DE Assignment 4 and 5 — Page 1

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Q1

$$\frac{dy}{dx} = \frac{x^2}{y^2+1} \quad y(0) = 0 \quad \text{at } x = .5 \text{ find } y$$

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)\,dx$$

$$= 0 + \int_0^x \frac{x^2}{0+1}\,dx = \left.\frac{x^3}{3}\right]_0^x = \frac{x^3}{3}$$

Now

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)\,dx$$

$$= 0 + \int_0^x \frac{x^2}{\left(\frac{x^3}{3}\right)^2+1}\,dx = \int \frac{9x^2}{x^6+9}\,dx$$

Let $x^3 = t$
$3x^2 dx = dt$

$$\Rightarrow y_2 = \int_0^x \frac{3\,dt}{t^2+3^2}$$

$$y_2 = \left.\tan^{-1}\left(\frac{t}{3}\right)\right]_0^x \Rightarrow y_2 = \left.\tan^{-1}\left(\frac{x^3}{3}\right)\right]_0^x$$

$$y_2 = \tan^{-1}\left(\frac{x^3}{3}\right)$$

when $x = 0.5$, $\tan y = \frac{x^3}{3}$

$$\Rightarrow y_2 = 2.385$$

Using Exact Method

$$\frac{dy}{dx} = \frac{x^2}{y^2+1}$$

$$\int (y^2+1)\,dy = \int x^2\,dx$$

$$\frac{y^3}{3} + y = \frac{x^3}{3} + c$$

$x=0, y=0, \Rightarrow c=0$

$$\frac{y^3}{3} + y = \frac{x^3}{3}$$

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from picard's method,

$$\tan y = \frac{x^3}{3}$$

Also $\tan y = y + \frac{y^3}{3} + \frac{2y^5}{15} + \cdots$

Compare

$$y + \frac{y^3}{3} + \underbrace{\frac{2y^5}{15} + \cdots}_{\text{Neglecting higher terms}} = \frac{x^3}{3}$$

$$y + \frac{y^3}{3} = \frac{x^3}{3}$$

After expanding $\tan y$ and neglecting higher term, we get exact solution.

So, same y in both solutions.

Q-2 Taylor series $\frac{dy}{dx} = 1 - 2xy$ $\quad y(0) = 0 \quad y(0.4) = ?$

$$y_1 = y_0 + \frac{h}{1!} y'(0) + \frac{h^2}{2!} y''(0) + \frac{h^3}{3!} y'''(0) + \cdots$$

Given $y_0 = 0$, $h = x - x_0 = 0.4$, $y' = 1 - 2xy$, $y'(0) = 1$, $y''(0) = 0$, $y'''(0) = -4$

Getting value at $x = 0.4$

$$y_1 = 0 + \frac{0.4(1)}{1!} + \frac{(0.4)^3}{3!}(-4)$$

$$= 0.357$$

Truncating error $\propto h^5$ since we have considered upto 4th deriv.

Exact

$y' = 1 - 2xy$

$\frac{dy}{dx} + y(2x) = 1$

$y \int e^{2x\,dx} = \int e^{x^2} dx$

$y e^{x^2} = \int 1 + \frac{x^2}{1!} + \frac{x^4}{2!} + \cdots dx$

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$$y \cdot e^{x^2} = x + \frac{x^3}{3!} + \frac{x^5}{5.2!} + \dots C$$

$y(0) = 0$, $C = 0$ find $x = 0.4$, y?

$$y e^{0.16} = 0.4 + \frac{(0.4)^3}{3!} + \frac{(0.4)^5}{5.2!} + \frac{(0.4)^7}{7.3!}$$

$y = 0.3602$

We got close values of y using both methods.

Q-3 Given $\frac{dy}{dx} = x + y + xy$ $y(0) = 1$ $y(0.1) = ?$
$h = 0.025$ (use Euler Method)

$y_1 = y_0 + h f(x_0 + y_0)$

$y_1 = 1 + 0.025[1] = 1.025$

$y_2 = y_1 + hf(x_0 + h, y_1)$

$= 1.025 + 0.025[f(0.025, 1.025)]$

$= 1.025 + 0.025(1.0756) = 1.025 + 0.0268$

$= 1.051$ at $x = 0.025$

$y_3 = y_2 + hf(x_0 + 2h, y_2)$

$= 1.0518 + 0.025[f(0.05, 1.05189)] = 1.0807$ at $x = 0.05$

$y_4 = y_3 + hf(x_0 + 3h, y_3)$

$= 1.0807 + 0.025[f(0.075, 1.0807)] = 1.1167$ at $x = 0.075$

$y_5 = y_4 + hf(x_0 + 4h, y_4)$

$= 1.116 + 0.025[f(0.1, 1.116)] = 1.1447$ at $x = 0.1$

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Exact Solution

$$\frac{dy}{dx} = x + y + xy$$

$$\frac{dy}{dx} - y(1+x) = x$$

$$y \cdot e^{-x - x^2/2} = \int e^{-x-\frac{x^2}{2}} dx$$

$$= \int \left(1 + \frac{\left(-x-\frac{x^2}{2}\right)}{1!} + \frac{\left(-x-\frac{x^2}{2}\right)^2}{2!} + \ldots\right) dx$$

$$= \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^6}{48} + \frac{x^5}{10} + C$$

$x = 0, y = 1 \Rightarrow \boxed{C = 1}$

At $x = 0.1$,

$$y e^{-0.105} = \frac{(0.1)^2}{2} - \frac{(0.1)^3}{3} + \frac{(0.1)^5}{10} + 1 + \frac{(0.1)^6}{48}$$

$\boxed{y = 1.1158}$ $\boxed{\text{Euler}} \rightarrow y = 1.1447$

We got similar values of y from both methods.

Modified Euler

Q-4 $y' = x + \sin y$ $y(0) = 1$, $h = 0.2$ $y(0.4) = ?$

x	y	y'	Mean Slope	$y_n = y_{n-1} + h(\text{Mean Slope})$
0	1	0.8414	-	$y_1 = 1.1682$
0.2	1.1682	1.12	0.9807	$y_1(1) = 1.1961$
0.2	1.1962	1.13	0.986	$y_1(2) = 1.1972$
0.2	1.972	1.131	0.9862	$y_1(3) = 1.1972$
0.2	1.972	1.131	-	$y_2 = 1.4234$
0.4	1.423	1.381	1.26	$y_2(1) = 1.4492$
0.4	1.4492	1.392	1.26	1.44956
0.4	1.4495	1.392	1.26	1.44956

Same

$y_2 = 1.44956$ | at $x = 0.4$ - Ans

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Q-5 $y' + y = 0$ $y(0) = 1$ $y(0.2) = ?$

$h = 0.1$

Using 3rd order Runge Kutta Method

$k_1 = h f(x_0, y_0)$

$k_2 = h f(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1)$

$k_3 = h f(x_0 + h, y_0 + 2k_2 - k_1)$

$y_1 = y_0 + \frac{1}{6}(k_1 + 4k_2 + k_3)$

Now

$k_1 = -0.1$

$k_2 = 0.1 f(0.05, 0.95)$

$= 0.095$

$k_3 = 0.1 f(0.1, 0.91)$

$= -0.091$

$y_1 = 1 + \frac{1}{6}(-0.1 - 0.38 - 0.091)$

$\boxed{= 0.90483}$ at $x = 0.1$

Now y_1 as y_0 and x_1 as x_0

$k_1 = h f(x_1, y_1)$

$= 0.1 f(0.1, 0.904) = -0.09$

$k_2 = -0.08595$

$k_3 = -0.082341$

$y_2 = y_1 + \frac{1}{6}(k_1 + 4k_2 + k_3)$

$\boxed{= 0.818760}$ at $x = 0.2$

Exact form $\frac{dy}{dx} = y$

$\ln y = -x + c$

$x = 0, y = 1$

$\Rightarrow c = 1$

$y = c e^{-x}$

$\boxed{y = e^{-x}}$

at $x = 0.2$

$y = e^{-0.2} = (0.818730)$

We got similar values of 'y' from both the methods

Q-6 $\frac{dy}{dx} = \sqrt{x+y}$ $y(0.4) = 0.41$ $h = 0.4$

4th order RK

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find $y(0.8)$

$K_1 = h f(x_0, y_0)$
$= 0.4 f(0.4, 0.4)$
$= 0.36$

$K_2 = h f(x_0 + h/2, y_0 + k_1/2)$
$= 0.4 f(0.6, 0.59) = 0.4363$

$K_3 = h f(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2})$
$= 0.4 f(0.6, 0.628)$
$= 0.4432$

$K_4 = h f(x_0 + h, y_0 + k_3)$
$= 0.4 f(0.8, 0.8532)$
$= 0.5143$

$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$
$= 0.41 + \frac{1}{6}(0.36 + 0.8726 + 0.88 + 0.51)$

$\boxed{= 0.8488}$ at $x = 0.8$

Exact Solution

$\frac{dy}{dx} = \sqrt{x+y}$ $x + y = t$

$\frac{dy}{dx} = \frac{dt}{dx} - 1$

$\frac{dt}{dx} - 1 = \sqrt{t}$

$\int \frac{dt}{1+\sqrt{t}} = \int dx$ $1 + \sqrt{t} = p$

$\frac{dt}{2\sqrt{t}} = dp$

$\int \frac{2(p-1)}{p} dp = x + c$

$2p - 2\log p = x + C$

$2(1+\sqrt{t}) - 2\log(1+\sqrt{t}) = x + C$

$2(1+\sqrt{x+y}) - 2\log(1+\sqrt{x+y}) = x + C$

At $y = 0.41$ and $x = 0.41$

$C = 2.8426$

find at $x = 0.8$

$2[(1+\sqrt{y+0.8} + \log(1+\sqrt{y+0.8})]$
$= x + 2.8426$

$\Rightarrow \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$

$\Rightarrow 2\left[1 + \sqrt{x+y} - \sqrt{x+y} + \left(\frac{x+y}{2}\right)\right] = x + 2.8426$

$\Rightarrow$ $y = 0.8426$

We got similar values of y

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Q-7 $\cancel{z = (\tilde{x} + \tilde{a})(\tilde{y} + b}$

$z = (x^2 + a^2)(y^2 + b^2) = x^2y^2 + x^2b^2 + a^2y^2 + a^2b^2$

$\frac{\partial z}{\partial x} = 2xy^2 + 2xb^2$

$\frac{\partial z}{\partial x} = 2x(y^2 + b^2)$ — ①

Now $\frac{\partial z}{\partial y} = 2yx^2 + 2ya^2$

$\frac{\partial z}{\partial y} = 2y(x^2 + a^2)$ — ②

Multiply ① and ②

$\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} = 4xy\,(x^2 + a^2)(y^2 + b^2)$

$$\boxed{\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} = 4xyz}$$

order = 1
degree = 2

It is non-linear degree is 2
It is homogeneous because every term has a dependant variable 'z'.

Q-8 $x + y + z = f(x^2 + y^2 + z^2)$

$1 + \frac{\partial z}{\partial x} = f'(x^2 + y^2 + z^2)\,2x$ — ①

$1 + \frac{\partial y}{\partial z} = f'(x^2 + y^2 + z^2)\,2y$ — ②

$\Rightarrow \cancel{y + y\frac{\partial z}{\partial y} = x + x}$

$\Rightarrow y + y\frac{\partial z}{\partial x} = x + x\frac{\partial z}{\partial y}$

$\Rightarrow \boxed{x\frac{dz}{dy} - y\frac{dz}{dx} = y - x}$ → reqd partial diff

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Q-9 (a) $\frac{\partial z}{\partial x} + \frac{\partial^2 z}{\partial y^2} = 2y$

→ It is linear

→ It is homogeneous.

(b) $\sqrt{x}\frac{\partial z}{\partial x} + \sqrt{y}\frac{\partial z}{\partial y} = \sqrt{z}$

→ Semi Linear and homogeneous

(c) $(x^2+y^2)\frac{\partial z}{\partial x} + z\frac{\partial z}{\partial y} = z^3 x + y$

→ QuasiLinear and non-homogeneous

(d) $\frac{\partial^2 z}{\partial x^2} + \left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right) = 2y$

→ Non Linear and homogeneous

(e) $\frac{\partial^2 z}{\partial x^2} + \left(\frac{\partial^2 z}{\partial y^2}\right)^3 = z + 2y$

→ Non Linear and non-homogeneous

Q-10 $\frac{\partial^2 z}{\partial x \partial y} = \sin x \sin y$ $\quad \frac{\partial z}{\partial y} = -2\sin y \quad x=0,\ z=0$

y is odd $\pi/2$

Integrate with x

$\frac{\partial z}{\partial y} = -\cos x \sin y + f(y)$

$-2\sin y = -\cos 0 \sin y + f(y)$ [substituting the given values]

$f(y) = -\sin y$

Also $\frac{\partial z}{\partial y} = -\cos x \sin y - \sin y$

Integrate with y

$z = -\cos x(-\cos y) + \cos y + g(x)$

$z = \cos x \cos y + \cos y + g(x)$

$0 = 0 + 0 + g(x)$ [given]

$\boxed{g(x) = 0}$

$\Rightarrow z = \cos x \cos y + \cos y$

$z = \cos y(\cos x + 1)$

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Q-11 $\frac{y^2 z}{x}\frac{\partial z}{\partial x} + xy\frac{\partial z}{\partial y} = y^2$

$\frac{\partial x}{P} = \frac{\partial y}{Q} = \frac{\partial z}{R} \Rightarrow P = \frac{y^2 z}{x}, \; Q = xz, \; R = y^2$

Auxiliary $\frac{\partial x}{y^2 z / x} = \frac{\partial y}{xz} = \frac{\partial z}{y^2}$

∂x and $\partial z \Rightarrow$

$\frac{x\,dx}{y^2 z} = \frac{\partial z}{y^2}$

$\int x\,dx = \int z\,dz$

$x^2/2 = z^2/2 + c$

$\boxed{x^2 - z^2 = c}$

∂x and ∂y

$\frac{x\,dx}{y^2 z} = \frac{dy\,\cancel{(y)}}{xyz}$

$\int x^2\,dx = \int y^2\,dy$

$\frac{x^3}{3} = \frac{y^3}{3} + c$

$\boxed{x^3 - y^3 = c}$

final solution $F(x^2 - y^2,\; x^3 - y^3) = 0$

Q-12 $z(z^2 + xy)(px - qy) = x^4$ $\qquad p = \frac{dz}{dx}, \; q = \frac{dz}{dy}$

$(z^3 + xyz)(px - qy) = x^4$

$(xz^3 + x^2yz)\frac{dz}{dx} - (yz^3 + xy^2z)\frac{\partial z}{\partial y} = x^4$

Auxiliary $\frac{\partial x}{x(z^3 + xyz)} = \frac{\partial y}{-y(z^3 + xyz)} = \frac{\partial z}{x^4}$

∂x and ∂y

$\frac{\partial x}{x(z^3 + xyz)} = \frac{\partial y}{-y(z^3 + xyz)}$

$\frac{dx}{x} = -\frac{dy}{y}$

$\ln x = -\ln y + c$

$\boxed{xy = c}$

∂x and ∂z

$\frac{dx}{x(z^3 + xyz)} = \frac{\partial z}{x^4}$

$\frac{dx}{(z^3 + c_1 z)x} = \frac{dz}{x^4}$

$\int x^3\,dx = \int (z^3 + c_1 z)\,dz$

$\frac{x^4}{4} = \frac{z^4}{4} + \frac{xyz^2}{2} + c_2$

$\boxed{\text{final } F(xy,\; x^4 - z^4 - 2xyz^2) = 0}$

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Q-13 $x(y^2-z^2)\frac{dz}{\partial x} + y(z^2-x^2)\frac{dz}{\partial y} = z(x^2-y^2)$

$$\frac{\partial x}{x(x^2-y^2)} = \frac{\partial y}{y(z^2-x^2)} = \frac{\partial z}{z(x^2-y^2)}$$

$$\frac{\partial x}{p} = \frac{\partial y}{q} = \frac{\partial z}{r} = \frac{P_1 dx + q_1 dy + r_1 dz}{pP_1 + q q_1 + r r_1}$$

Such that $pP_1 + q q_1 + r r_1 = 0 \Rightarrow \int P_1 dx + \int Q_1 dy + \int r_1 dz = C$

$P = x(y^2-z^2)$, $q = y(z^2-x^2)$,
$r = z(x^2-y^2)$

$P_1 = x$, $y = q_1$, $r_1 = z$

$\Rightarrow \int P_1 dx + \int q_1 dy + \int r_1 dz = C$
$\int x dx + \int y dy + \int z dz = C$
$\boxed{x^2+y^2+z^2 = C}$

then

Set 2 $\quad P_1 = yz$, $q_1 = xz$, $r_1 = xy$
$\int yz dx + \int xz dy + \int xy dz = 0$
$\boxed{3xyz = C}$

final solution $F(x^2+y^2+z^2, xyz) = 0$