# Continuous random Variables

## Answer Key

1. $f(x) = c(1-x^2) \quad -1 < x < 1$

we know that $\int_{-\infty}^{\infty} f(x) = 1$

(a) $\int_{-1}^{1} c(1-x^2) = 1 \Rightarrow c\left[x - \frac{x^3}{3}\right]_{-1}^{1} = 1$

$c\left[(2) - \frac{2}{3}\right] = 1$

$c\left[\frac{4}{3}\right] = 1$

$\boxed{c = 3/4}$

(b) CDF

$F_X(x) = P(X \le x) = \int_{-1}^{x} \frac{3}{4}(1-t^2)\,dt$

$= \frac{3}{4}\left(t - \frac{t^3}{3}\right)\Big]_{-1}^{x}$

$= \frac{3}{4}\left[x - \frac{x^3}{3} - \left(-1 + \frac{1}{3}\right)\right)$

$= \frac{3}{4}\left(x - \frac{x^3}{3} + \frac{2}{3}\right) = \frac{1}{4}\left(3x - x^3 + 2\right)$

$$F_X(x) = \begin{cases} 0 & x < -1 \\ \frac{3x - x^3 + 2}{4} & -1 \le x \le 1 \\ 1 & x > 1 \end{cases}$$

(c) $P(0.3 \le x \le 0.9) = \int_{0.3}^{0.9} \frac{3}{4}(1-x^2)\,dx =$ ______

$P(x \le 0.5) = \int^{0.5} \frac{3}{4} f(x)\,dx = \int^{0.5} \frac{3}{4}(1-x^2)\,dx =$ ______

$P(x=0.5) = 0$ as ($x$ is the Continuous RV)

$$F_x(0.5) = \left.\frac{3x - x^3 + 2}{4}\right|_{at\, x = 0.5}$$

$$= \frac{3(0.5) - 0.5^3 + 2}{4} =$$

(2) $F \cdot (v) = 1 - e^{-0.5v}$

$$PDF = f(v) = f'(v) = 0.5 e^{-0.5v} \quad \frac{d}{dv}(1 - e^{-0.5v})$$

$$P(V > 2) = \int_2^{\infty} \underbrace{0.5 e^{-0.5v}}_{f(v)}\, dv = e^{-1}$$

$$P(V < 1) = \int_0^1 0.5 e^{-0.5v}\, dv = 1 - e^{-0.5}$$

$P(V = 0.8) = 0$ (as $V$ is continuous r.v)

(3) $f(x) = \frac{1}{18}(x^2 - 2x) \qquad 2 \le x \le 5$

$$E[x] = \int x f(x)\, dx = \int_2^5 x \cdot \frac{1}{18}(x^2 - 2x) = \frac{1}{18}\left(\frac{x^4}{4} - \frac{2x^3}{3}\right)_2^5$$

$$= \underline{\underline{4.125}}$$

$$Var[x] = E[(x - \mu)^2] = E[x^2] - (E[x])^2$$

$$= \int x^2 f(x)\, dx - (4.125)^2$$

$$= \int_2^5 x^2 \cdot \frac{1}{18}(x^2 - 2x)\, dx - (4.125)^2$$

$$= \frac{1}{18}\left(\frac{x^5}{5} - \frac{2x^4}{4}\right)_2^5 - (4.125)^2 = \_\_\_\_\_$$

④ $f(y) = \frac{3}{y^4} \quad y \geq 1$

$$E[Y] = \int y f(y)\,dy = \int_1^\infty y \cdot \frac{3}{y^4}\,dy = \int_1^\infty \frac{3}{y^3}\,dy = 3\left(\frac{y^{-2}}{-2}\right)\Big|_1^\infty$$

$$= 0 + \frac{3}{2} = 3/2$$

iii) $E[Y^2] = \int_1^\infty y^2 \cdot \frac{3}{y^4}\,dy = 3$

$$E[5Y-2] = 5E[Y] - 2 = 5 \cdot \frac{3}{2} - 2 = 11/2$$

$$E\left[\frac{5Y+1}{6}\right] = \frac{5}{6}E[Y] + 1/6 = \frac{5}{6} \cdot \frac{3}{2} + 1/6 = 17/12$$

⑤ $f(x) = \frac{3}{4}(1-x^2) \quad -1 \leq x \leq 1$

$$E[X] = \int x f(x)\,dx = \int_{-1}^{1} \frac{3}{4}x(1-x^2)\,dx = \frac{3}{4}\int_{-1}^{1} x - x^3 = 0$$

To know about symmetry of distribution we have to find skew(x)

$$\text{Skew}(x) = E[(\underbrace{X-\mu_x}_{E[x]=0})^3] = E[(x-0)^3] = E[x^3]$$

$$= \int x^3 f(x)\,dx$$

$$= \int_{-1}^{1} \frac{3x^3 \cdot (1-x^2)}{4}\,dx$$

$$= \frac{3}{4}\left[\frac{x^4}{4} - \frac{x^6}{6}\right]_{-1}^{1} = 0$$

$\therefore$ Skew$(x) = 0$ It has symmetrical distribution

(6) $f(v) = \frac{1}{2} v \quad 0 \le v \le 2$

(a) $4^{th}$ moment $= E[V^4] = \int_0^2 v^4 \cdot \frac{1}{2} v \, dv = \frac{v^6}{12}\Big|_0^2$

$= \frac{16}{3}$

(b) $3^{rd}$ Central moment $= E[(V-\mu)^3]$

$E[V] = \int v f(v) dv = \int_0^2 v \cdot \frac{1}{2} v = \frac{4}{3}$

$E[(V-\mu)^3] = \int_0^2 (v - \frac{4}{3})^3 \cdot \frac{1}{2} v \, dv =$ ______

(c) Second moment about 1

$E[(V-1)^2] = \int (v-1)^2 f(v) \, dv = \int_0^2 (v-1)^2 \cdot \frac{1}{2} v \, dv$

$=$ ______

(7) $X \sim unif(-2, 2)$ X is Continuous uniform R.V.

PDF

$f(x) = \frac{1}{\beta - \alpha}$ if $\alpha < x < \beta$

Here $\boxed{f(x) = \frac{1}{2-(-2)} = 1/4}$ → PDF

$p(x<1) = \int_{-\infty}^{\infty} f(x) \, dx = \int_{-2}^{1} \frac{1}{4} dx = \frac{x}{4}\Big|_{-2}^{1} = \frac{1}{4}(1-(-2))$

$= 3/4$

$P(|x-1| \ge 1/2) = P\left(-\frac{1}{2} \ge x-1 \ge 1/2\right) = P\left(\frac{1}{2} \ge x \ge 3/2\right)$

i.e $x \le 1/2$ and $x \le 3/2$ in two intervals

$= P(-2 \le x \le 1/2) + P(3/2 \le x \le 2)$

$$= \int_{-2}^{1/2} \frac{1}{4}\,dx + \int_{3/2}^{2} 1/4\,dx$$

$$= \frac{1}{4}(1/2+2) + \frac{1}{4}(2-3/2) = \frac{5}{8} + 1/8 = 6/8 = \underline{3/4}$$

(8) $X \sim$ unit$(-\alpha, \alpha)$ where $\alpha > 0$

$P(X>1) = 1/3$ $\qquad \alpha = ??$

Here if $\alpha < 1$ then $P(X>1)$ will be zero since $x$ lies outside the given interval $(-\alpha, \alpha)$.

$\therefore$ $\alpha$ must be $> 1$.

$P(x>1) = 1/3$ give


$$\int_1^{\alpha} f(x)\,dx = 1/3$$

pdf of unit distribution

$$f(x) = \frac{1}{\beta-\alpha} = \frac{1}{\alpha-(-\alpha)} = 1/2\alpha$$

formula

$$\int_1^{\alpha} \frac{1}{2\alpha}\,dx = 1/3$$

$$\frac{1}{2\alpha}(\alpha-1) = 1/3$$

$$3(\alpha-1) = 2\alpha$$

$$\boxed{\alpha = 3}$$

(9) X ~ uniformly distributed continuous R.V.

X - waiting time

$P(X>7) = ?$

X ~ unif(0, 10)

Bus arrives every 10 minutes

$$P(X>7) = \int_7^{10} \frac{1}{10}\,dx$$

$$f(x) = \frac{1}{10-0} = 1/10$$

$$0 \le x \le 10$$

$$= \frac{1}{10}(x)\Big|_7^{10}$$

$$= \underline{\underline{3/10}}$$

(10) X ~ exp(0.005)   $\lambda = 0.005$

PDF of exp distribution

$$f(x) = \lambda e^{-\lambda x} \text{ for } x \ge 0$$

$$E(X) = 1/\lambda$$

$$Var(X) = 1/\lambda^2$$

$$P(X>210) = \int_{210}^{\infty} f(x)\,dx$$

$$= \int_{210}^{\infty} (0.005)\,e^{-0.005x}\,dx$$

$$= \frac{0.005\,e^{-0.005}}{-0.005}\Bigg|_{210}^{\infty} = e^{-1.05} = \underline{0.3499}$$