(r - \varepsilon)}} dr$$

$$= \sqrt{\frac{L^2/c^2}{1 - E^2/c^4}} \int_P^A \frac{1}{\sqrt{r^2(A - r)(r - P)\left(1 - \frac{\varepsilon}{r}\right)}} dr$$

$$= \sqrt{\frac{L^2/c^2}{1 - E^2/c^4}} \int_P^A \frac{1}{r\sqrt{(A - r)(r - P)}} \left(1 - \frac{\varepsilon}{r}\right)^{-1/2} dr$$

Now use the Taylor series expansion$\left(1 - \frac{\varepsilon}{r}\right)^{-1/2} \approx 1 + \frac{\varepsilon}{2r}$, with an error ε bounded by $|\varepsilon| \leq \frac{3}{8}\left(1 - \frac{\varepsilon}{r}\right)^{-5/2} \left(\frac{\varepsilon}{r}\right)^2 \leq \frac{3}{8}\left(1 - \frac{\varepsilon}{A}\right)^{-5/2} \left(\frac{\varepsilon}{P}\right)^2$ which produces:

$$= \sqrt{\frac{L^2/c^2}{1 - E^2/c^4}} \int_P^A \left[\frac{1}{r\sqrt{(A - r)(r - P)}} + \frac{\frac{\varepsilon}{2}}{r^2\sqrt{(A - r)(r - P)}}\right] dr \tag{40}$$

Note: In his article, “The Precession of Mercury’s Perihelion” from Owen Biesel, (January 25, 2008), (on page 8, the second formula from the bottom up), the left integral contains in the numerator $1 + \varepsilon$, however we are of the opinion that it should be 1 only and have adapted the formula accordingly.

The **first integral of (40)** (work-out see **3.2.3.1** and **3.2.3.3**) in closed form:

$$= \int_P^A \frac{1}{r\sqrt{(A - r)(r - P)}} dr$$

$$= \frac{1}{\sqrt{AP}} \arctan\left[\frac{(A - r)(r - P) + r^2 - AP}{2\sqrt{(A - r)(r - P)AP}}\right]_P^A$$

$$\rightarrow \frac{1}{\sqrt{AP}}[\arctan[+\infty] - \arctan[-\infty]] = \frac{1}{\sqrt{AP}}\left[\frac{\pi}{2} + \frac{\pi}{2}\right] = \frac{1}{\sqrt{AP}}\pi$$

The **second integral** (work-out see **3.2.3.2**) is trickier, but can be evaluated in closed form:

$$\int_P^A \frac{\varepsilon/2}{r^2\sqrt{(A - r)(r - P)}} dr = \frac{\pi\varepsilon/2}{2\sqrt{AP}} \frac{A + P}{AP} = \frac{1}{\sqrt{AP}} \frac{\pi\varepsilon}{4D}$$

Then if we recognize that $\frac{L^2/c^2 AP}{1 - E^2/c^4} = \frac{1}{1 - R_s/D}$ and $\varepsilon = \frac{R_s}{1 - R_s/D}$ (see (38) and (39) above), we find that

$$\emptyset_{\boldsymbol{A}}-\emptyset_{\boldsymbol{P}}=\frac{1}{\sqrt{AP}}\pi\sqrt{\frac{L^2/c^2}{1-E^2/c^4}}+\frac{1}{\sqrt{AP}}\frac{\pi\varepsilon}{4D}\sqrt{\frac{L^2/c^2}{1-E^2/c^4}}$$

$$\emptyset_{\boldsymbol{A}}-\emptyset_{\boldsymbol{P}}=\pi\sqrt{\frac{L^2/c^2AP}{1-E^2/c^4}}+\frac{\pi\varepsilon}{4D}\sqrt{\frac{L^2/c^2AP}{1-E^2/c^4}}=\pi\sqrt{\frac{1}{1-R_s/D}}+\frac{\pi\varepsilon}{4D}\sqrt{\frac{1}{1-R_s/D}}$$

$$=\frac{\pi}{\sqrt{1-R_s/D}}\left(1+\frac{\varepsilon}{4D}\right)=\frac{\pi}{\sqrt{1-R_s/D}}\left(1+\frac{1}{4}\frac{1}{D}\frac{R_s}{1-R_s/D}\right)$$

$$=\frac{\pi}{\sqrt{1-R_s/D}}\left(1+\frac{1}{4}\frac{R_s/D}{1-R_s/D}\right)$$

Using the observed values $A(phelion)=69.8.10^6\text{km}, P(erihelion)=46.0.10^6\text{km}$ (from which we obtain $=27.7.10^6\text{km}$, and $R_s=\frac{2GM}{c^2}=2.95km$. So the term $\frac{\pi}{\sqrt{1-R_s/D}}\left(1+\frac{1}{4}\frac{R_s/D}{1-R_s/D}\right)\approx\pi+2.512\cdot10^{-7}$ a trustworthy estimate of $\emptyset_{\boldsymbol{A}}-\emptyset_{\boldsymbol{P}}$ (half a revolution, in radians). This is $\Delta\emptyset=2.512\cdot10^{-7}$ radians for half a revolution and $\Delta\emptyset=5.024\cdot10^{-7}$ for one complete revolution.

The orbit period of Mercury is 87.969 days so Mercury completes 415.2 revolutions each century. There are $360\cdot60\cdot60/2\pi$ arcseconds per radian, we find that Mercury's perihelion advances by

$$\Delta\emptyset=(5.024\cdot10^{-7})\left(\frac{360\cdot60\cdot60}{2\pi}\right)\cdot415.2=43.027\ \text{arcseconds per century.}$$

$$\Delta\emptyset=\boldsymbol{43.027\ arcseconds\ per\ century}.$$

Note: From Asaf Pe'er: For small deflection angle, the result is (see equation 6 chapter 3.5)

$$\delta\phi_{prec}=\frac{6\pi\mathrm{GM_{sun}}}{c^2a(1-\varepsilon^2)}\tag{41}$$

Where *a* is the semi-major axis and ε is the eccentricity. Obviously, this effect is largest for small a. For mercury, it predicts 43 arc-secs per century, which is consistent with observations.

3.2.3.1. *Checking the first integral*

Check integrand

$$\frac{d}{dr}\left\{\frac{1}{\sqrt{AP}}\arctan\left[\frac{(A-r)(r-P)+r^2-AP}{2\sqrt{(A-r)(r-P)AP}}\right]\right\}?=?\ \frac{1}{r\sqrt{(A-r)(r-P)}}$$

$$\frac{\text{darctanx}}{dx}=\frac{1}{1+x^2}$$

$$\frac{1}{\sqrt{AP}}\frac{d}{dr}\left\{\arctan\left[\frac{(A-r)(r-P)+r^2-AP}{2\sqrt{(A-r)(r-P)AP}}\right]\right\}=\frac{1}{\sqrt{AP}}\frac{1}{1+\left[\frac{(A-r)(r-P)+r^2-AP}{2\sqrt{(A-r)(r-P)AP}}\right]^2}\frac{d}{dr}\left[\frac{(A-r)(r-P)+r^2-AP}{2\sqrt{(A-r)(r-P)AP}}\right]$$

$$=\frac{1}{\sqrt{AP}}\frac{4(A-r)(r-P)AP}{4(A-r)(r-P)AP+[(A-r)(r-P)+r^2-AP]^2}\frac{d}{dr}\left[\frac{(A-r)(r-P)+r^2-AP}{2\sqrt{(A-r)(r-P)AP}}\right]$$

$$=\frac{1}{\sqrt{AP}}\frac{4(A-r)(r-P)AP}{4(A-r)(r-P)AP+[(A-r)(r-P)+r^2-AP]^2}\left[\frac{-(r-P)+(A-r)+2r}{2\sqrt{(A-r)(r-P)AP}}\right.$$

$$\left.-\frac{AP\{(A-r)(r-P)+r^2-AP\}\{-(r-P)+(A-r)\}}{4\{(A-r)(r-P)AP\}^{3/2}}\right]$$

$$=\frac{1}{\sqrt{AP}}\frac{4(A-r)(r-P)AP}{4(A-r)(r-P)AP+[(A-r)(r-P)+r^2-AP]^2}\left[\frac{A+P}{2\sqrt{(A-r)(r-P)AP}}\right.$$

$$\left.-\frac{AP\{Ar-AP-r^2+rP+r^2-AP\}\{-r+P+A-r\}}{4\{(A-r)(r-P)AP\}^{3/2}}\right]$$

$$=\frac{1}{\sqrt{AP}}\frac{4(A-r)(r-P)AP}{4(A-r)(r-P)AP+[(A-r)(r-P)+r^2-AP]^2}\left[\frac{A+P}{2\sqrt{(A-r)(r-P)AP}}-\frac{AP\{Ar-2AP+rP\}\{P+A-2r\}}{4\{(A-r)(r-P)AP\}^{3/2}}\right]$$

$$=\frac{1}{\sqrt{AP}}\frac{4(A-r)(r-P)AP}{4(A-r)(r-P)AP+[(A-r)(r-P)+r^2-AP]^2}\frac{1}{\sqrt{(A-r)(r-P)AP}}\left[\frac{2(A+P)}{4}\right.$$

$$\left.-\frac{AP\{Ar-2AP+rP\}\{P+A-2r\}}{4(A-r)(r-P)AP}\right]$$

$$=\frac{1}{\sqrt{AP}}\frac{4(A-r)(r-P)AP}{4(A-r)(r-P)AP+[(A-r)(r-P)+r^2-AP]^2}\frac{1}{\sqrt{(A-r)(r-P)AP}}*$$

$$\left[\frac{2(A+P)(A-r)(r-P)AP-AP\{Ar-2AP+rP\}\{P+A-2r\}}{4(A-r)(r-P)AP}\right]$$

$$=\frac{1}{\sqrt{AP}}\frac{2(A+P)(A-r)(r-P)AP-AP\{Ar-2AP+rP\}\{P+A-2r\}}{4(A-r)(r-P)AP+[(A-r)(r-P)+r^2-AP]^2}\frac{1}{\sqrt{(A-r)(r-P)AP}}$$

$$=\frac{1}{AP\sqrt{(A-r)(r-P)}}\frac{2(A+P)(A-r)(r-P)AP-AP\{Ar-2AP+rP\}\{P+A-2r\}}{4(A-r)(r-P)AP+[(A-r)(r-P)+r^2-AP]^2}$$

$$=\frac{1}{\sqrt{(A-r)(r-P)}}\frac{2(A+P)(A-r)(r-P)-\{Ar-2AP+rP\}\{P+A-2r\}}{4(A-r)(r-P)AP+[(A-r)(r-P)+r^2-AP]^2}$$

$$=\frac{1}{\sqrt{(A-r)(r-P)}}\frac{(2A^2-2Ar+2AP-2Pr)(r-P)-\{APr-2AP^2+rP^2+A^2r-2A^2P+APr-2Ar^2+4APr-2Pr^2\}}{4A^2Pr-4APr^2-4A^2P^2+4AP^2r+[Ar-r^2-AP+Pr+r^2-AP]^2}$$

$$=\frac{1}{\sqrt{(A-r)(r-P)}}\frac{(2A^2-2Ar+2AP-2Pr)(r-P)-\{6APr-2AP^2+P^2r+A^2r-2A^2P-2Ar^2-2Pr^2\}}{4A^2Pr-4APr^2-4A^2P^2+4AP^2r+[Ar-2AP+Pr]^2}$$

$$=\frac{1}{\sqrt{(A-r)(r-P)}}\frac{2A^2r-2Ar^2+4APr-2Pr^2-2A^2P-2AP^2+2P^2r-6APr+2AP^2-P^2r-A^2r+2A^2P+2Ar^2+2Pr^2}{4A^2Pr-4APr^2-4A^2P^2+4AP^2r+[Ar-2AP+Pr]^2}$$

$$=\frac{1}{\sqrt{(A-r)(r-P)}}\frac{A^2r-2APr++P^2r}{4A^2Pr-4APr^2-4A^2P^2+4AP^2r+[Ar-2AP+Pr]^2}$$

$$=\frac{1}{\sqrt{(A-r)(r-P)}}\frac{r(A^2-2AP++P^2)}{4A^2Pr-4APr^2-4A^2P^2+4AP^2r+[Ar-2AP+Pr]^2}$$

$$=\frac{1}{\sqrt{(A-r)(r-P)}}\frac{r(A-P)^2}{4A^2Pr-4APr^2-4A^2P^2+4AP^2r+A^2r^2+4A^2P^2+P^2r^2-4A^2Pr+2APr^2-4AP^2r}$$

$$=\frac{1}{\sqrt{(A-r)(r-P)}}\frac{r(A-P)^2}{-2APr^2+A^2r^2+P^2r^2}=\frac{1}{\sqrt{(A-r)(r-P)}}\frac{r(A-P)^2}{r^2(-2AP+A^2+P^2)}$$

$$=\frac{1}{\sqrt{(A-r)(r-P)}}\frac{r(A-P)^2}{r^2(A-P)^2}$$

$$= \frac{1}{r\sqrt{(A-r)(r-P)}}$$

Thus

$$\frac{1}{r\sqrt{(A-r)(r-P)}}$$

Thus the integrand operation is correct!

3.2.3.2. *Work-out of the second integral in the previous chapter*

We have derived the expression for the second integral:
General form:

$$\int \frac{1}{x^2\sqrt{ax^2+bx+c}}dx = -\frac{\sqrt{ax^2+bx+c}}{cx} - \frac{b}{2c}\int \frac{1}{x\sqrt{ax^2+bx+c}}dx$$

(See also next chapter for the work-out of the right hand side integral)

$$\int \frac{1}{x^2\sqrt{ax^2+bx+c}}dx = -\frac{\sqrt{ax^2+bx+c}}{cx} - \frac{b}{2c\sqrt{-c}} arcsin \frac{bx+2c}{|x|\sqrt{b^2-4ac}} \quad ,(c<o)$$

Now with $a = -1, b = A + P \; and \; c = -AP$

$$\emptyset_{\boldsymbol{A}} - \emptyset_{\boldsymbol{P}} = \int_P^A \frac{\varepsilon/2}{r^2\sqrt{(A-r)(r-P)}}dr = \int_P^A \frac{\varepsilon/2}{r^2\sqrt{-r^2+(A+P)r-AP}}dr$$

$$= -\varepsilon/2\left[\frac{\sqrt{-r^2+(A+P)r-AP}}{-APr}\right]_P^A + \varepsilon/2\frac{(A+P)}{2AP\sqrt{AP}}\left[arcsin\frac{(A+P)r-2AP}{|r|\sqrt{(A+P)^2-4AP}}\right]_P^A$$

$$= 0 + \varepsilon/2\frac{(A+P)}{2AP\sqrt{AP}}\left\{arcsin\frac{(A+P)A-2AP}{|A|\sqrt{(A+P)^2-4AP}} - arcsin\frac{(A+P)P-2AP}{|P|\sqrt{(A+P)^2-4AP}}\right\}$$

$$= \varepsilon/2\frac{(A+P)}{2AP\sqrt{AP}}\left\{arcsin\frac{(A-P)A}{|A|(A-P)} - arcsin\frac{(P-A)P}{|P|(A-P)}\right\}$$

$$= \varepsilon/2\frac{(A+P)}{2AP\sqrt{AP}}\{\arcsin(1) - \arcsin(-1)\}$$

$$= \varepsilon/2\frac{(A+P)}{2AP\sqrt{AP}}\left\{\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right\} = \varepsilon/2\frac{\pi(A+P)}{2AP\sqrt{AP}} = \frac{\pi\varepsilon}{4D\sqrt{AP}}$$

This is in accordance with the calculations.

3.2.3.3. *Alternative solution for integral 1*

According to the solutions given in https://nl.wikipedia.org/wiki/Lijst_van_integralen

$$\int \frac{1}{x\sqrt{ax^2+bx+c}}dx = \frac{1}{\sqrt{-c}} arcsin\frac{bx+2c}{|x|\sqrt{b^2-4ac}} + C, (c<0)$$

Thus

$$\emptyset_{\boldsymbol{A}} - \emptyset_{\boldsymbol{P}} = \int_P^A \frac{1}{r\sqrt{(A-r)(r-P)}}dr = \quad \int_P^A \frac{1}{r\sqrt{-r^2+(A+P)r-AP}}dr$$

$$= \frac{1}{\sqrt{AP}}\arcsin\left[\frac{(A+P)r-2AP}{|r|\sqrt{(A+P)^2-4AP}}\right]_P^A$$

$$= \frac{1}{\sqrt{AP}}\left\{\arcsin\frac{(A+P)A-2AP}{|A|\sqrt{(A+P)^2-4AP}} - \arcsin\frac{(A+P)P-2AP}{|P|\sqrt{(A+P)^2-4AP}}\right\}$$

$$= \frac{1}{\sqrt{AP}}\left\{\arcsin\frac{(A-P)A}{|A|(A-P)} - \arcsin\frac{(P-A)P}{|P|(A-P)}\right\}$$

$$= \frac{1}{\sqrt{AP}}\{\arcsin(1) - \arcsin(-1)\}$$

$$= \frac{1}{\sqrt{AP}}\left\{\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right\} = \frac{\pi}{\sqrt{AP}}$$

3.2.3.4. *Detailed calculation of the time T of a revolution*

$$L = r^2\frac{d\emptyset}{d\tau} => d\tau = \frac{r^2}{L}d\emptyset => T = \int d\tau = \int_{\boldsymbol{0}}^{\boldsymbol{2\pi}} \frac{r^2}{L}d\emptyset$$

Using equation 40

$$d\emptyset = \sqrt{\frac{L^2/c^2}{1-E^2/c^4}}\left[\frac{1}{r\sqrt{(A-r)(r-P)}} + \frac{\frac{\varepsilon}{2}}{r^2\sqrt{(A-r)(r-P)}}\right]dr \tag{40}$$

$$d\tau = \frac{r^2}{L}\sqrt{\frac{L^2/c^2}{1-E^2/c^4}}\left[\frac{1}{r\sqrt{(A-r)(r-P)}} + \frac{\frac{\varepsilon}{2}}{r^2\sqrt{(A-r)(r-P)}}\right]dr$$

$$d\tau = \frac{1}{L}\sqrt{\frac{L^2/c^2}{1-E^2/c^4}}\left[\frac{r}{\sqrt{(A-r)(r-P)}} + \frac{\frac{\varepsilon}{2}}{\sqrt{(A-r)(r-P)}}\right]dr$$

$$\Delta T = \int d\tau = \frac{2}{L}\sqrt{\frac{L^2/c^2}{1-E^2/c^4}}\int_P^A \left[\frac{r}{\sqrt{(A-r)(r-P)}} + \frac{\frac{\varepsilon}{2}}{\sqrt{(A-r)(r-P)}}\right]dr$$

First work out of the left integral

$$\int_P^A \frac{r}{\sqrt{(A-r)(r-P)}}dr = \int_P^A \frac{r}{\sqrt{-r^2+(A+P)r-AP}}dr \tag{41}$$

According to list of integrals (https://nl.wikipedia.org/wiki/Lijst_van_integralen)

$$\int \frac{x}{\sqrt{ax^2+bx+c}}dx = \frac{\sqrt{ax^2+bx+c}}{a} - \frac{b}{2a}\int \frac{1}{\sqrt{ax^2+bx+c}}dx \tag{42}$$

And

$$\int \frac{1}{\sqrt{ax^2+bx+c}}dx = \frac{1}{\sqrt{-a}}\arcsin\frac{-2ax-b}{\sqrt{b^2-4ac}} + C, (a<0)$$

Convert left integrand to the integral formula

$$\int_P^A \frac{r}{\sqrt{-r^2+(A+P)r-AP}}dr =$$

$$= \left[\frac{\sqrt{-r^2+(A+P)r-AP}}{-1}\right]_P^A - \frac{(A+P)}{-2}\int_P^A \frac{1}{\sqrt{-r^2+(A+P)r-AP}}dr$$

$$= -\sqrt{-A^2+(A+P)A-AP} + \sqrt{-P^2+(A+P)P-AP} + \frac{(A+P)}{2}\int_P^A \frac{1}{\sqrt{-r^2+(A+P)r-AP}}dr$$

$$= -0+0+\frac{(A+P)}{2}\int_P^A \frac{1}{\sqrt{-r^2+(A+P)r-AP}}dr$$

Now only the integral

$$\int_P^A \frac{1}{\sqrt{-r^2+(A+P)r-AP}}dr = \left[\arcsin\frac{2r-(A+P)}{\sqrt{(A+P)^2-4AP}} + C\right]_P^A =$$

$$\arcsin\frac{2A-(A+P)}{\sqrt{(A+P)^2-4AP}} + C - \arcsin\frac{2P-(A+P)}{\sqrt{(A+P)^2-4AP}} - C$$

$$= \arcsin\frac{A-P}{A-P} - \arcsin\frac{-A+P}{A-P}$$

$$\frac{\pi}{2}+\frac{\pi}{2}=\pi$$

So the left integral produces

$$\frac{(A+P)\pi}{2}$$

So the right integral produces

$$\pi\frac{\varepsilon}{2}$$

The sum is

$$\frac{\pi}{2}\big((A+P)+\varepsilon\big)$$

So the total integral for a complete revolution

$$\Delta T = 2\frac{1}{L}\sqrt{\frac{L^2/c^2}{1-E^2/c^4}}\frac{\pi}{2}\big((A+P)+\varepsilon\big)$$

With

$$\varepsilon = \frac{R_s}{1-\frac{R_s}{D}}$$

$$\Delta T = 2\frac{1}{L}\sqrt{\frac{L^2/c^2}{1-E^2/c^4}}\frac{\pi}{2}\left((A+P)+\frac{R_s}{1-\frac{R_s}{D}}\right)$$

$$\Delta T = 2\pi \frac{A+P}{2L}\sqrt{\frac{L^2/c^2}{1-E^2/c^4}}\left(1+\frac{R_s}{(A+P)\left(1-\frac{R_s}{D}\right)}\right)$$

$$\Delta T = 2\pi \frac{A+P}{2L}\sqrt{\frac{AP}{1-R_s/D}}\left(1+\frac{R_s}{(A+P)\left(1-\frac{R_s}{D}\right)}\right)$$

For Mercury

$$A = 6.98 * 10^{10}, P = 4.60 * 10^{10}, D = 2.77 * 10^{10}, R_{s(sun)} = 2953.25, L = 2.71 * 10^{15},$$

Time of one revolution: $\Delta \boldsymbol{T} = 7598744sec => \frac{7598744}{24*3600} = \mathbf{87.95}\boldsymbol{days}$

Derived in chapter 3.6.2 equation 1, the instantaneous rotation velocity of Mercury as function of Ø:

$$\boldsymbol{v} = \left\{\frac{\boldsymbol{GM_{sun}}}{\boldsymbol{a}(\mathbf{1}-\boldsymbol{e}^{\mathbf{2}})}\left(\mathbf{1}+\mathbf{2}\boldsymbol{ecos}[\emptyset(\mathbf{1}-\boldsymbol{\epsilon})]+\boldsymbol{e}^{\mathbf{2}}\right)\right\}^{\frac{\mathbf{1}}{\mathbf{2}}} \tag{42a}$$

3.3. Experiment 5 - Shapiro Time Delay – Hobson et al

In the Shapiro experiment, radar signals were sent from the Earth to a planet, which was at that moment opposite the Sun, and back to Earth. According to the General Relativity Theory the signal just grazing the Sun will be deflected due to the Sun's gravity. The experiment was done in 1964 and after that various times verified. This experiment is sometimes called the *fourth* classical test of General Relativity.

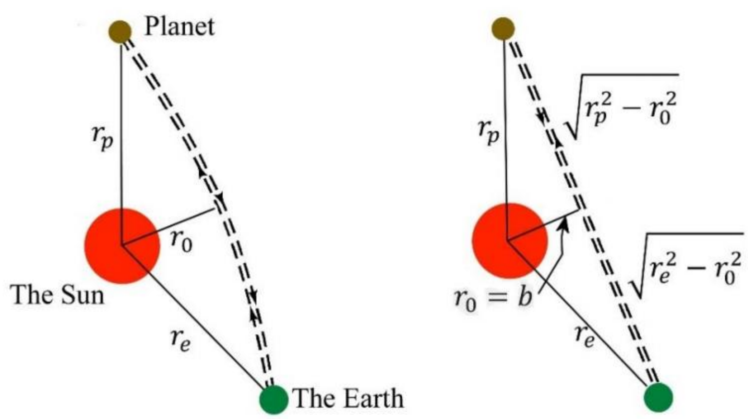


Figure 1. The radar reflection of photons from the Earth to a planet and back. The left image is the actual path, exaggerated. The right image is the Euclidean form

From Tests of General Relativity: A Review By Estelle Asmodelle (Asmodelle, 2017)

To define Shapiro delay, assume the Earth and the planet are stationary; while the total time for the round trip of the radar signal is Δt, in coordinate time. The value of *t* must be represented in terms of *r* along the entire pathway, while r_0 is the closest approach to the Sun.
For the calculation of the Shapiro delay, the Schwarzschild equation is applied.
Schwarzschild:

$$ds^2 = c^2 d\tau^2 = \sigma^2 c^2 dt^2 - \frac{dr^2}{\sigma^2} - r^2 d\theta^2 - r^2 \sin^2\theta^2\, d\emptyset^2$$

$$\sigma = \sqrt{1 - \frac{2GM_{sun}}{\boldsymbol{c^2} r}} = \sqrt{1 - \frac{R_s}{r}}$$

where $R_s = \frac{2GM_{sun}}{\boldsymbol{c^2}}$ (the Schwarzschild radius, here, of the Sun)

We choose the frame such that it is matching the equatorial plane $\theta = \pi/2$. So:

$$c^2 d\tau^2 = \sigma^2 c^2 dt^2 - \frac{dr^2}{\sigma^2} - r^2 d\emptyset^2$$

For photon or radar echoes holds that $d\tau = 0$. Then:

$$\sigma^2 c^2 dt^2 = \frac{dr^2}{\sigma^2} + r^2 d\emptyset^2$$

Derivation to the affine parameter λ

$$\sigma^2 c^2 \left(\frac{dt}{d\lambda}\right)^2 = \frac{1}{\sigma^2}\left(\frac{dr}{d\lambda}\right)^2 + r^2 \left(\frac{d\emptyset}{d\lambda}\right)^2$$

As derived in formula (21.1.40) the angular momentum is

$$L = r^2 \frac{d\emptyset}{d\lambda}$$

$$\sigma^2 c^2 \left(\frac{dt}{d\lambda}\right)^2 = \frac{1}{\sigma^2}\left(\frac{dr}{d\lambda}\right)^2 + \frac{L^2}{r^2}$$

Multiply with σ^2:

$$\sigma^4 c^2 \left(\frac{dt}{d\lambda}\right)^2 = \left(\frac{dr}{d\lambda}\right)^2 + \frac{L^2}{r^2}\sigma^2$$

Pose

$$k^2 = \sigma^4 \left(\frac{dt}{d\lambda}\right)^2$$

Note: this is also $k = \frac{E}{c^2}$ as is seen in formula (21.1.38)

$$\left(\frac{dr}{d\lambda}\right)^2 + \frac{L^2}{r^2}\sigma^2 = k^2 c^2$$

The "energy" equation for a photon orbit in the Schwarzschild geometry is:

$$\left(\frac{dr}{d\lambda}\right)^2 + \frac{L^2}{r^2}\left(1 - \frac{R_s}{r}\right) = k^2 c^2 \tag{42b}$$

As seen above:

$$k^2 = \sigma^4 \left(\frac{dt}{d\lambda}\right)^2 \ \Rightarrow \left(\frac{dt}{d\lambda}\right)^2 = \frac{k^2}{\sigma^4}$$

Using

$$\left(\frac{dt}{d\lambda}\right)^2 = \frac{k^2}{\sigma^4} = \frac{k^2}{\left(1 - \frac{R_s}{r}\right)^2}$$

Now:

$$\left(\frac{dr}{d\lambda}\right)^2 = \left(\frac{dr}{dt}\frac{dt}{d\lambda}\right)^2 = \frac{k^2}{\left(1-\frac{R_s}{r}\right)^2}\left(\frac{dr}{dt}\right)^2 \tag{42c}$$

We can rewrite the energy equation (42b)

$$\left(\frac{dr}{d\lambda}\right)^2 + \frac{L^2}{r^2}\left(1-\frac{R_s}{r}\right) = k^2c^2$$

Replace $\left(\frac{dr}{d\lambda}\right)^2$ with (42c)

$$\frac{k^2}{\left(1-\frac{R_s}{r}\right)^2}\left(\frac{dr}{dt}\right)^2 + \frac{L^2}{r^2}\left(1-\frac{R_s}{r}\right) = k^2c^2$$

Divide by $\left(1-\frac{R_s}{r}\right)$:

$$\frac{k^2}{\left(1-\frac{R_s}{r}\right)^3}\left(\frac{dr}{dt}\right)^2 + \frac{L^2}{r^2} - \frac{k^2c^2}{\left(1-\frac{R_s}{r}\right)} = 0$$

Divide by k^2:

$$\frac{1}{\left(1-\frac{R_s}{r}\right)^3}\left(\frac{dr}{dt}\right)^2 + \frac{L^2}{k^2r^2} - \frac{c^2}{1-\frac{R_s}{r}} = 0 \tag{43}$$

Now consider a photon path from Earth to another planet (say Venus, with $r_p = r_V$), as shown in Figure 2. Evidently the photon path will be deflected by the gravitational field of the Sun (assuming that the planets are in a configuration like that shown in the figure, where the photon has to pass close to the Sun in order to reach Venus). Let r_0 be the coordinate distance of the closest approach of the photon to the Sun; then

$$\left(\frac{dr}{dt}\right)_{r_0} = 0$$

And so from (43) we find the relation of the constants

$$\frac{L^2}{k^2{r_0}^2} = \frac{c^2}{1-\frac{R_s}{r_0}}$$

Thus, after rearrangement, we can write (43) as

$$\left(\frac{dr}{dt}\right)^2 = \left(1-\frac{R_s}{r}\right)^3\left(-\frac{L^2}{k^2r^2} + \frac{c^2}{1-\frac{R_s}{r}}\right) = \left(1-\frac{R_s}{r}\right)^3\left(\frac{c^2}{1-\frac{R_s}{r}} - \frac{L^2{r_0}^2}{k^2{r_0}^2r^2}\right)$$

$$= \left(1-\frac{R_s}{r}\right)^3\left(\frac{c^2}{1-\frac{R_s}{r}} - \frac{{r_0}^2c^2}{r^2\left(1-\frac{R_s}{r_0}\right)}\right)$$

$$= \left(1-\frac{R_s}{r}\right)^2\left(c^2 - \frac{{r_0}^2c^2\left(1-\frac{R_s}{r}\right)}{r^2\left(1-\frac{R_s}{r_0}\right)}\right) = c^2\left(1-\frac{R_s}{r}\right)^2\left(1 - \frac{{r_0}^2\left(1-\frac{R_s}{r}\right)}{r^2\left(1-\frac{R_s}{r_0}\right)}\right)$$

$$\Rightarrow \; \frac{dr}{dt} = c\left(1-\frac{R_s}{r}\right)\left[1-\frac{{r_0}^2\left(1-\frac{R_s}{r}\right)}{r^2\left(1-\frac{R_s}{r_0}\right)}\right]^{\frac{1}{2}}$$

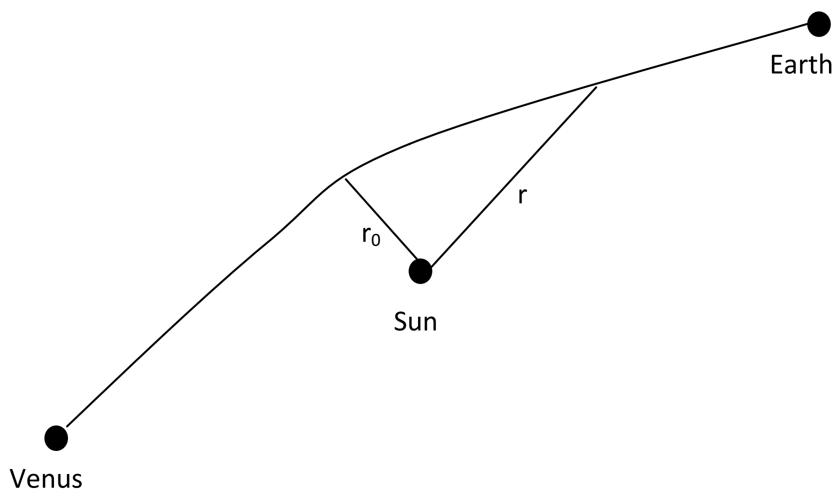


Figure 2 Photon path from Earth to Venus deflected by the Sun.

This can be integrated to give for the time taken to travel between point r_0 and r

$$t(r,r_0) = \int_{r_0}^{r} \frac{1}{c\left(1-\frac{R_s}{r}\right)\left[1-\frac{{r_0}^2\left(1-\frac{R_s}{r}\right)}{r^2\left(1-\frac{R_s}{r_0}\right)}\right]^{\frac{1}{2}}} dr$$

Because $R_s \ll r_0$ we can take the first order Taylor expansion of

$$\frac{\left(1-\frac{R_s}{r}\right)}{\left(1-\frac{R_s}{r_0}\right)} \approx \left(1-\frac{R_s}{r}\right)\left(1+\frac{R_s}{r_0}\right) = 1-\frac{R_s}{r}+\frac{R_s}{r_0}-\frac{{R_s}^2}{rr_0}$$

So the integrand can be expanded to the first order in R_s/r to obtain

$$t(r,r_0) = \int_{r_0}^{r} \frac{1}{c\left(1-\frac{R_s}{r}\right)\left[1-\frac{{r_0}^2}{r^2}\left(1-\frac{R_s}{r}+\frac{R_s}{r_0}-\frac{{R_s}^2}{rr_0}\right)\right]^{\frac{1}{2}}} dr$$

Multiply numerator and denominator by r

$$t(r,r_0) = \int_{r_0}^{r} \frac{r}{c\left(1-\frac{R_s}{r}\right)\left[r^2-{r_0}^2\left(1-\frac{R_s}{r}+\frac{R_s}{r_0}-\frac{{R_s}^2}{rr_0}\right)\right]^{\frac{1}{2}}} dr$$

$$t(r,r_0)=\int_{r_0}^{r}\frac{r}{c\left(1-\frac{R_s}{r}\right)\left[r^2-{r_0}^2-R_sr_0+\frac{R_s{r_0}^2}{r}+\frac{{R_s}^2r_0}{r}\right]^{\frac{1}{2}}}dr$$

$$t(r,r_0)=\int_{r_0}^{r}\frac{r}{c\sqrt{r^2-{r_0}^2}\left(1-\frac{R_s}{r}\right)\left[1-\frac{R_sr_0(1-\frac{r_0}{r}-\frac{R_s}{r}}{r^2-{r_0}^2}\right]^{\frac{1}{2}}}dr$$

$$t(r,r_0)=\int_{r_0}^{r}\frac{r}{c\sqrt{r^2-{r_0}^2}\left[\left(1-\frac{2R_s}{r}+\frac{{R_s}^2}{r^2}\right)\left(1-\frac{R_sr_0(1-\frac{r_0}{r}-\frac{R_s}{r}}{r^2-{r_0}^2}\right)\right]^{\frac{1}{2}}}dr$$

First work out the right hand-side of the de-numerator

$$\left(1-\frac{2R_s}{r}+\frac{{R_s}^2}{r^2}\right)\left(1-\frac{R_sr_0(1-\frac{r_0}{r}-\frac{R_s}{r}}{r^2-{r_0}^2}\right)$$

$$=\mathbf{1}-\frac{\mathbf{2R_s}}{\boldsymbol{r}}+\frac{{R_s}^2}{r^2}-\frac{\boldsymbol{R_sr_0\left(1-\frac{r_0}{r}-\frac{R_s}{r}\right)}}{\boldsymbol{r^2-{r_0}^2}}+\frac{2{R_s}^2r_0(1-\frac{r_0}{r}-\frac{R_s}{r}}{r(r^2-{r_0}^2)}-\frac{{R_s}^3r_0\left(1-\frac{r_0}{r}-\frac{R_s}{r}\right)}{r^2(r^2-{r_0}^2)}$$

After ignoring smallest

$$\left(1-\frac{2R_s}{r}+\frac{{R_s}^2}{r^2}\right)\left(1-\frac{R_sr_0(1-\frac{r_0}{r}-\frac{R_s}{r}}{r^2-{r_0}^2}\right)=\mathbf{1}-\frac{\mathbf{2R_s}}{\boldsymbol{r}}-\frac{\boldsymbol{R_sr_0\left(1-\frac{r_0}{r}\right)}}{\boldsymbol{r^2-{r_0}^2}}$$

$$\left(1-\frac{2R_s}{r}+\frac{{R_s}^2}{r^2}\right)\left(1-\frac{R_sr_0(1-\frac{r_0}{r}-\frac{R_s}{r}}{r^2-{r_0}^2}\right)=\mathbf{1}-\frac{\mathbf{2R_s}}{\boldsymbol{r}}-\frac{\boldsymbol{R_sr_0(r-r_0)}}{\boldsymbol{r(r+r_0)(r-r_0)}}$$

$$\left(1-\frac{2R_s}{r}+\frac{{R_s}^2}{r^2}\right)\left(1-\frac{R_sr_0(1-\frac{r_0}{r}-\frac{R_s}{r}}{r^2-{r_0}^2}\right)=\mathbf{1}-\frac{\mathbf{2R_s}}{\boldsymbol{r}}-\frac{\boldsymbol{R_sr_0}}{\boldsymbol{r(r+r_0)}}$$

Fill in the denominator

$$t(r,r_0)=\int_{r_0}^{r}\frac{r}{c\sqrt{r^2-{r_0}^2}\left[\mathbf{1}-\frac{\mathbf{2R_s}}{\boldsymbol{r}}-\frac{\boldsymbol{R_sr_0}}{\boldsymbol{r(r+r_0)}}\right]^{\frac{1}{2}}}dr$$

Approximation with first order Taylor expansion

$$t(r,r_0)=\int_{r_0}^{r}\frac{r}{c\sqrt{r^2-{r_0}^2}}\left[1+\frac{R_s}{r}+\frac{R_sr_0}{2r(r+r_0)}\right]dr$$

This can be evaluated (see check below) to give

$$t(r,r_0)=\frac{(r^2-{r_0}^2)^{\frac{1}{2}}}{c}+\frac{R_s}{c}ln\left[\frac{r+(r^2-{r_0}^2)^{\frac{1}{2}}}{r_0}\right]+\frac{R_s}{2c}\left(\frac{r-r_0}{r+r_0}\right)^{\frac{1}{2}}$$

We can check the formula above, by taking the derivative of the formula; this shall be equal to the integrand.

$$\frac{dt(r,r_0)}{dr} = \frac{r}{c(r^2-{r_0}^2)^{\frac{1}{2}}} + \frac{R_s}{c}\frac{\left(\frac{1}{r_0}+\frac{r}{r_0(r^2-{r_0}^2)^{\frac{1}{2}}}\right)}{\frac{r+(r^2-{r_0}^2)^{\frac{1}{2}}}{r_0}} + \frac{R_s}{4c}\frac{\left(\frac{1}{r+r_0}-\frac{(r-r_0)}{(r+r_0)^2}\right)}{\left(\frac{r-r_0}{r+r_0}\right)^{\frac{1}{2}}}$$

$$\frac{dt(r,r_0)}{dr} = \frac{r}{c(r^2-{r_0}^2)^{\frac{1}{2}}} + \frac{R_s}{c}\frac{\left(1+\frac{r}{(r^2-{r_0}^2)^{\frac{1}{2}}}\right)}{r+(r^2-{r_0}^2)^{\frac{1}{2}}} + \frac{R_s}{4c}\frac{\left(\frac{r+r_0-r+r_0}{(r+r_0)^2}\right)}{\left(\frac{r-r_0}{r+r_0}\right)^{\frac{1}{2}}}$$

$$\frac{dt(r,r_0)}{dr} = \frac{r}{c(r^2-{r_0}^2)^{\frac{1}{2}}} + \frac{R_s}{c}\frac{\left(r+(r^2-{r_0}^2)^{\frac{1}{2}}\right)}{\left(r+(r^2-{r_0}^2)^{\frac{1}{2}}\right)(r^2-{r_0}^2)^{\frac{1}{2}}} + \frac{R_s}{4c}\frac{(r+r_0-r+r_0)}{\left(\frac{r-r_0}{r+r_0}\right)^{\frac{1}{2}}(r+r_0)^2}$$

$$\frac{dt(r,r_0)}{dr} = \frac{r}{c(r^2-{r_0}^2)^{\frac{1}{2}}} + \frac{R_s}{c}\frac{1}{(r^2-{r_0}^2)^{\frac{1}{2}}} + \frac{R_s}{2c}\frac{r_0}{\frac{(r^2-{r_0}^2)^{\frac{1}{2}}}{r+r_0}(r+r_0)^2}$$

$$\frac{dt(r,r_0)}{dr} = \frac{r}{c(r^2-{r_0}^2)^{\frac{1}{2}}} + \frac{R_s}{c}\frac{1}{(r^2-{r_0}^2)^{\frac{1}{2}}} + \frac{R_s}{2c}\frac{r_0}{(r^2-{r_0}^2)^{\frac{1}{2}}(r+r_0)}$$

$$\frac{dt(r,r_0)}{dr} = \frac{r}{c(r^2-{r_0}^2)^{\frac{1}{2}}}\left[1+\frac{R_s}{r}+\frac{R_s r_0}{2r(r+r_0)}\right]$$

Thus the formula is correct!

So

$$t(r,r_0) = \frac{(r^2-{r_0}^2)^{\frac{1}{2}}}{c} + \frac{R_s}{c}ln\left[\frac{r+(r^2-{r_0}^2)^{\frac{1}{2}}}{r_0}\right] + \frac{R_s}{2c}\left(\frac{r-r_0}{r+r_0}\right)^{\frac{1}{2}}$$

The first term on the right-hand side is just what we would have expected if the light had been travelling in a straight line. The second and third terms give us the extra coordinate time taken for the photon to travel along the *curved* path to the point *r*. So, you can see from Figure 2 that if we bounce a radar beam to Venus and back then the excess coordinate-time delay over a straight-line path is

$$\Delta t = 2\left[t(r_E,r_0)+t(r_V,r_0)-\frac{({r_E}^2-{r_0}^2)^{\frac{1}{2}}}{c}-\frac{({r_V}^2-{r_0}^2)^{\frac{1}{2}}}{c}\right]$$

As was mentioned above, the first two terms between these brackets form the relativistic time from Earth to Venus and the two right hand side terms form the time when the path was just a straight line. Where the factor 2 is included because the photon has to go to Venus and back to Earth.

Since $r_{E\gg}r_0$ and $r_{v\gg}r_0$ we have

$$t(r_E,r_0)-\frac{({r_E}^2-{r_0}^2)^{\frac{1}{2}}}{c} \approx \frac{R_s}{c}ln\left(\frac{r_E+r_E}{r_0}\right)+\frac{R_s}{2c} = \frac{R_s}{c}ln\left(\frac{2r_E}{r_0}\right)+\frac{R_s}{2c}$$

$$t(r_V, r_0) - \frac{(r_V{}^2 - r_0{}^2)^{\frac{1}{2}}}{c} \approx \frac{R_s}{c} ln\left(\frac{r_V + r_V}{r_0}\right) + \frac{R_s}{2c} = \frac{R_s}{c} ln\left(\frac{2r_V}{r_0}\right) + \frac{R_s}{2c}$$

Summation

$$\frac{R_s}{c} ln\left(\frac{2r_E}{r_0}\right) + \frac{R_s}{c} ln\left(\frac{2r_V}{r_0}\right) + \frac{R_s}{c} = \frac{2GM}{c^3}\left[ln\left(\frac{4r_E r_V}{r_0{}^2}\right) + 1\right]$$

Thus to go to Venus and back, the excess coordinate-time delay is

$$\boxed{\boldsymbol{\Delta t \approx \frac{4GM_{sun}}{c^3}\left[ln\left(\frac{4r_E r_V}{r_0{}^2}\right) + 1\right]}}$$

For Venus, when it is opposite to the Earth on the far side of the Sun,

$$\Delta t \approx 252\mu s.$$

While for Mercury

$$\Delta t \approx 240\mu s.$$

The total time (Earth, Sun and Venus and back) without delay is 1720sec.

Of course, clocks on Earth do not measure coordinate time, due to the **rotation of the Earth around its own axis** and the effect of the **rotation of the Earth around the Sun.**
Due to the **rotation of the Earth around its own axis**, the corresponding proper time of the signal is given by

$$\Delta\tau = \left(1 - \frac{2GM_E}{c^2 r_E}\right)^{\frac{1}{2}} \Delta t$$

Thus the effect is

$$\Delta t - \Delta\tau = \Delta t - \left(1 - \frac{2GM_E}{c^2 r_E}\right)^{\frac{1}{2}} \Delta t$$

$$=> 6.98 * 10^{-10}\Delta t \quad for\ 252\mu s => 1.76 * 10^{-13} sec = 0.176 ps$$
$$p = 10^{-12}$$

So since $r_E \gg \frac{GM}{c^2}$, and thus $0.176ps \ll 252\mu s$ we can ignore this effect on the accuracy of our calculation.

The effect of the **rotation of the Earth around the Sun** gives a delay of 15*nsec/sec*, as is shown in chapter (3.4). So for the excess time delay $\Delta t \approx 252\mu s$. from Venus, the effect of the rotation of the Earth around the Sun gives a small effect of $252 * 10^{-6} * 15 * 10^{-9} = 3.78 * 10^{-12}\ sec = 3.78\ ps$ which can also be ignored.

3.4. Time relation between Earth observer and universal frame with centre of Sun

When, in other chapters, the deflection of light or orbits of planets around the Sun are considered, a frame is used with the centre in the middle of the Sun, while we observe the phenomenon from the Earth and have a **rotation velocity with respect to the Sun.** Here we consider this effect and calculate the correction factor.
Starting point is the Schwarzschild metric:

$$ds^2 = \sigma^2 c^2 dt^2 - \frac{dr^2}{\sigma^2} - r^2 d\theta^2 - r^2 \sin^2\theta^2\, d\emptyset^2$$

With:

$$\sigma = \sqrt{1 - \frac{2GM_{sun}}{c^2 r}} \qquad R_s = \frac{2GM_{sun}}{c^2}$$

Centre of the frame is the Sun centre. The orbit of the Earth around the Sun is assumed to be a circle. The observed physical movement is in the equatorial plane of the frame. Thus the radius is constant and theta is $\pi/2$.

The equation simplifies then to:

$$ds^2 = c^2 d\tau^2 = \sigma^2 c^2 dt^2 - r^2 d\emptyset^2$$

τ Is the proper time of the observer on Earth (on North or South Pole), while t is the coordinate time of the universal Sun frame. So everything, the Earth observer inclusive, is related to the universal Sun frame.

$$d\tau^2 = \sigma^2 dt^2 - \frac{r^2}{c^2}\left(\frac{d\emptyset}{dt}\right)^2 dt^2 = \left(1 - \frac{R_s}{r} - \frac{r^2}{c^2}\left(\frac{d\emptyset}{dt}\right)^2\right) dt^2$$

$$d\tau^2 = \left(1 - \frac{R_s}{r} - \frac{v^2}{c^2}\right) dt^2$$

$$d\tau = \sqrt{\left(1 - \frac{R_s}{r} - \frac{v^2}{c^2}\right)} dt$$

First order Taylor expansion:

$$d\tau = \left(1 - \frac{R_s}{2r} - \frac{v^2}{2c^2}\right) dt$$

$R_s = 2950m$, this is the Schwarzschild radius of the Sun. The rotation velocity of the Earth around the sun is $v = 30{,}000 m/s$. The distance from the observer to the sun is $r \approx 150 * 10^9$m.

The second term at the right hand side is due to the Sun gravity and the third term is due to the Earth velocity around the Sun.

$$d\tau = (1 - 99.10^{-10} - 50.10^{-10})dt$$

$$\boldsymbol{d\tau \approx \left(1 - 15.10^{-9}\right)dt}$$

$$\Delta t - \Delta\tau = 15.10^{-9}\Delta t$$

This is the relation between the time of the Earth observer and the universal Sun frame time t.

As the Earth observer is also influenced by the **gravity of the Earth**, while standing on one of the poles, $dr = d\theta = d\emptyset = 0$ then

$$d\tau = \sqrt{1 - \frac{2GM_E}{c^2 r}} dt = \sqrt{1 - 1.3908 * 10^{-9}} dt = (1 - 0.6954 * 10^{-9}) dt$$

At the equator is the radius r_e=6,378,137 m. In addition, the rotation of the Earth needs to be taken into account. This imparts on an observer an angular velocity of $\frac{d\emptyset}{dt}$ of 2π divided by the sidereal period of the Earth's rotation, 86162.4 seconds. So $d\emptyset = 7.2923 * 10^{-5} dt$. The proper time equation then produces

$$d\tau = \sqrt{(1 - 1.3908 * 10^{-9}) - 2.4059 * 10^{-12}} dt = (1 - 0.6966 * 10^{-9}) dt.$$

M_E=5.9742x10^{24}kg, r_e=6,356,752 m, G=6.674x10^{-11}N$kg^{-2}m^2$, c=299,792,458 m/s.

3.5. Trajectories of massive particles-Second Derivation

We do this second derivation because the solution brings us close to the original formula of an ellipse $\boldsymbol{r}(\emptyset)=\frac{\boldsymbol{a}(\mathbf{1}-\boldsymbol{e}^{2})}{\mathbf{1}+\boldsymbol{ecos}[\emptyset-\theta]}$ compared with the relativistic outcome at the end of this chapter.

$$\boldsymbol{r}=\frac{\boldsymbol{a}(\mathbf{1}-\boldsymbol{e}^{\mathbf{2}})}{\mathbf{1}+\boldsymbol{ecos}[\emptyset-\epsilon\emptyset]}$$

Here we see that θ is a function of $\emptyset$ and changes slightly with a factor ϵ.

From "General Relativity an introduction for Physicits" by M.P. Hobson, G. Efstathlou A.N. Lasenby Pag. 230 (M.P Hobson, 2006).

As derived above there are the following equations available:

$$\left(1-\frac{2GM}{c^2r}\right)\frac{dt}{d\lambda}=\frac{E}{c^2}$$

$$c^2\left(1-\frac{2GM}{c^2r}\right)\left(\frac{dt}{d\lambda}\right)^2-\left(1-\frac{2GM}{c^2r}\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2-r^2\left(\frac{d\emptyset}{d\lambda}\right)^2=c^2$$

$$r^2\frac{d\emptyset}{d\lambda}=L$$

By substituting the first and the third equation into the second equation:

$$c^2\left(1-\frac{2GM}{c^2r}\right)^2\left(\frac{dt}{d\lambda}\right)^2-\left(\frac{dr}{d\lambda}\right)^2-r^2\left(\frac{d\emptyset}{d\lambda}\right)^2\left(1-\frac{2GM}{c^2r}\right)=c^2\left(1-\frac{2GM}{c^2r}\right)$$

$$\left(\frac{dr}{d\lambda}\right)^2+r^2\left(\frac{d\emptyset}{d\lambda}\right)^2\left(1-\frac{2GM}{c^2r}\right)-c^2\left(1-\frac{2GM}{c^2r}\right)^2\left(\frac{dt}{d\lambda}\right)^2=c^2\left(\frac{2GM}{c^2r}-1\right)$$

$$\left(\frac{dr}{d\lambda}\right)^2+\frac{L^2}{r^2}\left(1-\frac{2GM}{c^2r}\right)-\frac{E^2}{c^2}=c^2\left(\frac{2GM}{c^2r}-1\right)=\frac{2GM}{r}-c^2$$

$$\left(\frac{dr}{d\lambda}\right)^2+\frac{L^2}{r^2}\left(1-\frac{2GM}{c^2r}\right)-\frac{2GM}{r}=c^2\left(\frac{E^2}{c^4}-1\right)$$

$$\left(\frac{dr}{d\lambda}\right)^2+\frac{L^2}{r^2}=c^2\left(\frac{E^2}{c^4}-1\right)+\frac{2GM}{r}+\frac{2GML^2}{c^2r^3}$$

Now:

$$\frac{dr}{d\lambda}=\frac{dr}{d\emptyset}\frac{d\emptyset}{d\lambda}=\frac{L}{r^2}\frac{dr}{d\emptyset}$$

Filled in the previous equation:

$$\left(\frac{dr}{d\lambda}\right)^2+\frac{L^2}{r^2}=c^2\left(\frac{E^2}{c^4}-1\right)+\frac{2GM}{r}+\frac{2GML^2}{c^2r^3}$$

$$\left(\frac{L}{r^2}\frac{dr}{d\emptyset}\right)^2+\frac{L^2}{r^2}=c^2\left(\frac{E^2}{c^4}-1\right)+\frac{2GM}{r}+\frac{2GML^2}{c^2r^3}$$

$$\left(\frac{1}{r^2}\frac{dr}{d\emptyset}\right)^2+\frac{1}{r^2}=\frac{c^2}{L^2}\left(\frac{E^2}{c^4}-1\right)+\frac{2GM}{rL^2}+\frac{2GM}{c^2r^3}$$

Substitute by $u=1/r$

$$\frac{du}{d\emptyset}=\frac{du}{dr}\frac{dr}{d\emptyset}=\frac{-1}{r^2}\frac{dr}{d\emptyset}=>\frac{1}{r^2}\frac{dr}{d\emptyset}=-\frac{du}{d\emptyset}$$
$$\left(\frac{du}{d\emptyset}\right)^2+u^2=\frac{c^2}{L^2}\left(\frac{E^2}{c^4}-1\right)+\frac{2GMu}{L^2}+\frac{2GMu^3}{c^2}$$

Now we differentiate this equation with respect to Ø to obtain:

$$2\frac{du}{d\emptyset}\frac{d^2u}{d\emptyset^2}+2u\frac{du}{d\emptyset}=\frac{2GM}{L^2}\frac{du}{d\emptyset}+\frac{6GMu^2}{c^2}\frac{du}{d\emptyset}$$

Divide by $2\frac{du}{d\emptyset}$:

$$\frac{d^2u}{d\emptyset^2}+u=\frac{GM}{L^2}+\frac{3GMu^2}{c^2} \tag{44}$$

If we ignore the last term, we get the equation according to the Newtonian theory, the solution is:

$$u=\frac{GM}{L^2}(1+e\cos\emptyset) \quad or \quad r=\frac{L^2}{GM(1+e\cos\emptyset)} \tag{45}$$

Which describes an ellipse, the parameter *e* measures the *eccentricity* of the orbit. Thus, for example, we can draw the orbit of a planet around the Sun as in the figure below. We can write the distance of closest approach (*perihelion*) as $r_1=a(1-e)$ and the distance of furthest approach (*aphelion*) as $r_2=a(1+e)$.

Derived from (45) with r=1/u:

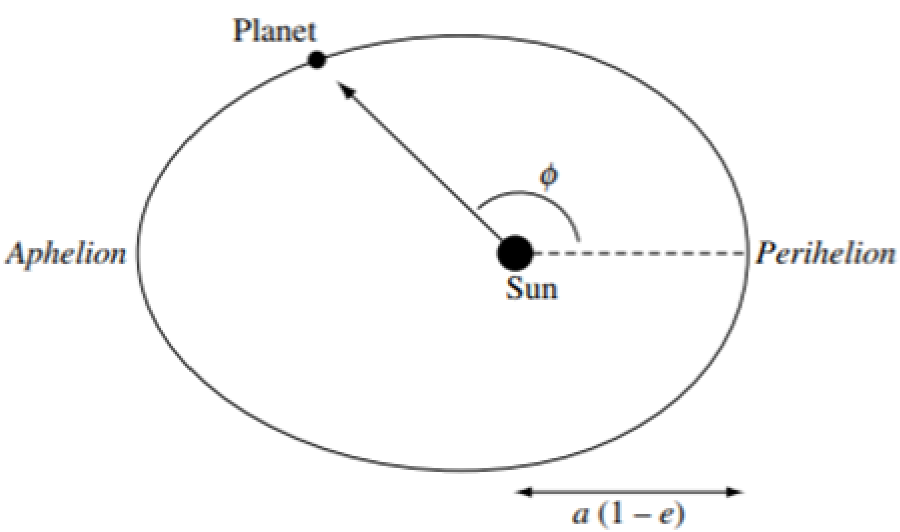
$$r=\frac{L^2}{GM(1+e\cos\emptyset)} \;=>\; r_{max}=\frac{L^2}{GM(1-e)} \;\; and \;\; r_{min}=\frac{L^2}{GM(1+e)}$$
$$\mathrm{a}=\frac{r_{max}+r_{min}}{2}=\frac{L^2}{2GM}\left(\frac{1}{(1-e)}+\frac{1}{(1+e)}\right)=\frac{L^2}{2GM}\left(\frac{1+e+1-e}{(1-e)(1+e)}\right)$$

So the equation of motion then requires that the semi-major axis is given by:

$$\boldsymbol{a}=\frac{\boldsymbol{L^2}}{\boldsymbol{GM(1-e^2)}} \tag{46}$$

Hence

$$r_{max}=\frac{L^2}{GM(1-e)}=a(1+e) \;\; and \;\; r_{min}=\frac{L^2}{GM(1+e)}=a(1-e)$$



The elliptical orbit of a planet around the Sun; *e* is the eccentricity of the orbit

Now to include the third term as well the solution looks like:

$$u = \frac{GM}{L^2}(1 + e\cos\emptyset) + \Delta\mathrm{u} \tag{47}$$

$$\frac{du}{d\emptyset} = -\frac{GM}{L^2}e\sin\emptyset + \frac{d\Delta u}{d\emptyset}$$

$$\frac{d^2u}{d\emptyset^2} = -\frac{GM}{L^2}e\cos\emptyset + \frac{d^2\Delta u}{d\emptyset^2}$$

Substitute this in formula (44):

$$\frac{d^2u}{d\emptyset^2} + u = \frac{GM}{L^2} + \frac{3GMu^2}{c^2} \tag{44}$$

$$\frac{d^2u}{d\emptyset^2} + u = \frac{GM}{L^2}(1 + e\cos\emptyset - e\cos\emptyset) + \frac{d^2\Delta u}{d\emptyset^2} + \Delta\mathrm{u} = \frac{GM}{L^2} + \frac{d^2\Delta u}{d\emptyset^2} + \Delta\mathrm{u}$$

$$\frac{d^2\Delta u}{d\emptyset^2} + \Delta\mathrm{u} = -\frac{GM}{L^2} + \frac{d^2u}{d\emptyset^2} + u = -\frac{GM}{L^2} + \frac{GM}{L^2} + \frac{3GMu^2}{c^2} = \frac{3GMu^2}{c^2}$$

$$\frac{d^2\Delta u}{d\emptyset^2} + \Delta\mathrm{u} = \frac{3GM}{c^2}\left(\left(\boldsymbol{\frac{GM}{L^2}}\right)^{\mathbf{2}} + \left(\boldsymbol{\frac{GM}{L^2}e\cos\emptyset}\right)^{\mathbf{2}} + (\Delta\mathrm{u})^2 + \mathbf{2}\left(\boldsymbol{\frac{GM}{L^2}}\right)^{\mathbf{2}}\boldsymbol{e\cos\emptyset} + 2\frac{GM}{L^2}\Delta\mathrm{u} + 2\frac{GM}{L^2}e\cos\emptyset.\Delta\mathrm{u}\right)$$

We find that, to first-order in Δu,

$$\frac{d^2\Delta u}{d\emptyset^2} + \Delta\mathrm{u} = \frac{3(GM)^3}{c^2L^4}\left(\mathbf{1} + (\boldsymbol{e}\,\mathbf{cos}\,\emptyset)^{\mathbf{2}} + \mathbf{2}\boldsymbol{e}\,\mathbf{cos}\,\emptyset\right)$$

A particular integral of the equation is found to be:

$$\boldsymbol{\Delta\mathrm{u}} = \boldsymbol{\frac{3(GM)^3}{c^2L^4}}\left[\mathbf{1} + \boldsymbol{e^2}\left(\frac{\mathbf{1}}{\mathbf{2}} - \frac{\mathbf{1}}{\mathbf{6}}\mathbf{cos2}\,\emptyset\right) + \mathbf{e}\,\emptyset\,\mathbf{sin}\,\emptyset\right] \tag{48}$$

This can be checked by direct differentiation of (48):

$$\frac{d\Delta u}{d\emptyset} = \frac{3(GM)^3}{c^2L^4}\left[\frac{1}{3}e^2\sin2\emptyset + \mathrm{e}\sin\emptyset + \mathrm{e}\,\emptyset\cos\emptyset\right]$$

$$\frac{d^2\Delta u}{d\emptyset^2} = \frac{3(GM)^3}{c^2L^4}\left[\frac{2}{3}e^2\cos2\emptyset + \mathrm{e}\cos\emptyset + \mathrm{e}\cos\emptyset - \mathrm{e}\,\emptyset\sin\emptyset\right]$$

$$\frac{d^2\Delta u}{d\emptyset^2} = \frac{3(GM)^3}{c^2L^4}\left[\frac{2}{3}e^2\cos2\emptyset + 2\mathrm{e}\cos\emptyset - \mathrm{e}\,\emptyset\sin\emptyset\right]$$

Fill in (48):

$$\frac{d^2\Delta u}{d\emptyset^2} + \Delta\mathrm{u} = \frac{3(GM)^3}{c^2L^4}\left[\frac{2}{3}e^2\cos2\emptyset + 2\mathrm{e}\cos\emptyset - \mathrm{e}\,\emptyset\sin\emptyset + 1 + e^2\left(\frac{1}{2} - \frac{1}{6}\cos2\emptyset\right) + \mathrm{e}\,\emptyset\sin\emptyset\right]$$

$$\frac{d^2\Delta u}{d\emptyset^2} + \Delta\mathrm{u} = \frac{3(GM)^3}{c^2L^4}\left[1 + \frac{1}{2}e^2 + \frac{1}{2}e^2\cos2\emptyset + 2\mathrm{e}\cos\emptyset\right]$$

$$\frac{d^2\Delta u}{d\emptyset^2} + \Delta\mathrm{u} = \frac{3(GM)^3}{c^2L^4}\left[1 + \frac{1}{2}e^2(1 + \cos2\emptyset) + 2\mathrm{e}\cos\emptyset\right]$$

$$\frac{d^2\Delta u}{d\emptyset^2} + \Delta\mathrm{u} = \frac{3(GM)^3}{c^2L^4}\left[1 + \frac{1}{2}e^2(\sin^2\emptyset + \cos^2\emptyset + \cos^2\emptyset - \sin^2\emptyset) + 2\mathrm{e}\cos\emptyset\right]$$

$$\frac{d^2\Delta u}{d\emptyset^2} + \Delta\mathrm{u} = \frac{3(GM)^3}{c^2L^4}[1 + e^2\cos^2\emptyset + 2\mathrm{e}\cos\emptyset]$$

So equation (4) is correct.

Now fill Δu in equation (3):

$$u = \frac{GM}{L^2}(1 + e\cos\emptyset) + \Delta\mathrm{u} = \frac{GM}{L^2}(1 + e\cos\emptyset) + \frac{3(GM)^3}{c^2L^4}\left[1 + e^2\left(\frac{1}{2} - \frac{1}{6}\cos 2\,\emptyset\right) + \mathrm{e}\,\emptyset\sin\emptyset\right]$$

$$u = \frac{GM}{L^2}(1 + e\cos\emptyset) + \frac{3(GM)^3}{c^2L^4}\mathrm{e}\,\emptyset\sin\emptyset + \frac{3(GM)^3}{c^2L^4}\left[1 + e^2\left(\frac{1}{2} - \frac{1}{6}\cos 2\,\emptyset\right)\right]$$

Since the constant $\frac{3(GM)^3}{c^2L^4}$ is very small, the last three terms on the right-hand side are tiny, and of no use in testing the theory. However, the last term $e\frac{3(GM)^3}{c^2L^4}\emptyset\sin\emptyset$ might be tiny at first but will gradually grow with time, since the factor $\emptyset$ means that it is cumulative. We must therefore retain it.

$$u = \frac{GM}{L^2}\left[1 + e\left(\cos\emptyset + \frac{3(GM)^2}{c^2L^2}\emptyset\sin\emptyset\right)\right] + \frac{3(GM)^3}{c^2L^4}\left[1 + e^2\left(\frac{1}{2} - \frac{1}{6}\cos 2\,\emptyset\right)\right]$$

So our approximate solution reads:

$$u = \frac{GM}{L^2}\left[1 + e\left(\cos\emptyset + \frac{3(GM)^2}{c^2L^2}\emptyset\sin\emptyset\right)\right]$$

Using the relation

$$cos\left[\emptyset\left(1 - \frac{3(GM)^2}{c^2L^2}\right)\right] = cos\left(\emptyset - \frac{3(GM)^2}{c^2L^2}\emptyset\right) = \cos\emptyset\cos\frac{3(GM)^2}{c^2L^2}\emptyset + \sin\emptyset\sin\frac{3(GM)^2}{c^2L^2}\emptyset$$

$$\approx \cos\emptyset + \frac{3(GM)^2}{c^2L^2}\emptyset\sin\emptyset \qquad \text{for } \frac{3(GM)^2}{c^2L^2} \ll 1\,,$$

We can therefore write

$$u \approx \frac{GM}{L^2}\left\{1 + ecos\left[\emptyset\left(1 - \frac{3(GM)^2}{c^2L^2}\right)\right]\right\} = \frac{GM}{L^2}\{1 + ecos[\emptyset(1-\epsilon)]\}$$

r=1/u gives:

$$r = \frac{L^2}{GM\{1 + ecos[\emptyset(1-\epsilon)]\}} \tag{5}$$

Here is

$$\epsilon = \frac{3(GM)^2}{c^2L^2}$$

From this expression, we see the orbit is periodic, but with a period $2\pi/(1-\epsilon)$, i.e. the *r*-values repeat on a cycle that is larger than 2π. The result is that the orbit cannot ‘close’, and so the ellipse *precesses* (see figure below). In one revolution, the ellipse will rotate around the focus by an amount

$$\Delta\emptyset = \frac{2\pi}{1-\epsilon} - 2\pi = \frac{2\pi\epsilon}{1-\epsilon} \approx 2\pi\epsilon = \frac{6\pi(GM)^2}{c^2L^2}$$

Substituting for *L* from (2)

$$a = \frac{L^2}{GM(1-e^2)} \tag{2}$$

Substituting in (5)

$$r = \frac{L^2}{GM\{1 + ecos[\emptyset(1-\epsilon)]\}}$$

Trajectory of orbit

$$\boldsymbol{r} = \frac{\boldsymbol{a}(\boldsymbol{1}-\boldsymbol{e^2})}{\boldsymbol{1}+\boldsymbol{ecos}[\emptyset(\boldsymbol{1}-\epsilon)]}$$

With

$$\epsilon = \frac{3(GM)^2}{c^2L^2} \; or \; \epsilon = \frac{3(GM)^2}{c^2GMa(1-e^2)} = \frac{3GM}{c^2a(1-e^2)}$$

Derived from Kepler's third law:

$$T^2 = \frac{4\pi^2a^3}{G(M+m)} \approx \frac{4\pi^2a^3}{GM} => T = 2\pi a\sqrt{\frac{a}{GM}}$$

And

$$v = \frac{L}{r\cos\alpha} = \frac{\left(aGM(1-e^2)\right)^{1/2}}{\boldsymbol{a}(\boldsymbol{1}-\boldsymbol{e^2})}\frac{(\boldsymbol{1}+\boldsymbol{ecos}[\emptyset(\boldsymbol{1}-\epsilon)])}{\cos\alpha}$$

$$v = \left(\frac{GM}{a(1-e^2)}\right)^{1/2}\frac{(\boldsymbol{1}+\boldsymbol{ecos}[\emptyset(\boldsymbol{1}-\epsilon)])}{\cos\alpha}$$

$$\epsilon = \frac{3(GM)^2}{c^2L^2} = \frac{3(GM)^2}{c^2aGM(1-e^2)} = \frac{3GM}{c^2a(1-e^2)}$$

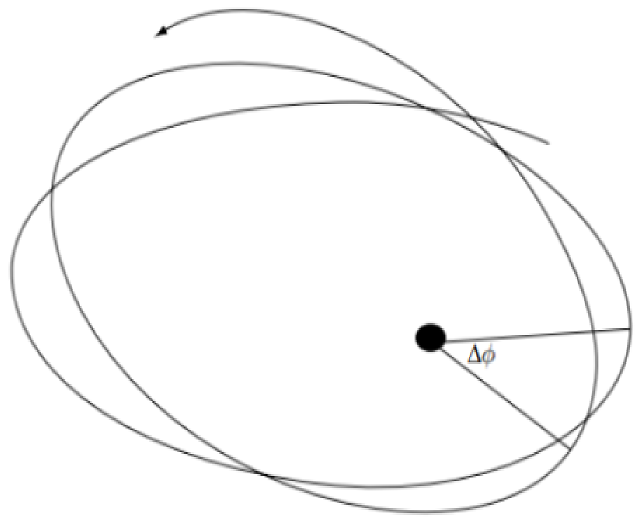
$$v = \left(\frac{GM}{a(1-e^2)}\right)^{1/2}\frac{(1+ecos[\emptyset(1-\epsilon)])}{\cos\alpha}$$

$$L^2 = aGM(1-e^2)$$

$$\Delta\emptyset = \frac{6\pi(GM)^2}{c^2aGM(1-e^2)}$$

We finally obtain

$$\boldsymbol{\Delta}\emptyset = \frac{\boldsymbol{6\pi GM}}{\boldsymbol{a}(\boldsymbol{1}-\boldsymbol{e^2})\boldsymbol{c^2}} \tag{6}$$



Precession of an elliptical orbit (greatly exaggerated)

Let us apply equation (3) to the orbit of Mercury, which has the following parameters: period=88 days, a=5.8x10^{10}m, e=0.2. Using M_s=2x10^{30}kg, we find

$$T = \sqrt{\frac{4\pi^2 a^3}{GM}} = 87.95\ days$$

$$\Delta\emptyset = \frac{6\pi GM}{a(1-e^2)c^2} = 5.02 * 10^{-7}\ \ per\ revolution$$

Thus per century

$$\Delta\emptyset = 5.02 * 10^{-7}\ * \left(100 * \frac{365.25}{88}\right) * \left(\frac{360 * 60 * 60}{2\pi}\right)$$

$$\Delta\emptyset = \mathbf{43''}\ \mathbf{per\ century}.$$

In fact, the measured precession is:

$$5599''.7 \pm 0''.4 \text{ per century},$$

But almost all of this is caused by perturbations from other planets. The residual, after taking perturbations into account, is in remarkable agreement with general relativity. The residuals for a number of planets (and Icarus, which is a large asteroid with a perihelion that lies within the orbit of Mercury) may also be calculated (in arcseconds per century):

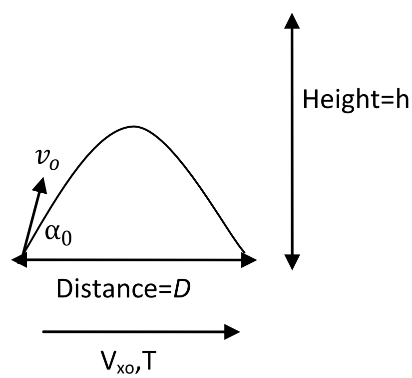
	Observed residual	Predicted residual
Mercury	43.1+/-0.5	43.03
Venus	8+/-5	8.6
Earth	5+/-1	3.8
Icarus	10+/-1	10.3

In each case, the results are in excellent agreement with the predictions of general relativity. Einstein included this calculation regarding Mercury in his 1915 paper on general relativity. He had solved one of the major problems of celestial mechanics in the very first application of his complicated theory to an empirically testable problem. As you can imagine, this gave him tremendous confidence in his new theory.

3.6. Experiment 6 - Calculation of trajectory of a bullet

As an exercise we are interested in calculating the trajectory of a bullet by means of the rules of General Relativity as opposed to the Newton approach.
For the General Relativity approach we assume that the trajectory of the bullet is forced by the mass of the Earth to follow an elliptic form. For the calculation we make use of the Schwarzschild equation. But first we start with the Newton approach.

3.6.1. Via Newton approach



The time for the bullet to cover the distance *D* with the initial horizontal velocity v_{xo} is:

$$v_{xo} = \frac{D}{T} \;=>\; T = \frac{D}{v_{xo}}$$

To cover the distance *D* the bullet needs to have also an upward velocity otherwise it hits the Earth to early. This requires an initial velocity component in the *y* direction v_{yo}. This velocity is determined by the horizontal distance *D* and the time *T*. So *T* is also the time that it takes from the ground upwards and the fall to the ground. The time upwards is the same as the time downwards.
To reach the highest point with respect to Earth takes T/2 seconds:

$$v_y = v_{yo} - gt$$

When at T/2 the highest point is reached then

$$v_y = 0$$

$$=> v_{yo} = gt = g\frac{T}{2} = g\frac{D}{2v_{xo}}$$

When the bullet falls from its highest point *h* it takes T/2 to reach the ground:

$$h - \frac{g}{2}\left(\frac{T}{2}\right)^2 = 0 \quad => \quad \frac{T}{2} = \sqrt{\frac{2h}{g}}$$

$$\frac{D}{2v_{xo}} = \sqrt{\frac{2h}{g}} \quad => \quad v_{xo} = D\sqrt{\frac{g}{8h}}$$

To reach the highest point:

$$v_{yo} = g\frac{T}{2} = g\sqrt{\frac{2h}{g}} = \sqrt{2hg}$$

$${v_o}^2 = {v_{xo}}^2 + v_{yo}^2 = \frac{gD^2}{8h} + 2hg = g\left(\frac{D^2 + 16h^2}{8h}\right)$$

$$v_o = \sqrt{g\left(\frac{D^2 + 16h^2}{8h}\right)}$$

$$v_{xo} = v_o \cos \alpha_0$$

$$\tan \alpha_0 = \frac{4\mathrm{h}}{\mathrm{D}}$$

Trajectory:

$$y(t) = v_{yo}\,\mathrm{t} - \frac{1}{2}\mathrm{g}t^2 = g\frac{T}{2}\mathrm{t} - \frac{1}{2}\mathrm{g}t^2 = \frac{1}{2}\mathrm{gt}(\mathrm{T} - \mathrm{t}) =$$

$$y(t) = \frac{1}{2}\mathrm{gt}\left(\frac{D}{v_{xo}} - \mathrm{t}\right)$$

Or:

$$y(x) = \frac{1}{2}\frac{\mathrm{g}}{{v_{xo}}^2}\mathrm{x}(\mathrm{D} - \mathrm{x})$$

So the trajectory *y(x)* is a function of the required distance *D* when the initial horizontal velocity component is v_{xo}.

Example:

Assume g=	9.87	9.87	9.87	9.87
Horizontal Distance (m)	10	10	100	100
Horizontal velocity (m/sec)	5	500	5	50
Time *T* (sec)	2	0.02	20	2
Height (m)	**4.93**	**4.93E-04**	**493**	**4.93**
Total velocity (m/sec)	**11.06**	**500**	**99**	**51**

3.6.2. Via Schwarzschild approach

For this approach we consider the bullet trajectory as a part of an ellipse with the earth centre as one of the foci. We use the results derived in the Schwarzschild equation in chapter Trajectories of massive particles-Second Derivation and Trajectory of orbit

The semi-major axis is:

$$a = \frac{L^2}{GM(1 - e^2)} \tag{2}$$

The parameter e measures the eccentricity of the orbit. The perihelion is $r_1 = a(1-e)$ and the aphelion is $r_2 = a(1+e)$

$$e = \sqrt{1-\frac{b^2}{c^2}} = \frac{r_2 - r_1}{r_2 + r_1}$$

So for a circle e=0 and $r = r_1 = r_2$=a.
To get an ellipse, as in the drawing below, where the earth centre coincides with the left focus of the ellipse, the equation looks like:

$$\boldsymbol{r}(\emptyset) = \frac{\boldsymbol{a}(\mathbf{1}-\boldsymbol{e}^{\mathbf{2}})}{\mathbf{1}-\boldsymbol{ecos}[\emptyset(\mathbf{1}-\epsilon)]} \qquad (2a)$$

Now we will derive the angle α between v, the velocity tangential to the ellipse, and the v_{per} , perpendicular with *r,* to find the angular momentum. So in this experiment v is the total velocity of the bullet along the ellipse, while v_{per} is the component of velocity v with respect to the surface of the earth and as mentioned perpendicular to $r(\emptyset)$.

$$\tan\alpha = \frac{dr}{rd\emptyset} = \frac{\{1-ecos[\emptyset(1-\epsilon)]\}}{a(1-e^2)}\frac{\{a(1-e^2)(1-\epsilon)(esin[\emptyset(1-\epsilon)])\}}{\{1-ecos[\emptyset(1-\epsilon)]\}^2}$$

$$\tan\alpha = \frac{dr}{rd\emptyset} = \frac{e(1-\epsilon)sin[\emptyset(1-\epsilon)]}{1-ecos[\emptyset(1-\epsilon)]}$$

$$\boldsymbol{\alpha} = \boldsymbol{arctan}\left\{\frac{\boldsymbol{e}(\mathbf{1}-\boldsymbol{\epsilon})\boldsymbol{sin}[\emptyset(\mathbf{1}-\boldsymbol{\epsilon})]}{\mathbf{1}-\boldsymbol{ecos}[\emptyset(\mathbf{1}-\boldsymbol{\epsilon})]}\right\}$$

As

$$\cos\alpha = \frac{1}{\sqrt{1+\tan\alpha}}$$

Gives

$$\cos\alpha = \left[1+\left(\frac{e(1-\epsilon)sin[\emptyset(1-\epsilon)]}{1-ecos[\emptyset(1-\epsilon)]}\right)^2\right]^{-\frac{1}{2}}$$

$$\cos\alpha = \left[\frac{1-2ecos[\emptyset(1-\epsilon)]+e^2cos^2[\emptyset(1-\epsilon)]+\{e(1-\epsilon)sin[\emptyset(1-\epsilon)]\}^2}{\{1-ecos[\emptyset(1-\epsilon)]\}^2}\right]^{-\frac{1}{2}}$$

Because of the negative square root sign we flip the equation

$$\cos\alpha = \frac{1-ecos[\emptyset(1-\epsilon)]}{\left[1-2ecos[\emptyset(1-\epsilon)]+e^2cos^2[\emptyset(1-\epsilon)]+(1-2\epsilon+\epsilon^2)e^2sin^2[\emptyset(1-\epsilon)]\right]^{1/2}}$$

$$\cos\alpha = \frac{1-ecos[\emptyset(1-\epsilon)]}{\left[1-2ecos[\emptyset(1-\epsilon)]+e^2cos^2[\emptyset(1-\epsilon)]+e^2sin^2[\emptyset(1-\epsilon)]-\epsilon(2-\epsilon)e^2sin^2[\emptyset(1-\epsilon)]\right]^{\frac{1}{2}}}$$

$$\boldsymbol{\cos\alpha} = \frac{\mathbf{1}-\boldsymbol{ecos}[\emptyset(\mathbf{1}-\boldsymbol{\epsilon})]}{[\mathbf{1}-\mathbf{2}\boldsymbol{ecos}[\emptyset(\mathbf{1}-\boldsymbol{\epsilon})]+\boldsymbol{e}^{\mathbf{2}}(\mathbf{1}-\boldsymbol{\epsilon}(\mathbf{2}-\boldsymbol{\epsilon})\boldsymbol{sin}^{\mathbf{2}}[\emptyset(\mathbf{1}-\boldsymbol{\epsilon})])]^{\mathbf{1/2}}} \qquad (2b)$$

The momentum L is constant over the whole ellipse. The momentum is the velocity perpendicular to r multiplied with r (assuming a unity mass):

$$L = v_{per} \cdot r = v \cdot \cos\alpha \cdot r$$

So here is:

$$\boldsymbol{L = v_{x0} \cdot R_{earth}}$$

According to (2):

$$L = \sqrt{aGM(1-e^2)}$$

$$v = \frac{L}{r\cos\alpha} = \frac{\left(aGM(1-e^2)\right)^{1/2}}{a(1-e^2)\cos\alpha}(1 - ecos[\emptyset(1-\epsilon)])$$

$$v = \left(\frac{GM}{a(1-e^2)}\right)^{\frac{1}{2}} \frac{(1 - ecos[\emptyset(1-\epsilon)])}{\cos\alpha} \qquad (2c)$$

Fill $\cos\alpha$ of (2b) into (2c):

$$v = \left(\frac{GM}{a(1-e^2)}\right)^{\frac{1}{2}} \frac{(1 - ecos[\emptyset(1-\epsilon)])}{1 - ecos[\emptyset(1-\epsilon)]}[1 - 2ecos[\emptyset(1-\epsilon)] + e^2(1 - \epsilon(2-\epsilon)sin^2[\emptyset(1-\epsilon)])]^{1/2}$$

Instantaneous velocity as function of $\emptyset$:

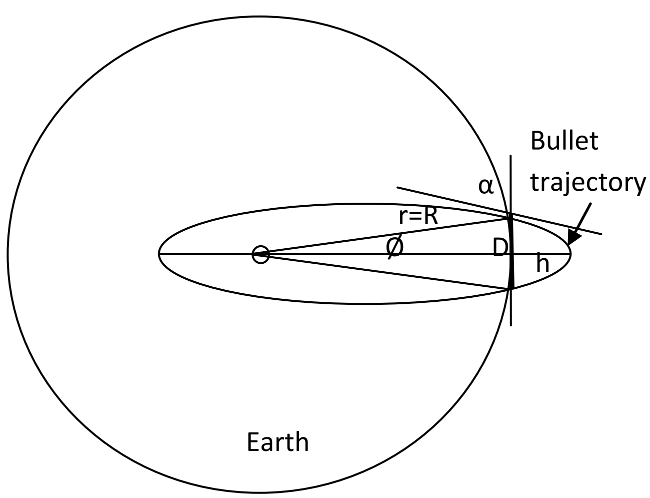
$$\boldsymbol{v = \left(\frac{GM}{a(1-e^2)}\left(1 - 2ecos[\emptyset(1-\epsilon)] + e^2\{1 - \epsilon(2-\epsilon)sin^2[\emptyset(1-\epsilon)]\}\right)\right)^{\frac{1}{2}}} \qquad (\boldsymbol{2d})$$

From previous chapter ϵ

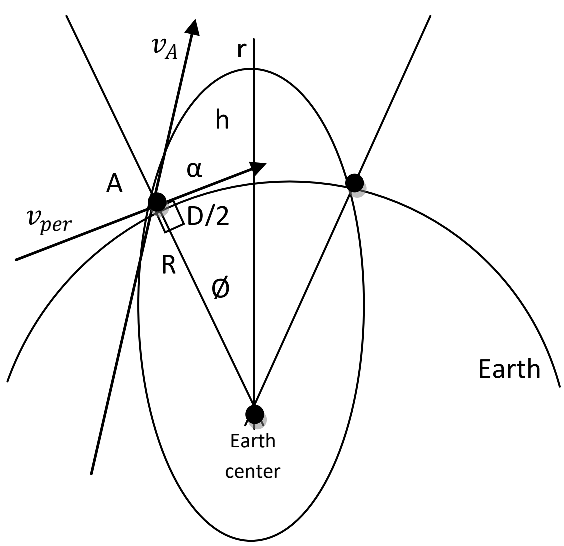
$$\epsilon = \frac{3(GM)^2}{c^2L^2} = \frac{3(GM)^2}{c^2aGM(1-e^2)} = \frac{3GM}{c^2a(1-e^2)}$$

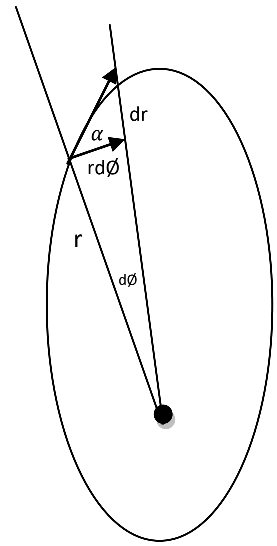
Here is:

$$\boldsymbol{\epsilon} = \frac{3(GM)^2}{c^2L^2} = \frac{3(GM)^2}{c^2(v_{x0}R_{earth})^2} = \boldsymbol{\frac{3c^2}{{v_{x0}}^2}\left(\frac{GM}{c^2R_{earth}}\right)^2} \text{ which is dimensionless} \quad (2e)$$



To zoom a bit in:





$$\emptyset R = \frac{D}{2} => \ \emptyset = \frac{D}{2R}$$

$$v_{per} \ = v_{x\mathrm{A}} = v\cos(\alpha) \ and \ v_{y\mathrm{A}} = v\sin(\alpha)$$

From (2a)

$$a(1-e^2) = r\{1 - ecos[\emptyset(1-\epsilon)]\}$$

From (2d)

$$v = \left(\frac{GM}{a(1-e^2)}(1 - 2ecos[\emptyset(1-\epsilon)] + e^2\{1 - \epsilon(2-\epsilon)sin^2[\emptyset(1-\epsilon)]\})\right)^{\frac{1}{2}}$$

$$v^2 = \frac{GM}{a(1-e^2)}(1 - 2ecos[\emptyset(1-\epsilon)] + e^2\{1 - \epsilon(2-\epsilon)sin^2[\emptyset(1-\epsilon)]\})$$

$$v^2 = GM\frac{(1 - 2ecos[\emptyset(1-\epsilon)] + e^2\{1 - \epsilon(2-\epsilon)sin^2[\emptyset(1-\epsilon)]\})}{r\{1 - ecos[\emptyset(1-\epsilon)]\}}$$

$$\frac{v^2r}{GM}\{1 - ecos[\emptyset(1-\epsilon)]\} = 1 - 2ecos[\emptyset(1-\epsilon)] + e^2\{1 - \epsilon(2-\epsilon)sin^2[\emptyset(1-\epsilon)]\}$$

$$\boldsymbol{e^2}\{1 - \epsilon(2-\epsilon)sin^2[\emptyset(1-\epsilon)]\} - \boldsymbol{e}cos[\emptyset(1-\epsilon)]\left(2 - \frac{v^2r}{GM}\right) + \left(1 - \frac{v^2r}{GM}\right) = 0$$

$$e = \frac{cos[\emptyset(1-\epsilon)]\left(2 - \frac{v^2r}{GM}\right) \pm \sqrt{\left[cos[\emptyset(1-\epsilon)]\left(2 - \frac{v^2r}{GM}\right)\right]^2 - 4\left(1 - \frac{v^2r}{GM}\right)\{1 - \epsilon(2-\epsilon)sin^2[\emptyset(1-\epsilon)]\}}}{2\{1 - \epsilon(2-\epsilon)sin^2[\emptyset(1-\epsilon)]\}}$$

For the starting point at the intersection of Earth and trajectory goes that *r=R*. (*R* is here the radius of the Earth) and $\emptyset = \frac{D}{2R}$

From (2a):

$$r = \frac{a(1-e^2)}{1 - ecos[\emptyset(1-\epsilon)]}$$

$$a(1-e^2) = R\left\{1 - e\cos\left[\frac{D}{2R}(1-\epsilon)\right]\right\}$$

$$a = \frac{R\left\{1 - e\cos\left[\frac{D}{2R}(1-\epsilon)\right]\right\}}{(1-e^2)} \tag{3}$$

$$\boldsymbol{e} = \frac{\boldsymbol{cos}\left[\frac{\boldsymbol{D}}{\boldsymbol{2R}}(\boldsymbol{1}-\boldsymbol{\epsilon})\right]\left(\boldsymbol{2}-\frac{\boldsymbol{v^2R}}{\boldsymbol{GM}}\right) \pm \sqrt{\left[\boldsymbol{cos}\left[\frac{\boldsymbol{D}}{\boldsymbol{2R}}(\boldsymbol{1}-\boldsymbol{\epsilon})\right]\left(\boldsymbol{2}-\frac{\boldsymbol{v^2R}}{\boldsymbol{GM}}\right)\right]^2 - \boldsymbol{4}\left(\boldsymbol{1}-\frac{\boldsymbol{v^2R}}{\boldsymbol{GM}}\right)\{\boldsymbol{1}-\boldsymbol{\epsilon}(\boldsymbol{2}-\boldsymbol{\epsilon})\boldsymbol{sin^2}[\emptyset(\boldsymbol{1}-\boldsymbol{\epsilon})]\}}}{\boldsymbol{2}\{\boldsymbol{1}-\boldsymbol{\epsilon}(\boldsymbol{2}-\boldsymbol{\epsilon})\boldsymbol{sin^2}[\emptyset(\boldsymbol{1}-\boldsymbol{\epsilon})]\}}$$

Or here from the equations (2), (2e) and (3):

$$R\left\{1 - e\cos\left[\frac{D}{2R}(1-\epsilon)\right]\right\} = \mathrm{a}(1-e^2) = \frac{L^2}{GM}$$

$$\mathrm{e} = \frac{1-\frac{L^2}{RGM}}{\cos\left[\frac{D}{2R}(1-\epsilon)\right]} = \frac{1-\frac{L^2}{RGM}}{\cos\left[\frac{D}{2R}\left(1-\frac{3c^2}{{v_{x0}}^2}\left(\frac{GM}{c^2R_{eart\,h}}\right)^2\right)\right]} = \frac{1-\frac{(v_{x0}R)^2}{RGM}}{\cos\left[\frac{D}{2R}\left(1-\frac{3c^2}{{v_{x0}}^2}\left(\frac{GM}{c^2R}\right)^2\right)\right]}$$

$$\mathrm{e} = \frac{1-\frac{{v_{x0}}^2R}{GM}}{\cos\left[\frac{D}{2R}\left(1-\frac{3c^2}{{v_{x0}}^2}\left(\frac{GM}{c^2R}\right)^2\right)\right]}$$

The given velocity at the *r=R* point is v. Thus for a given velocity there are two solutions for *e*.

Here is *h* the highest point of the bullet trajectory

$$h = a(1+e) - R$$

Together with (3):

$$h = \frac{R\left\{1 - e\cos\left[\frac{D}{2R}(1-\epsilon)\right]\right\}}{(1-e^2)}(1+e) - R = R\left\{\frac{1 - e\cos\left[\frac{D}{2R}(1-\epsilon)\right]}{1-e} - 1\right\}$$

$$h = R\left\{\frac{1 - e\cos\left[\frac{D}{2R}(1-\epsilon)\right] - 1 + e}{1-e}\right\} = R\frac{e\left(1-\cos\left[\frac{D}{2R}(1-\epsilon)\right]\right)}{1-e}$$

Here D is the horizontal distance of the bullet on Earth, v is the starting velocity of the bullet and *R* is the Earth radius. As seen above $\emptyset = \frac{D}{2R}$

Or more pragmatic here in our bullet example with v_{x0} and D as starting points:

$$h = a(1+e) - R = \frac{a(1-e^2)}{(1-e)} - R = \frac{L^2}{GM(1-e)} - \mathrm{R} = \frac{(v_{x0}R)^2}{GM(1-e)} - \mathrm{R}$$

$$h = \frac{(v_{x0}R)^2}{GM(1-e)} - \mathrm{R} = \frac{(v_{x0}R)^2}{GM(1-e)} - \mathrm{R}$$

Where:

$$e = \frac{1 - \frac{{v_{x0}}^2 R}{GM}}{cos\left[\frac{D}{2R}\left(1 - \frac{3c^2}{{v_{x0}}^2}\left(\frac{GM}{c^2 R}\right)^2\right)\right]}$$

Derivation of the circumference of an ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$x = acos\beta \text{ and } y = bsin\beta$$

$$Circumference = 4a\int_0^{\pi/2} \sqrt{\left(\frac{dx}{d\beta}\right)^2 + \left(\frac{dy}{d\beta}\right)^2}\, d\beta$$

$$= 4a\int_0^{\pi/2} \sqrt{a^2 sin^2\beta + b^2 cos^2\beta}\, d\beta$$

$$= 4a\int_0^{\pi/2} \sqrt{a^2(1 - cos^2\beta) + b^2 cos^2\beta}\, d\beta$$

$$= 4a\int_0^{\pi/2} \sqrt{a^2 - (a^2 - b^2)cos^2\beta}\, d\beta$$

$$Circumference = 4a\int_0^{\pi/2} \sqrt{1 - e^2 cos^2\beta}\, d\beta$$

For the **circumference of an ellipse** there is no simple closed solution.
There are approximations, for instance the Ramanujan approximation:

$$Circumference \approx \pi a\left[3\left(1 + \sqrt{1 - e^2}\right) - \sqrt{10\sqrt{1 - e^2} + 3(2 - e^2)}\right]$$

To summarize the used formulas:

The departure points for this derivation are the velocity of the bullet along the earth surface ($v_{x0} = v_{per}\ perpendicular\ to\ r$) and the required distance *D*. So at the starting point where the bullet was launched we know the position and the momentum of the bullet and should be able to calculate its trajectory.

$$L = v_{x0} \cdot R_{earth} \qquad so\ \epsilon\ is\ L\ function\ of\ L(v_{x0})$$

$$\epsilon = \frac{3(GM)^2}{c^2 L^2} \qquad \text{so } \epsilon(v_{x0})$$

$$\emptyset = \frac{D}{2R} \qquad \text{so } \emptyset(D)$$

$$e = \frac{1 - \frac{L^2}{RGM}}{cos[\emptyset(1 - \epsilon)]} \qquad so\ e(v_{x0}, D)$$

$$\alpha = arctan\left\{\frac{e(1 - \epsilon)sin[\emptyset(1 - \epsilon)]}{1 - ecos[\emptyset(1 - \epsilon)]}\right\} \qquad so\ e(v_{x0}, D)$$

$$a = \frac{L^2}{GM(1-e^2)} \qquad so\ a(v_{x0}, D)$$
$$h = a(1+e) - R \qquad so\ h(v_{x0}, D)$$

With these formulas we get our results as shown in the Excel table below

Detailed results of calculations on the example mentioned above.
The starting points are the (perpendicular to *r*) velocity of the bullet and the distance to be covered.

	Newton				Schwarschild			
Vper0(m/s)	**5**	**500**	**500**	**1000**	**5**	**500**	**500**	**1000**
Distance(m)	**10**	**10**	**2000**	**2000**	**10**	**10**	**2000**	**2000**
Vr0(m/s)	9.87	0.10	19.73	9.87	9.76	0.10	19.66	9.71
velocity(m/s)	11	500	500	1000	11	500	500	1000
epsilon					5.25E-03	5E-07	5.25E-07	1E-07
e(centricity)					1.000	0.996	0.996	0.984
a(m)					3.18E+06	3.18E+06	3.18E+06	3.20E+06
h(m)	4.93	4.93E-04	19.73	4.93	4.88	4.91E-04	19.66	4.85
alpha(rad)	1.10	0.000	0.04	0.010	1.10	0.000	0.04	0.010
alpha(deg)	63.13	0.0113	2.26	0.565	62.88	0.0113	2.25	0.556
Phi(rad)					7.87E-07	7.87E-07	1.57E-04	1.57E-04
L (ang. mom.)	3.18E+07	3.18E+09	3.18E+09	6.36E+09	3.18E+07	3.18E+09	3.18E+09	6.36E+09
cos(alpha)	0.4520	1.0000	0.9992	1.0000	0.4558	1.0000	0.9992	1.0000
cos(alpha+phi)					0.4558	1.000	0.9992	1.000
Circ.(km)					12662	12894	12894	13346

4. Coordinate systems

4.1. Rectangular coordinate system

In order to distinguish between points in space a coordinate system is created. The main characteristics of a coordinate system are an origin and the coordinate axis. The origin may be chosen what is most practical and for the axis mostly a Cartesian system is chosen because of its simplicity.

In a Cartesian coordinate system:

- The axes are perpendicular to one another.
- The axes are independent of each other. i.e. changing the size of one coordinate does not influence the others.
- The axes have a direction and size and therefore they could be considered as vectors.

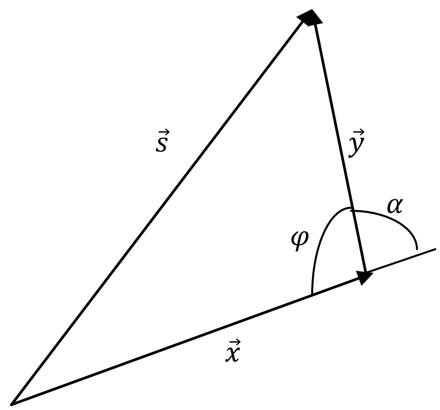
A point in space is represented by its coordinates, for instance $A(x_a, y_a)$. The x_a can be found by drawing a line, parallel to the y-ax; where that line intersects with the x-ax that point is x_a The same for the y_a.
The distance of point A to the origin can be found by Pythagoras. $(A\text{-origin})^2 = x_a^2 + y_a^2$.

In case one works with a line segment between A en B then the size is: $(A-B)^2=(x_a-x_b)^2+(y_a-y_b)^2$. The advantage here is that the length of the line segment is independent of the arbitrary chosen origin; i.e. the sizes of x_a, y_a, x_b, y_b do change but the difference A-B, which is the size of the line segment, does not change.

4.2. Non-rectangular coordinate system

Because of practical reasons also a coordinate system can be chosen of which the axis are not orthogonal. Now again we have to be aware that the segment *s* is build up out of vectors:

$$\vec{s} = \vec{x} + \vec{y}$$



The size *s* of $\vec{s}$ can be found by the in-product of $\vec{s}$ with itself:

$$\vec{s}.\vec{s} = (\vec{x}+\vec{y}).(\vec{x}+\vec{y}) = \vec{x}.\vec{x}+\vec{x}.\vec{y}+\vec{y}.\vec{x}+\vec{y}.\vec{y}$$
$$s^2 = x^2 + (2\cos\alpha)xy + y^2$$
$$\cos\alpha = \cos(180^0-\varphi) = -\cos\varphi$$
$$s^2 = x^2 + y^2 - 2xy\cos\varphi$$

This is the well known law of cosines.
So apart from the squares of the coordinates also the product of the coordinates are part of the equation.

4.3. Curved coordinates

Instead of coordinate axes that are not orthogonal it could also be practical to have curved coordinates. To work with these is obviously more complicated but Einstein had the following approach:

A curved line could be considered as a line build up out of infinitesimal straight lines. Looking at an infinitesimal area these curved coordinates could be considered as a local coordinate system with straight (linear) coordinates; but not necessarily rectangular.
Because the coordinate system here concerns infinitesimal coordinates, the coordinates are denoted as *dx*, *dy* etcetera. Furthermore these coordinates have coefficients and these coefficients contain information about the curvature of the coordinate lines. So the coefficients, in case of curvature, are not constants anymore but parameters depending on their location along the coordinate lines.

It is said that the gravity bend the coordinate lines and so deforms the space-time and creates gravitational force and thus acceleration. However by choosing a curved coordinate system in such a way that it moves and curves according to the direction of the gravity field, no force or gravity is experienced; in the same way as a moving coordinate system was chosen, in case of special relativity, to nullify the speed of the moving object.

4.4. General form for a coordinate system

Let us derive an equation for the relation between a line segment and its curved coordinate system. As mentioned before an infinitesimal line segment d$\vec{s}$ is a vector and the size can be calculated as shown above:

$$\overrightarrow{ds}.\overrightarrow{ds} = \left(\overrightarrow{dx} + \overrightarrow{dy}\right).\left(d\vec{x} + \overrightarrow{dy}\right)$$
$$= \overrightarrow{dx}.\overrightarrow{dx} + \overrightarrow{dx}.\overrightarrow{dy} + \overrightarrow{dy}.\overrightarrow{dx} + \overrightarrow{dy}.\overrightarrow{dy}\ (for\ a\ linear, non - orthogonal\ system)$$

In order to have a more general form (not necessarily orthogonal) it is assumed that each term has a coefficient $g_{\mu\nu}$:

$$ds^2 = g_{xx}\,dxdx + g_{xy}\,dxdy + g_{yx}\,dydx + g_{yy}\,dydy$$

Here, in the example of the cosine rule above, $g_{xx} = g_{yy} = 1 and\ \ g_{xy} = g_{yx} = -\cos\varphi$

The $g_{\mu\nu}$ is called the **metric tensor** and could be regarded, in this two dimensional coordinate system, as a matrix of 2x2 elements.

$$\begin{pmatrix} 1 & -\cos\varphi \\ -\cos\varphi & 1 \end{pmatrix}$$

For a general form:

$$ds^2 = \sum_{\mu\nu} g_{\mu\nu}\ dx^\mu\,dx^\nu$$

In Einstein notation:

$$\boldsymbol{ds^2 = g_{\mu\nu}dx^\mu dx^\nu}$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} \end{pmatrix}$$

For a space-time four dimensional coordinate system $\mu\ and\ \nu$ can be 0,1,2,3 or ct, x, y, z. So this formula shows the product of each coordinate and the cross products between each coordinate pair. In case the coordinate system is orthogonal then $\mu = \nu$. As said before, this local coordinate system consists of straight, linear, lines but the information about the curvature may not be lost and will be part of the $g_{\mu\nu}$ elements.

In case a different coordinate system is used then it still describes the same line segment. In that case the relation between the two coordinate systems is shown in:

$$\boldsymbol{ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu = g_{\alpha\beta}(y)dy^\alpha dy^\beta}$$

4.5. Transformation between two coordinate systems

As is mentioned before, that in case of a curved coordinate system "locally", in an infinitesimal area, a coordinate system with straight lines can be used. For a four dimensional coordinate system then each new coordinate, in the new *x*-frame, has a linear relation with all old coordinates, in the old *y*-frame, according to the equation:

$$dx^0 = \frac{\partial x^0}{\partial y^0} dy^0 + \frac{\partial x^0}{\partial y^1} dy^1 + \frac{\partial x^0}{\partial y^2} dy^2 + \frac{\partial x^0}{\partial y^3} dy^3$$

The same goes for the three other coordinates and leads to the general formula:

$$\boldsymbol{dx^m = \frac{\partial x^m}{\partial y^r} dy^r}$$

The summation is done over the repeated index *r*.

4.5.1. Extended elucidation of the metric tensor

We start from a Cartesian coordinate frame, in this case comparable with the Minkowski equation (see appendix 7.1 equation $11a$) in special relativity.

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$$

Now we call:

$$cdt = dx^0, dx = dx^1, dy = dx^2, dz = dx^3$$

$$(all\ have\ the\ dimension\ of\ meter)$$

The coordinates are indicated by indices.
In a more general form:

$$ds^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 = \eta_{\mu\nu}\, dx^\mu dx^\nu$$

The metric tensor here is:

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

So

$$ds^2 = \eta_{\mu\nu}\, dx^\mu dx^\nu$$

Now we go to an arbitrary coordinate system dy^α and dy^β $=> (dy^0, dy^1, dy^2, dy^3)$:

$$dx^\mu = \frac{\partial x^\mu}{\partial y^0} dy^0 + \frac{\partial x^\mu}{\partial y^1} dy^1 + \frac{\partial x^\mu}{\partial y^2} dy^2 + \frac{\partial x^\mu}{\partial y^3} dy^3 = \frac{\partial x^\mu}{\partial y^\alpha} dy^\alpha$$

And

$$dx^\nu = \frac{\partial x^\nu}{\partial y^\beta} dy^\beta$$

According to the chain rule as dx^μ (and dx^ν) is a function of dy^0, dy^1, dy^2, dy^3 :

$$ds^2 = \eta_{\mu\nu}\, dx^\mu dx^\nu$$

$$=> \; ds^2 = \boldsymbol{\eta_{\mu\nu}} \frac{\boldsymbol{\partial x^\mu}}{\boldsymbol{\partial y^\alpha}} \frac{\boldsymbol{\partial x^\nu}}{\boldsymbol{\partial y^\beta}} dy^\alpha dy^\beta$$

Here is:

$$g_{\alpha\beta} = \eta_{\mu\nu} \frac{\partial x^\mu}{\partial y^\alpha} \frac{\partial x^\nu}{\partial y^\beta}$$

So:

$$=> \; ds^2 = g_{\alpha\beta} \, dy^\alpha dy^\beta$$

Next we go to another arbitrary coordinate system and follow the same approach:

$$ds^2 = g_{\alpha\beta} \frac{\partial y^\alpha}{\partial x^\mu} \frac{\partial y^\beta}{\partial x^\nu} dx^\mu dx^\nu = g_{\mu\nu} \, dx^\mu dx^\nu$$

This results in a general transformation form between arbitrary coordinates systems or between the metric tensors:

$$\boldsymbol{g_{\mu\nu}(x) = \frac{\partial y^\alpha}{\partial x^\mu} \frac{\partial y^\beta}{\partial x^\nu} g_{\alpha\beta}}(\mathbf{y})$$

4.6. Transformation between Cartesian and polar (infinitesimal) coordinates

As an example we will now perform the transformation between Cartesian and polar coordinates.

It is assumed that the reader knows the following relation between polar and Cartesian coordinates (see figure below):

$$x = r \sin\theta \cos\varphi \qquad y = r \sin\theta \sin\varphi \qquad z = r \cos\theta$$

Derivation of the *dx*, *dy* en *dz*:

$$\overrightarrow{dx} = \sin\theta \cos\varphi \, \overrightarrow{dr} + r\cos\theta \cos\varphi \, \overrightarrow{d\theta} - r \sin\theta \sin\varphi \, \overrightarrow{d\varphi}$$
$$\overrightarrow{dy} = \sin\theta \sin\varphi \, \overrightarrow{dr} + r\cos\theta \sin\varphi \, \overrightarrow{d\theta} + r \sin\theta \cos\varphi \, \overrightarrow{d\varphi}$$
$$\overrightarrow{dz} = \cos\theta \, \overrightarrow{dr} - r \sin\theta \, \overrightarrow{d\theta}$$

To determine the size of *dx, dy* and *dz* we take the dot product of each.

$$dx^2 = \overrightarrow{dx} \cdot \overrightarrow{dx}; \quad dy^2 = \overrightarrow{dy} \cdot \overrightarrow{dy}; \qquad dz^2 = \overrightarrow{dz} \cdot \overrightarrow{dz}$$

Because the coordinates $r, \theta \; and \; \varphi$ are perpendicular the cross terms are zero, thus remaining:

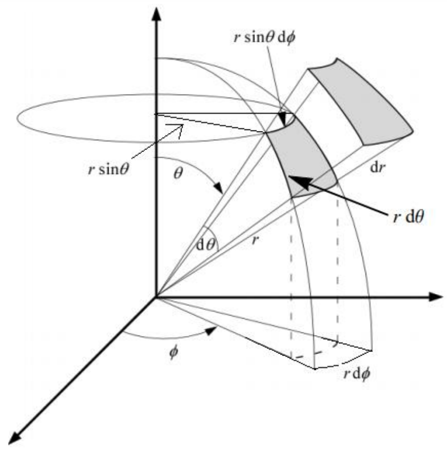
$$dx^2 = \sin^2\theta \cos^2\varphi \, dr^2 + r^2 \cos^2\theta \cos^2\varphi \, d\theta^2 + r^2 \sin^2\theta \sin^2\varphi \, d\varphi^2$$
$$dy^2 = \sin^2\theta \sin^2\varphi \, dr^2 + r^2 \cos^2\theta \sin^2\varphi \, d\theta^2 + r^2 \sin^2\theta \cos^2\varphi \, d\varphi^2$$
$$dz^2 = \cos^2\theta \, dr^2 + r^2 \sin^2\theta \, d\theta^2$$

Now summation of $dx^2 + dy^2 + dz^2$:

$$dx^2 + dy^2 + dz^2 = \sin^2\theta \cos^2\varphi \, dr^2 + \sin^2\theta \sin^2\varphi \, dr^2 + \cos^2\theta \, dr^2 + r^2 \cos^2\theta \cos^2\varphi \, d\theta^2 + r^2 \cos^2\theta \sin^2\varphi \, d\theta^2 + r^2 \sin^2\theta \, d\theta^2 + r^2 \sin^2\theta \sin^2\varphi \, d\varphi^2 + r^2 \sin^2\theta \cos^2\varphi \, d\varphi^2$$

$$=> \boldsymbol{dx^2 + dy^2 + dz^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\varphi^2}$$

This depicts the transformation from a frame with Cartesian coordinates to a frame with polar coordinates.



Volume element *dxdydz*:

$$\boldsymbol{dV = dxdydz} = dr.r\,d\theta.\mathrm{r}\sin\theta\,d\varphi = \boldsymbol{r^2\sin\theta\,drd\theta d\varphi}$$

$$V = \iiint r^2\sin\theta\,drd\theta d\varphi = \int_0^R r^2 dr\int_0^\pi \sin\theta d\theta\int_0^{2\pi} d\varphi$$

$$V = \frac{1}{3}r^3|_0^R.(-\cos\theta)\,|_0^\pi.\varphi|_0^{2\pi} = \frac{1}{3}R^3.2.2\pi = \frac{4}{3}\pi R^3$$

4.7. Exercise to formally apply the metric transformation formula

$$The\ relation\ between\ the\ \text{"new"}\ and\ \ \text{"old"}\ coordinates: dx^m = \frac{\partial x^m}{\partial y^r}dy^r$$

$$The\ relation\ between\ a\ line\ segment\ and\ its\ (Cartesian) coordinates: ds^2 = \eta_{mn}\,d\xi^m d\xi^n$$

$$The\ relation\ between\ two\ different\ coordinate\ systems: ds^2 = g_{mn}(x)dx^m dx^n = g_{pq}(y)dy^p dy^q$$

$$The\ relation\ between\ the\ \text{"new"}\ metric\ tensor\ and\ the\ \text{"old"}: \boldsymbol{g_{pq}(y) = g_{mn}(x)\frac{dx^m}{dy^p}\frac{dx^n}{dy^q}}$$

For the exercise we consider again the transformation between a Cartesian- and a polar system.

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$$

$$ds^2 = c^2dt^2 - dr^2 - r^2d\theta^2 - r^2\sin^2\theta\,d\varphi^2$$

The old metric tensor has $g_{00} = 1$, $g_{11} = -1$, $g_{22} = -1$, $g_{33} = -1$ as, Cartesian, elements and the rest is zero.

Now we have to find, via the formula, the new polar metric tensor elements,

$$g_{00} = 1,\ g_{11} = -1,\ g_{22} = -r^2\ \ ,g_{33} = -r^2\sin^2\theta$$

As mentioned in a previous chapter, the relationship between polar and Cartesian coordinates is:

$$x = r\sin\theta\cos\varphi \qquad y = r\sin\theta\sin\varphi \qquad z = r\cos\theta$$

In general:

$$dx^m = \frac{\partial x^m}{\partial y^r} dy^r$$

Worked out for this example:

$$dt = \frac{\partial t}{\partial t} dt + \frac{\partial t}{\partial r} dr + \frac{\partial t}{\partial \theta} d\theta + \frac{\partial t}{\partial \varphi} d\varphi$$
$$dx = \frac{\partial x}{\partial t} dt + \frac{\partial x}{\partial r} dr + \frac{\partial x}{\partial \theta} d\theta + \frac{\partial x}{\partial \varphi} d\varphi$$
$$dy = \frac{\partial y}{\partial t} dt + \frac{\partial y}{\partial r} dr + \frac{\partial y}{\partial \theta} d\theta + \frac{\partial y}{\partial \varphi} d\varphi$$
$$dz = \frac{\partial z}{\partial t} dt + \frac{\partial z}{\partial r} dr + \frac{\partial z}{\partial \theta} d\theta + \frac{\partial z}{\partial \varphi} d\varphi$$

Derivation of the *dx*, *dy* en *dz*:

$$dt = dt$$
$$dx = \sin\theta\cos\varphi\, dr + r\cos\theta\cos\varphi\, d\theta - r\sin\theta\sin\varphi\, d\varphi$$
$$dy = \sin\theta\sin\varphi\, dr + r\cos\theta\sin\varphi\, d\theta + r\sin\theta\cos\varphi\, d\varphi$$
$$dz = \cos\theta\, dr - r\sin\theta\, d\theta$$

Thus the metric tensor elements are:

$$\frac{\partial t}{\partial t} = 1 \qquad \frac{\partial t}{\partial r} = 0 \qquad \frac{\partial t}{\partial \theta} = 0 \qquad \frac{\partial t}{\partial \varphi} = 0$$
$$\frac{\partial x}{\partial t} = 0 \qquad \frac{\partial x}{\partial r} = +\sin\theta\cos\varphi \qquad \frac{\partial x}{\partial \theta} = +r\cos\theta\cos\varphi \qquad \frac{\partial x}{\partial \varphi} = -r\sin\theta\sin\varphi$$
$$\frac{\partial y}{\partial t} = 0 \qquad \frac{\partial y}{\partial r} = +\sin\theta\sin\varphi \qquad \frac{\partial y}{\partial \theta} = +r\cos\theta\sin\varphi \qquad \frac{\partial y}{\partial \varphi} = +r\sin\theta\cos\varphi$$
$$\frac{\partial z}{\partial t} = 0 \qquad \frac{\partial z}{\partial r} = +\cos\theta \qquad \frac{\partial z}{\partial \theta} = -r\sin\theta \qquad \frac{\partial z}{\partial \varphi} = 0$$

Now we apply:

$$g_{pq}(y) = g_{mn}(x)\frac{dx^m}{dy^p}\frac{dx^n}{dy^q}$$

Worked out for the metric tensor element:

$$g_{00}(y) = g_{00}(x)\frac{dx^0}{dy^0}\frac{dx^0}{dy^0} + g_{01}(x)\frac{dx^0}{dy^0}\frac{dx^1}{dy^0} + g_{02}(x)\frac{dx^0}{dy^0}\frac{dx^2}{dy^0} + g_{03}(x)\frac{dx^0}{dy^0}\frac{dx^3}{dy^0} +$$
$$g_{10}(x)\frac{dx^1}{dy^0}\frac{dx^0}{dy^0} + g_{11}(x)\frac{dx^1}{dy^0}\frac{dx^1}{dy^0} + g_{12}(x)\frac{dx^1}{dy^0}\frac{dx^2}{dy^0} + g_{13}(x)\frac{dx^1}{dy^0}\frac{dx^3}{dy^0} +$$
$$g_{20}(x)\frac{dx^2}{dy^0}\frac{dx^0}{dy^0} + g_{21}(x)\frac{dx^2}{dy^0}\frac{dx^1}{dy^0} + g_{22}(x)\frac{dx^2}{dy^0}\frac{dx^2}{dy^0} + g_{23}(x)\frac{dx^2}{dy^0}\frac{dx^3}{dy^0} +$$
$$g_{30}(x)\frac{dx^3}{dy^0}\frac{dx^0}{dy^0} + g_{31}(x)\frac{dx^3}{dy^0}\frac{dx^1}{dy^0} + g_{32}(x)\frac{dx^3}{dy^0}\frac{dx^2}{dy^0} + g_{33}(x)\frac{dx^3}{dy^0}\frac{dx^3}{dy^0}$$

Now we fill in, as an example, the appropriate, polar and Cartesian, coordinates in the element g_{11}

$$g_{rr} = g_{tt}\frac{dt}{dr}\frac{dt}{dr} + g_{tx}\frac{dt}{dr}\frac{dx}{dr} + g_{ty}\frac{dt}{dr}\frac{dy}{dr} + g_{tz}\frac{dt}{dr}\frac{dz}{dr} +$$
$$g_{xt}\frac{dx}{dr}\frac{dt}{dr} + g_{xx}\frac{dx}{dr}\frac{dx}{dr} + g_{xy}\frac{dx}{dr}\frac{dy}{dr} + g_{xz}\frac{dx}{dr}\frac{dz}{dr} +$$
$$g_{yt}\frac{dy}{dr}\frac{dt}{dr} + g_{yx}\frac{dy}{dr}\frac{dx}{dr} + g_{yy}\frac{dy}{dr}\frac{dy}{dr} + g_{yz}\frac{dy}{dr}\frac{dz}{dr} +$$
$$g_{zt}\frac{dz}{dr}\frac{dt}{dr} + g_{zx}\frac{dz}{dr}\frac{dx}{dr} + g_{zy}\frac{dz}{dr}\frac{dy}{dr} + g_{zz}\frac{dz}{dr}\frac{dz}{dr}$$

Because the coordinate system is an orthogonal system, only the elements with equal indices are non zero. Thus the matrix above boils down to:

$$g_{tt} = g_{tt}\frac{dt}{dt}\frac{dt}{dt} + g_{xx}\frac{dx}{dt}\frac{dx}{dt} + g_{yy}\frac{dy}{dt}\frac{dy}{dt} + g_{zz}\frac{dz}{dt}\frac{dz}{dt}$$
$$g_{tt} = 1 + 0 + 0 + 0 = 1$$
$$g_{rr} = g_{tt}\frac{dt}{dr}\frac{dt}{dr} + g_{xx}\frac{dx}{dr}\frac{dx}{dr} + g_{yy}\frac{dy}{dr}\frac{dy}{dr} + g_{zz}\frac{dz}{dr}\frac{dz}{dr}$$
$$g_{rr} = 0 - 1(+\sin\theta\cos\varphi)^2 - 1(+\sin\theta\sin\varphi)^2 - 1(+\cos\theta)^2 = -\sin^2\varphi - \cos^2\varphi = -1$$
$$g_{\theta\theta} = g_{tt}\frac{dt}{d\theta}\frac{dt}{d\theta} + g_{xx}\frac{dx}{d\theta}\frac{dx}{d\theta} + g_{yy}\frac{dy}{d\theta}\frac{dy}{d\theta} + g_{zz}\frac{dz}{d\theta}\frac{dz}{d\theta}$$
$$g_{\theta\theta} = 0 - 1(+r\cos\theta\cos\varphi)^2 - 1(+r\cos\theta\sin\varphi)^2 - 1(-r\sin\theta)^2 = -r^2\cos^2\theta - r^2\sin^2\theta = -r^2$$
$$g_{\varphi\varphi} = g_{tt}\frac{dt}{d\varphi}\frac{dt}{d\varphi} + g_{xx}\frac{dx}{d\varphi}\frac{dx}{d\varphi} + g_{yy}\frac{dy}{d\varphi}\frac{dy}{d\varphi} + g_{zz}\frac{dz}{d\varphi}\frac{dz}{d\varphi}$$
$$g_{\varphi\varphi} = 0 - 1(-r\sin\theta\sin\varphi)^2 - 1(+r\sin\theta\cos\varphi)^2 - 0 = -r^2\sin^2\varphi$$

Thus the transformation from Cartesian to polar metric tensor elements is:

$$\begin{array}{llll} \boldsymbol{g_{00} = 1} & \boldsymbol{g_{11} = -1} & \boldsymbol{g_{22} = -1} & \boldsymbol{g_{33} = -1} \\ \boldsymbol{g_{tt} = 1} & \boldsymbol{g_{rr} = -1} & \boldsymbol{g_{\theta\theta} = -r^2} & \boldsymbol{g_{\varphi\varphi} = -r^2\sin^2\varphi} \end{array}$$

4.8. Deliberations on the Minkowski and Schwarzschild formula

4.8.1. Minkowski

The Minkowski formula is used in Special Relativity where the effects of mass and acceleration is ignored. So frames move uniformly, with constant speed, relatively to each other. So the coordinate frame is linear.

Assume a point K in space-time with its own coordinate system. The point K stays in the origin of its coordinate system. The only thing that moves, i.e., progresses, is the time and, because it is in space-time, the distance, or interval, is ***s=cτ.*** An observer is at another location with his/her own coordinate system but there is a relative movement between the two coordinate systems. The relation between the two systems is:

$$v^2 = \frac{(x^2 + y^2 + z^2)}{t^2}$$

This means the observer sees K moving with a speed v.
In the Minkowski formula:

$$s^2 = c^2t^2 - x^2 - y^2 - z^2$$

For a small segment:

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$$

It has to be realized that t, x, y and z have a size and direction, they are vectors. Thus finding the size of ***s*** is adding the four vectors. If this coordinate system is an orthogonal system than Pythagoras theorem can be applied; this goes for the space part. If we consider the time part as complex *icdt* and for the left part of the formula *ds=icdτ*, then by squaring the coordinates we get the Minkowski formula.
So the formula signifies at the left hand side the distance covered by K in its K frame and this distance is the same as the distance covered by K as considered by the observer in the moving frame at the right hand side of the formula.

To find a general form for the relation between line segment *s* and its coordinates:

$$\vec{s} = a_1\vec{x}_1 + a_2\vec{x}_2$$

To find the size of s we find the in-product of s by multiplying s with itself:

$$\vec{s}.\vec{s} = (a_1\vec{x}_1 + a_2\vec{x}_2).(a_1\vec{x}_1 + a_2\vec{x}_2)$$
$$s^2 = a_1{}^2x_{1+}^2a_1a_2\vec{x}_1.\vec{x}_2 + a_1a_2\vec{x}_2.\vec{x}_{1+}a_2{}^2x_2^2$$

This was for two dimensions but to generalize this to four dimensions:

$$s^2 = \sum_{\mu}\sum_{\nu} g_{\mu\nu}\, x^{\mu}x^{\nu}$$

Or in Einstein notation (summation over the repeated, low and high, indices):

$$s^2 = g_{\mu\nu}\, x^{\mu}x^{\nu}$$

When an orthogonal coordinate system is used then all products where $\mu \neq \nu$ vanish. When only an infinitesimal small local "area" is considered *dx* is used instead of *x* etcetera.
Finally, when an orthogonal coordinate system is used, the equation results in a Minkowski or Schwarzschild form:

$$ds^2 = (cdx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2$$
$$ds^2 = g_{00}(cdx^0)^2 + g_{11}(dx^1)^2 + g_{22}(dx^2)^2 + g_{33}(dx^3)^2$$

For Minkowski the coefficients (tensor elements) are $g_{00} = 1\ and\ g_{11} = g_{22} = g_{33} = -1$

What does the Minkowski formula actually mean?

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2 = c^2dt'^2 - dx'^2 - dy'^2 - dz'^2$$

The *ds* term signifies an object which is in its own coordinate system where only the time ***τ*** progresses. An observer is in the origin of the system, t, x, y, z, he perceives that *ds* moves with a velocity of

$$v^2 = \frac{(dx^2 + dy^2 + dz^2)}{dt^2}$$

with respect to the origin of the observer's coordinate system. Another observer in the $t^{'}, x^{'}, y^{'}, z^{'}$ perceives *ds* moving with a velocity of

$$v^{'2} = \frac{(dx^{'2} + dy^{'2} + dz^{'2})}{dt^{'2}}$$

Thus if the observer is in *t, x, y, z* then when *s* changes with *ds* then the effect for the observer is *dt, dx, dy, dz.* If we jump back to t, x, y, z axis then x, y, z are the distances to s and t is the time in the t, x, y, z system while the time of *ds=cd*τ can change differently with respect to *cdt*:

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$$

$$c^2 d\tau^2 = c^2 dt^2 \left(1 - \frac{v^2}{c^2}\right) = \frac{c^2 dt^2}{\gamma^2}$$

Here is τ the so-called proper time that is the time of a moving clock which is and stays in the origin of its own co-moving coordinate frame.
The relation between the proper time τ in the *ds* system with respect to the observer:

$$d\tau^2 = \frac{dt^2}{\gamma^2}$$

$$dt^2 = \gamma^2 d\tau^2$$

As γ is 1 or greater, then $d\tau$ is always equal or smaller than dt. Thus the clock of *ds* goes slower than the clock of the observer.

4.8.2. Transformations performed by Schwarzschild

The Schwarzschild equation is comparable with the Minkowski equation in the sense that now the effects of mass and acceleration are also taken into account. This leads to a curved space-time and practically to a non linear coordinate frame considering this curved space-time geography.

Now we consider Schwarzschild equation and the transformation to new t, x, y and z coordinates:
He starts from Cartesian coordinates and transforms to polar coordinates according to the method followed above and resulting in:

$$ds^2 = \sigma^2 c^2 dt^2 - \frac{dr^2}{\sigma^2} - r^2 d\theta^2 - r^2 \sin^2\theta^2\, d\emptyset^2$$

Here he realizes that the product of the metric tensor elements, the g determinant, is not -1 as wished by Einstein.
Because $g = \sigma^2 . \left(\frac{-1}{\sigma^2}\right) . (-r^2) . (-r^2 \sin^2\theta) = -r^4 \sin^2\theta$
To meet the desire to have g=-1, he wants to perform the transformation with $\frac{dr}{dx_1} = \frac{1}{r^2}$ *and* $\frac{d\theta}{dx_2} = \frac{1}{\sin\theta}$ *and* $\frac{d\emptyset}{dx_3} = 1$
And as Schwarzschild mentioned: "The new variables are the *polar coordinates with the determinant 1".*
In order to get these derivatives he finds the relations $x_1 = \frac{r^3}{3}$, $x_2 = -\cos\theta$, $x_3 = \emptyset$ and transforms accordingly.

$$ds^2 = \sigma^2 c^2 dt^2 - \frac{dx_1{}^2}{r^4\sigma^2} - r^2 \frac{1}{\sin^2\theta} dx_2{}^2 - r^2 \sin^2\theta\, dx_3{}^2$$

As $x_2 = -\cos\theta$ then $x_2{}^2 = \cos^2\theta = 1 - \sin^2\theta \Rightarrow \sin^2\theta = 1 - x_2{}^2$

$$ds^2 = \sigma^2 c^2 dt^2 - \frac{dx_1{}^2}{r^4\sigma^2} - r^2 \frac{1}{1 - x_2{}^2} dx_2{}^2 - r^2(1 - x_2{}^2) dx_3{}^2$$

Thus the metric tensor elements are:

$$\boldsymbol{g_{00}} = \sigma^2 \quad \boldsymbol{g_{11}} = -\frac{1}{r^4\sigma^2} \quad \boldsymbol{g_{22}} = \frac{-r^2}{1 - x_2{}^2} \quad \boldsymbol{g_{33}} = -r^2(1 - x_2{}^2)$$

So indeed now g=-1 and the performed transformations are legitimate. In the special case $\theta = 90^0$ then $x_2 = 0$

4.9. Summary on Schwarzschild's: "On the Gravitational Field of a Mass Point According to Einstein's Theory

Schwarzschild aim was to find an equation that satisfies Einstein's field equations in vacuum. The equation depicts a point that moves along a geodesic line in a manifold characterized by the line element *ds*.
The conditions that must be fulfilled as well are:

1. All the components are independent of the time x_4.
2. The equations $g_{\rho 4} = g_{4\rho} = 0$ hold exactly for $\rho = 1,2,3$.
3. The solution is spatially symmetric with respect to the origin of the coordinate system in the sense that one finds again the same solution when x_1, x_2, x_3 are subjected to an orthogonal transformation (rotation).
4. The $g_{\mu\nu}$ vanish at infinity with the exception of the following limits different from zero:
$$g_{44} = 1,\ g_{11} = g_{22} = g_{33} = -1$$

The initial equation was based on rectangular coordinates:

$$ds^2 = Fdt^2 - G(dx^2 + dy^2 + dz^2) - H(xdx + ydy + zdz)^2$$

Now he changes to polar coordinates according to $x = r\sin\vartheta\cos\varphi,\ y = r\sin\vartheta\sin\varphi,\ \mathrm{r}\cos\vartheta$; the same element reads:

$$ds^2 = Fdt^2 - G(dr^2 + r^2 d\vartheta^2 + r^2\sin\vartheta^2\, d\varphi^2) - Hr^2dr^2$$
$$= Fdt^2 - (G + Hr^2)dr^2 - Gr^2(d\vartheta^2 + \sin\vartheta^2\, d\varphi^2)$$

As the determinant of the metric is unequal to -1 a following transformation is done:
With new variables and polar coordinates with determinant 1:

$$x_1 = \frac{r^3}{3}, x_2 = -\cos\vartheta, x_3 = \varphi,$$

$$ds^2 = Fdt^2 - \left(\frac{G}{r^4} + \frac{H}{r^2}\right) dx_1{}^2 - Gr^2 \left[\frac{dx_2{}^2}{1 - x_2{}^2} + dx_3{}^2(1 - x_2{}^2)\right]$$

Via the Einstein field equations the coefficients are found and this results in the following, mostly used, Schwarzschild equation:

$$ds^2 = \boldsymbol{c^2 d\tau^2} = \left(\boldsymbol{1} - \frac{\boldsymbol{2GM}}{\boldsymbol{c^2 r}}\right)\boldsymbol{c^2 dt^2} - \frac{\boldsymbol{1}}{\left(\boldsymbol{1} - \frac{\boldsymbol{2GM}}{\boldsymbol{c^2 r}}\right)} \boldsymbol{dr^2} - \boldsymbol{r^2 d\theta^2} - \boldsymbol{r^2 \sin^2\theta^2\, d\emptyset^2} \qquad (12)$$

The original approach of Schwarzschild was in Cartesian coordinates. The derivation of the equation resulted in the equation (2.16.1 8) in polar coordinates but this could also be transformed to the original Cartesian coordinates as follows:

$$ds^2 = c^2 d\tau^2 = \left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 - (dx^2 + dy^2 + dz^2) - \frac{\frac{2GM}{c^2 r}}{\left(1 - \frac{2GM}{c^2 r}\right) r^2} (xdx + ydy + zdz)^2 \qquad (13)$$

However this form is hardly used.

For derivation of the Schwarzschild equation: (Schwarzschild, On the Gravitational Field of a Point-Mass, According to Einstein's Theory, 13 January 1916) and (Oas):
(Schwarzschild, On the Gravitational Field of a Point-Mass, According to Einstein's Theory, 13 January 1916)
(Oas)

5. Check whether the Schwarzschild elements meet the Einstein field equations

The general form of the Einstein field equations is:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

In general λ is very small and is only relevant when calculations are done on the total universe. So generally the following form is used:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}$$

The left part of the formula depicts the geometry and the right part the mass and energy. When the calculation is done in a vacuum, so outside a mass, the right side becomes zero. In that case the formula becomes:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0$$

μ and ν depicts the four dimensions of space and time. This means that the Einstein formula consists of 16 equations.
The field equations are totally dependent on metric tensor elements $g_{\mu\nu}$ and its first and second derivatives.
Schwarzschild derived a formula that meets the Einstein field equations in vacuum.

$$\boldsymbol{ds^2 = \sigma^2 c^2 dt^2 - \frac{dr^2}{\sigma^2} - r^2 d\theta^2 - r^2 \sin^2 \theta^2 \, d\emptyset^2}$$

The general form to find the metric tensor elements:

$$ds^2 = g_{00} dt^2 + g_{11} dr^2 + g_{22} d\theta^2 + g_{33} d\emptyset^2$$

As can be seen from the formula only four of the sixteen metric tensor elements are relevant; the rest is zero. As a consequence of the 16 field equations only four are relevant: R_{00}, R_{11}, R_{22} and R_{33}.

$\mathrm{R}_{\mu\nu}$ is called the Ricci tensor and consists of sixteen elements. The general form of the Ricci tensor elements is:

$$\mathrm{R}_{\mu\nu} = \mathrm{R}^{\rho}_{\mu\rho\nu} = \Gamma^{\rho}_{\mu\nu,\rho} - \Gamma^{\rho}_{\rho\mu,\nu} + \Gamma^{\rho}_{\rho\lambda}\,\Gamma^{\lambda}_{\nu\mu} - \Gamma^{\rho}_{\nu\lambda}\,\Gamma^{\lambda}_{\rho\mu}$$

Or written differently

$$\mathrm{R}_{\mu\nu} = \mathrm{R}^{\rho}_{\mu\rho\nu} = \frac{\partial\Gamma^{\rho}_{\mu\nu}}{\partial x^{\rho}} - \frac{\partial\Gamma^{\rho}_{\rho\mu}}{\partial x^{\nu}} + \Gamma^{\rho}_{\rho\lambda}\,\Gamma^{\lambda}_{\nu\mu} - \Gamma^{\rho}_{\nu\lambda}\,\Gamma^{\lambda}_{\rho\mu}$$

In this formula there are the so called Christoffel symbols. The first two left items at the right hand side are the derivatives of the Christoffel symbols. The general form of the Christoffel symbol is:

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\alpha}\left\{\frac{\partial g_{\nu\alpha}}{\partial x^{\mu}} + \frac{\partial g_{\mu\alpha}}{\partial x^{\nu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\alpha}}\right\}$$

The Schwarzschild equation is for situations in vacuum and in that case:

$$\mathrm{R}_{\mu\nu} - \frac{1}{2}\mathrm{g}_{\mu\nu}\mathrm{R} = 0$$

Here is R the Ricci scalar and stands for the curvature of the local space-time

$$R = g^{\mu\nu}\,\mathrm{R}_{\mu\nu}$$

So multiplying the formula above with $g^{\mu\nu}$ gives:

$$g^{\mu\nu}\,\mathrm{R}_{\mu\nu} - \frac{1}{2} g^{\mu\nu}\,g_{\mu\nu}\,\mathrm{R} = 0 \;\;=> R - \frac{1}{2}4\mathrm{R} = 0$$

This can only be true if R=0 and thus $\mathrm{R}_{\mu\nu} = 0$.

So because of the relation of R with $\mathrm{R}_{\mu\nu}$, it is obvious that

$$\mathrm{R}_{\mu\nu} - \frac{1}{2}\mathrm{g}_{\mu\nu}\mathrm{R} = 0$$

Can be limited to:

$$\mathrm{R}_{\mu\nu} = 0$$

By analyzing the general form of the Ricci elements and the Christoffel symbols the simplification could go even further. First we derived a program so that by computer and numeric application of the equations, we found the relevant form of the Ricci elements. Also by theoretical analysis of the Ricci elements the simplification could be derived.(See OAS page **204)**

This resulted in the following formulas with the only relevant Christoffel symbols:

$$\mathrm{R}_{00} = \boldsymbol{\Gamma^{1}_{00,1}} + \Gamma^{1}_{00}\,\Gamma^{1}_{11} + \Gamma^{1}_{00}\,\Gamma^{2}_{12} + \Gamma^{1}_{00}\,\Gamma^{3}_{13} - \Gamma^{0}_{01}\,\Gamma^{1}_{00}$$

$$\mathrm{R}_{11} = -\boldsymbol{\Gamma^{0}_{10,1}} - \boldsymbol{\Gamma^{2}_{12,1}} - \boldsymbol{\Gamma^{3}_{13,1}} + \Gamma^{1}_{11}\Gamma^{0}_{10} + \Gamma^{1}_{11}\,\Gamma^{2}_{12} + \Gamma^{1}_{11}\,\Gamma^{3}_{13} - \Gamma^{0}_{10}\Gamma^{0}_{01} - \Gamma^{2}_{12}\,\Gamma^{2}_{21} - \Gamma^{3}_{13}\,\Gamma^{3}_{31}$$

$$\mathrm{R}_{22} = \boldsymbol{\Gamma^{1}_{22,1}} - \boldsymbol{\Gamma^{3}_{23,2}} + \Gamma^{1}_{22}\,\Gamma^{0}_{10} + \Gamma^{1}_{22}\,\Gamma^{1}_{11} + \Gamma^{1}_{22}\,\Gamma^{3}_{13} - \Gamma^{2}_{21}\,\Gamma^{1}_{22} - \Gamma^{3}_{23}\,\Gamma^{3}_{32}$$

$$\mathrm{R}_{33} = +\boldsymbol{\Gamma^{1}_{33,1}} + \boldsymbol{\Gamma^{2}_{33,2}} + \Gamma^{1}_{33}\,\Gamma^{0}_{10} + \Gamma^{1}_{33}\,\Gamma^{1}_{11} + \Gamma^{1}_{33}\,\Gamma^{2}_{12} - \Gamma^{3}_{31}\,\Gamma^{1}_{33} - \Gamma^{3}_{32}\,\Gamma^{2}_{33}$$

First the spherical coordinates are tested. The elements in the 4 formulas above are filled in with Christoffel symbols which are derived and summarized in de table below. (See Appendix 1.2)

In the literature the Christoffel symbol formula is sometimes shown with the first element -1/2 and sometimes +1/2.

Because of the method of deriving our formulas, the Christoffel formula has a leading +1/2. After some calculations the formula with +1/2 achieved the result of R_{11}=R_{22}=R_{33}=R_{44}=0, which is required by the Einstein field equations in vacuum. Thus the formula in the following format has been applied:

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\alpha} \left\{ \frac{\partial g_{\nu\alpha}}{\partial x^{\mu}} + \frac{\partial g_{\mu\alpha}}{\partial x^{\nu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\alpha}} \right\}$$

5.1. Checking of R_{00}, R_{11}, R_{22} and R_{33} with spherical coordinates Schwarzschild

$$ds^2 = \sigma^2 c^2 dt^2 - \frac{dr^2}{\sigma^2} - r^2 d\theta^2 - r^2 \sin^2\theta \, d\emptyset^2$$

In order to proof the validity of this Schwarzschild formula, we will check whether $R_{\mu\nu}$ and in particular

R_{11}, R_{22}, R_{33} and R_{44} are zero in vacuum as is required by the theory of General Relativity.

The Christoffel symbols and its derivatives are used from the table below. (See Appendix 1.2)

$$\mathbf{R_{00}} = \Gamma^1_{00,1} + \Gamma^1_{00}\,\Gamma^1_{11} + \Gamma^1_{00}\,\Gamma^2_{12} + \Gamma^1_{00}\,\Gamma^3_{13} - \Gamma^0_{01}\,\Gamma^1_{00}$$
$$R_{00} = \frac{R_s(3R_s - 2r)}{2r^4} + \frac{\sigma^2 R_s}{2r^2}\frac{-R_s}{2r^2\sigma^2} + \frac{\sigma^2 R_s}{2r^2}\frac{1}{r} + \frac{\sigma^2 R_s}{2r^2}\frac{1}{r} - \frac{R_s}{2r^2\sigma^2}\frac{\sigma^2 R_s}{2r^2}$$
$$R_{00} = \frac{R_s(3R_s - 2r)}{2r^4} - \frac{{R_s}^2}{2r^4} + \frac{2R_s(r - Rs)}{2r^4} = \frac{3{R_s}^2 - 2rRs - {R_s}^2 + 2R_s r - 2{R_s}^2}{2r^4} = 0$$
$$\mathbf{R_{00} = 0} \quad \boldsymbol{q.e.d.}$$

$$\mathbf{R_{11}} = -\boldsymbol{\Gamma^0_{10,1}} - \boldsymbol{\Gamma^2_{12,1}} - \boldsymbol{\Gamma^3_{13,1}} + \Gamma^1_{11}\Gamma^0_{10} + \Gamma^1_{11}\,\Gamma^2_{12} + \Gamma^1_{11}\,\Gamma^3_{13} - \Gamma^0_{10}\Gamma^0_{01} - \Gamma^2_{12}\,\Gamma^2_{21} - \Gamma^3_{13}\,\Gamma^3_{31}$$
$$R_{11} = -\frac{R_s(R_s - 2r)}{2r^4\sigma^4} - \frac{-1}{r^2} - \frac{-1}{r^2} + \frac{-R_s}{2r^2\sigma^2}\frac{R_s}{2r^2\sigma^2} + \frac{-R_s}{2r^2\sigma^2}\frac{1}{r} + \frac{-R_s}{2r^2\sigma^2}\frac{1}{r} - \frac{R_s}{2r^2\sigma^2}\frac{R_s}{2r^2\sigma^2} - \frac{1}{r}\frac{1}{r} - \frac{1}{r}\frac{1}{r}$$
$$R_{11} = -\frac{R_s(R_s - 2r)}{2r^4\sigma^4} + \frac{1}{r^2} + \frac{1}{r^2} - \frac{{R_s}^2}{4r^4\sigma^4} - \frac{R_s}{2r^3\sigma^2} - \frac{R_s}{2r^3\sigma^2} - \frac{{R_s}^2}{4r^4\sigma^4} - \frac{1}{r^2} - \frac{1}{r^2}$$
$$R_{11} = -\frac{R_s(R_s - 2r)}{2r^4\sigma^4} - \frac{{R_s}^2}{2r^4\sigma^4} - \frac{2R_s r(1 - \frac{Rs}{r})}{2r^4\sigma^4} = -\frac{R_s(R_s - 2r)}{2r^4\sigma^4} - \frac{{R_s}^2}{2r^4\sigma^4} - \frac{2R_s r - 2{R_s}^2}{2r^4\sigma^4}$$
$$R_{11} = \frac{-{R_s}^2 + 2rRs}{2r^4\sigma^4} + \frac{-{R_s}^2}{2r^4\sigma^4} + \frac{-2R_s r + 2{R_s}^2}{2r^4\sigma^4} = 0$$
$$\mathbf{R_{11} = 0} \quad \boldsymbol{q.e.d.}$$

$$\mathbf{R_{22}} = \boldsymbol{\Gamma^1_{22,1}} - \boldsymbol{\Gamma^3_{23,2}} + \Gamma^1_{22}\,\Gamma^0_{10} + \Gamma^1_{22}\,\Gamma^1_{11} + \Gamma^1_{22}\,\Gamma^3_{13} - \Gamma^2_{21}\Gamma^1_{22} - \Gamma^3_{23}\,\Gamma^3_{32}$$
$$R_{22} = -1 + 1 - r\sigma^2\,\frac{R_s}{2r^2\sigma^2} + r\sigma^2\,\frac{+R_s}{2r^2\sigma^2} - r\sigma^2\,\frac{1}{r} + \frac{1}{r}\,r\sigma^2 - \;0 = 0$$
$$\mathbf{R_{22} = 0} \quad \boldsymbol{q.e.d.}$$

$$\mathbf{R_{33}} = +\boldsymbol{\Gamma^1_{33,1}} + \boldsymbol{\Gamma^2_{33,2}} + \Gamma^1_{33}\,\Gamma^0_{10} + \Gamma^1_{33}\,\Gamma^1_{11} + \Gamma^1_{33}\,\Gamma^2_{12} - \Gamma^3_{31}\,\Gamma^1_{33} - \Gamma^3_{32}\,\Gamma^2_{33}$$
$$R_{33} = -1 + 1 - r\sigma^2\,\frac{R_s}{2r^2\sigma^2} + r\sigma^2\,\frac{R_s}{2r^2\sigma^2} - r\sigma^2\,\frac{1}{r} + \frac{1}{r}r\sigma^2 - \;0 = 0$$
$$\mathbf{R_{33} = 0} \quad \boldsymbol{q.e.d.}$$

So it is shown that all $R_{\mu\nu}$ are zero and consequently that the Schwarzschild equation meets the Einstein equation in vacuum.

5.2. Checking of R_{00}, R_{11}, R_{22} and R_{33} with t, x, y and z (adapted polar) coordinates Schwarzschild

$$ds^2 = \sigma^2 c^2 dt_\infty{}^2 - \frac{dx_1{}^2}{r^4\sigma^2} - \frac{r^2 dx_2{}^2}{\sin^2\theta} - r^2\sin^2\theta\, dx_3{}^2$$

The Christoffel symbols and its derivatives are used from the table below. (See Appendix 1.3)

$$\mathbf{R_{00}} = \Gamma^1_{00,1} + \Gamma^1_{00}\,\Gamma^1_{11} + \Gamma^1_{00}\,\Gamma^2_{12} + \Gamma^1_{00}\,\Gamma^3_{13} - \Gamma^0_{01}\,\Gamma^1_{00}$$

$$R_{00} = \frac{R_s{}^2}{2r^4} + \frac{R_s\sigma^2}{2}\frac{3R_s - 4r}{2r^4\sigma^2} + \frac{R_s\sigma^2}{2}\frac{1}{r^3} + \frac{R_s\sigma^2}{2}\frac{1}{r^3} - \frac{R_s}{2r^4\sigma^2}\frac{R_s\sigma^2}{2}$$

$$R_{00} = \frac{2R_s{}^2}{4r^4} + \frac{3R_s{}^2 - 4rR_s}{4r^4} + \frac{4R_s r\sigma^2}{4r^4} - \frac{R_s{}^2}{4r^4}$$

$$R_{00} = \frac{2R_s{}^2 + 3R_s{}^2 - 4rR_s - R_s{}^2}{4r^4} + \frac{4R_s(r - R_s)}{4r^4} = \frac{2R_s{}^2 + 3R_s{}^2 - 4rR_s - R_s{}^2}{4r^4} + \frac{4R_s r - 4R_s{}^2}{4r^4}$$

$$R_{00} = \frac{4R_s{}^2 - 4rR_s}{4r^4} + \frac{4R_s r - 4R_s{}^2}{4r^4} = 0$$

$$\mathbf{R_{00} = 0 \quad \boldsymbol{q.e.d.}}$$

$$\mathbf{R_{11}} = -\boldsymbol{\Gamma^0_{10,1}} - \boldsymbol{\Gamma^2_{12,1}} - \boldsymbol{\Gamma^3_{13,1}} + \Gamma^1_{11}\Gamma^0_{10} + \Gamma^1_{11}\,\Gamma^2_{12} + \Gamma^1_{11}\,\Gamma^3_{13} - \Gamma^0_{10}\Gamma^0_{01} - \Gamma^2_{12}\,\Gamma^2_{21} - \Gamma^3_{13}\,\Gamma^3_{31}$$

$$R_{11} = -\frac{R_s(3R_s - 4r)}{2r^8\sigma^4} - \frac{-3}{r^6} - \frac{-3}{r^6} + \frac{3R_s - 4r}{2r^4\sigma^2}\frac{R_s}{2r^4\sigma^2} + \frac{3R_s - 4r}{2r^4\sigma^2}\frac{1}{r^3} + \frac{3R_s - 4r}{2r^4\sigma^2}\frac{1}{r^3} - \frac{R_s}{2r^4\sigma^2}\frac{R_s}{2r^4\sigma^2} - \frac{1}{r^3}\frac{1}{r^3} - \frac{1}{r^3}\frac{1}{r^3}$$

$$R_{11} = -\frac{2R_s(3R_s - 4r)}{4r^8\sigma^4} + \frac{4}{r^6} + \frac{R_s(3R_s - 4r)}{4r^8\sigma^4} + \frac{4(3R_s - 4r)r(1 - \frac{Rs}{r})}{4r^8\sigma^4} - \frac{R_s{}^2}{4r^8\sigma^4}$$

$$R_{11} = \frac{-6R_s{}^2 + 8rR_s + 3R_s{}^2 - 4rR_s + 12R_s r - 16r^2 - 12R_s{}^2 + 16rRs - R_s{}^2}{4r^8\sigma^4} + \frac{4}{r^6}$$

$$R_{11} = \frac{-16R_s{}^2 + 32rR_s - 16r^2}{4r^8\sigma^4} + \frac{4}{r^6} = \frac{-16R_s{}^2 + 32rR_s - 16r^2}{4r^8\sigma^4} + \frac{16r^2\left(1 - \frac{Rs}{r}\right)^2}{4r^8\sigma^4}$$

$$R_{11} = \frac{-16R_s{}^2 + 32rR_s - 16r^2}{4r^8\sigma^4} + \frac{4}{r^6} = \frac{-16R_s{}^2 + 32rR_s - 16r^2}{4r^8\sigma^4} + \frac{16r^2(1 - 2\frac{Rs}{r} + \frac{R_s{}^2}{r^2})}{4r^8\sigma^4} =$$

$$R_{11} = \frac{-16R_s{}^2 + 32rR_s - 16r^2}{4r^8\sigma^4} + \frac{4}{r^6} = \frac{-16R_s{}^2 + 32rR_s - 16r^2}{4r^8\sigma^4} + \frac{16r^2 - 32rRs + 16R_s{}^2}{4r^8\sigma^4} = 0$$

$$\mathbf{R_{11} = 0 \quad \boldsymbol{q.e.d.}}$$

$$\mathbf{R_{22}} = \boldsymbol{\Gamma^1_{22,1}} - \boldsymbol{\Gamma^3_{23,2}} + \Gamma^1_{22}\,\Gamma^0_{10} + \Gamma^1_{22}\,\Gamma^1_{11} + \Gamma^1_{22}\,\Gamma^3_{13} - \Gamma^2_{21}\,\Gamma^1_{22} - \Gamma^3_{23}\,\Gamma^3_{32}$$

$$R_{22} = -3 + \frac{2R_s}{r} + 1 - r^3\sigma^2\frac{R_s}{2r^4\sigma^2} - r^3\sigma^2\frac{3R_s - 4r}{2r^4\sigma^2} - r^3\sigma^2\frac{1}{r^3} + \frac{1}{r^3}r^3\sigma^2 - 0$$

$$R_{22} = -3 + \frac{2R_s}{r} + 1 - \frac{R_s}{2r} - \frac{3R_s - 4r}{2r} - r^3\sigma^2\frac{1}{r^3} + \frac{1}{r^3}r^3\sigma^2 - 0$$

$$R_{22} = \frac{-4r}{2r} + \frac{4R_s}{2r} - \frac{R_s}{2r} - \frac{3R_s - 4r}{2r} = 0$$

$$\mathbf{R_{22} = 0 \quad \boldsymbol{q.e.d.}}$$

$$R_{33} = +\boldsymbol{\Gamma^{1}_{33,1}} + \boldsymbol{\Gamma^{2}_{33,2}} + \Gamma^{1}_{33}\,\Gamma^{0}_{10} + \Gamma^{1}_{33}\,\Gamma^{1}_{11} + \Gamma^{1}_{33}\,\Gamma^{2}_{12} - \Gamma^{3}_{31}\,\Gamma^{1}_{33} - \Gamma^{3}_{32}\,\Gamma^{2}_{33}$$
$$R_{33} = -3 + \frac{2R_s}{r} + 1 - r^3\sigma^2\,\frac{R_s}{2r^4\sigma^2} - r^3\sigma^2\,\frac{3R_s - 4r}{2r^4\sigma^2} - r^3\sigma^2\,\frac{1}{r^3} + \frac{1}{r^3}\,r^3\sigma^2 - 0$$
$$R_{33} = -3 + \frac{2R_s}{r} + 1 - \frac{R_s}{2r} - \frac{3R_s - 4r}{2r} - r^3\sigma^2\,\frac{1}{r^3} + \frac{1}{r^3}\,r^3\sigma^2 - 0$$
$$R_{33} = \frac{-4r}{2r} + \frac{4R_s}{2r} - \frac{R_s}{2r} - \frac{3R_s - 4r}{2r} = 0$$
$$\mathbf{R_{33} = 0}\quad \boldsymbol{q.e.d.}$$

6. Check whether the Schwarzschild elements meet the Einstein field equations according the limited formula

In this chapter we will check the Schwarzschild solution with the limited original Einstein formula, which is only valid when, the trace of the metric tensor $t(g_{\mu\nu}) = -1$:

$$G_{\mu\nu} = \frac{\partial \Gamma^{\alpha}_{\mu\nu}}{\partial x_{\alpha}} + \Gamma^{\alpha}_{\mu\beta}\;\Gamma^{\beta}_{\nu\alpha}$$

Here we use the Christoffel symbol with negative sign as Schwarzschild applied in his derivation.

$$\Gamma^{\rho}_{\mu\nu} = -\frac{1}{2} g^{\rho\alpha}\left\{\frac{\partial g_{\nu\alpha}}{\partial x^{\mu}} + \frac{\partial g_{\mu\alpha}}{\partial x^{\nu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\alpha}}\right\}$$

In that case the formulas of the Christoffel symbols and its derivatives, in the table below, shall change sign as well.

In the derivation of his solution Schwarzschild used the t, x, y, z coordinates, so let us first start with these coordinates.

We first derived the relevant Ricci elements:

$$R_{00} = \boldsymbol{\Gamma^{1}_{00,1}} + \Gamma^{0}_{01}\,\Gamma^{1}_{00} + \Gamma^{1}_{00}\,\Gamma^{0}_{10}$$
$$R_{11} = \boldsymbol{\Gamma^{1}_{11,1}} + \Gamma^{0}_{10}\Gamma^{0}_{01} + \Gamma^{1}_{11}\,\Gamma^{1}_{11} + \Gamma^{2}_{12}\,\Gamma^{2}_{21} + \Gamma^{3}_{13}\,\Gamma^{3}_{31}$$
$$R_{22} = \boldsymbol{\Gamma^{1}_{22,1}} + \boldsymbol{\Gamma^{2}_{22,2}} + \Gamma^{1}_{22}\,\Gamma^{2}_{12} + \Gamma^{2}_{21}\,\Gamma^{1}_{22} + \Gamma^{2}_{22}\,\Gamma^{2}_{22} + \Gamma^{3}_{23}\,\Gamma^{3}_{23}$$
$$R_{33} = +\boldsymbol{\Gamma^{1}_{33,1}} + \boldsymbol{\Gamma^{2}_{33,2}} + \Gamma^{1}_{33}\,\Gamma^{3}_{13} + \Gamma^{2}_{33}\,\Gamma^{3}_{23} + \Gamma^{3}_{31}\,\Gamma^{1}_{33} + \Gamma^{3}_{32}\,\Gamma^{2}_{33}$$

First:

6.1. t,x,y,z (adapted polar) coordinates

$$R_{00} = \boldsymbol{\Gamma^{1}_{00,1}} + \Gamma^{0}_{01}\,\Gamma^{1}_{00} + \Gamma^{1}_{00}\,\Gamma^{0}_{10}$$
$$R_{00} = \frac{-{R_s}^2}{2r^4} + \frac{R_s}{2r^4\sigma^2}\,\frac{R_s\sigma^2}{2} + \frac{R_s\sigma^2}{2}\,\frac{R_s}{2r^4\sigma^2} = \frac{-{R_s}^2}{2r^4} + \frac{{R_s}^2}{2r^4} = 0$$
$$\mathbf{R_{00} = 0\ q.e.d.}$$

$$R_{11} = \boldsymbol{\Gamma^{1}_{11,1}} + \Gamma^{0}_{10}\Gamma^{0}_{01} + \Gamma^{1}_{11}\,\Gamma^{1}_{11} + \Gamma^{2}_{12}\,\Gamma^{2}_{21} + \Gamma^{3}_{13}\,\Gamma^{3}_{31}$$
$$R_{11} = \frac{-6}{r^6\sigma^4} + \frac{10R_s}{r^7\sigma^4} - \frac{4.5{R_s}^2}{r^8\sigma^4} + \frac{R_s}{2r^4\sigma^2}\frac{R_s}{2r^4\sigma^2} + \frac{3R_s - 4r}{2r^4\sigma^2}\,\frac{3R_s - 4r}{2r^4\sigma^2} + \frac{1}{r^3}\,\frac{1}{r^3} + \frac{1}{r^3}\,\frac{1}{r^3}$$
$$R_{11} = \frac{-6}{r^6\sigma^4} + \frac{10R_s}{r^7\sigma^4} - \frac{4.5{R_s}^2}{r^8\sigma^4} + \frac{{R_s}^2}{4r^8\sigma^4} + \frac{9{R_s}^2 + 16r^2 - 24rR_s}{4r^8\sigma^4} + \frac{2}{r^6}$$

$$R_{11} = \frac{-24r^2}{4r^8\sigma^4} + \frac{40rR_s}{4r^8\sigma^4} - \frac{18{R_s}^2}{4r^8\sigma^4} + \frac{{R_s}^2}{4r^8\sigma^4} + \frac{9{R_s}^2 + 16r^2 - 24rR_s}{4r^8\sigma^4} + \frac{2}{r^6}$$

$$R_{11} = \frac{-8{R_s}^2 - 8r^2 + 16rR_s}{4r^8\sigma^4} + \frac{2}{r^6}$$

$$R_{11} = \frac{-8{R_s}^2 - 8r^2 + 16rR_s}{4r^8\sigma^4} + \frac{8r^2\sigma^4}{4r^8\sigma^4}$$

$$R_{11} = \frac{-8{R_s}^2 - 8r^2 + 16rR_s}{4r^8\sigma^4} + \frac{8r^2\left(1 - \frac{R_s}{r}\right)^2}{4r^8\sigma^4}$$

$$R_{11} = \frac{-8{R_s}^2 - 8r^2 + 16rR_s}{4r^8\sigma^4} + \frac{8\left(r^2 + {R_s}^2 - 2rR_s\right)}{4r^8\sigma^4} = 0$$

$$\mathbf{R_{11} = 0 \quad q.e.d.}$$

$$R_{22} = \boldsymbol{\Gamma^1_{22,1}} + \boldsymbol{\Gamma^2_{22,2}} + \Gamma^1_{22}\,\Gamma^2_{12} + \Gamma^2_{21}\,\Gamma^1_{22} + \Gamma^2_{22}\,\Gamma^2_{22} + \Gamma^3_{23}\,\Gamma^3_{23}$$

$$R_{22} = \frac{-2R_s + 3r}{r\sin^2\theta} + \frac{-1 - \cos^2\theta}{\sin^4\theta} + \frac{-r^3\sigma^2}{\sin^2\theta}\,\frac{1}{r^3} + \frac{1}{r^3}\,\frac{-r^3\sigma^2}{\sin^2\theta} + \frac{-\cos(\theta)}{\sin^2(\theta)}\,\frac{-\cos(\theta)}{\sin^2(\theta)} + \frac{\cos\theta}{\sin^2(\theta)}\,\frac{\cos\theta}{\sin^2(\theta)}$$

$$R_{22} = \frac{-2R_s + 3r}{r\sin^2\theta} + \frac{-1 - \cos^2\theta}{\sin^4\theta} + \frac{-2r^3\sigma^2}{r^3\sin^2\theta} + \frac{2\cos^2\theta}{\sin^4\theta}$$

$$R_{22} = \frac{-2R_s + 3r}{r\sin^2\theta} + \frac{-1 - \cos^2\theta}{\sin^4\theta} + \frac{-2(r - R_s)}{r\sin^2\theta} + \frac{2\cos^2\theta}{\sin^4\theta}$$

$$R_{22} = \frac{1}{\sin^2\theta} + \frac{-1 - \cos^2\theta}{\sin^4\theta} + \frac{2\cos^2\theta}{\sin^4\theta}$$

$$R_{22} = \frac{\sin^2\theta}{\sin^4\theta} + \frac{-\sin^2\theta - \cos^2\theta - \cos^2\theta}{\sin^4\theta} + \frac{2\cos^2\theta}{\sin^4\theta} = 0$$

$$\mathbf{R_{22} = 0 \quad q.e.d.}$$

$$R_{33} = +\boldsymbol{\Gamma^1_{33,1}} + \boldsymbol{\Gamma^2_{33,2}} + \Gamma^1_{33}\,\Gamma^3_{13} + \Gamma^2_{33}\,\Gamma^3_{23} + \Gamma^3_{31}\,\Gamma^1_{33} + \Gamma^3_{32}\,\Gamma^2_{33}$$

$$R_{33} = \left(3 - \frac{2R_s}{r}\right).\sin^2\theta + 3\cos^2\theta - 1 - r^3\sigma^2\sin^2\theta\,\frac{1}{r^3} + (-\sin^2\theta\cos\theta)\,\frac{\cos\theta}{\sin^2(\theta)}$$
$$- \frac{1}{r^3}\,r^3\sigma^2\sin^2\theta + \frac{\cos\theta}{\sin^2(\theta)}(-\sin^2\theta\cos\theta)$$

$$R_{33} = \left(3 - \frac{2R_s}{r}\right).\sin^2\theta + 3\cos^2\theta - 1 - 2\sigma^2\sin^2\theta - 2\sin^2(\theta)\cos\theta\,\frac{\cos\theta}{\sin^2(\theta)}$$

$$R_{33} = \left(3 - \frac{2R_s}{r}\right).\sin^2\theta + 3\cos^2\theta - 1 - 2\left(1 - \frac{R_s}{r}\right).\sin^2\theta - 2\cos^2\theta$$

$$R_{33} = \left(3 - \frac{2R_s}{r}\right).\sin^2\theta + 3\cos^2\theta - 1 + \left(-2 + \frac{2R_s}{r}\right).\sin^2\theta - 2\cos^2\theta$$

$$R_{33} = \sin^2\theta + 3\cos^2\theta - 1 - 2\cos^2\theta$$

$$R_{33} = \sin^2\theta + 3\cos^2\theta - \sin^2\theta - \cos^2\theta - 2\cos^2\theta = 0$$

$$\mathbf{R_{33} = 0 \quad q.e.d.}$$

6.2. Spherical coordinates

$$R_{00} = \boldsymbol{\Gamma^1_{00,1}} + \Gamma^0_{01}\,\Gamma^1_{00} + \Gamma^1_{00}\,\Gamma^0_{10}$$

$$\mathrm{R}_{00} = \frac{-\mathrm{R_s}(3\mathrm{R_s} - 2\mathrm{r})}{2\mathrm{r}^4} + \frac{R_s}{2r^2\sigma^2}\,\frac{\sigma^2 R_s}{2r^2} + \frac{\sigma^2 R_s}{2r^2}\,\frac{R_s}{2r^2\sigma^2}$$

$$\mathrm{R}_{00} = \frac{-\mathrm{R_s}(3\mathrm{R_s} - 2\mathrm{r})}{2\mathrm{r}^4} + \frac{\mathrm{R_s}^2}{2r^4} =$$

$$\mathrm{R}_{00} = \frac{-\mathrm{R_s}(2\mathrm{R_s} - 2\mathrm{r})}{2\mathrm{r}^4} = \frac{-\mathrm{R_s}(\mathrm{R_s} - \mathrm{r})}{\mathrm{r}^4}$$

$$\mathbf{R_{00} \neq 0} \quad ??$$

$$\mathrm{R}_{11} = \boldsymbol{\Gamma^1_{11,1}} + \Gamma^0_{10}\Gamma^0_{01} + \Gamma^1_{11}\,\Gamma^1_{11} + \Gamma^2_{12}\,\Gamma^2_{21} + \Gamma^3_{13}\,\Gamma^3_{31}$$

$$\mathrm{R}_{11} = \frac{-\mathrm{R_s}(2\mathrm{r} - \mathrm{Rs})}{2\mathrm{r}^4\sigma^4} + \frac{R_s}{2r^2\sigma^2}\frac{R_s}{2r^2\sigma^2} + \frac{-R_s}{2r^2\sigma^2}\,\frac{-R_s}{2r^2\sigma^2} + \frac{1}{r}\,\frac{1}{r} + \frac{1}{r}\,\frac{1}{r}$$

$$\mathrm{R}_{11} = \frac{-\mathrm{R_s}(2\mathrm{r} - \mathrm{Rs})}{2\mathrm{r}^4\sigma^4} + \frac{R_s{}^2}{2\mathrm{r}^4\sigma^4} + \frac{2}{r^2}$$

$$\mathrm{R}_{11} = \frac{-\mathrm{R_s}(2\mathrm{r} - \mathrm{Rs})}{2\mathrm{r}^4\sigma^4} + \frac{R_s{}^2}{2\mathrm{r}^4\sigma^4} + \frac{4\left(r^2 + \mathrm{R_s}^2 - 2rR_s\right)}{2\mathrm{r}^4\sigma^4}$$

$$\mathrm{R}_{11} = \frac{-2\mathrm{rR_s} + \mathrm{R_s}^2}{2\mathrm{r}^4\sigma^4} + \frac{R_s{}^2}{2\mathrm{r}^4\sigma^4} + \frac{4\left(r^2 + \mathrm{R_s}^2 - 2rR_s\right)}{2\mathrm{r}^4\sigma^4}$$

$$\mathrm{R}_{11} = \frac{-2\mathrm{rR_s} + 2\mathrm{R_s}^2}{2\mathrm{r}^4\sigma^4} + \frac{4\left(r^2 + \mathrm{R_s}^2 - 2rR_s\right)}{2\mathrm{r}^4\sigma^4}$$

$$\mathrm{R}_{11} = \frac{-2\mathrm{rR_s} + 2\mathrm{R_s}^2 + 4r^2 + 4\mathrm{R_s}^2 - 8rR_s}{2\mathrm{r}^4\sigma^4}$$

$$\mathrm{R}_{11} = \frac{-10\mathrm{rR_s} + 6\mathrm{R_s}^2 + 4r^2}{2\mathrm{r}^4\sigma^4} = \frac{3\mathrm{R_s}^2 + 2r^2 - 5\mathrm{rR_s}}{\mathrm{r}^4\sigma^4}$$

$$\mathrm{R}_{11} = \frac{-10\mathrm{rR_s} + 6\mathrm{R_s}^2 + 4r^2}{2\mathrm{r}^4\sigma^4} = \frac{3\mathrm{R_s}^2 + 2r^2 - 5\mathrm{rR_s}}{\mathrm{r}^2(\mathrm{R_s}^2 + \mathrm{r}^2 - 2\mathrm{rR_s})}$$

$$\mathbf{R_{11} \neq 0} \quad ??$$

$$\mathrm{R}_{22} = \boldsymbol{\Gamma^1_{22,1}} + \boldsymbol{\Gamma^2_{22,2}} + \Gamma^1_{22}\,\Gamma^2_{12} + \Gamma^2_{21}\,\Gamma^1_{22} + \Gamma^2_{22}\,\Gamma^2_{22} + \Gamma^3_{23}\,\Gamma^3_{23}$$

$$\mathrm{R}_{22} = 1 + 0 + (-r\sigma^2)\,\frac{1}{r} + \frac{1}{r}\,(-r\sigma^2) + 0 + \frac{\cos\theta}{\sin\theta}\,\frac{\cos\theta}{\sin\theta}$$

$$\mathrm{R}_{22} = 1 - 2\sigma^2 + \frac{\cos^2\theta}{\sin^2\theta} = \frac{\sin^2\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta} - 2\sigma^2$$

$$\mathrm{R}_{22} = \frac{1}{\sin^2\theta} - 2\sigma^2$$

$$\mathbf{R_{22} \neq 0} \quad ??$$

$$\mathrm{R}_{33} = +\boldsymbol{\Gamma^1_{33,1}} + \boldsymbol{\Gamma^2_{33,2}} + \Gamma^1_{33}\,\Gamma^3_{13} + \Gamma^2_{33}\,\Gamma^3_{23} + \Gamma^3_{31}\,\Gamma^1_{33} + \Gamma^3_{32}\,\Gamma^2_{33}$$

$$\mathrm{R}_{33} = 1 + \cos^2\theta - \sin^2\theta \;\; - r\sigma^2\sin^2\theta\,\frac{1}{r} - \cos\theta\sin\theta.\,\frac{\cos\theta}{\sin\theta} + \frac{1}{r}\,(-r\sigma^2\sin^2\theta) + \frac{\cos\theta}{\sin\theta}\,(-\cos\theta\sin\theta)$$

$$\mathrm{R}_{33} = 1 + \cos^2\theta - \sin^2\theta \;\; - 2\sigma^2\sin^2\theta \;\; - 2\cos\theta\sin\theta.\,\frac{\cos\theta}{\sin\theta}$$

$$\mathrm{R}_{33} = 1 + \cos^2\theta - \sin^2\theta \;\; - 2\sigma^2\sin^2\theta \;\; - 2\cos^2\theta$$

$$\mathrm{R}_{33} = 1 - \cos^2\theta - \sin^2\theta \;\; - 2\sigma^2\sin^2\theta$$

$$\mathrm{R}_{33} = -2\sigma^2\sin^2\theta$$

$$\mathbf{R_{33} \neq 0} \quad ??$$

Thus the Schwarzschild formula with spherical/polar coordinates does not meet the limited formula of Einstein. This is not surprising because the determinant of g for the spherical coordinates is not -1, which is a requirement in order to use the limited formula.
However, as for the complete formula for the Einstein field equations is concerned, the Schwarzschild's spherical/polar coordinate equation is in agreement, as was shown above.

Note:
The limited formula was the result caused by the extra requirement that Einstein added which means that the product of the elements of the trace of the metric tensor is g=-1 ($g = \mathrm{g}_{00}.\mathrm{g}_{11}.\mathrm{g}_{22}.\mathrm{g}_{33} = -1$). This extra requirement was introduced by Einstein to make the calculations easier and to simplify his general formula. However the limited formula is in my view a limitation that ignores a number of possible solutions. So applying the general formula is in my view the best approach. This is also supported by the fact that the practical Schwarzschild equation

$$ds^2 = \sigma^2 c^2 dt^2 - \frac{dr^2}{\sigma^2} - r^2 d\theta^2 - r^2 \sin^2 \theta \, d\emptyset^2$$

has a *g* which is unequal to -1 and so does not meet the limited Einstein formula but it meets the general formula. With this formula all kind of practical general relativity problems, like bending of light, the trajectory of Mercury and so on, could be solved with this equation.

7. Answers on Questions

7.1. Derivation of the Schwarzschild formula to tau (proper time)

Question: *What is hard for me to accept in General Relativity is the differentiation to "ds". The line element is nothing else then the speed of light, times the locally measured time difference "dt_0" (ds = c. dt_0).*
I still can get dt/ds (difference in clock speed) but what signifies dx/ds ?

Answer:
We should realize that $ds = cd\tau$ and not t_0. τ is the time measured on a clock travelling with the speed of its frame. So the clock is at rest in its own frame. The time of the travelling clock with respect to a universal frame *dt* is a hypothetical time at the origin of the considered universal frame so for instance the middle of the Earth. So *dt* cannot be measured but only derived from $d\tau$, via the relation mentioned below $d\tau = \frac{\sigma}{\gamma} dt$.

The Schwarzschild formula can be split up in partial derivatives as follows:

Assume the metric tensor components as general components A, B, D and E.

$$c^2 d\tau^2 = Ac^2 dt^2 - Bdx^2 - Ddy^2 - Edz^2$$

Divide by $c^2 d\tau^2$:

$$1 = A\left(\frac{dt}{d\tau}\right)^2 - \frac{B}{c^2}\left(\frac{dx}{dt}\right)^2\left(\frac{dt}{d\tau}\right)^2 - \frac{D}{c^2}\left(\frac{dy}{dt}\right)^2\left(\frac{dt}{d\tau}\right)^2 - \frac{E}{c^2}\left(\frac{dz}{dt}\right)^2\left(\frac{dt}{d\tau}\right)^2$$

Subsequently x, y and z are divided in their own frame (here the universal frame) and appear to be velocities in that frame.

$$1 = A\left(\frac{dt}{d\tau}\right)^2\left\{1 - \frac{B}{Ac^2}\left(\frac{dx}{dt}\right)^2 - \frac{D}{Ac^2}\left(\frac{dy}{dt}\right)^2 - \frac{E}{Ac^2}\left(\frac{dz}{dt}\right)^2\right\}$$

$$v^2 = \frac{B}{A}\left(\frac{dx}{dt}\right)^2 - \frac{D}{A}\left(\frac{dy}{dt}\right)^2 - \frac{E}{A}\left(\frac{dz}{dt}\right)^2$$

$$1 = A\left(\frac{dt}{d\tau}\right)^2\left\{1 - \frac{v^2}{c^2}\right\} = \frac{A}{\gamma^2}\left(\frac{dt}{d\tau}\right)^2$$

$$\left(\frac{dt}{d\tau}\right)^2 = \frac{\gamma^2}{A}$$

Or:

$$d\tau^2 = \frac{A}{\gamma^2}dt^2 = \frac{\sigma^2}{\gamma^2}dt^2$$

$$d\tau = \frac{\sigma}{\gamma}dt$$

This is the relation between the time of the measuring clock and the time at the origin of the universal frame.
Where:

$$\sigma = \sqrt{1 - \frac{R_s}{r}} \quad and \quad \gamma = \frac{1}{\sqrt{\left(1 - \frac{v^2}{c^2}\right)}} \; and \; R_s = \frac{2\mathrm{GM}}{c^2}$$

7.2. Elucidation of translation formula of Einstein

The formula stands for the translation between two coordinate systems. The old system is denoted by x_β, so with coordinate axes x_0, x_1, x_2, x_3. The new system x_α', by x_0', x_1', x_2', x_3'. The relation between these two systems is denoted by the following formula (with covariant components):

$$dx_\alpha' = \frac{\partial x_\beta}{\partial x_\alpha'}dx_\beta$$

This formula is written according to the Einstein notation, which means a summation overβ.
This actually stands for:

$$dx_\alpha' = \frac{\partial x_0}{\partial x_\alpha'}dx_0 + \frac{\partial x_1}{\partial x_\alpha'}dx_1 + \frac{\partial x_2}{\partial x_\alpha'}dx_2 + \frac{\partial x_3}{\partial x_\alpha'}dx_3$$

Thus each new coordinate is expressed in all the old coordinates.
In total:

$$dx_0' = \frac{\partial x_0}{\partial x_0'}dx_0 + \frac{\partial x_1}{\partial x_0'}dx_1 + \frac{\partial x_2}{\partial x_0'}dx_2 + \frac{\partial x_3}{\partial x_0'}dx_3$$

$$dx_1' = \frac{\partial x_0}{\partial x_1'}dx_0 + \frac{\partial x_1}{\partial x_1'}dx_1 + \frac{\partial x_2}{\partial x_1'}dx_2 + \frac{\partial x_3}{\partial x_1'}dx_3$$

$$dx_2' = \frac{\partial x_0}{\partial x_2'}dx_0 + \frac{\partial x_1}{\partial x_2'}dx_1 + \frac{\partial x_2}{\partial x_2'}dx_2 + \frac{\partial x_3}{\partial x_2'}dx_3$$

$$dx_3' = \frac{\partial x_0}{\partial x_3'} dx_0 + \frac{\partial x_1}{\partial x_3'} dx_1 + \frac{\partial x_2}{\partial x_3'} dx_2 + \frac{\partial x_3}{\partial x_3'} dx_3$$

Could also be denoted as a tensor (tensor notation):

$$\begin{pmatrix} dx_0' \\ dx_1' \\ dx_2' \\ dx_3' \end{pmatrix} = \begin{bmatrix} \frac{\partial x_0}{\partial x_0'} & \frac{\partial x_1}{\partial x_0'} & \frac{\partial x_2}{\partial x_0'} & \frac{\partial x_3}{\partial x_0'} \\ \frac{\partial x_0}{\partial x_1'} & \frac{\partial x_1}{\partial x_1'} & \frac{\partial x_2}{\partial x_1'} & \frac{\partial x_3}{\partial x_1'} \\ \frac{\partial x_0}{\partial x_2'} & \frac{\partial x_1}{\partial x_2'} & \frac{\partial x_2}{\partial x_2'} & \frac{\partial x_3}{\partial x_2'} \\ \frac{\partial x_0}{\partial x_3'} & \frac{\partial x_1}{\partial x_3'} & \frac{\partial x_2}{\partial x_3'} & \frac{\partial x_3}{\partial x_3'} \end{bmatrix} \cdot \begin{pmatrix} dx_0 \\ dx_1 \\ dx_2 \\ dx_3 \end{pmatrix}$$

So it is only a translation from one coordinate system to another. You could use this e.g. to transpose from Schwarzschild *t, r, theta, phi* to Schwarzschild *t, x, y, z.*

7.3. Answer on question concerning Schwarzschild

Question 1: *Where is the General Relativity formula after 1916 coming from, the one with the Ricci tensor?*

In various literature $G_{\mu\nu}$ is called the Einstein tensor, but Einstein liked to keep things as simple as possible and meant with $G_{\mu\nu}$ nothing else then

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$$

In this formula the Ricci tensor was always there. The Ricci scalar *R* is related to $R_{\mu\nu}$ according to:

$$R = g^{\mu\nu} R_{\mu\nu} = g^{00} R_{00} + g^{11} R_{11} + g^{22} R_{22} + g^{33} R_{33}$$

By multiplying $G_{\mu\nu}$ with $g^{\mu\nu}$ we get:

$$g^{\mu\nu} G_{\mu\nu} = g^{\mu\nu} R_{\mu\nu} - \frac{1}{2} g^{\mu\nu} g_{\mu\nu} R = R - \frac{1}{2} 4R = -R$$

The total Einstein formula is

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}$$

Here is $R_{\mu\nu}$ the Ricci tensor, $g_{\mu\nu}$ the metric tensor, *G* is the gravitational constant and $T_{\mu\nu}$ the energy-momentum tensor.

When we are outside a sphere then there is no mass and energy of matter, in that case the mass-energy-momentum tensor $T_{\mu\nu} = 0$ and consequently

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0$$

We know that

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} g^{\mu\nu} R_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} 4R_{\mu\nu} = -R_{\mu\nu}$$

So

$$\boldsymbol{G_{\mu\nu} = 0 \; only \; if \; R = 0 \; and \; consequently \; R_{\mu\nu} = 0}$$

Einstein tried to describe the curvature of space-time and used the work of Riemann who has done this for curved surfaces. The Riemann tensor, for instance, is: $R_{\mu\beta\rho\nu}$. This is a rank four tensor and is hardly possible to imagine (a rank three tensor is a cube with elements, but four becomes more difficult to imagine). As the mass-energy-momentum $T_{\mu\nu}$ has two indices the Riemann tensor has also to be converted from four indices to two. With the aid of the metric tensor the covariant Riemann tensor can be changed to a partial contra-variant form:

$$R^{\beta}{}_{\mu\rho\nu} = g^{\beta\beta} R_{\mu\beta\rho\nu}$$

This is necessary in order to perform the desired contraction. By posing $\beta = \rho$ the contraction can be done with the result that we get the Ricci tensor $R_{\mu\nu}$.

$$R^{\beta}{}_{\mu\beta\nu} = R_{\mu\nu}$$

So here the Ricci tensor is the trace of the Riemann tensor and apparently many elements of the Riemann tensor are superfluous. This step is not very clear, the fact that without consequences these elements can be ignored. The relation with Riemann can still be seen in the Ricci tensor elements and Christoffel symbols:

Ricci: $$\mathrm{R}_{\mu\nu} = \mathrm{R}^{\rho}_{\mu\rho\nu} = \Gamma^{\rho}_{\mu\nu,\rho} - \Gamma^{\rho}_{\rho\mu,\nu} + \Gamma^{\rho}_{\rho\lambda}\,\Gamma^{\lambda}_{\nu\mu} - \Gamma^{\rho}_{\nu\lambda}\,\Gamma^{\lambda}_{\rho\mu}$$

Christoffel symbol: $$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\alpha}\left\{\frac{\partial g_{\nu\alpha}}{\partial x^{\mu}} + \frac{\partial g_{\mu\alpha}}{\partial x^{\nu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\alpha}}\right\}$$

Derivative of Christoffel symbol: $$\Gamma^{\rho}_{\mu\nu,\gamma} = \frac{\partial \Gamma^{\rho}_{\mu\nu}}{\partial x^{\gamma}} = -g^{\rho\alpha}\cdot\frac{\partial g_{\rho\alpha}}{\partial x^{\gamma}}\cdot\Gamma^{\rho}_{\mu\nu} + \frac{1}{2} g^{\rho\alpha}\left\{\frac{\partial^2 g_{\nu\alpha}}{\partial x^{\mu}\partial x^{\gamma}} + \frac{\partial^2 g_{\mu\alpha}}{\partial x^{\nu}\partial x^{\gamma}} - \frac{\partial^2 g_{\mu\nu}}{\partial x^{\alpha}\partial x^{\gamma}}\right\}$$

When I calculate, by means of my program, whether the Ricci elements $R_{00}, R_{11}, R_{22}, R_{33}$ are zero, with the general Einstein formula, then the result is correct, also theoretically, but with the formula, with the limited formula of the Einstein field equations (*g=-1*) then the result is not correct. I have not worked it out yet, but I am convinced that it will be correct because Schwarzschild derived his equation from, so based it on, Einstein's general formula for the field equations. Thus by filling in the Schwarzschild result in the general Einstein formula the result shall be correct.

Further investigated:

Schwarzschild uses the well known polar equation. The determinant of the metric tensor (here the product of the coefficients) is not -1. This polar equation meets the Einstein field equations but not the limited version, because in the latter *g=-1* is required. Schwarzschild has derived a transformation, based on adapted polar coordinates, where he choose the transformation such that the *g=-1* is met. In that case the equation also meets the limited Einstein field equations. Although Schwarzschild tried to meet the wish of Einstein to have the metric trace *g=-1* in my opinion the only relevant issue is that the Einstein Field equations, where $T_{\mu\nu} = 0$, and thus $R_{00} = R_{11} = R_{22} = R_{33} = 0$ is met regardless of whether $g = -1$ *or* $g \neq -1$. So the requirement of $g = -1$ is an unnecessary limitation

Question 2: *The consequence of the difference in formulae is big. In your document I count nine Christoffel symbols, while Karl Schwarzschild found 10. In yours 222 seems to be absent. This is because your definition of the metric tensor g differs from the one of Schwarzschild,* g_{22} *and* g_{33} *are -1 for Schwarzschild, while you add the coordinate r (e.g.* $g_{22} = -r^2$*). Also Droste (1917), Eddington (1921), MWT (1975) and OAS (2007) kept themselves to g=-1 for the Schwarzschild solution so that:* $g_{22} = g_{33} = -1$*.This raises the question for me to you: do you think that g=-1 is required for the Schwarzschild solution?*

In first instance, Schwarzschild derived his equation from the Cartesian axes system x, y, z. In that case the result is a metric tensor with the following term:

$$g_{00} = \sigma^2 \quad g_{11} = -\frac{1}{r^4\sigma^2} \quad g_{22} = -\frac{r^2}{\sin^2\theta} \quad g_{33} = -r^2\sin^2\theta$$

In that case 10 (14) relevant Christoffel symbols are created.
Also in my formulae overview you see that I derived formulae for the spherical form as well as for the x, y, z form. In the x, y, z form the 222 does exist.
However for the spherical form this is different, there the metric tensor elements are:

$$g_{00} = \sigma^2 \quad g_{11} = \frac{-1}{\sigma^2} \quad g_{22} = -r^2 \quad g_{33} = -r^2\sin^2\theta$$

This holds for Schwarzschild as well! The elements g_{22} and g_{33} cannot be -1 because in that case $\frac{\partial g_{22}}{\partial r}, \frac{\partial g_{22}}{\partial \theta}, \frac{\partial g_{33}}{\partial r}, \frac{\partial g_{33}}{\partial \theta}$ would be zero and the number of Christoffel symbols would be limited to 001 (and 010), 100 and 111.
In case of spherical 222 is indeed zero because g_{22} is independent of θ and so the derivative is zero:

$$\Gamma_{22}^{2} = \frac{1}{2}g^{22}\left\{\frac{\partial g_{22}}{\partial x^2}\right\} = 0$$

A remark has to be made that in case it is stated that $\theta = 90^0$this should be done at the end of the calculations. For instance:

$$\Gamma_{33}^{2} = \frac{1}{2}g^{22}\left\{-\frac{\partial g_{33}}{\partial \theta}\right\} = -\boldsymbol{\cos\theta\sin\theta = 0} \ \ when\ \theta = 90^0$$

But for the Ricci element also the derivative of this Christoffel symbol is needed and that is:

$$\frac{\partial \Gamma_{33}^{2}}{\partial \theta} = -\boldsymbol{\cos^2\theta + \sin^2\theta = 1} \ \ when\ \theta = 90^0$$

And not zero! And that goes for some other terms as well.
Why Einstein introduced the limitation of det(*g*)=-1 I have no idea, except that over the whole the calculations become simpler in form and more symmetrical. But it leads, in my opinion, to an unnecessary limitation. But it also depends on what kind of coordinate system is chosen. For instance the metric tensor element of t, x, y, z produces indeed a det(g) of -1.

$$\sigma^2.\left(-\frac{1}{r^4\sigma^2}\right).\left(-\frac{r^2}{\sin^2\theta}\right).(-r^2\sin^2\theta) = -1$$

But with spherical coordinates it is:

$$\sigma^2.\frac{-1}{\sigma^2}.(-r^2).(-r^2\sin^2\theta) = -\ r^4\sin^2\theta$$

And thus $\det(g) \neq -1$.

Question 3: *The field equations in your document on page 2 and 3 on basis of the Ricci tensor differ strongly from those that we (and Karl Schwarzschild) in appendix E on basis of the G tensor have used. You also mentioned the G tensor in your document on page 9. My question is: should the result not the same?*

I mentioned the G tensor formula also in my formulae overview to have it by hand and for comparison reason, but I have not checked it theoretically. In my calculations with my program in Excel, I also used the G formula but it never yielded $\mathrm{R}_{00} = 0$ a.o. But I have to try it out any further. As I said before it has to and it certainly will be correct because otherwise Schwarzschild would never have come to this configuration.

Question 4: *I still have some difficulty in understanding the Schwarzschild equation and Einstein's field equations. Could you elaborate a bit on this issue?*

I am afraid we will get entangled into the same discussion as last time. I am not trying to defend the Schwarzschild/Einstein solution and criticize your approach on the proposed modification of the Schwarzschild equation. But I am just trying to understand it all and if I do not completely understand Schwarzschild then I keep on striving for finding the right answer before starting to modify his solution. I only modify when I see and understand a possible error in his equation.
So let us first scrutinize the Schwarzschild equation before we dive into Einstein. I do not pretend to know it all but here I will elucidate how I understand it up till now.

Einstein endeavored to find a coordinate system such that no gravitational forces were felt. So instead of a Cartesian coordinate system, where there are gravitational forces due to mass and consequently acceleration of a particle, Einstein wanted a curved coordinate frame in such a way that the particle follows a trajectory as if there was no force present. And if there is no force then the particle follows a voluntary trajectory, a geodesic, or perhaps you may call this a "straight" line in this new frame. According to Newton that if something has a momentum it will keep on moving along a straight line. And also in the Einstein curved frame, where there is no force but there is momentum, the particle follows a geodesic and that has to be, in that frame, a straight" line.

Einstein has tried to find an equation for every arbitrary coordinate frame. He found for a space-time outside a mass the following formula:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0$$

Here is the formula independent of the chosen coordinate frame. But for instance the term $R_{\mu\nu}$ is independent of the chosen frame but is built up out of the squares of its coordinates (times the coefficients). And these coordinates and its coefficients do change in accordance with the chosen frame, though the total $R_{\mu\nu}$ stays the same.

So in general one could chose any frame one likes as long as the coordinates together with their appropriate coefficients leads to the same result.

It is comparable with a line segment that can be calculated by using the Pythagoras formula $c^2 = a^2 + b^2$. Here is *c* constant and one could change *a* and *b* as much as one wishes as long as the sum of the squares stays the same.
Also coordinate frames could be used of which the coordinates are not rectangular but form an angle *phi* with each other, but that has to be expressed in the coefficients of that frame. As example we could use $c^2 = a^2 + b^2 - 2ab \cos \emptyset$ the cosine rule where we get a cross product *ab*. Here the coordinates are not curved but straight lines but are not mutual rectangular.

Einstein depicts now the curved space-time as built up out of infinitesimal small linear line segments that form together a line or coordinate frame. So he sees every, infinitesimal, location in space-time as consisting of a rectangular coordinate system. However this relation of coordinates and coefficients is for every shift of location again depending on the curvature at that new location.
Due to the fact that Einstein looked for a solution for every possibly curved and non rectangular coordinate frame, there arises not only a relation of the line segment with the coordinates but also with all possible cross products of the coordinates.
Although the coordinate frame is limited to one location the relation between line segment and coordinate frame comprises all the information over all space-time. This information is contained in the so called metric that are the coefficients that in this relation are attached to every coordinate and the possible cross products of these coordinates.
By working in such an infinitesimal way the total, curved line is split up in small linear segments and in this way there arises locally a linear relation.

Schwarzschild tried to find a solution that was more practical and decided that the local coordinate frame would consist of only coordinates that were mutually rectangular. In this way the cross products disappeared. Subsequently he came to the following polar equation for a line segment in vacuum:

$$ds^2 = \left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta^2\, d\emptyset^2$$

So we see here a small line segment *ds* that has a relation with the four coordinates $(t, r, \theta, \emptyset)$. The size of dependency of the line segment with the coordinates is determined by the coefficients. If we consider here the effect of the sun (M) on the space-time then it can be seen that the coefficients are dependent on the location by means of *r* and θ. So the small line segment on every location is determined by the coordinates; and the weight of this determinacy is different for each location, but the total sum always leads to the same small line segment.

By taking an integral over the line segment we get a summation of all those infinitesimal small line segments depicting the total trajectory.

7.4. Detailed derivation of the Einstein equation (57) from equation (53)

Question:
I am reading Einstein's original GR paper. I've attached it as a PDF to this email. (Einstein, Relativity: The Special and General Theory, 1916 (this revised edition: 1924)) (Einstein, The Collected Papers of Albert Einstein, 1997)
In section 18, at the bottom of page 186 of the paper (bottom left of page 22 of the PDF), there is an equation that I am trying to derive using the method Einstein suggests in the paper (multiplying Eq. 53 by the derivative of the metric tensor, and using the methods in section 15). Would you be able to derive this equation in the specific way Einstein suggests, and based solely on the previous material in Einstein's paper? Could you show me the detailed steps you took for getting to that equation using the method Einstein indicates?

Answer:

Note: the equation numbers refer to the original work of Einstein on General Relativity.

Einstein equation (53)

$$\boldsymbol{\frac{\partial \Gamma^{\alpha}_{\mu\nu}}{\partial x^{\alpha}} + \Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\alpha} = -\kappa\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)}$$

$$\boldsymbol{\sqrt{-g} = 1}$$

Multiply (53) with $\frac{\partial g^{\mu\nu}}{\partial x^{\sigma}}$:

$$\frac{\partial g^{\mu\nu}}{\partial x^{\sigma}}\left(\frac{\partial \Gamma^{\alpha}_{\mu\nu}}{\partial x^{\alpha}} + \Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\alpha}\right) = \frac{\partial g^{\mu\nu}}{\partial x^{\sigma}}\left(-\kappa\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)\right) =$$

$$= -\kappa\left(\frac{\partial g^{\mu\nu}}{\partial x^{\sigma}}T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\frac{\partial g^{\mu\nu}}{\partial x^{\sigma}}T\right)$$

From equation Einstein (29)

$$\frac{1}{\sqrt{-g}}\frac{\partial\sqrt{-g}}{\partial x^{\sigma}} = -\frac{1}{2}g_{\mu\nu}\frac{\partial g^{\mu\nu}}{\partial x^{\sigma}}$$

In case g=-1 then:

$$-\frac{1}{2}g_{\mu\nu}\frac{\partial g^{\mu\nu}}{\partial x^{\sigma}} = 0$$

So filled in:

$$= -\kappa\left(\frac{\partial g^{\mu\nu}}{\partial x^{\sigma}}T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\frac{\partial g^{\mu\nu}}{\partial x^{\sigma}}T\right) = -\kappa\left(\frac{\partial g^{\mu\nu}}{\partial x^{\sigma}}T_{\mu\nu} - 0\right) = -\kappa\frac{\partial g^{\mu\nu}}{\partial x^{\sigma}}T_{\mu\nu}$$

$$\frac{\partial g^{\mu\nu}}{\partial x^{\sigma}}\frac{\partial \Gamma^{\alpha}_{\mu\nu}}{\partial x^{\alpha}} + \frac{\partial g^{\mu\nu}}{\partial x^{\sigma}}\Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\alpha} + \kappa\frac{\partial g^{\mu\nu}}{\partial x^{\sigma}}T_{\mu\nu} = 0$$

$$\frac{1}{2\kappa}\left(\frac{\partial g^{\mu\nu}}{\partial x^{\sigma}}\frac{\partial \Gamma^{\alpha}_{\mu\nu}}{\partial x^{\alpha}} + \frac{\partial g^{\mu\nu}}{\partial x^{\sigma}}\Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\alpha}\right) + \frac{1}{2}\frac{\partial g^{\mu\nu}}{\partial x^{\sigma}}T_{\mu\nu} = 0 \tag{1}$$

$$\frac{1}{2\kappa}\left(\frac{\partial}{\partial x^{\alpha}}(-2\kappa t^{\alpha}_{\sigma})\right) + \frac{1}{2}\frac{\partial g^{\mu\nu}}{\partial x^{\sigma}}T_{\mu\nu} = 0$$

See for the work out of the yellow step, under the dotted line below.

$$-\frac{\partial t^{\alpha}_{\sigma}}{\partial x^{\alpha}} + \frac{1}{2}\frac{\partial g^{\mu\nu}}{\partial x^{\sigma}}T_{\mu\nu} = 0 \tag{2}$$

Equation from Einstein (56):

$$\boldsymbol{\frac{\partial\left(t^{\sigma}_{\mu} + T^{\sigma}_{\mu}\right)}{\partial x^{\sigma}} = 0}$$

$$\frac{\partial t^{\sigma}_{\mu}}{\partial x^{\sigma}} = -\frac{\partial T^{\sigma}_{\mu}}{\partial x^{\sigma}}$$

Replace σ with α, and replace μ with σ:

$$\frac{\partial t^{\alpha}_{\sigma}}{\partial x^{\alpha}} = -\frac{\partial T^{\alpha}_{\sigma}}{\partial x^{\alpha}}$$

The equation (2) becomes:

$$\boldsymbol{\frac{\partial T^{\alpha}_{\sigma}}{\partial x^{\alpha}} + \frac{1}{2}\frac{\partial g^{\mu\nu}}{\partial x^{\sigma}}T_{\mu\nu} = 0} \tag{57}$$

==

Work-out yellow step:
To proof that:

$$\frac{\partial}{\partial x^\alpha}(-2\kappa t_\sigma^\alpha) = \frac{\partial g^{\mu\nu}}{\partial x^\sigma}\frac{\partial \Gamma_{\mu\nu}^\alpha}{\partial x^\alpha} + \frac{\partial g^{\mu\nu}}{\partial x^\sigma}\Gamma_{\mu\beta}^\alpha \Gamma_{\nu\alpha}^\beta$$

Einstein equation (48):

$$\frac{\partial H}{\partial g^{\mu\nu}} = -\Gamma_{\mu\beta}^\alpha \Gamma_{\nu\alpha}^\beta$$

$$\frac{\partial H}{\partial g_\sigma^{\mu\nu}} = \Gamma_{\mu\nu}^\sigma$$

Einstein equation (47b):

$$\frac{\partial}{\partial x^\alpha}\left(\frac{\partial H}{\partial g_\alpha^{\mu\nu}}\right) - \frac{\partial H}{\partial g^{\mu\nu}} = 0 => \frac{\partial}{\partial x^\alpha}\left(\frac{\partial H}{\partial g_\alpha^{\mu\nu}}\right) = \frac{\partial H}{\partial g^{\mu\nu}}$$

$$\frac{\partial g^{\mu\nu}}{\partial x^\sigma}\frac{\partial \Gamma_{\mu\nu}^\alpha}{\partial x^\alpha} + \frac{\partial g^{\mu\nu}}{\partial x^\sigma}\Gamma_{\mu\beta}^\alpha \Gamma_{\nu\alpha}^\beta$$

$$\frac{\partial g^{\mu\nu}}{\partial x^\sigma}\frac{\partial}{\partial x^\alpha}\left(\frac{\partial H}{\partial g_\alpha^{\mu\nu}}\right) - \frac{\partial g^{\mu\nu}}{\partial x^\sigma}\frac{\partial H}{\partial g^{\mu\nu}}$$

$$\frac{\partial}{\partial x^\alpha}\left(g_\sigma^{\mu\nu}\frac{\partial H}{\partial g_\alpha^{\mu\nu}}\right) - \frac{\partial\left(g_\sigma^{\mu\nu}\right)}{\partial x^\alpha}\frac{\partial H}{\partial g_\alpha^{\mu\nu}} - \frac{\partial g^{\mu\nu}}{\partial x^\sigma}\frac{\partial H}{\partial g^{\mu\nu}}$$

Here is:

$$\frac{\partial\left(g_\sigma^{\mu\nu}\right)}{\partial x^\alpha} = \frac{\partial^2 g^{\mu\nu}}{\partial x^\alpha \partial x^\sigma} = \frac{\partial\left(g_\alpha^{\mu\nu}\right)}{\partial x^\sigma}$$

Fill in:

$$\frac{\partial}{\partial x^\alpha}\left(g_\sigma^{\mu\nu}\frac{\partial H}{\partial g_\alpha^{\mu\nu}}\right) - \frac{\partial\left(g_\alpha^{\mu\nu}\right)}{\partial x^\sigma}\frac{\partial H}{\partial g_\alpha^{\mu\nu}} - \frac{\partial H}{\partial g^{\mu\nu}}\frac{\partial g^{\mu\nu}}{\partial x^\sigma}$$

As mentioned in Einstein's document under equation (47a) is *H* regarded as a function of $g^{\mu\nu}$ and $g_\sigma^{\mu\nu}$ $\left(= \frac{\partial g^{\mu\nu}}{\partial x^\sigma}\right)$ thus:

$$\frac{\partial H}{\partial x^\sigma} = \frac{\partial H}{\partial g_{,\alpha}^{\mu\nu}}\frac{\partial g_{,\alpha}^{\mu\nu}}{\partial x^\sigma} + \frac{\partial H}{\partial g^{\mu\nu}}\frac{\partial g^{\mu\nu}}{\partial x^\sigma}$$

Fill in:

$$\frac{\partial}{\partial x^\alpha}\left(g_\sigma^{\mu\nu}\frac{\partial H}{\partial g_\alpha^{\mu\nu}}\right) - \frac{\partial H}{\partial x^\sigma}$$

$$\frac{\partial}{\partial x^\alpha}\left(g_\sigma^{\mu\nu}\frac{\partial H}{\partial g_\alpha^{\mu\nu}}\right) - \frac{\partial}{\partial x^\sigma}(H)$$

$$\frac{\partial}{\partial x^\alpha}\left(g_\sigma^{\mu\nu}\frac{\partial H}{\partial g_\alpha^{\mu\nu}}\right) - \frac{\partial}{\partial x^\alpha}(\delta_\sigma^\alpha H)$$

$$\frac{\partial}{\partial x^\alpha}\left(g_\sigma^{\mu\nu}\frac{\partial H}{\partial g_\alpha^{\mu\nu}} - \delta_\sigma^\alpha H\right)$$

Einstein equation (49):

$$-2\kappa t_\sigma^\alpha = g_\sigma^{\mu\nu} \frac{\partial H}{\partial g_\alpha^{\mu\nu}} - \delta_\sigma^\alpha H$$

Fill this in (1):

$$\frac{1}{2\kappa}\left(\frac{\partial g^{\mu\nu}}{\partial x^\sigma}\frac{\partial \Gamma_{\mu\nu}^\alpha}{\partial x^\alpha} + \frac{\partial g^{\mu\nu}}{\partial x^\sigma}\Gamma_{\mu\beta}^\alpha \Gamma_{\nu\alpha}^\beta\right) + \frac{1}{2}\frac{\partial g^{\mu\nu}}{\partial x^\sigma} T_{\mu\nu} = 0$$

$$\frac{1}{2\kappa}\left\{\frac{\partial}{\partial x^\alpha}(-2\kappa t_\sigma^\alpha)\right\} + \frac{1}{2}\frac{\partial g^{\mu\nu}}{\partial x^\sigma} T_{\mu\nu} = 0$$

q.e.d.

7.5. Question on Equation in Einstein original work (English version)

Question:

I'm attaching the PDF of Einstein's paper again for your reference. (Einstein, Relativity: The Special and General Theory, 1916 (this revised edition: 1924)) (Einstein, The Collected Papers of Albert Einstein, 1997)
On the bottom line of page 191, there are three terms separated by equal signs. I can't justify the first equals sign, i.e., I can't see how the first term is equal to the second term. Einstein says to use equation (60), but I'm not having any luck.
Are you able to figure out why those two terms are equal?

Answer:

First we check the equation (60) in the original (German) Einstein paper.

On page 812 of the original, German, Einstein paper there is probably an error in equation (60):

$$\frac{\partial F_{\varrho\sigma}}{\partial x^\tau} + \frac{\partial F_{\sigma\tau}}{\partial x^\varrho} + \frac{\partial F_{\tau\varrho}}{\partial x^\varrho} = 0$$

This should most likely be:

$$\frac{\partial F_{\varrho\sigma}}{\partial x^\tau} + \frac{\partial F_{\sigma\tau}}{\partial x^\varrho} + \frac{\partial F_{\tau\varrho}}{\partial x^\sigma} = 0 \qquad (60)$$

In the English translation (page 189) it is already corrected.
The last equation on page 191 (page 814 of original German version):

$$F^{\mu\nu}\frac{\partial F_{\sigma\mu}}{\partial x^\nu} = -\frac{1}{2}F^{\mu\nu}\frac{\partial F_{\mu\nu}}{\partial x^\sigma} = -\frac{1}{2}g^{\mu\alpha} g^{\nu\beta}\frac{\partial F_{\mu\nu}}{\partial x^\sigma} \qquad (1)$$

To proof the validity of the equal sign between the two left terms leads to:
According to equation (60):

$$\frac{\partial F_{\mu\nu}}{\partial x^\sigma} + \frac{\partial F_{\nu\sigma}}{\partial x^\mu} + \frac{\partial F_{\sigma\mu}}{\partial x^\nu} = 0$$

$$\frac{\partial F_{\sigma\mu}}{\partial x^\nu} = -\frac{\partial F_{\mu\nu}}{\partial x^\sigma} - \frac{\partial F_{\nu\sigma}}{\partial x^\mu}$$

Together with (1):

$$F^{\mu\nu}\frac{\partial F_{\sigma\mu}}{\partial x^\nu} = F^{\mu\nu}\left(-\frac{\partial F_{\mu\nu}}{\partial x^\sigma} - \frac{\partial F_{\nu\sigma}}{\partial x^\mu}\right) = -F^{\mu\nu}\frac{\partial F_{\mu\nu}}{\partial x^\sigma} - F^{\mu\nu}\frac{\partial F_{\nu\sigma}}{\partial x^\mu}$$

$$= -\frac{1}{2}F^{\mu\nu}\frac{\partial F_{\mu\nu}}{\partial x^{\sigma}} - \frac{1}{2}F^{\mu\nu}\frac{\partial F_{\mu\nu}}{\partial x^{\sigma}} - \frac{1}{2}F^{\mu\nu}\frac{\partial F_{\nu\sigma}}{\partial x^{\mu}} - \frac{1}{2}F^{\mu\nu}\frac{\partial F_{\nu\sigma}}{\partial x^{\mu}}$$

$$= -\frac{1}{2}F^{\mu\nu}\frac{\partial F_{\mu\nu}}{\partial x^{\sigma}} - \frac{1}{2}\left(F^{\mu\nu}\frac{\partial F_{\mu\nu}}{\partial x^{\sigma}} + F^{\mu\nu}\frac{\partial F_{\nu\sigma}}{\partial x^{\mu}} + F^{\mu\nu}\frac{\partial F_{\nu\sigma}}{\partial x^{\mu}}\right)$$

Swapping of the dummy indices of the term at the right hand side:

$$= -\frac{1}{2}F^{\mu\nu}\frac{\partial F_{\mu\nu}}{\partial x^{\sigma}} - \frac{1}{2}\left(F^{\mu\nu}\frac{\partial F_{\mu\nu}}{\partial x^{\sigma}} + F^{\mu\nu}\frac{\partial F_{\nu\sigma}}{\partial x^{\mu}} + F^{\nu\mu}\frac{\partial F_{\mu\sigma}}{\partial x^{\nu}}\right)$$

Interchanging the indices of $F^{\nu\mu}$ and changing sign:

$$= -\frac{1}{2}F^{\mu\nu}\frac{\partial F_{\mu\nu}}{\partial x^{\sigma}} - \frac{1}{2}\left(F^{\mu\nu}\frac{\partial F_{\mu\nu}}{\partial x^{\sigma}} + F^{\mu\nu}\frac{\partial F_{\nu\sigma}}{\partial x^{\mu}} - F^{\mu\nu}\frac{\partial F_{\mu\sigma}}{\partial x^{\nu}}\right)$$

Interchanging the indices of $\frac{\partial F_{\sigma\mu}}{\partial x^{\nu}}$ and changing sign:

$$= -\frac{1}{2}F^{\mu\nu}\frac{\partial F_{\mu\nu}}{\partial x^{\sigma}} - \frac{1}{2}\left(F^{\mu\nu}\frac{\partial F_{\mu\nu}}{\partial x^{\sigma}} + F^{\mu\nu}\frac{\partial F_{\nu\sigma}}{\partial x^{\mu}} + F^{\mu\nu}\frac{\partial F_{\sigma\mu}}{\partial x^{\nu}}\right)$$

$$= -\frac{1}{2}F^{\mu\nu}\frac{\partial F_{\mu\nu}}{\partial x^{\sigma}} - \frac{1}{2}F^{\mu\nu}\left(\frac{\partial F_{\mu\nu}}{\partial x^{\sigma}} + \frac{\partial F_{\nu\sigma}}{\partial x^{\mu}} + \frac{\partial F_{\sigma\mu}}{\partial x^{\nu}}\right)$$

The right hand side is equation (60) and is zero. This leaves:

$$F^{\mu\nu}\frac{\partial F_{\sigma\mu}}{\partial x^{\nu}} = -\frac{1}{2}F^{\mu\nu}\frac{\partial F_{\mu\nu}}{\partial x^{\sigma}}$$

q.e.d.

7.6. Question on Einstein equation (69)

Question:

Confusion about Einstein equation (69):

$$k = \frac{8\pi K}{c^2} = 1.87.10^{-27} \tag{E69}$$

Answer:

Einstein worked with centimeters and grams (CGS units). Now we use meters and kg (MKS units). So there is a discrepancy with the units and shall be corrected. Furthermore Einstein is here not consistent because he has set c=1 but in the formula there is a c^2.

If we correct this to current customary units, we get:

$$k = \frac{8\pi G}{c^4} \approx 2.07.10^{-43}$$

So K and G are both the gravitational constant but with different units

Appendix 1 General Relativity formulae

Below, in appendix 1.1 we summarize a number of previously derived formulae. In appendix 1.2 and 1.3 we derive all the formulae that are relevant for calculations in various chapters.

Appendix 1.1 Overview General Relativity - and Schwarzschild formulae

Einstein notation is applied.

$$dx^m = \frac{\partial x^m}{\partial y^r} dy^r$$
$$ds^2 = \eta_{mn} d\xi^m d\xi^n$$
$$ds^2 = g_{mn}(x) dx^m dx^n = g_{pq}(y) dy^p dy^q$$
$$g_{pq}(y) = g_{mn}(x) \frac{dx^m}{dy^p} \frac{dx^n}{dy^q}$$
$$V'^n(y) = \frac{\partial y^n}{\partial x^m} V^m(x)$$
$$W'_p(y) = \frac{\partial x^q}{\partial y^p} W'_q$$
$$T_{mn}(x) = \frac{\partial V^m(x)}{\partial x^n}$$
$$T_{mn}(y) = \frac{\partial x^r}{\partial y^m} \frac{\partial x^s}{\partial y^n} T_{rs}(x)$$
$$T^{mn}(y) = \frac{\partial y^m}{\partial x^r} \frac{\partial y^n}{\partial x^s} T^{rs}(x)$$
$$T^{rs}(x) = A^r_x B^s_x$$
$$E_\mu = g_{\mu\vartheta} E^\vartheta$$
$$E^\mu = g^{\mu\vartheta} E_\vartheta = g^{\mu\vartheta} g_{\vartheta\rho} E^\rho = \delta^\mu_\rho E^\rho = E^\mu$$

Lijnsegment in klein gebied geldt: Pythagoras:

$$ds^2 = \delta_{mn} \frac{\partial x^m}{\partial y^n} dy^n . \frac{\partial x^n}{\partial y^s} dy^s$$

Transformeren naar ander frame:

$$ds^2 = \boxed{\delta_{mn} \frac{\partial x^m}{\partial y^r} . \frac{\partial x^n}{\partial y^s}} dy^r dy^s$$

$$\boldsymbol{metric\ tensor}: \ g_{mn} = \delta_{mn} \frac{\partial x^m}{\partial y^r} . \frac{\partial x^n}{\partial y^s}$$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

Geodesic equation:

$$\frac{d^2 x^\lambda}{d\tau^2} + \Gamma^\lambda_{\mu\nu} . \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = 0 \qquad \Gamma^\lambda_{\mu\nu} \equiv \frac{\partial x^\lambda}{\partial \xi^\alpha} \frac{\partial^2 \xi^\alpha}{\partial x^\mu \partial x^\nu}$$
$$T'_{\mu\vartheta}(y) = \frac{\partial x^\alpha}{\partial y^\mu} \frac{\partial x^\beta}{\partial y^\vartheta} T_{\alpha\beta}$$
$$T'^{\mu\vartheta}(y) = \frac{\partial y^\mu}{\partial x^\alpha} \frac{\partial y^\vartheta}{\partial x^\beta} T_{\alpha\beta}$$
$$T'^{\vartheta}_{\mu}(y) = \frac{\partial x^\alpha}{\partial y^\mu} \frac{\partial y^\vartheta}{\partial x^\beta} T^\beta_\alpha$$
$$g_{\mu\alpha} g^{\alpha\vartheta} = \delta^\vartheta_\mu$$

Contraction:

$$A^\mu = g^{\mu\vartheta} A_\vartheta$$
$$A_\mu = g_{\mu\vartheta} A^\vartheta$$
$$so: \ A.B = g_{\mu\vartheta} A^\mu B^\vartheta \equiv A_\vartheta B^\vartheta$$

Ricci Tensor:

$$\mathrm{R}_{\mu\nu} = \mathrm{R}^\rho_{\mu\rho\nu} = \Gamma^\rho_{\mu\nu,\rho} - \Gamma^\rho_{\rho\mu,\nu} + \Gamma^\rho_{\rho\lambda} \Gamma^\lambda_{\nu\mu} - \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\rho\mu}$$
$$G_{\mu\nu} = \Gamma^\rho_{\mu\nu,\rho} - \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\rho\mu} \ only\ if\ g = \det(g_{\mu\nu}) = -1$$

Christoffel symbol:

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\alpha} \left\{ \frac{\partial g_{\nu\alpha}}{\partial x^\mu} + \frac{\partial g_{\mu\alpha}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\alpha} \right\}$$

Ricci scalar:

$$g^{\mu\nu} \mathrm{R}_{\mu\nu} = R^\mu_\mu$$
$$R = g^{ab} (\Gamma^c_{ab,c} - \Gamma^c_{ac,b} + \Gamma^d_{ab} \Gamma^c_{cd} - \Gamma^d_{ac} \Gamma^c_{bd})$$
$$R = 2 g^{ab} (\Gamma^c_{a[b,c]} + \Gamma^d_{a[b} \Gamma^c_{c]d})$$

Appendix 1.2 Schwarzschild metric - polar coordinates

Schwarzschild metric

$$ds^2 = \sigma^2 c^2 dt^2 - \frac{dr^2}{\sigma^2} - r^2 d\theta^2 - r^2 \sin^2\theta\, d\emptyset^2$$

$\sigma^2 = 1 - \frac{R_s}{r}$ *here is*: $R_s = \frac{2GM}{c^2}$

$g_{00} = g_{tt}$ $\quad g_{22} = g_{\theta\theta}$

$g_{11} = g_{rr}$ $\quad g_{33} = g_{\emptyset\emptyset}$

Schwarzschild on polar coordinates (in plane $\theta = 90^0$)

$g_{00} = \boldsymbol{\sigma^2}$ $\quad g^{00} = \frac{\mathbf{1}}{\boldsymbol{\sigma^2}}$

$g_{11} = \frac{\mathbf{-1}}{\boldsymbol{\sigma^2}}$ $\quad g^{11} = -\boldsymbol{\sigma^2}$

$g_{22} = -\boldsymbol{r^2}$ $\quad g^{22} = \frac{-1}{\boldsymbol{r^2}}$

$g_{33} = -\boldsymbol{r^2 \sin^2\theta} = -r^2$ $\quad g^{33} = \frac{-1}{\boldsymbol{r^2 \sin^2\theta}} = \frac{-1}{r^2}$

$$\frac{d\sigma}{dr} = \frac{\boldsymbol{R_s}}{\boldsymbol{2r^2\sigma}}$$

Metric first derivative on spherical coordinates

$\frac{\partial g_{00}}{\partial r} = \frac{\boldsymbol{R_s}}{\boldsymbol{r^2}}$ $\quad \frac{\partial g_{11}}{\partial r} = \frac{\boldsymbol{R_s}}{\boldsymbol{r^2\sigma^4}}$

$\frac{\partial g_{22}}{\partial r} = -\boldsymbol{2r}$ $\quad \frac{\partial g_{33}}{\partial r} = -\boldsymbol{2r\sin^2\theta} = -\boldsymbol{2r}$

$\frac{\partial g_{33}}{\partial \theta} = -\boldsymbol{2r^2.\, sin(\theta) cos(\theta)} = \mathbf{0}$

Metric second derivative on spherical coordinates

$\frac{\partial^2 g_{00}}{\partial r^2} = \frac{\boldsymbol{-2R_s}}{\boldsymbol{r^3}}$ $\quad \frac{\partial^2 g_{11}}{\partial r^2} = \frac{\boldsymbol{-2R_s}}{\boldsymbol{r^3\sigma^6}}$

$\frac{\partial^2 g_{22}}{\partial r^2} = -\mathbf{2}$ $\quad \frac{\partial^2 g_{33}}{\partial r^2} = -\boldsymbol{2\sin^2\theta} = -\mathbf{2}$

$\frac{\partial^2 g_{33}}{\partial\theta\partial r} = -\boldsymbol{4r.\, sin(\theta) cos(\theta)} = \mathbf{0}$

$\frac{\partial^2 g_{33}}{\partial \theta^2} = \boldsymbol{2r^2.(\sin^2(\theta) - \cos^2(\theta))} = \boldsymbol{2r^2}$

Schwarzschild polar coordinates:

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\alpha}\left\{\frac{\partial g_{\nu\alpha}}{\partial x^{\mu}} + \frac{\partial g_{\mu\alpha}}{\partial x^{\nu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\alpha}}\right\}$$

$\Gamma^0_{01} = \Gamma^0_{10} = \frac{1}{2} g^{00}\left\{\frac{\partial g_{00}}{\partial r}\right\} = \frac{\boldsymbol{R_s}}{\boldsymbol{2r^2\sigma^2}}$

$\Gamma^1_{00} = \frac{1}{2} g^{11}\left\{-\frac{\partial g_{00}}{\partial r}\right\} = \frac{\boldsymbol{\sigma^2 R_s}}{\boldsymbol{2r^2}}$

$\Gamma^1_{11} = \frac{1}{2} g^{11}\left\{\frac{\partial g_{11}}{\partial r}\right\} = \frac{\boldsymbol{-R_s}}{\boldsymbol{2r^2\sigma^2}}$

$\Gamma^1_{22} = \frac{1}{2} g^{11}\left\{-\frac{\partial g_{22}}{\partial r}\right\} = -\boldsymbol{r\sigma^2}$

$\Gamma^1_{33} = \frac{1}{2} g^{11}\left\{-\frac{\partial g_{33}}{\partial r}\right\} = -\boldsymbol{r\sigma^2\sin^2\theta} = -\boldsymbol{r\sigma^2}$

$\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{2} g^{22}\left\{\frac{\partial g_{22}}{\partial r}\right\} = \frac{\mathbf{1}}{\boldsymbol{r}}$

$\Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{2} g^{33}\left\{\frac{\partial g_{33}}{\partial r}\right\} = \frac{\mathbf{1}}{\boldsymbol{r}}$

$\Gamma^2_{33} = \frac{1}{2} g^{22}\left\{-\frac{\partial g_{33}}{\partial \theta}\right\} = -\boldsymbol{\cos\theta\sin\theta} = \mathbf{0}$

$\Gamma^3_{23} = \Gamma^3_{32} = \frac{1}{2} g^{33}\left\{\frac{\partial g_{33}}{\partial \theta}\right\} = \frac{\boldsymbol{\cos\theta}}{\boldsymbol{\sin\theta}} = \mathbf{0}$

For **r, theta, phi** coordinates:

$\frac{\partial\Gamma^0_{01}}{\partial r} = \frac{\partial\Gamma^0_{10}}{\partial r} = \frac{\mathbf{R_s(R_s - 2r)}}{\mathbf{2r^4\sigma^4}}$

$\frac{\partial\Gamma^1_{00}}{\partial r} = \frac{\mathbf{R_s(3R_s - 2r)}}{\mathbf{2r^4}}$

$\frac{\partial\Gamma^1_{11}}{\partial r} = \frac{\mathbf{R_s(2r - Rs)}}{\mathbf{2r^4\sigma^4}}$

$\frac{\partial\Gamma^1_{22}}{\partial r} = -\mathbf{1}$

$\frac{\partial\Gamma^1_{33}}{\partial r} = -\boldsymbol{\sin^2\theta}$

$\frac{\partial\Gamma^2_{12}}{\partial r} = \frac{\partial\Gamma^2_{21}}{\partial r} = \frac{\partial\Gamma^3_{13}}{\partial r} = \frac{\partial\Gamma^3_{31}}{\partial r} = \frac{\mathbf{-1}}{\mathbf{r^2}}$

$\frac{\partial\Gamma^2_{33}}{\partial \theta} = -\boldsymbol{\cos^2\theta} + \boldsymbol{\sin^2\theta} = \mathbf{1}$

$\frac{\partial\Gamma^3_{23}}{\partial \theta} = \frac{\partial\Gamma^3_{32}}{\partial \theta} = \frac{-1}{\boldsymbol{\sin^2\theta}} = -\mathbf{1}$

Schwarzschild on r, theta, phi coordinates:

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\alpha}\left\{\frac{\partial g_{\nu\alpha}}{\partial x^{\mu}} + \frac{\partial g_{\mu\alpha}}{\partial x^{\nu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\alpha}}\right\}$$

First derivative of Christoffel symbol:

$$\frac{\partial \Gamma^{\rho}_{\mu\nu}}{\partial x^{\delta}} = \frac{1}{2}\frac{\partial g^{\rho\alpha}}{\partial x^{\delta}}\left\{\frac{\partial g_{\nu\alpha}}{\partial x^{\mu}} + \frac{\partial g_{\mu\alpha}}{\partial x^{\nu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\alpha}}\right\} + \frac{1}{2} g^{\rho\alpha}\left\{\frac{\partial^2 g_{\nu\alpha}}{\partial x^{\mu}\partial x^{\delta}} + \frac{\partial^2 g_{\mu\alpha}}{\partial x^{\nu}\partial x^{\delta}} - \frac{\partial^2 g_{\mu\nu}}{\partial x^{\alpha}\partial x^{\delta}}\right\}$$

$$\frac{\partial \Gamma^{\rho}_{\mu\nu}}{\partial x^{\delta}} = \frac{-1}{2}(g^{\rho\alpha})^2\frac{\partial g_{\rho\alpha}}{\partial x^{\delta}}\left\{\frac{\partial g_{\nu\alpha}}{\partial x^{\mu}} + \frac{\partial g_{\mu\alpha}}{\partial x^{\nu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\alpha}}\right\} + \frac{1}{2} g^{\rho\alpha}\left\{\frac{\partial^2 g_{\nu\alpha}}{\partial x^{\mu}\partial x^{\delta}} + \frac{\partial^2 g_{\mu\alpha}}{\partial x^{\nu}\partial x^{\delta}} - \frac{\partial^2 g_{\mu\nu}}{\partial x^{\alpha}\partial x^{\delta}}\right\}$$

==

Appendix 1.3 Schwarzschild metric - x,y,z coordinates

$x_0 = t_\infty \qquad dx_0 = dt_\infty$

$x_1 = \frac{r^3}{3} \qquad dx_1 = r^2.dr \qquad \frac{\boldsymbol{dr}}{\boldsymbol{dx_1}} = \frac{\mathbf{1}}{\boldsymbol{r^2}}$

$x_2 = -\cos\theta = 0 \quad dx_2 = \sin\theta\,.d\theta = d\theta \quad \frac{\boldsymbol{d\theta}}{\boldsymbol{dx_2}} = \frac{\mathbf{1}}{\mathbf{sin}\,\boldsymbol{\theta}}$

$x_3 = \emptyset \qquad dx_3 = d\emptyset$

$$ds^2 = \sigma^2 c^2 dt_\infty{}^2 - \frac{dx_1{}^2}{r^4\sigma^2} - \frac{r^2 dx_2{}^2}{\sin^2\theta} - r^2\sin^2\theta\, dx_3{}^2$$

Assume at equator level $\theta = 90^0 \Rightarrow sin\,\theta = 1$

$$\boxed{ds^2 = \sigma^2 c^2 dt_\infty{}^2 - \frac{dx_1{}^2}{r^4\sigma^2} - r^2 dx_2{}^2 - r^2 dx_3{}^2}$$

Schwarzschild metric on x,y,z

$g_{00} = \sigma^2 \qquad g^{00} = \frac{1}{\sigma^2}$

$g_{11} = -\frac{1}{r^4\sigma^2} \qquad g^{11} = -r^4\sigma^2$

$g_{22} = -\frac{r^2}{\sin^2\theta} \qquad g^{22} = -\frac{\sin^2\theta}{r^2}$

$g_{33} = -r^2\sin^2\theta = -r^2 \qquad g^{33} = \frac{\mathbf{-1}}{r^2\sin^2\theta} = \frac{-1}{r^2}$

g's are dependent on r (so x_1) and θ (so x_2):

$$\frac{dr}{dx_1} = \frac{\mathbf{1}}{\boldsymbol{r^2}} \qquad \frac{d\sigma}{dx_1} = \frac{\boldsymbol{R_s}}{\boldsymbol{2r^4\sigma}} \qquad \frac{d\theta}{dx_2} = \frac{\mathbf{1}}{\mathbf{sin}\,\boldsymbol{\theta}}$$

Metric derivative on x,y,z

$$\frac{\partial g_{00}}{\partial x_1} = \frac{\partial g_{00}}{\partial r}\frac{dr}{dx_1} = 2\sigma\frac{R_s}{2r^4\sigma} = \frac{\boldsymbol{R_s}}{\boldsymbol{r^4}}$$

$$\frac{\partial g_{11}}{\partial x_1} = \frac{\boldsymbol{4r - 3R_s}}{\boldsymbol{r^8\sigma^4}}$$

$$\frac{\partial g_{22}}{\partial x_1} = \frac{\partial g_{22}}{\partial r}\frac{dr}{dx_1} = r^{-2}\left(\frac{-2r}{\sin^2\theta}\right) = \frac{\mathbf{-2}}{\boldsymbol{r}\,\mathbf{sin^2}\,\boldsymbol{\theta}} = \frac{\mathbf{-2}}{\boldsymbol{r}}$$

$$\frac{\partial g_{33}}{\partial x_1} = r^{-2}(-2r\sin^2\theta) = \frac{\mathbf{-2\,sin^2}\,\boldsymbol{\theta}}{\boldsymbol{r}} = \frac{\mathbf{-2}}{\boldsymbol{r}}$$

$$\frac{\partial g_{22}}{\partial x_2} = \frac{2r^2\cos(\theta)}{sin^3(\theta)}.\frac{1}{sin\,\theta} = \frac{\boldsymbol{2r^2\cos(\theta)}}{\boldsymbol{sin^4(\theta)}} = \mathbf{0}$$

$$\frac{\partial g_{33}}{\partial x_2} = \frac{\partial g_{33}}{\partial\theta}\frac{d\theta}{dx_2} = \left(\boldsymbol{-2r^2.sin(\theta)co\,s(\theta)}\right)\frac{1}{sin\,\theta}$$

$$= \boldsymbol{-2.r^2.co\,s(\theta)} = \mathbf{0}$$

Metric second derivative on x,y,z coordinates

$$\frac{\partial^2 g_{00}}{\partial x_1{}^2} = \frac{\boldsymbol{-4R_s}}{\boldsymbol{r^7}} \qquad \frac{\partial^2 g_{11}}{\partial x_1{}^2} = \frac{\boldsymbol{-2\left(14r^2 + 9R_s{}^2 - 22rR_s\right)}}{\boldsymbol{r^{12}\sigma^6}}$$

$$\frac{\partial^2 g_{22}}{\partial x_1{}^2} = \frac{\mathbf{2}}{\boldsymbol{r^4 sin^2(\theta)}} = \frac{\mathbf{2}}{\boldsymbol{r^4}}$$

$$\frac{\partial^2 g_{22}}{\partial x_2{}^2} = \frac{\boldsymbol{-2r^2\left(1 + 3\cos^2(\theta)\right)}}{\boldsymbol{sin^6(\theta)}} = \boldsymbol{-2r^2}$$

$$\frac{\partial^2 g_{22}}{\partial x_1\partial x_2} = \frac{\partial^2 g_{22}}{\partial x_2\partial x_1} = \frac{\boldsymbol{4co\,s(\theta)}}{\boldsymbol{rsin^4(\theta)}} = \mathbf{0}$$

$$\frac{\partial^2 g_{33}}{\partial x_1{}^2} = \frac{\boldsymbol{2sin^2(\theta)}}{\boldsymbol{r^4}} = \frac{\mathbf{2}}{\boldsymbol{r^4}}$$

$$\frac{\partial^2 g_{33}}{\partial x_1\partial x_2} = \frac{\partial^2 g_{33}}{\partial x_2\partial x_1} = \frac{\boldsymbol{-4co\,s(\theta)}}{\boldsymbol{r}} = \mathbf{0}$$

$$\frac{\partial^2 g_{33}}{\partial x_2{}^2} = \boldsymbol{2r^2.\sin\theta}\frac{\mathbf{1}}{\mathbf{sin}\,\boldsymbol{\theta}} = \boldsymbol{2r^2}$$

Schwarzschild on x,y,z

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2}g^{\rho\alpha}\left\{\frac{\partial g_{\nu\alpha}}{\partial x^\mu} + \frac{\partial g_{\mu\alpha}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\alpha}\right\}$$

$$\Gamma^0_{01} = \Gamma^0_{10} = \frac{1}{2}g^{00}\left\{\frac{\partial g_{00}}{\partial x^1}\right\} = \frac{1}{2}\frac{1}{\sigma^2}\frac{R_s}{r^4} = \frac{\boldsymbol{R_s}}{\boldsymbol{2r^4\sigma^2}}$$

$$\Gamma^1_{00} = \frac{1}{2}g^{11}\left\{-\frac{\partial g_{00}}{\partial x^1}\right\} = \frac{1}{2}(-r^4\sigma^2)\frac{-R_s}{r^4} = \frac{\boldsymbol{R_s\sigma^2}}{\mathbf{2}}$$

$$\Gamma^1_{11} = \frac{1}{2}g^{11}\left\{\frac{\partial g_{11}}{\partial x^1}\right\} = \frac{1}{2}(-r^4\sigma^2)\frac{4r - 3R_s}{r^8\sigma^4} = \frac{\boldsymbol{3R_s - 4r}}{\boldsymbol{2r^4\sigma^2}}$$

$$\Gamma^1_{22} = \frac{1}{2}g^{11}\left\{-\frac{\partial g_{22}}{\partial x^1}\right\} = \frac{1}{2}(-r^4\sigma^2)\frac{2}{r\sin^2\theta} = \frac{\boldsymbol{-r^3\sigma^2}}{\mathbf{sin^2}\,\boldsymbol{\theta}}$$

$$= \boldsymbol{-r^3\sigma^2}$$

$$\Gamma^1_{33} = \frac{1}{2}g^{11}\left\{-\frac{\partial g_{33}}{\partial x^1}\right\} = \frac{1}{2}(-r^4\sigma^2)\frac{2\sin^2\theta}{r}$$

$$= \boldsymbol{-r^3\sigma^2}\,\mathbf{sin^2}\,\boldsymbol{\theta} = \boldsymbol{-r^3\sigma^2}$$

$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{2}g^{22}\left\{\frac{\partial g_{22}}{\partial x^1}\right\} = \frac{1}{2}\left(-\frac{\sin^2\theta}{r^2}\right)\frac{-2}{r\sin^2\theta} = \frac{\mathbf{1}}{\boldsymbol{r^3}}$$

$$\Gamma^2_{33} = \frac{1}{2}g^{22}\left\{-\frac{\partial g_{33}}{\partial x^2}\right\} = \frac{1}{2}\left(-\frac{\sin^2\theta}{r^2}\right)(2.r^2.\cos(\theta))$$

$$= -\mathbf{sin^2}\,\boldsymbol{\theta}\,\mathbf{cos}\,\boldsymbol{\theta} = 0$$

$$\Gamma^2_{22} = \frac{1}{2}g^{22}\left\{\frac{\partial g_{22}}{\partial x^2}\right\} = \frac{1}{2}\left(-\frac{\sin^2\theta}{r^2}\right)\frac{2r^2\cos(\theta)}{sin^4(\theta)}$$

$$= \frac{\boldsymbol{-co\,s(\theta)}}{\mathbf{sin^2(\boldsymbol{\theta})}} = \mathbf{0}$$

$$\Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{2} g^{33} \left\{\frac{\partial g_{33}}{\partial x^1}\right\} = \frac{1}{2}\left(\frac{-1}{r^2 \sin^2\theta}\right)\frac{-2\sin^2\theta}{r}$$

$$= \boldsymbol{\frac{1}{r^3}}$$

$$\Gamma^3_{23} = \Gamma^3_{32} = \frac{1}{2} g^{33} \left\{\frac{\partial g_{33}}{\partial x^2}\right\}$$

$$= \frac{1}{2}\left(\frac{-1}{r^2 \sin^2\theta}\right)(-2.r^2.\cos(\theta))$$

$$= \boldsymbol{\frac{\cos\theta}{\sin^2(\theta)} = 0}$$

For x,y,z coordinates:

$$\frac{\partial \Gamma^0_{01}}{\partial x_1} = \frac{\partial \Gamma^0_{10}}{\partial x_1} = \mathbf{\frac{R_s(3R_s - 4r)}{2r^8\sigma^4}}$$

$$\frac{\partial \Gamma^1_{00}}{\partial x_1} = \mathbf{\frac{{R_s}^2}{2r^4}}$$

$$\frac{\partial \Gamma^1_{11}}{\partial x_1} = \mathbf{\frac{6}{r^6\sigma^4} - \frac{10R_s}{r^7\sigma^4} + \frac{4.5{R_s}^2}{r^8\sigma^4}}$$

$$\frac{\partial \Gamma^1_{22}}{\partial x_1} = \mathbf{\frac{2R_s - 3r}{r\sin^2\boldsymbol{\theta}} = -3 + \frac{2R_s}{r}}$$

$$\frac{\partial \Gamma^1_{33}}{\partial x_1} = \mathbf{\left(-3 + \frac{2R_s}{r}\right).\sin^2\boldsymbol{\theta} \;\; = -3 + \frac{2R_s}{r}}$$

$$\frac{\partial \Gamma^2_{12}}{\partial x_1} = \frac{\partial \Gamma^2_{21}}{\partial x_1} = \frac{\partial \Gamma^3_{13}}{\partial x_1} = \frac{\partial \Gamma^3_{31}}{\partial x_1} = \mathbf{\frac{-3}{r^6}}$$

$$\frac{\partial \Gamma^2_{33}}{\partial x_1} = \frac{\partial \Gamma^2_{22}}{\partial x_1} = \frac{\partial \Gamma^3_{23}}{\partial x_1} = \frac{\partial \Gamma^3_{32}}{\partial x_1} = \mathbf{0}$$

$$\frac{\partial \Gamma^1_{22}}{\partial x_2} = \mathbf{\frac{2r^3\sigma^2\cos\boldsymbol{\theta}}{\sin^4\boldsymbol{\theta}} = 0}$$

$$\frac{\partial \Gamma^1_{33}}{\partial x_2} = \mathbf{-2r^3\sigma^2\cos\boldsymbol{\theta} = 0}$$

$$\frac{\partial \Gamma^2_{33}}{\partial x_2} = -3\,\mathbf{\cos^2\boldsymbol{\theta} + 1 = 1}$$

$$\frac{\partial \Gamma^2_{22}}{\partial x_2} = \mathbf{\frac{1 + \cos^2\boldsymbol{\theta}}{\sin^4\boldsymbol{\theta}} = 1}$$

$$\frac{\partial \Gamma^3_{23}}{\partial x_2} = \frac{\partial \Gamma^3_{32}}{\partial x_2} = \mathbf{\frac{-1 - \cos^2\boldsymbol{\theta}}{\sin^4\boldsymbol{\theta}} = -1}$$

$$R^{i}_{jkl} = \Gamma^{i}_{jl,k} - \Gamma^{i}_{jk,l} + \Gamma^{u}_{jl}\,\Gamma^{i}_{uk} - \Gamma^{u}_{jk}\,\Gamma^{i}_{ul}$$
$$\mathbf{R_{\mu\nu} = R^{\rho}_{\mu\rho\nu} = \Gamma^{\rho}_{\mu\nu,\rho} - \Gamma^{\rho}_{\mu\rho,\nu} + \Gamma^{\lambda}_{\mu\nu}\,\Gamma^{\rho}_{\lambda\rho} - \Gamma^{\lambda}_{\mu\rho}\,\Gamma^{\rho}_{\lambda\nu}}$$
$$R_{\mu\nu} = R^{\rho}_{\mu\nu\rho} = -\Gamma^{\rho}_{\mu\nu,\rho} + \Gamma^{\rho}_{\mu\rho,\nu} - \Gamma^{\lambda}_{\mu\nu}\,\Gamma^{\rho}_{\lambda\rho} + \Gamma^{\lambda}_{\mu\rho}\,\Gamma^{\rho}_{\lambda\nu}$$

After some calculations the conclusion was that in order to achieve all Ricci tensor elements being zero in vacuum the Christoffel symbol formula should start with a positive +1/2:

$$\Gamma^{\rho}_{\mu\nu} = +\frac{1}{2} g^{\rho\alpha}\left\{\frac{\partial g_{\nu\alpha}}{\partial x^{\mu}} + \frac{\partial g_{\mu\alpha}}{\partial x^{\nu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\alpha}}\right\}$$

The start sign of the Christoffel symbols has no influence on the product of the Christoffel symbols in the Ricci tensor element but only on the sign of the first two terms: the derivatives of the Christoffel symbols.

Schwarzschild symmetry

$$\begin{aligned} R_{\mu\nu} = \ & \Gamma^{0}_{\mu\nu,0} - \Gamma^{0}_{0\mu,\nu} + \Gamma^{0}_{0\lambda}\,\Gamma^{\lambda}_{\nu\mu} - \Gamma^{0}_{\nu\lambda}\,\Gamma^{\lambda}_{0\mu} \\ & + \Gamma^{1}_{\mu\nu,1} - \Gamma^{1}_{1\mu,\nu} + \Gamma^{1}_{1\lambda}\,\Gamma^{\lambda}_{\nu\mu} - \Gamma^{1}_{\nu\lambda}\,\Gamma^{\lambda}_{1\mu} \\ & + \Gamma^{2}_{\mu\nu,2} - \Gamma^{2}_{2\mu,\nu} + \Gamma^{2}_{2\lambda}\,\Gamma^{\lambda}_{\nu\mu} - \Gamma^{2}_{\nu\lambda}\,\Gamma^{\lambda}_{2\mu} \\ & + \Gamma^{3}_{\mu\nu,3} - \Gamma^{3}_{3\mu,\nu} + \Gamma^{3}_{3\lambda}\,\Gamma^{\lambda}_{\nu\mu} - \Gamma^{3}_{\nu\lambda}\,\Gamma^{\lambda}_{3\mu} \end{aligned}$$

$$R_{\mu\nu} = \Gamma^{\rho}_{\mu\nu,\rho} - \Gamma^{\rho}_{\rho\mu,\nu} + \Gamma^{\rho}_{\rho\lambda}\,\Gamma^{\lambda}_{\nu\mu} - \Gamma^{\rho}_{\nu\lambda}\,\Gamma^{\lambda}_{\rho\mu}$$
$$R_{00} = \Gamma^{1}_{00,1} + \Gamma^{1}_{11}\,\Gamma^{1}_{00} + \Gamma^{2}_{21}\,\Gamma^{1}_{00} + \Gamma^{3}_{31}\,\Gamma^{1}_{00} - \Gamma^{1}_{00}\,\Gamma^{0}_{10} = \frac{{R_s}^2}{2r^4} - \frac{1}{2}\frac{4r-3R_s}{r^4\sigma^2}\frac{1}{2}R_s\sigma^2 - \frac{1}{2}R_s\sigma^2\frac{1}{2}\frac{R_s}{r^4\sigma^2} - \frac{1}{2}\frac{R_s}{r^4\sigma^2}\frac{1}{2}R_s\sigma^2$$
$$R_{11} = -\Gamma^{0}_{01,1} - \Gamma^{2}_{21,1} - \Gamma^{3}_{31,1} + \Gamma^{0}_{01}\,\Gamma^{1}_{11} + \Gamma^{2}_{21}\,\Gamma^{1}_{11} + \Gamma^{3}_{31}\,\Gamma^{1}_{11} - \Gamma^{0}_{10}\,\Gamma^{0}_{01} - \Gamma^{2}_{12}\,\Gamma^{2}_{21} - \Gamma^{3}_{13}\,\Gamma^{3}_{31}$$
$$R_{22} = \Gamma^{1}_{22,1} - \Gamma^{3}_{32,2} + \Gamma^{0}_{01}\,\Gamma^{1}_{22} + \Gamma^{1}_{11}\,\Gamma^{1}_{22} + \Gamma^{2}_{21}\,\Gamma^{1}_{22} + \Gamma^{3}_{31}\,\Gamma^{1}_{22} - \Gamma^{1}_{22}\,\Gamma^{2}_{12} - \Gamma^{2}_{21}\,\Gamma^{1}_{22}$$
$$R_{33} = \Gamma^{1}_{33,1} + \Gamma^{0}_{01}\,\Gamma^{1}_{33} + \Gamma^{1}_{11}\,\Gamma^{1}_{33} + \Gamma^{2}_{21}\,\Gamma^{1}_{33} - \Gamma^{1}_{33}\,\Gamma^{3}_{13}$$

For spherical coordinates and Schwarzschild configuration with theta is 90^0, the following Ricci tensor elements are relevant:

$$R_{00} = \mathbf{\Gamma^{1}_{00,1}} + \Gamma^{1}_{00}\,\Gamma^{1}_{11} + \Gamma^{1}_{00}\,\Gamma^{2}_{12} + \Gamma^{1}_{00}\,\Gamma^{3}_{13} - \Gamma^{0}_{01}\,\Gamma^{1}_{00}$$
$$R_{11} = -\mathbf{\Gamma^{0}_{10,1}} - \mathbf{\Gamma^{2}_{12,1}} - \mathbf{\Gamma^{3}_{13,1}} + \Gamma^{1}_{11}\Gamma^{0}_{10} + \Gamma^{1}_{11}\,\Gamma^{2}_{12} + \Gamma^{1}_{11}\,\Gamma^{3}_{13} - \Gamma^{0}_{10}\Gamma^{0}_{01} - \Gamma^{2}_{12}\,\Gamma^{2}_{21} - \Gamma^{3}_{13}\,\Gamma^{3}_{31}$$
$$R_{22} = \mathbf{\Gamma^{1}_{22,1}} - \mathbf{\Gamma^{3}_{23,2}} + \Gamma^{1}_{22}\,\Gamma^{0}_{10} + \Gamma^{1}_{22}\,\Gamma^{1}_{11} + \Gamma^{1}_{22}\,\Gamma^{3}_{13} + \Gamma^{2}_{22}\,\Gamma^{3}_{32} - \Gamma^{2}_{21}\,\Gamma^{1}_{22} - \Gamma^{3}_{23}\,\Gamma^{3}_{32}$$
$$R_{33} = +\mathbf{\Gamma^{1}_{33,1}} + \mathbf{\Gamma^{2}_{33,2}} + \Gamma^{1}_{33}\,\Gamma^{0}_{10} + \Gamma^{1}_{33}\,\Gamma^{1}_{11} + \Gamma^{1}_{33}\,\Gamma^{2}_{12} + \Gamma^{2}_{33}\,\Gamma^{2}_{22} - \Gamma^{3}_{31}\,\Gamma^{1}_{33} - \Gamma^{3}_{32}\,\Gamma^{2}_{33}$$
$$R_{33} = \sin^2\theta\,.\,R_{22}$$

When theta is not 90^0 then for R_{22} and R_{33} there is an extra term $\Gamma^{2}_{22}\,\Gamma^{3}_{32}$ and respectively $+\Gamma^{2}_{33}\,\Gamma^{2}_{22}$.

Appendix 2 Derivation of derivative of Christoffel symbol in general form

It is shown how the Christoffel symbol only depends on the metric tensor elements and its derivatives. This is handy when used in a spreadsheet or program.

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\alpha} \left\{ \frac{\partial g_{\nu\alpha}}{\partial x^{\mu}} + \frac{\partial g_{\mu\alpha}}{\partial x^{\nu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\alpha}} \right\}$$

$$\frac{\partial \Gamma^{\rho}_{\mu\nu}}{\partial x^{\gamma}} = \frac{1}{2} \frac{\partial g^{\rho\alpha}}{\partial x^{\gamma}} \left\{ \frac{\partial g_{\nu\alpha}}{\partial x^{\mu}} + \frac{\partial g_{\mu\alpha}}{\partial x^{\nu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\alpha}} \right\} + \frac{1}{2} g^{\rho\alpha} \left\{ \frac{\partial^2 g_{\nu\alpha}}{\partial x^{\mu} \partial x^{\gamma}} + \frac{\partial^2 g_{\mu\alpha}}{\partial x^{\nu} \partial x^{\gamma}} - \frac{\partial^2 g_{\mu\nu}}{\partial x^{\alpha} \partial x^{\gamma}} \right\}$$

$$\frac{\partial g^{\rho\alpha}}{\partial x^{\gamma}} = \frac{\partial \frac{1}{g_{\rho\alpha}}}{\partial x^{\gamma}} = \frac{-1}{{g_{\rho\alpha}}^2} . \frac{\partial g_{\rho\alpha}}{\partial x^{\gamma}} = -(g^{\rho\alpha})^2 . \frac{\partial g_{\rho\alpha}}{\partial x^{\gamma}}$$

$$\frac{\partial \Gamma^{\rho}_{\mu\nu}}{\partial x^{\gamma}} = \frac{-1}{2} (g^{\rho\alpha})^2 . \frac{\partial g_{\rho\alpha}}{\partial x^{\gamma}} \left\{ \frac{\partial g_{\nu\alpha}}{\partial x^{\mu}} + \frac{\partial g_{\mu\alpha}}{\partial x^{\nu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\alpha}} \right\} + \frac{1}{2} g^{\rho\alpha} \left\{ \frac{\partial^2 g_{\nu\alpha}}{\partial x^{\mu} \partial x^{\gamma}} + \frac{\partial^2 g_{\mu\alpha}}{\partial x^{\nu} \partial x^{\gamma}} - \frac{\partial^2 g_{\mu\nu}}{\partial x^{\alpha} \partial x^{\gamma}} \right\}$$

$$\boldsymbol{\frac{\partial \Gamma^{\rho}_{\mu\nu}}{\partial x^{\gamma}} = \frac{1}{2} g^{\rho\alpha} \left[-g^{\rho\alpha} . \frac{\partial g_{\rho\alpha}}{\partial x^{\gamma}} \left\{ \frac{\partial g_{\nu\alpha}}{\partial x^{\mu}} + \frac{\partial g_{\mu\alpha}}{\partial x^{\nu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\alpha}} \right\} + \left\{ \frac{\partial^2 g_{\nu\alpha}}{\partial x^{\mu} \partial x^{\gamma}} + \frac{\partial^2 g_{\mu\alpha}}{\partial x^{\nu} \partial x^{\gamma}} - \frac{\partial^2 g_{\mu\nu}}{\partial x^{\alpha} \partial x^{\gamma}} \right\} \right]}$$

Or:

$$\frac{\partial \Gamma^{\rho}_{\mu\nu}}{\partial x^{\gamma}} = -g^{\rho\alpha} \cdot \frac{\partial g_{\rho\alpha}}{\partial x^{\gamma}} \cdot \Gamma^{\rho}_{\mu\nu} + \frac{1}{2} g^{\rho\alpha} \left\{ \frac{\partial^2 g_{\nu\alpha}}{\partial x^{\mu} \partial x^{\gamma}} + \frac{\partial^2 g_{\mu\alpha}}{\partial x^{\nu} \partial x^{\gamma}} - \frac{\partial^2 g_{\mu\nu}}{\partial x^{\alpha} \partial x^{\gamma}} \right\}$$

Appendix 3 Mathematical elaboration of Schwarzschild

Here we will work out the Christoffel symbols for the metric tensor of the Schwarzschild configuration.

Schwarzschild on r, theta, phi coordinates:

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\alpha} \left\{ \frac{\partial g_{\nu\alpha}}{\partial x^{\mu}} + \frac{\partial g_{\mu\alpha}}{\partial x^{\nu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\alpha}} \right\}$$

$\Gamma^{0}_{01} = \Gamma^{0}_{10} = \frac{1}{2} g^{00} \left\{ \frac{\partial g_{00}}{\partial r} \right\}$ $\Gamma^{1}_{00} = \frac{1}{2} g^{11} \left\{ -\frac{\partial g_{00}}{\partial r} \right\}$ $\Gamma^{1}_{11} = \frac{1}{2} g^{11} \left\{ \frac{\partial g_{11}}{\partial r} \right\}$ $\Gamma^{1}_{22} = \frac{1}{2} g^{11} \left\{ -\frac{\partial g_{22}}{\partial r} \right\}$

$\Gamma^{1}_{33} = \frac{1}{2} g^{11} \left\{ -\frac{\partial g_{33}}{\partial r} \right\}$ $\Gamma^{2}_{12} = \Gamma^{2}_{21} = \frac{1}{2} g^{22} \left\{ \frac{\partial g_{22}}{\partial r} \right\}$ $\Gamma^{3}_{13} = \Gamma^{3}_{31} = \frac{1}{2} g^{33} \left\{ \frac{\partial g_{33}}{\partial r} \right\}$ $\Gamma^{2}_{33} = \frac{1}{2} g^{22} \left\{ -\frac{\partial g_{33}}{\partial \theta} \right\}$

$\Gamma^{3}_{23} = \Gamma^{3}_{32} = \frac{1}{2} g^{33} \left\{ \frac{\partial g_{33}}{\partial \theta} \right\}$

All elements in the metric tensor are zero apart from the elements in the trace. This means that the contravariant elements are the directly inverse of the covariant components. Thus e.g. $g^{00} = \frac{1}{g_{00}}$ etcetera.

For **r, theta, phi** coordinates:

Derivatives of gamma to x_1=r:

$$\mathbf{0011} = \mathbf{0101} = \boldsymbol{\frac{\partial \Gamma^{0}_{01}}{\partial r}} = \boldsymbol{\frac{\partial \Gamma^{0}_{10}}{\partial r}} = \frac{\mathbf{1}}{\mathbf{2}} \left\{ \frac{-1}{{g_{00}}^2} \left(\frac{\partial g_{00}}{\partial r} \right)^2 + \frac{1}{g_{00}} \frac{\partial^2 g_{00}}{\partial r^2} \right\} = \boldsymbol{\frac{1}{2 g_{00}} \left\{ \frac{-1}{g_{00}} ({g_{00}}')^2 + {g_{00}}'' \right\}}$$

$$1001 = \frac{\partial \Gamma^1_{00}}{\partial r} = \frac{-1}{2}\left\{\frac{-1}{{g_{11}}^2}\frac{\partial g_{11}}{\partial r}\frac{\partial g_{00}}{\partial r} + \frac{1}{g_{11}}\frac{\partial^2 g_{00}}{\partial r^2}\right\} = \frac{-1}{2g_{11}}\left\{\frac{-1}{g_{11}}{g_{11}}'{g_{00}}' + {g_{00}}''\right\}$$

$$1111 = \frac{\partial \Gamma^1_{11}}{\partial r} = \frac{1}{2}\left\{\frac{-1}{{g_{11}}^2}\left(\frac{\partial g_{11}}{\partial r}\right)^2 + \frac{1}{g_{11}}\frac{\partial^2 g_{11}}{\partial r^2}\right\} = \frac{1}{2g_{11}}\left\{\frac{-1}{g_{11}}({g_{11}}')^2 + {g_{11}}''\right\}$$

$$1221 = \frac{\partial \Gamma^1_{22}}{\partial r} = \frac{-1}{2}\left\{\frac{-1}{{g_{11}}^2}\frac{\partial g_{11}}{\partial r}\frac{\partial g_{22}}{\partial r} + \frac{1}{g_{11}}\frac{\partial^2 g_{22}}{\partial r^2}\right\} = \frac{-1}{2g_{11}}\left\{\frac{-1}{g_{11}}{g_{11}}'{g_{22}}' + {g_{22}}''\right\}$$

$$1331 = \frac{\partial \Gamma^1_{33}}{\partial r} = \frac{-1}{2}\left\{\frac{-1}{{g_{11}}^2}\frac{\partial g_{11}}{\partial r}\frac{\partial g_{33}}{\partial r} + \frac{1}{g_{11}}\frac{\partial^2 g_{33}}{\partial r^2}\right\} = \frac{-1}{2g_{11}}\left\{\frac{-1}{g_{11}}{g_{11}}'{g_{33}}' + {g_{33}}''\right\}$$

$$2121 = 2211 = \frac{\partial \Gamma^2_{12}}{\partial r} = \frac{\partial \Gamma^2_{21}}{\partial r} = \frac{1}{2}\left\{\frac{-1}{{g_{22}}^2}\left(\frac{\partial g_{22}}{\partial r}\right)^2 + \frac{1}{g_{22}}\frac{\partial^2 g_{22}}{\partial r^2}\right\} = \frac{1}{2g_{22}}\left\{\frac{-1}{g_{22}}({g_{22}}')^2 + {g_{22}}''\right\}$$

$$3131 = 3311 = \frac{\partial \Gamma^3_{13}}{\partial r} = \frac{\partial \Gamma^3_{31}}{\partial r} = \frac{1}{2}\left\{\frac{-1}{{g_{33}}^2}\left(\frac{\partial g_{33}}{\partial r}\right)^2 + \frac{1}{g_{33}}\frac{\partial^2 g_{33}}{\partial r^2}\right\} = \frac{1}{2g_{33}}\left\{\frac{-1}{g_{33}}({g_{33}}')^2 + {g_{33}}''\right\}$$

$$2331 = \frac{\partial \Gamma^2_{33}}{\partial r} = \frac{-1}{2}\left\{\frac{-1}{{g_{22}}^2}\frac{\partial g_{22}}{\partial r}\frac{\partial g_{33}}{\partial \theta} + \frac{1}{g_{22}}\frac{\partial^2 g_{33}}{\partial r\partial \theta}\right\} = \frac{-1}{2g_{22}}\left\{\frac{-1}{g_{22}}{g_{22}}'\frac{\partial g_{33}}{\partial \theta} + \frac{\partial^2 g_{33}}{\partial r\partial \theta}\right\}$$

$$3231 = 3321 = \frac{\partial \Gamma^3_{23}}{\partial r} = \frac{\partial \Gamma^3_{32}}{\partial r} = \frac{1}{2}\left\{\frac{-1}{{g_{33}}^2}\frac{\partial g_{33}}{\partial r}\frac{\partial g_{33}}{\partial \theta} + \frac{1}{g_{33}}\frac{\partial^2 g_{33}}{\partial r\partial \theta}\right\} = \frac{1}{2g_{33}}\left\{\frac{-1}{g_{33}}{g_{33}}'\frac{\partial g_{33}}{\partial \theta} + \frac{\partial^2 g_{33}}{\partial r\partial \theta}\right\}$$

Derivatives of gamma to $x_2=\theta$:

$$1222 = \frac{\partial \Gamma^1_{22}}{\partial \theta} = \frac{-1}{2g_{11}}\frac{\partial^2 g_{22}}{\partial r\partial \theta}$$

$$1332 = \frac{\partial \Gamma^1_{33}}{\partial \theta} = \frac{-1}{2g_{11}}\frac{\partial^2 g_{33}}{\partial r\partial \theta}$$

$$2332 = \frac{\partial \Gamma^2_{33}}{\partial \theta} = \frac{-1}{2}\left\{\frac{-1}{{g_{22}}^2}\frac{\partial g_{22}}{\partial \theta}\frac{\partial g_{33}}{\partial \theta} + \frac{1}{g_{22}}\frac{\partial^2 g_{33}}{\partial \theta^2}\right\} = \frac{-1}{2g_{22}}\left\{\frac{-1}{g_{22}}\frac{\partial g_{22}}{\partial \theta}\frac{\partial g_{33}}{\partial \theta} + \frac{\partial^2 g_{33}}{\partial \theta^2}\right\}$$

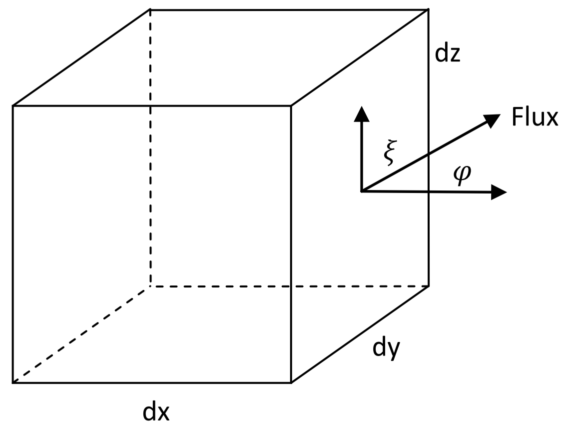
$$2222 = \frac{\partial \Gamma^2_{22}}{\partial \theta} = \frac{1}{2}\left\{\frac{-1}{{g_{22}}^2}\left(\frac{\partial g_{22}}{\partial \theta}\right)^2 + \frac{1}{g_{22}}\frac{\partial^2 g_{22}}{\partial \theta^2}\right\} = \frac{1}{2g_{22}}\left\{\frac{-1}{g_{22}}\left(\frac{\partial g_{22}}{\partial \theta}\right)^2 + \frac{\partial^2 g_{22}}{\partial \theta^2}\right\}$$

$$3312 = 3132 = \frac{\partial \Gamma^3_{31}}{\partial \theta} = \frac{\partial \Gamma^3_{13}}{\partial \theta} = \frac{1}{2}\left\{\frac{-1}{{g_{33}}^2}\frac{\partial g_{33}}{\partial r}\frac{\partial g_{33}}{\partial \theta} + \frac{1}{g_{33}}\frac{\partial^2 g_{33}}{\partial r\partial \theta}\right\} = \frac{1}{2g_{33}}\left\{\frac{-1}{g_{33}}{g_{33}}'\frac{\partial g_{33}}{\partial \theta} + \frac{\partial^2 g_{33}}{\partial r\partial \theta}\right\}$$

$$3232 = 3322 = \frac{\partial \Gamma^3_{23}}{\partial \theta} = \frac{\partial \Gamma^3_{32}}{\partial \theta} = \frac{1}{2}\left\{\frac{-1}{{g_{33}}^2}\left(\frac{\partial g_{33}}{\partial \theta}\right)^2 + \frac{1}{g_{33}}\frac{\partial^2 g_{33}}{\partial \theta^2}\right\} = \frac{1}{2g_{33}}\left\{\frac{-1}{g_{33}}\left(\frac{\partial g_{33}}{\partial \theta}\right)^2 + \frac{\partial^2 g_{33}}{\partial \theta^2}\right\}$$

Appendix 4 Derivation of the Gauss theorem

We start from a cube:



Through this infinitesimal small cube flows a flux F. This flux is not everywhere the same and therefore F is a function of *x, y, z* and *t.* The flux is a vector because it has a magnitude and a direction.

$$Flux = \vec{F}(x, y, z, t) \tag{1}$$

The flux that flows through the right hand side is:

$$Flux_{rightside} = \vec{F}\sin\xi dydz \tag{2}$$

Here is ξ the angle between the flux direction and the surface. The flux component perpendicular to the surface flows really through that surface. The surface is the cross product of *dx* and *dy* and is a new vector:

$$d\vec{A} = \overrightarrow{dy} \times \overrightarrow{dz} \quad with\ size \quad dA = \sin\xi dydz \tag{3}$$

So the flux that flows through the right hand side:

$$Flux_{rightside} = \vec{F}\sin\xi dydz = \vec{F}\cos\left(\frac{1}{2}\pi - \xi\right)d\vec{A} = \vec{F}\cos\varphi\, d\vec{A} = \vec{F}d\vec{A}\ \cos\varphi \tag{4}$$

The vector *dA* is perpendicular to the surface and φ is here the complementary angle of ξ. So now we see here the dot product:

$$Flux_{rightside} = \vec{F}d\vec{A}\ \cos\varphi = \vec{F}\cdot d\vec{A} \tag{5}$$

In case the cube is not infinitesimal then we could integrate:

$$Flux_{cube} = \iint\limits_{right}\vec{F}\cdot d\vec{A} + \iint\limits_{left}\vec{F}\cdot d\vec{A} + \iint\limits_{front}\vec{F}\cdot d\vec{A} + \iint\limits_{back}\vec{F}\cdot d\vec{A} + \iint\limits_{under}\vec{F}\cdot d\vec{A} + \iint\limits_{above}\vec{F}\cdot d\vec{A} \tag{6}$$

We can also write this as an integral over the total surface of the cube:

$$Flux_{cube} = \oiint\limits_{cube}\vec{F}\cdot d\vec{A} \tag{7}$$

Now we use another approach. First we consider the x direction. The flux enters the cube from the left:

$$Flux_{left} = F_x dydz \tag{8}$$

This flux leaves the right hand side increased or decreased with $d\phi$ from y or z direction:

$$Flux_{right} = (F_x + dF_x)dydz \tag{9}$$

So the net flux in the x direction becomes

$$Flux_x = Flux_{right} - Flux_{left} = (F_x + dF_x)dydz - F_x dydz = dF_x dydz \tag{10}$$

The same goes for the y and z direction:

$$Flux_y = dF_y dxdz \tag{11}$$

$$Flux_z = dF_z dxdy \tag{12}$$

The total flux through the cube:

$$Flux_{cube} = Flux_x + Flux_y + Flux_z = dF_x dydz + dF_y dxdz + dF_z dxdy \tag{13}$$

$$= \frac{\partial F_x}{\partial x} dxdydz + \frac{\partial F_y}{\partial y} dxdydz + \frac{\partial F_z}{\partial z} dxdydz = \left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\right) dxdydz$$

$$= \left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\right) dV$$

The operator $\vec{\nabla}$ is:

$$\vec{\nabla} = \frac{\partial}{\partial x}\vec{e}_x + \frac{\partial}{\partial y}\vec{e}_y + \frac{\partial}{\partial z}\vec{e}_z = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \tag{14}$$

Thus equation (13) becomes:

$$Flux_{cube} = \left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\right) dV = \vec{\nabla} \cdot \vec{F}\ \mathrm{dV} \tag{15}$$

By integrating over the total cube we find the net flux through the cube:

$$Flux_{cube} = \iiint_{cube} \vec{\nabla} \cdot \vec{F}\ \mathrm{dV} \tag{16}$$

The equations (7) and (16) both present the same flux through the cube so:

$$\oiint_{cube} \vec{F} \cdot d\vec{A} = \iiint_{cube} \vec{\nabla} \cdot \vec{F}\ \mathrm{dV} \tag{17}$$

We started with an infinitesimal cube and because integration was done it is irrelevant whether it is a cube or any other arbitrary form so we can skip the cube term:

$$\oiint \boldsymbol{\vec{F} \cdot d\vec{A}} = \iiint \boldsymbol{\vec{\nabla} \cdot \vec{F}\ \mathrm{dV}} \tag{18}$$

This equation is known as the Gauss theorem.

In the special case where the net flux through the closed surface is zero (so nothing is generated or disappeared inside the volume):

$$\oiint \vec{F} \cdot d\vec{A} = 0 \quad => \quad \iiint \vec{\nabla} \cdot \vec{F}\ \mathrm{dV} = 0 \tag{19}$$

Thus

$$\vec{\nabla} \cdot \vec{F} = 0 \tag{20}$$

Can also be written as:

$$\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} = 0 \tag{21}$$

Or in Einstein notation:

$$\frac{\partial F^\alpha}{\partial x^\alpha} = 0 \tag{22}$$

Appendix 5 Derivation of the Laplace and Poisson equations

A vector field for which it is irrelevant what trajectory has been taken to go from one arbitrary point to another, i.e. each chosen route costs the same amount of energy; such a field is called a *conservative field.* Let us call this field *F*. For a conservative field there exists a scalar function φ with the following relation:

$$\vec{F} = \vec{\nabla}\varphi \tag{1}$$

Where $\vec{\nabla}$ is the operator:

$$\vec{\nabla} = \frac{\partial}{\partial x}\vec{e}_x + \frac{\partial}{\partial y}\vec{e}_y + \frac{\partial}{\partial z}\vec{e}_z = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \tag{2}$$

The gravitational field F_g is a conservative field:

$$\vec{F}_g = \vec{\nabla}\varphi \tag{3}$$

According to the Gauss theorem:

$$\oiint_A \vec{F}_g \cdot d\vec{A} = \iiint_V \vec{\nabla} \cdot \vec{F}_g \, dV \tag{4}$$

With result in vacuum:

$$\vec{\nabla} \cdot \vec{F}_g = 0 \tag{5}$$

We conclude now from (5) and (3) that:

$$\vec{\nabla} \cdot \vec{F}_g = 0 \quad => \quad \vec{\nabla} \cdot \vec{\nabla}\varphi = 0 \tag{6}$$

To work this further out:

$$\vec{\nabla} \cdot \vec{\nabla}\varphi = 0 \tag{7}$$

$$\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\varphi = 0$$

As x, y, z are orthogonal:

$$\left(\frac{\partial}{\partial x}\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\frac{\partial}{\partial y} + \frac{\partial}{\partial z}\frac{\partial}{\partial z}\right)\varphi = 0$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\varphi = 0$$

Also written as:

$$\boldsymbol{\vec{\nabla}^2 \varphi = 0} \quad \textbf{or} \quad \boldsymbol{\Delta\varphi = 0} \tag{8}$$

The operator ∇^2 is called the Laplacian:

$$\boldsymbol{\nabla^2} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

So for *vacuum holds:* $\Delta\varphi = 0$.

Now we do the calculation within a mass.

According to Newton the field of gravity is:

$$\vec{F}_g = G\frac{m}{r^2}\hat{r} \tag{9}$$

Here $\hat{r}$ signifies the unit vector.

Now use again the theorem of Gauss:

$$\iiint_V \vec{\nabla}\cdot\vec{F}_g dV = \oiint_A \vec{F}_g \cdot d\vec{A} \tag{10}$$

$$\iiint_V \Delta\varphi dV = \oiint_A G\frac{m}{r^2}\hat{r}\cdot d\vec{A} = \oiint_A G\frac{m}{r^2}dA$$

Considering the volume as a sphere then

$$A = 4\pi r^2 \tag{11}$$

$$V = \frac{4}{3}\pi r^3 \tag{12}$$

As the radius *r* of the sphere stays constant over the total surface of the sphere then equation (10) becomes:

$$\iiint_V \Delta\varphi dV = \oiint_A G\frac{m}{r^2}dA = G\frac{m}{r^2}\oiint_A dA = G\frac{m}{r^2}4\pi r^2 = 4\pi Gm \tag{13}$$

With ρ as the mass density:

$$\rho = \frac{m}{V} \tag{14}$$

So (13) becomes:

$$\iiint_V \Delta\varphi dV = 4\pi Gm = 4\pi G\iiint_V \rho dV = \iiint_V 4\pi G\rho dV \implies \Delta\varphi = 4\pi G\rho \tag{15}$$

Thus for a **volume where flux** is created, i.e. mass creates gravity, the **equation of Poisson** is relevant:

$$\boldsymbol{\Delta\varphi = 4\pi G\rho} \tag{16}$$

Or:

$$\nabla^2\varphi = 4\pi G\rho$$

Thus for an **empty space** holds the **Laplace equation**:

$$\boldsymbol{\Delta\varphi = 0} \tag{17}$$

Or:

$$\nabla^2\varphi = 0$$

Consideration:
The existence of mass causes gravitational flux. When you are inside a mass-sphere and move outwards, the amount of enclosed mass changes and so the total flux changes $(\Delta\varphi = 4\pi G\rho)$ as well. When finally outside the mass-sphere, but the mass is still enclosed, the total flux stays constant $(\Delta\varphi = 0)$.

Appendix 5.1 The Laplace *operator* applied on the gravitational potential outside and inside a static sphere

Next we will apply the Laplace operator on the gravitational potential outside a sphere (Appendix 5.1.1) and inside a static sphere (Appendix 5.1.2)
The gravitational force according to Newton is:

$F = mg = \frac{mMG}{r^2} \Longrightarrow$ gravitational field: $g = \frac{MG}{r^2} \Rightarrow$ gravitational potential: $\emptyset_{newton} = \frac{-MG}{r}$ where $g = \frac{d\emptyset_{newton}}{dr}$

Here is *r* the distance with respect to the center of the sphere and *R* is the radius of the sphere. *M* is the mass of the sphere and *m* is the mass of a particle.

Gravitational potential **outside** a sphere in General Relativity (GR) is (chapter 2.7 equation 5):

$$\emptyset = g_{00} = 1 - \frac{2GM}{c^2 r} = 1 + \frac{2\emptyset_{newton}}{c^2}$$
$$=> \emptyset_{newton\ \_outside} = -\frac{GM}{r} \quad (1)$$

Gravitational potential **inside** a sphere (see derivation below)

$$\emptyset = 1 - \frac{3GM}{c^2 R} + \frac{GM}{c^2}\frac{r^2}{R^3} = 1 + \frac{2}{c^2}.\left(-\frac{3GM}{2R} + \frac{GM}{2}\frac{r^2}{R^3}\right)$$
$$=> \emptyset_{newto\ n_{inside}} = -\frac{3GM}{2R} + \frac{GM}{2}\frac{r^2}{R^3} \quad (2)$$

See Appendix 5.1.4 formula 3

Next the application of the Laplace operator on the gravitational potential outside and inside a sphere, where:

$$r^2 = x^2 + y^2 + z^2$$

Appendix 5.1.1 Outside a sphere (Laplace)

$$r^2 = x^2 + y^2 + z^2$$
$$\frac{\partial r}{\partial x} \Longrightarrow \quad 2r\frac{\partial r}{\partial x} = 2\mathrm{x} \Longrightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$$

See equation (1) in Appendix 5.1

$$\emptyset_{newton\ \_outside} = -\frac{GM}{r}$$
$$\frac{\partial \emptyset_{newton}}{\partial x} = \frac{\partial \emptyset_{newton}}{\partial r}\frac{\partial r}{\partial x} = \frac{GM}{r^2}.\frac{x}{r} = \frac{GMx}{r^3}$$
$$\frac{\partial^2 \emptyset_{newton}}{\partial x^2} = \frac{-3GMx}{r^4}.\frac{x}{r} + \frac{GM}{r^3} = \frac{-3GMx^2}{r^5} + \frac{GM}{r^3}$$

The same goes for *y* and *z*, so total:

$$\Delta\emptyset_{newton} = \frac{\partial^2 \emptyset_{newton}}{\partial x^2} + \frac{\partial^2 \emptyset_{newton}}{\partial y^2} + \frac{\partial^2 \emptyset_{newton}}{\partial z^2}$$
$$\Delta\emptyset_{newton} = \frac{-3GM}{r^3}.\frac{x^2 + y^2 + z^2}{r^2} + 3\frac{GM}{r^3} = \frac{-3GM}{r^3} + 3\frac{GM}{r^3} = 0$$

Thus

Thus

$$\boldsymbol{\Delta\emptyset_{newton} = 0}.$$

Appendix 5.1.2 Inside a sphere (Poisson)

$$\frac{\partial r}{\partial x} = \frac{x}{r}$$

See equation (2) in Appendix 5.1

$$\emptyset_{newton\ \_inside} = -\frac{3GM}{2R} + \frac{GM}{2}\frac{r^2}{R^3}$$
$$\frac{\partial \emptyset_{newton}}{\partial x} = \frac{\partial \emptyset_{newton}}{\partial r}\frac{\partial r}{\partial x} = \frac{2GM}{2}\frac{r}{R^3}\frac{x}{r} = \frac{GMx}{R^3}$$
$$\frac{\partial^2 \emptyset_{newton}}{\partial x^2} = \frac{GM}{R^3}$$

The same goes for y and z:

$$\Delta\emptyset_{newto\ n} = \frac{\partial^2 \emptyset_{newton}}{\partial x^2} + \frac{\partial^2 \emptyset_{newton}}{\partial y^2} + \frac{\partial^2 \emptyset_{newton}}{\partial z^2} = \frac{3GM}{R^3} = \frac{3G.\frac{4}{3}\pi R^3 \rho}{R^3} = 4\pi G\rho$$

Thus

$$\Delta\emptyset_{\boldsymbol{newton}} = \boldsymbol{4\pi G\rho}.$$

This is in agreement with the Poisson equation.
Hence:

$$\emptyset = 1 + \frac{2\emptyset_{newton}}{c^2} \Longrightarrow \Delta\emptyset = \frac{2}{c^2}\Delta\emptyset_{newton} = \frac{2}{c^2}4\pi G\rho = \frac{8\pi G\rho}{c^2}$$
$$\Delta\emptyset = \frac{\boldsymbol{8\pi G\rho}}{\boldsymbol{c^2}}$$

Appendix 5.1.3 Simplification of the application of the Laplace/Poisson operator

Let us assume a function f(r) on which the Laplace operator will be applied

$$r^2 = x^2 + y^2 + z^2$$

Gradient of $f(r)$:

$$\nabla f(r) = \left(\frac{\partial f(r)}{\partial x}, \frac{\partial f(r)}{\partial y}, \frac{\partial f(r)}{\partial z}\right)$$
$$\frac{\partial f(r)}{\partial x} = \frac{\partial f(r)}{\partial r}.\frac{\partial r}{\partial x} = \frac{\partial f(r)}{\partial r}.\frac{\vec{x}}{r} \qquad (1)$$

Gradient of $f(r)$:

$$\nabla f(r) = \frac{\partial f(r)}{\partial r}.\left(\frac{\vec{x}}{r} + \frac{\vec{y}}{r} + \frac{\vec{z}}{r}\right) = \frac{\partial f(r)}{\partial r}.\frac{\vec{r}}{r} = \frac{\partial f(r)}{\partial r}.\hat{r}$$

Further differentiation of (1):

$$\frac{\partial^2 f(r)}{\partial x^2} = \frac{\partial^2 f(r)}{\partial r^2}.\frac{x}{r}.\frac{x}{r} + \frac{\partial f(r)}{\partial r}.\frac{1}{r} - \frac{\partial f(r)}{\partial r}.\frac{x}{r^2}.\frac{x}{r}$$
$$\frac{\partial^2 f(r)}{\partial x^2} = \frac{\partial^2 f(r)}{\partial r^2}.\frac{x^2}{r^2} + \frac{\partial f(r)}{\partial r}.\frac{1}{r}.(1 - \frac{x^2}{r^2})$$

Now for x, y, en z:

$$\frac{\partial^2 f(r)}{\partial x^2} + \frac{\partial^2 f(r)}{\partial y^2} + \frac{\partial^2 f(r)}{\partial z^2} = \frac{\partial^2 f(r)}{\partial r^2}.\frac{x^2+y^2+z^2}{r^2} + \frac{\partial f(r)}{\partial r}.\frac{1}{r}.(3 - \frac{x^2+y^2+z^2}{r^2})$$

$$\boldsymbol{Laplace/Poisson\ equation:} \qquad \Delta \boldsymbol{f}(\boldsymbol{r}) = \frac{\boldsymbol{\partial^2 f(r)}}{\boldsymbol{\partial r^2}} + \frac{\boldsymbol{2}}{\boldsymbol{r}}.\frac{\boldsymbol{\partial f(r)}}{\boldsymbol{\partial r}} \qquad (\mathbf{2})$$

Let the general form of $\emptyset_{newton}$ be:

$$\emptyset_{newton} = L + Kr^n \qquad (3)$$

With L and K being constants.

$$\frac{\partial \emptyset_{newton}}{\partial r} = nKr^{n-1}$$
$$\frac{\partial^2 \emptyset_{newton}}{\partial r^2} = n(n-1)Kr^{n-2}$$

Hence from equation (2):

$$\Delta\emptyset_{newton} = n(n-1)Kr^{n-2} + \frac{2}{r}.nKr^{n-1} = n(n-1)Kr^{n-2} + 2nKr^{n-2}$$

$$\Delta\emptyset_{\boldsymbol{newton}} = \boldsymbol{n(n+1)Kr^{n-2}} \qquad (\mathbf{4})$$

Let us apply this formula on the gravitational potentials on the outside and inside of a sphere.

Outside a sphere:

$$\emptyset_{\boldsymbol{newton}} = -\frac{\boldsymbol{GM}}{\boldsymbol{r}}$$

So with (3)

$$\emptyset_{newton} = L + Kr^n$$

Thus n= -1, L=0 and K= -GM. Then with (4):

$$\Delta\emptyset_{newton} = -1(-1+1)GMr^{-1-2} = 0.GMr^{-3} = 0$$

Inside a sphere:

$$\emptyset_{\boldsymbol{newton}} = -\frac{\boldsymbol{3GM}}{\boldsymbol{2R}} + \frac{\boldsymbol{GM}}{\boldsymbol{2}}\frac{\boldsymbol{r^2}}{\boldsymbol{R^3}}$$

So with (3)

$$\emptyset_{newton} = L + Kr^n$$

Thus n=+2, L=-3GM/2R and K=GM/2R^3

$$\Delta\emptyset_{newton} = +2(2+1)\frac{GM}{2R^3}r^{2-2} = 6\frac{GM}{2R^3} = \frac{\boldsymbol{3GM}}{\boldsymbol{R^3}} = \frac{3G.\frac{4}{3}\pi R^3\rho}{R^3} = \boldsymbol{4\pi G\rho}$$

This is in accordance with the calculations in the previous chapter.
Furthermore it can be seen that $\Delta\emptyset_{newton}$ is zero when n=0 or -1, and obvious when r goes to infinity while n<2.

Appendix 5.1.4 Derivation of the gravitational potential inside a static sphere

The gravitational potential inside a static sphere will be derived based on the Poisson equation:

$$\Delta\emptyset_{\boldsymbol{newton}} = \boldsymbol{4\pi G\rho}.$$

And the general form of $\emptyset_{newton}$

$$\emptyset_{newton} = L + Kr^n$$

With formula (4) derived above

$$\Delta\emptyset_{\boldsymbol{newton}} = \boldsymbol{n(n+1)Kr^{n-2}} \quad (\boldsymbol{2})$$

$$=> 4\pi G\rho = n(n+1)Kr^{n-2}$$

$$=> n = 2\ so\ 6K = 4\pi G\rho => K = \frac{2}{3}\pi G\rho = \frac{2}{3}\pi G\frac{M}{\frac{4}{3}\pi R^3} = \frac{1}{2}\frac{GM}{R^3}$$

Thus the gravitational potential inside a static sphere:

$$=> \emptyset_{newton} = L + \frac{2}{3}\pi G\rho r^2$$

On the surface of the sphere where r=R

$$\emptyset_{newton} = -\frac{GM}{r}$$

For a continuous transition of $\emptyset$ at the surface of the sphere (r=R) the outside gravitational potential shall be equal to the inside gravitational potential:

$$\emptyset_{newton} = -\frac{GM}{R} = -\frac{4}{3}\pi\frac{R^3}{R}G\rho = -\frac{4}{3}\pi R^2 G\rho = L + \frac{2}{3}\pi G\rho R^2$$

$$=> L = -\frac{4}{3}\pi R^2 G\rho - \frac{2}{3}\pi G\rho R^2 = -\frac{6}{3}\pi R^2 G\rho = -\frac{6}{3}\pi R^2 G\frac{M}{\frac{4}{3}\pi R^3} = -\frac{3}{2}\frac{MG}{R}$$

$$\emptyset_{newton} = L + \frac{2}{3}\pi G\rho r^2 = L + \frac{2}{3}\pi G r^2\frac{M}{\frac{4}{3}\pi R^3} = L + \frac{1}{2}\frac{GM}{R^3}r^2$$

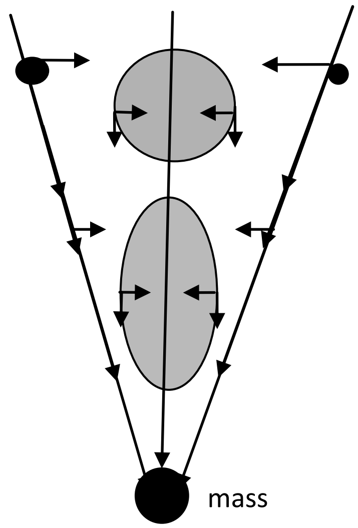
$$=> \emptyset_{newton} = -\frac{3}{2}\frac{MG}{R} + \frac{1}{2}\frac{GM}{R^3}r^2$$

$$acceleration: g_r = \frac{d\emptyset_{newton}}{dr} = \frac{GM}{R^3}r$$

At r=0 $acceleration: g_r = 0$ And at r=R then $acceleration: g_r = \frac{GM}{R^2}$.

$\boldsymbol{Gravitational\ \ potential\ inside\ sphere}$:	$\emptyset = 1 + \frac{2\emptyset_{newton}}{c^2} = \boldsymbol{1 - \frac{3MG}{c^2R} + \frac{GM}{c^2R^3}r^2}$		(3)

Appendix 6 Tidal Forces



Tidal forces

The lines of the gravitational field, caused by a mass, are not parallel but directed towards the center of the mass. The size of the force is inversely proportional with the square of the distance towards the center of the mass. The gravitational force on the grey body can be split in horizontal- and vertical components. The grey body is squeezed because of the horizontal components of the force and, as the gravitational field is increasing towards the mass, the body is vertically stretched.

Thus as the lines of the gravitational field are radially directed the force is called a tidal force.

In case of a “black hole” the forces are so huge that the grey body is so much stretched that the phenomenon is called spaghettification.

Appendix 7 Special Relativity

In Special Relativity, Einstein only considered coordinate frames that moved uniformly, thus with constant speed with respect to each other; the effect of masses, and thus gravitation, was not taken into account. The presumptions where the Special Relativity is based on are:

- The maximum possible speed, in each coordinate frame, is the velocity of light c=299792458 m/s.
- The laws of physics are valid in each uniformly moving coordinate frame.

In Newton's approach the increments of time were equal in the "rest" frame and the moving frame. However via the Special Relativity Theory it was revealed that the **increments of time** in a moving frame were different and smaller than in a frame at "rest". Furthermore the **length of an object** is influenced by its velocity and decreases, with respect to the "rest" frame, in the moving direction.
These were both consequences of the observation that the velocity of light in vacuum was always the same in each frame independent of its speed.

In this chapter we summarize a number of items which are often used in Special Relativity (SR) and which are relevant for the application in General Relativity (GR).

We first establish the relation between two coordinate frames moving with a constant speed with respect to each other. This relation is known as the Lorentz Transformation of which the derivation is shown below.

Appendix 7.1 Simple Derivation of the Lorentz Transformation

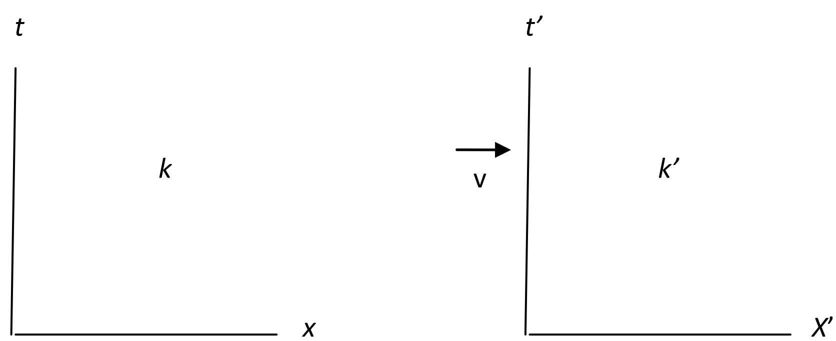


Fig. 1

Coordinate system k' moves uniformly with a velocity v with respect to coordinate system k.

We take two coordinate systems of which the origins move with a constant velocity *v,* with respect to each other, in the x respectively x' direction. Although the coordinate systems are four dimensional (*t, x, y, z*) only the t an x axes are drawn for simplicity reasons and because there is no movement in the y and z directions.

A light-signal is transmitted on time $t = t^{'} = 0$ in the direction of the positive x-axis, according the equation

$$x = ct$$

Or

$$x - ct = 0 \tag{1}$$

Since the same light-signal has to be transmitted relative to *k′* with the velocity *c,* the propagation relative to the system *k′* will be represented by the analogous formula

$$x' - ct' = 0 \tag{2}$$

Those space-time points (events) which satisfy (1) must also satisfy (2). Obviously this will be the case when the relation

$$(x' - ct') = \lambda(x - ct) \tag{3}$$

is fulfilled in general, where λ indicates a constant; for, according to (3), the disappearance of (*x* – *ct*) involves the disappearance of (*x′* – *ct′*).

If we apply quite similar considerations to light rays which are being transmitted along the negative *x*-axis, we obtain the condition

$$(x' + ct') = \mu(x + ct) \tag{4}$$

By adding (or subtracting) equations (3) and (4), and introducing for convenience the constants *a* and *b* in place of the constants λ and μ where

$$a = \frac{\lambda + \mu}{2}$$

And

$$b = \frac{\lambda - \mu}{2}$$

We obtain the equations

$$\begin{aligned} x' &= ax - bct \\ ct' &= act - bx \end{aligned} \tag{5}$$

We should thus have the solution of our problem, if the constants *a* and *b* were known. These result from the following discussion.

For the origin of *k′* we have permanently *x′* = 0, and hence according to the first of the equations (5)

$$x = \frac{bc}{a}t$$

If we call *v* the velocity with which the origin of *k′* is moving relative to *K,* we then have

$$v = \frac{bc}{a} \tag{6}$$

The same value *v* can be obtained from equation (5), if we calculate the velocity of another point of *k′* relative to *K,* or the velocity (directed towards the negative *x*-axis) of a point of *K* with respect to *K′.* In short, we can designate v as the relative velocity of the two systems.

Furthermore, the principle of relativity teaches us that, as judged from *K,* the length of a unit measuring-rod which is at rest with reference to *k′* must be exactly the same as the length, as judged from *K′,* of a unit measuring-rod which is at rest relative to *K.* In order to see how the points of the *x′*-axis appear as viewed from *K,* we only require to take a “snapshot” of *k′* from *K;* this means that we have to insert a particular value of *t* (time of *K*), e.g. *t* = 0. For this value of *t* we then obtain from the first of the equations (5)

$$x' = ax$$

Two points of the *x′*-axis which are separated by the distance *x′*=*L* when measured in the *k′* system are thus separated in our instantaneous photograph by the distance

$$\Delta x = \frac{L}{a} \tag{7}$$

But if the snapshot be taken from $K'(t' = 0)$, and if we eliminate t from the equations (5), taking into account the expression (6), we obtain

$$0 = act - bx$$
$$t = \frac{b}{ac}x$$
$$x' = ax - bct = ax - \frac{b^2}{a}x = ax\left(1 - \frac{b^2}{a^2}\right)$$

From (6) we get

$$\frac{b}{a} = \frac{v}{c}$$
$$=> x' = a\left(1 - \frac{v^2}{c^2}\right)x \tag{7a}$$

From this we conclude that two points on the x-axis and separated by the distance L (relative to K) will be represented on our snapshot by the distance

$$\Delta x' = a\left(1 - \frac{v^2}{c^2}\right)L \tag{7b}$$

But from what has been said, the two snapshots must be identical; hence Δx in (7) must be equal to $\Delta x'$ in (7*b*), so that we obtain

$$\Delta x = \frac{L}{a} = \Delta x' = a\left(1 - \frac{v^2}{c^2}\right)L$$
$$\frac{1}{a} = a\left(1 - \frac{v^2}{c^2}\right)$$
$$=> a^2 = \frac{1}{1 - \frac{v^2}{c^2}} \tag{7c}$$

The equations (6) and (7*c*) determine the constants *a* and *b.* By inserting the values of these constants in (5), we obtain the equations:

$$x' = ax - bct = ax - avt = a(x - vt)$$
$$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}}$$
$$ct' = act - bx = act - \frac{av}{c}x = ac\left(t - \frac{v}{c^2}x\right)$$
$$t' = \frac{t - \frac{v}{c^2}x}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{8}$$

Thus we have obtained the Lorentz transformation for events on the x-axis. It satisfies the condition

$$x'^2 - c^2t'^2 = x^2 - c^2t^2 \tag{8a}$$

The extension of this result, to include events which take place outside the x-axis, is obtained by retaining equations (8) and supplementing them by the relations

$$y' = y$$
$$z' = z \tag{9}$$

In this way we satisfy the postulate of the constancy of the velocity of light *in vacuo* for rays of light of arbitrary direction, both for the system *K* and for the system *K′*. This may be shown in the following manner.

We suppose a light-signal sent out from the origin of *K* at the time *t* = 0. It will be propagated according to the equation

$$r = \sqrt{x^2 + y^2 + z^2} = ct$$

or, if we square this equation, according to the equation

$$x^2 + y^2 + z^2 - c^2t^2 = 0 \qquad (10)$$

It is required by the law of propagation of light, in conjunction with the postulate of relativity, that the transmission of the signal in question should take place—as judged from *K′*—in accordance with the corresponding formula

$$r' = ct'$$

Or,

$$x'^2 + y'^2 + z'^2 - c^2t'^2 = 0 \qquad (10a)$$

In order that equation (10a) may be a consequence of equation (10), we must have

$$(x'^2 + y'^2 + z'^2 - c^2t'^2) = \sigma(x^2 + y^2 + z^2 - c^2t^2) \qquad (11)$$

Since equation (8*a*) must hold for points on the x-axis, we thus have $\sigma = 1$; for (11) is a consequence of (8*a*) and (9), and hence also of (8) and (9). We have thus derived the Lorentz transformation.

The Lorentz transformation represented by (8) and (9) still requires to be generalized. Obviously it is immaterial whether the axes of *K′* be chosen so that they are spatially parallel to those of *K.* It is also not essential that the velocity of translation of *K′* with respect to *K* should be in the direction of the *x*-axis. A simple consideration shows that we are able to construct the Lorentz transformation in this general sense from two kinds of transformations, viz. from Lorentz transformations in the special sense and from purely spatial transformations, which corresponds to the replacement of the rectangular co-ordinate system by a new system with its axes pointing in other directions.

Mathematically, we can characterize the generalized Lorentz transformation thus: It expresses *x', y', z', t',* in terms of linear homogeneous functions of *x, y, z, t,* of such a kind that the relation

$$x'^2 + y'^2 + z'^2 - c^2t'^2 = x^2 + y^2 + z^2 - c^2t^2 \qquad (11a)$$

is satisfied identically. That is to say: If we substitute their expressions in *x, y, z, t,* in place of *x', y', z', t',* on the left-hand side, then the left-hand side of (11*a*) agrees with the right-hand side.

We can characterize the Lorentz transformation still more simply if we introduce the imaginary

$$\sqrt{-1}ct$$

in place of *ct,* as time-variable. If, in accordance with this, we insert

$$\begin{aligned} x_1 &= x \\ x_2 &= y \\ x_3 &= z \\ x_4 &= \sqrt{-1}.ct \end{aligned}$$

and similarly for the accented system *K′,* then the condition which is identically satisfied by the transformation can be expressed thus:

$$x_1'^2 + x_2'^2 + x_3'^2 + x_4'^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2 \qquad (12)$$

That is, by the afore-mentioned choice of “co-ordinates” (11*a*) is transformed into this equation.

We see from (12) that the imaginary time co-ordinate x_4 enters into the condition of transformation in exactly the same way as the space co-ordinates x_1, x_2, x_3. It is due to this fact that, according to the theory of relativity, the "time" x_4 enters into natural laws in the same form as the space co-ordinates x_1, x_2, x_3.

A four-dimensional continuum described by the "co-ordinates" x_1, x_2, x_3, x_4, was called "world" by Minkowski, who also termed a point-event a "world-point." From a "happening" in three-dimensional space, physics becomes, as it were, an "existence" in the four-dimensional "world."

This four-dimensional "world" bears a close similarity to the three-dimensional "space" of (Euclidean) analytical geometry. If we introduce into the latter a new Cartesian co-ordinate system (x'_1, x'_2, x'_3) with the same origin, then x'_1, x'_2, x'_3, are linear homogeneous functions of x_1, x_2, x_3, which identically satisfy the equation

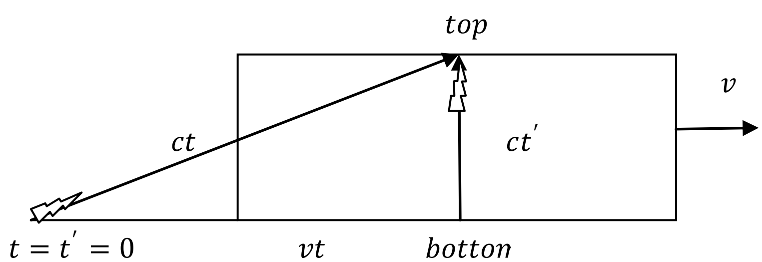
$${x_1}'^2 + {x_2}'^2 + {x_3}'^2 = {x_1}^2 + {x_2}^2 + {x_3}^2$$

The analogy with (12) is a complete one. We can regard Minkowski's "world" in a formal manner as a four-dimensional Euclidean space (with imaginary time co-ordinate); the Lorentz transformation corresponds to a "rotation" of the co-ordinate system in the four-dimensional "world."

Appendix 7.2 Alternative derivation of time dilation and length contraction.

First we derive the relation between the time t in our coordinate frame and the time t' in a system moving with a velocity v. We take the origin of our frame equal to the origin of the moving frame at the time zero where $t = t' = 0$. As was stated by Einstein the velocity of light in our frame is the same as in the moving frame.

We consider a person in a fast moving object, for instance a rocket, who switches on a light flash in the direction perpendicular to the velocity direction of the rocket.



The time in our frame is depicted with t and the time in the moving rocket with t'.

In the drawing above, the flash goes straight to the "top" and the height when reaching the other side of the rocket is ct'. While the light flash moves from the "bottom" to the "top" in the rocket then, from the moment that the flash was initiated the rocket moves, in our not moving frame, horizontally to the right and so the flash moves also from the left to the right while it also moves to "top". Now we consider the relation between the covered distances of the flash in the rocket and of the flash in our frame. So the next calculation can be done:

$$c^2t^2 = c^2t'^2 + v^2t^2$$

$$c^2t^2 - v^2t^2 = c^2t'^2$$

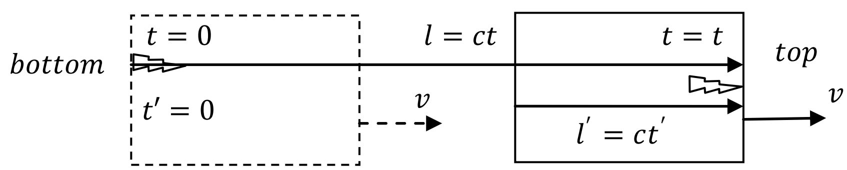
$$t^2(c^2 - v^2) = c^2t'^2$$

$$t^2\left(1 - \frac{v^2}{c^2}\right) = t'^2$$

$$t' = t\sqrt{1-\frac{v^2}{c^2}}$$

So from this calculation it has been shown, seen from our reference frame, that the time *t'* in the rocket is always less than the time *t* in our frame.

Next a flash is sent in the horizontal direction. We consider the distance the flash has covered in our frame and in the rocket. As the rocket is moving horizontally to the right there is no movement in the vertical direction. So the vertical direction should not be affected. The horizontal direction is influenced due to the horizontal velocity.



We consider the length of the distance the flash has covered from the moment it started from the left side, bottom, of the rocket till it reached the right side, top, of the rocket. From the starting point in our frame the distance is *l=ct* and in the rocket *l'=ct'*.

The light velocity in our frame is equal to the light velocity in the rocket as seen from our point of view.

Thus:

$$c = \frac{l}{t} = \frac{l'}{t'}$$

From the first part above we know:

$$t' = t\sqrt{1-\frac{v^2}{c^2}}$$

So

$$\frac{l}{t} = \frac{l'}{t'} = \frac{l'}{t\sqrt{1-\frac{v^2}{c^2}}}$$

$$l = \frac{l'}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$l' = l\sqrt{1-\frac{v^2}{c^2}}$$

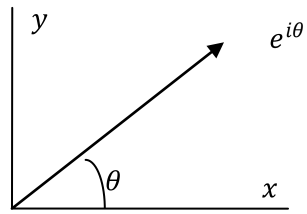
Thus the resulting relations are:

$$t' = t\sqrt{1-\frac{v^2}{c^2}} \quad and \quad l' = l\sqrt{1-\frac{v^2}{c^2}}$$

So seen from our reference frame the time in the rocket frame is less than our time and the length of the rocket is shorter.

Appendix 7.3 Goniometric Tools

As goniometric formulas are frequently used in Special Relativity we give a brief overview over a number of them and how they can easily be derived.



Per definition:

$$e^{i\theta} = \cos\theta + i\sin\theta \qquad (1)$$

$$i = \sqrt{-1}$$

Substantiation of this equation:

First we consider a function $F(x) = e^{\alpha x}$. The derivative is $\frac{d\, e^{\alpha x}}{dx} = \alpha e^{\alpha x}$ so $\frac{dF(x)}{dx} = \alpha F(x)$

Thus the derivative of a function $F(x) = e^{\alpha x}$ is a factor times that function.

Next we consider a function $F(x) = \cos\alpha x + i\sin\alpha x$ of which the derivative is:

$$\frac{d(\cos\alpha x + i\sin\alpha x\,)}{dx} = -\alpha\sin\alpha x + i\alpha\cos\alpha x = i\alpha(\cos\alpha x + i\sin\alpha x\,)$$

Here we see again that $\frac{dF(x)}{dx} = i\alpha F(x)$ where

$$F(x) = e^{i\alpha x} = \cos\alpha x + i\sin\alpha x\,.$$

From this we can deduct:

$$e^{i\theta} = \cos\theta + i\sin\theta \qquad (1)$$

From this, all goniometric formulas can be derived:

Like:

$$e^{-i\theta} = \cos\theta - i\sin\theta \qquad (2)$$

(1)+(2) gives:

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

And (1)-(2) gives:

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$e^{i\theta}\cdot e^{-i\theta} = e^{i\theta - i\theta} = e^0 = 1 = (\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta) = \sin^2\theta + \cos^2\theta = 1$$

Next we define the hyperbolic functions:

$$\cosh \mathrm{x} = \frac{e^x + e^{-x}}{2}$$

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

From these equations we can derive:

$$\cosh(\mathrm{x}) = \cosh(-\mathrm{x})$$
$$\sinh(x) = -\sinh(-x)$$
$$\cosh(\mathrm{ix}) = \cos(\mathrm{x})$$
$$-\mathrm{isinh}(ix) = \sin(x)$$

With these tools we should be able to derive all goniometric equations needed.

Appendix 7.4 Adding of Velocities

We consider two coordinate systems A and B that move with a constant velocity *v* m/s with respect to each other. The coordinate systems are chosen such that the relative movement between the systems is along their x-axes. In A we move an object with velocity V' with its components in all directions. So now we have to consider the velocity of the object with respect to system B. According to Newton the added velocity with respect to the B system is V_x'+v. Now according to the Special Relativity theory:

First we start with the equations for the Lorentz transformation derived in the previous chapters:

$$ct^{'} = \frac{ct - \frac{v}{c}x}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma(ct - \beta x) \qquad (1)$$
$$x^{'} = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma(x - \beta ct) \qquad (2)$$
$$y^{'} = y$$
$$z^{'} = z$$

Here is

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \quad and\ \beta = \frac{v}{c}$$

The velocity *v* of the origin of frame A with respect to frame B is here in the *x*-direction.

The relation between system B and A:

$$ct = \gamma(ct^{'} + \beta x^{'}) \qquad (1a)$$
$$x = \gamma(x^{'} + \beta ct^{'}) \qquad (2a)$$
$$y = y^{'}$$
$$z = z^{'}$$

The velocity in the x'direction in system A can be found by taken the derivative of (2):

$$V_x^{'} = \frac{dx'}{dt'} = \gamma\left(\frac{\partial x}{\partial t}\frac{dt}{dt'} - \beta c\frac{dt}{dt'}\right) = \gamma\left(\frac{\partial x}{\partial t} - \beta c\right)\frac{dt}{dt'} = \gamma(V_x - \beta c)\frac{dt}{dt'} \qquad (3)$$

The derivative of (1):

$$c\frac{dt'}{dt'} = c = \gamma\left(c\frac{dt}{dt'} - \beta\frac{\partial x}{\partial t}\frac{dt}{dt'}\right) = \gamma\left(c - \beta\frac{\partial x}{\partial t}\right)\frac{dt}{dt'} = \gamma(c - \beta V_x)\frac{dt}{dt'}$$
$$\frac{dt}{dt'} = \frac{1}{\gamma\left(1 - \frac{\beta V_x}{c}\right)} \qquad (4)$$

Then fill (4) in (3):

$$\boldsymbol{V_x'} = \frac{\gamma(V_x - \beta c)}{\gamma\left(1 - \frac{\beta V_x}{c}\right)} = \frac{\boldsymbol{V_x - \beta c}}{\boldsymbol{1 - \frac{\beta V_x}{c}}} \quad (5)$$

Velocity in the y' direction:

$$V_y' = \frac{\partial y'}{\partial t'} = \frac{\partial y}{\partial t'} = \frac{\partial y}{\partial t}\frac{dt}{dt'} = V_y \frac{dt}{dt'} = \frac{V_y}{\gamma\left(1 - \frac{\beta V_x}{c}\right)}$$

$$=> \quad \boldsymbol{V_y'} = \frac{\boldsymbol{V_y}}{\boldsymbol{\gamma\left(1 - \frac{\beta V_x}{c}\right)}} \qquad (6)$$

Similar for the z' direction:

$$\boldsymbol{V_z'} = \frac{\boldsymbol{V_z}}{\boldsymbol{\gamma\left(1 - \frac{\beta V_x}{c}\right)}} \qquad (\boldsymbol{7})$$

Second look at equations (4):

$$\frac{dt}{dt'} = \frac{1}{\gamma\left(1 - \frac{\beta V_x}{c}\right)} = \frac{\sqrt{1 - \frac{v^2}{c^2}}}{1 - \frac{vV_x}{c^2}}$$

In the special case when $V_x' = 0$ then $V_x = v$:

$$\frac{dt}{dt'} = \frac{\sqrt{1 - \frac{v^2}{c^2}}}{1 - \frac{v^2}{c^2}} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} => dt' = \sqrt{1 - \frac{v^2}{c^2}}\, dt \text{ thus } dt' \ll dt$$

Back to the general case:

For the X component (5):

$$V_x' = \frac{V_x - \beta c}{1 - \frac{\beta V_x}{c}} = \frac{V_x - v}{1 - \frac{vV_x}{c^2}}$$

Or:

$$V_x = \frac{V_x' + v}{1 + \frac{vV_x'}{c^2}}$$

Now a similar derivation via equations (1a) and (2a) gives:

$$\boldsymbol{V_x} = \frac{\boldsymbol{V_x' + \beta c}}{\boldsymbol{1 + \frac{\beta V_x'}{c}}} \qquad (\boldsymbol{5a})$$

$$\boldsymbol{V_y} = \frac{\boldsymbol{V_y'}}{\boldsymbol{\gamma\left(1 + \frac{\beta V_x'}{c}\right)}} \qquad (\boldsymbol{6a})$$

$$\boldsymbol{V_z} = \frac{\boldsymbol{V_z'}}{\boldsymbol{\gamma\left(1 + \frac{\beta V_x'}{c}\right)}} \qquad (\boldsymbol{7a})$$

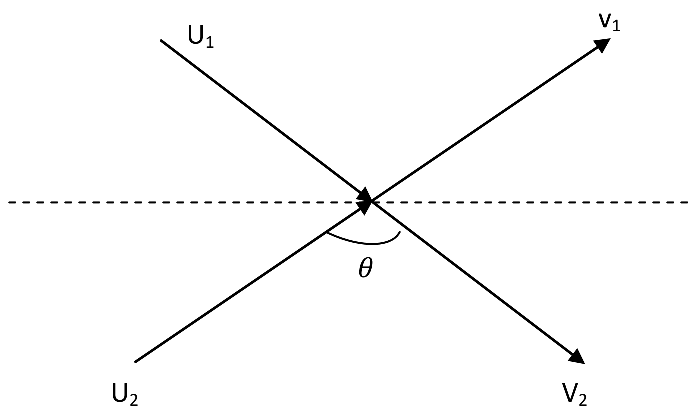
So via Newton we would have got in the x-direction an added velocity of $V_x^{'} + v$, but according to the Special Relativity theory, the Newton result is corrected to $\frac{V_x^{'}+v}{1+\frac{vV'_x}{c^2}}$. In general when the term $vV_x^{'}$ is much smaller than c^2 we can approach the result to the Newton outcome $V_x^{'} + v$.

Appendix 7.5 Collisions

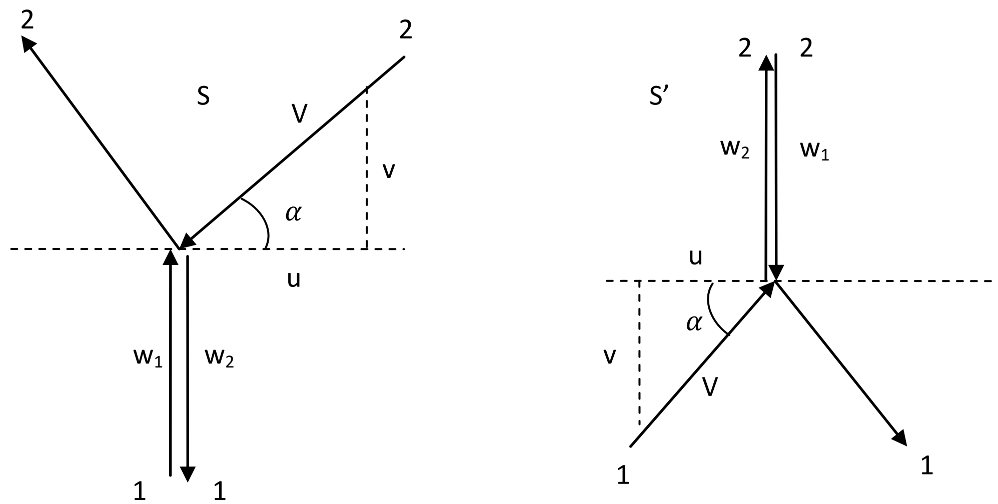
Assume a perfect elastic collision between two identical particles; an elastic collision is a collision with no loss of kinetic energy. The initial velocities of the particles are $\vec{u}_1$ *and* $\vec{u}_2$ respectively and after the collision $\vec{v}_1$ *and* $\vec{v}_2$. Because of conservation of momentum:

$$m_{1u}u_1 + m_{2u}u_2 = m_{1v}v_1 + m_{2v}v_2$$

Here are m_{1u} *and* m_{2u} the two masses before the collision and m_{1v} *and* m_{2v} after the collision.



First we consider the collision from a coordinate system that moves with particle one. Then particle 1 moves upwards with velocity w_1 and downwards with w_2. These velocities are equal but opposite.
Particle 2 has velocity $\vec{V}$ with a x-component *u* and a y-component *v.*



Left: Collision between two identical particles in a coordinate system S co-moving with particle 1. Right: The same but now S' co-moving with particle 2.

Now we have to find the relationship between the y-components of the momentum of particle 1 and 2 in system S, thus *w* and *v*. In the previous chapter there we found the following relation:

$$V_y' = \frac{V_y}{\gamma\left(1-\frac{\beta V_x}{c}\right)}$$

As we have:

$$V_y = w$$
$$V_x = 0$$

Then:

$$v = \frac{w}{\gamma}$$

Because of the symmetry is w here the velocity of particle 1 in system S and the velocity of particle 2 in S'; v is the y-component of particle 2 in S and particle 1 in S'.

The total velocity of the moving particle 1 in S and of the moving particle in S' is the same i.e.: $V = \sqrt{v^2+u^2}$.
The conservation of momentum in the y-direction gives now:

$$m_w w - m_V v = -m_w w + m_V v$$
$$=> \ m_w w = m_V v$$
$$=> \frac{m_V}{m_w} = \frac{w}{v} = \frac{w}{w/\gamma} = \gamma \qquad (1)$$

Assume that the velocity w is very small. In this limit $lim_{w\to 0} v = 0$ and $lim_{w\to 0} V = u$. In that case the relativistic effects can be neglected and the classical expression for momentum can be retrieved.
Thus

$$lim_{w\to 0} m_w = m$$

Fill this in (1):

$$lim_{w \to 0} m_V = \gamma m = \frac{m}{\sqrt{1-\frac{u^2}{c^2}}}$$

Because of conservation of momentum the definition of momentum shall be adapted. This relativistic extension is:

$$\vec{p} = \gamma m \vec{v}$$

Appendix 7.6 The Derivation of E=mc2

Einstein found the equation $E=mc^2$ by means of his so-called thought experiments:
There is a stationary box floating in space, not influenced by any gravitational forces. When from the left a photon is emitted and travels towards the right, the box will move a bit to the left because of the conservation of momentum. At some time the photon collides with the right hand side of the box whereby all of its momentum is transferred to the box. Because of the conservation of momentum the box stops moving.
The photon has moved and the box has moved while there were no external forces. Thus the centre of the mass of the system should stay fixed.
As we know the energy of a photon is E=hν, where ν is the frequency of the light. The momentum of a photon is inversely proportional to the wavelength λ and is given by $p = h/\lambda$. The wavelength is $\lambda = cT = \frac{c}{\nu} => \lambda\nu = c$. (T is the time of one period).

$$E = h\nu = p\lambda\nu = pc$$

So the momentum of the photon:

$$p_{photon} = \frac{E}{c}$$

The box with mass M will move a bit in the opposite direction with speed *v.*
The momentum of the box is:

$$p_{box} = Mv$$

In the time Δt the photo will reach the other side. In this time the box has moved Δx. The speed of the box is:

$$v = -\frac{\Delta x}{\Delta t}$$

Because of the conservation of momentum $p_{photon} + p_{box} = 0 \; => p_{box} = -p_{photon}$. Thus:

$$M\frac{\Delta x}{\Delta t} = \frac{E}{c}$$

The length of the box is *L* and the time for the photon to reach the other side of the box is:

$$\Delta t = \frac{L}{c}$$

So

$$M\Delta x = \frac{EL}{c^2}$$

Suppose, hypothetically, that the photon has some mass *m*. Then the centre of mass of the whole system can be calculated. If the position of the box is x_1 and the photon has position x_2, then the centre for the whole system is

$$\bar{x} = \frac{Mx_1 + mx_2}{M+m}$$

It is required that the centre of the whole system does not change. So the centre of the mass must be the same at the end of the experiment as at the start:

$$\frac{Mx_1 + mx_2}{M+m} = \frac{M(x_1 - \Delta x) + mL}{M+m}$$

The photon starts at $x_2 = 0$ so we get:

$$mL = M\Delta x$$

Now we get:

$$mL = \frac{EL}{c^2}$$

With some rearrangement:

$$\boldsymbol{E = mc^2}$$

Note:

It seems that in this derivation an approximation is made, because when the photon reaches the other side of the box, the box has moved Δx in opposite direction so that the total path of the photon is $L - \Delta x$, and not just L. Moreover there is also a relativistic effect, the Lorentz contraction due to the velocity *v* of the box. So the path becomes:

$$L\sqrt{\left(1 - \frac{v^2}{c^2}\right)} - \Delta x$$

This leads to:

$$\Delta t = \frac{L\sqrt{\left(1 - \frac{v^2}{c^2}\right)} - \Delta x}{c}$$

$$M\frac{\Delta x}{\Delta t} = \frac{E}{c}$$

$$M\Delta x = \frac{E}{c}\,\Delta t$$

So

$$M\Delta x = \frac{E\left(L\sqrt{\left(1 - \frac{v^2}{c^2}\right)} - \Delta x\right)}{c^2}$$

Now:

$$\frac{Mx_1 + mx_2}{M + m} = \frac{M(x_1 - \Delta x) + m\left(L\sqrt{\left(1 - \frac{v^2}{c^2}\right)} - \Delta x\right)}{M + m}$$

$$=> -M\Delta x + m\left(L\sqrt{\left(1 - \frac{v^2}{c^2}\right)} - \Delta x\right) = 0$$

$$m\left(L\sqrt{\left(1 - \frac{v^2}{c^2}\right)} - \Delta x\right) = M\Delta x = \frac{E\left(L\sqrt{\left(1 - \frac{v^2}{c^2}\right)} - \Delta x\right)}{c^2}$$

$$\frac{E\left(L\sqrt{\left(1 - \frac{v^2}{c^2}\right)} - \Delta x\right)}{c^2} = m\left(L\sqrt{\left(1 - \frac{v^2}{c^2}\right)} - \Delta x\right)$$

$$\boldsymbol{E = mc^2}$$

Luckily it ends up in the same equation.

Appendix 7.7 The Energy of a Moving Object

With the thought experiment Einstein showed that energy and mass are equivalent via the relation $E = mc^2$. We have shown that for an object that moves with a velocity the momentum has to be adapted to the relativistic description

$$\vec{p} = \gamma m \vec{v}$$

So it can be postulated that the energy of an object is equal to

$$E = \gamma mc^2.$$

Thus

$$E = \frac{mc^2}{\sqrt{1 - \frac{u^2}{c^2}}}$$

With Taylor's expansion:

$$E = \gamma mc^2 \approx mc^2 \left(1 + \frac{v^2}{2c^2} - \frac{3v^4}{8c^4} \dots \dots.\right)$$

If v is much smaller than c, then the third and subsequent terms, within the bracket, could be neglected. This will lead to:

$$E \approx mc^2 + \frac{1}{2} mv^2$$

So this is the kinetic energy $\frac{1}{2} mv^2$ plus a constant mc^2.

Appendix 7.8 Energy Momentum Vector

As found by Minkowski:

$$c^2 d\tau^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 \tag{1}$$

$$c^2 d\tau^2 = c^2 dt^2 \left(1 - \frac{dx^2 + dy^2 + dz^2}{c^2 dt^2}\right)$$

$$d\tau^2 = dt^2 \left(1 - \frac{v^2}{c^2}\right)$$

As

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \;=>\; 1 - \frac{v^2}{c^2} = \frac{1}{\gamma^2}$$

$$d\tau^2 = dt^2 \left(1 - \frac{v^2}{c^2}\right) = \frac{dt^2}{\gamma^2}$$

$$=> \boldsymbol{\gamma = \frac{dt}{d\tau}}$$

From (1):

$$c^2 = c^2 \frac{dt^2}{d\tau^2} - \frac{dx^2}{dt^2}\frac{dt^2}{d\tau^2} - \frac{dy^2}{dt^2}\frac{dt^2}{d\tau^2} - \frac{dz^2}{dt^2}\frac{dt^2}{d\tau^2}$$

$$m^2 c^2 = m^2 c^2 \frac{dt^2}{d\tau^2} - m^2 \frac{dx^2}{dt^2}\frac{dt^2}{d\tau^2} - m^2 \frac{dy^2}{dt^2}\frac{dt^2}{d\tau^2} - m^2 \frac{dz^2}{dt^2}\frac{dt^2}{d\tau^2}$$

$$m^2 c^2 = \gamma^2 m^2 c^2 - \gamma^2 m^2 {v_x}^2 - \gamma^2 m^2 {v_y}^2 - \gamma^2 m^2 {v_z}^2$$

$$p^2 = \left(\frac{E}{c}\right)^2 - {p_x}^2 - {p_y}^2 - {p_z}^2$$

$$p_0 = \frac{E}{c}$$

$$p_1 = p_x$$

$$p_2 = p_y$$

$$p_3 = p_z$$

$$p^2 = \left(\frac{E}{c}\right)^2 - |\vec{p}|^2 = m^2c^2$$

$$E^2 - c^2|\vec{p}|^2 = m^2c^4$$

$$E = \pm\sqrt{m^2c^4 + c^2|\vec{p}|^2}$$

Or

$$E^2 = m^2c^4 + c^2|\vec{p}|^2$$

Where

$$p = \gamma mv = \frac{mv}{\sqrt{1 - \frac{v^2}{c^2}}}$$

And m is the rest mass (mass at zero velocity)

Or via the relation:

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$m^2\left(1 - \frac{v^2}{c^2}\right) = {m_0}^2$$

$$m^2 - \frac{m^2v^2}{c^2} = {m_0}^2$$

$$m^2 - \frac{p^2}{c^2} = {m_0}^2$$

Multiply with c^4:

$$m^2c^4 - p^2c^2 = {m_0}^2c^4$$
$$m^2c^4 = p^2c^2 + {m_0}^2c^4$$
$$E^2 = p^2c^2 + {m_0}^2c^4$$

Or:

$$E^2 = p^2c^2 + {E_0}^2$$
$$E_0 \; is\; the\; rest\; energy \;\; (v = 0)$$

Appendix 7.8.1 Alternative derivation of the Energy-Momentum-Mass relation

$$p = mv$$

$$p = \gamma m_0 v = \gamma m_0 c^2 \frac{v}{c^2}$$

$$pc = \gamma m_0 c^2 \frac{v}{c} = \beta\gamma m_0 c^2$$

Here is

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \ and\ \beta = \frac{v}{c}$$

Now, with use of the above, let's see what is:

$$(pc)^2 + (m_0c^2)^2 = (\beta\gamma m_0 c^2)^2 + (m_0c^2)^2$$
$$(pc)^2 + (m_0c^2)^2 = (m_0c^2)^2(\beta^2\gamma^2 + 1)$$
$$(pc)^2 + (m_0c^2)^2 = (m_0c^2)^2\left(1 + \beta^2\frac{1}{1-\beta^2}\right)$$
$$(pc)^2 + (m_0c^2)^2 = (m_0c^2)^2\left(\frac{1-\beta^2+\beta^2}{1-\beta^2}\right) = (m_0c^2)^2\left(\frac{1}{1-\beta^2}\right) = (m_0c^2)^2\gamma^2$$

$$(pc)^2 + (m_0c^2)^2 = (m_0c^2)^2\gamma^2 = (\gamma m_0c^2)^2 = E^2$$

So:

$$\boldsymbol{E^2 = (pc)^2 + \left(m_0c^2\right)^2}$$

Appendix 7.8.2 Classical proof of Energy Conservation

Energy is the sum of kinetic *K* and potential energy *U*:

$$E = \frac{1}{2}mv^2 + U$$

Time derivative and partial derivative for a one dimensional case:

$$\frac{dE}{dt} = mv\frac{dv}{dt} + \frac{dU}{dx}\frac{dx}{dt} = mva + v\frac{dU}{dx}$$

The force on a particle, according to the potential energy principle, is related to the derivative of the potential energy, *U(x)*:

$$F = -\frac{dU}{dx}$$
$$\frac{dE}{dt} = v\left(ma + \frac{dU}{dx}\right) = v(ma - F)$$

We know that according to Newton:

$$F = ma$$

So

$$\frac{dE}{dt} = 0 => E = constant$$

So the energy *E* is conserved.

Appendix 7.9 Application

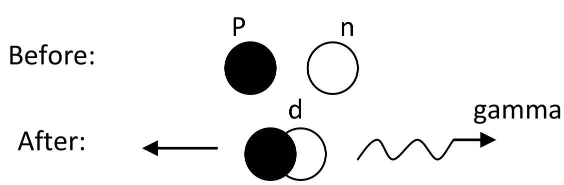
Appendix 7.9.1 Nuclear Fusion and Nuclear Fission

When a proton and a neutron *n* are brought together they could merge and form a core (nucleus) of deuterium (a.k.a. heavy hydrogen) *d*. The masses of *p*, *n*, and *d* are:

$$m_p = 938.27231\ MeV/c^2$$
$$m_n = 939.56563\ MeV/c^2$$
$$m_d = 1875.61339\ MeV/c^2$$

The used unit $\frac{MeV}{c^2}$, needs some elucidation. From the relation $E = mc^2$we can see that mass could be expressed in units of energy divided by a constant c^2. In 'MKSA' units is the unit of energy the Joule, but it is also possible, and customary in high-energy physics, to choose the electron-volt, *eV*. An electron-volt is the amount of energy that a unit-charge gets when it passes a potential difference of 1 Volt. The unit-charge (charge of the electron) is equal to $1.6.10^{-19}$ Coulomb, so $1eV = 1.6.10^{-19}J, 1\ MeV = 10^6 eV$.
Because the mass of the deuteron (=deuterium-core) is smaller than the sum of the masses of the component parts, proton and neutron, energy must have been released! If *p* and *n* are brought together with negligible velocity, then the released energy is equal to:

$$E = m_p c^2 + m_n c^2 - m_d c^2$$
$$= 2.22455\ MeV$$



This energy is released in the form of a photon:

$$p + n \rightarrow d + \gamma$$

A photon is massless; it is a quantum of the electro-magnetic field, introduced by Einstein to explain the photo-electric effect; it is given the symbol γ. Not all the missing mass goes to the energy of the photon. Even if before the reaction *p* and *n* are at rest with respect to each other, then after the reaction the γ will move away with the light velocity. And to warrant the conservation of momentum, *d* shall move in the opposite direction with the same momentum (see figure above). Because of the size of the mass of *d*, is the, with this momentum related, energy very small.

$$Because\ if\ pc \ll mc^2, then\ E = \sqrt{p^2c^2 + m^2c^4} \cong mc^2$$

The above described reaction is an example of nuclear fusion. In general it appears that light cores (nuclei) could merge to heavier cores while energy is released as in the example above. All cores up to and included iron could be produced via fusion while releasing energy.
The opposite effect is that heavier cores, like the well known example of Uranium, are heavier than the sum of the component parts of the core. In that case, energy is only released when the cores are split (nuclear fission).

Appendix 7.10 Relativistic electromagnetism

(Calculations based on Richard Feynman https://www.feynmanlectures.caltech.edu/II_13.html)

Appendix 7.10.1 Introduction

The word electromagnetism assumes there is an electric field and a magnetic field and consequently it suggests that there are sources for both fields. However we know that the electric charge is the source for the electric field and up till now there are no magnetic sources found for the magnetic field. It seems that a magnetic field is always caused by an in time varying electric field. Even on microscopic scale, the quantum scale, the magnetic field is caused by electric spins of electrons or atoms. The electric field has as sources the electrons, -1e, and the protons, +1e.

So perhaps we may go so far that we can say that the magnetic field model is only a very useful mathematical tool to describe the electromagnetic phenomenon; but the only thing there is, is the electric field and the varying of the electric field based on accumulations of electrons and of protons.

Appendix 7.10.2 Calculations

If we take as an example a current carrying wire, then normally we can calculate with the Maxwell equations the electric and magnetic field. An alternative approach is to do the calculation totally based on the electric field and bypassing the magnetic part.

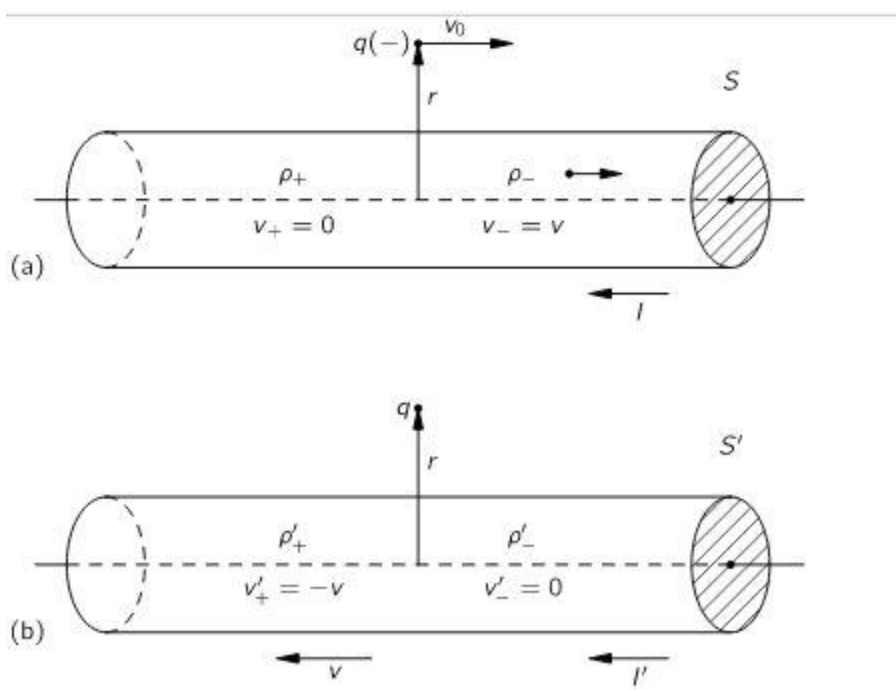


Fig. 1.The interaction of a current-carrying wire and a particle with the charge q as seen in two frames. In frame S (part a), the wire is at rest; in frame S' (part b), the charge is at rest.

We will derive some formulae for later use.

The current density is the average flow velocity of the charges. Assume that there is a distribution of charges with an average motion with velocity *v*. The distribution passes over a surface element ΔS, the charge Δq passing through the surface element in a time Δt is equal to the charge

$$\Delta q = \rho \boldsymbol{v} \cdot \boldsymbol{n} \Delta S \Delta t \quad (1)$$

Here is ρ the charge density: the charge per unit volume.

Here $\boldsymbol{v}\Delta t \cdot \Delta \boldsymbol{S}$ could be considered as a volume. So the charge is the charge density times the volume.

The charge per unit time is then $\rho \boldsymbol{v} \cdot \boldsymbol{n} \Delta S$,which gives:

$$\boldsymbol{j} = \rho \boldsymbol{v} \quad (\mathbf{2})$$

The total current through the surface *S* is:

$$\boldsymbol{i} = \boldsymbol{j} \cdot \boldsymbol{S} \quad (3)$$

We now consider a current-carrying wire at rest and electrons, negative charged particles, moving with a velocity *v* to the right. The protons, positive charged particles, stay at rest in the wire. A test particle, with a negative charge q_- , is moving with the same velocity as the electrons to the right. We observe the whole at rest relative to the wire. The total wire distributes all the charges such that it is neutral.

Let us consider the external, from the wire, force which can be caused by the electric and magnetic fields:

$$F = q(E + v \times B)$$

$$B = \mu_0 H$$

The magnetic field around the wire is:

$$H = \frac{i}{2\pi r}$$

Consider the force on the test particle where the electric field is zero because the total charge in the wire is neutral:

$$F = q(v \times B) = qvB \sin \varphi$$

As *v* is perpendicular with *B* then $\sin \varphi = 1$

$$F = qvB = qv\mu_0 H = \frac{qv\mu_0 i}{2\pi r}$$

The charge density ρ is defined as the total charge in a volume divide by the size of the volume V:

$$\rho = \frac{q}{V}$$

If *A* is the cross-sectional area of the wire and *L* is the arbitrarily length of the volume, at rest, along the wire then:

$$q = \rho A L$$

When wire is at rest

$$\rho_+ + \rho_- = 0$$

If we consider now the situation from the test particle perspective the test particle is at rest and the wire is moving to the left with a velocity *v*.

The volume is determined by *A* and its length *L*. The length between a moving volume and a volume at rest is

$$L_{moving} = L_{rest}\sqrt{1-\frac{v^2}{c^2}}$$

As the velocity of the electrons is the same as the chosen velocity of the test particle the electrons are also now at rest. This means that

$$L_{rest} = \frac{L_{moving}}{\sqrt{1-\frac{v^2}{c^2}}}$$

Because the wire has now a velocity of *v* to the left the positive particle also move with *v* to the left and the length *L* of the volume changes with the factor:

$$\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

When the wire was at rest the external electric field outside the wire was

$$\rho_+ + \rho_- = \rho_+ - \rho_+ = 0$$

Because the moving length is smaller than the rest length, the moving volume is smaller as well. So consequently the density of the charged particle is bigger. So if we consider the charge density when the test particle is at rest then we should multiply the moving density ρ_- with $\sqrt{1-\frac{v^2}{c^2}}$.

So now the electric field outside the wire is determined by the total charge density:

$$\rho_{netto} = \frac{\rho_+}{\sqrt{1-\frac{v^2}{c^2}}} + \rho_-\sqrt{1-\frac{v^2}{c^2}} = \frac{\rho_+}{\sqrt{1-\frac{v^2}{c^2}}} - \rho_+\sqrt{1-\frac{v^2}{c^2}} = \rho_+\frac{1-1+\frac{v^2}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$\rho_{netto} = \rho_+\frac{\frac{v^2}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}$$

The volume of a length *L* of the wire gives a charge of:

$$q = \rho_+\frac{\frac{v^2}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}AL$$

So the electric field outside the wire is not zero and perpendicular at the wire. If we consider a tube around the wire of length *L* and distanced from the axis of the wire of *r* the volume is

$$E = \rho_+\frac{\frac{v^2}{c^2}}{2\pi\varepsilon_0 rL\sqrt{1-\frac{v^2}{c^2}}}AL = \rho_+\frac{\frac{v^2}{c^2}}{2\pi\varepsilon_0 r\sqrt{1-\frac{v^2}{c^2}}}A$$

So

$$F' = qE = q\rho_+ \frac{\frac{v^2}{c^2}}{2\pi\varepsilon_0 r\sqrt{1-\frac{v^2}{c^2}}} A$$

For *v* much smaller than *c*:

$$F' = qE = q\rho_+ \frac{1}{2\pi\varepsilon_0 r}\frac{v^2}{c^2} A$$

For the magnetic field in the rest situation was:

$$F = qvB = qv\mu_0 H = \frac{qv\mu_0 i}{2\pi r}$$

This gives, when *J* is the current density through the wire and $J = \rho v$:

$$F = \frac{qv\mu_0 i}{2\pi r} = \frac{qv\mu_0 JA}{2\pi r}$$

$$c^2 = \frac{1}{\varepsilon_0\mu_0} \;=>\mu_0 = \frac{1}{\varepsilon_0 c^2}$$

$$F = \frac{qv\mu_0 JA}{2\pi r} = \frac{qv\rho vA}{2\pi r\varepsilon_0 c^2} = \frac{qv\rho vA}{2\pi r\varepsilon_0 c^2} = q\rho\frac{1}{2\pi\varepsilon_0 r}\frac{v^2}{c^2}A$$

So

$$F' = \frac{F}{\sqrt{1-\frac{v^2}{c^2}}}$$

The forces act in the transverse *y* direction so the momentum in the y direction and the y' direction shall be the same because the transversal velocity is zero.

Now we compare the momentum in y and the y' direction:

$$\Delta p_y = F\Delta t$$

And

$$\Delta p'_y = F'\Delta t'$$

As we know the time for a moving particle appear to be slower than those in the rest system of the particle, so:

$$\Delta t = \frac{\Delta t'}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$\Delta p_y = \Delta p'_y = F\Delta t = F\frac{\Delta t'}{\sqrt{1-\frac{v^2}{c^2}}} = F'\Delta t'$$

$$=> F' = \frac{F}{\sqrt{1-\frac{v^2}{c^2}}}$$

So now has been shown that:

$$F = qvB = \sqrt{1-\frac{v^2}{c^2}}F^{'} = \sqrt{1-\frac{v^2}{c^2}}qE$$

Appendix 7.10.3 Conclusion

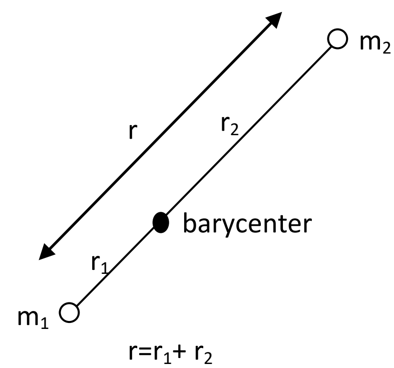
We have found that we get the same physical result whether we analyze the motion of a particle moving along a wire in a coordinate system at rest with respect to the wire, or in a system at rest with respect to the particle. In the first instance, the force was purely “magnetic,” while in the second, it was purely “electric.” It also shows that magnetism is actually a relativistic effect.

Appendix 8 Specific Angular Momentum

In this document, and especially where we use the Schwarzschild equation, the term angular moment is used. It is denoted by the form $L = mr^2 \frac{d\phi}{dt}$. $\left(Because\colon L = mvr = mrv = mr\frac{rd\phi}{dt} = mr^2\frac{d\phi}{dt}\right)$

However it is not the actual angular momentum but an approximation. Next follows an elucidation.

In the Schwarzschild formula there is a relationship between a particle and big massive body. The chosen reference frame is the center of the big massive body. So it is a kind of two body problem. Let us now scrutinize the angular momentum for a two body problem.



The two bodies circle around each other and the center of gravity is called the barycenter. The condition of the circling bodies is that

$$\frac{m_1 v_1^2}{r_1} = \frac{m_2 v_2^2}{r_2} \tag{1}$$

For force symmetry reasons the masses must remain on opposite sides of the barycenter. Thus the periods of the orbits must be equal.

$$T = \frac{2\pi r_1}{v_1} = \frac{2\pi r_2}{v_2} \quad \Longrightarrow \frac{\boldsymbol{v_1}}{\boldsymbol{v_2}} = \frac{\boldsymbol{r_1}}{\boldsymbol{r_2}} \tag{2}$$

$$v_1 = \frac{r_1}{r_2} v_2 = \frac{r_1}{r_2}(v - v_1) \tag{3}$$

$$v_1\left(1 + \frac{r_1}{r_2}\right) = \frac{r_1}{r_2} v \quad \Longrightarrow \boldsymbol{v_1} = \frac{\boldsymbol{r_1}}{\boldsymbol{r}} \boldsymbol{v} \tag{4}$$

And in the same way:

$$v_2 = \frac{r_2}{r_1} v_1 \quad \Rightarrow \boldsymbol{v_2} = \frac{\boldsymbol{r_2}}{\boldsymbol{r}} \boldsymbol{v} \tag{5}$$

The velocity of m_2 with respect to m_1 is:

$$v = v_1 + v_2 \tag{6}$$

Fill (3) in (1):

$$\frac{m_1 v_1^2}{r_1} = \frac{m_1 v_2^2}{r_1}\left(\frac{r_1}{r_2}\right)^2 = \frac{m_2 v_2^2}{r_2} \Longrightarrow \frac{m_1}{r_1}\left(\frac{r_1}{r_2}\right)^2 = \frac{m_2}{r_2}$$

$$\Longrightarrow \boldsymbol{m_1 r_1} = \boldsymbol{m_2 r_2} \tag{7}$$

$$m_2 r_2 = m_1(r - r_2) = m_1 r - m_1 r_2$$

$$r_2(m_1+m_2)=m_1 r \quad \Rightarrow \boldsymbol{r_2}=\frac{\boldsymbol{m_1}}{\boldsymbol{m_1}+\boldsymbol{m_2}}\boldsymbol{r} \tag{8}$$

Let us calculate the angular momentum of m_2 with respect to m_1.

$$L_2=m_2 v_2 r_2=m_2\frac{r_2}{r}v\frac{m_1}{m_1+m_2}r=m_2\frac{m_1}{m_1+m_2}r_2 v=m_2 \quad \left(\frac{m_1}{m_1+m_2}\right)^2 vr \tag{9}$$

$$L_2=\frac{1}{m_2}\left(\frac{m_1 m_2}{m_1+m_2}\right)^2 \omega r^2 \tag{10}$$

$$L_1=\frac{1}{m_1}\left(\frac{m_1 m_2}{m_1+m_2}\right)^2 \omega r^2 \tag{11}$$

The total angular momentum of the two bodies:

$$L=L_2+L_1=\left(\frac{1}{m_2}+\frac{1}{m_1}\right)\left(\frac{m_1 m_2}{m_1+m_2}\right)^2 \omega r^2=\frac{m_1+m_2}{m_1 m_2}\left(\frac{m_1 m_2}{m_1+m_2}\right)^2 \omega r^2=\frac{m_1 m_2}{m_1+m_2}\ \omega r^2$$

To make it in line with the Schwarzschild equation:

$$L=\frac{m_1 m_2}{m_1+m_2}r^2\frac{d\Phi}{d\tau} \tag{12}$$

We call *m* the reduced mass.

$$m=\frac{m_2 m_1}{m_1+m_2} \tag{13}$$

The specific angular momentum *h* is:

$$h=\frac{L}{m}=r^2\frac{d\Phi}{d\tau} \tag{14}$$

In case m_1 stands for a big mass *M* and m_2 the mass of a particle then:

$$\frac{m_2 M}{M+m_2} \Rightarrow m_2 \tag{15}$$

Thus, if $M \gg m_2$ then the mass in the angular momentum equation is determined by the mass of the particle only.

Appendix 9 Deliberations on Rotation

Appendix 9.1 Introduction

Below we will give an elucidation on the centrifugal and centripetal force firstly based on Newton and later we will expand it to general relativity. The centrifugal force is the force from the center of rotation outwards. The centripetal force is directed towards the center.

Appendix 9.2 Momentum

According to Newton a moving particle, with mass *m* and a velocity *v*, has a momentum *mv;* if there are no forces on the particle the particle will move continuously in a straight line with velocity *v*. With respect to a point, with distance *r*, the particle has an angular momentum $m\vec{v} \times \vec{r}$.

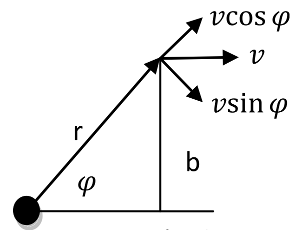


Fig. 1

In the picture above the angular momentum is $L = m\,\mathrm{v}\sin\varphi\,.\,r = mvr\sin\varphi$ or $L = mvb$.

Appendix 9.3 Circle

As is said above the particle will move uniformly in one straight line, so if the trajectory of the particle is a circle, a force is needed.

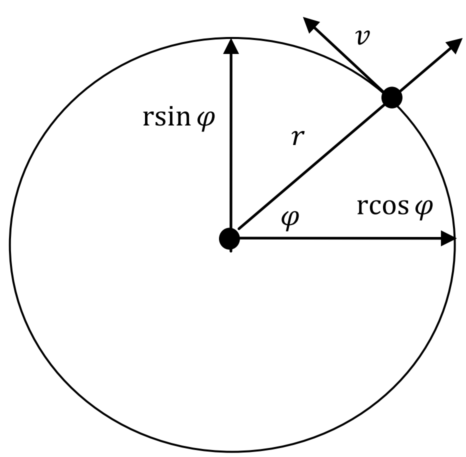


Fig. 2

We start from a constant radius *r* and split this in its *x*- and *y*- component. From there we calculate the circular velocity and acceleration.

$$x = r\cos\varphi = r\cos\omega t$$

$$y = r\sin\varphi = r\sin\omega t$$

$$v_x = \frac{dx}{dt} = -\omega r\sin\omega t$$

$$v_y = \frac{dy}{dt} = \omega r\cos\omega t$$

$$a_x = \frac{d^2x}{dt^2} = -\omega^2 r\cos\omega t$$

$$a_y = \frac{d^2y}{dt^2} = -\omega^2 r\sin\omega t$$

$$F = m\sqrt{{a_x}^2 + {a_y}^2} = -m\omega^2 r$$

Thus the particle wants to move along a straight line but because of its rotation it feels a perpendicular outward force $F = m\omega^2 r$; this force, as is shown above, needs to be compensated with a centripetal reaction force $F = -m\omega^2 r$ towards the center in order to keep the particle on its circular trajectory.

Appendix 9.4 Rotation of a Sphere

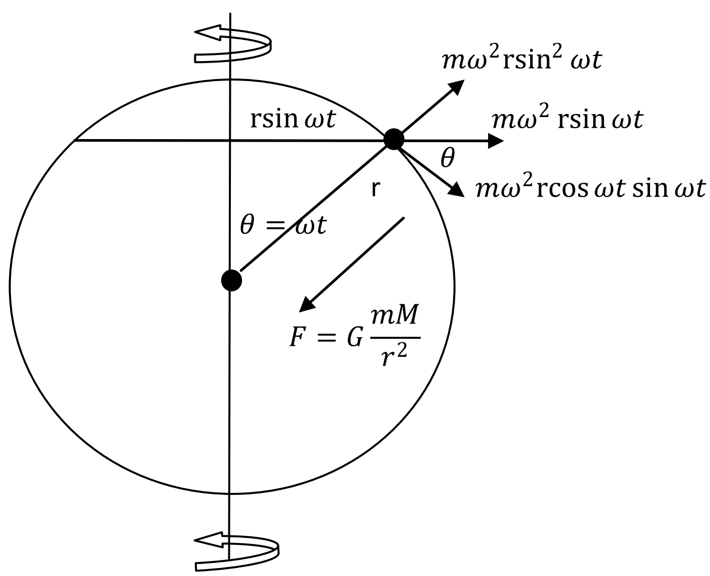


Fig. 3

The particle is rotating around the vertical axis and has a centrifugal force of:

$$m\omega^2\,\mathrm{rsin}\,\omega t$$

This gives a centrifugal force along the radial direction of the sphere:

$$m\omega^2 \mathrm{rsin}^2\,\omega t$$

Together with the centripetal force the resulting force is:

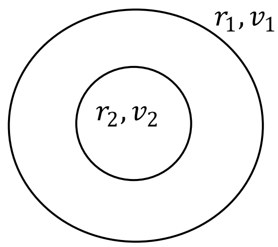
$$G\frac{mM}{r^2} - m\omega^2 \mathrm{rsin}^2\,\omega t = m\left(\frac{GM}{r^2} - \omega^2 \mathrm{rsin}^2\,\omega t\right)$$

And there is a tangential force towards the equator of:

$$m\omega^2 \mathrm{r}\cos\omega t\sin\omega t$$

So particles will feel a force towards the equator and will cause that the sphere will be transformed towards an ellipsoid. This means that the distance from the center to the particle is shortest at the poles and longest at the equator; consequently the gravitational force differs per location. The gravitational force is also depending on the enclosed amount of mass; as the distance from the poles to the center is smallest the enclosed mass is smallest. So the gravitational force at the poles increases because of the smaller distance but decreases because of the enclosed mass. The transformation of the sphere will result in an ellipsoid where there is equilibrium.
(see also: http://farside.ph.utexas.edu/teaching/336k/Newton/node109.html).

Appendix 9.5 Relation between Angular Momentum and Energy



Difference in kinetic energy of the two circles:

$$\Delta K = \frac{1}{2}m{v_1}^2 - \frac{1}{2}m{v_2}^2 \tag{1}$$

The angular moment is constant so:

$$mv_1r_1 = mv_2r_2$$
$$v_2 = \frac{v_1r_1}{r_2} \tag{2}$$

Now (2) in (1):

$$\Delta K = \frac{1}{2}m{v_1}^2 - \frac{1}{2}m\left(\frac{v_1r_1}{r_2}\right)^2 = \frac{1}{2}m{v_1}^2\left(1 - \frac{{r_1}^2}{{r_2}^2}\right) \tag{3}$$

This energy difference ΔK has to be delivered by the centripetal force:

$$F = -\frac{mv^2}{r}$$

Energy is:

$$\int_{r_2}^{r_1} F dr = -\int_{r_2}^{r_1} \frac{mv^2}{r} dr$$

The angular momentum is constant so:

$$mvr = Const$$
$$v = \frac{Const}{mr}$$
$$\int_{r_2}^{r_1} F dr = -\int_{r_2}^{r_1} \frac{m}{r}\frac{Const^2}{m^2r^2} dr = -\int_{r_2}^{r_1} \frac{Const^2}{mr^3} dr = \frac{Const^2}{2mr^2}\Big|_{r_2}^{r_1} = \frac{Const^2}{2m}\left(\frac{1}{{r_1}^2} - \frac{1}{{r_2}^2}\right)$$
$$= \frac{m^2{v_1}^2{r_1}^2}{2m}\left(\frac{1}{{r_1}^2} - \frac{1}{{r_2}^2}\right) = \frac{1}{2}m{v_1}^2\left(1 - \frac{{r_1}^2}{{r_2}^2}\right) \tag{4}$$

It can be seen that formula (3) and (4) are equal so:

$$\Delta K = \int_{r_2}^{r_1} F dr = \frac{1}{2} m {v_1}^2 \left(1 - \frac{{r_1}^2}{{r_2}^2}\right)$$

Appendix 10 Derivation of the Euler-Lagrange equation.

The starting point is a function f_1 which depends on three variables t, x_1 and $\frac{dx_1}{dt}$:

$$f_1 = f\left(t, x_1(t), \frac{dx_1(xt)}{dt}\right) \quad or \quad f_1 = f(t, x_1, \dot{x}_1) \tag{1}$$

Here is x_1 a function of *t,* hence $\frac{dx_1(t)}{dt}$ is not zero. So in principle *t* is the only variable that determines the function f_1. So f_1 is a function of a function.

Now the function f_1 will be considered between the points $t_1$and t_2. An integration of the function f_1 will be made between the points $t_1$and t_2.

$$I_1 = \int_{t_1}^{t_2} f_1 \, dt \tag{2}$$

$$I_1 = \int_{t_1}^{t_2} f\left(t, x_1(t), \frac{dx_1(t)}{dt}\right) dt$$

To find the extreme value, maximum, saddle point or minimum value, of I_1 then:

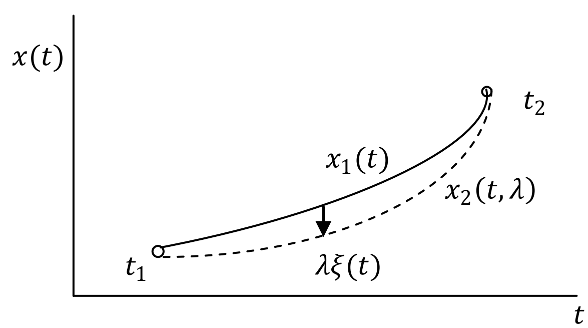
$$\delta I_1 = 0 \tag{3}$$

To proof that I_1 is an extreme we consider a curve $x_2(t)$ that is slightly shifted to show that I_2 is not extreme:

$$x_2(t, \lambda) = x_1(t) + \lambda \xi(t) \tag{4}$$

Here is λ a parameter that is independent of t. As we consider a curve that goes from t_1 to t_2 then $x_2(t)$ differs from $x_1(t)$ between these points but at the points $t_1$and t_2 goes that $x_1(t) = x_2(t)$. Thus:

$$\xi(t_1) = 0 \; and \; \xi(t_2) = 0 \tag{5}$$



Now the integral I_2 for the adjacent curve is:

$$I_2 = \int_{t_1}^{t_2} f_2 \, dt \tag{6}$$

$$I_1 = \int_{t_1}^{t_2} f\left(t, x_2(t,\lambda), \frac{dx_2(t,\lambda)}{dt}\right) dt$$

Filling in (6) the equation (4) will result in:

$$I_2 = \int_{t_1}^{t_2} f\left(t, x_2(t,\lambda), \frac{dx_2(t,\lambda)}{dt}\right) dt \tag{7}$$

$$= \int_{t_1}^{t_2} f\left(t, x_1(t) + \lambda\xi(t), \frac{d(x_1(t) + \lambda\xi(t))}{dt}\right) dt$$

$$= \int_{t_1}^{t_2} f\left(t, x_1(t) + \lambda\xi(t), \frac{dx_1(t)}{dt} + \lambda\frac{d\xi(t)}{dt}\right) dt$$

As I_1 is an extreme value then I_2 is also extreme for $\lambda = 0$:

$$\frac{lim}{\lambda \to 0} I_2 = minimum, saddle\ point\ or\ maximum \tag{8}$$

The extreme value can be found by determining the derivative and set to zero:

$$\frac{lim}{\lambda \to 0} \frac{dI_2}{d\lambda} = 0 \tag{9}$$

Combined with equation (6):

$$\frac{lim}{\lambda \to 0} \frac{d}{d\lambda}\left(\int_{t_1}^{t_2} f_2 \, dt\right) = 0 \tag{10}$$

$$\frac{lim}{\lambda \to 0} \int_{t_1}^{t_2} \frac{d}{d\lambda}(f_2 dt) \ = 0$$

Here we have a product of two functions so we will apply the rule of differentiation by parts:

$$\frac{lim}{\lambda \to 0} \int_{t_1}^{t_2} \left(\frac{df_2}{d\lambda} dt + f_2 \frac{d(dt)}{d\lambda}\right) \ = 0 \tag{11}$$

Because the variables t and λ are mutually independent then the derivative of t to λ , or vice versa, is equal to zero:

$$\frac{lim}{\lambda \to 0} \int_{t_1}^{t_2} \left(\frac{df_2}{d\lambda} dt + f_2 . 0 \right) = 0 \tag{12}$$

$$\frac{lim}{\lambda \to 0} \int_{t_1}^{t_2} \frac{df_2}{d\lambda} dt = 0$$

Next the rule of differentiation by parts will be applied:

$$\frac{lim}{\lambda \to 0} \int_{t_1}^{t_2} \left(\frac{\partial f_2}{\partial t} \frac{dt}{d\lambda} + \frac{\partial f_2}{\partial x_2} \frac{dx_2}{d\lambda} + \frac{\partial f_2}{\partial \left(\frac{dx_2}{dt}\right)} \frac{d\left(\frac{dx_2}{dt}\right)}{d\lambda} \right) dt = 0 \tag{13}$$

Due to the mutual independency of t and λ the first term is zero:

$$\frac{lim}{\lambda \to 0} \int_{t_1}^{t_2} \left(\frac{\partial f_2}{\partial t} . 0 + \frac{\partial f_2}{\partial x_2} \frac{dx_2}{d\lambda} + \frac{\partial f_2}{\partial \left(\frac{dx_2}{dt}\right)} \frac{d\left(\frac{dx_2}{dt}\right)}{d\lambda} \right) dt = 0 \tag{14}$$

$$\frac{lim}{\lambda \to 0} \int_{t_1}^{t_2} \left(\frac{\partial f_2}{\partial x_2} \frac{dx_2}{d\lambda} + \frac{\partial f_2}{\partial \left(\frac{dx_2}{dt}\right)} \frac{d\left(\frac{dx_2}{dt}\right)}{d\lambda} \right) dt = 0$$

Also goes:

$$\frac{d\left(\frac{dx_2}{dt}\right)}{d\lambda} = \frac{d^2 x_2}{dt d\lambda} = \frac{d\left(\frac{dx_2}{d\lambda}\right)}{dt} \tag{15}$$

So equation (14) together with (15) leads to:

$$\frac{lim}{\lambda \to 0} \int_{t_1}^{t_2} \left(\frac{\partial f_2}{\partial x_2} \frac{dx_2}{d\lambda} + \frac{\partial f_2}{\partial \left(\frac{dx_2}{dt}\right)} \frac{d\left(\frac{dx_2}{d\lambda}\right)}{dt} \right) dt = 0 \tag{16}$$

$$\frac{lim}{\lambda \to 0} \left(\int_{t_1}^{t_2} \frac{\partial f_2}{\partial x_2} \frac{dx_2}{d\lambda} dt + \int_{t_1}^{t_2} \frac{\partial f_2}{\partial \left(\frac{dx_2}{dt}\right)} \frac{d\left(\frac{dx_2}{d\lambda}\right)}{dt} dt \right) = 0$$

We will now integrate by parts the right hand side of this equation:

$$\int_{t_1}^{t_2} \frac{\partial f_2}{\partial \left(\frac{dx_2}{dt}\right)} \frac{d\left(\frac{dx_2}{d\lambda}\right)}{dt} dt = \int_{t_1}^{t_2} \frac{\partial f_2}{\partial \left(\frac{dx_2}{dt}\right)} d\left(\frac{dx_2}{d\lambda}\right) \tag{17}$$

$$= \left[\frac{\partial f_2}{\partial \left(\frac{dx_2}{dt}\right)} \frac{dx_2}{d\lambda} \right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{dx_2}{d\lambda} \frac{\partial}{\partial t} \left(\frac{\partial f_2}{\partial \left(\frac{dx_2}{dt}\right)} \right) dt$$

The derivative of x_2 to λ is found to differentiate equation (4):

$$\frac{dx_2(t,\lambda)}{d\lambda} = \frac{d(x_1(t) + \lambda\xi(t))}{d\lambda} = 0 + \xi(t) = \xi(t) \tag{18}$$

As the function $\xi(t)$ is zero at the limits of the integral, see equation (5), the left hand term of the right hand part of the equation (17) disappears:

$$\int_{t_1}^{t_2} \frac{\partial f_2}{\partial\left(\frac{dx_2}{dt}\right)} \frac{d\left(\frac{dx_2}{d\lambda}\right)}{dt} dt = \left[\frac{\partial f_2}{\partial\left(\frac{dx_2}{dx}\right)} \frac{dx_2}{d\lambda}\right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{dx_2}{dt} \frac{\partial}{\partial t}\left(\frac{\partial f_2}{\partial\left(\frac{dx_2}{dt}\right)}\right) dt$$

$$= -\int_{t_1}^{t_2} \frac{dx_2}{dt} \frac{\partial}{\partial t}\left(\frac{\partial f_2}{\partial\left(\frac{dx_2}{dt}\right)}\right) dt \tag{19}$$

This result is combined with equation (16) and results in:

$$\frac{lim}{\lambda \to 0}\left(\int_{t_1}^{t_2} \frac{\partial f_2}{\partial x_2} \frac{dx_2}{d\lambda} dt + \int_{t_1}^{t_2} \frac{\partial f_2}{\partial\left(\frac{dx_2}{dt}\right)} \frac{d\left(\frac{dx_2}{d\lambda}\right)}{dt} dt\right) = 0 \tag{20}$$

$$\frac{lim}{\lambda \to 0}\left(\int_{t_1}^{t_2} \frac{\partial f_2}{\partial x_2} \frac{dx_2}{d\lambda} dt - \int_{t_1}^{t_2} \frac{dx_2}{d\lambda} \frac{d}{dt}\left(\frac{\partial f_2}{\partial\left(\frac{dx_2}{dt}\right)}\right) dt\right) = 0$$

$$\frac{lim}{\lambda \to 0}\left(\int_{t_1}^{t_2}\left(\frac{\partial f_2}{\partial x_2} \frac{dx_2}{d\lambda} - \frac{dx_2}{d\lambda} \frac{d}{dt}\left(\frac{\partial f_2}{\partial\left(\frac{dx_2}{dt}\right)}\right)\right) dt\right) = 0$$

$$\frac{lim}{\lambda \to 0}\left(\int_{t_1}^{t_2}\left(\frac{\partial f_2}{\partial x_2} - \frac{d}{dt}\left(\frac{\partial f_2}{\partial\left(\frac{dx_2}{dt}\right)}\right)\right) \frac{dx_2}{d\lambda} dt\right) = 0$$

In order to have this integral zero we next we state that:

$$\frac{lim}{\lambda \to 0}\left(\frac{\partial f_2}{\partial x_2} - \frac{d}{dt}\left(\frac{\partial f_2}{\partial\left(\frac{dx_2}{dt}\right)}\right)\right) = 0$$

$$\frac{\partial f_1}{\partial x_1} - \frac{d}{dt}\left(\frac{\partial f_1}{\partial\left(\frac{dx_1}{dt}\right)}\right) = 0 \tag{21}$$

Now λ completely disappeared and we achieved a general expression for the condition that a function needed to have such that integral I is an extreme value.

We started with equation (1) for our derivation, but we could make this starting point even more general by taking a function like:

$$f_1 = f\left(t, x_1(t), \frac{dx_1(t)}{dt}, x_2(t), \frac{dx_2(t)}{dt}, \dots \dots \ldots, , x_n(t), \frac{dx_n(t)}{dt}\right) \qquad 22$$

This would have led to a more general form of equation (21):

$$\frac{\partial f}{\partial x_n} - \frac{d}{dt}\left(\frac{\partial f}{\partial\left(\frac{dx_n}{dt}\right)}\right) = 0 \qquad (23)$$

Or in a different notation:

$$\frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x}_n}\right) = \frac{\partial f}{\partial x_n} \qquad (24)$$

The equation (24) is the **Euler-Lagrange equation**. It expresses the condition for a function such that integral I is an extreme value.

8. Bibliography

Anderson, P. B. (2008, October 16). *A Hafele & Keating like thought experiment*. Retrieved from https://paulba.no/pdf/H&K like.pdf

Asmodelle, E. (2017). *Tests of General Relativity: A Review.* University of Central Lancashire (UCLAN).

Bentvelsen, S. (September 2005 version 3.0). *Speciale Relativiteitstheorie.* NIKHEF/Onderzoeksinstituut HEF.

Biesel, O. (2008, January 25). *The Precession of Mercury's Perihelion*. Retrieved from https://sites.math.washington.edu/~morrow/papers/Genrel.pdf

Blau, M. (11 june 2018). *Lecture Notes on General Relativity.*

Blote, H. W. *Introduction to General Relativity.* http://wwwhome.lorentz.leidenuniv.nl/~henk/grh.pdf.

Charles W Misner, K. S. (1932). *Gravitation.* Princeton and Oxford: Princeton University Press.

Crowell, B. (Mar 11, 2018). *Time distortion arising from motion and gravity.* Libretexts.

Einstein Relatively Easy. http://einsteinrelativelyeasy.com/index.php/general-relativity.

Einstein, A. (1920). *RELATIVITY - The special and General Relativity.* created by Jose Menendez.

Einstein, A. (1916 (this revised edition: 1924)). *Relativity: The Special and General Theory.* Methuen & Vo Ltd.

Einstein, A. (1997). *The Collected Papers of Albert Einstein.* Princeton University Press.

Heinicke, Christian and Hehl, Friedrich W. (2015). *Schwarzschild and Kerr Solutions of Einstein's Field Equation.* Dept. Physics & Astron., Univ. of Missouri, Columbia,.

Hooft, G. '. *INTRODUCTION TO GENERAL RELATIVITY.* Institute for Theoretical Physics Utrecht University and Spinoza Institute.

http://adamauton.com/warp/emc2.html.

Janssen, M. *Einstein's First Systematic Exposition of General Relativity.* https://philpapers.org/rec/JANEFS.

M.P Hobson, G. E. (2006). *General Relativity an introduction for Physics.* Cambridge University Press.

Magnan, C. (n.d.). *Complete calculations of the perihelion precession of Mercury an the degelection of light by the Sun in General Relativity*. Retrieved from https://arxiv.org/pdf/0712.3709.pdf

Oas, G. *Full derivation of the Schwarzschild solution.* EPGY Summer Institute SRGR.

Pe'er1, A. (2014, February 19). *Schwarzschild Solution and Black holes*. Retrieved from http://www.physics.ucc.ie/apeer/PY4112/Sch.pdf

Salas, d. C. *Geometric interpretation of Christoffel symbols and some alternative approaches to calculating them.*

Schwarzschild, K. (13 January 1916). *On the Gravitational Field of a Point-Mass, According to Einstein's Theory.*

Schwarzschild, K. (24 February 1916). *On the Gravitational Field of a Sphere of Incompressible Liquid, According to Einstein's Theory.*

Susskind, L. (2012). *General Relativity.* Stanford University.

Visser, M. (30 June 2007). *The Kerr spacetime: A brief introduction.* https://arxiv.org/pdf/0706.0622.pdf.

Vlieger, K. d. (2018). *http://voorbijeinstein.nl/html/artikel_art.htm.* Heeten, Netherlands: Karel de Vlieger.

9. Interesting sites:

http://voorbijeinstein.nl/html/artikel_art_paragraaf_05.htm
http://star-www.st-and.ac.uk/~hz4/gr/HeavensGR.pdf
https://einsteinrelativelyeasy.com/index.php/general-relativity/22-geodesics-and-christoffel-symbols
http://jamesowenweatherall.com/SCPPRG/LehmkuhlDennis2008Man_MassEnergyMomentumGR.pdf
http://www.damtp.cam.ac.uk/user/reh10/lectures/nst-mmii-chapter2.pdf Chapter 2 Poisson's Equation - University of Cambridge
http://www.mathpages.com/home/kmath711.htm Poisson's Equation and the Universe
https://web.stanford.edu/~oas/SI/SRGR/notes/SchwarzschildSolution.pdf Gary Oas
https://ned.ipac.caltech.edu/level5/March01/Carroll3/Carroll7.html
http://www.feynmanlectures.caltech.edu/
http://www.blau.itp.unibe.ch/newlecturesGR.pdf#page577 (Blau, 11 june 2018)
http://adamauton.com/warp/emc2.html
https://web.stanford.edu/~oas/SI/SRGR/notes/SchwarzschildSolution.pdf
https://drchristiansalas.com/2017/03/11/geometric-interpretation-of-christoffel-symbols-and-some-alternative-approaches-to-calculating-them/
https://stanfordvideo.stanford.edu/leonard-susskinds-general-relativity-lecture-1/
https://www.youtube.com/playlist?list=PLQrxduI9Pds1fm91Dmn8x1lo-O_kpZGk8
https://arxiv.org/pdf/0706.0622.pdf

Bundling of My Derivations and Deliberations

On

General Relativity

END

Your feedback is welcome at:
aprins@hotmail.com