Summation of Series via the Residue Theorem

R. Herman

Let $f(z)$ to be a meromorphic function with a finite set of poles that are not integers. Furthermore, let's assume that

$$|f(z)| < \frac{M}{|z|^k}$$

for $k > 1$ and M a constant. We seek to show that

$$\sum_{n=-\infty}^{\infty} f(n) = -\sum \operatorname{Res}\left[\pi \cot \pi z f(z); z_k\right]$$

where the second sum is over the poles of $f(z)$.

We consider the integral

$$I = \oint_{C_N} \pi \cot \pi z f(z)\, dz, \tag{1}$$

where the contour C_N is shown in Figure 1.

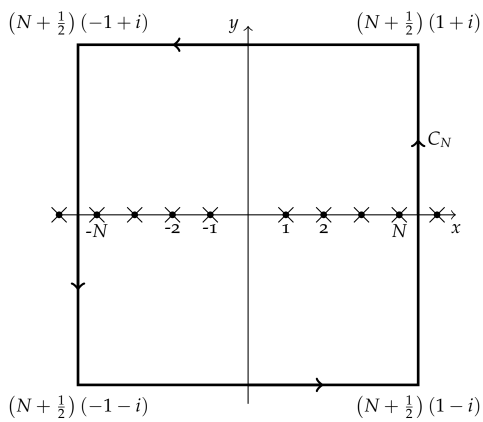

Figure 1: Contour for integration.

There are simple poles of $g(z) = \pi \cot \pi z f(z)$ at $z = 0, \pm 1, \pm 2, \ldots$. The residues are found as

$$\operatorname{Res}[g(z), z = n] = \lim_{z \to n} \frac{(z-n)\pi \cos \pi z}{\sin \pi z} f(z) = f(n).$$

Thus, for the above contour, the Residue theorem gives

$$\oint_{C_N} \pi \cot \pi z f(z)\, dz = 2\pi i \left[\sum_{n=-N}^{N} f(n) + \sum_k \operatorname{Res}\left[\pi \cot \pi z f(z); z_k\right]\right], \tag{2}$$

where the second sum is over the poles of $f(z)$.

We eventually will let $N \to \infty$. We need to consider the value of the contour integral around the rectangle and equate it to this result. We will show that

$$\lim_{N \to \infty} \oint_{C_N} \pi \cot \pi z f(z)\, dz = 0,$$

leading to the desired result.

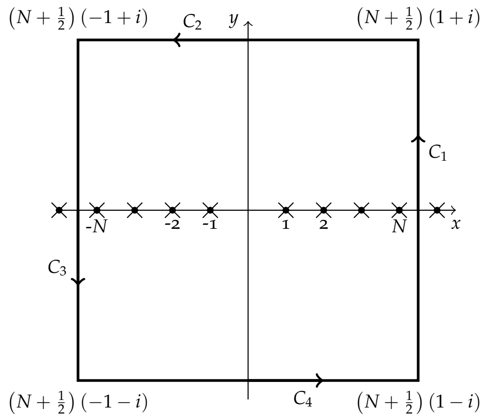

Figure 2: Contour for integration.

The contour C_N can be broken into four pieces, as noted in Figure 2.

$$
\begin{aligned}
|\cot \pi z| &= \left|\frac{e^{i\pi z}+e^{-i\pi z}}{e^{i\pi z}-e^{-i\pi z}}\right| \\
&= \left|\frac{e^{i\pi x-\pi y}+e^{-i\pi x+\pi y}}{e^{i\pi x-\pi y}-e^{-i\pi x+\pi y}}\right| \\
&\leq \frac{|e^{i\pi x-\pi y}|+|e^{-i\pi x+\pi y}|}{|e^{i\pi x+\pi y}|-|e^{-i\pi x-\pi y}|} \\
&= \frac{e^{-\pi y}+e^{+\pi y}}{e^{\pi y}-e^{-\pi y}} = \coth \pi y. \qquad (3)
\end{aligned}
$$

Along path C_2, $y > \frac{1}{2}$. So, we have $|\coth \pi z| \leq \coth \pi/2$. Along path C_4, $y > \frac{1}{2}$ and we also have $|\cot \pi z| \leq \coth \pi/2$.

For paths C_1 and C_3 we also have $-\frac{1}{2} \leq y \leq \frac{1}{2}$. In these cases $z = N + \frac{1}{2} + iy$. Then,

$$
\begin{aligned}
|\cot \pi z| &= \left|\cot\left(N+\frac{1}{2}+iy\right)\right| \\
&= |\tanh \pi y| \\
&\leq \tanh\frac{\pi}{2} < \coth\frac{\pi}{2}. \qquad (4)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\left|\oint_{C_N} \pi \cot \pi z f(z)\, dz\right| &\leq \pi \oint_{C_N} |\cot \pi z||f(z)|\, dz \\
&\leq \pi M \coth\frac{\pi}{2} \oint_{C_N} \frac{dz}{|z|^k} \\
&\leq \frac{\pi M}{N^k} \coth\frac{\pi}{2} \oint_{C_N} dz \\
&= \frac{\pi M}{N^k} \coth\frac{\pi}{2} 4(2N+1). \qquad (5)
\end{aligned}
$$

Therefore, we have shown that

$$\lim_{N\to\infty} \oint_{C_N} \pi \cot \pi z f(z)\, dz = 0.$$

From Equation (2) we have

$$2\pi i \left[\sum_{n=-N}^{N} f(n) + \sum_k \operatorname{Res}\left[\pi \cot \pi z f(z); z_k\right] \right] = 0.$$

This gives

$$\sum_{n=-N}^{N} f(n) = -\sum_k \operatorname{Res}\left[\pi \cot \pi z f(z); z_k\right].$$

Example 1. Prove

$$\sum_{n=-N}^{N} \frac{1}{n^2+a^2} = \frac{\pi}{a} \coth \pi a$$

for a>0.

Here $f(z) = \frac{1}{z^2+a^2}$. This function has simple poles at $z = \pm ia$. The residues of $\pi \cot \pi z f(z)$ are then

$$\begin{aligned}
\lim_{z\to\pm ia} (z \mp ia) \frac{\pi \cot \pi z}{z^2+a^2} &= \lim_{z\to\pm ia} (z \mp ia) \frac{\pi \cot \pi z}{(z+ia)(z-ia)} \\
&= \frac{\pi \cot \pi ia}{2ia} \\
&= \frac{\pi \cos i\pi a}{2ia \sin i\pi a} \\
&= -\frac{\pi \cosh \pi a}{2a \sinh \pi a} \\
&= -\frac{\pi}{2a} \coth \pi a. \qquad (6)
\end{aligned}$$

Adding these residues gives the result.

A similar series can be obtained under the same hypotheses:

$$\sum_{n=-\infty}^{\infty} (-1)^n f(n) = -\sum \operatorname{Res}\left[\pi \csc \pi z f(z); z_k\right]$$

where the second sum is over the poles of $f(z)$.