Differential Equations Assignment

Q-1 Equation of parabolas with focii as origin

$$y^2 = 4a(x+a)$$

diff

$$2yy' + (-4a) = 0$$

$$a = \frac{yy'}{2}$$

Put back

$$y^2 = 2yy'x + 4\frac{y^2}{4}(y')^2$$

$$y^2 = 2xy\frac{dy}{dx} + y^2\left(\frac{dy}{dx}\right)^2$$

Order = 1
Degree = 2

Q-2 (a) $2xe^{x^2+y^2} = (1+2y)e^{-y}\frac{dy}{dx}$ variable (Separable)

$$\int 2x\, e^{x^2} dx = \int (1+2y)\, e^{-y-y^2} dy$$

$x^2 = t \Rightarrow 2x\,dx = dt \quad | \quad -y-y^2 = v \Rightarrow -(1+2y)\,dy = dv$

$$e^{x^2} = -e^{-y-y^2} + c$$

$$e^{-(y+y^2)} + e^{x^2} = c$$

(4) $(x-y)^2 \frac{dy}{dx} = a^2$

$x-y=t \Rightarrow 1-y' = dt/dx \Rightarrow \frac{dy}{dx} = 1-\frac{dt}{dx}$

$t^2\left[1-\frac{dt}{dx}\right] = a^2 \Rightarrow 1-\frac{dt}{dx} = \frac{a^2}{t^2}$

$\frac{t^2-a^2}{t^2} = \frac{dt}{dx} \Rightarrow dx = \frac{t^2 dt}{t^2-a^2}$

$dx = \left(1+\frac{a^2}{t^2-a^2}\right) dt$

$\int dx = \int dt + \int \frac{a^2}{t^2-a^2} dt$

$x = t + \frac{a}{2} \log \frac{(t-a)}{(t+a)}$ $\boxed{t = x-y}$

$2y = a \log\left(\frac{x-y-a}{x+y+a}\right) + c$

Q-3 $x \tan\frac{y}{x} + x\sec^2\frac{y}{x} \frac{dy}{dx} = y \sec^2 \frac{y}{x}$

$\frac{dy}{dx} = \frac{y \sec^2 y/x - x \tan y/x}{x \sec^2(y/x)}$

Homogeneous

$x\frac{dv}{dx} = -\frac{\sin 2v}{2}$

$-2\int \frac{dv}{\sin 2v} = \int \frac{dx}{x}$

$-\log|\operatorname{cosec} v - \cot v| = \log x$

$\boxed{v = y/x}$

$\operatorname{cosec}(y/x) - \cot(y/x) = \frac{1}{cx}$

Ans

Q-4

$$y' = \frac{(e^x+2e^y-3)\, e^x}{(2e^x+e^y-3)\, e^y}$$

Let $e^x = X$
$e^y = Y$

$$\frac{dY}{dX} = \frac{X+2Y-3}{2X+Y-3}$$

Let $X = X'+h$
$Y = Y'+k$

$$\frac{dY'}{dX'} = \frac{X'+2Y'}{2X'+Y'}$$

$h=k=1$

$$V + X'\frac{dV}{dX'} = \frac{1+2V}{2+V}$$

$$X'\frac{dV}{dX} = \frac{1-V^2}{2\cancel{+}V} \Rightarrow \frac{2+V}{1-V^2}\, dV = \frac{dX'}{X'}$$

$$\log\left|\frac{1+V}{1-V}\right| + -\frac{1}{2}\log(1-V^2) = \log X'$$

Substitute back

$$\frac{e^x+e^y-2}{(e^x-e^y)^3} = c \quad \text{Ans}$$

Q-5 $(1+y^2)\dfrac{dx}{dy} = e^{\tan^{-1}y} - x$

$$\frac{dx}{dy} = -\frac{x}{(1+y^2)} + \frac{e^{\tan^{-1}y}}{(1+y^2)}$$

$$IF = e^{\int dy/1+y^2} = e^{\tan^{-1}y}$$

Solve

$$x\, e^{\tan^{-1}y} = \int \frac{e^{2\tan^{-1}y}}{1+y^2}\, dy$$

$\tan^{-1}y = t$

$$x \tan^{-1}y = \int e^{2t}\, dt + c$$

$$x \tan^{-1}y = \frac{e^{2t}}{2} + c$$

$$x \tan^{-1}y = \frac{e^{2\tan^{-1}y}}{2} + c$$

Q-6

$$y' - \frac{\tan y}{1+x} = (1+x)\, e^{x} \sec x$$

$$\cos x \frac{dy}{dx} - \frac{\sin y}{1+x} = (1+x)\, e^{x}$$

$\boxed{\sin y = t}$

$$\frac{dt}{dx} - \frac{t}{(1+x)} = (1+x)\, e^{x}$$

$$If = e^{-\int \frac{dx}{1+x}} = (1+x)^{-1}$$

Sol^n

$$\frac{t}{1+x} = \int e^{x}\, dx + C$$

$$\frac{t}{1+x} = e^{x} + C$$

$$\frac{\sin y}{1+x} = e^{x} + C \quad \text{Ans}$$

Q-7

$$y^4 dx - xy\, dx + xy^3 dy + x^2 dy = 0$$

$$(4y^3 - x)\, dx + (y^3 + 2x)\, dy = 0$$

$$If = e^{-3\int dy/y} = 1/y^3$$

Multiplying with If

$$\Rightarrow x + \frac{x^2}{2} \frac{(2)}{y^3} + h'(y) \qquad h'(y) = 0$$

Sol^n

$$yx - \frac{x^2}{2y^2} = C$$

Q-8 $(2xy^2+y)dx+(x+2x^2y-x^4y^3)dy=0$

$M = 2xy^2+y \qquad N = x+2x^2y-x^4y^3$

$$IF = \frac{1}{2x^2y^2+xy-xy-2x^2y^2+x^4y^4} = \frac{1}{x^4y^4}$$

Multiplying IF

$$\left(\frac{2x^2y+y}{x^4y^4}\right)dx + \left(\frac{x+2x^2y-x^4y^3}{x^4y^4}\right)dy = 0$$

$$\Rightarrow \quad \frac{-1}{x^2y^2} - \frac{1}{3x^3y^3} - \log y + C = 0$$

$$\Rightarrow \quad \log y + \frac{1}{(xy)^2} + \frac{1}{3(xy)^3} = 0 \quad \text{Ans}$$

Q-9 $x^3y^3(2y\,dx+x\,dy)-5y\,dx+7x\,dy=0$

Multiplying with $x^h y^k$

$$\frac{3+h+1}{2} = 3k+3+1-2k \quad \Rightarrow \quad 2k-h=-4$$

and $5k-7k=2$

$$\left(2x^{1/3}y^{2/3}-5x^{-8/3}y^{-1/3}\right)dx +$$

$$\left(x^{4/3}y^{-1/3}-7x^{-5/3}y^{-10/3}\right)dy = 0$$

$$\frac{\partial(M)}{dy} = \frac{\partial N}{\partial x} \quad \left(\text{Exact Equation Satisfied}\right)$$

$$\frac{3}{2}x^{4/3}y^{2/3} + 3x^{-5/3}y^{-1/3} + C = 0$$

Ans

Q-10 $(x^2+x)p^2+(x^2-x-2xy-y)p+y^2-xy=0$

Sol

$$(x^2+x)p^2+(x^2+x-xy-y)p-xyp+(y^2-xy)=0$$

$$[(x+1)p-y](xp+x-y)=0$$

$y=p(x+1)$

$p=\frac{y}{x+1} \Rightarrow \frac{dy}{dx}=\frac{y}{x+1}$

$\log y=\log(x+1)+C$

$y=C(x+1)$

$xp=y-x$

$p=\frac{y-x}{x}$

$\frac{dy}{dx}=\frac{y}{x}-1$

$v=\frac{y}{x}$

$v=-\log x+C$

$\frac{y}{x}=-\log x+c$

$$\left(\frac{y}{x+1}-c\right)\left(\frac{y}{x}+\log x-c\right)=0 \quad \text{Ans}$$

Q-11 $4y^2+p^3=2xyp$

Divide by yp

$2x=\frac{4y^2}{yp}+\frac{p^3}{yp}=\frac{p^2}{y}+\frac{4y}{p}$

diff w.r.t. y

$\frac{2}{p}=\frac{2dp}{dy}\left[\frac{p}{y}-\frac{2y}{p^2}\right]+\frac{4}{p}-\frac{p^2}{y^2}$

$\Rightarrow \frac{2dp}{dy}\left(\frac{p}{y}-\frac{2y}{p^2}\right)+\frac{4}{p}-\frac{2}{p}-\frac{p^2}{y^2}=0$

$\frac{2dp}{dy}\left(\frac{p^3-2y^2}{p^2y}\right)+\frac{2}{p}-\frac{p^2}{y^2}=0$

$2\int\frac{dp}{p}=\int\frac{dy}{y}$

$\log p^2=\log+c$

Substitute back

$4y^2+(cy)^{3/2}=2xy^{3/2}c^{1/2}$

Q.12 $p^2y + px^3 - x^2y = 0$ $\quad x^2 = u$, $y^2 = v$, $p = \frac{dv}{dx}$

$$\Rightarrow \frac{dx}{dy} = \sqrt{\frac{v}{u}}\frac{du}{dv} = \frac{1}{p}$$

$$p = \sqrt{\frac{u}{v}}\frac{dv}{du}$$

$$\left(\frac{\sqrt{u}\,dv}{\sqrt{v}\,du}\right)^2 y + \frac{\sqrt{u}}{\sqrt{v}}\frac{dv}{du}x^3 - x^2y = 0$$

$$\left(\frac{du}{dv}\right)^2 + \frac{u}{\sqrt{v}} + \frac{u^2}{\sqrt{v}}\frac{dv}{du} - u\sqrt{v} = 0$$

$$\left(\frac{dv}{du}\right)^2 + u\left(\frac{dv}{du}\right) - v = 0 \qquad \frac{dv}{du} = t$$

$$\Rightarrow t^2 + ut - v = 0$$

(a) $v = ut + v^2$

diff u $\quad \frac{dv}{du} = t$

$$t = 2t\frac{dt}{du} + t + u\frac{dt}{du}$$

$$(2t + u)\frac{dt}{du} = 0$$

$\frac{dt}{du} = 0$

$\Rightarrow v = cu + c^2$

$$\boxed{y^2 = cx^2 + c^2}$$

$2t + u = 0$

$v = -\frac{u^2}{4} + C$

$$\boxed{4y^2 + x^4 = C}$$

Multiply and answer together

Q-13 $y^2 = 2cx + c^2$

$2y\frac{dy}{dx} = 2c + 0$

$c = yy'$

Put c back

$y^2 = 2yy'x + (yy')^2$ — ② Put $y' = -1/y'$

(Checking orthogonal)

$y^2 = -\frac{2y}{y'} + \frac{y^2}{y'^2} \Rightarrow y^2 = \frac{-2yy_1 + y^2}{y_1^2}$

$y^2y_1^2 = -2yy_1 + y^2$ — ①

Eq ① and Eq ② are same. Hence self orthogonal

Q-14 Slope $\frac{dy}{dx}$

$(y - y_1) = \frac{dy}{dx}(x - x_1)$ $\boxed{\text{Put } y_1 = 0 \mid x_1 = x/2}$

$-y = \frac{dy}{dx}\left(\frac{x}{2} - x\right)$

$2\int\frac{dx}{x} = \int\frac{dy}{y} \Rightarrow 2\ln x + C = \ln y$

$\Rightarrow y = cx^2$ Ans

Q-15 $\frac{dNi}{dt} \propto Ni(4000 - Ni)$

$\Rightarrow \frac{dNi}{Ni(4000-Ni)} = k\,dt$

$\Rightarrow \frac{1}{40000}\int \frac{dNi}{4000-Ni} + \int \frac{dNi}{Ni} = \int k\,dt$

$\Rightarrow \frac{Ni}{4000-Ni} = c\,e^{4000kt}$

Given $t=0$, $Ni=1$

$c = 1/3999$

$t=5$, $Ni = 1000$

$\Rightarrow e^{(5\times4000)k} = \frac{3999}{39}$

$t=10$

$\frac{Ni}{4000-Ni} = \frac{1}{3999} \times \frac{3999}{39} \times \frac{3999}{39}$

$1521\,Ni = 15996\times10^3 - 3999Ni$

$5520Ni = 15996\times10^3$

$\boxed{Ni = 2898}$

$\boxed{2898}$ Ans