DE Assignment - 3

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Q-1

$$\Delta^3 y_{n+1} - \Delta y_{n-1} = 3n$$

$$\Delta^2 (\Delta y_{n+1}) - \Delta y_{n-1} = 3n$$

$$\Delta^2 (y_{n+2} - y_{n+1}) - (y_n - y_{n-1}) = 3n$$

$$\Delta (y_{n+3} - y_{n+2}) - \Delta (y_{n+2} - y_{n+1}) - y_n + y_{n-1} = 3n$$

$$\Rightarrow \Delta y_{n+3} - \Delta y_{n+2} - \Delta y_{n+2} + \Delta y_{n+1} - y_n + y_{n-1} = 3n$$

$$\Rightarrow y_{n+4} - 3y_{n+3} + 3y_{n+2} - y_{n+1} - y_n + y_{n-1} = 3n \quad (1)$$

$$\text{order} = \frac{(n+4)-(n-1)}{1} = 5$$

The equation (1) is linear as derivatives are of power 1 and are not multiplied together.

The equation is non-homogeneous as there is an extra 3n term at RHS.

Q-2 Given

$$y(x) = ax + b2^x \quad \text{— } (1)$$

$$y_{x+1} = a(x+1) + b2^{x+1}$$

$$= ax + 2b^x + a \quad - (2)$$

$$y_{x+2} = a(x+2) + b2^{x+2}$$

$$= ax + 4b2^x + 2a \quad - (3)$$

So, eq 3 - 2 (eq 2) + 1

$$y_{x+2} - 2y_{x+1} + y_x = b2^x$$

$$b = \frac{y_{x+2} - 2y_{x+1} + y_n}{2^x}$$

Subtract eq ② and 1

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$$y_{x+1} - y_x = a + (y_{x+2} - 2y_{x+1} + y_x)$$

$$a = 3y_{x+1} - y_{x+2} - 2y_x$$

Substitute back values of a and b

$$y_x = ax + b2^x$$

$$y_x = (3y_{x+1} - 2y_x - y_{x+2})x + (y_{x+2} - 2y_{x+1} + y_x)$$

$$y_x = x3y_{x+1} - 2y_x x - y_{x+2}x + y_{x+2} - 2y_{x+1} + y_x$$

$$2xy_x = (1-x)y_{x+2} + 2y_{x+1}(x-1) + xy_{x+1}$$

$$y_x + 2xy_x - xy_{x+1} = (1-x)[y_{x+2} - 2y_{x+1}]$$

$$y_x = (1-x)[y_{x+2} - 2y_{x+1} + y_x) + xy_{x+1} - xy_x$$

$$\Rightarrow y_x = (1-x)[y_{x+2} - 2y_{x+1} + y_x] + x[y_{x+1} - y_x))$$

$$\boxed{y_x = (1-x)[y_{x+2} - 2y_{x+1} + y_x] + x[y_{x+1} - y_x]}$$

Ans

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Q-3 $y_0 = 2,\ y_1 = 5,\ y_2 = 15$

$y_{n+3} - 6y_{n+2} + 11y_{n+1} - 6y_n = 0$

Using $y_{n+3} = E^3 y_n$

$y_{n+2} = E^2 y_n$

$y_{n+1} = E y_n$

So, $(E^3 - 6E^2 + 11E - 6)\, y_n = 0$

$(E-1)(E-2)(E-3)\, y_n = 0$

roots $1, 2, 3$

$y_{cf} = c_1(1)^n + c_2(2)^n + c_3(3)^n$

$y_0 = c_1 + c_2 + c_3$

$\boxed{c_1 + c_2 + c_3 = 2}$ — (1)

$y_1 = c_1 + 2c_2 + 3c_3$

$\boxed{c_1 + 2c_2 + 3c_3 = 5}$ — (2)

$y_2 = c_1 + 4c_2 + 9c_3 \Rightarrow \boxed{c_1 + 4c_2 + 9c_3 = 15}$ — (3)

Solving eq (1), (2), (3) we get

$\boxed{c_1 = 1, \quad c_2 = -1, \quad c_3 = 2}$

Substitute them back

$y_n = 1^n - 2^n + 2 \cdot 3^n$

$\boxed{y_n = 1 - 2^n + 2 \cdot 3^n}$ Ans

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Q-4 $(E^2 + 2E + 4)\, y_n = 0$

roots $\dfrac{-2 \pm \sqrt{4-16}}{2}$

$= \dfrac{-2 \pm 2\sqrt{3}i}{2} = -1 \pm \sqrt{3}i$

Complementary solution for imaginary roots is

$r^n (c_1 \cos n\theta + c_2 \sin n\theta)$

$\theta = \tan^{-1}(\beta/\alpha)$ and $r = \sqrt{\alpha^2+\beta^2}$

$\theta = \tan^{-1}(\sqrt{3}/-1)$ $\quad r = \sqrt{1^2+3}$

$\theta = \tan^{-1}(-\sqrt{3})$ $\quad r = 2$

$\theta = 2\pi/3$

Complementary solution

$y_n = 2^n \left(c_1 \cos \frac{2\pi}{3} n + c_2 \sin \frac{2\pi}{3}\right)$

Q-5 $(E^2+2E-8)\, a_n = 5n + 2^n \cdot 3$

Roots $(E^2 + 2E - 8) = 0$

$(E+4)(E-2)\, a_n = 0$

Roots $2, -4$

~~fo~~ $y_{cf} = c_1 2^n + c_2(-4)^n$ (Complementary Solution)

Trial Solution for $5n + 3(2)^n$

$y_{trial} = (c_0 + c_1 n) + c_2(2)^n n$ ← (n for being in complementary as well)

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$$a_n = (C_0 + C_1 n) + C_2 2^n n$$

$$a_{n+1} = (C_0 + C_1) + C_1 n + 2C_2 2^n + 2C_2 2^n n$$

$$a_{n+2} = (C_0 + 2C_1) + nC_1 + 4C_2 2^n n + 8C_2 2^n$$

<u>Given</u> $a_{n+2} + 2a_{n+1} - 8a_n = 5n + 3(2)^n$

$$(C_0 + 2C_1) + C_1 n + 4C_2 2^n n + 8C_2 2^n$$
$$+ 2\left[(C_0 + C_1) + C_1 n + 2C_2 2^n + 2C_2 2^n n\right] = 5n + 3(2)^n$$
$$-8\left[(C_0 + C_1)n + (-C_0 n) + C_0 + C_2 2^n n\right]$$

Solving the equation above

$$(-5C_0 + 4C_1) - 5C_1 n + 12C_2 2^n = 5n + 3(2)^n$$

$$-5C_0 + 4C_1 = 0 \qquad -5C_1 = 5$$

$$12C_2 = 3 \qquad \boxed{C_1 = -1}$$

$$\boxed{C_2 = 1/4} \qquad \boxed{C_0 = -4/5}$$

$$a_n = -\frac{4}{5} - n + \frac{1}{4} 2^n n$$

Particular Solution $= -\frac{4}{5} - n + \frac{1}{4} 2^n n$

Total Solution $= C_1 2^n + C_2(-4)^n + \left(-\frac{4}{5} - n\right) + \left(\frac{1}{4}\right) 2^n n$

<u>Ans</u>

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Q-6

$$x_{n+1} - 3x_n - 2y_n = -n$$

$$y_{n+1} - 2y_n - x_n = n$$

Also $x_0 = 0$ | $y_0 = 3$

$\Rightarrow$ Using $Ey_n = y_{n+1}$

$$(E-3)\,x_n - 2y_n = -n \quad \text{— (1)}$$

$$(E-2)\,y_n - x_n = n \quad \text{— (2)} \Big] \times (E-3)$$

$$(E-3)(E-2)\,y_n - (E-3)\,x_n = (E-3)\,n$$

$$-2y_n + (E-3)\,x_n = -n$$

$$(E-3)(E-2)\,y_n - 2y_n = (E-4)\,n$$

$$(E^2 - 5E + 4)\,y_n = n+1-4n$$

$$(E^2 - 5E + 4)\,y_n = 1-3n$$

$y_{CF} \Rightarrow E^2 - 5E + 4 = 0$

$$(E-4)(E-1) = 0 \Rightarrow E = 1, 4$$

$$y_{CF} = c_1 + c_2 (4)^n$$

Trial Solution

$$y_{Trial} = (c_0 + c_1 n)\,n$$

$$y_{n+1} = (c_0 + c_1 + c_1 n)(n+1)$$

$$y_{n+2} = (c_0 + 2c_1 + c_1 n)(n+2)$$

Solving and putting back

$$-3c_0 + c_1 - 6c_1 n = 1 - 3n$$

$$-6c_1 = -3$$

$$\boxed{c_1 = 1/2}$$

$$-3c_0 = 1 + c_1$$

$$-3c_0 = \frac{3}{2}$$

$$c_0 = -1/2$$

$$y_{Particular} = \left(-\frac{1}{2} + \frac{1}{2}n\right)n$$

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Total Soln => $y_n = c_1 + c_2 4^n + \left(-\frac{1}{2} + \frac{1}{2}n\right)n$

Now

$$(E-2)(E-3)x_n - 2(E-2)y_n = (E-2)(-n)$$

$$2(E-2)y_n - 2x_n = 2n$$

$$(E-2)(E-3)x_n - 2x_n = 4n - En$$

$$(E^2 - 5E + 4)x_n = 4n - n - 1$$

$$(E^2 + 5E + 4)x_n = 3n - 1$$

Complementary Solution

$$(E-4)(E-1)x_n = 0$$

$$\Rightarrow \text{roots } 4, 1$$

$$~~y_{cf}~~ \quad x_n = c_3 + c_4(4)^n$$

Now, finding particular soln

$$x_{trial} = (c_0 + c_1 n)n$$

$$x_n = (c_0 + c_1 n)n \qquad \Rightarrow (-3c_0 + c_1) - 6c_1 n = 3n - 1$$

$$x_{n+1} = (c_0 + c_1 n + c_1)(n+1)$$

$$x_{n+2} = (c_0 + 2c_1 + c_1 n)(n+2)$$

$$\Rightarrow \quad c_0 = 1/2 \quad \text{and} \quad c_1 = -1/2$$

Total soln $\quad x_n = c_3 + c_4(4)^n + (1/2 - 1/2\, n)n$

Putting these values back

$$x_1 - 3x_0 - 2y_0 = 0 \qquad (\because n = 0)$$

$$y_1 - 2y_0 - x_0 = 0$$

$$x_1 = 6 \quad \text{and} \quad y_1 = 6$$

$$x_n = c_3 + c_4 4^n + \frac{n(1-n)}{2}$$

$$y_n = c_1 + c_2 4^n + \frac{n(n-1)}{2}$$

$$y_0 = c_1 + c_2 \qquad \Rightarrow c_1 = 2$$

$$y_1 = c_1 + 4c_2 \qquad c_2 = 1$$

$$x_0 = c_3 + c_4 \qquad \Rightarrow c_3 = -2, \; c_4 = +2$$

$$c_1 = c_3 + 4c_4$$

Ans

$$x_n = -2 + 2(4)^n + \frac{n(1-n)}{2}$$

$$y_n = 2 + (4)^n + \frac{n(n-1)}{2}$$

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Q-7 Find formula for sum of squares of first 'n' natural numbers

$$(n+1)^3 - n^3 = 3n^2 + 3n + 1$$

If we consider

$$y_n = n^3$$

$$\left.\begin{aligned} y_2 - y_1 &= 2^3 - 1^3 = 3(1)^2 + 3(1) + 3 \\ y_3 - y_2 &= 3^3 - 2^3 = 3(2)^2 + 3(2) + 1 \\ y_4 - y_3 &= 4^3 - 3^3 = 3(3)^2 + 3(3) + 1 \end{aligned}\right\}$$ add upto m

$$\Rightarrow y_{m+1} - y_m = (m+1)^3 - 1^3 = 3(1^2 + 2^2 + 3^2 + \dots m^2) + 3(1 + 2 + 3 + \dots m) + \underbrace{(1 + 1 + 1 \dots 1)}_{m \text{ times}}$$

Sum of 'n' natural numbers $= \frac{n(n+1)}{2}$

Let sum of squares of 'n' natural numbers be 'S'.

$$y_{m+1} - y_m = (m+1)^3 - 1^3 = 3S + 3\frac{m(m+1)}{2} + m$$

$$\Rightarrow \frac{2m^3 + 6m^2 + 4m - 3m^2 - 3m}{2} = 3S$$

$$3S = \frac{2m^3 + 3m^2 + m}{2}$$

$$3(S) = 2m^3 + 2m^2 + m^2 + m$$

$$3S = \frac{m(2m+1)(m+1)}{2}$$

$$S = \frac{m(m+1)(2m+1)}{6}$$

Sum of 'n' natural numbers $= \frac{n(n+1)(2n+1)}{6}$ ← Ans

Q8 Given $P = 10,000/-$

$R = 6\%$ compounded quartely

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y_n = amount initially

y_{n+1} = amount after interest

$\Rightarrow \quad y_{n+1} - y_n = \Delta y_n$

$y_{n+1} - y_n = 0.06 y_n$

$y_{n+1} = 1.06 y_n \Rightarrow$

$(E - 1.06) y_n = 0$

$y_n = c_1 (1.06)^n$ (Complementary Solution)

$n=0, \Rightarrow y_n = y_0 \Rightarrow y_0 = c_1 \times 1$

So $c_1 = 10,000$

$y_n = 10,000(1.06)^n$

Double the amount, 1unit = 3 months

$20,000 = 10,000 (1.06)^n$

$2 = (1.06)^n$

$n = \frac{\log 2}{\log 1.06} = 11.895$ units

So, number of months $= 11.895 \times 3 = 35.685$

(No of months taken $= 35.685$ approx)

Ans