Lecture III : Understanding & Measuring a Qubit

26/07/19

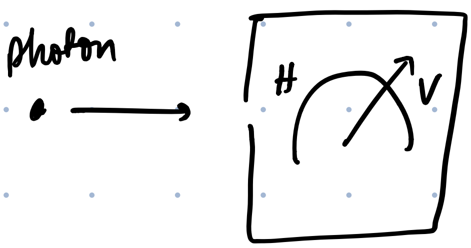
binary bits = {0, 1}

↳ can be represented by a stream of 10s of millions of electrons (say low and high voltages).

electron "spin" = {"up", "down"}
(0) (1)

photon polarisation = {"horizontal", "vertical"}
($\leftrightarrow$) ($\updownarrow$)

"measuring" device for polarisation



H ← say 0
V ← say 1

QM. Law I:

- If a "particle" can be in one of 2 basic states $|0\rangle$ or $|1\rangle$, it can also be in a superposition of these states
  ↳ ie α amplitude on $|0\rangle$ and β on $|1\rangle$ where α and β are complex numbers satisfying
  $$|\alpha|^2 + |\beta|^2 = 1$$
- Such a particle is a QUBIT

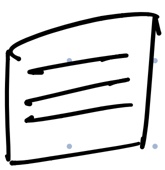
eg: photon with amplitude → 0.8 on $|0\rangle$ and 0.6 on $|1\rangle$
→ 0.8 on $|0\rangle$ and −0.6 on $|1\rangle$

QM. Law II :

- For a particle with α amp on $|0\rangle$ and β amp on $|1\rangle$ if it is measured, the readout shows $|0\rangle$ with prob. $|\alpha|^2$ and $|1\rangle$ with prob. $|\beta|^2$
- measuring device (like a black box)
  ↳ you know it when you see it
- If the readout showed $|0\rangle$, the particle state changes to 1 amp on $|0\rangle$ and 0 on $|1\rangle$ and vice versa.

Storing 2 numbers → $f(0) = \alpha$ and $f(1) = \beta$ (stored in QUBITs)
↳ 1 QUBIT stores 2 numbers

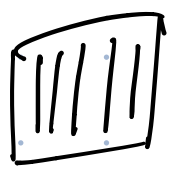
{Quantum Zeno effect}
↳ hold the state of a particle by repeatedly measuring it



horizontal filter

① measures photon
↳ $|0\rangle$ or $|1\rangle$

② if measurement was
$|0\rangle \equiv \leftrightarrow$
photon flies through
else : photon $\longrightarrow$ heat



vertical filter

① measures photon
↳ $|0\rangle$ or $|1\rangle$

② if measurement was
$|0\rangle \equiv \updownarrow$
photon flies through
else : photon $\longrightarrow$ heat

Particle can have 3 possible states too :

↳ $|1\rangle, |2\rangle, |3\rangle$: qutrit

4 possible states :

↳ $|1\rangle, |2\rangle, |3\rangle, |4\rangle$: qudit (for $d \geq 4$)

2 photons :

$\leftrightarrow \; \leftrightarrow$: $|00\rangle \equiv |1\rangle$

$\leftrightarrow \; \updownarrow$: $|01\rangle \equiv |2\rangle$

$\updownarrow \; \leftrightarrow$: $|10\rangle \equiv |3\rangle$

$\updownarrow \; \updownarrow$: $|11\rangle \equiv |4\rangle$

<u>QM. law I for a qudit</u> :

ampl α_1 on $|1\rangle$, α_2 on $|2\rangle$ .... α_d on $|d\rangle$

then $\sum_d |\alpha_k|^2 = 1$

<u>QM. law II for a qudit</u> :

reads out $|i\rangle$ with prob. $|\alpha_i|^2$

and then states change to $\alpha_i = 1$, $\alpha_j = 0$; $j \neq i$

$$\text{Qudit state} \equiv \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \vdots \\ \alpha_d \end{bmatrix} \in \mathbb{C}^d \qquad \leftarrow \vec{V}$$

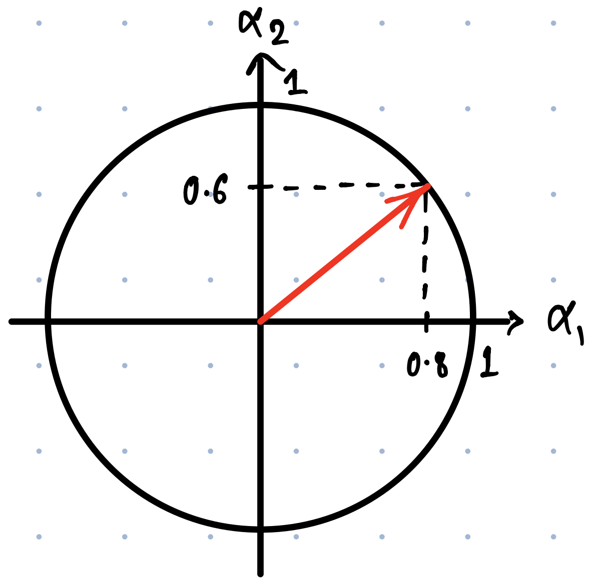
unit, column vector

unit vector

∵ $\sum_d |\alpha_i|^2 = 1$

Let a qubit be

$$\begin{bmatrix} 0.8 \\ 0.6 \end{bmatrix} \equiv$$



every vector on this unit circle is a qubit
(but here only real amplitudes are being considered)

CONFUSION:

- One qubit defined by 2 complex numbers
- One complex number $\leftarrow$ 2 real numbers

Math review:

- $z = x + iy$

$$\|z\|^2 = x^2 + y^2 = (x+iy)(x-iy) = z \cdot z^*$$

- unit vector : $1 = \|\vec{v}\|^2 = \langle \vec{v}, \vec{v} \rangle$

inner product (dot)

- inner product (real numbers) : $\|\vec{u}\| \cdot \|\vec{v}\| \cdot \cos(\theta) = \langle \vec{u}, \vec{v} \rangle$

$$= u_1 v_1 + u_2 v_2 + \dots + u_d v_d$$

- $\langle \vec{u}, \vec{v} \rangle = [u_1^* \ u_2^* \ \ldots \ u_d^*] \cdot \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_d \end{bmatrix} = u^\dagger v$

Inner product for complex vectors:

$$\langle \vec{u}, \vec{v} \rangle = u_1^* v_1 + u_2^* v_2 + \ldots + u_d^* v_d$$

$$= u^\dagger v$$

dagger : conjugate + transpose

Dirac's Bra-Ket Notation: (for vectors)

- high school : $\vec{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \vec{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \vec{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$
- college : $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}; \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \ldots ; \vec{e}_d \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$

  ↳ d standard basis vectors for d-dimensions
- quantum : $\vec{e}_1 \equiv |1\rangle ; \vec{e}_2 \equiv |2\rangle \ldots ; \vec{e}_d \equiv |d\rangle$

<u>exception</u> (for $d=2$)

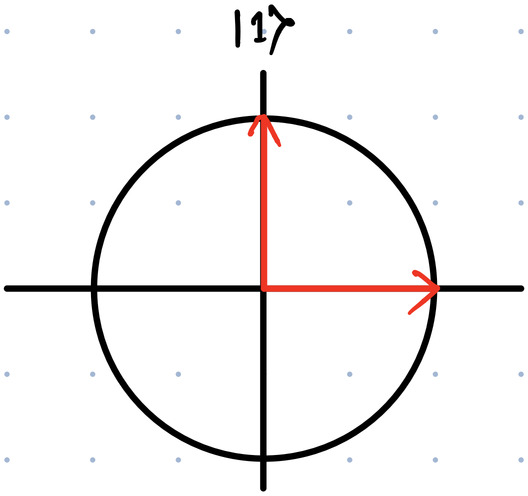
↳ we $|0\rangle$ and $|1\rangle$

$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\quad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

only for a qubit

qubit state $= \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$

$= \alpha |0\rangle + \beta |1\rangle$



$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

<u>notation:</u> $|\text{blah}\rangle$ ← column vector named "blah"!

$|\quad\rangle$ type = col. vector

"ket"

<u>notation:</u> $\langle \text{blah}|$ ← conjugate transpose of $|\text{blah}\rangle$

"bra"

$\langle \text{blah}| \equiv |\text{blah}\rangle^{\dagger}$

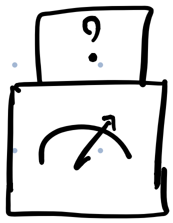
<u>notation:</u> $\langle \vec{u}, \vec{v} \rangle = u^{\dagger} v = \langle u| \cdot |v\rangle$

$=$ "$\langle u|v\rangle$"

eg qubits:

① $\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle \equiv$ "$|+\rangle$"

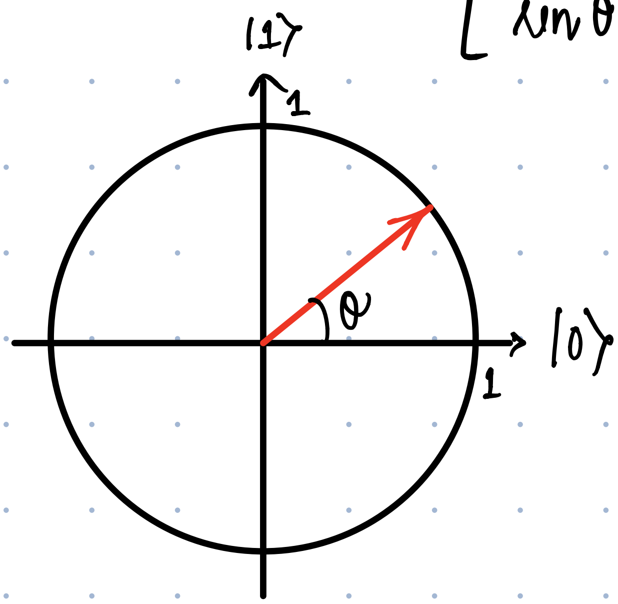
② $\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle \equiv$ "$|-\rangle$"

$|\pm\rangle \longrightarrow$ 

: $\begin{cases} |0\rangle \text{ with prob } 1/2 \\ |1\rangle \text{ with prob } 1/2 \end{cases}$

$|\psi\rangle \leftarrow$ traditional notation for a qubit

$\hookrightarrow |\psi\rangle = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} = \cos(\theta)\cdot|0\rangle + \sin(\theta)\cdot|1\rangle$



$\therefore \Pr[\text{see } |0\rangle] = \cos^2(\theta)$

$\therefore \Pr[\text{see } |1\rangle] = \sin^2(\theta)$

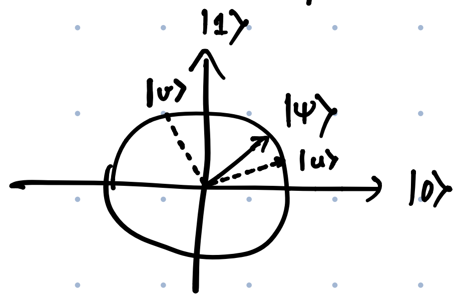
$$|\psi\rangle = \underline{\qquad\quad}\,|0\rangle + \underline{\qquad\quad}\,|1\rangle$$

$\langle 1|\psi\rangle$

$$\left\langle |\psi\rangle, \begin{bmatrix}1\\0\end{bmatrix}\right\rangle = \langle 0|\psi\rangle$$

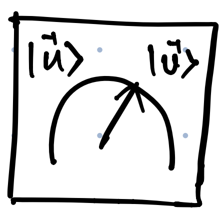
$$\therefore \Pr\left[\text{measure } \text{“}|0\rangle\text{”}\right] = \left|\langle 0|\psi\rangle\right|^2$$

Measuring in a diff. basis:



For any orthonormal basis (ie. any pair of perpendicular unit vectors), a measuring device can be built for this basis.

say $\{\vec{u}, \vec{v}\}$

$|\psi\rangle \longrightarrow$ 

$$\therefore \Pr\left[\text{measure } \text{“}|\vec{u}\rangle\text{”}\right] = |\langle u|\psi\rangle|^2$$

$$\Pr\left[\text{measure } \text{“}|\vec{v}\rangle\text{”}\right] = |\langle v|\psi\rangle|^2$$

After measurement, state changes to measured state