DIFFERENTIAL EQUATIONS

UNIT I

FIRST ORDER DIFFERENTIAL EQUATIONS

What is a differential Equation?

- An equation expressing a relation between functions, their derivatives and the variables is called a differential equation
- Classification of differential equations
  - Ordinary differential equations
  - Partial differential equations

Ordinary differential equations

- An ordinary differential equation is a differential equation in which the dependent variable (say y) depends only on one independent variable(say x)
- F(x, y , y ’, y’’ , y’’’………….)= 0
- <u>Example :</u>
- $\frac{dy}{dx}$= sinx+cosx

Partial differential equation

- A partial differential equation is the differential equation in which y depends on two or more independent variables x,t…

- F(x,t,y,$\frac{\partial y}{\partial x}$, $\frac{\partial y}{\partial t}$, $\frac{\partial^2 y}{\partial x^2}$, $\frac{\partial^2 y}{\partial t^2}$, $\frac{\partial^2 y}{\partial x\,\partial t}$ ……………)=0
- Example :

$$\frac{\partial y}{\partial x} + t\frac{\partial y}{\partial t} = 2y$$

- **<u>Order of the differential equation</u>**
- The order of the highest derivative involved in a differential equation is called the order of the differential equation.

- **<u>Degree of the differential equation</u>**- The degree of a differential equation is the degree or power of the highest ordered derivative present in the equation, after the equation is made free from radicals and fractions in respect of derivatives

Find order and degree of the following differential equations:

1. $\dfrac{d^2y}{dx^2} = 0$

2. $y = x\dfrac{dy}{dx} + \dfrac{a}{\dfrac{dy}{dx}}$

3. $\dfrac{d^2y}{dx^2} + m^2 y = 0$

4. $y\dfrac{d^2y}{dx^2} - \left(\dfrac{dy}{dx}\right)^2 = y^2 \log y$

5. $\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{3/2} = a\dfrac{d^2y}{dx^2}$

6. $y = x\dfrac{dy}{dx} + a\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}$

Practical approach to differential equations

Differential equations arise in many engineering and physical problems. The approach of an engineer to the study of differential equations has to be practical, and so, it consists of

(1) formation of a differential equation from the physical conditions, called modelling;

(2) solution of a differential equation under the initial/boundary conditions; and

(3) the physical interpretation of the results.

Formation of the differential equation

- An ordinary differential equation is obtained when we eliminate arbitrary constants (also called parameters) from a given relation involving the variables
- <u>Example</u>
- Form differential equation for the following

1) y= $ax^2 + bx$

2)y= $Ae^{2x} + Be^{-2x}$

- Solution:

1.y= $ax^2 + bx$, y is the dependent variable

x is independent variable

a,b arbitrary constants or parameters

Y'= 2ax+b------→1

Y''= 2a we say that a= ½ y''--→2

b= y'-2(1/2y'')x----→y'-y''x=b

X^2y''-2xy'+2y=0 order= 2 and degree=1

Note: order of the differential equation formed depends on the number of arbitrary constants eliminated

- 2)y= $Ae^{2x} + Be^{-2x}$
- Y'=2$Ae^{2x} - 2Be^{-2x}$
- Y''= 4$Ae^{2x} + 4Be^{-2x}$=4($Ae^{2x} + Be^{-2x}$)=4y
- Y''-4y=0

Solution of a differential equation:

- A solution of differential equation is the relation between the variables which satisfies the given differential equation.
- Example:
- The solution of the differential equation $\frac{dy}{dx} - 2\mathrm{y} = 0$ is given by $\mathrm{y} = \mathrm{c}e^{2x}$

- **General solution**: The general solution of the differential equation is that in which the number of arbitrary constants is equal to the order of the given differential equation. y= c e^2x
- **Particular solution:** A particular solution is that solution which can be obtained from the general solution by giving particular values to the arbitrary constants. Ex- ---y= 2e^2x
- **Singular solution:** A singular solution is the solution of the differential equation which is neither a general solution nor a particular solution. Only some equations have singular solution.

- Example: The solution of the DE

  $(\frac{dy}{dx})^2 - 4\mathrm{y} = 0$ is
- y=$(x + a)^2$ *is the general solution*
- y=x^2 *is the particular solution*
- singular solution ???
- Y=0 is a singular solution

First order first degree differential equations

- The first order first degree differential equations are in the form of
- $$F(\ x\ ,\ y(x)\ ,\frac{dy}{dx}) = 0$$

- <u>Classification of first order first degree differential equations</u>

1. Variable separable equations
2. Reducible to variable separable
3. Homogeneous equations
4. Non homogeneous but reducible to homogeneous equations
5. Linear DE
6. Non Linear but reducible to linear DE
7. Exact equations
8. Inexact equations but reducible to exact equation

Lecture 2

Grading policy

- Assignments +quizzes=40%

- Assignments=25%(5 assignments each 5%)
- Quizzes=15%(3 quizzes each 5%)
- Class participation=10%
- Midterm=20%
- Endsem=30%

I. Variable Separable Equations

- If in an equation it is possible to separate all the functions of x and dx to one side and all the functions of y and dy to the other side then the variables are said to be separable. The general form of such equations is
- $f(x)dx = g(y)dy$
- Solution of variable separable equations is obtainable by integrating on both sides of equation

$$\int f(x)dx = \int g(y)dy + c$$ is the solution

- Example: solve (1-x)dy+(1-y)dx=0
- (1-x)dy=(y-1)dx
- $\frac{1}{(1-x)}$dx=$\frac{1}{(y-1)}$dy→ variable and separable equation
- $\int \frac{1}{(1-x)}$dx =$\int \frac{1}{(y-1)}$dy
- -log(1-x)=log(y-1)+c
- log(y-1)+log(1-x)+c=0
- log(1-x)(y-1)=-c
- (1-x)(y-1)=e^{-c}
- (1-x)(1-y)=A where A=-e^{-c}→ general solution

II.Equations reducible to variable separable

- By choosing a suitable substitution some first order first degree differential equations are reducible to variable separable equation.Then we follow the same solution method of variable separable equations(i.e by integrating the equation)
- Example: Solve $\frac{dy}{dx}$ =$(4x + y + 1)^2$

III. Homogeneous equations

- A first order first degree differential equation is classified as homogeneous equation if they are in the form of

$$\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$$

where f(x,y) and g(x,y) are homogeneous functions of the same degree in x and y.

Method of finding a solution for Homogeneous equations

- Substitute **$y=vx$** and solve
- It reduces the equation to a variable separable equation
- Use the solution method of variable separable equations i.e by integrating

- Example: Solve $(x^2 - y^2)dx - xydy = 0$

Lecture 3

- Review: solution methods of first order first degree DE’s

1. Variable separable equations
2. Equations reducible to variable separable
3. Homogeneous equations

IV. Equations reducible to homogeneous form

- If the equation is of the form

$$\frac{dy}{dx} = \frac{ax + by + c}{a'x + b'y + c'}$$

It can be reduced to a homogeneous differential equation by making a substitution.

Check : whether the ratio $\frac{a}{a'}$ and $\frac{b}{b'}$ are equal?

- **Case 1**: If $\frac{a}{a'} \neq \frac{b}{b'}$ then make a substitution

  x=X+h and y=Y+k   where X and Y are new variables and h and k are constants yet to be chosen.

$$\frac{dy}{dx} = \frac{ax + by + c}{a'x + b'y + c'}$$

Substitute   x=X+h and y=Y+k ,dx=dX and dy=dY

$$\frac{dY}{dX} = \frac{a(X + h) + b(Y + k) + c}{a'(X + h) + b'(Y + k) + c'}$$

- $\frac{dY}{dX} = \frac{aX+bY+(ah+bk+c)}{a'X+b'Y+(a'h+b'k+c')}$ → (1)
- In order to have equation (1) as homogeneous equation ,choose h and k such that the following equations are satisfied
- $ah + bk + c=0$ and $a'h + b'k + c'=0$ such that
- $h=\frac{bc'-b'c}{ab'-a'b}$ and $k=\frac{ca'-c'a}{ab'-a'b}$
- Now it becomes $\frac{dY}{dX} = \frac{aX+bY}{a'X+b'Y}$

Example:

- $$\frac{dy}{dx} = \frac{x+y-2}{-x+y-4}$$

Lecture 4(28-8-2020)

- Case 2: If $\frac{a}{a'} = \frac{b}{b'}$
- Substitute ax+by=t reduces to a homogeneous variable separable equation.

- Examples

1. $\frac{\mathrm{dy}}{\mathrm{dx}} = \frac{\mathrm{x+y-2}}{\mathrm{-x+y-4}}$
2. (3y+2x+4)dx=(4x+6y+5)dy

2. $(3y+2x+4)dx=(4x+6y+5)dy$

Refer to pdf notes of lecture 4 for the method of solution

V. Linear Differential Equation

- A differential equation is said to be linear if the dependent variable y and its derivatives appear only in the first degree and are also not multiplied together. The linear differential equation of the first order is of the form

$$\frac{dy}{dx} + Py = Q$$

Where P and Q are constants or functions of X alone.

- The First order linear differential equation is also called as Leibnitz equation.

Solution Method for linear differential equations

- $$\frac{dy}{dx} + Py = Q$$

Example: Solve xdy-(y-3x^2)dx=0

- Refer to lecture 4 notes

Lecture 5(29-8-2020)

VI. Non Linear equations Reducible to Linear form

- An equation of the form $\frac{dy}{dx} + Py = Qy^n$

where P and Q are the functions of x alone and n is a constant except 0 and 1 is called a Bernoulli's equation.

Method of solution for Bernoulli’s equation

- Use the substitution y^{1-n}=t reduces it to the linear differential equation.
- Example: $x\frac{dy}{dx} + y = x^3 y^6$

VII. Exact Differential equation

- A differential equation is said to be exact if it can be derived from its primitive(general solution) directly by differentiation without any subsequent multiplication ,elimination
- The necessary and the sufficient condition for the differential equation Mdx+Ndy=0 to be exact is that $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

solution of exact differential equation

- In order to obtain the solution of an exact DE

($Mdx+Ndy=0$)we have proceed as below

1. Integrate M with respect to x keeping y as constant
2. Integrate with respect to y only those terms of N which do not contain x
3. Add 1 and 2 and equate the result to arbitrary constant

- Example: solve the following DE

  1. $y\sin 2x\,dx=(y^2+\cos^2 x)dy$