1.1.1. Via Schwarzschild approach

A summary of the results derived in the Schwarzschild equation in chapter Trajectories of massive particles-Second Derivation and **Error! Reference source not found.**The semi-major axis is:

$$a = \frac{L^2}{GM(1-e^2)} \tag{2}$$

The parameter *e* measures the eccentricity of the orbit. The perihelion is $r_1 = a(1-e)$ and the aphelion is $r_2 = a(1+e)$

$$e = \sqrt{1 - \frac{b^2}{c^2}} = \frac{r_2 - r_1}{r_2 + r_1}$$

So for a circle *e*=0 and r=a.

$$\boldsymbol{r} = \frac{\boldsymbol{a}(\boldsymbol{1} - \boldsymbol{e^2})}{\boldsymbol{1} - \boldsymbol{ecos}[\emptyset(\boldsymbol{1} - \epsilon)]} \tag{2a}$$

When $\emptyset = 0$ then

$$r = \frac{a(1-e^2)}{1 - ecos[\emptyset(1-\epsilon)]} = \frac{a(1-e^2)}{1-e} = \frac{a(1-e)(1+e)}{1-e} = a(1+e)$$

This is the aphelion.

Find the angle between *v* and the v_{tan} (perpendicular with *r*, to find the angular momentum).

$$\tan\alpha = \frac{dr}{rd\emptyset} = \frac{\{1 - ecos[\emptyset(1-\epsilon)]\}}{a(1-e^2)} \frac{\{a(1-e^2)(esin[\emptyset(1-\epsilon)])\}}{\{1 - ecos[\emptyset(1-\epsilon)]\}^2}$$

$$\tan\alpha = \frac{dr}{rd\emptyset} = \frac{esin[\emptyset(1-\epsilon)]}{1 - ecos[\emptyset(1-\epsilon)]}$$

$$\boldsymbol{\alpha} = \boldsymbol{arctan}\left\{\frac{\boldsymbol{esin}[\emptyset(\boldsymbol{1} - \boldsymbol{\epsilon})]}{\boldsymbol{1} - \boldsymbol{ecos}[\emptyset(\boldsymbol{1} - \boldsymbol{\epsilon})]}\right\}$$

Gives

$$\cos\alpha = \left[1 + \left(\frac{esin[\emptyset(1-\epsilon)]}{1 - ecos[\emptyset(1-\epsilon)]}\right)^2\right]^{-\frac{1}{2}} \tag{2b}$$

$$\cos\alpha = \left[\frac{1 - 2ecos[\emptyset(1-\epsilon)] + \{ecos[\emptyset(1-\epsilon)]\}^2 + \{esin[\emptyset(1-\epsilon)]\}^2}{\{1 - ecos[\emptyset(1-\epsilon)]\}^2}\right]^{-\frac{1}{2}}$$

$$\cos\alpha = \frac{1 - ecos[\emptyset(1-\epsilon)]}{[1 - 2ecos[\emptyset(1-\epsilon)] + e^2]^{1/2}}$$

$$v = \frac{L}{r} = \frac{\left(aGM(1-e^2)\right)^{1/2}}{a(1-e^2)}(1 - ecos[\emptyset(1-\epsilon)])$$

$$v = \frac{\left(aGM(1-e^2)\right)^{1/2}}{(a^2(1-e^2)^2)^{1/2}}(1 - ecos[\emptyset(1-\epsilon)]) = \frac{(GM)^{1/2}\,(1 - ecos[\emptyset(1-\epsilon)])}{\left(a(1-e^2)\right)^{1/2}}$$

$$\mathbf{v} = (\mathbf{1} - \boldsymbol{ecos}[\emptyset(\mathbf{1} - \boldsymbol{\epsilon})]) \cdot \sqrt{\frac{\boldsymbol{GM}}{\boldsymbol{a}(\mathbf{1} - \boldsymbol{e}^{\mathbf{2}})}} \qquad (\mathbf{2}\boldsymbol{d})$$

Velocity along the earth surface:

$$\mathrm{v}_{\tan} = \mathrm{v}\cos\alpha = \frac{(GM)^{1/2}\,(1 - ecos[\emptyset(1-\epsilon)])}{\left(a(1-e^2)\right)^{1/2}} \frac{1 - ecos[\emptyset(1-\epsilon)]}{[1 - 2ecos[\emptyset(1-\epsilon)] + e^2]^{1/2}}$$

$$= \frac{(1 - ecos[\emptyset(1-\epsilon)])^2}{[1 - 2ecos[\emptyset(1-\epsilon)] + e^2]^{1/2}} \cdot \sqrt{\frac{GM}{a(1-e^2)}}$$

$$\mathrm{v}_{\tan} = \frac{\left(1 - ecos\left[\frac{d}{2R}(1-\epsilon)\right]\right)^2}{\left[1 - 2ecos\left[\frac{d}{2R}(1-\epsilon)\right] + e^2\right]^{1/2}} \cdot \sqrt{\frac{GM}{a(1-e^2)}}$$

Or:

$$\mathrm{v} = \frac{\mathrm{v}_{\tan}}{\cos\alpha} = \mathrm{v}_{\tan} \frac{[1 - 2ecos[\emptyset(1-\epsilon)] + e^2]^{1/2}}{1 - ecos[\emptyset(1-\epsilon)]}$$

$$\mathrm{v} = \frac{\mathrm{v}_{\tan}}{\cos\alpha} = \mathrm{v}_{\tan} \frac{\left[1 - 2ecos\left[\frac{d}{2R}(1-\epsilon)\right] + e^2\right]^{1/2}}{1 - ecos\left[\frac{d}{2R}(1-\epsilon)\right]}$$

$$v = \frac{L}{r\cos\alpha} = \frac{\left(aGM(1-e^2)\right)^{1/2}}{a(1-e^2)\cos\alpha}(1 + ecos[\emptyset(1-\epsilon)])$$

$$v = \left(\frac{GM}{a(1-e^2)}\right)^{\frac{1}{2}} \frac{(1 + ecos[\emptyset(1-\epsilon)])}{\cos\alpha} \qquad (2c)$$

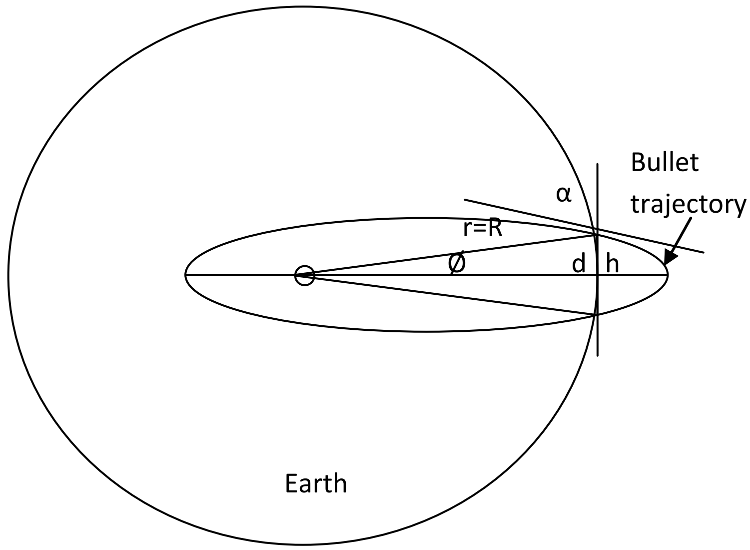
$$v = \left(\frac{GM}{a(1-e^2)}\right)^{\frac{1}{2}} \frac{(1 + ecos[\emptyset(1-\epsilon)])}{1 + ecos[\emptyset(1-\epsilon)]} [1 + 2ecos[\emptyset(1-\epsilon)] + e^2]^{1/2}$$

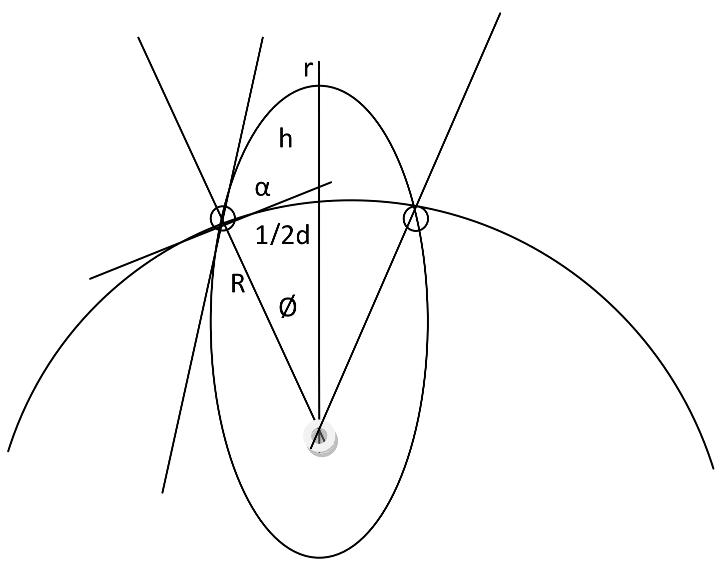
Instantaneous velocity as function of $\emptyset$:

$$\boldsymbol{v} = \left(\frac{\boldsymbol{GM}}{\boldsymbol{a}(\mathbf{1} - \boldsymbol{e}^{\mathbf{2}})}(\mathbf{1} + \mathbf{2}\boldsymbol{ecos}[\emptyset(\mathbf{1} - \boldsymbol{\epsilon})] + \boldsymbol{e}^{\mathbf{2}})\right)^{\frac{\mathbf{1}}{\mathbf{2}}} \qquad (\mathbf{2}\boldsymbol{d})$$

From previous chapter ϵ

$$\epsilon = \frac{3(GM)^2}{c^2L^2} = \frac{3(GM)^2}{c^2aGM(1-e^2)} = \frac{3GM}{c^2a(1-e^2)}$$





From (2d) we will derive *e*:

$$\mathrm{v} = (1 - e\cos[\emptyset(1-\epsilon)]) \cdot \sqrt{\frac{GM}{a(1-e^2)}}$$

$$\mathrm{v}^2 a(1-e^2) = GM(1 - e\cos[\emptyset(1-\epsilon)])^2$$

$$\mathrm{v}^2 a - \mathrm{v}^2 a e^2 = GM(1 - 2e\cos[\emptyset(1-\epsilon)] + e^2\cos^2[\emptyset(1-\epsilon)])$$

$$\mathrm{v}^2 a - \mathrm{v}^2 a e^2 = GM - 2GMe\cos[\emptyset(1-\epsilon)] + GMe^2\cos^2[\emptyset(1-\epsilon)]$$

$$GMe^2cos^2[\emptyset(1-\epsilon)] + \mathrm{v}^2ae^2 - 2GMecos[\emptyset(1-\epsilon)] + GM - \mathrm{v}^2a = 0$$

$$e^2(GMcos^2[\emptyset(1-\epsilon)] + \mathrm{v}^2a) - e2GMcos[\emptyset(1-\epsilon)] + GM - \mathrm{v}^2a = 0$$

$$\boldsymbol{e} = \frac{\boldsymbol{2GMcos[\emptyset(1-\epsilon)] \pm \sqrt{(2GMcos[\emptyset(1-\epsilon)])^2 - 4(GMcos^2[\emptyset(1-\epsilon)] + \mathrm{v}^2a)(GM - \mathrm{v}^2a)}}}{\boldsymbol{(GMcos^2[\emptyset(1-\epsilon)] + \mathrm{v}^2a)}}$$

For the starting point, the intersection of Earth and trajectory *r=R*. (*R* is here the radius of the Earth) and $\emptyset = \frac{d}{2R}$

The given velocity at the *r=R* point is v. Thus for a velocity there are two solutions of *e*.

Here is *h* the highest point of the bullet trajectory

$$h = a(1+e) - R$$

Here *d* is the distance on Earth, *v* is the starting velocity of the bullet and *R* is the Earth radius. As seen above $\emptyset = \frac{d}{2R}$

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$$\emptyset R = \frac{d}{2} => \ \emptyset = \frac{d}{2R}$$

$$v_{x0} = v\cos(\alpha + \emptyset) \ and \ v_{y0} = v\sin(\alpha + \emptyset)$$

From (2a)

$$a(1-e^2) = r\{1 + ecos[\emptyset(1-\epsilon)]\}$$

From (2c)

$$v = \left(\frac{GM}{a(1-e^2)}\right)^{1/2} \frac{(1 + ecos[\emptyset(1-\epsilon)])}{\cos\alpha}$$

$$a(1-e^2) = \frac{GM}{v^2}\frac{(1 + ecos[\emptyset(1-\epsilon)])^2}{(\cos\alpha)^2}$$

Together with (2a)

$$\frac{GM}{v^2}\frac{(1 + ecos[\emptyset(1-\epsilon)])^2}{(\cos\alpha)^2} = \ r\{1 + ecos[\emptyset(1-\epsilon)]\}$$

From (2b)

$$(\cos\alpha)^2 = \frac{1}{1 + \left(\frac{esin[\emptyset(1-\epsilon)]}{1 + ecos[\emptyset(1-\epsilon)]}\right)^2}$$

$$= \frac{(1+ecos[\emptyset(1-\epsilon)])^2}{(1+ecos[\emptyset(1-\epsilon)])^2+(esin[\emptyset(1-\epsilon)])^2}$$

$$= \frac{(1+ecos[\emptyset(1-\epsilon)])^2}{1+2ecos[\emptyset(1-\epsilon)]+(ecos[\emptyset(1-\epsilon)])^2+(esin[\emptyset(1-\epsilon)])^2}$$

$$(\cos\alpha)^2 = \frac{(1+ecos[\emptyset(1-\epsilon)])^2}{1+2ecos[\emptyset(1-\epsilon)]+e^2}$$

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$$\frac{GM}{v^2}(1+ecos[\emptyset(1-\epsilon)])^2\frac{1+2ecos[\emptyset(1-\epsilon)]+e^2}{(1+ecos[\emptyset(1-\epsilon)])^2} = r\{1+ecos[\emptyset(1-\epsilon)]\}$$

$$\frac{GM}{v^2}(1+2ecos[\emptyset(1-\epsilon)]+e^2) = r\{1+ecos[\emptyset(1-\epsilon)]\}$$

$$(1+2ecos[\emptyset(1-\epsilon)]+e^2) = \frac{v^2r}{GM}\{1+ecos[\emptyset(1-\epsilon)]\}$$

$$e^2+ecos[\emptyset(1-\epsilon)]\left(2-\frac{v^2r}{GM}\right)+1-\frac{v^2r}{GM}=0$$

$$e = \frac{-cos[\emptyset(1-\epsilon)]\left(2-\frac{v^2r}{GM}\right)\pm\sqrt{\left[cos[\emptyset(1-\epsilon)]\left(2-\frac{v^2r}{GM}\right)\right]^2-4\left(1-\frac{v^2r}{GM}\right)}}{2}$$

For the starting point, the intersection of Earth and trajectory *r=R*. (*R* is here the radius of the Earth) and $\emptyset = \frac{d}{2R}$

$$a(1-e^2) = R\left\{1+ecos\left[\frac{d}{2R}(1-\epsilon)\right]\right\}$$

$$a = \frac{R\left\{1+ecos\left[\frac{d}{2R}(1-\epsilon)\right]\right\}}{(1-e^2)}$$

$$\boldsymbol{e = \frac{-cos\left[\frac{d}{2R}(1-\epsilon)\right]\left(2-\frac{v^2R}{GM}\right)\pm\sqrt{\left[cos\left[\frac{d}{2R}(1-\epsilon)\right]\left(2-\frac{v^2R}{GM}\right)\right]^2-4\left(1-\frac{v^2R}{GM}\right)}}{2}}$$

The given velocity at the *r=R* point is v. Thus for a velocity there are two solutions of *e*.

Here is *h* the highest point of the bullet trajectory

$$h = a(1-e)-R$$

$$h = \frac{R\left\{1+ecos\left[\frac{d}{2R}(1-\epsilon)\right]\right\}}{(1-e^2)}(1-e)-R = R\left\{\frac{1+ecos\left[\frac{d}{2R}(1-\epsilon)\right]}{1+e}-1\right\}$$

$$h = R\left\{\frac{1+ecos\left[\frac{d}{2R}(1-\epsilon)\right]-1-e}{1+e}\right\} = R\frac{e\left(cos\left[\frac{d}{2R}(1-\epsilon)\right]-1\right)}{1+e}$$

Here *d* is the distance on Earth, *v* is the starting velocity of the bullet and *R* is the Earth radius. As seen above $\emptyset = \frac{d}{2R}$

Result of the example:

	Conservative	Schwarzschild

Horizontal distance (m)	10	10		
Horizontal velocity (m/sec)	5	500		
Total velocity (m/sec)	11.06	500	11.06	500
time	2	$2*10^{-4}$		
Height (m)	4.93	$4.93*10^{-4}$	4.93	$4.91*10^{-4}$

Note: ϵ is hard to solve. By iterative processing this can be approximated or by “seek goal” in the program Excel.

Derivation of the circumference of an ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$$x=acos\beta \text{ and } y=bsin\beta$$

$$Circ.=4a\int_0^{\pi/2}\sqrt{\left(\frac{dx}{d\beta}\right)^2+\left(\frac{dy}{d\beta}\right)^2}\,d\beta$$

$$=4a\int_0^{\pi/2}\sqrt{a^2sin^2\beta+b^2cos^2\beta}\,d\beta$$

$$=4a\int_0^{\pi/2}\sqrt{a^2(1-cos^2\beta)+b^2cos^2\beta}\,d\beta$$

$$=4a\int_0^{\pi/2}\sqrt{a^2-(a^2-b^2)cos^2\beta}\,d\beta$$

$$Circ.=4a\int_0^{\pi/2}\sqrt{1-e^2cos^2\beta}\,d\beta$$

For the **circumference of an ellipse** there is no simple closed solution.
There are approximations, for instance the Ramanujan approximation:

$$Circ.\approx\pi a\left[3\left(1+\sqrt{1-e^2}\right)-\sqrt{10\sqrt{1-e^2}+3(2-e^2)}\right]$$

Detailed results of calculations on the example mentioned above, showing that there are two solutions per initial velocity.

The starting points are the velocity of the bullet and the distance to be covered.

velocity(m/s)	**11**		**500**	
d(m)	**10**	**10**	**10**	**10**
epsilon	5E-03	1E-03	5E-07	3E+00
e(centricity)	-1.00000E+00	-9.99998E-01	-9.96014E-01	-1.00000E+00
a(km)	3178	3178	3185	3185
h(m)	4.9522	1.2624	4.91E-04	12693.05
alpha(deg)	-63.3	-26.8	-0.01	89.98
L (ang. mom.)	3.E+07	6.E+07	3.E+09	1.E+06
cos(alpha)	0.449	0.892	1.000	0.000
cos(alpha+phi)	0.449	0.892	1.000	0.000
vx0(m/s)	4.96	9.87	500.00	0.21
vy0(m/s)	-9.89	-4.99	-0.10	500.00
Circ.(km)	12662	12663	12894	12686