Probability Theory

Assignment Solutions

1. X - minimum of two number that appear when pair of dice is rolled.

   x can be 1, 2, 3, 4, 5, 6

   6: 6 appears in only in one case (6, 6)

   $P(X=6) = \frac{1}{36}$

   5: 5 appears (5,5), (5,6), (6,5)

   $P(X=5) = \frac{3}{36}$

   4: 4 appears (4,4) (4,5), (5,4), (4,6), (6,4)

   $P(X=4) = \frac{5}{36}$

   3: 3 appears (3,3) (3,4) (4,3) (3,5) (5,3) (3,6), (6,3)

   $P(X=3) = 7/36$

   2: 2 appears (2,2) (2,3) (3,2) (2,4) (4,2) (2,5) (5,2) (2,6) (6,2)

   $P(X=2) = 9/36$

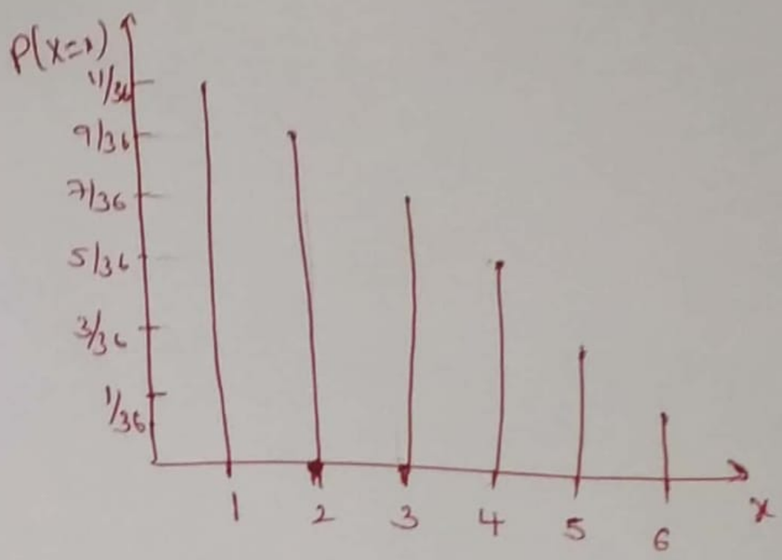
   1: 1 appears (1,1) (1,2) (2,1) (1,3) (3,1) (1,4) (4,1) (1,5) (5,1) (1,6) (6,1)

   $P(X=1) = 11/36$

PMF:

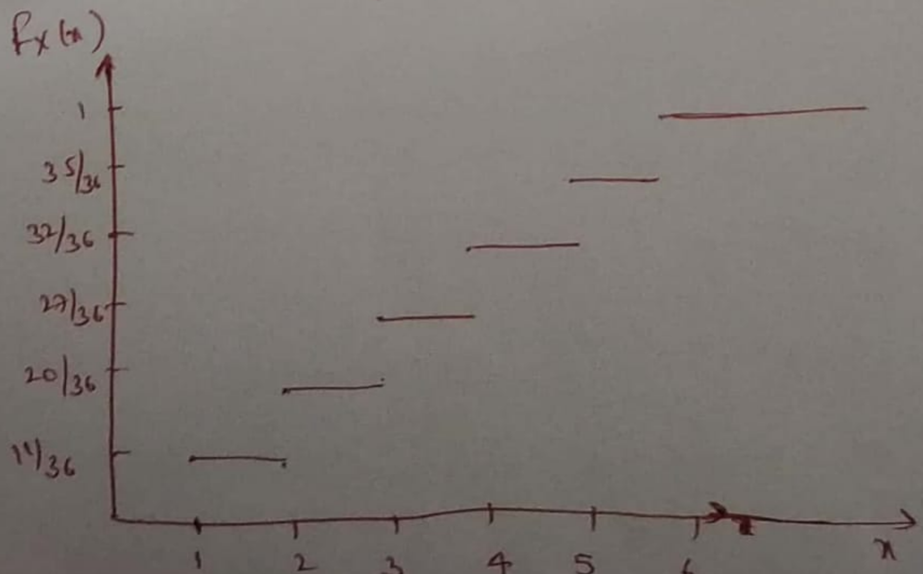
x	1	2	3	4	5	6
$P(X=x)$	$\frac{11}{36}$	$\frac{9}{36}$	$\frac{7}{36}$	$\frac{5}{36}$	$\frac{3}{36}$	$\frac{1}{36}$

PMF (graphically)



CDF

$$F_X(x) = \begin{cases} 0 & x < 1 \\ \frac{11}{36} & 1 \le x < 2 \\ \frac{20}{36} & 2 \le x < 3 \\ \frac{27}{36} & 3 \le x < 4 \\ \frac{32}{36} & 4 \le x < 5 \\ \frac{35}{36} & 5 \le x < 6 \\ 1 & x \ge 6 \end{cases}$$



$$E[X] = \sum x\, P(x)$$

$$= 1 \cdot \frac{11}{36} + 2 \cdot \frac{9}{36} + 3 \cdot \frac{7}{36} + 4 \cdot \frac{5}{36} + 5 \cdot \frac{3}{36} + 6 \cdot \frac{1}{36} = \underline{\underline{2.5}}$$

$$\boxed{E[X] = 2.5}$$

$$Var(X) = E[X^2] - (E[X])^2$$

$$= \sum x^2 f(x) - (E[X])^2$$

$$= 1 \cdot \frac{11}{36} + 2^2 \cdot \frac{9}{36} + 3^2 \cdot \frac{7}{36} + 4^2 \cdot \frac{5}{36} + 5^2 \cdot \frac{3}{36} + 6^2 \cdot \frac{1}{36} - (2.5)^2$$

$$= \underline{\underline{1.9745}}$$

Standard deviation $= \sigma = \sqrt{Var(X)} = \sqrt{1.9745} = \underline{\underline{1.4}}$.

Skewness $= E[(X-\mu)^3] = \sum (x-\mu)^3 P(X = x)$

$$= (1-2.5)^2 \cdot \frac{11}{36} + (2-2.5)^2 \cdot \frac{9}{36} + (0.5)^2 \cdot \frac{7}{36} + (1.5)^2 \cdot \frac{5}{36} + (2.5)^2 \cdot \frac{3}{36} + (3.5)^2 \cdot \frac{1}{36}$$

$$= \underline{\underline{1.972}}$$

Skewness > 0 positively skewed

2(a) Find K.

We know that $\sum P(X = x_i) = 1$

$$= 0 + K + 2K + 2K + 3K + K^2 + 2K^2 + 7K^2 + K = 1$$

$$= 10K^2 + 9K = 1 \Rightarrow 10K^2 + 9K - 1 = 0$$

$$\frac{-9 \pm \sqrt{9^2 - 4 \times 10 \times (-1)}}{2(10)} = \frac{-9 \pm \sqrt{121}}{20} = \frac{-9 \pm 11}{20}$$

$K = 1/10$ (or) -1

$P(X=x) \geq 0 \; \forall \, x$ satisfied only when $K = 1/10$

$\therefore \boxed{K = 1/10}$

(b) $P(X<6) = P(X=0) + P(X=1) + P(X=2) + P(X=3) + P(X=4) + P(X=5)$

(or) $1 - P(X \geq 6)$

$= 1 - P(X=6) - P(X=7)$

$= 1 - 2(1/10)^2 - (7(1/10)^2 + 1/10)$

$= 1 - \frac{2}{100} - \frac{7}{100} - \frac{1}{10} = 0.81$

$P(X \geq 6) = P(X=6) + P(X=7)$

$= \underline{0.19}$

$P(0<X<5) = P(X=1) + P(X=2) + P(X=3) + P(X=4)$

$= 0.1 + 0.2 + 0.2 + 0.3 +$

$= \underline{0.8}$

(c) $P(X \leq a) > 0.5)$ find Minimum Value of a.

$$F_X(x) = \begin{cases} 0 & x \\ 0.1 & 1 \leq x \leq 2 \\ 0.3 & 2 \leq x < 3 \\ 0.5 & 3 \leq x < 4 \\ 0.7 & 4 \leq x < 5 \\ 0.71 & 5 \leq x < 6 \\ 0.73 & 6 \leq x < 7 \\ 1 & x \geq 7 \end{cases}$$

The minimum Value of a for $P(X \leq a) > 0.5$ is $\underline{4}$ as $F_X(4) = \underline{0.7}$

(3.) Let X be discrete random variable denoting number of heads occuring in 3 tosses of a fair coin.

$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT)$

x can take values 0, 1, 2, 3

$X = x_i$	1000 $\to$ 1500 0	wins 100 1	wins 300 2	3 wins 500
$P(X = x_i)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

Expected value of the game $= 500 \times \frac{1}{8} + 300 \times \frac{3}{8} + 100 \times \frac{3}{8} - 1500 \times \frac{1}{8}$

$= \frac{200}{8} = 25$ rupees.

as $E[X] > 0$ game is favourable to the player.

(4) Given $F_X(0) = 1/2$ $F_X(1) = 3/5$ $F_X(2) = 4/5$, $F_X(3) = 9/10$

$F_X(3.5) = 1$

$P(X = 0) = F_X(0) = 1/2$

To find $P(X=1) = F_X(1) - F_X(0)$

$= 3/5 - 1/2 = 1/10$

we know
$F_X(1) = P(X \le 1) = P(X=0) + P(X=1)$
$F_X(1) = F_X(0) + P(X=1)$

III^{ly}

$P(X=2) = F_X(2) - F_X(1)$

$= \frac{4}{5} - 3/5 = 1/5$

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$P(X=3) = F_X(3) - F_X(2)$

$= \frac{9}{10} - \frac{4}{5} = 1/10$

$P(X=\cancel{3}.5) = F_X(3.5) - F_X(3)$

$= 1 - 9/10 = 1/10$

x_i	0	1	2	3	3.5
$P(X=x_i)$	1/2	1/10	1/5	1/10	1/10

5(a) $E[X] = 10$, $Var[X] = 1$

$Y = 2X(X+20)$

$E[Y] = E[2X(X+20)] = E[2X^2 + 40X]$

$= 2E[X^2] + 40E[X]$

we know $Var[X] = E[X^2] - (E[X])^2$

$1 = E[X^2] - (10)^2$

$E[X^2] = 1 + 100 = 101$

$\boxed{E[X^2] = 101}$

$\therefore E[Y] = 2E[X^2] + 40E[X]$

$= 2 \times 101 + 40 \times 10 = \underline{\underline{602}}$

5 (b) $E[X]=1$, $E[X^2]=4$ $Y=2X-3$

find second moment of Y about $c=3$

i.e $E[(Y-3)^2]$

$E[Y] = E[2X-3] = 2E[X]-3$

$= 2(1)-3 = -1$

$Var[X] = E[X^2] - (E[X])^2$

$= 4-1 = 3$

$Var[Y] = Var[2X-3] = 4\,Var[X] = 4 \times 3 = 12$

$Var[Y] = 12$

$\downarrow$

$E[Y^2] - (E[Y])^2 = 12$

$E[Y^2] - (-1)^2 = 12 \Rightarrow E[Y^2] = 12+1 = 13$

$E[Y^2] = 13$

Second moment of Y about $c=3$

$E[(Y-3)^2] = E[Y^2+9-6Y] = E[Y^2] - 6E[Y] + E[9]$

$= 13 - 6(-1) + 9$

$= \underline{\underline{28}}$

(6) Since each ticket has same probability for being drawn (equally likely). The probability distribution is discrete uniform distribution.

PMF

$$P(X=x) = \frac{1}{k} = \frac{1}{10} \text{ for } x = 1, 2, 3, \dots 10$$

(b) $P(X = \text{even number}) = \{2, 4, 6, 8, 10\}$ are even numbers each with $\frac{1}{10}$ probability

$$= \frac{5}{10} = 1/2$$

→ $P(X = \text{prime number}) = \{2, 3, 5, 7\}$ prime numbers each with $\frac{1}{10}$ probability

$$= \frac{4}{10} = \frac{2}{5}$$

→ $P(X<4) = \cancel{P(X=0)} + P(X=1) + P(X=2) + P(X=3)$

$$= \frac{1}{10} + \frac{1}{10} + \frac{1}{10} = \frac{3}{10}$$

(c) $E[X] = \frac{k+1}{2}$ (for uniform distribution)

$$= \frac{10+1}{2} = \underline{\underline{5.5}}$$

$$Var[X] = \frac{k^2-1}{12} = \frac{10^2-1}{12} = \frac{99}{12} = \underline{\underline{8.25}}$$

7) $X[\text{accepted}] = 1$

$X[\text{rejected}] = 0 \qquad n = 5$

Only two possible outcomes with $n = 5$.
It has Binomial distribution

$X \sim Bin(5, 0.3)$ $\quad x$ Can take $0,1,2,3,4,5$ Values

PMF:- $P(X=x) = {}^nC_x\, p^x (1-p)^{n-x}$

(b) $P(X \le 2) = P(X=0) + P(X=1) + P(X=2)$

$= 5c_0 (0.3)^0 (0.7)^5 + 5c_1 (0.3)^1 (0.7)^4 + 5c_2 (0.3)^2 (0.7)^3$

$= \underline{0.8369}$

(c) $E[X] = np$ (For Binomial dist)

$= 5 \times 0.3 = 1.5$

$Var[X] = npq = np(1-p)$

$= 5 \times (0.3)(0.7)$

$= \underline{1.05}$

(8) X - Random Variable denotes number of forms containing a numerical error.

This is Binomial dist (2 outcomes) with

$p = \frac{1}{1000} = 0.001$ and $n = 10000$

Since n is large and p is small we approximate by Poisson distribution with

$\lambda = np = 10000 \times \frac{1}{1000} = 10$

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$X \sim Poi(\lambda = 10)$ PMF $P(X=x) = \frac{e^{-\lambda}\lambda^x}{x!}$ $x=0,1,2\ldots$

(b) $P(X = 6, 7 \text{ or } 8) = P(X=6) + P(X=7) + P(X=8)$

$$= \frac{e^{-10} \times 10^6}{6!} + \frac{e^{-10} \times 10^7}{7!} + \frac{e^{-10} \times 10^8}{8!}$$

$$= \underline{\underline{0.2657}}$$

(c) $E[X] = \lambda = 10$

$Var[X] = \lambda = 10$

(9) X: Number of failures before first success.

$x = 0, 1, 2 \ldots$

$P(X = x) = (1-p)^x p$, $p = 0.25$

we know that geometric variable Y is defined as

Y: Number of trials required for first success

if 1st success occurs at n^{th} trial. i.e there must be $(n-1)$ failures.

PMF $P(Y=n) = (1-p)^{n-1} p$, $E[Y] = 1/p$

Here $X = Y - 1$

$E[X] = E[Y-1] = E[Y] - E[1]$

$= \frac{1}{p} - 1 = \frac{1-p}{p}$

$E[X] = \frac{1-p}{p} = \frac{1-0.25}{0.25} = \frac{0.75}{0.25} = 3$

On average man would fail 3 times before he hits the target

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(10) $X \sim NegBin(25, 8, 0.35)$ It is a negative binomial distribution

$P(X=8) = {}^{24}C_{7} (0.35)^{7} (0.65)^{17}$ PMF $= {}^{n-1}C_{r-1}\, p^{r} (1-p)^{n-r}$

(3) $E[X] = \frac{r}{p} = \frac{8}{0.35} = 22.85$

$Var[X] = \frac{r(1-p)}{p^2} = \frac{8(1-0.35)}{(0.35)^2} \approx \cancel{22.8571}$

$= \underline{\underline{42.448}}$